Michael Tsamparlis

Special Relativity

An Introduction with 200 Problems and Solutions

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Omnia mea mecum fero
Whatever I possess I bear with me

Preface

Writing a new book on the classic subject of Special Relativity, on which numerous important physicists have contributed and many books have already been written, can be like adding another epicycle to the Ptolemaic cosmology. Furthermore, it is our belief that if a book has no new elements, but simply repeats what is written in the existing literature, perhaps with a different style, then this is not enough to justify its publication. However, after having spent a number of years, both in class and research with relativity, I have come to the conclusion that there exists a place for a new book. Since it appears that somewhere along the way, mathematics may have obscured and prevailed to the degree that we tend to teach relativity (and I believe, theoretical physics) simply using "heavier" mathematics without the inspiration and the mastery of the classic physicists of the last century. Moreover current trends encourage the application of techniques in producing quick results and not tedious conceptual approaches resulting in long-lasting reasoning. On the other hand, physics cannot be done á la carte stripped from philosophy, or, to put it in a simple but dramatic context

> A building is not an accumulation of stones!

As a result of the above, a major aim in the writing of this book has been the distinction between the mathematics of Minkowski space and the physics of relativity. This is necessary for one to understand the physics of the theory and not stay with the geometry, which by itself is a very elegant and attractive tool. Therefore in the first chapter we develop the mathematics needed for the statement and development of the theory. The approach is limited and concise but sufficient for the purposes it is supposed to serve. Having finished with the mathematical concepts we continue with the foundation of the physical theory. Chapter 2 sets the framework on the scope and the structure of a theory of physics. We introduce the principle of relativity and the covariance principle, both principles being keystones in every theory of physics. Subsequently we apply the scenario first to formulate Newtonian Physics (Chap. 3) and then Special Relativity (Chap. 4). The formulation of Newtonian Physics is done in a relativistic way, in order to prepare the ground for a proper understanding of the parallel formulation of Special Relativity.

Having founded the theory we continue with its application. The approach is systematic in the sense that we develop the theory by means of a stepwise introduction of new physical quantities. Special Relativity being a kinematic theory forces us to consider as the fundamental quantity the position four-vector. This is done in Chap. 5 where we define the relativistic measurement of the position four-vector by means of the process of chronometry. To relate the theory with Newtonian reality, we introduce rules, which identify Newtonian space and Newtonian time in Special Relativity.

In Chaps. 6 and 7 we introduce the remaining elements of kinematics, that is, the four-velocity and the four-acceleration. We discuss the well-known relativistic composition law for the three-velocities and show that it is equivalent to the Einstein relativity principle, that is, the Lorentz transformation. In the chapter of four-acceleration we introduce the concept of synchronization which is a key concept in the relativistic description of motion. Finally, we discuss the phenomenon of acceleration redshift which together with some other applications of four-acceleration shows that here the limits of Special Relativity are reached and one must go over to General Relativity.

After the presentation of kinematics, in Chap. 8 we discuss various paradoxes, which play an important role in the physical understanding of the theory. We choose to present paradoxes which are not well known, as for example, it is the twin paradox.

In Chap. 9 we introduce the (relativistic) mass and the four-momentum by means of which we distinguish the particles in massive particles and luxons (photons).

Chapter 10 is the most useful chapter of this book, because it concerns relativistic reactions, where the use of Special Relativity is indispensible. This chapter contains many examples in order to familiarize the student with a tool, that will be necessary to other major courses such as particle physics and high energy physics.

In Chap. 11 we commence the dynamics of Special Relativity by the introduction of the four-force. We discuss many practical problems and use the tetrahedron of Frenet–Serret to compute the generic form of the four-force. We show how the well-known four-forces comply with the generic form.

In Chap. 12 we introduce the concept of covariant decomposition of a tensor along a vector and give the basic results concerning the $1 + 3$ decomposition in Minkowski space. The mathematics of this chapter is necessary in order to understand properly the relativistic physics. It is used extensively in General Relativity but up to now we have not seen its explicit appearance in Special Relativity, even though it is a powerful and natural tool both for the theory and the applications.

Chapter 13 is the next pillar of Special Relativity, that is, electromagnetism. We present in a concise way the standard vector form of electromagnetism and subsequently we are led to the four formalism formulation as a natural consequence. After discussing the standard material on the subject (four-potential, electromagnetic field tensor, etc.) we continue with lesser known material, such as the tensor formulation of Ohm's law and the $1+3$ decomposition of Maxwell's equations. The reason why we introduce these more advanced topics is that we wish to prepare the student for courses on important subjects such as relativistic magnetohydrodynamics (RMHD).

The rest of the book concerns topics which, to our knowledge, cannot be found in the existing books on Special Relativity yet. In Chap. 14 we discuss the concept of spin as a natural result of the generalization of the angular momentum tensor in Special Relativity. We follow a formal mathematical procedure, which reveals what "the spin is" without the use of the quantum field theory. As an application, we discuss the motion of a charged particle with spin in a homogeneous electromagnetic field and recover the well-known results in the literature.

Chapter 15 deals with the covariant Lorentz transformation, a form which is not widely known. All four types of Lorentz transformations are produced in covariant form and the results are applied to applications involving the geometry of three-velocity space, the composition of Lorentz transformations, etc.

Finally, in Chap. 16 we study the reaction $A + B \longrightarrow C + D$ in a fully covariant form. The results are generic and can be used to develop software which will solve such reactions directly, provided one introduces the right data.

The book includes numerous exercises and solved problems, plenty of which supplement the theory and can be useful to the reader on many occasions. In addition, a large number of problems, carefully classified in all topics accompany the book.

The above does not cover all topics we would like to consider. One such topic is relativistic waves, which leads to the introduction of De Broglie waves and subsequently to the foundation of quantum mechanics. A second topic is relativistic hydrodynamics and its extension to RMHD. However, one has to draw a line somewhere and leave the future to take care of things to be done.

Looking back at the long hours over the many years which were necessary for the preparation of this book, I cannot help feeling that, perhaps, I should not have undertaken the project. However, I feel that it would be unfair to all the students and colleagues, who for more that 30 years have helped me to understand and develop the relativistic ideas, to find and solve problems, and in general to keep my interest alive. Therefore the present book is a collective work and my role has been simply to compile these experiences. I do not mention specific names – the list would be too long, and I will certainly forget quite a few – but they know and I know, and that is enough.

I close this preface, with an apology to my family for the long working hours; that I was kept away from them for writing this book and I would like to thank them for their continuous support and understanding.

Athens, Greece Michael Tsamparlis
October 2009

CONSEIL DE PHYSIQUE SOLVAY

BRUXELLES 1911

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Chapter 1
Mathematical Part

1.1 Introduction

As will become clear from the first chapters of the book, the theories of physics which study motion have a common base and structure and they are not independent and unrelated considerations, which at some limit simply produce the same numerical results. The differentiation between the theories of motion is either due to the mathematical quantities they use or in the way they describe motion or both.

The Theory of Special Relativity was the first theory of physics which introduced different mathematics from those of Newtonian Physics and a new way of describing motion. A result of this double differentiation was the creation of an obscurity concerning the "new"mathematics and the "strange"physical considerations, which often led to mistaken understandings of both.

For this reason the approach in this book is somewhat different from the one usually followed in the literature. That is, we present first the necessary mathematics per se without any reference to the physical ideas. Then the physical principles of the physical theory of Special Relativity are stated and the theory is developed conceptually. Finally the interrelation of the two parts is done via the position vector and the description of motion in spacetime. In this manner the reader avoids the "paradoxes"and other misunderstandings resulting partially from the "new" mathematics and partially from remnants of Newtonian ideas in the new theory.

Following the above approach, in the first chapter we present, in a concise manner, the main elements of the mathematical formalism required for the development and the discussion of the basic concepts of Special Relativity. The discussion gives emphasis to the geometric role of the new geometric objects and their relation to the mathematical consistency of the theory rather than to the formalism. Needless to say that for a deeper understanding of the Theory of Special Relativity and the subsequent transition to the Theory of General Relativity, or even to other more specialized areas of relativistic physics, it is necessary that the ideas discussed in this chapter be enriched and studied in greater depth.

The discussion in this chapter is as follows. At first we recall certain elements from the theory of linear spaces and coordinate transformations. We define the concepts of dual space and dual basis. We consider the linear coordinate transformations

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and the group $GL(n, R)$, whose action preserves the linear structure of the space. Subsequently we define the inner product and produce the general isometry equation (orthogonal transformations) in a real linear metric space. Up to this point the discussion is common for both Euclidean geometry and Lorentzian geometry (i.e., Special Relativity). The differentiation starts with the specification of the inner product. The Euclidean inner product defines the Euclidean metric, the Euclidean space, and introduces the Euclidean Cartesian coordinate systems, the Euclidean orthogonal transformations, and finally the Euclidean tensors. Similarly the Lorentz inner product defines the Lorentz metric, the Minkowski space or spacetime (of Special Relativity), the Lorentz Cartesian coordinate systems, the Lorentz transformations, and finally the Lorentz tensors. The parallel development of Newtonian Physics and the Theory of Special Relativity in their mathematical and physical structure is at the root of our approach and will be followed throughout the book.

1.2 Elements From the Theory of Linear Spaces

Although the basic notions of linear spaces are well known it would be useful to refer to some basic elements from an angle suitable to the physicist. In the following we consider a linear (real) space of dimension three, but the results apply to any (real) linear space of finite dimension. The elements of a linear space V^3 can be described in terms of the elements of the linear space $\mathbf{R}^3$ if we define in V^3 a *basis*. Indeed if $\{\mathbf{e}_\mu\}$ $(\mu = 1, 2, 3)$ is a basis in V^3 then the vector $\mathbf{v} \in V^3$ can be written as

$$\mathbf{v} = \sum_{\mu=1}^{3} v^\mu \mathbf{e}_\mu, \tag{1.1}$$

where $v^\mu \in R$ $(\mu = 1, 2, 3)$ are the components of $\mathbf{v}$ in the basis $\{\mathbf{e}_\mu\}$. If in the space V^3 there are functions $\{x^\mu\}$ $(\mu = 1, 2, 3)$ such that $\frac{\partial}{\partial x^\mu} = \mathbf{e}_\mu$ $(\mu = 1, 2, 3)$ then the functions $\{x^\mu\}$ are called *coordinate functions* in V^3 and the basis $\{\mathbf{e}_\mu\}$ is called a *holonomic basis*. In V^n there are holonomic and non-holonomic bases. In the following by the term basis we shall always mean a holonomic basis. Furthermore the Greek indices $\mu, \nu, \ldots$ take the values 1, 2, 3 except if specified differently.

1.2.1 Coordinate Transformations

In a linear space there are infinitely many coordinate systems or, equivalently, bases. If $\{\mathbf{e}_\mu\}$,$\{\mathbf{e}_{\mu'}\}$ are two bases, a vector $\mathbf{v} \in V^3$ is decomposed as follows:

$$\mathbf{v} = \sum_{\mu=1}^{3} v^\mu \mathbf{e}_\mu = \sum_{\mu'=1}^{3} v^{\mu'} \mathbf{e}_{\mu'}, \tag{1.2}$$

where v^{μ}, $v^{\mu'}$ are the components of $\mathbf{v}$ in the bases $\{\mathbf{e}_{\mu}\}$, $\{\mathbf{e}_{\mu'}\}$, respectively. The bases $\{\mathbf{e}_{\mu}\}$, $\{\mathbf{e}_{\mu'}\}$ are related by a *coordinate transformation* or change of basis defined by the expression

$$\mathbf{e}_{\mu'} = \sum_{\mu=1}^{3} a^{\mu}_{\mu'} \mathbf{e}_{\mu} \qquad \mu' = 1, 2, 3. \tag{1.3}$$

The nine numbers $a^{\mu}_{\mu'}$ define a 3×3 matrix $A = (a^{\mu}_{\mu'})$ whose determinant does not vanish. The non-singular matrix A is called the *transformation matrix* from the basis $\{\mathbf{e}_{\mu}\}$ to the basis $\{\mathbf{e}_{\mu'}\}$. The matrix A is not in general symmetric and we must consider a convention as to which index of $a^{\mu}_{\mu'}$ counts rows and which columns. In this book we make the following convention concerning the indices of matrices:

Notation 1.21 (Convention of matrix indices) *(1) We shall write the basis vectors as the* 1×3 *matrix* $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \equiv [\mathbf{e}_{\mu}]$. *In general the lower indices will count columns.*

(2) We shall write the components v^1, v^2, v^3 *of a vector* $\mathbf{v}$ *in the basis* $[\mathbf{e}_{\mu}]$ *as the* 3×1 *matrix:*

$$\begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix} = [v^{\mu}]_{\{\mathbf{e}_{\mu}\}} = [v^{\mu}].$$

In general the upper indices will count rows.

According to the above convention we write the 3×3 matrix $A = [a^{\mu}_{\mu'}]$ as follows:

$$A = \begin{pmatrix} a^1_{1'} & a^1_{2'} & a^1_{3'} \\ a^2_{1'} & a^2_{2'} & a^2_{3'} \\ a^3_{1'} & a^3_{2'} & a^3_{3'} \end{pmatrix}. \tag{1.4}$$

In this notation the vector $\mathbf{v}$ is written as a product of matrices as follows:

$$\mathbf{v} = [\mathbf{e}_{\mu}][v^{\mu}].$$

We note that this form is simpler than the previous $\mathbf{v} = \sum_{\mu=1}^{3} v^{\mu}\mathbf{e}_{\mu}$ because it does not have the $\sum$ symbol. However, it is still elaborate because of the brackets of the matrices. In order to save writing space and time and make the expressions simpler and more functional, we make the following further convention:

Notation 1.22 (Einstein's summation convention) *When in a mononym (i.e., a simple term) an index is repeated as upper index and lower index, then it will be* *understood that the index is summed over all its values and will be called a dummy index. If we do not want summation over a particular repeated index, then we must specify this explicitly.*

Therefore instead of $\sum_{\mu=1}^{3} a^{\mu} b_{\mu}$ we shall simply write $a^{\mu} b_{\mu}$. According to the convention of Einstein relation (1.3) is written as

$$\mathbf{e}_{\mu'} = a^{\mu}_{\mu'} \mathbf{e}_{\mu}. \tag{1.5}$$

Similarly for the vector $\mathbf{v}$ we have $\mathbf{v} = v^{\mu} \mathbf{e}_{\mu} = v^{\mu'} \mathbf{e}_{\mu'} = v^{\mu'} a_{\mu'}^{\ \mu} \mathbf{e}_{\mu}$ which gives

$$v^{\mu} = a^{\mu}_{\ \mu'} v^{\mu'}.$$

We note that the matrices of the coordinates transform differently from the matrices of the bases (in the left-hand side we have v^{μ} and not $v^{\mu'}$). This leads us to consider two types of vector quantities: the *covariant* with low indices (which transform as the matrices of bases) and the *contravariant* with upper indices (which transform as the matrices of the coordinates). Furthermore we name the upper indices *contravariant indices* and the lower indices *covariant indices*.

The sole difference between covariant and contravariant indices is their behavior under successive coordinate transformations. Indeed let $(a^{\mu}_{\ \mu'})$, $(a^{\mu'}_{\ \mu''})$ be two successive changes of basis. The composite transformation $(a^{\mu}_{\ \mu''})$ is defined by the product of matrices:

$$[a^{\mu}_{\ \mu''}] = [a^{\mu}_{\ \mu'}][a^{\mu'}_{\ \mu''}]$$

or simply

$$a^{\mu}_{\ \mu''} = a^{\mu}_{\ \mu'} a^{\mu'}_{\ \mu''}. \tag{1.6}$$

Concerning the bases we have, in a profound notation,

$$\begin{gathered} [\mathbf{e}'] = [\mathbf{e}]A \ , \ [\mathbf{e}''] = [\mathbf{e}']A' \Rightarrow \\ [\mathbf{e}''] = [\mathbf{e}]AA' \end{gathered} \tag{1.7}$$

whereas for the coordinates,

$$\begin{gathered} [v] = A[v'], \quad [v'] = A'[v''] \Rightarrow \\ [v''] = A'^{-1}A^{-1}[v] = (AA')^{-1}[v]. \end{gathered} \tag{1.8}$$

Relations (1.7) and (1.8) show the difference in the behavior of the two types of indices under composition of coordinate transformations.

Let V^3 be a linear space and V^{3*} the set of all *linear* maps U of V^3 into R:

$$U(a\mathbf{u} + b\mathbf{v}) = aU(\mathbf{u}) + bU(\mathbf{v}) \qquad \forall a, b \in R,\ \mathbf{u}, \mathbf{v} \in V^3.$$

The set V^{3*} becomes a linear space if we define the R-linear operation as

$$(aU + bV)(\mathbf{u}) = aU(\mathbf{u}) + bV(\mathbf{u}) \qquad \forall a, b \in R,\ \mathbf{u} \in V^3, U, V \in V^{3*}.$$

This new linear space V^{3*} will be called the *dual space* of V^3. The dimension of V^{3*} equals the dimension of V^3. In every basis $[\mathbf{e}_\mu]$ of V^3 there corresponds a unique basis of V^{3*}, which we call the *dual basis* of $[\mathbf{e}_\mu]$, write[1] as $[\mathbf{e}^\mu]$, and define as follows:

$$\mathbf{e}^\mu(\mathbf{e}_\nu) = \delta^\mu_\nu,$$

where δ^μ_ν is the delta of Kronecker. It is easy to show that to every coordinate transformation $A = [\alpha^\nu_{\mu'}]$ of V^3 there corresponds a unique coordinate transformation of V^{3*}, which is represented by the inverse matrix $A^{-1} = [B^{\mu'}_\nu]$. We agree to write this matrix as $\alpha^{\mu'}_\nu$. Then the corresponding transformation of the dual basis is written as

$$\mathbf{e}^{\mu'} = \alpha^{\mu'}{}_\nu \mathbf{e}^\nu.$$

As a result of this last convention we have the following "orthogonality relation" for the matrix of a coordinate transformation A:

$$\alpha^\mu{}_{\mu'} \alpha^{\mu'}{}_{\mu''} = \delta^\mu_{\mu''}.$$

As we remarked above in a linear space there are linear and non-linear coordinate transformations. The linear transformations $f : V^3 \to V^3$ are defined as follows:

$$f(a\mathbf{u} + b\mathbf{v}) = af(\mathbf{u}) + bf(\mathbf{v}) \qquad a, b \in R,\ \mathbf{u},\ \mathbf{v} \in V^3.$$

They preserve the linear structure of V^3 and they are described by matrices with constant coefficients (i.e., real numbers). Geometrically they preserve the straight lines and the planes of V^3. The matrices defining the non-linear transformations are not constant and do not preserve (in general) the straight lines and the planes.

As a rule in the following we shall consider the linear transformations only.[2] We shall write the set of all linear transformations of V^3 as $L(V^3)$.

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[1] The notation of the dual basis with the same symbol as the corresponding basis of V^3 but with upper index is justified by the fact that the dual space of the dual space is the initial space, that is, $(V^{3*})^* = V^3$. Therefore we need only two positions in order to differentiate the bases and this is achieved with the change of the position of the corresponding indices.

[2] However, we shall consider non-linear transformations when we study the four-acceleration (see Sect. 7.11).

In a basis $[\mathbf{e}_\mu]$ of V^3 a linear map $f \in L(V^3)$ is represented by a 3×3 matrix $(f^{\mu'}_{\ \nu})$, which we call the *representation* of f in the basis $[\mathbf{e}_\mu]$. There are two ways to relate the matrix $(f^{\mu'}_{\ \nu})$ with a transformation: either to consider that $(f^{\mu'}_{\nu})$ defines a transformation of coordinates $[v^{\mu'}] \to [v^\mu]$ or to assume that it defines a transformation of bases $[\mathbf{e}_\mu] \to [\mathbf{e}_{\mu'}]$. In the first case we say that we have an *active interpretation* of the transformation and in the second case a *passive interpretation* of the transformation. For finite-dimensional spaces the two interpretations are equivalent. In the following we shall follow the trend in the literature and select the active interpretation.

The set $L(V^3)$ of all linear maps of a linear space V^3 into itself becomes a linear space of dimension $3^2 = 9$, if we define the operation

$$(\lambda f + \mu g)(\mathbf{v}) = \lambda f(\mathbf{v}) + \mu g(\mathbf{v}) \qquad \forall f,\ g \in L(V^3),\ \lambda,\ \mu \in R,\ \mathbf{v} \in V^3.$$

Furthermore the set $L(V^3)$ has the structure of a group with operation the composition of transformations (defined by the multiplication of the representative matrices).[3] We denote this group by $GL(3, R)$ and call the *general linear group* in three-dimensions.

1.3 Inner Product – Metric

The general linear transformations are not useful in the study of linear spaces in practice, because they describe nothing more but the space itself. For this reason we use the space as the substratum onto which we define various geometric (and non-geometric) mathematical structures, which can be used in various applications. These new structures inherit the linear structure of the background space in the sense (to be clarified further down) that they transform in a definite manner under the action of certain subgroups of transformations of the general linear group $GL(3, R)$.

The most fundamental new structure on a linear space is the *inner product* and is defined as follows:

Definition 1 A map $\rho : V^3 \times V^3 \to R$ is an inner product on the linear space V^3 if it satisfies the following properties:

α. $\rho(\mathbf{u}, \mathbf{v}) = \rho(\mathbf{v}, \mathbf{u})$.

β. $\rho(\mu\mathbf{u} + \nu\mathbf{v}, \mathbf{w}) = \mu\rho(\mathbf{u}, \mathbf{w}) + \nu\rho(\mathbf{v}, \mathbf{w}),\ \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V^3,\ \mu,\ \nu \in R.$

---

[3] A group G is a set of elements $(a, b, \ldots)$, in which we have defined a binary operation "$\circ$" which satisfies the following relations:

a) $\forall b, c \in G$ the element $b \circ c \in G$.

b) There exists a unique unit element $e \in G$ such that $\forall a \in G : e \circ a = a \circ e = a$.

c) $\forall b \in G$ there exists a unique element $c \in G$ such that $b \circ c = c \circ b = e$. We call the element c the inverse of b and write it as b^{-1}.

Obviously $\rho(\ ,\ )$ is symmetric[4] and R-linear. A linear space endowed with an inner product will be called a *metric space* or a space with a metric. We shall indicate the inner product in general with a dot $\rho(\mathbf{u},\mathbf{v}) = \mathbf{u}\cdot\mathbf{v}$. The length of the vector $\mathbf{u}$ with reference to the inner product "$\cdot$" is defined by the relation $u^2 \equiv \mathbf{u}\cdot\mathbf{u} = \rho(\mathbf{u},\mathbf{u})$.

In a linear space it is possible to define many inner products. The inner product for which $u^2 > 0\ \forall\mathbf{u} \in V^3$ and $u^2 = 0 \Rightarrow \mathbf{u} = \mathbf{0}$ we call the *Euclidean inner product* and denote by $\cdot_E$ or by a simple dot if it is explicitly understood. The Euclidean inner product is unique in the property $u^2 > 0\ \forall\mathbf{u} \in V^3 - \{\mathbf{0}\}$. In all other inner products the length of a vector can be positive, negative, or zero.

To every inner product in V^3 we can associate in each basis $[\mathbf{e}_\mu]$ of V^3 the 3×3 symmetric matrix:

$$g_{\mu\nu} \equiv \mathbf{e}_\mu \cdot \mathbf{e}_\nu. \tag{1.9}$$

We call the matrix $g_{\mu\nu}$ the *representation of the inner product* in the basis $[\mathbf{e}_\mu]$. The inner product will be called *non-degenerate* if $\det[g_{\mu\nu}] \neq 0$. We assume all inner products in this book to be non-degenerate.

A basis $[\mathbf{e}_\mu]$ of V^3 will be called *g-Cartesian* or *g-orthonormal* if the representation of the inner product in this basis is $g_{\mu\nu} = \pm 1$. Obviously there are infinite g-orthonormal bases for every inner product. We have the following result.

Proposition 1 *For every (non-degenerate) inner product there exist g-orthonormal bases (Gram – Schmidt Theorem) and furthermore the number of* $+1$ *and* -1 *is the same for every g-orthonormal basis and it is characteristic of the inner product (Theorem of Sylvester). If r is the number of* -1 *and s the number of* $+1$. *We call the number* $r - s$ *the character of the inner product.*

As is well known from algebra a non-degenerate symmetric matrix with distinct real eigenvalues can always be brought to diagonal form with elements ± 1 by means of a *similarity transformation*.[5] This form of the matrix is called the *canonical form*. The similarity transformation which brings a non-degenerate symmetric matrix to its canonical form is not unique. In fact for each non-degenerate symmetric matrix there is a group of similarity transformations which brings the matrix into its canonical form. Under the action of this group the reduced form of the matrix remains the same.

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[4] The inner product is not necessarily symmetric. In this book all inner products are symmetric.

[5] Two $n \times n$ matrices A and B over the same field K are called *similar* if there exists an invertible $n \times n$ matrix P over K such that $P^{-1}AP = B$. A similarity transformation is a mapping whose transformation matrix is P. Similar matrices share many properties: they have the same rank, the same determinant, the same trace, the same eigenvalues (but not necessarily the same eigenvectors), the same characteristic polynomial, and the same minimal polynomial. Similar matrices can be thought of as describing the same linear map in different bases. Because of this, for a given matrix A, one is interested in finding a simple "normal form" B which is similar to A and reduce the study of the matrix A. This form depends on the type of eigenvalues (real or complex) and the dimension of the corresponding invariant subspaces. In case all eigenvalues are real and distinct then the canonical form is of the diagonal form with entries ± 1.

Let $g_{\mu\nu}$ be the representation of the inner product in a general basis and $G_{\rho\sigma}$ the representation in a g-orthonormal basis. If f_{ρ}^{μ} is the transformation which relates the two bases then it is easy to show the relation $g_{\mu\nu} = (f^{-1})_{\mu}{}^{\rho} G_{\rho\sigma} f^{\sigma}{}_{\nu}$. This relation implies that the transformation $f^{\mu}{}_{\rho}$ is a similarity transformation, therefore it always exists and can be found with well-known methods.

We conclude that in a four-dimensional space we can have at most three inner products:

- The Euclidean inner product with character -4 and canonical form $g_{ij} = diag(1, 1, 1, 1)$
- The Lorentz inner product with character -2 and canonical form $g_{ij} = diag(-1, 1, 1, 1)$
- The inner product (without a specific name) with character 0 and canonical form $g_{ij} = diag(-1, -1, 1, 1)$

A linear space (of any finite dimension n) endowed with the Euclidean inner product is called *Euclidean space* and is denoted by E^n. A linear space of dimension four endowed with the Lorentz inner product is called *spacetime* or *Minkowski space* and written as M^4. Newtonian Physics uses the Euclidean space E^3 and the Theory of Special Relativity the Minkowski space M^4.

Every inner product in the space V^3 induces a unique inner product in the dual space V^{3*} as follows. Because the inner product is a linear function it is enough to define its action on the basis vectors. Let $[\mathbf{e}_\mu]$ be a basis of V^3 and $[\mathbf{e}^\mu]$ its dual in V^{3*}. We define the matrix $g^{\mu\nu}$ of the induced inner product in V^{3*} by the requirement

$$g^{\mu\nu} \equiv \mathbf{e}^\mu \cdot \mathbf{e}^\nu := [g_{\mu\nu}]^{-1}$$

or equivalently

$$g_{\mu\nu} g^{\nu\rho} = \delta^{\rho}_{\mu}. \tag{1.10}$$

For the definition to be acceptable it must be independent of the particular basis $[\mathbf{e}^\mu]$. To achieve that we demand that in any other coordinate system $[\mathbf{e}^{\mu'}]$ holds:

$$g_{\mu'\nu'} g^{\nu'\rho'} = \delta^{\rho'}_{\mu'}. \tag{1.11}$$

In order to find the transformation of the matrices $g_{\mu\nu}, g^{\mu\nu}$ under coordinate transformations we consider two bases $[\mathbf{e}_\mu]$ and $[\mathbf{e}_{\mu'}]$, which are related with the transformation $f^{\mu}{}_{\mu'}$:

$$\mathbf{e}_{\mu'} = f_{\mu'}{}^{\mu} \mathbf{e}_\mu.$$

Using relation (1.9) and the linearity of the inner product we have

$$g_{\mu'\nu'} = \mathbf{e}_{\mu'} \cdot \mathbf{e}_{\nu'} = f^{\mu}_{\mu'} f^{\nu}_{\nu'} \mathbf{e}_{\mu} \cdot \mathbf{e}_{\nu} \Rightarrow$$
$$g_{\mu'\nu'} = f_{\mu'}{}^{\mu} f^{\nu}_{\nu'} g_{\mu\nu}. \tag{1.12}$$

We conclude that under coordinate transformations the matrix $g_{\mu\nu}$ is transformed homogeneously (there is no constant term) and linearly (there is only one $g_{\mu\nu}$ in the rhs). We call this type of transformation *tensorial* and this shall be used extensively in the following.

Exercise 1 Show that the induced matrix $g^{\mu\nu}$ in the dual space transforms tensorially under coordinate transformations in that space.

Because the transformation matrices $f^{\rho'}_{\mu}$, $f^{\mu}_{\mu'}$ are inverse to one another they satisfy the relation

$$\delta^{\rho'}_{\mu'} = f^{\rho'}{}_{\mu} f^{\mu}{}_{\mu'}, \tag{1.13}$$

which can be written as

$$\delta^{\rho'}_{\mu'} = f^{\rho'}{}_{\sigma} \delta^{\sigma}{}_{\tau} f^{\tau}{}_{\mu'}. \tag{1.14}$$

The last equation indicates that the 3×3 matrices δ^{σ}_{τ} under coordinate transformations transform tensorially. The representation of the inner product in the dual spaces V^3 and V^{3*} with the inverse matrices $g_{\mu\nu}$, $g^{\mu\nu}$, respectively, the relation of the dual bases with the matrix δ^{μ}_{ν}, and the fact that the matrices $g_{\mu'\rho'}$, $g^{\mu'\rho'}$, and $\delta^{\rho'}_{\mu'}$ transform tensorially under coordinate transformations lead us to the conclusion that the three matrices $g_{\mu\nu}$, $g^{\mu\nu}$, δ^{μ}_{ν} are the representations of one and the same geometric object. We call this new geometric object the *metric* of the inner product. Thus the Euclidean inner product corresponds/leads to the *Euclidean metric* g_E, and the Lorentz inner product to the Lorentz metric g_L.

The main role of the metric in a linear space is the selection of the g-*Cartesian* or g-*orthonormal* bases and coordinate systems, in which the metric has its canonical form $g_{\mu\nu} = diag(\pm 1, \pm 1, \ldots)$. For the Euclidean metric g_E we have the *Euclidean Cartesian* frames and for the Lorentz metric g_L we have the *Lorentz Cartesian* frames. In the following in order to save writing we shall refer to the first as ECF and the second as LCF.

As we shall see later the ECF correspond to the Newtonian inertial frames and the LCF to the relativistic inertial frames of Special Relativity.

Let $K(g, \mathbf{e})$ be the set of all g-Cartesian bases of a metric g. This set can be generated from any of its elements by the action of a proper coordinate transformation. These coordinate transformations are called g-*isometries* or g-*orthogonal transformations*. The g-isometries are all linear[6] transformations of the linear space which leave the canonical form of the metric the same. If $[f] = (f^{\mu'}_{\ \ \mu})$ is a g-isometry between the bases $\mathbf{e}_{\mu}, \mathbf{e}_{\mu'} \in K(g, \mathbf{e})$ and $[g]$ is the canonical form of g, we have the relation (similarity transformation)

[6] Not necessarily linear but we shall restrict our considerations to linear transformations only.

$$[f^{-1}]^t[g][f^{-1}] = [g']. \tag{1.15}$$

Equation (1.15) is important because it is a matrix equation whose solution gives the totality (of the group) of isometries of the metric g. For this reason we call (1.15) the *fundamental equation of isometry*.[7] The set of all (linear) isometries of a metric of a linear metric space V^3 is a closed subgroup of the general linear group $GL(3, R)$. From (1.15) it follows that the determinant of a g-orthogonal transformation equals ± 1. Due to that we distinguish the g-orthogonal transformations in two large subsets: The *proper g-orthogonal transformations* with determinant $+1$ and the *improper g-orthogonal transformations* with determinant -1. The proper g-orthogonal transformations form a group whereas the improper do not (why?). Every metric has its own group of proper orthogonal transformations. For the Euclidean metric this group is the *Galileo group* and for the Lorentz metric the *Poincaré group*.

The group of g-orthogonal transformations in a linear space of dimension n has dimension $n(n+1)/2$. This means that any element of the group can be described in terms of $\frac{n(n+1)}{2}$ parameters, called *group parameters*.[8] From these $n(= \dim V)$ refer to "translations"along the coordinate lines and the rest $\frac{n(n+1)}{2} - n = \frac{n(n-1)}{2}$ to "rotations" in the corresponding planes of the coordinates. How these parameters are used to describe motion in a given theory is postulated by the kinematics of the theory. The Galileo group has dimension 6 (3 translations and 3 rotations) and the Poincaré group dimension 10 (4 translations and 6 rotations). The rotations of the Poincaré group form a closed subgroup called the *Lorentz group*.

We conclude that the role of a metric is manifold. From the set of all bases of the space selects the g-Cartesian bases and from the set of all linear (non-degenerate) automorphisms of the space the g-orthogonal transformations. Furthermore the metric specifies the g-tensorial behavior which will be used in the next section to define geometric objects more general than the metric and compatible with the linear and the metric structure of the space.

1.4 Tensors

The vectors and the metric of a linear space are geometric objects, with one and two indices, respectively, which transform tensorially.[9] The question arises if we need to consider geometric objects with more indices, which under the action of $GL(n, R)$

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[7] This is essentially Killing's equation in a linear space.

[8] We recall that translation is a transformation of the form $\mathbf{u} \to \mathbf{u}' = \mathbf{u} + \mathbf{a}\ \forall \mathbf{u},\ \mathbf{u}' \in V^3$ and rotation is a transformation of the form $\mathbf{u} \to \mathbf{u}' = A\mathbf{u}$ where the matrix A satisfies the fundamental equation of isometry (1.15) and $\mathbf{u},\ \mathbf{u}' \in V^3$. The translations follow in an obvious way and we need only to compute the rotations.

[9] A geometric object is any quantity whose components have a specific transformation law (not necessarily linear and homogeneous, i.e., tensorial) under coordinate transformations.

or of a given subgroup G of $GL(n, R)$ transform tensorially. The study of geometry and physics showed that this is imperative. For example the curvature tensor is a geometric object with four indices. We call these new geometric objects with a collective name *tensors* and they are the basic tools of both geometry and physics. Of course in mathematical physics one needs to use geometric objects which are not tensors, but this will not concern us in this book.

We start our discussion of tensors without specifying either the dimension of the space or the subgroup G of $GL(n, R)$, therefore the results apply to both Euclidean tensors and Lorentz tensors. Let G be a group of linear transformations of a linear space V^n and let $F(V^n)$ be the set of all bases of V^n. We consider an arbitrary basis in $F(V^n)$ and construct all the bases which are obtained from that basis under the action of the group G. This action selects in general a subset of bases in the set $F(V^n)$. We say that the bases in this set are G-related and we call them *G-bases*. From the subset of remaining bases we select a new basis and repeat the procedure and so on. Eventually we end up with a set of sets of bases so that the bases in each set are G-related and bases in different sets are not related with an element of G.[10] We conclude that the existence of a group G of coordinate transformations in V^n makes possible the division of the bases and the coordinate systems V^n in classes of G-equivalent elements. The choice of G-bases in a linear space makes possible the definition of G-equivalent geometric objects in that space, by means of the following definition.

Definition 2 Let G be a group of linear coordinate transformations of a linear space $\mathrm{V^n}$ and suppose that the element of G which relates the G-bases $[\mathbf{e}] \to [\mathbf{e}']$ is represented by the $n \times n$ matrix $A_n^{m'}$. We define a G-tensor of order (r, s) and write as $T^{i_1\ldots i_r}_{j_1\ldots j_s}$ a geometric object which

1. Has n^{r+s} components of which n^r are G-contravariant (upper indices $i_1 \ldots i_r$) and n^s are G-covariant (lower indices $j_1 \ldots j_s$).
2. The n^{r+s} components under the action of the element $A_n^{m'}$ of the group G transform tensorially, that is:

$$T^{i'_1\ldots i'_r}_{j'_1\ldots j'_s} = A^{i'_1}_{i_1} \ldots A^{i'_r}_{i_r} A^{j_1}_{j'_1} \ldots A^{j_s}_{j'_s} T^{i_1\ldots i_r}_{j_1\ldots j_s}, \tag{1.16}$$

where $T^{i_1\ldots i_r}_{j_1\ldots j_s}$ are the components of the tensor in the basis $[\mathbf{e}]$ and $T^{i'_1\ldots i'_r}_{j'_1\ldots j'_s}$ the components in the G-related basis $[\mathbf{e}']$.

From the definition it follows that if we know a tensor in one G-basis then we can compute it in any other G-basis using (1.16) and the matrix $A_n^{m'}$ representing the element of G relating the two bases. Therefore we can divide the set of all tensors, $T(V^n)$ say, on the linear space V^n, in sets of G-tensors in the same manner

[10] To be precise using G we define an equivalence relation in $F(V^n)$ whose classes are the subsets mentioned above.

we divided the set of bases and the set of coordinate systems in G-bases and G-coordinate systems, respectively. In conclusion,

For every subgroup G of $GL(n, R)$ we have G-bases, G-coordinate systems, and G-tensors.

The discussion so far has not specified either the group G or the dimension of the space n. In order to define a specific group G of transformations in a linear space we must introduce a geometric structure in the space whose symmetry group will be the group G. Without getting into details, we consider in the linear space V^n the structure inner product. As we have shown, the inner product defines the group of isometries in V^n, which can be used as the group G. In that spirit the Euclidean inner product defines the *Euclidean tensors* and the Lorentz inner product the *Lorentz tensors*.

The selection of the dimension of the linear space and the group of coordinate transformations G by a theory of physics is made by means of principles which satisfy certain physical criteria:
a. The group of transformations G attains physical meaning only after a correspondence has been defined between the G-coordinate systems and the characteristic frames of reference of the theory.
b. The physical quantities of the theory are described in terms of G-tensors.

As a result of the tensorial character it is enough to give a physical quantity in one characteristic frame of the theory[11] and compute it in any other frame (without any further experimentation of measurements!) using the appropriate element of G relating the two frames. This procedure achieves the "de-personalization" of physics, that is, all frames (observers) are "equal," and defines the "objectivity" of the theories of physics. More on that topic is in Chap. 2, when we discuss the covariance principle.

An important class of G-tensors which deserves special reference is the G-invariants. These are tensors of class (0, 0), so that they have no indices and under G-transformations retain their value, that is, their transformation is

$$a' = a. \tag{1.17}$$

It is important to note that a scalar is not necessarily invariant under a group G whereas an invariant is always a scalar. Scalar means one component whereas G-invariant means one component and in addition this component must transform tensorially under the action of G. Furthermore a G-invariant is not necessarily a G'-invariant for $G \neq G'$. For example the Newtonian time is invariant under the Galileo group (i.e., $t = t'$) but not invariant under the Lorentz group (as we shall see $t' = \gamma(t - \beta x/c)$).

A question we have to answer at this point is

Given a G-tensor how we can construct/define new G-tensors?

[11] In the Newtonian theory and the Theory of Special Relativity these frames/observers are the inertial frames/observers.

The answer to this question is the following simple rule.

Proposition 2 (Construction of G-tensors) *There are two methods to construct G-tensors from given G-tensors:*

(1) Differentiation of a G-tensor with a G-invariant
(2) Multiplication of a G-tensor with a G-invariant

1.4.1 Operations of Tensors

The G-tensors in V^n are linear geometric objects which can be combined with algebraic operations.

Let $T = T^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s}$ and $S = S^{k_1 k_2 \ldots k_r}_{l_1 l_2 \ldots l_s}$ be two G-tensors of order (r, s) and let $R = R^{t_1 t_2 \ldots t_m}_{s_1 s_2 \ldots s_n}$ be a G-tensor of order (m, n). (All components refer to the same G-basis!) We define

(1) Addition (subtraction) of tensors
The sum (difference) $T \pm S$ of the G-tensors T, S is defined to be the G-tensor of type (r, s) whose components are the sum (difference) of the components of the tensors $T = T^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s}$ and $S = S^{k_1 k_2 \ldots k_r}_{l_1 l_2 \ldots l_s}$.
(2) Multiplication of tensors (tensor product)
The tensor product $T \otimes S$ is defined to be the G-tensor of type $(r + m, s + n)$ whose components are the product of the components of the tensors T, R.
(3) Contraction of indices
When in a G-tensor of order (r, s) we sum over a contravariant and a covariant index then we obtain a G-tensor of type $(r - 1, s - 1)$.

In this book we shall use the tensor operations mainly for vectors and tensors of second order, therefore we shall not pursue the study of these operations further.

There is a final important point concerning tensors in a metric space. Indeed in such a space the metric tensor can be used to raise and lower an index as follows:

$$g_{a_0 i_1} T^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s} = T^{i_2 \ldots i_r}_{a_0 j_1 j_2 \ldots j_s},$$
$$g^{a_0 j_1} T^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s} = T^{a_0 i_1 i_2 \ldots i_r}_{j_2 \ldots j_s}.$$

This implies that in a metric space the contravariant and the covariant indices lose their character and become equivalent. Caution must be paid when we change the relative position of the indices in a tensor, because that may effect the symmetries of the tensor. However, let us not worry about that for the moment.

In Fig. 1.1 we show the role of the Euclidean and the Lorentz inner product on

(1) The definition of the subgroups $E(n)$ and $L(4)$ of $GL(n, R)$
(2) The definition of the ECF and the LCF in the set $F(V^n)$
(3) The definition of the $TE(n)$-tensors and the $TL(4)$-tensors in $TGL(n, R)$-tensors on V^n

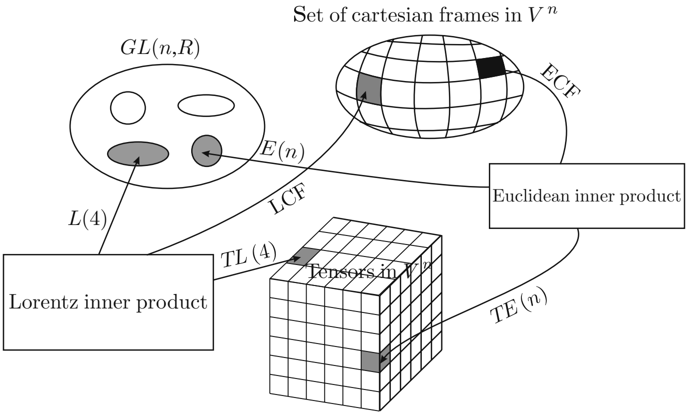

Fig. 1.1 The role of the Euclidean and the Lorentz inner products

1.5 The Case of Euclidean Geometry

We consider the Euclidean space[12] E^3 and the group g_E of all coordinate transformations, which leave the canonical form of the Euclidean metric, the same, that is, $g_{E\mu'\nu'} = g_{E\mu\nu} = diag(1, 1, 1) = I$. These coordinate transformations are the g_E-canonical transformations, which in Sect. 1.4 we called Euclidean orthogonal transformations (EOT) and the group they form is the Galileo group. The g_E-coordinate systems are the Euclidean coordinate systems (ECF) mentioned in Sect. 1.3. In an ECF, and only in these coordinate systems, the expression of the Euclidean inner product and the Euclidean length are written as follows:

$$\mathbf{u} \cdot \mathbf{v} = u^\mu v_\mu = u^1 v^1 + u^2 v^2 + u^3 v^3, \tag{1.18}$$

$$\mathbf{u} \cdot \mathbf{u} = u^\mu u_\mu = (u^1)^2 + (u^2)^2 + (u^3)^2, \tag{1.19}$$

where $(u^1, u^2, u^3)^t$ and $(v^1, v^2, v^3)^t$ are the components of the vectors $\mathbf{u}, \mathbf{v}$ in the same ECF and t indicates the transpose of a matrix.

In order to compute the explicit form of the elements of the Galileo group we consider the fundamental equation of isometry (1.15) and make use of the definition $g_{E\mu'\nu'} = g_{E\mu\nu} = diag(1, 1, 1) = I$ of ECF. We find immediately that the transformation matrix A satisfies the relation

$$A^t A = I. \tag{1.20}$$

Relation (1.20) means that the inverse of a matrix representing an EOT equals its transpose. In order to compute all EOT it is enough to solve the matrix equation

[12] The following holds for a Euclidean space of any finite dimension.

(1.20). This equation is solved as follows. We consider first the simple case of a two-dimensional ($n = 2$) Euclidean space in which the dimension of the group of Galileo is three $\left(\frac{2(2+1)}{2}\right)$, therefore we need three parameters to describe the general element of the isometry group. Two of them ($n = 2$) are used to describe translations and the rest $\frac{2(2+1)}{2} - 2 = \frac{2(2-1)}{2} = 1$ is used for the description of rotations.

In order to compute the rotations we write

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, \tag{1.21}$$

where a_1, a_2, a_3, a_4 are functions of the (group) parameter ϕ (say). Replacing in (1.20) we find that the functions a_1, a_2, a_3, a_4 must satisfy the following relations:

$$a_1^2 + a_2^2 = 1, \tag{1.22}$$

$$a_1 a_3 + a_2 a_4 = 0, \tag{1.23}$$

$$a_3^2 + a_4^2 = 1. \tag{1.24}$$

This is a system of four simultaneous equations in three unknowns, therefore the solution is expressed in terms of a parameter, as expected. It is easy to show that the solution of the system is

$$a_1 = a_4 = \pm\cos\phi, \;\; a_2 = -a_3 = \pm\sin\phi, \tag{1.25}$$

where ϕ is the group parameter.

The determinant of the transformation $a_1 a_4 - a_2 a_3 = \pm 1$. This means that the Euclidean isometry group has two subsets. The first defined by the value $\det A = +1$ is called the *proper Euclidean group of rotations* and it is a subgroup of $E(3)$. The other set defined by the value $\det\ A = -1$ is not a group (because it does not contain the identity element). We conclude that the elements of the proper two-dimensional rotational Euclidean group have the general form

$$A = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}. \tag{1.26}$$

In order to give the parameter ϕ a geometric meaning we consider the space E^2 to be the plane x–y of the ECF $\Sigma(xyz)$. Then the parameter ϕ is the (left-hand) rotation of the plane x–y around the z-axis.

From the general expression of an EOT in two dimensions we can produce the corresponding expression in three dimensions as follows. First we note that the number of required parameters is $\frac{3(3+1)}{2} = 6$, three for translations along the coordinate axes and three for rotations. There are many sets of three parameters for the rotations, the commonest being the Euler angles. The steps for the computation of the general EOT in terms of these angles are the following (see also Fig. 1.2)[13]:

(i) Rotation of the x–y plane about the z-axis with angle ϕ:

$$\begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.27}$$

Let X, Y, z be the new coordinate axes.

(ii) Rotation of the Y–z plane about the X-axis with angle θ:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}. \tag{1.28}$$

Let X, Y', z' be the new axes.

(iii) Rotation of the X–Y' plane about the z'-axis with angle ψ:

$$\begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.29}$$

Let x', y', z' be the new axes.

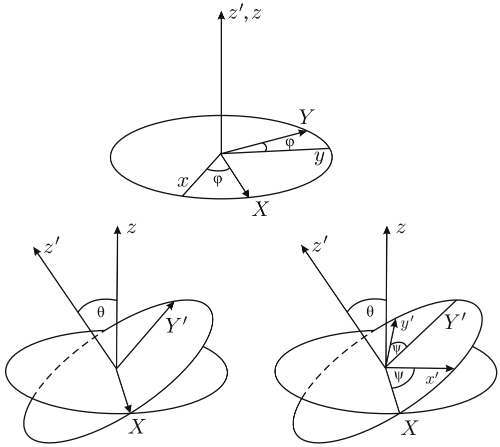

Fig. 1.2 Euler angles

[13] For more details see H. Goldstein, C. Poole, J. Safko *Classical Mechanics* Third Edition. Addison Wesley (2002).

Multiplication of the three matrices gives the total EOT $\{x,\ y,\ z\} \to \{x', y', z'\}$:

$$\begin{pmatrix} \cos\psi\cos\phi - \sin\phi\cos\theta\sin\phi & \cos\psi\sin\phi + \sin\psi\cos\theta\cos\phi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\psi\cos\theta\sin\phi & -\sin\psi\sin\phi + \cos\psi\cos\theta\cos\phi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix}. \tag{1.30}$$

The three angles θ, ϕ, ψ (Euler angles) are the three group parameters which describe the general rotation of a Euclidean isometry. We recall at this point that the full Euclidean isometry is given by the product of a general translation and a general rotation in any order, because the two actions commute. This result is the geometric explanation of the statement of Newtonian mechanics that any motion of a rigid body can be described in terms of one rotation and one translation in any order.

1.6 The Lorentz Geometry

The Theory of Special Relativity is developed on Minkowski space M^4. The Minkowski space is a flat linear space of dimension $n = 4$ endowed with the Lorentz inner product or metric. The term flat[14] means that we can employ a unique coordinate system to cover *all* M^4 and map all M^4 in R^4. We call the vectors of the space M^4 *four-vectors* and denote as $u^i, v^i, \ldots$ where the index $i = 0, 1, 2, 3$. The component which corresponds to the value $i = 0$ shall be called *temporal or zeroth component* and the rest three *spatial components*. The spacetime indices shall be denoted with small Latin letters $i, j, k, a, b, c \ldots$ and will be assumed to take the values 0, 1, 2, 3. The Greek indices will be used to indicate the spatial components and take the values 1, 2, 3.

The group of g_L-isometries of spacetime is the Poincaré group. The elements of this group are *linear transformations* of M^4 which preserve the Lorentz inner product and consequently the lengths of the four-vectors. The Lorentz metric is not positive definite and the Lorentz length $u^2 = g_{Lij}u^i u^j$ of a four-vector can be positive, negative, or zero. Because the length of a four-vector is invariant under a Lorentz isometry we can divide the four-vectors in M^4 in three large and non-intersecting sets:

Null four-vectors:	$u^2 = 0$
Timelike four-vectors:	$u^2 < 0$
Spacelike four-vectors:	$u^2 > 0$

Considering an arbitrary point O of spacetime as the origin we can describe any other point by its position vector wrt this origin. Applying the above classification of four-vectors we can divide M^4 in three large and non-intersecting parts. The first part consists of the points whose position vector wrt O is null. This is a three-dimensional subspace (a hypersurface) in M^4, which we call the *null cone* at O.

[14] The spacetime used in General Relativity has curvature and in general there does not exist a unique chart to cover all spacetime.

The second (resp. third) part consists of the points inside (resp. outside) the null cone and consists of all points with a timelike (resp. spacelike) position vector wrt the selected origin O of M^4. We note that the null cone is characteristic of the point O, which has been selected as the origin of M^4. That is, at every point there exists a *unique* null cone associated with that point and different points have different null cones. From the three parts in spacetime we shall use mostly the timelike and the null. The reason is that the null cone concerns events associated with light rays and the timelike part events which describe the motion of massive particles and observers.

Concerning the geometry of M^4 we have the following simple but important result.[15]

Proposition 3 *The sum of timelike or timelike and null four-vectors is a timelike four-vector except if, and only if, all four-vectors are null and parallel in which case the sum is a null four-vector parallel to the null four-vectors.*

This result is important because it allows us to study reactions of elementary particles including photons. Indeed as we shall see the elementary particles are characterized with their four-momenta, which is null for photons and timelike for the rest of the particles. Then Proposition 3 means that the interaction of particles and photons results again in particles and photons, and in the case of a light beam consisting only of parallel photons, this stays a light beam as it propagates. As we shall see, the existence of light beams in Minkowski space is vital to Special Relativity because they are used for the measurement of the position vector (= coordinatization)in spacetime.

1.6.1 Lorentz Transformations

The Poincaré group consists of all linear transformations of M^4, which satisfy the matrix equation $[f^{-1}]^t[\eta][f^{-1}] = [\eta]$ where $[\eta] = diag(-1, 1, 1, 1)$ is the canonical form of the Lorentz metric. The dimension of the Poincaré group is $\frac{4(4+1)}{2} = 10$, therefore an arbitrary element of the group is described in terms of 10 parameters. Four of these parameters ($n = 4$) concern the closed (Abelian) subgroup of translations and the rest six the subgroup of rotations. This later subgroup is called the Lorentz group and the resulting coordinate transformations the Lorentz transformations. As was the case with the Galileo group every element of the Poincaré group is decomposed as the product of a translation and a Lorentz transformation (rotation) about a characteristic direction. Therefore in order to compute the general Poincaré transformation it is enough to compute the Lorentz transformations (rotations) defined by the matrix equation

$$[L]^t[\eta][L] = [\eta]. \tag{1.31}$$

[15] For a proof see Theorem 1, Sect. 16.4.

In Sect. 1.7 we solve this equation directly using formal algebra and compute all Lorentz transformations. However, this solution lacks the geometric insight and does not make clear their relation with the Euclidean transformations. Therefore at this point we work differently and compute the Lorentz transformations in the same way we did for the Euclidean rotations. For that reason we shall use the Euler angles (see Sect. 1.5) and in addition three more spacetime rotations, that is, rotations of two-dimensional planes (l, x), (l, y), (l, z) (to be defined below) about the spatial axes. These planes have a Lorentz metric and we call them *hyperbolic planes*. Let us see how the method works.

We consider first a 3×3 EOT, E say, in three-dimensional Euclidean spatial space and the 4×4 block matrix[16]:

$$R = \begin{pmatrix} 1 & 0 \\ 0 & E \end{pmatrix}. \tag{1.32}$$

The matrix E satisfies $E^t E = I_3$ and it is described in terms of three parameters (e.g., the Euler angles), therefore the same holds for the matrix R. It is easy to show that the matrix R satisfies the equation

$$R^t \eta R = \eta,$$

therefore it is a Lorentz transformation. Furthermore one can show that the set of all matrices R of the form (1.32) is a closed subgroup of the group of Lorentz transformations. This means that the general Lorentz transformation can be written as the product of two matrices as follows:

$$L(\boldsymbol{\beta}, \Phi) = L(\boldsymbol{\beta}) R(\Phi), \tag{1.33}$$

where $L(\boldsymbol{\beta})$ is a transformation we must find and $R(\Phi)$ is a Lorentz transformation of the form (1.32). The vector $\boldsymbol{\beta}(= \beta^\mu)$ involves three independent parameters and the symbol Φ refers to the three parameters of the Euclidean rotations (e.g., the Euler angles). We demand that the general Lorentz transformation $L(\boldsymbol{\beta}, \Phi)$ satisfies the defining equation (1.31) and get

$$R^t L^t \eta L R = \eta \Rightarrow L^t \eta L = \eta, \tag{1.34}$$

which implies that $L(\boldsymbol{\beta})$ is also a Lorentz transformation.

The Lorentz transformation $L(\boldsymbol{\beta})$ contains the non-Euclidean part of the general Lorentz transformation, therefore it contains all spacetime rotations, that is, rotations which involve in some way the zeroth component l. There can be two types of such transformations: Euclidean rotations about the l-axis and rotation of one of the

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[16] A block matrix is a matrix whose elements are matrices. With block matrices we can perform all matrix operations provided the element matrices are of a suitable dimension. Here the element $(1, 2)$ is a 1×3 matrix and the element $(2, 1)$ a 3×1 matrix.

planes $(l,\ x)$, $(l,\ y)$, $(l,\ z)$ about the corresponding normal spatial axes x, y, z. This latter type of Lorentz transformations is called *boosts*. We have already computed the Euclidean rotation in Sect. 1.5. Therefore the rotation of the hyperbolic planes remains to be computed.

We consider the linear transformation (boost)

$$\begin{aligned} l' &= al + bx, \\ x' &= cl + dx, \end{aligned}$$

which defines the transformation matrix

$$L(\psi) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The parameter ψ is a real parameter whose geometric significance has to be determined. We demand the matrix $L(\psi)$ to satisfy the equation of isometry (1.34) and find the following conditions on the coefficients a, b, c, d:

$$\begin{aligned} a^2 - c^2 &= 1, \\ d^2 - b^2 &= 1, \\ ab &= cd. \end{aligned}$$

This system of simultaneous equations has two solutions:

$$\begin{aligned} a = d &= \cosh\psi, \\ c = b &= \sinh\psi \end{aligned} \tag{1.35}$$

and

$$\begin{aligned} a = -d &= \cosh\psi, \\ c = -b &= \sinh\psi. \end{aligned} \tag{1.36}$$

It follows that geometrically ψ is the hyperbolic angle of rotation in the plane (l, x). We call ψ the *rapidity* of the boost. The solution (1.35) has det $L(\psi) = +1$ and leads to the subgroup of *proper Lorentz transformations*. The solution (1.36) has det $L(\psi) = -1$ and leads to the improper Lorentz transformations which do not form a group. Because we have also two types of Euclidean transformations eventually we have four classes of general Lorentz transformations. We choose the solution (1.35) and write

$$L(\psi) = \begin{pmatrix} \cosh\psi & \sinh\psi \\ \sinh\psi & \cosh\psi \end{pmatrix}. \tag{1.37}$$

It is easy to prove the relation

$$L(\psi)L(-\psi) = 1,$$

that is the inverse of $L(\psi)$ is found if we replace ψ with $-\psi$. We introduce the new parameter β ($|\beta| \leq 1$) with the relation

$$\cosh\psi = \gamma = \frac{1}{\sqrt{1-\beta^2}} \quad \beta \in (-1, 1). \tag{1.38}$$

We compute $\sinh\psi = \pm\beta\gamma$ and finally we have for a boost in the $(l,\ x)$ plane

$$L(\beta) = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix}. \tag{1.39}$$

We have the obvious relation $L^{-1}(\beta) = L(-\beta)$.

We continue with the computation of the general Lorentz transformation. We assume that the three parameters $(\beta_1, \beta_2, \beta_3)$ ($|\beta_\mu| \leq 1$) which concern the general Lorentz transformation $L(\boldsymbol{\beta})$ define in the three-dimensional spatial space the direction cosines of a characteristic direction specified uniquely by the LCFs $(l,\ x,\ y,\ z)$, $(l',\ x',\ y',\ z')$. The rotations involved in $L(\boldsymbol{\beta})$ must be rotations of hyperbolic planes about a spatial axis, therefore there is no room for Euclidean rotations. This implies that we *must* consider the axes (xyz), $(x'y'z')$ of the two LCFs as being "parallel." This parallelism is of a Euclidean nature, therefore it is not Lorentz invariant and *must* be defined. We do this below.

Without restricting the generality of our considerations, and in order to have the possibility of visual representation, we suppress one dimension and consider the LCFs $(l,\ x,\ y)$, $(l',\ x',\ y')$. In order to calculate $L(\boldsymbol{\beta})$ we shall use the angles of rotation of the Euclidean case with the difference that they will be treated as hyperbolic instead of Euclidean. Following the discussion of Sect. 1.5 we have the three rotations (see Fig. 1.3):

(A1) Rotation of the Euclidean plane $(x,\ y)$ about the l-axis with Euclidean angle ϕ so that the new coordinate Y will be parallel to the characteristic spatial direction defined by the direction cosines $\beta_x,\ \beta_y$. The transformation is

$$(l,\ x,\ y) \rightarrow (l, X, Y)$$

and has matrix

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix}. \tag{1.40}$$

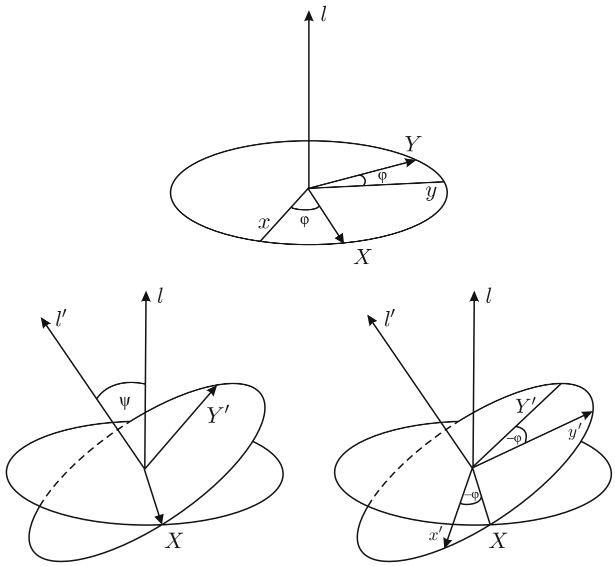

Fig. 1.3 The Lorentz transformation

(L1) Rotation of the hyperbolic plane $(l,\ y)$ about the X-axis with hyperbolic angle ψ, which is fixed by the parameter $\beta = \sqrt{\beta_x^2 + \beta_y^2}$ via the relations $\cosh\psi = \gamma$, $\sinh\psi = \beta\gamma$. The transformation is

$$(l, X, Y) \to (l', X, Y')$$

and has matrix

$$L_1(\beta) = \begin{pmatrix} \gamma & \beta\gamma & 0 \\ \beta\gamma & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.41}$$

(A2) Rotation of the Euclidean plane (X, Y') about the l'-axis with Euclidean angle $-\phi$ in order to reverse the initial rotation of the spatial axes and make them (by definition!) "parallel." The transformation is

$$(l', X, Y') \to (l', x', y')$$

and has matrix

$$A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}. \tag{1.42}$$

We note that $A_1 A_2 = I$. This relation *defines* the parallelism of the spatial axes ("Euclidean parallelism") in the geometry of the Minkowski space M^4. We note that in M^4 the initial and the final axes do not coincide and are not parallel in the Euclidean sense, as someone might have expected. Indeed the rotation $-\phi$ takes place in the plane (X', Y), which is normal to l' whereas the rotation ϕ in the plane (x, y), which is normal to the l-axis. Therefore the axes (x', y') (in the space M^4) are on a different plane from the initial axes (x, y).

Obviously we must expect a different and "strange" behavior of the Euclidean "parallelism" in the geometry of M^4. For example if the spatial axes of the LCFs (l, x, y) and (l', x', y') are parallel and the same holds for the LCFs (l', x', y') and (l'', x'', y'') then the spatial axes of the LCFs (l, x, y) and (l'', x'', y'') *are not* in general parallel in the Euclidean sense. We shall discuss the consequences of the Euclidean parallelism in Minkowski space when we consider the Thomas precession.

The general Lorentz transformation $L(\boldsymbol{\beta})$ is the combination of these three rotations in the same way the general Euclidean transformation is derived from the composition of the three Euler rotations. That is we have

$$L(\boldsymbol{\beta}) = A_2(-\phi) L_1(\beta) A_1(\phi). \tag{1.43}$$

Replacing the matrices from relations (1.40), (1.41), and (1.42) and writing $\beta_x = \beta \cos\phi$, $\beta_y = \beta \sin\phi$ we compute easily the following result:

$$L(\beta_\mu) = \begin{pmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y \\ -\gamma\beta_x & 1 + \frac{(\gamma-1)}{\beta^2}\beta_x^2 & \frac{(\gamma-1)}{\beta^2}\beta_x\beta_y \\ -\gamma\beta_y & \frac{\gamma-1}{\beta^2}\beta_x\beta_y & 1 + \frac{(\gamma-1)}{\beta^2}\beta_y^2 \end{pmatrix}. \tag{1.44}$$

This matrix can be written in a more compact form which holds generally, that is including the ignored coordinate z, as follows:

$$L(\beta_\mu) = \begin{pmatrix} \gamma & -\gamma\beta_\mu \\ -\gamma\beta^\mu & \delta^\mu_\nu + \frac{(\gamma-1)}{\beta^2}\beta^\mu\beta_\nu \end{pmatrix}. \tag{1.45}$$

The transformation (1.45) holds only when the axes of the initial and the final LCF are parallel and have the same orientation (that is both left-handed or both right-handed). According to (1.43) the most general Lorentz transformation is given by the product of this transformation and a transformation of the form (1.32) defined by a Euclidean rotation.

Exercise 2 Multiply the matrices in (1.43) and show that the matrix (1.44) and the matrix (1.45) satisfy the isometry equation $L(\boldsymbol{\beta})^t \eta L(\boldsymbol{\beta}) = \eta$. Compute the determinant of this transformation and show that equals $+1$. Conclude that the transformation (1.45) describes indeed the general proper Lorentz transformation.

It can be shown[17] that the general Lorentz transformation (not only the proper) has the form

$$L(\boldsymbol{\beta}) = \begin{pmatrix} \pm\gamma & \mp\gamma\boldsymbol{\beta}^t \\ \mp\gamma\boldsymbol{\beta} & \pm[I + \frac{\det L(\gamma-1)}{\beta^2}\boldsymbol{\beta}\boldsymbol{\beta}^t] \end{pmatrix}. \tag{1.46}$$

The transformation (1.46) has four families depending on the sign of the (00) element and the $\det L = \pm 1$. In order to distinguish the four cases we use for the sign of the term 00 the arrows $\uparrow$ / $\downarrow$ as follows:

$$L_{\uparrow}(\beta) : A^0 A^{0'} > 0,$$
$$L_{\downarrow}(\beta) : A^0 A^{0'} < 0.$$

Concerning the sign of the determinant we use one further index $\pm$. As a result of these conventions we have the following four families of Lorentz transformations[18]:

(1) Proper Lorentz transformations ($\det\ L = 1$):

$$L_{+\uparrow}(\boldsymbol{\beta}) = \begin{pmatrix} \gamma & -\gamma\boldsymbol{\beta}^t \\ -\gamma\boldsymbol{\beta} & I + \frac{\gamma-1}{\boldsymbol{\beta}^2}\boldsymbol{\beta}\boldsymbol{\beta}^t \end{pmatrix}. \tag{1.47}$$

(2) Space inversion Lorentz transformations ($\det\ L = 1$):

$$L_{-\uparrow}(\boldsymbol{\beta}) = \begin{pmatrix} \gamma & -\gamma\boldsymbol{\beta}^t \\ -\gamma\boldsymbol{\beta} & -I - \frac{\gamma-1}{\boldsymbol{\beta}^2}\boldsymbol{\beta}\boldsymbol{\beta}^t \end{pmatrix}. \tag{1.48}$$

(3) Time-inversion Lorentz transformations ($\det\ L = -1$):

$$L_{+\downarrow}(\boldsymbol{\beta}) = \begin{pmatrix} -\gamma & \gamma\boldsymbol{\beta}^t \\ \gamma\boldsymbol{\beta} & I - \frac{\gamma+1}{\boldsymbol{\beta}^2}\boldsymbol{\beta}\boldsymbol{\beta}^t \end{pmatrix}. \tag{1.49}$$

(4) Spacetime-inversion Lorentz transformations ($\det\ L = -1$):

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[17] See Sect. 15.2 for the derivation of the general forms of Lorentz transformation.

[18] Note that

$$\frac{\gamma-1}{\beta^2} = \gamma^2\frac{\gamma-1}{\gamma^2\beta^2} = \gamma^2\frac{\gamma-1}{\gamma^2-1} = \frac{\gamma^2}{\gamma+1}.$$

Therefore it is possible to write the Lorentz transformation in terms of the parameter γ only.

$$L_{-\downarrow}(\boldsymbol{\beta}) = \begin{pmatrix} -\gamma & \gamma\boldsymbol{\beta}^t \\ \gamma\boldsymbol{\beta} & -I + \frac{\gamma+1}{\beta^2}\boldsymbol{\beta}\boldsymbol{\beta}^t \end{pmatrix}. \tag{1.50}$$

Only the proper Lorentz transformations form a group (a closed subgroup of the Lorentz group). All four types of Lorentz transformations are important in physics, but in the present book (and in most applications) we use only the proper Lorentz transformations.

In order to express the Lorentz transformations in terms of components we consider a four-vector A^i which in the LCFs Σ and Σ' has components $\begin{pmatrix} A^0 \\ \mathbf{A} \end{pmatrix}_\Sigma$, $\begin{pmatrix} A^{0'} \\ \mathbf{A}' \end{pmatrix}_{\Sigma'}$ which are related by the proper Lorentz transformation $L_{+\uparrow}(\boldsymbol{\beta})$ relating Σ, Σ' :

$$\begin{pmatrix} A^{0\prime} \\ \mathbf{A}' \end{pmatrix}_{\Sigma'} = L_{+\uparrow}(\beta_\mu) \begin{pmatrix} A^0 \\ \mathbf{A} \end{pmatrix}_\Sigma .$$

Replacing $L_{+\uparrow}(\boldsymbol{\beta})$ from (1.47) we find

$$A^{0'} = \gamma(A^0 - \boldsymbol{\beta}\cdot\mathbf{A}), \tag{1.51}$$

$$\mathbf{A}' = \mathbf{A} + \frac{\gamma-1}{\boldsymbol{\beta}^2}(\boldsymbol{\beta}\cdot\mathbf{A})\boldsymbol{\beta} - \gamma A^0\boldsymbol{\beta}. \tag{1.52}$$

A special type of proper Lorentz transformations are the boosts, defined by the requirement that two of the three direction cosines vanish. In this case we say that the LCFs Σ and Σ' are moving in the *standard configuration* along the axis specified by the remaining direction cosine. For example if $\beta_y = \beta_z = 0$ we have the boost along the x-axis with factor β and the boost is

$$\begin{aligned} A^{0'} &= \gamma\left(A^0 - \beta A^x\right), \\ A^{x'} &= \gamma\left(A^x - \beta A^0\right), \\ A^{y'} &= A^y, \\ A^{z'} &= A^z. \end{aligned} \tag{1.53}$$

Exercise 3 Show that the proper Lorentz transformation for the position four-vector $\begin{pmatrix} l \\ \mathbf{r} \end{pmatrix}$ is

$$l' = \gamma(l - \boldsymbol{\beta}\cdot\mathbf{r}), \tag{1.54}$$

$$\mathbf{r}' = \mathbf{r} + \frac{\gamma-1}{\boldsymbol{\beta}^2}(\boldsymbol{\beta}\cdot\mathbf{r})\boldsymbol{\beta} - \gamma l\boldsymbol{\beta}. \tag{1.55}$$

In the special case of the boost along the x-axis show that the transformation of the position four-vector is

$$\begin{aligned} l' &= \gamma(l - \beta x), \\ x' &= \gamma(x - \beta l), \\ y' &= y, \\ z' &= z. \end{aligned} \tag{1.56}$$

Example 1 Prove that the proper Lorentz transformation does not change the sign of the zero component of a timelike four-vector.
Solution

Let A^i be a timelike four-vector which in the LCF Σ has components $A^i = (l, \mathbf{r})^t$, and let that in its proper frame Σ^+ has components $A^i = (l^+, \mathbf{0})^t$. From the proper Lorentz transformation we have for the zeroth coordinate $l = \gamma l^+$, which proves that l, l^+ have the same sign.

The result of Example 1 allows us to consider in a covariant manner the timelike vectors in *future directed* which have $l > 0$ and are directed in the "upper part" of the null cone and the *past directed* which have $l < 0$ and are directed in the "lower part" of the null cone.

1.7 Algebraic Determination of the General Vector Lorentz Transformation

It is generally believed that the determination of the analytical form of the Lorentz transformation in an LCF requires the use of Special Relativity. This is wrong. Lorentz transformations are the solutions of the mathematical equation $\eta = L^t \eta L$ where η is the 4×4 matrix $diag(-1,\ 1,\ 1,\ 1)$ and L is a matrix to be determined. For that reason in this section we solve this equation using pure algebra and produce the so-called *vector Lorentz transformation*. In a subsequent chapter (see Chap. 15), when the reader will be more experienced, we shall derive the covariant form of the Lorentz transformation.

In order to solve (1.34) we consider an arbitrary LCF and write L as the block matrix:

$$L = \begin{bmatrix} D & C \\ B & A \end{bmatrix}, \tag{1.57}$$

where the submatrices A, B, C, D are as follows:

$D : 1 \times 1$ (a function but not necessarily an invariant!)
$B : 3 \times 1$
$C : 1 \times 3$
$A : 3 \times 3$

Equation (1.34) is written as the following matrix equation:

$$\begin{bmatrix} D & B^t \\ C^t & A^t \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & I_3 \end{bmatrix} \begin{bmatrix} D & C \\ B & A \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & I_3 \end{bmatrix}.$$

Multiplying the block matrices we find the matrix equations

$$A^t A - C^t C = I_3, \tag{1.58}$$

$$B^t A - DC = 0, \tag{1.59}$$

$$B^t B - D^2 = -1, \tag{1.60}$$

whose solution determines the explicit form of Lorentz transformation. Before we attempt the general solution we look at two special solutions of physical importance.

Case 1. $C = 0,\ A \neq 0$

In this case (1.58) implies $A^t A = I_3$, therefore A is a Euclidean matrix. Then (1.59) implies $B = 0$ and (1.60) $D = \pm 1$. Therefore we have the two special Lorentz transformations:

$$R_+(E) = \begin{bmatrix} 1 & 0 \\ 0 & E \end{bmatrix}, \quad R_-(E) = \begin{bmatrix} -1 & 0 \\ 0 & E \end{bmatrix}. \tag{1.61}$$

It follows that the EOTs are impeded in a natural manner in the Lorentz transformations. We note the relations

$$\det\ R_+(E) = +1, \qquad \det\ R_-(E) = -1 \tag{1.62}$$

and

$$R^t_+(E)R_+(E) = I_4, \quad R^t_-(E)R_-(E) = I_4, \quad R^t_+(E)R_-(E) = \eta. \tag{1.63}$$

Case 2. $A = diag(K, 1, 1) \quad (K \in (-1, 1) - \{0\})$.

Let $B^t = (B_1,\ B_2,\ B_3)$, $C = (C_1,\ C_2,\ C_3)$. Then (1.58) gives

$$diag\left(K^2,\ 1,\ 1\right) - diag\left(C_1^2,\ C_2^2,\ C_3^2\right) = diag(1, 1, 1)$$

from which follows

$$C_1 = \pm\sqrt{K^2 - 1}, \quad C_2 = C_3 = 0.$$

Equation (1.59) gives

$$(KB_1,\ B_2,\ B_3) = (\pm D\sqrt{K^2 - 1}, 0, 0)$$

from which follows

$$B_1 = \pm \frac{D}{K}\sqrt{K^2 - 1}, \ B_2 = B_3 = 0.$$

Finally relation (1.60) implies

$$D = \pm K.$$

We conclude that in Case 2 we have the solution ($K \in (-1, 1) - \{0\}$)

$$\begin{aligned} A &= diag(K, 1, 1), \qquad C = \left(\pm\sqrt{K^2 - 1}, 0, 0\right), \\ B &= \left(\pm\sqrt{K^2 - 1}, 0, 0\right)^t, \quad D = \pm K, \end{aligned} \tag{1.64}$$

which defines the following sixteen Lorentz transformations ($C_1 = \pm\sqrt{K^2 - 1}$):

$$L_+ = \begin{bmatrix} K & C_1 & 0 & 0 \\ C_1 & K & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ L_2 = \begin{bmatrix} -K & -C_1 & 0 & 0 \\ C_1 & K & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$L_3 = \begin{bmatrix} K & C_1 & 0 & 0 \\ -C_1 & K & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ L_4 = \begin{bmatrix} -K & C_1 & 0 & 0 \\ -C_1 & K & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The special solutions L_+ are called *boosts*.

General solution

We define the 3×1 matrix β with the relation

$$B = -D\beta.$$

Replacing in (1.60) we find

$$D^{-1} = \pm\sqrt{1 - \beta^2},$$

where $\beta^2 = \beta^t \beta$ is a scalar and we assume $0 < \beta^2 < 1$ in order $D \in R$. Equation (1.59) implies

$$-D\beta^t A - DC = 0 \Rightarrow C = -\beta^t A.$$

Replacing in (1.58) we find[19]

$$A^t A - A^t \beta\beta^t A = I_3 \Rightarrow A^t(I_3 - \beta\beta^t)A = I_3,$$
$$A^t(I_3 - \beta\beta^t) = A^{-1} \Rightarrow AA^t(I_3 - \beta\beta^t) = I_3.$$

To continue we need the inverse of the matrix $(I - \beta\beta^t)$. We claim that $(I - \beta\beta^t)^{-1} = I + D^2\beta\beta^t$. Indeed

$$\begin{aligned}(I - \beta\beta^t)(I + D^2\beta\beta^t) &= I - \beta\beta^t + D^2\beta\beta^t - D^2\beta\beta^t\beta\beta^t \\ &= I + (-1 + D^2 - D^2\beta^2)\beta\beta^t \\ &= I + \left[-1 + D^2(1 - \beta^2)\right]\beta\beta^t = I.\end{aligned}$$

Therefore $AA^t = I + D^2\beta\beta^t$. In order to compute the matrix A we note that

$$\begin{aligned}\left(I + \frac{D-1}{\beta^2}\beta\beta^t\right)^2 &= I + 2\frac{D-1}{\beta^2}\beta\beta^t + \left(\frac{D-1}{\beta^2}\right)^2 \left(\beta\beta^t\right)^2 \\ &= I + \frac{D-1}{\beta^2}(2 + D - 1)\beta\beta^t = I + \frac{(D-1)(D+1)}{\beta^2}\beta\beta^t \\ &= I + \frac{D^2-1}{\beta^2}\beta\beta^t = I + D^2\beta\beta^t.\end{aligned}$$

Using the fact that $\beta\beta^t$ is a symmetric matrix one can easily show that

$$I_3 + \frac{D^2-1}{\beta^2}\beta\beta^t = \left(I_3 + \frac{D^2-1}{\beta^2}\beta\beta^t\right)^t.$$

This implies

$$AA^t = I_3 + D^2\beta\beta^t = \left(I + \frac{D-1}{\beta^2}\beta\beta^t\right)\left(I + \frac{D-1}{\beta^2}\beta\beta^t\right)$$

$$= \left(I + \frac{D-1}{\beta^2}\beta\beta^t\right)\left(I + \frac{D-1}{\beta^2}\beta\beta^t\right)^t \Rightarrow$$

$$A = \pm\left(I + \frac{D-1}{\beta^2}\beta\beta^t\right)E,$$

[19] Note that $\beta\beta^t$ is a symmetric 3×3 matrix.

where the 3×3 matrix E is an EOT, that is, satisfies the property $E^t E = I_3$. Replacing in $C = -\beta^t A$ we find for the matrix C,

$$C = \mp\beta^t \left(I + \frac{D-1}{\beta^2}\beta\beta^t \right) E = \mp \left(\beta^t + (D-1)\beta^t\right) E = \mp D\beta^t E.$$

We conclude that the general Lorentz transformation is

$$L(\boldsymbol{\beta}, E) = L(\boldsymbol{\beta})R(E), \tag{1.65}$$

where the matrix $R(E)$ is one of the solutions of Case 1 (Euclidean solution) and the matrix $L(\beta)$ (relativistic solution) is defined by the block matrix

$$L(\boldsymbol{\beta}) = \begin{bmatrix} D & \mp D\beta^t \\ -D\beta & \pm\left(I + \frac{D-1}{\beta^2}\beta\beta^t\right) \end{bmatrix}, \tag{1.66}$$

where $D = \pm\gamma,\ \gamma = \sqrt{1-\beta^2}$.

There result 16 different Lorentz transformations which are defined by the signs of the terms with D, the sign of the term $I + \frac{D-1}{\beta^2}\beta\beta^t$. If we take into consideration the rotation matrices $R_\pm(E)$ we have in total 32 cases. In the following with the term Lorentz transformation we shall mean the matrix $L(\boldsymbol{\beta})$, which is the relativistic part of the transformation. The role of the Euclidean part $R(E)$ will be discussed in Sect. 1.8.1. From (1.66) we find four disjoint subsets of Lorentz transformations classified by the sign of the determinant of the transformation and the sign of the components[20] with D.

Based on the above results we have the following four classes of Lorentz transformations:

(a) Proper Lorentz transformation ($D = \gamma$):

$$L_{+\uparrow}(\boldsymbol{\beta}) = \begin{bmatrix} \gamma & -\gamma\beta^t \\ -\gamma\beta & I + \frac{\gamma-1}{\beta^2}\beta\beta^t \end{bmatrix}. \tag{1.67}$$

(b) Lorentz transformation with space inversion ($D = \gamma$):

$$L_{+\downarrow}(\boldsymbol{\beta}) = \begin{bmatrix} \gamma & \gamma\beta^t \\ -\gamma\beta & -I - \frac{\gamma-1}{\beta^2}\beta\beta^t \end{bmatrix}. \tag{1.68}$$

(c) Lorentz transformation with time inversion ($D = -\gamma$):

[20] The first parameter has to do with the orientability of M^4 and the second with the preservation of the sense of direction of the timelike curves.

$$L_{-\uparrow}(\boldsymbol{\beta}) = \begin{bmatrix} -\gamma & \gamma\beta^t \\ \gamma\beta & I - \frac{\gamma+1}{\beta^2}\beta\beta^t \end{bmatrix}. \tag{1.69}$$

(d) Lorentz transformation with spacetime inversion ($D = -\gamma$):

$$L_{-\downarrow}(\boldsymbol{\beta}) = \begin{bmatrix} -\gamma & -\gamma\beta^t \\ \gamma\beta & -I + \frac{\gamma+1}{\beta^2}\beta\beta^t \end{bmatrix}. \tag{1.70}$$

All four forms of the Lorentz transformation are useful in the study of physical problems. But as a rule we use the proper Lorentz transformations because they form a (continuous) group of transformations. A closed subgroup in this group is the *boosts* which are the proper Lorentz transformations defined by $\boldsymbol{\beta}_x = (1, 0, 0)$, $\boldsymbol{\beta}_y = (0, 1, 0)$, $\boldsymbol{\beta}_z = (0, 0, 1)$. For example the boost along the x-axis and along the y-axis, respectively, is

$$L_{+\uparrow,\,x}(\boldsymbol{\beta}) = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad L_{+\uparrow,\,y}(\boldsymbol{\beta}) = \begin{bmatrix} \gamma & 0 & -\gamma\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\gamma\beta & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{1.71}$$

Example 2 Compute det $L(\beta)$ and comment on the result.
Solution

We set $\epsilon = \pm 1$, $k = \frac{D-1}{\beta^2}$ and have

$$L(\boldsymbol{\beta}) = \begin{bmatrix} D & -\varepsilon D\beta_1 & -\varepsilon D\beta_2 & -\varepsilon D\beta_3 \\ -D\beta_1 & \varepsilon + \varepsilon k\beta_1^2 & \varepsilon k\beta_1\beta_2 & \varepsilon k\beta_1\beta_3 \\ -D\beta_2 & \varepsilon k\beta_1\beta_2 & \varepsilon + \varepsilon k\beta_2^2 & \varepsilon k\beta_2\beta_3 \\ -D\beta_3 & \varepsilon k\beta_1\beta_3 & \varepsilon k\beta_2\beta_3 & \varepsilon + \varepsilon k\beta_3^2 \end{bmatrix}. \tag{1.72}$$

A standard computation gives

$$\begin{aligned} \det\ L(\boldsymbol{\beta}) &= -\epsilon^2 D\left[(D - k\epsilon)\beta_1^2 + (D - k\epsilon)\beta_2^2 + (D - k\epsilon)\beta_1^2 - \epsilon\right] \\ &= -D\left[(D - k\varepsilon)\beta^2 - \varepsilon\right] = 1 + D(\varepsilon - 1). \end{aligned} \tag{1.73}$$

We consider the following cases:

(1) If $\epsilon = +1 \Rightarrow$ det $L(\boldsymbol{\beta}) = +1$ and the resulting matrices $L(\boldsymbol{\beta})$ form a group with group operation the matrix multiplication. This class contains 16 cases.

(2) If $\epsilon = -1 \Rightarrow \det\ L(\boldsymbol{\beta}) = 1 - 2D \neq +1$ and the resulting transformations do not form a group. This class contains the remaining 16 cases.

In order to write the proper Lorentz transformation as a coordinate transformation in M^4 we consider an arbitrary four-vector and write[21]

$$\begin{pmatrix} l' \\ \mathbf{r}' \end{pmatrix} = L(\boldsymbol{\beta}) \begin{pmatrix} l \\ \mathbf{r} \end{pmatrix}. \tag{1.74}$$

Then we find the following "vector expressions" of the Lorentz transformation:

(a) Proper Lorentz transformation:

$$\mathbf{r}' = \mathbf{r} + \frac{\gamma - 1}{\beta^2}(\boldsymbol{\beta} \cdot \mathbf{r})\boldsymbol{\beta} - \gamma l \boldsymbol{\beta}, \qquad l' = \gamma(1 - \boldsymbol{\beta} \cdot \mathbf{r}). \tag{1.75}$$

(b) Lorentz transformation with space inversion:

$$\mathbf{r}' = -\mathbf{r} - \frac{\gamma - 1}{\beta^2}(\boldsymbol{\beta} \cdot \mathbf{r})\boldsymbol{\beta} - \gamma l \boldsymbol{\beta}, \quad l' = \gamma(1 + \boldsymbol{\beta} \cdot \mathbf{r}). \tag{1.76}$$

(c) Lorentz transformation with time inversion:

$$\mathbf{r}' = \mathbf{r} - \frac{\gamma + 1}{\beta^2}(\boldsymbol{\beta} \cdot \mathbf{r})\boldsymbol{\beta} + \gamma l \boldsymbol{\beta}, \qquad l' = -\gamma(1 - \boldsymbol{\beta} \cdot \mathbf{r}). \tag{1.77}$$

(d) Lorentz transformation with spacetime inversion:

$$\mathbf{r}' = -\mathbf{r} + \frac{\gamma + 1}{\beta^2}(\boldsymbol{\beta} \cdot \mathbf{r})\boldsymbol{\beta} + \gamma l \boldsymbol{\beta}, \qquad l' = -\gamma(1 + \boldsymbol{\beta} \cdot \mathbf{r}). \tag{1.78}$$

In the following example we give a simpler version of the derivation of Lorentz transformations using simple algebra.

Example 3 Consider in space R^4 the *linear* transformations

$$R^4 \rightarrow R^4,$$
$$(l, x, y, z) \rightarrow (l', x', y', z'),$$

which

---

[21] The Euclidean part of the transformation E is ignored because we assume that it defines the relative orientation of the axes of the two coordinate frames related by the Lorentz transformation and does not effect the transformation of the four-vectors (and more generally tensors) in M^4. Moreover in Sect. 1.8.1 we shall define that the space axes of two LCFs are parallel if $E = I_3$. In other words the Lorentz transformation $L(\boldsymbol{\beta})$ relates the coordinates of two LCFs whose spatial axes are parallel. This has always to be kept in mind.

(i) Are defined by the transformation equations

$$\begin{aligned} l' &= \gamma_1 x + \gamma_2 y + \gamma_3 z + \beta_4 l, \\ x' &= \alpha_1 x + \beta_1 l, \\ y' &= \alpha_2 y + \beta_2 l, \\ z' &= \alpha_3 z + \beta_3 l, \end{aligned}$$

where the 10 parameters $\alpha_i, \beta_i, \gamma_i \quad (i = 1, 2, 3)$ are such that $\alpha_i \neq 0$, $\beta_4 \neq 0$ and at least one of the $\gamma_i \neq 0$.

(ii) Satisfy the relation

$$x'^2 + y'^2 + z'^2 - l'^2 = x^2 + y^2 + z^2 - l^2, \tag{1.79}$$

that is, leave the Lorentz length (in an LCF!) invariant.

(a) Show that the transformations defined by requirements (i), (ii) are 32 and can be classified by means of one parameter only. Write these transformations in terms of a general expression.
(b) Compute the determinant of the general transformation and show that 16 of them have determinant equal to $+1$ and the rest 16 have determinant equal to -1.
(c) Demand further that when $l = l' = 0$ then $x = x'$, $y = y'$, $z = z'$ and show that with this requirement only two transformations survive.
(d) Give a kinematic interpretation of these two transformations.

Solution

(a) From the transformation equations we have

$$\begin{aligned} &(x')^2 + (y')^2 + (z')^2 - (l')^2 \\ &= (\alpha_1 x + \beta_1 l)^2 + (\alpha_2 y + \beta_2 l)^2 + (\alpha_3 z + \beta_3 l)^2 - (\gamma_1 x + \gamma_2 y + \gamma_3 z + \beta_4 l)^2 \\ &= (\alpha_1^2 - \gamma_1^2)x^2 + (\alpha_2^2 - \gamma_2^2)y^2 + (\alpha_3^2 - \gamma_3^2)z^2 + (\beta_1^2 + \beta_2^2 + \beta_3^2 - \beta_4^2)l^2 \\ &\quad + 2xl(\alpha_1\beta_1 - \gamma_1\beta_4) + 2yl(\alpha_2\beta_2 - \gamma_2\beta_4) + 2zl(\alpha_3\beta_3 - \gamma_3\beta_4) \\ &\quad - 2\gamma_1\gamma_2 xy - 2\gamma_2\gamma_3 yz - 2\gamma_3\gamma_1 zx. \end{aligned} \tag{1.80}$$

Comparison of (1.79) and (1.80) implies the relations ($i = 1, 2, 3$)

$$\alpha_i^2 - \gamma_i^2 = 1, \tag{1.81}$$

$$\gamma_1\gamma_2 = \gamma_2\gamma_3 = \gamma_3\gamma_1 = 0, \tag{1.82}$$

$$\beta_1^2 + \beta_2^2 + \beta_3^2 - \beta_4^2 = -1, \tag{1.83}$$

$$\alpha_i\beta_i - \gamma_i\beta_4 = 0. \tag{1.84}$$

Relations (1.81), (1.82), (1.83) and (1.84) constitute a system of nine simultaneous equations which can be used to express nine parameters in terms of one. Indeed (1.82) implies that two of the γ_i are equal to zero. Without restricting the generality we assume $\gamma_1 \neq 0 \Rightarrow \gamma_2 = \gamma_3 = 0$. Then (1.81) gives

$$\alpha_1 = \pm\sqrt{1+\gamma_1^2}, \tag{1.85}$$

$$\alpha_2 = \pm 1, \qquad \alpha_3 = \pm 1. \tag{1.86}$$

Then from (1.83) and (1.84) it follows that

$$\beta_1 = \frac{\gamma_1}{\alpha_1}\beta_4, \tag{1.87}$$

$$\beta_2 = \beta_3 = 0, \tag{1.88}$$

$$\beta_4 = \pm\sqrt{1+\gamma_1^2}. \tag{1.89}$$

We note that all coefficients of the transformation (1.79) have been expressed in terms of the parameter γ_1. The general form of the transformation is

$$\begin{pmatrix} l' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{bmatrix} \beta_4 & \gamma_1 & 0 & 0 \\ \frac{\gamma_1\beta_4}{\alpha_1} & \alpha_1 & 0 & 0 \\ 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & 0 & \alpha_3 \end{bmatrix} \begin{pmatrix} l \\ x \\ y \\ z \end{pmatrix} = L(\gamma_1) \begin{pmatrix} l \\ x \\ y \\ z \end{pmatrix},$$

where the quantities α_1, β_4 are defined in terms of the parameter γ_1 via relations (1.85) and (1.89), respectively. It follows that we have in total $2^5 = 32$ possible one-parameter transformations, $L(\gamma_1)$ say, which are specified by the different combinations of signs of the components of the transformation. Because the parameter γ_1 appears as $1+\gamma_1^2$ we introduce a new parameter γ with the relation

$$1+\gamma_1^2 = \frac{1}{1-\beta^2} = \gamma^2 \qquad (-1 < \beta < 1, \quad \gamma > 0) \tag{1.90}$$

and have

$$\gamma_1 = \pm\gamma\beta. \tag{1.91}$$

Then the expression of the various parameters in terms of the new parameter γ is

$$\alpha_1 = \pm\gamma,\ \alpha_2 = \pm 1,\ \alpha_3 = \pm 1,\ \beta_4 = \pm\gamma.$$

The 32 transformations are written in terms of the parameter β (or γ):

$$
\begin{aligned}
l' &= \varepsilon_1 \gamma \beta x + \varepsilon_2 \gamma l, \\
x' &= \varepsilon_3 \gamma x + \varepsilon_1 \varepsilon_2 \varepsilon_3 \beta \gamma l, \\
y' &= \varepsilon_4 y, \\
z' &= \varepsilon_5 z,
\end{aligned} \tag{1.92}
$$

where the quantities $\varepsilon_i = \pm 1 \quad (i = 1, 2, 3, 4, 5)$ are defined by the relations

$$
\alpha_1 = \varepsilon_3 \gamma,\ \alpha_2 = \varepsilon_4,\ \alpha_3 = \varepsilon_5,\ \beta_4 = \varepsilon_2 \gamma,\ \gamma_1 = \varepsilon_1 \gamma \beta. \tag{1.93}
$$

(b) The determinant of the matrix $L(\beta)$ equals

$$
\det\ L(\beta) = \begin{vmatrix} \varepsilon_2 \gamma & \varepsilon_1 \gamma \beta & 0 & 0 \\ \varepsilon_1 \varepsilon_2 \varepsilon_3 \beta \gamma & \varepsilon_3 \gamma & 0 & 0 \\ 0 & 0 & \varepsilon_4 & 0 \\ 0 & 0 & 0 & \varepsilon_5 \end{vmatrix} = \varepsilon_2 \varepsilon_3 \varepsilon_4 \varepsilon_5. \tag{1.94}
$$

The possible values of the determinant are ± 1. The requirement $\det\ L(\beta) = +1$ is equivalent to the condition

$$
\varepsilon_2 \varepsilon_3 \varepsilon_4 \varepsilon_5 = 1, \tag{1.95}
$$

which gives $2^4 = 16$ cases. Therefore, the condition that the determinant equals $+1$ or -1 selects 16 cases, respectively. We select the first set because it contains the identity transformation (whose determinant equals $+1$). The transformations in this set form a group and their general form is given by the transformation equations (1.92).
(c) We consider next the transformations which in addition to the condition $\det L = +1$ satisfy the condition that when $l' = l = 0$ then $x = x'$, $y = y'$, $z = z'$. The new condition implies the equations

$$
\beta \gamma x = 0, \quad x = \varepsilon_3 \gamma x, \quad y = \varepsilon_4 y, \quad z = \varepsilon_5 z, \tag{1.96}
$$

which are satisfied by the following values of the parameters:

$$
\beta = 0 \quad (\text{or } \gamma = 1), \quad \varepsilon_3 \varepsilon_4 \varepsilon_5 = 1.
$$

The last equation and (1.95) imply

$$
\varepsilon_2 = 1. \tag{1.97}
$$

There remains only one parameter free, the ε_1, therefore there are two families of single-parametric Lorentz transformations with $\det\ L(\beta) = +1$. The transformations in these families constitute a group and they are called *boosts* with parameter

$\beta \quad (\varepsilon_1 = -1)$ and $-\beta \quad (\varepsilon_1 = 1)$, respectively. The transformation equations for each case are as follows:
$\varepsilon_1 = 1$

$$\begin{aligned} l' &= \gamma(l + \beta x), \\ x' &= \gamma(x + \beta l), \\ y' &= y, \ z' = z. \end{aligned} \tag{1.98}$$

$\varepsilon_1 = -1$

$$\begin{aligned} l' &= \gamma(l - \beta x), \\ x' &= \gamma(x - \beta l), \\ y' &= y, \ z' = z. \end{aligned} \tag{1.99}$$

In the following table we summarize the results concerning the number of free parameters and the number of the corresponding transformations for each requirement.[22]

Requirements	Free parameter	Possible transformations
Conditions (1.79,1.80)	1	32
det $L(\beta) = +1$ or -1	1	16
$l = l' = 0 \Rightarrow \mathbf{r}' = \mathbf{r}$	1	2

(d) We shall give the geometric interpretation of the transformation in Sect. 1.8.1. Concerning the kinematic interpretation (that is interpretation involving the time and the space) of the transformation we note the following:

- The transformation is single-parametric, therefore it must be related with one scalar kinematic quantity only.
- The transformation is symmetric in the sense that if the coordinates (l', x', y', z') are expressed in terms of (l, x, y, z) with β then the (l, x, y, z) are expressed in terms of (l', x', y', z') with $-\beta$ (prove this).
- The transformation must satisfy the initial condition $l = l' = 0$, then $x = x'$, $y = y'$, $z = z'$ (or $\mathbf{r} = \mathbf{r}'$).

The above results lead to the following kinematic interpretation of the transformation. The coordinates l, l' concern the time and the coordinates x, x', y, y', z, z' the orthonormal (in the Euclidean sense!) spatial axes of two Relativistic inertial observers. At the time moment $l = l' = 0$ the spatial axes of the observers coincide and subsequently they are moving so that the plane $y - z$ is parallel transported with

[22] We note that a pure Lorentz transformation (i.e., Euclidean rotations and translations excluded) depends on six parameters. Here we are left with one parameter due to the simplified form of the transformation.

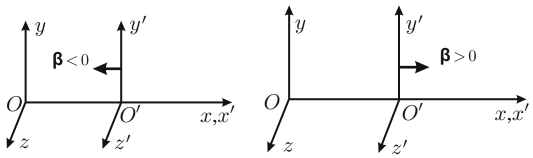

Fig. 1.4 Kinematic interpretation of Lorentz transformation

respect to the plane $x' - z'$ (because during the motion $y = y'$) and similarly the plane $x - y$ is parallel transported with respect to the plane $x' - y'$ (because during the motion $z = z'$).

We conclude that there are only two motions possible:

- One motion in which the x'-axis slides along the x-axis in the direction $x > 0$
- One motion in which the x'-axis slides along the x-axis in the direction $x < 0$

We consider that the first type of motion corresponds to the values $1 > \beta > 0$ while the second to the values $0 > \beta > -1$. We identify the parameter β with the quotient v/c where $|v| < c$ (because $\beta < 1$) is the speed of the relative velocity of the parallel axes x, y, z and x', y', z' and c is a universal constant, which we identify with the speed of light in vacuum (see Fig. 1.4).

Exercise 4 Prove the identities[23]

$$\gamma^2 = \gamma^2\beta^2 + 1, \tag{1.100}$$

$$\gamma\left(\frac{\beta_1 \pm \beta_2}{1 \pm \beta_1\beta_2}\right) = \gamma(\beta_1)\gamma(\beta_2)\,(1 \pm \beta_1\beta_2)\,, \tag{1.101}$$

$$d\gamma = \gamma^3\beta d\beta, \qquad d(\gamma\beta) = \gamma^3 d\beta, \tag{1.102}$$

$$\gamma = 1 + \frac{1}{2}\beta^2 + \frac{3}{8}\beta^4 + \dots\,. \tag{1.103}$$

Exercise 5 Consider the matrix

$$L^{i'}_{j}(\beta) = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where $\gamma = 1/\sqrt{1-\beta^2}$, $\beta \in [0, 1)$.

(a) Compute the inverse $L^{i}_{j'}(\beta)$.
(b) Prove that $L^{i'}_{j}(\beta)$ is a Lorentz transformation.

---

[23] Identity (1.100) is used extensively in the calculations. Identity (1.101) expresses the relativistic composition of three-velocities under successive boosts.

(c) Define the hyperbolic angle ϕ with the relation $\cosh\phi = \gamma$ and prove that the boost along the x-axis is written as

$$L^i_j(\beta) = \begin{bmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where $\sinh\phi = \beta\gamma$. Also show that $\tanh\phi = \beta$ and $e^\phi = \sqrt{\frac{1+\beta}{1-\beta}}$. The parameter ϕ is called the *rapidity* of the transformation.

[Hint: (a) The inverse is

$$L^i_j(\beta) = \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(b) It is enough to show that $\eta_{ij} = \eta_{rs} L^r_i L^s_j$ or in the form of matrices $\eta = L^t \eta L$ where $\eta = diag(-1, 1, 1, 1)$.]

Example 4 Consider the LCFs $\{K', x^{i'}, \mathbf{E}_{i'}\}, \{K, x^i, \mathbf{E}_i\}$ which are related by the transformation

$$\begin{aligned} x^{1'} &= -\sinh\phi x^0 + \cosh\phi x^1, \\ x^{2'} &= x^2, \qquad x^{3'} = x^3, \\ x^{0'} &= -\sinh\phi x^1 + \cosh\phi x^0, \end{aligned}$$

where ϕ is a real parameter.

(a) Prove that the transformation $K \to K'$ is a Lorentz transformation.
(b) A four-vector V^i in K has components $(0, 1, 0, 1)^t$. Compute the components of V^i in K'. A Lorentz tensor T_{ij} of type $(0, 2)$ in K' has all its components equal to zero except the $T_{1'1'} = T_{3'3'} = 1$. Compute the components of T_{ij} in K. Compute in K the covariant vector $T_{ij}V^i$ and the invariant $T_{ij}V^iV^j$.

Solution

(a) The matrix transformation between K, K' is

$$L^i_j(\beta) = \begin{bmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

It is easy to show that this is a Lorentz transformation (see Exercise 5).

(b) For the four-vector V^i we have

$$V^{i'} = \frac{\partial x^{i'}}{\partial x^i} V^i = L_i^{i'} V^i.$$

Hence

$$\begin{aligned} V^{0'} &= L_i^{0'} V^i = L_0^{0'} V^0 + L_1^{0'} V^1 + L_2^{0'} V^2 + L_3^{0'} V^3 = -\sinh\phi, \\ V^{1'} &= L_i^{1'} V^i = L_0^{1'} V^0 + L_1^{1'} V^1 + L_2^{1'} V^2 + L_3^{1'} V^3 = \cosh\phi, \\ V^{2'} &= L_i^{2'} V^i = L_0^{2'} V^0 + L_1^{2'} V^1 + L_2^{2'} V^2 + L_3^{2'} V^3 = 0, \\ V^{3'} &= L_i^{3'} V^i = L_0^{3'} V^0 + L_1^{3'} V^1 + L_2^{3'} V^2 + L_3^{3'} V^3 = 1. \end{aligned}$$

Therefore $[V^i] = (-\sinh\phi, \cosh\phi, 0, 1)^t$.

Similarly for the tensor T_{ij} we have

$$\begin{aligned} T_{ij} &= \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^{j'}}{\partial x^j} T_{i'j'} = L_i^{i'} L_j^{j'} T_{i'j'} = L_i^{1'} L_j^{1'} T_{1'1'} + L_i^{3'} L_j^{3'} T_{3'3'} \\ &= L_i^{1'} L_j^{1'} + L_i^{3'} L_j^{3'}, \end{aligned}$$

where we have used the fact that in K' only the components $T_{1'1'}$, $T_{3'3'}$ do not vanish and are equal to 1. From this relation we compute the components of T_{ij} in K using the standard method. For example for the T_{00} component we have

$$T_{00} = L_0^{1'} L_0^{1'} + L_0^{3'} L_0^{3'} = \sinh^2\phi.$$

In the form of a matrix the components of T_{ij} in K are

$$[T_{ij}] = \begin{bmatrix} \sinh^2\phi & -\sinh\phi\cosh\phi & 0 & 0 \\ -\sinh\phi\cosh\phi & \cosh^2\phi & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_K.$$

In K we have for the vector $a_i = T_{ij} V^j$:

$$\begin{aligned} a_0 &= T_{00} V^0 + T_{01} V^1 + T_{02} V^2 + T_{03} V^3 = -\cosh\phi\sinh\phi, \\ a_1 &= T_{10} V^0 + T_{11} V^1 + T_{12} V^2 + T_{13} V^3 = \cosh^2\phi, \\ a_2 &= T_{20} V^0 + T_{21} V^1 + T_{22} V^2 + T_{23} V^3 = 0, \\ a_3 &= T_{30} V^0 + T_{31} V^1 + T_{32} V^2 + T_{33} V^3 = 1. \end{aligned}$$

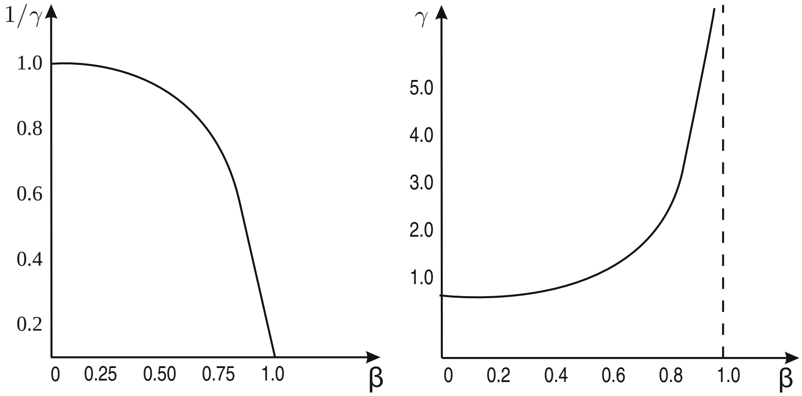

Fig. 1.5 The factors γ and $\frac{1}{\gamma}$

Finally for the invariant $a_i V^i = T_{ij} V^i V^j$ we have (in all LCFs!)

$$a_i V^i = a_0 V^0 + a_1 V^1 + a_2 V^2 + a_3 V^3 = \cosh^2 \phi + 1.$$

Exercise 6 Derive the results of Example 4 using matrix multiplication as follows:

$$[\,V]_{K'} = [L][V]_K, \ [T]_{K'} = [L^{-1}]^t[T]_K[L^{-1}], \ [a]_K = ([T]^t[V]_K)^t,$$
$$a_i V^i = [a]_K[V]_K.$$

Example 5 Compute the values of the functions $\gamma = \left(1 - \beta^2\right)^{-\frac{1}{2}}$ and γ^{-1} for the values $\beta = 0.100, 0.300, 0.600, 0.800, 0.900, 0.950, 0.990$. Plot the results.
Solution

β	0.100	0.300	0.600	0.800	0.900	0.950	0.990
$1/\gamma$	0.995	0.954	0.800	0.600	0.436	0.312	0.141
γ	1.005	1.048	1.250	1.667	2.294	3.203	7.089

Using the figures of the table we draw the curves of Fig. 1.5 from which we note that the relativistic effects become significant for values $\beta > 0.8$. This is the reason why we expect the relativistic effects to appear at high relative velocities.

1.8 The Kinematic Interpretation of the General Lorentz Transformation

1.8.1 Relativistic Parallelism of Space Axes

In Sect. 1.7 we have shown that the general Lorentz transformation $L(\boldsymbol{\beta}, E)$ can be written as the product of two transformations:

$$L(\boldsymbol{\beta}, E) = L(\boldsymbol{\beta})R(E). \tag{1.104}$$

The transformation $L(\boldsymbol{\beta})$ depends on three parameters $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3)$, which may be considered as the components of a vector in a linear space R^3. The transformation $R(E)$ is an EOT and it is also defined uniquely in terms of three parameters, e.g., the Euler angles. Each part of the general Lorentz transformation has a different kinematical meaning.

In an LCF the transformation $R(E)$ is (see (1.61))

$$R(E) = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & E \end{array}\right], \tag{1.105}$$

therefore it effects the spatial part of the four-vectors only. This leads to the following geometric and kinematic interpretation of the transformation $R(E)$. The transformation $R(E)$

(1) Distinguishes the components of a four-vector A^i in two groups: the temporal or zeroth component, which is not affected by the action of $R(E)$, and the spatial part, which is transformed as a three-vector under the action of the Euclidean transformation E. This grouping of the components of a four-vector is very important and it is used extensively in the study of Relativity Physics (Special and General). It is known as the $1 + 3$ decomposition and it can be extended to apply to any tensor (see Sect. 12.2).
(2) Geometrically describes the relative orientation of the spatial axes of the LCF it relates. That is if $K(E_0, E_1, E_2)$ and $K'(E'_0, E'_1, E'_2)$ are two LCFs related by the general Lorentz transformation $L(\boldsymbol{\beta}, E)$, then the matrix E relates the spatial bases of K, K' according to the equation

$$(E'_1, E'_2, E'_3) = (E_1, E_2, E_3)E. \tag{1.106}$$

This interpretation of the transformation $R(E)$ is Euclidean in the sense that it refers to the relative orientation of the axes of the frames in Euclidean space E^3. This means that we may distinguish the action of the general Lorentz transformation in two parts: a Euclidean part encountered by the action of the transformation $R(E)$ and a relativistic part expressed by $L(\boldsymbol{\beta})$.

Therefore without restricting the generality we can get rid of the transformation $R(E)$ by simply considering the general Lorentz transformation for $E = I_3$. Then one is left with the relativistic part of the transformation. Now $E = I_3$ means that the axes of the frames K, K' have the same orientation, that is, they are parallel. This parallelism is understood in the *Euclidean sense and it is not Lorentz invariant*. Therefore it has to be defined in a covariant manner in order to attain objectivity in the world of M^4. This leads us to the following definition:

Definition 3 Let K, K' be two LCFs which are related by the general Lorentz transformation $L(\boldsymbol{\beta}, E) = L(\boldsymbol{\beta})R(E)$. We say that the spatial axes of K, K' are relativistically parallel if, and only if, $R(E) = I_4$.

This concept of parallelism is directly comparable to the Euclidean concept of parallelism in E^3 because the latter can be defined as follows:

Definition 4 Let Σ, Σ' be two ECF which are related with the EOT E. We say that the axes of Σ, Σ' are parallel if, and only if, $E = I_3$ where $I_3 = diag(1, 1, 1)$ is the identity matrix.

The fact that the Euclidean and the relativistic parallelism of space axes are closely related is frequently misunderstood and has led to erroneous conclusions. On the other hand it has important applications as, for example, the Thomas rotation. The difference between the two types of parallelism is that they coincide for two LCFs K, K' but not necessarily for more. That is, if K, K', K'' are three LCFs such that the axes of the pairs K, K' and K', K'' are relativistically parallel then it does not follow (in general) that the space axes of the LCFs K, K'' are relativistically parallel. Of course this is not the case for the Euclidean parallelism.

The "broken parallelism" is due to the fact that the action of the second general Lorentz transformation $L(\boldsymbol{\beta}_{K''})$ acts on the Euclidean part $R(E_{K''})$ of the first Lorentz transformation so that the latter becomes $\neq I_4$, which "brakes" the relativistic parallelism of the space axes of K, K''.

The interpretation of the transformation $R(E)$ in terms of the relative direction of the space axes is possible because we assume isotropy of three-space, therefore all directions in space are equivalent. From the point of view of physics this means that the orientation of space axes in the geometric space does not affect the physical properties of physical systems. Such conditions are known as gauge conditions and play an important role in theoretical physics.

1.8.2 The Kinematic Interpretation of Lorentz Transformation

In order to give a kinematic interpretation of the (pure) Lorentz transformation $L(\boldsymbol{\beta})$ we use the fact that this transformation depends on the vector $\boldsymbol{\beta}$ only and not on the matrix E. Therefore we identify

$$L(\boldsymbol{\beta}) = L(\boldsymbol{\beta}, I_3),$$

that is, we consider that the Lorentz transformation is the general Lorentz transformation when the axes of Σ and Σ' are relativistically parallel. We identify the vector $\boldsymbol{\beta}$ with the relative velocity of Σ and Σ'. This kinematic interpretation is shown in Fig. 1.6.

With the above interpretation, the general Lorentz transformation in M^4 is decomposed into two general Lorentz transformations: one transformation $L(\boldsymbol{\beta}, I_3)$ in which the space axes of the related LCF are parallel and their relative velocity $\boldsymbol{\beta}$ and a second transformation $L(\mathbf{0}, E)$ which rotates the axes in the Euclidean space E^3 about the (fixed) origin in order to make them parallel. The relativistic part of the transformation is what we call the (pure) Lorentz transformation. This interpretation implies that the relativistic effects show up only when we have relative motion!

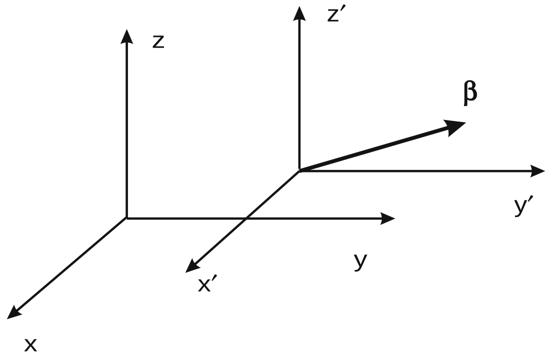

Fig. 1.6 The kinematic interpretation of Lorentz transformation

Geometrically it is possible to view the Lorentz transformation as a "hyperbolic rotation" around the space direction $\boldsymbol{\beta}$ with a hyperbolic angle ψ, defined by the relation

$$\cosh\psi = |\gamma|. \tag{1.107}$$

We call the hyperbolic angle ψ *rapidity*.

1.9 The Geometry of the Boost

As we have shown in Sect. 1.7 the Lorentz transformation $L(\boldsymbol{\beta})$[24] can be expressed as the product of a boost and two Euclidean rotations. These rotations concern the direction of the relative velocity in the (parallel) axes of the two LCFs related by the Lorentz transformation. This decomposition is helpful in practice because we solve a specific problem for a boost and then we use the Euclidean rotations to get the (usually more complicated) answer. Therefore it is important to study the geometric structure of the boost.

We recall that if (l, x, y, z) and (l', x', y', z') are the coordinates of a point P in M^4 in the LCF Σ and Σ', respectively, then the boost $L(\beta)$ (to be precise we should write $L(\boldsymbol{\beta})_x$) along the direction of the x-axis is the transformation

$$x' = \gamma(x - \beta l), \quad y' = y, \quad z' = z, \quad l' = \gamma(l - \beta x). \tag{1.108}$$

Let us represent this transformation on the *Euclidean* plane[25] (l, x) – the y, z coordinates are not affected by the transformation and we ignore them. The boost

[24] Not the general Lorentz transformation $L(\boldsymbol{\beta}, E)$!

[25] It is generally believed that this plane is a two-dimensional Minkowski space, that is, a two-dimensional linear space endowed with the Lorentz metric. This is wrong and leads to many

$L(\beta)$ and its inverse $L^{-1}(\beta)$ are represented with the following matrices:

$$L(\beta) = \begin{bmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{bmatrix}, \quad L^{-1}(\beta) = \begin{bmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{bmatrix}. \tag{1.109}$$

Because $L(\beta)$ is a linear transformation it is enough to study its action on the basis vectors $\boldsymbol{e}_{0,\Sigma} = (1, 0)_\Sigma$, $\boldsymbol{e}_{1,\Sigma} = (0, 1)_\Sigma$ of the LCF Σ. We have

$$(\mathbf{e}_{0,\ \Sigma'}, \mathbf{e}_{1,\ \Sigma'}) = (\mathbf{e}_{0,\ \Sigma}, \mathbf{e}_{1,\ \Sigma})L^{-1} = (\mathbf{e}_{0,\ \Sigma}, \mathbf{e}_{1,\ \Sigma}) \begin{bmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{bmatrix} \Rightarrow$$
$$\mathbf{e}_{0,\ \Sigma'} = \gamma\mathbf{e}_{0,\ \Sigma} + \beta\gamma\mathbf{e}_{1,\ \Sigma}, \quad \mathbf{e}_{1,\ \Sigma'} = \beta\gamma\mathbf{e}_{0,\ \Sigma} + \gamma\mathbf{e}_{1,\ \Sigma}. \tag{1.110}$$

Concerning the *Euclidean length* and the *Euclidean angles* of the basis vectors we have

$$\begin{aligned} |\mathbf{e}_{0,\ \Sigma}| &= |\mathbf{e}_{1,\ \Sigma}| = 1, \\ |\mathbf{e}_{0,\ \Sigma'}| &= |\mathbf{e}_{1,\ \Sigma'}| = \gamma\sqrt{1+\beta^2} = \sqrt{\frac{1+\beta^2}{1-\beta^2}} > 1, \\ \mathbf{e}_{0,\ \Sigma} \cdot \mathbf{e}_{1,\ \Sigma} &= 0, \qquad \mathbf{e}_{0,\ \Sigma'} \cdot \mathbf{e}_{1,\ \Sigma'} = 2\beta\gamma^2, \\ \mathbf{e}_{0,\ \Sigma} \cdot \mathbf{e}_{0,\ \Sigma'} &= \mathbf{e}_{1,\ \Sigma} \cdot \mathbf{e}_{1,\ \Sigma'} = \gamma. \end{aligned} \tag{1.111}$$

These results lead to the representation of the basis vectors in the Euclidean plane (l, x) as shown in Fig. 1.7.

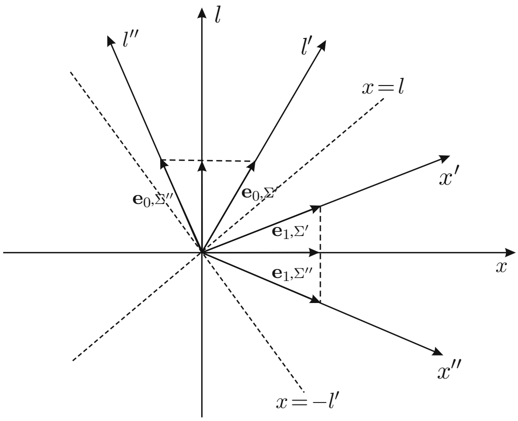

Fig. 1.7 The action of boost on the basis vectors

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misunderstandings in Special Relativity. It should always be borne in mind that the blackboard is and stays a two-dimensional Euclidean space no matter what we draw on it!

In Fig. 1.7 the Euclidean angle ϕ is defined by the relation

$$\tan\phi = \beta. \tag{1.112}$$

We note that the vectors $\boldsymbol{e}_{0,\Sigma'} = (1,0)_{\Sigma'},\ \boldsymbol{e}_{1,\Sigma'} = (0,1)_{\Sigma'}$ make equal angles with the basis vectors $\boldsymbol{e}_{0,\Sigma} = (1,0)_{\Sigma},\ \boldsymbol{e}_{1,\Sigma} = (0,1)_{\Sigma}$ and furthermore they are symmetric about the internal bisector. When $\beta = 1$ then $\tan\phi = 1$ hence $\phi = 45^o$ and the vectors $\boldsymbol{e}_{0,\Sigma'} = (1,0)_{\Sigma'},\ \boldsymbol{e}_{1,\Sigma'} = (0,1)_{\Sigma'}$ coincide with the internal bisector $l = x$ of the axes (l, x).

If the LCFs Σ and Σ' are moving with factor $-\beta$ then the vectors $\boldsymbol{e}_{0,\Sigma'} = (1,0)_{\Sigma'},\ \boldsymbol{e}_{1,\Sigma'} = (0,1)_{\Sigma'}$ make a common external angle ϕ with the vectors $\boldsymbol{e}_{0,\Sigma'} = (1,0)_{\Sigma'},\ \boldsymbol{e}_{1,\Sigma'} = (0,1)_{\Sigma'}$ (see Fig. 1.7).

Exercise 7 Prove that the Lorentz length of the basis vectors $\boldsymbol{e}_{0,\Sigma},\ \boldsymbol{e}_{1,\Sigma}, \boldsymbol{e}_{0,\Sigma'}, \boldsymbol{e}_{1,\Sigma'}$ equals ± 1. Also show that the vectors of each basis are Lorentz orthogonal and that the Lorentz angle between, e.g., the vectors $\boldsymbol{e}_{1,\Sigma}, \boldsymbol{e}_{1,\Sigma'}$ is given by the relation

$$\cosh\phi_L = \frac{\boldsymbol{e}_{1,\Sigma} \circ \boldsymbol{e}_{1,\Sigma'}}{|\boldsymbol{e}_{1,\Sigma}|_L |\boldsymbol{e}_{1,\Sigma'}|_L} = \gamma, \tag{1.113}$$

where $\circ$ indicates Lorentz product in the plane (l, x).

Note that in (1.113) we are using cosh and not cos because $\gamma > 1$.

Exercise 8 Show that the Euclidean lengths satisfy the relation

$$|\mathbf{e}_{i,\Sigma'}| = \frac{1}{\sqrt{\cosh 2\phi}} |\mathbf{e}_{i,\Sigma}| \qquad i = 0, 1 \tag{1.114}$$

and conclude that

$$|\mathbf{e}_{i,\Sigma'}| > |\mathbf{e}_{i,\Sigma}|. \tag{1.115}$$

The geometric meaning of the inequality (1.115) is that the *Euclidean* length of the unit rod along the x'-axis is larger than the Euclidean length of the unit rod along the x-axis. Therefore an object lying along the x-axis (e.g., a rod) when it is measured with the unit of Σ gives the number $d(\Sigma)$ and when it is measured with the unit of Σ' gives another number $d'(\Sigma')$ smaller than $d(\Sigma)$ because

$$d(\Sigma)|\mathbf{e}_{i,\Sigma}| = d'(\Sigma')|\mathbf{e}_{i,\Sigma'}|. \tag{1.116}$$

The physical meaning of the inequality $d'(\Sigma') < d(\Sigma)$ is that the (Euclidean!) length of the rod as measured in Σ (that is the number $d(\Sigma)$) is smaller than the (Euclidean!) length $d'(\Sigma')$ as measured in Σ'. This inequality of (Euclidean!) length measurements has been called length contraction. Concerning the unit along the timelike vector $\mathbf{e}_{0,\Sigma}$ we identify $d(\Sigma)$ with the negative of time duration. Then the inequality $d'(\Sigma') < d(\Sigma)$ means that the (Newtonian!) time duration of a phenomenon in Σ is smaller than the (Newtonian!) duration of the same phenomenon in Σ'. This result has been called time dilation effect. Both length contraction and time dilation will be discussed in Chap. 5.

A simple way to draw the vectors $\mathbf{e}_{0,L}$ and $\mathbf{e}_{1,L}$ on the *Euclidean plane* E^2 is the following. We consider a Lorentz unit vector (in E^2!) with components $\begin{pmatrix} x \\ y \end{pmatrix}$ and demand

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \pm 1$$

from which follows

$$-x^2 + y^2 = \pm 1.$$

We infer that the tip of the Lorentz unit vectors moves on hyperbolae with asymptotes $x = \pm y$ (see Fig. 1.8).

If we consider orthonormal coordinates (l, x) in the Euclidean plane then these hyperbolae are symmetric about the bisectors at the origin. For each value of the parameter β they make an angle ϕ with the l, x-axes. In order to compute this angle in terms of the parameter β we consider the Euclidean inner product of the vectors $\mathbf{e}_{0,\ L}, \mathbf{e}_{1,\ L}$ with the basis vectors $\mathbf{i}, \mathbf{j}$. For the vector $\mathbf{e}_{1,\ L}$ we have

$$\mathbf{e}_{1,L} \cdot \mathbf{i} = \cos\phi |\mathbf{e}_{1,L}|_E$$

from which follows

$$\cos\phi = \frac{\gamma}{\sqrt{2\gamma^2 - 1}}. \tag{1.117}$$

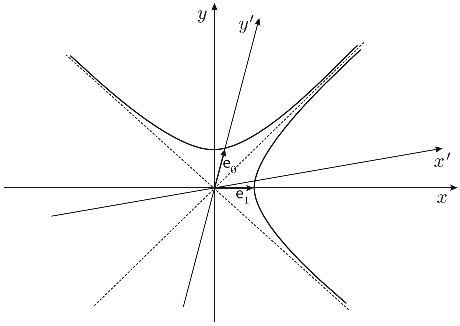

Fig. 1.8 The Lorentz transformation in the Euclidean plane

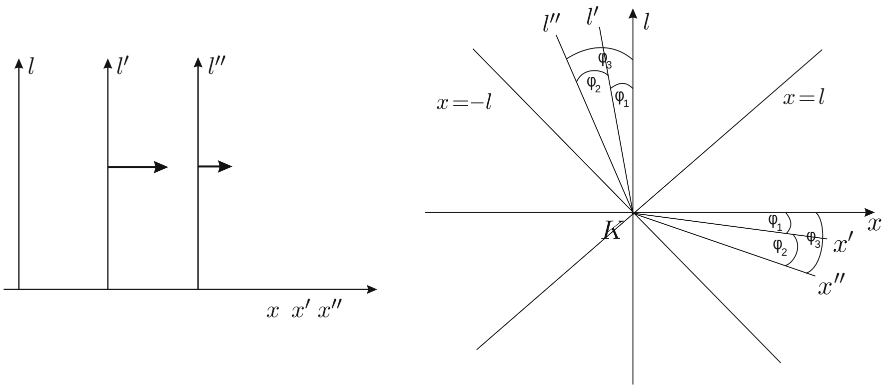

Fig. 1.9 Combination of β-factors under successive boosts

With this result we can draw in the Euclidean plane (l, x), the vectors $\mathbf{e}_{0,L}$ and $\mathbf{e}_{1,L}$ for every value of the parameter β.

Example 6 Consider three LCFs Σ, Σ', Σ'', which are moving in the standard configuration with factors β_1, β_2, respectively, along the common axes x, x', x''. Use the geometric representation of the boost to compute the factor β between the LCFs Σ, Σ'' (see also (1.101)).

Proof
From Fig. 1.9 we have $\tan\phi_3 = \tan(\phi_1 + \phi_2)$. Hence

$$\beta_3 = \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2}. \tag{1.118}$$

Another simple method to represent the boost in the Euclidean plane (l, x) is the following. We consider two orthogonal axes $l,\ x$ and consider the boost (1.108), which defines the new axes $l',\ x'$ on the same plane. The l'-axis is defined by the requirement

$$x' = 0 \Rightarrow x = \beta l,$$

therefore it is a straight line with an inclination β wrt the axis l. Similarly the x'-axis is defined by the requirement $l' = 0$, therefore it is the set of points

$$l = \beta x,$$

which is a straight line with inclination β wrt the axis x.

In Fig. 1.7 we have taken the axes $l,\ x$ to be orthogonal but this is not necessary and what we say holds for non-orthogonal axes. We have also drawn the axes x'', l'' with inclination $-\beta$.

1.10 Characteristic Frames of Four-Vectors

We have divided the four-vectors in M^4 in timelike, spacelike, and null according to if their length is < 0, > 0, or $= 0$, respectively. In this section we show that the first two types of four-vectors admit characteristic LCF in which they retain their reduced form, that is the timelike vectors have zero spatial components and the spacelike vectors have zero time component. As we shall see in the subsequent chapters the reduced form of the timelike and the spacelike vectors is used extensively in the study of Lorentz geometry and in the Theory of Special Relativity.

1.10.1 Proper Frame of a Timelike Four-Vector

Consider the timelike four-vector A^i which in the LCF Σ has decomposition $\begin{pmatrix} A^0 \\ \mathbf{A} \end{pmatrix}_\Sigma$. When the zeroth component $A^0 > 0$ (respectively, $A^0 < 0$) we say that the four-vector A^i is directed along the future (respectively, past) light cone. Because the proper Lorentz transformation preserves the sign of the zeroth component it is possible to divide the timelike four-vectors in *future directed* and *past directed*. For every timelike four-vector A^i we have $\mathbf{A}^2 - (A^0)^2 < 0$, therefore there exists always a unique LCF, Σ^+ say, in which the spatial components of A^i vanish and the four-vector takes its reduced or canonical form $\begin{pmatrix} A^{0+} \\ \mathbf{0} \end{pmatrix}_{\Sigma^+}$. We name the frame Σ^+ the *proper frame* of the four-vector A^i.

In order to determine Σ^+ when A^i is given in an arbitrary LCF Σ, we must find the β-factor $\boldsymbol{\beta}$ of Σ^+ with respect to Σ. This is done from the proper Lorentz transformation (1.51), (1.52) using the reduced form of A^i. We find

$$A^0 = \gamma (A^0)^+, \tag{1.119}$$

$$\mathbf{A} = \gamma \boldsymbol{\beta} (A^0)^+. \tag{1.120}$$

From these relations,

$$\gamma = \frac{A^0}{A^{0+}}, \tag{1.121}$$

$$\boldsymbol{\beta} = \frac{\mathbf{A}}{A^0}. \tag{1.122}$$

Equations (1.121) and (1.122) fix the proper frame Σ^+ of the four-vector A^i by giving the three parameters β^μ wrt an arbitrary LCF Σ in which we know the components of the four-vector A^i.

In the proper frame Σ^+ of A^i the length $A^i A_i = -(A^{0^+})^2$, therefore the component A^{0^+} is not simply a component but an invariant (tensor), so that it has the same value in all LCFs. We use this fact extensively in relativistic physics in order to define timelike four-vectors which have a definite physical meaning. Indeed we define the timelike four-vector in its proper frame and then we compute it in any other LCF using the appropriate Lorentz transformation. One such example is the four-velocity of a relativistic particle which, as we shall see, is a four-vector with constant length c, where c is the speed of light in empty space. Because in Special Relativity we consider c to be an invariant (in fact a universal constant) we define the four-velocity of the relativistic particle in its proper frame to be $u^i = \begin{pmatrix} c \\ \mathbf{0} \end{pmatrix}_{\Sigma^+}$. In any other frame with axes parallel to those of Σ^+ and with β-factor $\boldsymbol{\beta}$ relations (1.51), (1.52) give that the four-velocity is $u^i = \begin{pmatrix} \gamma c \\ \gamma c \boldsymbol{\beta} \end{pmatrix}_{\Sigma}$.

1.10.2 Characteristic Frame of a Spacelike Four-Vector

Let A^i be a spacelike four-vector which in an LCF Σ has decomposition $\begin{pmatrix} A^0 \\ \mathbf{A} \end{pmatrix}_{\Sigma}$. We are looking for another LCF Σ^- in which A^i has the reduced form $\begin{pmatrix} 0 \\ \mathbf{A}^- \end{pmatrix}_{\Sigma^-}$. The two decompositions of the four-vector are related with a proper Lorentz transformation, therefore the transformation equations (1.51), (1.52) give

$$B^0 = \gamma \boldsymbol{\beta} \cdot \mathbf{B}^-, \tag{1.123}$$

$$\mathbf{B} = \mathbf{B}^- + \frac{\gamma - 1}{\beta^2}(\boldsymbol{\beta} \cdot \mathbf{B}^-)\boldsymbol{\beta}. \tag{1.124}$$

We consider the Euclidean inner product of the second equation with $\boldsymbol{\beta}$ and get

$$\beta = \frac{B^0}{B_{||}},$$

where $B_{||} = \frac{\boldsymbol{\beta} \cdot \mathbf{B}}{\beta}$. It follows that it is not possible to define $\boldsymbol{\beta}$ completely (we can fix only the length of $\boldsymbol{\beta}$) as in the case of the timelike four-vectors. Therefore there are infinitely many LCFs in which a spacelike four-vector has its reduced form.

However, in the set of all these frames there is a unique LCF defined as follows. We consider the position four-vectors of the end points A, B of the spacelike four-vector AB^i and assume that in a characteristic frame Σ^- they are decomposed as $\begin{pmatrix} A^0 \\ A^- \hat{\mathbf{e}}_A \end{pmatrix}_{\Sigma^-}, \begin{pmatrix} A^0 \\ B^- \hat{\mathbf{e}}_B \end{pmatrix}_{\Sigma^-}$. Then we specify a unique characteristic frame by the condition $\hat{\mathbf{e}}_A = -\hat{\mathbf{e}}_B$. This particular characteristic frame of the spacelike four-vector AB^i is called the *rest frame* of AB^i. The rest frame is used to describe the motion of rigid rods in Special Relativity.

1.11 Particle Four-Vectors

The timelike and the null four-vectors play an important role in Special (and General) Relativity, because they are associated with physical quantities of particles and photons, respectively. Since in many problems the study of particles and photons is identical it is useful to introduce a new class of four-vectors, the particle four-vectors.

Definition 5 A four-vector is a *particle four-vector* if and only if

- It is a timelike or a null four-vector.
- The zeroth component in an LCF is positive (i.e., it is future directed).

For particle four-vectors we have the following result which is a consequence of Proposition 3.

Proposition 4 *The sum of particle four-vectors is a particle four-vector.*

This result indicates that geometry allows us to describe systems of particles with corresponding systems of particle four-vectors and, furthermore, to study the reaction of these particles by studying the sum of the corresponding particle four-vectors. We consider two cases: the case of parallel propagation of a beam of particles and the triangle inequality in M^4.

Proposition 5 *If two future-directed null vectors are parallel (antiparallel), then their spatial directions are parallel (antiparallel) for all observers. Equivalently the property of three-parallelism (three-antiparallelism) of null vectors is a covariant property.*

Proof

Consider two future-directed null vectors which in the LCF Σ have components

$$A^a_{(I)} = E_{(I)} \begin{pmatrix} 1 \\ \mathbf{e}_{(I)} \end{pmatrix} \qquad E_{(I)} > 0, \quad I = 1, 2.$$

Then by Theorem 1 we have $\left[\sum_{I=1}^n \left(A^a_{(I)}\right)\right]^2$ =0, hence

$$\begin{aligned} (E_1 + E_2, \mathbf{e}_1 E_1 + \mathbf{e}_2 E_2)^2 = 0 \Rightarrow \\ -(E_1 + E_2)^2 + (\mathbf{e}_1 E_1 + \mathbf{e}_2 E_2)^2 = 0 \Rightarrow \\ \mathbf{e}_1 \cdot \mathbf{e}_2 = 1 \Rightarrow \mathbf{e}_1 // \mathbf{e}_2 \Rightarrow \mathbf{e}_1 E_1 // \mathbf{e}_2 E_2. \end{aligned}$$

Second Proof

We have

$$A^a_{(I)} // A^a_{(J)} \Rightarrow A^a_{(I)} = \lambda A^a_{(J)} (\lambda \in R).$$

Considering the components of the four-vectors in Σ we have

$$E_{(I)} = \lambda E_{(J)}, \qquad \mathbf{e}_{(I)} = \lambda \mathbf{e}_{(J)} \quad I, J = 1, 2. \qquad \square$$

Proposition 6 *Let O, A, B be three points in M^4 such that the four-vectors OA^a and OB^a are future-directed timelike four-vectors. Then the four-vector AB^a is a future-directed four-vector and the absolute value of the Lorentz lengths (positive numbers) of the three four-vectors satisfy the relation*

$$(OB) \geq (OA) + (AB), \tag{1.125}$$

where the equality holds if, and only if, the three points O, A, B are on a straight line.

Proof

The linearity of M^4 implies

$$AB^a = -OA^a + OB^a.$$

From Proposition 4 we conclude that AB^a is a timelike four-vector. The length

$$-(AB)^2 = -(OA)^2 - (OB)^2 - 2OA^a OB_a. \tag{1.126}$$

In the proper frame of OA^a we have

$$OA^a = \begin{pmatrix} (OA) \\ \mathbf{0} \end{pmatrix}_{\Sigma(OA)}$$

($(OA) > 0$) and suppose that $OB^a = \begin{pmatrix} (OB)^0 \\ (\mathbf{OB}) \end{pmatrix}_{\Sigma(OA)}$ where $(OB)^0 > 0$ because OB^a is future directed. The invariant

$$OA^a OB_a = -(OA)(OB)^0.$$

But $-(OB)^2 = -(OB)^{0^2} + (\mathbf{OB})^2 < 0$ because OB^a is timelike, hence $(OB)^0 > (OB) > 0 \Longrightarrow (OA)(OB)^0 \geq (OA)(OB)$. Finally

$$OA^a OB_a \geq (OA)(OB),$$

where the equality holds only if $(\mathbf{OB}) = 0$, that is the four-vectors OA^a and OB^a are parallel, hence the three points O, A, B lie on the same straight line (in M^4). Replacing in (1.126) we find

$$(AB)^2 = (OA)^2 + (OB)^2 + 2OA^a OB_a \leq ((OA) - (OB))^2 \Longrightarrow$$
$$(AB) \leq (OA) - (OB) \Longrightarrow (OB) \geq (OA) + (AB),$$

which completes the proof. $\square$

The result of Proposition 6 can be generalized for a polygon consisting of $(n-1)$ future-directed timelike four-vectors $OA_1A_2\dots A_n$. In this case the inequality reads:

$$(OA_n) > (OA_1) + (A_1A_2) + \cdots + (A_{n-1}A_n). \tag{1.127}$$

We note that in M^4 the triangle inequality (1.125) is opposite to the corresponding inequality of the Euclidean geometry. As we shall see this geometric result is the reason for the mass loss in relativistic reactions.

1.12 The Center System (CS) of a System of Particle Four-Vectors

Let $A^a_{(I)}$, $I = 1, \dots, n$, be a finite set of future-directed particle vectors not all null and parallel. According to Proposition 3 their sum $A^a = \sum_{I=1}^n \left(A^a_{(I)}\right)$ is a timelike four-vector. The proper frame of the vector A^a is denoted with Σ^+ and called the *Center System*[26] (CS) of the set of the four-vectors $A^a_{(I)}$, $I = 1, \dots, n$. In the CS A^a has the following components:

$$A^a = \begin{pmatrix} (A^0)^+ \\ \mathbf{0} \end{pmatrix}_{\Sigma^+},$$

where $(A^0)^+ = \sqrt{-A^aA_a}$ is an invariant.

Exercise 9 Assume that in their CS Σ^+ the four-vectors $A^a_{(I)}$, $I = 1, \dots, n$, have components $A^a_{(I)} = \begin{pmatrix} E^+_I \\ \mathbf{A}^+_I \end{pmatrix}_{\Sigma^+}$. Show that

$$A^{0+} = \sum_{I=1}^n E^+_{(I)}, \quad \sum_{I=1}^n \mathbf{A}^+_I = 0,$$

where A^{0+} is the zeroth component of the sum A^a in Σ^+.

Exercise 10 Prove that the γ- and β-factors of the CS in Σ are given by the relations

$$\gamma = \frac{\sum_{I=1}^n E_{(I)}}{A^{0+}}, \tag{1.128}$$

[26] In case all four-vectors are null and parallel then A^a is also null and parallel to these four-vectors and in that case we do not define a CS.

$$\boldsymbol{\beta} = \frac{\sum_{I=1}^{n} \mathbf{A}_I}{\sum_{I=1}^{n} E_{(I)}}. \tag{1.129}$$

Verify that the above quantities satisfy the relation $\gamma^2 = \frac{1}{1-\beta^2}$.

[Hint: Recall that if a timelike four-vector in an LCF Σ has components $A^a = \begin{pmatrix} A^0 \\ \mathbf{A} \end{pmatrix}_\Sigma$ then the γ- and β-factors of the proper frame of A^a in Σ are given by the relations

$$\gamma = \frac{A^0}{\sqrt{-A^a A_a}}, \qquad \boldsymbol{\beta} = \frac{\mathbf{A}}{A^0}.\Bigg] \tag{1.130}$$

Chapter 2
The Structure of the Theories of Physics

2.1 Introduction

The mechanistic point of view that the ultimate scope of physics is to "explain" the whole of the physical world and the numerous phenomena in it is no longer widely acceptable and hides behind cosmogonic and theocratic beliefs. It is rather safer to say that nowadays we believe that physics describes the physical phenomena by means of "pictures", which are constructed according to strictly defined procedures. What a "picture" of a physical phenomenon is and how it is constructed is a very serious philosophical subject. Equally serious is the assessment of when these "pictures" are to be considered successful. Obviously the present book cannot address these difficult questions in depth or in extent. However, an "answer" to these questions must be given if we are going to develop Special Relativity (in fact any theory of physics) on a firm conceptual basis and avoid the many misunderstandings which have accompanied this theory for long periods in its history.

Consequently, avoiding difficult questions and obscure discussions, we demand that a physical phenomenon be described by a set of physical quantities which measure/characterize the organization of the physical systems participating in the phenomenon. These physical quantities are the elements comprising the "picture" of the physical phenomenon. Furthermore we agree that the "picture" of a physical phenomenon will not be *unsuccessful* if it predicts/explains/describes to within a "logical" accuracy and idealistic approximations, the physical phenomenon as the result of an organization of some relevant physical systems. For example, if we have a simple pendulum of length l in a gravitational field g, then the prediction that the period T of the pendulum, under specified experimental conditions and idealistic approximations, is given by the formula $2\pi\sqrt{l/g}$ can be verified within some acceptable limits of accuracy. The above lead to the following questions:

(a) How do we "specify/describe" a physical quantity?
(b) When shall the descriptions of a physical quantity be equal or "equivalent"? Are the "pictures" of a given physical quantity always the same ones?
(c) How do we differentiate or "measure" the differences between two descriptions of the same physical quantity?

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All these questions and many more of the same kind have been and will be posed throughout the course of human history. To these questions there are not definite and unique answers and this is the reason we have made extensive use of quotation marks. The answers given to such questions can only be beliefs and "logical" explanations which, in turn, are based on other more fundamental beliefs and explanations and so on. This infinite sequence of reasoning is the realm of philosophy. However, in science, and more specifically in physics, we cannot afford the luxury of an endless series of questions, beliefs, and explanations. This is because physics is an applied science, which acquires meaning and substance in the laboratory and in everyday practice. For example there is no room for beliefs and explanations concerning the take-off of an airplane, or the safety of a nuclear reactor. In conclusion, the practicalities of life itself impose upon us a definite "real" world of objective physical phenomena to which science is called upon to consider and offer systematically formed views and propositions corresponding to its purposes. This "imposed" reality is the realm of science and this is the "world" we want to "explain" in terms of the human concept of reality, created by our senses and perception.

In the following sections we develop *one* approach/explanation as to how the methodology of physics as a science is systematically formed. Obviously other writers have a different approach and the reader may have his/her own. However, no matter which approach one takes the *common agreement* is that

(a) There is not "the correct" approach but some of them are more "successful" than others in certain subsets of phenomena.
(b) Whichever approach is adopted, the final numeric answer to a given physical phenomenon or set of phenomena must be within the accuracy of the experimental measurements or observations, otherwise the approach is not acceptable.

2.2 The Role of Physics

Our interaction with the environment is always by means of some kind of sensors. These sensors can be either the sensors of the human body (direct action) or man-made sensors such as lenses, clocks, meters, etc. (indirect action). The first type of sensors shall be called *basic sensors* and the second type *measuring sensors*. Physics is the unique science which is concerned with the development, use, and study of the measuring sensors. This gives physics a distinct position against all other sciences in the study for an "objective" reality.

The basic function of the measuring sensors is the observation of a physical phenomenon or of a characteristic quantity(ies) of a physical phenomenon. The result of an observation is (as a rule) a set of numbers. Therefore observation is the procedure with which a physical quantity is described in the mathematical world of geometry. Because

- The result of an observation depends on the specific observer performing the observation.
- The nature of the physical quantity is independent of the observer observing it.

We are led to demand that the sets of numbers associated by the various observers to a given physical quantity shall be related by means of some transformations. Mathematically this means that the sets of numbers of each observer are the components of a *geometric object* for that observer. Equivalently we can say that the mathematical description of the physical quantities will be done in terms of geometric objects in a proper geometric space. We emphasize that the result of an observation *is not* an element of the objective world (reality), but an element of the world of geometry. The description of a phenomenon by means of a set of geometric objects is called the *image* of the observed physical phenomenon.

According to this view the sensors we use for the observation of the real world create another world of images. Science studies this world of images and tries to discover its internal structure, if any, or the internal structure of a subset of images of a given class. The mechanistic point of view, which prevailed at the end of the 19th century, can be understood as the belief that there exists *one and only one* internal structure of the world of images and the scope of science is to discover this structure. Today, as we remarked at the beginning of this section, this position is considered as being too strong. It is our belief that there exist internal structures in subsets of certain types of images that do not appear in the set of all images of the objective world, i.e., different structures for different subsets.[1]

In the game of science, physics enters with a special role. The fundamental requirement of physics is that whichever internal structure relates a subset of images, it must be "objective," or to put it in another way, non-personalized. Due to this requirement, in physics one uses (in general) images which are created by measuring sensors only.

Furthermore, it is required that physics be concerned only with images which have a definite qualitative and quantitative character. This means that the images studied in physics will be "geometric," that is they will be described by means of concrete mathematical quantities defined on mathematical sets with definite mathematical structure (linear spaces, manifolds, etc.). These spaces are the space of physics. The type and the quantity of the geometric objects which describe the images of a theory geometrize these images and define the "reality" of that theory of physics. For example, in Newtonian Physics the physical quantity "position" is associated with the geometric element position vector in the Euclidean three-dimensional space.

Physics has shown that indeed the set of images created by the measuring sensors is divided into more than one subset, with each subset having its own internal structure. This has resulted in many theories of physics, each theory having a different "reality." For example Newtonian Physics has one reality (that of the Newtonian Physical phenomena geometrized by Euclidean tensors in three-dimensional Euclidean space), Special Relativity another (the special relativistic phenomena geometrized by Lorentz tensors defined on a four-dimensional flat metric space endowed with a Lorentzian metric). The unification of these different realities as

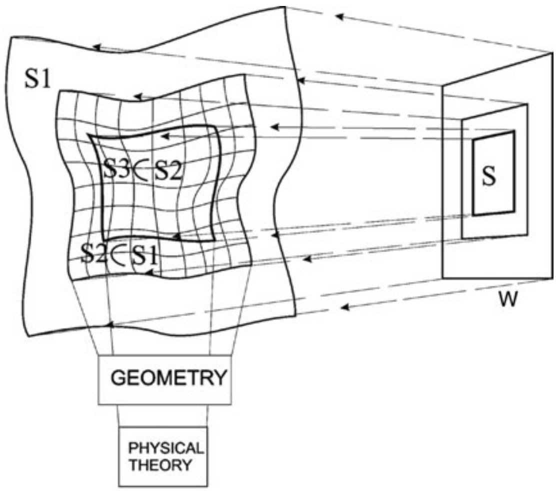
[1] In our opinion the question of the existence of a universal internal structure involving all images of the objective world is equivalent to the question if there is an omnipresent super power or not!

parts of a super or universal reality has raised the problem of the unification of the theories of physics in a Unified Theory, a problem which has occupied many distinguished physicists over the years and which remains open.

Each theory of physics is equally strong to any other within the subset of images to which it applies. Therefore the statement "this theory of physics is not valid" makes no sense. For example Newtonian Physics holds in the subset of Newtonian phenomena only and does not hold in the subset of relativistic phenomena. Similarly Special Relativity holds in the subset of special relativistic phenomena only.

The above analysis makes clear that every theory of physics is intimately and uniquely related to the geometrization of the subset of images of the world it studies. This geometrization has two branches:

(a) The correspondence of every image in the subset with a geometric object of specific type
(b) The description of the internal structure of the subset of images by means of geometric relations among the geometric objects of the theory (these include – among others – the laws of that theory of physics)



S = Set of physical quantities of a theory of Physics

W= Window of measuring sensors

S1= Set of images of measuring sensors

S2 = Set of images of measuring sensors which can be geometrized

S3 = Set of images of a theory of Physics

Fig. 2.1 The structure of a theory of physics

Figure 2.1 describes the above by means of a diagram. We note that in Fig. 2.1 every subset of images is defined by a different window of measuring sensors. In the following sections we discuss the general structure of a theory of physics in practical terms.

2.3 The Structure of a Theory of Physics

As we mentioned in the last section the images studied in physics are created by means of the measuring sensors and are described mathematically by geometric objects. The images of physical phenomena are generated by the *observers* by *observation*.

The observer in a theory of physics is the "window" of Fig. 2.1. Its sole role is to generate geometric images of the various physical phenomena, which are observed. In practical terms we may think the observer as a machine – robot (EGO does not exist in physics!) – which is equipped with the following:

- A set of specific measuring sensors, which we call *observation means* or *observation instruments*
- A definite *set of instructions* concerning the use of these sensors

Depending on the type of the observation instruments and the directions of use the observers create geometrized images of the physical phenomena. Every theory of physics has its own observers or, equivalently, there are as many types of observers as viable theories of physics. For each class of observers there corresponds a "reality", which is the world made up by the subset of images involved in the theory.

In a sense the observers operate as "generalized functions"[2] from the set of all physical phenomena to the space of images R^n (for a properly defined n). The value of n is fixed by the *characteristic quantity* of the theory. For the theories of physics studying motion the space of images is called *spacetime* and the characteristic quantity is the *position vector*.

For example in Newtonian Physics the characteristic quantity of the theory is motion and it is described with the image orbit in a linear space. Practical experience has shown that the orbit is fully described with three numbers at each time moment, therefore the spacetime is a four ($n = 4$) dimensional linear space (endowed with an extra structure, which we shall consider in the following). There is no point to consider a spacetime of higher dimension in Newtonian Physics, because the extra coordinates will always be zero, and therefore redundant. In general the dimension of the space of a theory of physics equals the minimum number of components required to describe completely the characteristic quantity of the theory. We call this number the *dimension of the space* of the theory.

[2] This type of functions are called functors in Category Theory, but this is something outside the scope of this book.

We summarize the above discussion as follows:

- The observer of a theory of physics is a machine (robot) which is equipped with specific instrumentation and directions of use, so that it can produce components for the characteristic quantities of the physical phenomena.
- Observation or measurement is the operation of the observer which has as a result the production of components for the characteristic quantities of a specific physical phenomenon or physical system.
- There exists neither a unique type of observers nor a unique type of observation. Every class of observers produces a set of images of the outside world, which is specific to that class of observers and type of observation. For example, we have the sequences: Newtonian observers – Newtonian Physics – Newtonian world, Relativistic observers – Relativistic physics – Relativistic world.

2.4 Physical Quantities and Reality of a Theory of Physics

From the previous considerations it becomes clear that the image of a physical phenomenon depends on the observer (i.e., the theory) describing it. However, the fundamental principle of physics is that the image of a single observer for a physical phenomenon has no objective value. The "reality" of one has no place in today's science and specifically in physics, where all observers are considered to be (within each theory!) equivalent and similar in all respects. The objectivity of the description of physical phenomena within a given theory of physics is achieved by means of the following methodology:

(1) We consider the infinity of the specific type of observers (i.e., identical robots equipped with the same instrumentation and the same programming) used by the theory.
(2) We define a specific and unique code of communication amongst these observers, so that each observer is able to communicate the image of any physical quantity to any other observer in that same class of observers.
(3) We define a procedure which we agree will "prove" that a given physical phenomenon is described successfully by a specific theory of physics.

Following the above approach, we define the "successfulness" of the image of a phenomenon within a (any) theory of physics by means of the procedure described in the general scheme of Fig. 2.2.

In Fig. 2.2 there are two major actions:

(1) The *observation* of the physical phenomenon by an observer with the measuring means and the procedures, defined by the specific theory of physics. The result of each observation is the creation of geometric objects (in general elements of $\mathbf{R}^n$) for the characteristic quantity(ies) of the observed physical phenomenon.
(2) The *verification* of the "objectivity" of the concerned physical phenomenon by the comparison of the geometric images created by procedure 1. This second

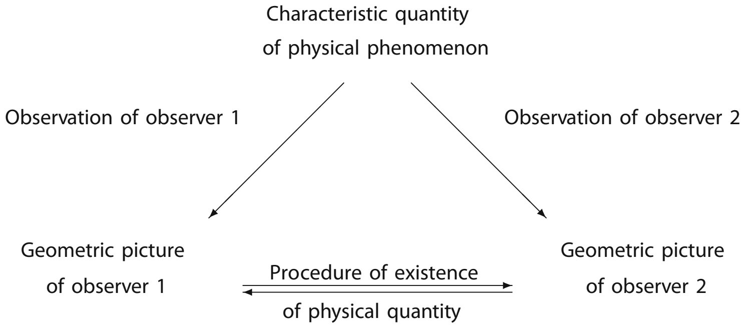


Fig. 2.2 Generating diagram of principles of relativity

activity involves the transfer of the image of a phenomenon observed by the observer 1 to observer 2 and vice versa, according to the esoteric code of communication of the observers specified by the specific theory of physics. More specifically, observer 2 compares the image of the phenomenon created by direct observation (*direct image*) with the communicated image of observer 1 (*communicated image*). If these two images do not coincide (within the specified limits of observation and possible idealizations) then the physical phenomenon is not a physical quantity for that specific theory of physics. If they do, then observer 2 communicates his direct image to observer 1, who accordingly repeats the same procedure. If the two images of observer 1 coincide then the concerned physical phenomenon is *potentially* a physical quantity for that theory of physics. If the coincidence of the images holds for any pair of observers of the specific theory then this physical phenomenon is a *physical quantity* for that theory of physics.

The set of all physical quantities of a theory comprises the *reality* of this theory of physics. We note that the reality for physics is relative to the theory used to "explain" the physical phenomena. That is, the reality of Newtonian Physics is different from the reality of the Theory of Special Relativity, in the sense that the physical quantities of Newtonian Physics are not physical quantities for Special Relativity and vice versa. For example the physical quantities time and mass of Newtonian Physics do not exist in the Theory of Special Relativity and, for the same reason, the speed of light and the four-momentum of Special Relativity do not exist (i.e., are not physical quantities) in Newtonian Physics.

It must be understood that the reality of the physical world we live in is beyond and outside the geometric reality of the theories of physics. On the other hand the reality of the world of images of a theory of physics is a property which can be established with concrete and well-defined procedures. This makes the theories of physics the most faithful creations of human intelligence for the understanding and the manipulation of the physical environment. The fact that these creations indeed work is an amazing fact which indicates that the basic human sensors and the human "operational system" (that is the brain) function in agreement with the laws of nature. This is the prime reason which led to the initial identification of the observer – robot with the actual human observer.

The procedure with which one assesses the existence of a physical quantity in the realm of the world of a theory of physics is called the *Principle of Relativity* of that theory. The Principle of Relativity involves transmission (i.e., communication) of *geometric* images among two observers of a theory of physics, therefore it is of a pure geometric nature and can be described mathematically in appropriate ways. It must be clear that it is not possible to have a theory of physics without a Principle of Relativity, because that will be a theory without any possibility of being tested against the real phenomena. In the sections to follow we shall show how these general considerations apply in the cases of Newtonian Physics and the Theory of Special Relativity.

From the above considerations it is wrong to conclude that the identification of a physical quantity is simply a comparison of images (direct and communicated image) based on the Principle of Relativity. Indeed, the identification of a physical quantity is a twofold activity:

(a) A measuring procedure (the observation, direct image) with which the geometric image of the quantity is created and involves one observer and one physical phenomenon
(b) A communication and comparison of images which involves two observers and a specified physical phenomenon

2.5 Inertial Observers

The Principle of Relativity of a theory of physics concerns the communication (exchange of information – images) between the observers of the theory. Geometrically this is expressed by groups of transformations in the geometric space of images of the theory. But how does one define a Principle of Relativity in practice?

In the present book we shall deal only with the cases of Newtonian theory and the Theory of Special Relativity. Therefore in the following when we refer to a theory of physics we shall mean one of these two theories only.

As we have said in the previous sections, the observers of a theory of physics are machines equipped with specific instrumentation and directions of use and produce for each physical quantity a set of numbers, which we consider to be the components of a geometric object. Two questions arise:

- In which coordinate system are these components measured?
- What kind of geometric objects will be associated with this set of components or, equivalently, which is the group of transformations associated with that theory?

Both these questions must be answered before the theory is ready to be used in practice.

The first question involves essentially the geometrization of the observers, that is, their correspondence with geometric objects in R^n. These objects we consider to be the coordinate systems. In a linear space there are infinitely many coordinate systems. One special class of coordinate systems is the Cartesian coordinate systems, which are defined by the requirement that their coordinate lines are straight lines, that is, they are described by equations of the form

$$\mathbf{r} = \mathbf{a}t + \mathbf{b},$$

where $\mathbf{a}$, $\mathbf{b}$ are constant vectors in R^n and $t \in R$.

If the Cartesian coordinate systems are going to have a physical meaning in a theory of physics (this is not necessary, but it is the case in Newtonian Physics and in the Theory of Special Relativity) the observers who are associated with these systems must have the ability to define straight lines[3] and in the real world by means of a physical process. The observers who have this ability are called *inertial observers*.

In a theory of physics the existence of these observers is guaranteed with an axiom. In Newtonian Physics this axiom is Newton's First Law and in the Theory of Special Relativity there exists a similar axiom (to be referred to later on). At this point it becomes clear that the inertial observers of Newtonian Physics are different from those of Special Relativity. For this reason we shall name the first *Newtonian inertial observers* (*NIO*) and the latter *Relativistic inertial observers* (*RIO*).

2.6 Geometrization of the Principle of Relativity

The Principle of Relativity is formulated geometrically by means of two new principles: the *Principle of Inertia* and the *Principle of Covariance*.

The first defines the inertial observers of the theory and the second the geometric nature (that is the type of the mathematical objects or the transformation group) of the theory. Let us examine these two principles in some detail.

2.6.1 Principle of Inertia

From the point of view of physics the concept "inertial observer" concerns the observer – robot as a physical system and not as a Cartesian coordinate system. There are not inertial coordinate systems, but only inertial observers who are related

[3] A straight line in a linear metric space can be defined in two different ways: either as the curve to which the tangent at every point lies on the curve (this type of lines is called autoparallels) or as the curve with extreme length between any of its points (these curves are called geodesics). These two types of curves need not coincide; however, they do so in Newtonian Physics, Special Relativity, and General Relativity.

to the Cartesian systems by means of a specified procedure. An inertial observer can use any coordinate system to refer his/her measurements (observations). For example, an inertial observer in R^2 can change from the Cartesian coordinates (x, y) to new coordinates u, v defined by the transformation $u = xy$, $v = x - y$ and subsequently express all his measurements in the new coordinate system (u, v). Obviously the coordinate system (u, v) is not Cartesian. However, this is not a problem because it is possible for the observer to transfer his data to a Cartesian coordinate system by means of the inverse transformation. (Think of a similar situation where a novel written in one language is translated either to other dialects of the same language or to other languages altogether. The novel in all these changes of language is the same.) The inertiality of an observer is a property which can be verified and tested experimentally. *The Principle of Inertia* specifies the *experimental* conditions, which decide on the inertiality or not of an observer. The Principle of Inertia is different in each theory of physics.

According to the above the question that a given observer – robot – is inertial makes sense only within a specified theory of physics. For example in Newtonian Physics it has been found that the coordinate system of the distant stars is inertial (because Newton's First Law is satisfied). Similarly it has been verified that (within specified experimental limits) the coordinate system based on the solar system is a Newtonian inertial observer, for restricted motions and small speeds (compared to the speed of light) within the solar system.

In order to prevent questions and misunderstandings which arise (as a rule) with the concept of the inertial observer in dynamics, we consider the Second Law of Newton. This law *does not* define the Newtonian inertial observers but concerns the study of the motion (all motions) in space by *Newtonian inertial observers*. That is, this law does not make sense for non-inertial Newtonian observers, for example for Relativistic inertial observers (to be considered below). The mathematical expression of this law is the same for all Newtonian inertial observers and defines the Newtonian physical quantity force. For non-Newtonian inertial observers this law makes no sense and the physical quantity Newtonian force is not defined as, for example, is the case with Special Relativity.

2.6.2 The Covariance Principle

After establishing the relation between the inertial observers of a theory with the Cartesian coordinate systems in R^n, it is possible to quantify geometrically the Relativity Principle of that theory. This is achieved with a new principle, which we call the *Covariance Principle* and is as follows.

Let K be the set of all Cartesian coordinate systems in R^n (for proper n), which correspond to the inertial observers of the theory. Let GK be the group of all (linear!) transformations between the Cartesian systems in K. We demand the geometric objects which describe the physical quantities of the theory to be invariant under the action of GK. This means that if Σ, Σ' are two elements of K with $\Sigma' = B\,\Sigma$ where B is an element ($n \times n$ matrix with coefficients parameterized only by the parameters which relate $\Sigma,\ \Sigma'$ in K) of GK and T the geometric object corresponding to a physical quantity of the theory with components $T^{i'\dots j'}_{r'\dots s'}$ in Σ' and $T^{i\dots j}_{r\dots s}$ in Σ, then the two sets of components are related by the following relation/transformation:

$$T^{i'\dots j'}_{r'\dots s'} = B^{i'}_{i} \dots B^{j'}_{j}\, B^{r}_{r'} \dots B^{s}_{s'}\, T^{i\dots j}_{r\dots s}.$$

In Newtonian Physics the Cartesian coordinate systems (that is, the set K) are the Newtonian Cartesian systems and the transformations (that is the group GK) is the group of Euclidean orthogonal transformations or the group of transformations of Galileo. In the Theory of Special Relativity the Cartesian systems (set K) are the Lorentz Cartesian coordinate systems and the corresponding set of transformations (i.e., the GK) is the group of Lorentz (or to be more precise the Poincaré) transformations. In both cases the transformation groups are subgroups of the group of linear transformations $GL(n, R)$ for $n = 3, 4$, respectively. In Newtonian Physics the geometric objects corresponding to physical quantities are the Newtonian (or Euclidean) tensors and in the Theory of Special Relativity the Lorentz tensors.

The Principle of Covariance in contrast to the Principle of Inertia does not involve experimental procedures and concerns only exchange of information (i.e., images) by means of transformations in the geometric space of a theory. We give the schematic diagram of Fig. 2.3 (compare with the corresponding diagram of Fig. 2.2). Note that the arrow which corresponds to the Principle of Covariance is bidirectional (refers to bidirectional communication and not one directional observation) and it is different from the single arrow of observation (the observer observes the observant but not vice versa!).

We also note that the physical quantities of a theory of physics are those for which the diagram of Fig. 2.3 commutes. This means that the image of a physical quantity for observer 1 (direct image) must coincide with the image received from observer 2 (communicated image) and vice versa. This "locking" of the diagram is the criterion that decides, in a unique way, the "reality" or "existence" of a physical quantity in a theory of physics.

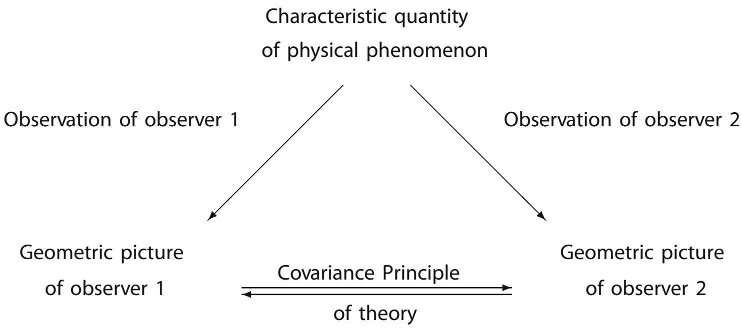


Fig. 2.3 Diagram of Principles of Covariance

2.7 Relativity and the Predictions of a Theory

The geometrization of the Principle of Relativity of a theory of physics allows the mathematical manipulation of the geometric objects, which describe the images of the physical quantities created by the theory. Therefore working in the geometric space of the images of the theory, it is possible to construct mathematically new images which satisfy the Principle of Covariance of the theory. The below question arises:

Do the new mathematically constructed images correspond to images of existing physical quantities/phenomena or are they simply consistent mathematical constructions?

The assessment that they do correspond is a *prediction* of the theory, which can be true or false. Therefore the verification of a prediction of a theory consists in the verification of a proposed experimental/measuring procedure (creation of a direct image) of a physical quantity by an inertial observer of the theory. In conclusion, the verification of a prediction of a theory of physics is an iterative process which follows the following algorithm:

(a) We choose a physical phenomenon.
(b) We propose an experimental procedure for its measurement by an inertial observer of the theory.
(c) We compare the image obtained through an inertial observer with the one constructed mathematically by the theory (for an arbitrary inertial observer of the theory).
(d) If the two images coincide the prediction is verified. If not, then we may propose a different method of measurement or discard the prediction as false.

The prediction and the verification of a prediction by experimental/measuring procedures is a powerful tool of physics and gives it a unique position against all other sciences. Essentially the verification of a prediction increases our *belief* that a given theory of physics is "correct," in the sense that it describes well the "outside world." At the same time predictions set the limits of the theories of physics, because as we have already remarked, no theory of physics appears to be *THE ULTIMATE THEORY*, which can explain and predict everything. However, this is a difficult and open issue, which need not, and should not, concern us further in this book.

Chapter 3
Newtonian Physics

3.1 Introduction

Newtonian Physics is the first theory of physics which was formulated scientifically and became the milestone for the Theory of Special (and General) Relativity. All relativistic ideas are hidden into the Newtonian structure, therefore it is imperative that before we proceed with the development of the Theory of Special Relativity, we examine Newtonian Physics from the relativistic point of view. In the standard treatment, the concepts and the structure of Newtonian Physics are distinguished in a fundamental (and not formal as it is often stated) way, in two parts: kinematics and dynamics. Kinematics considers (a) the various structures (e.g., mass point, rigid bodies) which experience motion and (b) the substratum (space and time) in which motion occurs. Dynamics involves the study of motion by means of equations of motion, which relate the causes (forces) with the development of motion (trajectory) in space and time.

The same kinematics supports many types of forces. For example Newtonian kinematics applies equally well to the motion of a mass point in the gravitational field and the motion of a charge in the electromagnetic field. In general kinematics lies in the foundations of a theory of motion and it is not possible to be changed unless the theory itself is changed. On the other hand dynamics is possible to be changed within a given theory of physics, in the sense that one can consider different equations for the laws of motion in a given scenario. One such example is the force law on an accelerating charge moving in an electromagnetic field, where various force equations have been suggested.

In what follows we present and discuss briefly the concepts which comprise the kinematics and the dynamics of Newtonian Physics. The presentation is neither detailed nor complete because its purpose is to prepare the ground for the introduction of the Theory of Special Relativity through the known "relativistic" environment of Newtonian Physics, and not the Newtonian Physics per se. Of course the presentation which follows is based on the general discussion concerning the theories of physics presented in Chap. 2.

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3.2 Newtonian Kinematics

The fundamental objects of Newtonian kinematics are

- Mass point
- Space
- Time

In the following we deal with each of these concepts separately starting with the mass point, which is common in Newtonian and relativistic kinematics.

3.2.1 Mass Point

The *mass point* in Newtonian kinematics is identified with a geometric point in space and its motion is characterized completely by the position vector $\mathbf{r}$ at every time moment t in some coordinate system. The change of position of a mass point in space is described with a curve $\mathbf{r}(t)$, which we call the *trajectory* of the mass point. The purpose and the role of kinematics is the geometric study of this curve. This study is achieved by the study of the first $d\mathbf{r}/dt$ and the second $d^2\mathbf{r}/dt^2$ derivative[1] along the trajectory. At each point, P say, of the trajectory we call these quantities the *velocity* and the *acceleration* of the mass point at P.

In dynamics the mass point attains internal characteristics, which correspond to Physical quantities, e.g., mass, charge. These new Physical quantities are used to define new mixed Physical quantities (e.g., linear momentum), which are used in the statement of the laws of dynamics. The laws of dynamics are mathematical expressions which relate the generators of motion (referred with the generic name *forces* or *dynamical fields*) with the form of the trajectory $\mathbf{r}(t)$ in space. Newton's Second Law

$$\mathbf{F} = m\frac{d^2\mathbf{r}}{dt^2}$$

relates the quantities

$$\begin{aligned} \mathbf{F} &= \text{Cause of motion,} \\ \frac{d^2\mathbf{r}}{dt^2} &= \text{Form (curvature) of the trajectory } \mathbf{r}(t) \text{ in space,} \\ m &= \text{Coupling coefficient between } \mathbf{F} \text{ and } \frac{d^2\mathbf{r}}{dt^2}. \end{aligned}$$

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[1] It is a standard result that given the first and the second derivative of a (smooth) curve in a Euclidean (finite-dimensional) space at all the points of the curve, it is possible to construct the curve through any of its points. This is the reason we do not need to consider higher derivatives along the trajectory.

In this law the force **F** is not specified and depends on the type of the dynamical field which causes the motion and the type of matter of the mass point. For example if the mass point has charge q and inertial mass m and moves in an electromagnetic field $(\mathbf{E}, \mathbf{B})$ then the force **F** has the form (Lorentz force)

$$\mathbf{F} = kq(\mathbf{E} + \mathbf{v} \times \mathbf{B}),$$

where k is a coefficient, which depends on the system of units, and $\mathbf{v}$ is the velocity of the mass point.

3.2.2 Space

In Newtonian Physics space is defined as follows[2]:

> Absolute space, in its own nature, without regard to anything external, remains always similar and immovable.

This definition was satisfactory at the time it was given, because it had the necessary philosophical support allowing a theoretical perception of the world. However, it is not satisfactory today, when science has made so many and important steps into the knowledge of the structure of matter and the processing of information. This fact was recognized by Einstein (and others), who emphasized that the concept of space must be brought "down to earth" and become a fundamental tool in the theory and the laboratory.[3] Therefore we have to give a "practical" answer to the question:

What properties does the physical system space have?

The answer we shall give will not conflict the previous "classical" definition, but will make it clear and place it in the chessboard of today's knowledge and experience.

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[2] This definition was given by Newton in his celebrated book "Principia." For a more recent reference see A. Sommerfeld, *Lectures in theoretical physics*. Vol. I, Mechanics, Academic Press, p. 9 (1964).

[3] In Einstein's own words "... The only justification for our concepts and system of concepts is that they serve to represent the complex of our experiences; beyond this they have no legitimacy. I am convinced that the philosophers have had a harmful effect upon the progress of scientific thinking in removing certain fundamental concepts from the domain of empiricism, where they are under our control, to the intangible heights of a priori. For even if it should appear that the universe of ideas cannot be deduced from experience by logical means, but is, in a sense, a creation of the human mind, without which no science is possible, nevertheless the universe of ideas is just as little independent of the nature of our experiences as clothes are of the form of the human body. This is particularly true of our concepts of time and space, which physicists have been obliged by the facts to bring them down from the Olympus of the a priori in order to adjust them and put them in a serviceable condition. ..." *Extract from the book The meaning of Relativity* by Albert Einstein Mathuen, Sixth Edition London (1967).

A Physical system can have two types of properties:

- Properties which refer to the character of the various fundamental units of the system
- Properties which concern the interaction among the fundamental units of the system and comprise what we call the structure of the system

For example according to this point of view, the simple pendulum of Fig. 3.1(a) is the following system:

Fundamental parts: (a) Mass point of mass m and (b) Supporting string of length $l = l_1 + l_2$.

Structure: The mass m is fixed at one end of the supporting string, whose other end is fixed.

The structure can be different although the fundamental parts remain the same. For example with the same parts we can construct the oscillator of Fig. 3.1(b), whose physical properties (that is, motion – e.g., period of oscillation – in the gravitational field) are different from those of the simple pendulum.

Let us consider now the system space, which is our concern.

We define the fundamental units of the system space to be the *points*. The points are not defined in terms of simpler entities and in this respect are self-sufficient with respect to their existence. According to Newtonian Physics the points have no quantity and quality and they do not interact in any of the known ways with the rest of the Physical systems. Due to this latter property, we say that the points of space are "absolute."

The structure of space is expressed with mathematical relations among its points. These relations constitute a complete and logically consistent structure, in which we give the generic name *geometry*. The set of points of a given space admits infinitely many geometries, in exactly the same way that the fundamental parts of the simple pendulum can create infinitely many systems by different combination of the lengths l_1, l_2.

Newtonian Physics demands that whichever geometry is selected for the structure of the space, this is also "absolute," in the sense that the relations which describe it do not change (are not affected) by the physical phenomena (not only the motion!) occurring in space. In conclusion,

> Space is comprised of absolute fundamental parts (the points) and has an absolute structure (geometry).

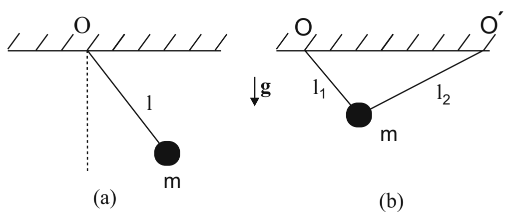

Fig. 3.1 Fundamental parts and structure

This is what we understand from the phrase of the classical definition "*space is absolute*."

Newtonian Physics makes fundamental assumptions, which concern the general characteristics of Newtonian observation. According to our discussion in Chap. 2, these assumptions are necessary for observation to be a well-defined procedure. In Newtonian Physics these assumptions are as follows:

(1) The position of a mass point in space is described by the *position vector*, which is a vector in the linear space R^3. Consequently the space of Newtonian Physics is a real three-dimensional space and the coordinate systems of Newtonian Physics involve three (real) coordinate functions.
(2) The position of the origin of a Newtonian coordinate system in space is immaterial or, equivalently, all points of space can serve as origin of a Newtonian coordinate system. In practice this means that the point of space where an experiment or observation is made does not affect the quality or the quantity of the physical phenomenon under consideration. This property of space is called *homogeneity* of space and it is fundamental in the development of Newtonian Physics.
(3) In space there exist motions whose trajectories are straight lines (in the sense of geometry) described by equations of the form[4]

$$\mathbf{r} = \mathbf{a}t + \mathbf{b},$$

where $t \in R$ and $\mathbf{a}, \mathbf{b} \in R^3$. This type of motions is called *inertial*.
(4) In Newtonian Physics inertial motions can occur in all directions. In practice this implies that the basis vectors of a Newtonian frame can have any direction in space. We call this property of space *isotropy*.
(5) An inertial motion is preserved in time if no external causes (i.e., forces) are exerted on the moving mass point. Therefore the straight lines extend endlessly and continuously in space. This property means that the space is flat or, equivalently, its curvature is constant and equals zero.

The above assumptions specify the environment where the Newtonian measurements and observations will take place. They do not define the geometry of the space, which will be defined by the Principle of Relativity of the theory.

We continue with the other fundamental concept of Newtonian Physics, the time.

3.2.3 Time

In Newtonian Physics time is understood as follows[5]:

> Absolute time, and mathematical time, by itself, and from its own nature, flows equally without regard to anything external and by another name it is called duration.

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[4] In geometry the linear spaces which admit straight lines are called *affine spaces*.

[5] See Footnote 2.

As was the case with the definition of space, the above definition of time is also not usable in practice. In order to arrive at a concrete and useful definition we follow the same path we took with the concept of space. That is, we are approaching time as a Physical system which has fundamental parts and a structure defined by relations among these parts. Such elements must exist because according to Newtonian Physics, time exists.

We consider the fundamental parts of time to be points, which we call "*time moments*" or simply "*moments*." They have no quantity or quality (as the points of space). Concerning the structure (that is, the geometry) of time, we demand that the set of moments has the structure of a one-dimensional real Euclidean space. Mathematically this assumption means that

(1) The time moments are described by one real number only.
(2) All real numbers, which correspond to the time moments, cover completely the real line with its standard Euclidean structure. We call this line "straight line of time" or "cosmic straight line of time."

In practice the above imply that in order to define a system for the measurement of time (that is, a coordinate system in the "space" of time) it is enough to consider an arbitrary Physical system which has the ability to produce numbers. We shall call these Physical systems with the generic name *clock*. There are numerous clocks in nature. Let us mention some of them:

(1) The sun, whose mass is continuously reduced. The value of the mass of the sun provides numbers in a natural manner and therefore can be considered as a clock.
(2) The position of the Earth on its orbit. Every point of the trajectory of the Earth along the ecliptic is specified by a single number (the arc length from a reference point). These numbers can be used to define the coordinate of time. This is done in practice and the resulting time is known as the *calendar time*. We define similar measurements of time with other elements of the motion of the Earth. For example, the daily rotation of the Earth defines the *astronomical time* and the average value of the calendar time of two successive passages of the sun over the same point of the Earth during a calendar year gives the *average solar time*.
(3) The mass of a radioactive material, e.g., the content in C^{14} is used in radiometry to measure the age of archaeological findings. Similarly, we have the atomic clocks which measure time with great accuracy.
(4) We can also define a time coordinate by mechanical systems. Such systems are all types of mechanical clocks used for many years in everyday life.
(5) Finally, we mention the quartz crystals which produce numbers by their oscillations under certain conditions.

From the above examples we see that clocks exist in all areas of physics, from the motion of the planets to the radioactivity of the nucleus or the solid state physics. This is a noticeable fact which emphasizes the universality of time in the cosmos of Newtonian Physics.

It is important that we differentiate between the concepts of clock and time. Indeed,

- A clock is *any* Physical (material) system, which produces single numbers. Time is not a Physical system.
- Clocks do not have an absolute character and they react with their environment. Time is absolute.
- Clocks are considered as good, bad, etc., whereas time has no quality. A "good" clock is one which produces numbers which *flow equally* and furthermore it is not effected substantially during its operation by the external environment. According to this view, an atomic clock is considered to be better than a mechanical clock. Obviously the better a clock is, the closer the concept of absolute time is represented.

We return to the definition of time and specifically to the part *flows* which we have not considered yet. The concept of flow is related to the concept of direction. This defines the so-called *arrow of time* and has the following meaning:

– Geometrically means that the cosmic line of time is oriented. Therefore the cosmic straight line of time is the real axis and not simply the real line.
– Physically means that in Newtonian Physics the Physical systems age with the same rate, irreversibly, irrevocably, and independently of their choice. This is so, because time being absolute and universal affects everything and it is affected by nothing. Therefore there exist the past and the future, and they are independent of each other, however common to all Physical systems in the cosmos. The demand of the direction in the "flow" of time restricts the Physical systems, which can be used as clocks, because the numbers they produce must appear continuously and in ascending order.

We examine now the concept *equally* in the definition of time.

Geometrically it means that the direction of the arrow of time is constant, that is, independent of the point of time. With this new assumption the cosmic straight line of time is identified naturally[6] with the line of real numbers as we use it, with origin at the number zero (which corresponds to the present). Then the positive numbers correspond to the "future" and the negative numbers to the "past." We note that the concepts future and past are relative to the present, which is specified arbitrarily.

Physically it means that the clocks we are using for the measurement of the Newtonian time must be unaffected by the environment and, furthermore, should work in a way so that they produce numbers continuously, in ascending order and with constant rate. The rate has to do with the structure of the clock and not with the actual value of the "time rate," which is the result of the operation of the clock. For example, for a mass clock the rate $\frac{dm}{dt}$ must be constant between two numbers, but the value of this constant is not prefixed.

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[6] That is without reference to coordinates, but directly point by point.

Obviously such clocks do not exist in nature, because no Physical system is energetically closed. Therefore the requirement for the measurement of Newtonian time takes us to the limit of the "ideal clock," which is independent of everything, effected by nothing, absolutely isochronous, and above all hypothetical. This ideal clock is the universal and absolute governor and in other words, it is the transfer of the divine concept in Newtonian Physics. It is important to note that the ideal clock does not have a universal beginning[7] but only a universal rate (rhythm). Furthermore, it is the same for all physical phenomena and for every point of the space ("everywhere present"). This clock is absolute and it is the clock of Newtonian Physics.

As we shall see presently, the Theory of Special Relativity abandoned the concept of Newtonian time – and accordingly the absolute universal clock – and replaced it with the concept of synchronization, which involves *two* clocks. In that theory, there is neither (universal) beginning nor (universal) rate (rhythm) common to all observers, but these concepts are absolute only for every and each relativistic observer. Naturally the Theory of Special Relativity introduced universal quantities with absolute character, but at another level. For the human intelligence, the absolute is always imperative for the definition of the relative, which is at the root of our perception and understanding of the world.

3.3 Newtonian Inertial Observers

As we remarked in Chap. 2 every theory of physics has its own "means" of observation and "directions" of use. It is due to these characteristics that the study of motion is differentiated from theory to theory. Furthermore, the "reality" of a theory of physics is determined by two fundamental principles:

(1) The prescription of the procedure(s) for the observation of the fundamental Physical quantities by the observers robots of the theory
(2) The "Physical interpretation" of the results of the measurements, expressed by the Principle of Relativity of the theory

In Newtonian Physics both these principles are self-evident, because Newtonian Physics has a direct relation with our sensory perception of the world (primary sensors), especially so for the space, the time, and the motion. In the Theory of Special Relativity we do not have this direct sensory feeling of the "reality" (secondary sensors), therefore we have to work with strictly prescribed procedures and

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[7] The clock and the Newtonian concept of time had been used by ancient Greeks. However, the clock of Aristotle had a fixed origin (beginning). Due to this, the Greeks were making Cosmogony and not Cosmology. Today we arrived again at the concept of the "beginning" of the Universe with the Big Bang theory. However, we do Cosmology because the cosmos of Aristotle was constructed once and for all, whereas in our approach the Universe is constantly changing as a dynamical system. The interested reader should look for these fascinating topics in special books on the subject. However, he/she should be cautious to distinguish between the "myth" and the "truth."

practices, which will define the relativistic Physical quantities. Let us discuss these two principles in the case of Newtonian Physics.

In Newtonian Physics it is assumed/postulated that the Newtonian observer is equipped with the following three measuring systems:

(1) An *ideal unit rod*, that is, a one-dimensional rigid body whose Euclidean length is considered to equal unity, which is unaffected in all its aspects (i.e., rigidity and length) by its motion during the measurement procedure.[8]
(2) A *Newtonian gun* for the determination of directions. This is a machine which fires rigid point mass bullets and it is fixed on a structure which returns the value of the direction (e.g., two angles on the unit sphere) of the gun at each instance.
(3) *An ideal replica* of the absolute clock, which has been set to zero at some moment during its existence.

In Newtonian Physics it is assumed that there is no interaction between the measurement of the length and the measurement of time, because the ideal unit rod, the Newtonian gun, and the ideal clock are considered to be absolute and closed to interactions with anything external.

The Newtonian observer uses this equipment in order to perform two operations:

- To testify that it is a Newtonian inertial observer
- If it is a Newtonian inertial observer, to measure the position vector of a mass point in the Physical space and associate with it a time moment

3.3.1 Determination of Newtonian Inertial Observers

In the last section we have seen that Newtonian Physics introduces a special class of observers (the Newtonian inertial observers) and the reality it creates (that is the set of all Newtonian Physical quantities) refers only to *these* observers. However, how does one Newtonian observer find out/determine that it is[9] a Newtonian inertial observer?

In order for a Newtonian observer to determine if it (no he/she!) is a Newtonian inertial observer the following procedure is applied:

At some nearby point in space a smooth rigid metallic plane is placed and an elastic mass bullet is shut toward the plane. Let us assume that the bullet hits the plane and it is reflected elastically. There are two options:

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[8] A prototype ideal unit rod was kept in Paris at the Institute of Standards. Today the standard unit of length and time are defined with atomic rather than with mechanical Physical systems. More specifically, the standard unit of 1 m is defined as the length of 1650763.73 wavelengths of the red line in the spectrum of ^{86}Kr and the standard unit of time interval (1 s) as the time required for 9192631770 periods of the microwave transfer between the two superfine levels of the ground state of the isotope of Cesium ^{133}Cs.

[9] We use "it" and not "he/she" because observers in physics are machines (robots), not humans. The identification of the observers with humans is a remnant of Newtonian Physics and the early anthropomorphic approach to science is due mainly to the close relation of Newtonian Physics and early science with the sensory perception of physical phenomena.

Either the direction of incidence coincides with the direction of reflection, in which case the mass bullet returns to the Newtonian gun or not. In case of the first option, we say that the direction of firing is an *instantaneous inertial direction*. In case the second option occurs, we say the Newtonian observer is not a Newtonian inertial observer.

Assume that the Newtonian observer finds an inertial direction. Then it registers this direction by the direction pointer of its Newtonian gun and repeats the same procedure with the purpose to find two more independent instantaneous inertial directions. If this turns out to be impossible then the Newtonian observer is not an inertial Newtonian observer. If such directions are found and they last for a period of time then the Newtonian observer is an *inertial Newtonian observer* for that period of time. Motions which define Newtonian inertial directions are called *inertial motions*.

The following questions emerge:

- Do Newtonian inertial directions exist and consequently Newtonian inertial observers?
- If they do, why are there at most three independent inertial directions, which a Newtonian observer can find experimentally?

The answer to both questions is given by the following axioms.

3.3.1.1 Newton's First Law

There do exist Newtonian inertial observers or, equivalently, there exist Newtonian inertial directions in space

3.3.1.2 Axiom on the Dimension of Space

Space has three dimensions

The above axioms assess the inertial Newtonian observers from the point of view of physics, that is, in terms of Physical measurements. However, in order for the Newtonian inertial observers to be used for the study of motion it is necessary that they be identified in geometry. Obviously the geometric determination of an inertial Newtonian observer cannot be done by means of unit rods, clocks, and guns but in terms of "geometric" objects. We define geometrically the Newtonian inertial observers by means of the following requirements/characteristics:

(1) The trajectory of a Newtonian inertial observer in the three-dimensional Euclidean space of Newtonian Physics is a straight line and its velocity is constant.
(2) The coordinate systems of Newtonian inertial observers are the Euclidean coordinate frames (ECF). Therefore the numbers which are derived by the observation/measurement of a Newtonian Physical quantity by a Newtonian inertial

observer are the components of the corresponding geometric object describing the quantity (i.e., the Newtonian tensor) in the ECF of the Newtonian inertial observer.

The existence of Newtonian inertial observers is of rather theoretical value. Indeed in practice most observers move non-inertially or, equivalently, most motions in practice are accelerated. Then how do we make physics for accelerated observers? To answer this question we generalize the concept of the Newtonian inertial observer as follows.

At a point P (say) along the trajectory of an accelerated Newtonian observer (this trajectory cannot be a straight line, except in the trivial case of one-dimensional accelerated motion) we consider the tangent line, which we identify with the trajectory of an inertial Newtonian observer. We call this Newtonian inertial observer the *instantaneous Newtonian inertial observer* at the point P. With this procedure a Newtonian accelerated observer is equivalent to (or defines) a continuous sequence of inertial Newtonian observers, each observer with a different velocity. Then the observations at each point along the trajectory of an accelerated observer are made by the corresponding Newtonian inertial observer at that point.

3.3.2 Measurement of the Position Vector

The primary element which is measured by all theories of physics studying motion is the position of the moving mass point at every instant in physical space. This position is specified by the position vector in the geometric space where the theory studies motion. In Newtonian Physics motion is studied in the three-dimensional Euclidean space, therefore the position vector is a vector $\mathbf{r}(t)$ in that space. The procedures of the observation/measurement of the instantaneous position in Newtonian Physics is defined only for the Newtonian inertial observers and it is the following.

Consider a Newtonian inertial observer who wishes to determine the position vector of a point mass moving in space. The observer points his Newtonian gun at the point P and fires an elastic bullet. The bullet moves inertially, because the observer is a Newtonian observer and it is assumed that it has infinite speed (action at a distance). Assume that the bullet hits the point P where it is reflected by means of some mechanism and returns to the gun of the observer along the direction of firing. The observer marks this direction and identifies it as the direction of the position vector of the point P. In order to determine the length of the position vector the observer draws the straight line connecting the origin of his coordinate system, O say, with the point P. Then the observer translates the ideal unit rod along this straight line OP (this procedure is called superposition) and measures (as it is done in good old Euclidean geometry) its length. The observer identifies the number resulting from this measurement with the length of the position vector of the point P.

There still remains the measurement of the time coordinate. Newtonian Physics assesses that the measurement of time is done simply by reading the ideal clock indication at the moment of completing the measurement of the position vector. Because in Newtonian theory time and space are assumed to be absolute, the independent measurement of the space coordinates and the time coordinate of the position vector is compatible and therefore acceptable. This completes the measurement of the position vector by the Newtonian inertial observer. In the following we shall write NIO for Newtonian inertial observer.

3.4 Galileo Principle of Relativity

With the measuring procedure discussed in the last section every NIO describes the position vector of any mass point with four coordinates in the coordinate frame it (not he/she) uses. However, as we have remarked in Chap. 2 the measurements of one NIO have no Physical significance if they are not verified with the corresponding measurements of another NIO. This verification is necessary in order for the measured Physical quantity to be "objective," that is, independent of the observer observing it. According to what has been said in Chap. 2, the procedure of verification is established by a Relativity Principle, which specifies

- An internal code of communication (= transformation of measurements/images) between NIO, which establishes the existence of Newtonian Physical quantities
- The type of geometric objects which will be used for the mathematical description of the images of the Newtonian Physical quantities

The Principle of Relativity in Newtonian Physics is the *Galileo Principle of Relativity* and it is described by the diagram of Fig. 3.2, which is a special case of the general diagram of Fig. 2.2.

The Galileo Principle of Relativity is of a different nature than the direct Newtonian observation, because it relates the images of one NIO with those of another NIO and not an observer with an observed Physical quantity. For this reason the arrow which represents the Galileo Principle is bidirectional (mutual exchange of information).

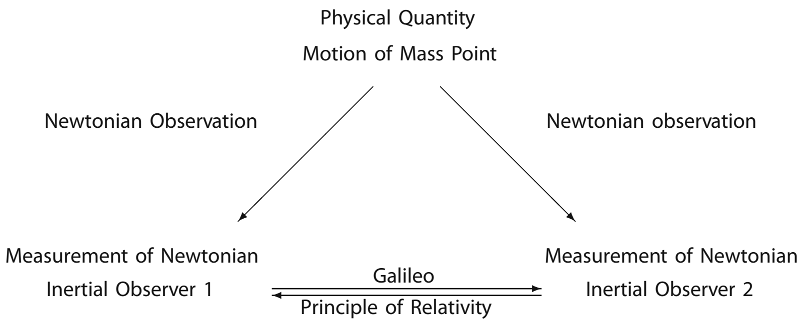

Fig. 3.2 The Galileo Relativity Principle

More specifically, let us consider the NIO O and let K be its Euclidean Cartesian coordinates in space. Consider a second NIO O' with the Euclidean Cartesian coordinate system K'. Then the Galileo Relativity Principle says that there exists a transformation which relates the coordinates of all points in space in the systems K, K'. We demand that this coordinate transformation be linear and depend only on the relative velocity of the coordinate frames K, K'.

We note that the triangle formed by the various arrows is closed, which means that if a Newtonian Physical quantity is observed by one NIO, then the Physical quantity can be observed by all NIOs. Finally, the triangle commutes, which means that in order to describe the motion of a mass particle, it is enough to use one NIO to perform the measurement of the position vector and then communicate the result to any other NIO by means of the proper Galileo coordinate transformation.

3.5 Galileo Transformations for Space and Time – Newtonian Physical Quantities

The transformation of coordinates specified by the Principle of Relativity of a theory of physics is a well-defined mathematical procedure if it leads to a set of transformations which follow the below rules:

- They form a group under the operation of composition of maps.
- Each transformation is specified uniquely in terms of the relative velocity of the observers it relates.

In Newtonian Physics the Galileo Principle of Relativity leads to the Galileo transformations. These transformations relate the coordinate systems of NIO and they define the Galileo group[10] under the composition of transformations.

In order to compute the Galileo transformations, one needs a minimum number of fundamental Newtonian Physical quantities, which will be the basis on which more Newtonian Physical quantities will be defined. The space and the time must be the first fundamental Newtonian Physical quantities, because they are the substratum on which motion is described and studied. Therefore the first demand/requirement of the Galileo Principle of Relativity is as follows:

3.5.1 Galileo Covariant Principle: Part I

For NIO the position vector and the time coordinate are Newtonian Physical quantities.

In order to compute the analytic expression of the Galileo transformations, we consider the ideal unit rod and let A, B be its end points whose position vectors

[10] This is the isometry group of the three-dimensional Euclidean space.

in $K,\ K'$ are, respectively, $\mathbf{r}_A,\ \mathbf{r}_B$ and $\mathbf{r}'_A,\ \mathbf{r}'_B$. Then the Galileo Relativity Principle specifies the exchange of information between the NIOs $K,\ K'$ by the requirements:

3.5.2 Galileo Principle of Communication

(a) The Euclidean distance of the points A and B is an invariant.
(b) The time moment of the points A, B is the same and it is also an invariant.

These two requirements when expressed mathematically lead to the following equations:

$$(AB) = (AB)',\ t = t'$$

or

$$(\mathbf{r}_A - \mathbf{r}_B)^2 = (\mathbf{r}'_A - \mathbf{r}'_B)^2, \tag{3.1}$$
$$t = t'. \tag{3.2}$$

Equation (3.1) is the equation of Euclidean isometry, which we studied in Sect. 1.5. Therefore Galileo transformations are the group $SO(3)$ of Euclidean orthogonal transformations (EOT), which in two general, not necessarily orthogonal, frames K_1, K_2 are given by the equation

$$A^t[g]_{K_1}A = [g]_{K_2}. \tag{3.3}$$

Especially for a Euclidean Cartesian coordinate system (and only there!) $[g]_{K_1} = [g]_{K_2} = I_3$, where I_3, is the unit 3×3 matrix $\delta_{\mu\nu}$, and the transformation matrix A satisfies the orthogonality relation

$$A^t A = I_3. \tag{3.4}$$

Under the action of the Galileo transformations the position vector transforms as follows:

$$\mathbf{r}' = A\mathbf{r} + \mathbf{O}'\mathbf{O}. \tag{3.5}$$

In this relation $\mathbf{r}$ is the position vector as measured by the NIO O, $\mathbf{r}'$ is the position vector (of the same mass point!) as measured by the NIO O', and A is the Galileo transformation relating the observers O, O'.

It remains the time transformation equation (3.2). However, this simply says that time is an invariant of the group of Galileo transformations. In conclusion, the Galileo Principle of Relativity provides us with one Newtonian vector quantity (the position vector) and one Newtonian scalar quantity (the time). In the next section we show how these two fundamental Newtonian Physical quantities are used to define new ones.

3.6 Newtonian Physical Quantities. The Covariance Principle

The Galileo Principle of Relativity introduced the Newtonian Physical quantities, position vector and time. However, in order to develop Newtonian Physics one needs many more Physical quantities. Therefore we have to have a procedure, which will allow us to define additional Newtonian Physical quantities. This procedure is established by a new principle, called the Principle of Covariance, whose general form has been given in Sect. 2.6.2 of Chap. 2.

3.6.1 Galileo Covariance Principle: Part II

The Newtonian Physical quantities are described with Newtonian tensors.

More specifically the Newtonian Physical quantities have 3^n, $n = 0, 1, 2, \ldots$, components, which under the Galileo transformations transform as follows:

$$T^{\mu_{1'}\mu_{2'}\mu_{3'}\cdots} = A^{\mu_{1'}}_{\mu_1} A^{\mu_{2'}}_{\mu_2} A^{\mu_{3'}}_{\mu_3} \cdots T^{\mu_1\mu_2\mu_3\cdots}, \tag{3.6}$$

where $T^{\mu_{1'}\mu_{2'}\mu_{3'}\cdots}$, $T^{\mu_1\mu_2\mu_3\cdots}$ are the components of the Newtonian Physical quantity T as measured by the NIOs O' and O, respectively, and $A^{\mu_{i'}}_{\mu_i}$ $(i, i' = 1, 2, 3)$ is the Euclidean orthogonal transformation relating $O', \ O$ and defined by the transformation of the connecting vector(!).

Obviously for $n = 0$ one gets the Newtonian invariants, for $n = 1$ the Newtonian vectors, etc. The Galileo Covariance Principle does not say that all Newtonian tensors are Newtonian Physical quantities. It says only that a Newtonian tensor is a *potential* Newtonian Physical quantity and it is physics which will decide if this quantity is indeed a Physical quantity or not! More on that delicate subject shall be dealt with when we discuss the Theory of Special Relativity.

From given Newtonian tensors we define new ones by the general rules stated in Sect. 2:

Rule I:

If we differentiate a Newtonian tensor of order (r, s) *wrt a Newtonian invariant then the new geometric object we find is a Newtonian tensor of order* (r, s).

Rule II:
If we multiply a Newtonian tensor of order (r, s) *with a Newtonian invariant then the new geometric object we find is a Newtonian tensor of order* (r, s).

3.7 Newtonian Composition Law of Vectors

The composition of Newtonian vectors is vital in the study of Newtonian Physics; however, many times it is approached as a case by case matter, thus losing its deeper geometric significance. The commonest rule of composing vectors in Newtonian Physics is the *composition of velocities*. Indeed the composition of velocities is a simple yet important issue of Newtonian kinematics and constitutes one of the reasons for the introduction of the Theory of Special Relativity.

We consider a point P with position vector $\mathbf{r}$ and $\mathbf{r}'$ wrt to the Cartesian coordinate systems K and K' of the NIOs O, O', respectively. The linearity of the space implies the relation

$$\mathbf{r} - \mathbf{OO}' = \mathbf{r}'. \tag{3.7}$$

The left-hand side of relation (3.7) contains the vectors $\mathbf{r}$, $\mathbf{OO}'$, which are measured by the observer O and in the right-hand side the vector $\mathbf{r}'$ which is measured by observer O'. The linearity of space implies that these two "different" conceptions of the point P are the same. We differentiate (3.7) wrt the time t of observer O and find

$$\frac{d\mathbf{r}}{dt} - \frac{d\mathbf{OO}'}{dt} = \frac{d\mathbf{r}'}{dt} \tag{3.8}$$

or, in terms of the velocities,

$$\mathbf{V}_P - \mathbf{V}_{OO'} = \frac{d\mathbf{r}'}{dt}. \tag{3.9}$$

In (3.9) $\mathbf{V}_P, \mathbf{V}_{OO'}$ are the velocities of P and O' as measured by the observer O. The quantity $\frac{d\mathbf{r}'}{dt}$ in the right-hand side *is not* a velocity, because the time t is not the time of the observer O'.

Let t' be the time of observer O'. Then (3.9) is written as

$$\mathbf{V}_P - \mathbf{V}_{OO'} = \frac{dt'}{dt}\mathbf{V}'_P, \tag{3.10}$$

where $\mathbf{V}'_P$ is the velocity of P as measured by observer O'. However, in Newtonian Physics $t' = t$ (because time is an invariant!), therefore $\frac{dt'}{dt} = 1$, and relation (3.10) gives the following law of composition of velocities in Newtonian Physics:

$$\mathbf{V}_P - \mathbf{V}_{OO'} = \mathbf{V}'_P. \tag{3.11}$$

Now relation (3.11) is the Galileo transformation of the vector $\mathbf{V}_P$ from the coordinate system K of O to the coordinate system K' of O', that is,

$$\begin{aligned} K &\to K', \\ \mathbf{V}_P \to \mathbf{V}'_P &= \mathbf{V}_P - \mathbf{V}_{OO'}, \end{aligned} \tag{3.12}$$

where $\mathbf{V}_{OO'}$ is the velocity of O' as measured by O.

Having the above as a guide we define the composition law of a Newtonian vector *quantity* $\mathbf{A}_P$, say, observed by the NIOs O and O' with the relation

$$\mathbf{A}_P \to \mathbf{A}'_P = \mathbf{A}_P - \mathbf{A}_{OO'}, \tag{3.13}$$

where $\mathbf{A}_{OO'}$ is the corresponding vector A of O' as it is measured by O. Equivalently, if $\Gamma(O, O')$ is the Galileo transformation relating O, O' we may define the composition law of the Newtonian vector $\mathbf{A}_T$ with the relation

$$\mathbf{A}'_T = \Gamma(O, O')\mathbf{A}_T. \tag{3.14}$$

For example if $\mathbf{a}_T$ (respectively, $\mathbf{a}'_T$) is the acceleration of the point P as measured by the observer O (respectively, O') and $\mathbf{A}_{OO'}$ is the acceleration of observer O' as measured by observer O then

$$\mathbf{a}'_P = \Gamma(O, O')\mathbf{a}_T = \mathbf{a}_T - \mathbf{a}_{OO'}. \tag{3.15}$$

We conclude that the law of composition of the Newtonian vector Physical quantities is equivalent to the Galileo transformation. This is the reason why the discovery that the velocity of light did not obey this composition law resulted in the necessity of the introduction of a new theory of motion, which was the Theory of Special Relativity.

3.8 Newtonian Dynamics

Newtonian kinematics is concerned with the geometric study of the trajectory of a mass point in space, without involving the mass point itself. For this reason the Newtonian Physical quantities which characterize the trajectory of a mass point are limited to the position vector, the velocity, and the acceleration. Newtonian dynamics is the part of Newtonian Physics which studies motion including the mass point. In order to make this possible, one has to introduce new Physical quantities, which characterize the mass point itself.

The simplest new Newtonian Physical quantities to be considered are the invariants, the most basic being the *inertial mass* m of the mass point. The mass (inertial) m is the first invariant Newtonian Physical quantity after the time and connects the kinematics with the dynamics of the theory.

We recall that a Newtonian invariant is a potentially Newtonian Physical quantity and becomes a Physical quantity only *after* a definite measuring or observational procedure has been given. Therefore if we wish to have a mass m associated with each mass point, we must specify an experimental procedure for its measurement. This is achieved by the introduction of a new Physical quantity, the linear momentum.

For a mass point of mass m we define the vector $\mathbf{p} = m\mathbf{v}$, which we name the *linear momentum* of the mass point. Because $\mathbf{v}$ is a Euclidean vector and m a Euclidean invariant the quantity $\mathbf{p}$ is a Euclidean vector, therefore a potential Newtonian Physical quantity. For this new vector we define a conservation law which distinguishes it from other Newtonian vectors (otherwise $\mathbf{p}$ would be an arbitrary vector without any Physical significance).

3.8.1 Law of Conservation of Linear Momentum

A Physical system consisting of Newtonian mass points will be called *closed* if the trajectories of all mass points comprising the system move inertially.[11] If $\mathbf{p}_i$ is the linear momentum of the ith particle of a closed system of mass points then we demand that the following equation holds:

$$\sum \mathbf{p}_i = constant. \tag{3.16}$$

Relation (3.16) is the mathematical expression of the law of conservation of linear momentum. The word "law" means that (3.16) has been proved true in every case we have applied so far; however, we cannot prove it in general. We remark that this law holds for each and all NIO and, furthermore, that it is compatible with the Galileo Principle of Relativity.

The deeper significance of the law of conservation of linear momentum is that it makes the linear momentum from a potentially Newtonian Physical quantity to a Newtonian Physical quantity. This is done as follows: We consider two solid bodies (properly chosen, e.g., spheres) which rest on a smooth frictionless surface connected with a spring under compression and a string which keeps them in place. At one moment we cut the string and the bodies are moving apart in opposite directions (we assume one-dimensional motion). We calculate the velocities of the bodies just after the string is cut.

We number the bodies by 1 and 2 and consider two NIOs O, O' with relative velocity $\mathbf{u}$. Let m_1, m_2 be the masses of the bodies wrt the observer O and m'_1, m'_2 wrt observer O'. We assume that the bodies 1,2 initially rest wrt observer O and that after the cutting of the string they have velocities $\mathbf{V}_1, \mathbf{V}_2$ wrt the observer O and $\mathbf{V}'_1, \mathbf{V}'_2$ wrt the observer O'. Conservation of the linear momentum for observers O and O' gives

[11] In which case every mass point can be associated with an NIO.

$$m_1\mathbf{V}_1 + m_2\mathbf{V}_2 = 0, \tag{3.17}$$
$$-(m_1' + m_2')\mathbf{u} = m_1'\mathbf{V}_1' + m_2'\mathbf{V}_2'. \tag{3.18}$$

From the Newtonian composition law for velocities we have

$$\mathbf{V}_1' = \mathbf{V}_1 - \mathbf{u}, \quad \mathbf{V}_2' = \mathbf{V}_2 - \mathbf{u}.$$

Replacing in (3.18) we find

$$m_1'\mathbf{V}_1 + m_2'\mathbf{V}_2 = 0. \tag{3.19}$$

From (3.17) and (3.19) it follows that:

$$\frac{m_1'}{m_2'} = \frac{m_1}{m_2}, \tag{3.20}$$

which implies that the quotient of the inertial masses of the bodies $\frac{m_1}{m_2}$ is a Euclidean invariant. This invariant is a Euclidean Physical quantity because it can be measured experimentally. Therefore if the mass of body 1 (say) is k times the mass of body 2 for one NIO then this is true for all NIOs. By choosing one body to have unit mass for one NIO (therefore for all NIOs because we assume mass to be an invariant) we have an experimental measurement for mass. Therefore mass is a Newtonian Physical quantity.

Using the mass of a mass point and the acceleration we define the new potentially Newtonian Physical quantity:

$$m\mathbf{a}.$$

The Second Law of Newton says that this new quantity is a Newtonian Physical quantity, which we call *force*:

$$\mathbf{F} = m\mathbf{a}. \tag{3.21}$$

A subtle point which was pointed out for the first time by Special Relativity is the covariant character of (3.21). That is if O' is another NIO then we demand that

$$\mathbf{F}' = \frac{d\mathbf{p}'}{dt}, \tag{3.22}$$

where $\mathbf{F}'$, $\mathbf{p}'$ (of course $t' = t$) are the Physical quantities $\mathbf{F}$, $\mathbf{p}$ as measured by O'. This requirement says that in addition to the demand that the Newtonian Physical quantities must be expressed in terms of Newtonian tensors, the dynamical equations must also be tensor equations, that is covariant under the action of the Galileo transformations. This requirement, which has been (unfortunately) called again Covariance Principle, has been considered at times as trivial, since the right-hand side of (3.22) defines a Newtonian Physical quantity. However, this point of view is not correct because at this elementary level everything is profound. However, in more general cases (field theory) the Principle of Covariance attains a practical significance in the formulation of the dynamical equations. Closing, we emphasize that what has been said for (3.22) applies to all dynamical equations of Newtonian Physics, whatever dynamical Newtonian Physical quantities they involve.

Chapter 4
The Foundation of Special Relativity

4.1 Introduction

In Chap. 3 we developed Newtonian Physics from the relativistic point of view. We have discussed the deeper role of the Galileo Principle of Relativity and the concept of Newtonian physical quantity. In this chapter we shall use these Newtonian relativistic concepts to formulate the Theory of Special Relativity. We emphasize that concerning their structure both Newtonian Physics and Special Relativity are similar, the difference between the two theories being in the method of position measurement.

Why had the Theory of Special Relativity to be introduced? How long did it take for the theory to be developed and when did these take place? For a detailed answer to these and similar questions the interested reader should consult the relevant literature. In the following we shall refer briefly some historic elements mainly for the conceptual understanding and the historic connection. Newtonian Physics was in its colophon by the end of the 19th century when the mechanistic (deterministic) conception of the world was the prevailing trend in science. Indeed it was believed that one could provide the future and the past of a system (not necessarily a system of physics) if one was given the present state of the system and its "equation of evolution." Consequently everything was predetermined concerning the development in space and time, a point of view which was in agreement with the view of absolute time and absolute space. More specifically it was believed that all physical phenomena were described by Newtonian physical quantities and interactions among these quantities.

During the second part of the 19th century a number of experiments, carried out mainly by Michelson and Morley, were indicating that there was a fundamental physical system which could not be described by a Newtonian physical quantity. That was the light whose kinematics was appearing to be at odds with Newtonian kinematics. Specifically the speed of light was appearing to be independent of the velocity of the emitter and the receiver.

As was expected this radical result could not be accepted by the established scientific community and physicists tried to manipulate the Newtonian theory with the view to explain the "disturbing" non-Newtonian behavior of the speed of light. However, the changes were not considered at the level of the foundations of

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Newtonian Physics (i.e., Galileo Principle of Relativity, Newtonian physical quantities, etc.) but instead they focussed on the modification of the properties of physical space. A new cosmic fluid was introduced, which was named *ether* to which were attributed as many properties as were required in order to explain the newly discovered non-Newtonian phenomena, with prevailing the constancy of the speed of light. Soon the ether attained too many and sometimes conflicting properties and it was becoming clear that simply the "new" phenomena were just not Newtonian.

Poincaré understood that and began to consider the existence of non-Newtonian physical quantities, a revolutionary point of view at that time. He reached very close to formulate the Theory of Special Relativity. However, the clear formulation of the theory was made by A. Einstein with his seminal work[1] (*'On the electrodynamics of moving bodies' Annalen der Physik,* ***17*** *(1905)).*

In the following we present the foundation of Special Relativity starting from the non-Newtonian character of the light and subsequently stating the Einstein's Principle of Relativity and the definition of the relativistic physical quantities.

4.2 Light and the Galileo Principle of Relativity

4.2.1 The Existence of Non-Newtonian Physical Quantities

In Newtonian Physics the Galileo Principle of Relativity is a consequence of the direct sensory experience we have for motion. Due to this, the principle is "obvious" and need not be considered at the foundations of Newtonian Theory but only after the theory has been developed and one wishes to differentiate it from another theory of physics.

A direct result of the assumptions of Newtonian Physics concerning the space and the time is the Newtonian law of composition of velocities. The special attention which has been paid to that law is due to historic reasons. Because due to this law it was shown for the first time that there are physical quantities which cannot be described with Newtonian physical quantities or, equivalently, which were not compatible with the Galileo Principle of Relativity. In practice this means that one cannot employ inertial Newtonian observers to measure the kinematics of light phenomena with ideal rods and ideal clocks and then compare the measurements of different observers using the Galileo transformation. One could argue that we already have a sound theory of optics which is widely applied in Newtonian Physics. This is true. However, one must not forget that the theory of Newtonian optics concerns observers and optical instruments which have very small relative velocities

[1] An English translation of this paper can be found in the book *Principle of relativity* by *H. Lorentz, A. Einstein, H. Minkowski, H. Weyl, Dover (1952)*. In this volume one can find more papers which lead to the development of the Theory of Special Relativity including the work of Einstein where the famous relation $E = mc^2$ appeared for the first time as well as the introduction of the term spacetime by H. Minkowski.

compared to the speed of light, therefore the divergency of the measurements are much smaller than the systematic errors of the instruments. For observers with relative speed comparable to the speed of light (e.g., greater than $0.8c$) the experimental results differentiate significantly and are not compatible with the Galileo Principle of Relativity. Due to the small speeds we are moving, these results do not trigger the direct sensory experience and one needs special devices to do that, which in Chap. 2 we called secondary sensors.

Finally, today we know that at high relative speeds it is not only the light phenomena which diverge from the Newtonian reality but all physical phenomena (e.g., the energy of elementary particles). It is this global change of physics at high speeds which is the great contribution of Einstein and not the Lorentz transformation, which was known (with a different context) to Lorentz and Poincaré. Normally only atomic and nuclear phenomena are studied at these speeds, therefore these are the phenomena which comprise the domain of Special Relativity.

4.2.2 The Limit of Special Relativity to Newtonian Physics

Our senses (primary sensors) work in the Newtonian world, therefore every theory of physics eventually must give answers in that world, otherwise it is of no use. For this reason we demand that the new theory we shall develop below, in the limit of small relative velocities, will give numerical results which coincide (within the experimental accuracy) with the corresponding Newtonian ones, provided that they exist. Indeed it is possible that some relativistic phenomena do not have Newtonian limit, as for example the radioactivity where the radioactive source can be at rest in the lab.

The coincidence of the results of Special Relativity with the corresponding results of Newtonian Physics in the limit of small relative velocities must not be understood in the context that at that limit the two theories coincide. Each theory has its own distinct principles and assumptions which are different from those of the other theory. The limit concerns only the *numerical values* of the components of common physical quantities.

The requirement of the limit justifies the point of view that there is a continuation in nature and the phenomena occurring in it and, in addition, enables one to identify the physical role (in our Newtonian environment) of the physical quantities of Special Relativity. Schematically these ideas are presented as follows:

$$\begin{bmatrix} \text{Physics} \\ \text{of high} \\ \text{speeds} \end{bmatrix} \xrightarrow{u \ll c} \begin{bmatrix} \text{Newtonian} \\ \text{Physics} \end{bmatrix}.$$

Exercise 11 Define the variable $l = ct$ and consider the D'Alembert operator

$$\Box^2 = \nabla^2 - \frac{\partial^2}{\partial l^2}. \tag{4.1}$$

Let V^4 be a four-dimensional linear space and let (l, x, y, z), (l', x', y', z') be two coordinate systems in V^4, which are related by the linear transformation

$$x = ax' + bl' \quad , \quad y = y' \ , \ \ z = z' \ , \qquad l = bx' + al'. \tag{4.2}$$

(1) Show that under this coordinate transformation the D'Alembert operator transforms as follows :

$$\Box'^2 = (a^2 - b^2)\Box^2.$$

(2) Consider two Newtonian inertial observers O and O' whose coordinate axes x, y, z and x', y', z' are parallel and the x'-axis moves parallel to the x-axis with velocity $\mathbf{u} = u\mathbf{i}$. Show that the Galileo transformation defined by the two observers is obtained from the general transformation (4.2) if one takes $a = 1$, $b = \beta = \frac{u}{c}$. Infer that under Galileo transformation the D'Alembert operator transforms as follows:

$$\Box'^2 = (1 - \beta^2)\Box^2 = \frac{1}{\gamma^2}\Box^2,$$

where $\gamma = (1 - \beta^2)^{-1/2}$. (What is the meaning of the case $\beta = 0$?) Conclude that the D'Alembert operator is not covariant (that is, its form changes) under a Galileo transformation.

(3) Show that the D'Alembert operator transforms covariantly (that is $\Box'^2 = \Box^2$) if the coefficients a, b satisfy the relation/constraint $a^2 - b^2 = 1$. One solution of this equation is $a = \gamma, b = \beta\gamma, \gamma = \left(1 - \beta^2\right)^{-1/2}, \ \beta \in (0, 1)$. Write the resulting coordinate transformation in the space V^4 using these values of the coefficients and the general form (4.2). As will be shown below this transformation is the boost along the x-axis and is a particular case of the Lorentz transformation.

(4) The wave equation for the electromagnetic field ϕ is $\Box^2\phi = 0$. Show that the wave equation is not covariant under a Galileo transformation and that it is covariant under a Lorentz transformation. This result shows that light waves are not Newtonian physical quantities.

[Hint: $\frac{\partial}{\partial x'} = \frac{\partial x}{\partial x'}\frac{\partial}{\partial x} + \frac{\partial l}{\partial x'}\frac{\partial}{\partial l} = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial l}$, etc.]

When Einstein presented Special Relativity people could not understand the new relativistic physical quantities, because they appeared to behave in a "crazy," i.e., non-Newtonian way. For that reason he devised simple but didactic arguments whose main characteristic is the simple mathematics and the essential physics. These arguments have been called *thought experiments* (or *Gedanken experimente* in German) and have contributed essentially to the comprehension of the theory. We present one such experiment in the following example.

Example 7 Consider a box of mass M which is resting on a smooth horizontal plane of a Newtonian inertial observer. Suddenly and without any external cause two small equal masses m ($< M/2$) are emitted from the box in opposite directions with equal speeds (see Fig. 4.1). Prove that this phenomenon cannot be explained if one assumes the conservation of mass, conservation of linear momentum, and conservation of energy. Furthermore show that the phenomenon can be explained provided one assumes

(a) The relation $E = \Delta mc^2$ where Δm is the reduction in mass, E is the kinetic energy of the fragments, and c is a universal constant (the speed of light) and
(b) The mass is not preserved

Assuming conservation of linear momentum we infer that the remaining part of the box will continue to be at rest after the emission of the masses m. Assuming conservation of mass we have that the mass of the box after emission equals $M-2m$.

Concerning the conservation of energy we have that before the emission the (kinetic) energy of the system equals zero whereas after the emission it equals $\frac{2p^2}{2m} = \frac{p^2}{m}$, where p is the measure of the linear momentum of each mass m. Therefore we have a violation of the law of conservation of energy. This violation would not bother us if no such phenomena exist in nature. However, observation has shown that this is not so and one such example is the spontaneous disintegration of a radioactive nucleus. Therefore we must be able to "explain" such phenomena with theory.

In order to give an "explanation" of the phenomenon we have to abandon one of the three conservation laws and, at the same time, abandon Newtonian Physics. Conservation of linear momentum is out of question, because we know that it works very well. Between the conservation of mass and the conservation of energy we prefer to abandon the first. Therefore we assume that after the emission the (relativistic, not the Newtonian!) mass of the remaining box equals $M - 2m - \Delta m$ where Δm is a correcting factor necessary to keep the energy balance. Still we need a relation which will relate the (relativistic, not Newtonian!) energy with the (relativistic) mass. Of all possible relations we select the simplest, that is $E = \lambda \Delta m$ where λ is a universal constant. A dimensional analysis shows that λ has dimensions $[L]^2[T]^{-2}$, that is, speed^2. But experiment has shown that a universal speed does exist and it is the speed c of light in vacuum. We identify λ with c and write

$$E = mc^2.$$

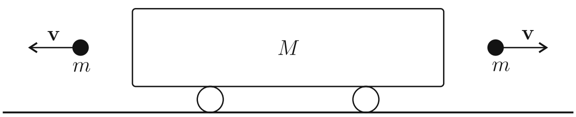

Fig. 4.1 Spontaneous emission

Using this relation the conservation of energy gives

$$\Delta mc^2 = \frac{p^2}{m} = 2E_K,$$

where E_K is the (relativistic) kinetic energy of each fragment in the coordinate system we are working. This equation shows that part of the mass of the box has been changed to kinetic energy of the masses m. This effect allows us to say that matter in relativity exists in two equivalent forms: (relativistic) mass and (relativistic) energy.

We emphasize that in the above example both mass and energy must be understood relativistically and not in the Newtonian context, otherwise one is possible to be led to erroneous conclusions.

4.3 The Physical Role of the Speed of Light

The speed of light has two characteristics which make it important in the study of physical phenomena. As we have already remarked the speed of light is not compatible with the velocity composition rule of Newtonian Physics, hence light is not a Newtonian physical quantity. This means that we must develop a new theory of physics in which the Newtonian method of measuring space and time intervals, that is absolute meter sticks and absolute clocks, does not exist.

The second property is that the speed of light in vacuum is constant and independent of the velocity of the emitter and the receiver. This is not a truly established experimental fact because all experiments measure the speed of light in two ways ("go" and "return"), therefore they measure the average speed. However, we accept this result as a law of physics and wait for the appropriate experiment which will validate this law.

The second property of light is equally important as the first because while the first prohibits the rigid rods and the absolute clocks, the second allows us to define new ways to measure space distances and consequently to propose procedures which will define Relativistic inertial observers and Relativistic physical quantities. These procedures are known with the collective name *chronometry*.

Chronometry defines a new arrow, in analogy with the Newtonian arrow of pre-relativistic physics:

$$\begin{bmatrix} \text{Physical Quantity} \\ \text{or} \\ \text{phenomenon} \end{bmatrix} \overset{\text{Chronometry}}{\longrightarrow} \left[\text{Observer} \right].$$

We call the new type of observation defined by chronometry *relativistic observation* and the new class of observers *relativistic observers*. The theory of these observers is the Theory of Special Relativity. The new theory needs a Relativity Principle to define its objectivity. We call the new principle the *Einstein Principle*

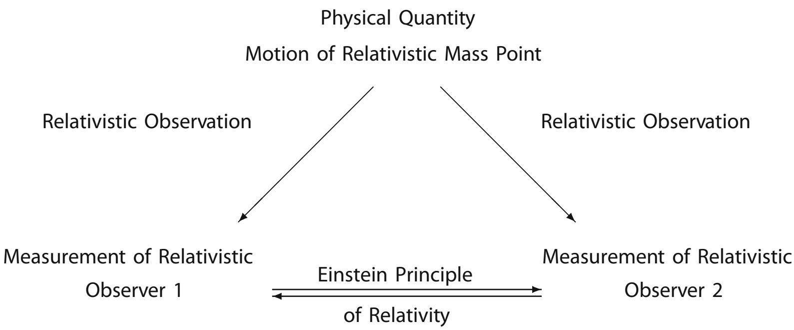

Fig. 4.2 The Einstein principle of relativity

of Relativity and it is shown schematically in Fig. 4.2 with a diagram similar to the one of the Galilean Principle of Relativity (see Fig. 3.2).

Concerning this diagram we make the following observations:

(a) The arrow between the two observers (although it has been drawn the same) is different from the corresponding arrow of Fig. 3.2. This is due to the fact that in Special Relativity there are events which are observable from the relativistic observer 1 but they are not so from the relativistic observer 2 and vice versa, contrary to what is the case with the Newtonian Physics.
(b) For the relativistic observation the sharp distinction between the observed and the observer still holds. This means that Special Relativity is a classical theory of physics. In the non-classical theories of physics this distinction does not hold and one speaks for quantum theories of physics, relativistic or not.

Finally in Special Relativity the Einstein Principle of Relativity concerns the *exchange of information* between the relativistic observers and it is foreign to the relativistic observation. This principle is quantified by a transformation group which defines the covariance group of the theory and subsequently (a) the mathematical nature of the relativistic physical quantities and (b) the mathematical form of the laws of Special Relativity.

4.4 The Physical Definition of Spacetime

Every theory of physics creates via its observers "images" of the physical phenomena in a geometric space, the "space" of the theory. In Newtonian Physics this space is the Euclidean space E^3. In Special Relativity this space is called *spacetime* and is fundamentally different from the space E^3 of Newtonian Physics.

For the first time the concept of spacetime was introduced by H. Minkowski in his seminal talk at the 80th Congress of the German Scientists of Physical Sciences, which took place in Cologne on 21 September 1908, three years after the celebrated work of Einstein on the Theory of Special Relativity. The words of H. Minkowski in that congress are considered to be classical and are as follows:

> The views of space and time which I wish to lay before you have sprung from the soil of experimental Physics, and therein lies their strength. They are radical. Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality.

In order to define the concept of spacetime (which apparently is a bad word because it has nothing to do with either space or time!) we use the same methodology we followed with the concepts of space and time in Newtonian Physics, that is, we develop the concept of spacetime by defining its parts (points) and its structure (geometry).

4.4.1 The Events

We call the points of spacetime *events*. In contrast to the points of Newtonian space the events have identity. Indeed we consider that each event refers to something that *happened* to a physical system or systems (the event does not characterize the physical system per se). For example consider the disintegration of a nucleus. The point of spacetime is the fact that the nucleus disintegrated not where and when this happened. The latter are the coordinates of the point of spacetime (the event) in some coordinate system. The coordinates can change depending on the coordinate system used; however, the event remains the same! In a sense spacetime is the aggregate of all facts which happened to all physical systems.

We infer immediately that Newtonian space and Newtonian time are not relativistic physical quantities, that is, they have no objectivity in Special Relativity as individual entities. Because no event is possible either for the (Newtonian) space or for the (Newtonian) time, because they are absolute!

4.4.2 The Geometry of Spacetime

In addition to its points, spacetime is characterized by its geometry. In Special Relativity it is assumed that the geometry of spacetime is absolute, in the sense that the relations which describe it remain the same – are independent of the various physical phenomena, which occur in physical systems. The geometry of spacetime is determined in terms of a number of assumptions which are summarized below:

(1) Spacetime is a four-dimensional real linear space.
(2) Spacetime is homogeneous. From the geometric point of view this means that every point in spacetime can be used equivalently as the origin of coordinates. Concerning physics, this means that where and when an experiment (i.e., event) takes place has no effect on the quality and the values of the dynamical variables describing the event.

(3) In spacetime there are straight lines, that is unbounded curves which are described geometrically with equations of the form

$$\mathbf{r} = \mathbf{a}t + \mathbf{b},$$

where $t \in R$, $\mathbf{a}, \mathbf{b} \in R^4$. From the physics point of view these curves are the trajectories of special motions of physical systems in R^4 which we call *relativistic inertial motions*. All motions which are not relativistic inertial motions are called accelerating motions. With each accelerated motion we associate a four-force in a manner to be defined later on.

(4) Spacetime is isotropic, that is all directions at any point are equivalent. The assumptions of homogeneity and isotropy imply that the spacetime of Special Relativity is a flat space or, equivalently, has zero curvature. In practice this means that one is possible to define a coordinate system that covers *all* spacetime or, equivalently, spacetime is diffeomorphic (i.e., looks like) to the linear space R^4.

(5) Spacetime is an affine space, that is, if one is given a straight line or a hyperplane (=three-dimensional linear subspace with zero curvature) in spacetime then (axiom!) there exists at least one hyperplane which is parallel to them, in the sense that it meets the straight line or the other hyperplane at infinity.

(6) If we consider a straight line in spacetime then there is a continuous sequence of parallel hyperplanes which cut the straight line once and fill up all spacetime. We say that these hyperplanes *foliate spacetime*. Due to the fact that there are not absolute straight lines in spacetime there are infinitely many foliations. In Newtonian space there is the absolute straight line of time. Therefore there is a preferred foliation (that of cosmic time; see Fig. 4.3).

(7) Spacetime is a metric vector space. The necessity of the introduction of the metric is twofold: (a) Selects a special type of coordinate systems (the Cartesian systems of the metric) which are defined by the requirement that in these coordinate systems the metric has its canonical form (i.e., diagonal with components ± 1). We associate these coordinate systems with the Relativistic inertial systems. (b) Each timelike straight line defines the foliation in which the parallel planes are normal to that line.

(8) The metric of spacetime is the Lorentz metric, that is, the metric of the four-dimensional real space whose canonical form is $(-1, 1, 1, 1)$. The selection of

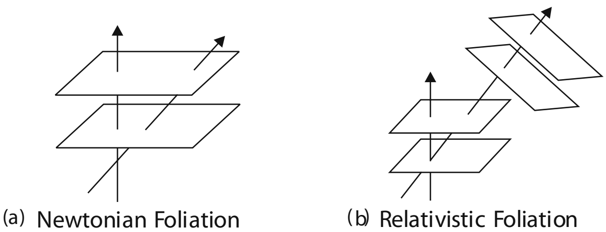

Fig. 4.3 Newtonian and (Special) relativistic foliations of spacetime

the Lorentz metric is a consequence of the Einstein Relativity Principle as will be shown in Sect. 4.7. The spacetime endowed with the Lorentz metric is called *Minkowski space* (see also Sect. 1.6). In the following we prefer to refer to Minkowski space rather than to spacetime, because Special Relativity is not a theory of space and time but a theory of many more physical quantities, which are described with tensors defined over Minkowski space. In addition it is best to reserve the word spacetime for General Relativity.

4.5 Structures in Minkowski Space

The Lorentz metric is very different from the Euclidean metric. The most characteristic difference is that the Lorentz distance of two different spacetime points can be $> 0, < 0$, or even $= 0$ whereas the Euclidean distance is always positive. Using this property we divide at any point spacetime into three regions.

4.5.1 The Light Cone

We consider all points Q of Minkowski space whose Lorentz distance from a fixed point P equals zero, that is,

$$(PQ)^2 = 0.$$

If the coordinates of the points P, Q in some coordinate system are $(ct(Q), \mathbf{r}(Q))$ and $(ct(P), \mathbf{r}(P))$, respectively, we have

$$(\mathbf{r}(Q) - \mathbf{r}(P))^2 - c^2(t(Q) - t(P))^2 = 0.$$

This equation defines a three-dimensional conic surface in Minkowski space whose top is at the point P. This surface is called *the light cone* at P and the vectors PQ *null vectors*. At each point in Minkowski space there exists *only one* null cone with vertex at that point.

The light cone divides the Minkowski space into three parts $(A), (B), (C)$ as shown in Fig. 4.4. Region (A) contains all points Q whose Lorentz distance from P is negative and the zeroth component is positive. This region is called the *future light cone* at P and the four-vectors PQ^i *future-directed timelike* four-vectors. Region (B) contains all points Q whose Lorentz distance from the reference point P is negative and their zeroth component is negative. This region is called the *past light cone* at P and the four-vectors PQ^i *past-directed timelike* four-vectors. Finally region (C) consists of all points whose Lorentz distance from the reference point P is positive.

In Special Relativity we consider that region (A) (respectively, (B)) contains all events from which one can receive (respectively, emit) information from

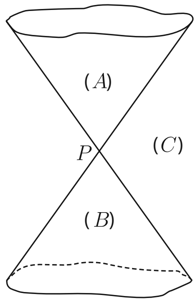

Fig. 4.4 The light cone

(respectively, to) the point P. We say that regions (A), (B) are in *causal relation* with the reference point P.

The light cone at P concerns all the light signals which arrive or are emitted from P. These signals are used to transfer information from and to P.

The events in the region (C) are not connected causally with the reference point P, in the sense that no light signals can reach P from these points. The *event horizon* at the point P is the null cone at P.

The fact that some events of Minkowski space do not exist (in the context that it is not possible to get information about these events via light signals) for a given point in Minkowski space sometimes is difficult to understand due to the absolute and instantaneous possibility of global knowledge of Newtonian Physics. However, because light transfers information with a finite speed it is possible that no information or interaction can reach points in Minkowski space beyond a distance.

4.5.2 World Lines

In Newtonian Physics the motion of a Newtonian mass point is represented by its trajectory in E^3. In Special Relativity the "motion" of a "relativistic mass point" is represented by a curve in Minkowski space. We call this curve the *world line* of the relativistic mass point.[2] The world line represents the history of the mass point, in the sense that each point of the curve contains information about the motion of the mass point collected by the proper observer of the mass point. The photons are not considered to be mass points in that approach. The events in spacetime are absolute in the sense that the information they contain is independent of who is observing the events. Because the description of the information depends on the observer describing the event we consider that the "absolute" (or reference) information is the one provided by the proper observer of the physical system (photons excluded).

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[2] A relativistic mass point should be understood as a particle with non-zero proper mass and speed $< c$. In Sect. 6.2 we shall give a geometric and precise definition of the relativistic mass point.

4.5.3 Curves in Minkowski Space

Let $x(\tau)$ be a curve in Minkowski space parameterized by the real parameter τ. The tangent vector to the curve at each point is the four-vector

$$u^i = dx^i/d\tau.$$

From the totality of curves in spacetime there are three types of curves which are used in Special Relativity.

(a) *Timelike curves*
These are the curves in Minkowski space whose tangent vector at *all their points* is timelike, that is,

$$\eta_{ij}u^i u^j < 0.$$

The geometric characteristic of these curves is that their tangent four-vector at every point lies in the timelike region of the null cone at that point. Concerning their physical significance we assume that the timelike curves are world lines of relativistic mass points, that is, particles with non-zero mass. Because the speed of particles with non-zero mass is less than c we call them *bradyons* (from the Greek word $\beta\rho\alpha\delta\upsilon s$). All known elementary particles with non-zero rest mass are bradyons, e.g., electron, proton, pion, lepton.

(b) *Null curves*
These are the curves in Minkowski space whose tangent vector p^i *at all points* is null:

$$\eta_{ij}p^i p^j = 0.$$

The null curves lie entirely on the light cone and we assume that they represent the world lines of relativistic systems of zero proper mass and speed c. One such system is the photon (and perhaps the neutrino). These particles are called luxons.

(c) *Spacelike curves*
These are the curves whose tangent vector n^i *at all their points* is spacelike, that is,

$$\eta_{ij}n^i n^j > 0.$$

The tangent vector of these curves at all their points lies outside the light cone. We assume that these curves represent the field lines of various dynamical vector fields, e.g., the magnetic field. The study of the geometry of these curves can be used to rewrite the dynamic equations of these fields in terms of geometry and then use geometrical methods to deal with physical problems. We shall not do this in the present book but the interested reader can look up information in the web on the term spacelike congruences. At some stage people attempted to associate with these curves particles with imaginary mass and speed greater than c. These "particles" have been named *tachyons* (from the Greek work $\tau\alpha\chi\upsilon s$ which means fast).

Although the theory does not exclude the existence of such particles, it is safer that we restrict the role of the spacelike curves to field lines of dynamic fields.

We emphasize that the three types of curves we considered do not exhaust the possible curves in Minkowski space. For example we do not consider curves which are in part spacelike and in part timelike. These curves do not interest us. Finally we note that each particular set of curves we considered is closed in the sense that a given curve is not possible to belong to two different sets. This corresponds to the fact that a bradyon can never be a luxon or a tachyon and vice versa.

4.5.4 Geometric Definition of Relativistic Inertial Observers (RIO)

In Newtonian Physics we divide the curves in E^3 into two classes: the straight lines and the rest. The straight lines are identified with the trajectories of the Newtonian inertial observers and the rest with the trajectories of the accelerated observers. Furthermore all linear coordinate transformations in E^3 which preserve the Euclidean distance define the Galileo transformations we considered in Sect. 3.5. The Galileo transformations relate the measurements of two Newtonian inertial observers.

In analogy with the above we define in Special Relativity the *Relativistic inertial observers (RIO)* as the observers whose world lines are straight lines in Minkowski space. The world lines which are not straight lines is assumed to correspond to accelerated relativistic mass points. Furthermore the group of linear transformations of Minkowski space which preserves the Lorentz metric is called the *Poincaré transformations* and a closed subgroup of them (to be defined properly later) is the Lorentz transformations. The Poincaré transformations relate the measurements of two RIOs.

The identification of the world lines of RIO with the timelike straight lines in Minkowski space is compatible with the concept of foliation of Minkowski space, which has been mentioned in Sect. 4.4.

Special Relativity and Lorentz transformations involve straight lines only. How can one study accelerated motions which are described by non-straight lines? This is done as in Newtonian Physics (see Sect. 3.3.1) by means of the *instantaneous Relativistic inertial observers (IRIO)* defined as follows. Each world line of an accelerated motion can be approximated by a great number of straight line segments as shown in Fig. 4.5. Each of these straight line segments defines the world line of a RIO, a different RIO at each point of the world line. The RIO at a point of the world line is called the IRIO at that point and it is assumed that an accelerated relativistic motion is equivalent to a continuous sequence of inertial motions. This is a point on which we shall return when we study four-acceleration.

4.5.5 Proper Time

The world lines are parameterized by a real parameter. Out of all possible parameterizations there is one class of parameters, called *affine parameters*, defined by the

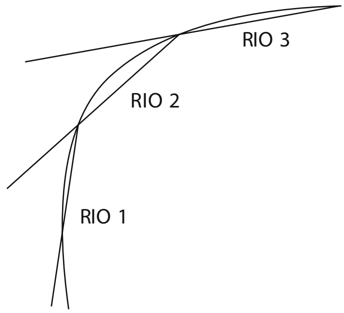

Fig. 4.5 World line accelerated observer

requirement that the tangent vector $u^i = \frac{dx^i}{d\tau}$ has fixed Lorentz length at all points of the world line . It is easy to show that if τ is an affine parameter, then $\tau' = \alpha\tau + \beta$ where $\alpha, \beta \in R$ are also affine parameters. Out of all possible affine parameters, we select the ones for which the constant length of the tangent vector is c^2, that is we demand

$$\eta_{ij} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau} = -c^2. \tag{4.3}$$

We identify this affine parameter with the time measured by the proper observer of the world line and call *proper time*. The demand that the length of the tangent vector equal a universal constant makes all proper clocks to have the same rate for all RIOs. Therefore, the only freedom left for the proper time is to set the "time zero" along the world line.

4.5.6 The Proper Frame of a RIO

Let $x^i(\tau)$ be the world line of a relativistic (not necessarily inertial) observer where τ is an affine parameter. At each point P along the world line there exists a comoving observer, that is an observer with respect to whom the three-velocity of the relativistic observer equals zero. From all LCF[3] systems in Minkowski space we select the one which satisfies the following conditions:

(a) Its origin is at P, that is $\mathbf{r}_P = 0$.
(b) The tangent vector of the world line at P has components $u_i = (c, 0)$.

We call this LCF *instantaneous proper frame* and denote by Σ_P^+ . If the proper observer is accelerating – equivalently the world line is not a straight line in

[3] Recall that an LCF is a coordinate system in which the Lorentz metric has its canonical form $diag(-1, 1, 1, 1)$.

Minkowski space – then the instantaneous proper frame is different from point to point along the world line. Every instantaneous proper frame corresponds to an instantaneous proper observer we described in Sect. 4.5.4. The aggregate of all these frames comprises the proper frame of the accelerated observer.

If the relativistic observer is a RIO then the instantaneous proper frame is the same all along the world line; we call it the *proper frame* of the RIO and denote with Σ^+.

4.5.7 Proper or Rest Space

At every point P along the world line of a relativistic observer, affinely parameterized with proper time τ, we consider the hyperplane (Lorentz) normal to the world line at the point P. This hyperplane is called the *proper space* of the observer at the point P. For the case of a RIO the proper space at every point along its world line is the space E^3. The proper spaces of a RIO are parallel and create a foliation of Minkowski space as shown in Fig. 4.3.

The proper spaces of an accelerated observer are not parallel (see Fig. 4.6).

If the proper spaces and the proper times of two RIOs Σ_1, Σ_2 coincide then we consider them to be the same observer. If two RIOs Σ_1, Σ_2 move with constant relative velocity then their world lines (which are straight lines) make an angle in Minkowski space and their proper spaces intersect as shown in Fig. 4.3. The proper spaces of each RIO give a different foliation of Minkowski space.[4]

The angle ϕ between the world lines is called *rapidity* and is given by the relation

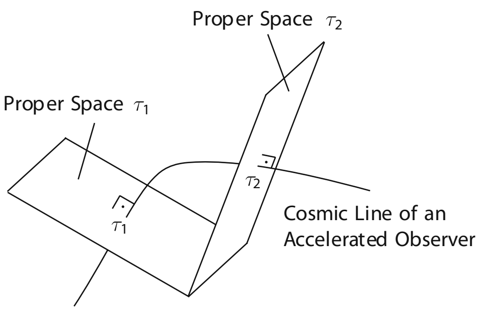

Fig. 4.6 Proper spaces of an accelerated observer

[4] It is instructive to mention at this point that Newtonian Physics can be formulated in a four-dimensional Euclidean space, where one dimension is for the time and three dimensions are for the Euclidean space E^3 in the same way it is done in Minkowski space. This four-dimensional space is foliated by the hyperplanes E^3; however, due to the absolute nature of time, this foliation is the *same for all* Newtonian inertial observers. The fundamental difference (apart from the character of the metric) between the two theories is the different foliations of Special Relativity and the unique foliation of Newtonian Physics.

$$\tanh\phi = \beta, \quad \cosh\phi = \gamma, \quad \sinh\phi = \beta\gamma, \tag{4.4}$$

where $\beta = \frac{u}{c}$ and u is the relative speed of Σ_1, Σ_2. We note that $\tanh\phi$ takes its maximum value when $\beta = 1$, i.e., $u = c$, that is when the relative speed of the two RIOs equals the speed of light in vacuum. This emphasizes the limiting character of c in Special Relativity.

4.6 Spacetime Description of Motion

The Theory of Special Relativity is a theory of motion, therefore the concept of position vector is fundamental. That is one has to specify the means and the procedures (= directions of use) which must be given to a RIO in order to determine the components of the position vector of an event in spacetime. The use of rigid rods is out of question because they take one back to Newtonian Physics. Instead the properties of light signals must be used because they are universal in Special Relativity and, furthermore, light is the fundamental relativistic physical system of that theory.

Following the above, we equip the relativistic observers with two measuring devices:

(a) *A photongun*
This is a device which emits beams of photons (e.g., a torch) and has a construction which makes possible the specification of the direction of the emitted beam. For example the photongun can be a monochromatic small laser emitter placed at the center of a transparent sphere on the surface of which there are marked equatorial coordinates which make possible the reading of the spherical coordinates of the emitted beam.

(b) *A personal clock (proper clock)*
This is every physical system which produces numbers in a specified way (see below) and it is used by the observer to associate a number with each distance measurement. This number is the *time or zeroth component* of the position four-vector. We demand that

(1) The proper clock will be the same for all RIOs.
(2) The rate of production of numbers by the proper clock will be constant and independent of the way a RIO moves.
(3) It will operate continuously.

We identify the numbers produced by the proper clock with the proper time of the RIO. Each RIO has its own proper clock and the clocks of two RIOs in relative motion cannot coincide except at one point, which is the event that both RIOs set the indication of their clocks to zero. This activity is called *synchronization of the clocks*. We see that in Special Relativity there is no meaning of speaking about "time" because there is no unique, universal clock which will measure it and the corresponding Newtonian inertial observers who will use it as a common reference.

From the definition of the proper clock it is seen that the proper time of a relativistic observer increases continuously. This means that the world line is oriented, that is one can have a sense of direction along this line. We say that this direction defines the *arrow of time* for each observer. It is accustomed to say that at each point of a world line there exists *past* and *future*. This terminology is not quite successful and can cause confusion because these terms refer to the Newtonian conception of the world, where time is absolute and universal for all Newtonian observers. In Special Relativity each RIO has its own past and its own future.

The photongun and the proper clock can be used by a relativistic observer in order to perform two fundamental operations:

(1) To determine if it (not he/she!) is a RIO.
(2) In case it is a RIO, and only then, to determine the coordinates of the position four-vector of events in spacetime following a measuring procedure to be specified below.

The two operations are different, independent, and equally important. We discuss each of them below.

4.6.1 The Physical Definition of a RIO

In order for a relativistic observer to testify that it is a RIO it must follow a procedure which is identical to the corresponding procedure of a Newtonian observer with the sole difference that the Newtonian gun is replaced with the photongun. There is no point repeating this procedure here and we refer the reader to Sect. 3.3.1 where the procedure is described in detail and ask him/her to simply change the word gun with photongun. The three relativistic inertial directions which specify a RIO, Σ say, are called *relativistic inertial directions* of Σ. If a relativistic observer cannot determine three independent inertial directions the observer is an accelerated relativistic observer.

The question which arises is: Do RIOs exist? The answer is given by the following axiom.

Axiom of Relativistic Inertia

There exist relativistic observers, whose world lines in Minkowski space are timelike straight lines. These lines are called lines of time. For these observers there are at most three independent relativistic inertial directions in physical space, which is equivalent to the fact that physical space is three dimensional.

This axiom has physical and geometric consequences.

Concerning the physical consequences the axiom declares that the RIOs perceive the physical space as continuous, isotropic, homogeneous, and three dimensional, that is, exactly as the Newtonian observers do. Furthermore their proper time is independent of their perception of space and it is described by a one-dimensional Euclidean space. In conclusion a RIO is identical with the typical Newtonian inertial observer, as far as the perception of space and time is concerned. However, different RIOs have different perceptions of space and time.

The three independent inertial directions determined by a RIO define in physical space a frame which we call a *Relativistic light frame*. A RIO can always find inertial directions which are mutually perpendicular (in the Euclidean sense). The frame defined by such directions is called a *Lorentz Cartesian frame (LCF)*.

Concerning the geometric implications of the axiom they are the following:

(a) The axiom associates each RIO with one timelike straight line in Minkowski space, therefore with a definite foliation of Minkowski space. The three-dimensional hyperplanes of this foliation (i.e., the proper spaces of the RIO) correspond to the perception of the physical space by that RIO.
(b) The LCFs correspond to the LCFs of Minkowski space, that is, the coordinate frames in which the Lorentz metric attains its canonical form $diag(-1, 1, 1, 1)$.

We note that the action of the axiom of relativistic inertia is similar to the Newton's First Law, that is, geometrizes the concept of RIO as well as the concept of LCF.

4.6.2 Relativistic Measurement of the Position Vector

Having established the concept of a RIO and that of an LCF we continue with the procedure of measurement of the coordinates of the position vector in spacetime by a RIO. This procedure is called *chronometry*.

Consider the RIO Σ and a point P in spacetime whose position vector is to be determined. We postulate the following operational procedure.

The RIO Σ places at the point P a small plane mirror and sends to P a light beam at the indication τ of the proper clock. There are two possibilities: either the light beam is reflected on the mirror and returns to the RIO along the same direction of emission or not. If the second case occurs Σ changes the direction of the photongun until the first case results. This is bound to happen because there are inertial directions for a RIO. Then the RIO fixes that direction of the photongun and reads

(1) The time interval $2\Delta\tau(P)$ between the emission and the reception of the light beam
(2) The direction $\mathbf{e}_r$ of emission of the light beam using the scale of measurement of directions of the photongun

Subsequently Σ defines the position vector $\mathbf{r}(P)$ of P in three-space as follows:

$$\mathbf{r}(P) = c\Delta\tau(P)\mathbf{e}_r,$$

where c is the speed of light in vacuum. Concerning the time coordinate of the point P, Σ sets the number $\tau + \Delta\tau(P)$ where τ is the indication (proper time) of the proper clock at the event of emission of the beam. Eventually the coordinates of the position four-vector of the event P are[5] $(c\tau + c\Delta\tau(P), c\Delta\tau(P)\mathbf{e}_r)$. We see that the measurement of the components of the position vector requires the measurement of two readings of the proper clock and one reading of the scale of directions of the photongun.

4.6.3 The Physical Definition of an LRIO

Since chronometry has been defined only for RIO, many times the erroneous point of view is created that Special Relativity cannot study accelerated motions. If that was true, then that theory would be of pure theoretical interest, because inertial motions are the exception rather than the rule.

The extension of chronometry to non-relativistic inertial observers is done via the concept of locally Relativistic inertial observer (LRIO) we defined geometrically in Sect. 4.5.4. In the present section we define the LRIO from the physics point of view.

Consider a relativistic mass which is accelerating wrt the RIO Σ. Let P be a point along the trajectory of the mass point where $\mathbf{r}(t)$, $\mathbf{v}(t)$, $\mathbf{a}(t)$ are the position, the velocity, and the acceleration vector at time t in Σ, respectively.

We consider another RIO Σ' which wrt Σ has velocity $\mathbf{u} = \mathbf{v}(t)$. This observer is the LRIO of the mass point at the event P. Due to the acceleration at time $t + dt$ the position of the moving mass will be $\mathbf{r}(t + dt)$ and its velocity $\mathbf{v}(t + dt)$, therefore at the point $\mathbf{r}(t + dt)$ there is a different LRIO. Because the orbit of the moving mass is independent of observation, the continuous sequence of LRIO is inherent to the motion of the mass. This sequence of LRIOs is called the proper observer of the accelerating mass point. With this mechanism we extend chronometry to the accelerated observers.

There is another way to understand accelerated relativistic observers by relaxing the condition that the Lorentz transformation is linear and that preserves the canonical form of the Lorentz metric. This approach takes us very close to the theory of General Relativity. More on this topic shall be said in Chap. 7.

4.7 The Einstein Principle of Relativity

The "world" of a theory of physics consists of all physical quantities of the theory which are determined by the Principle of Relativity of the theory as explained in Chap. 2. For example the Galileo Principle of Relativity determines the physical

[5] We write $c\tau$ instead of τ for the time component, because the components of a vector must have the same dimensions, that is space length $[L]^1[T]^0[M]^0$.

quantities of Newtonian Physics. In the same spirit the Einstein Principle of Relativity determines the physical quantities of Special Relativity. More specifically the Einstein Principle of Relativity acts at three levels:

(a) Defines a code of exchange of information (transformation of "pictures") between RIOs and determines the group of transformations of the theory (Poincaré group)
(b) Defines a Covariance Principle which determines the kind of geometric objects which will be used by the RIOs for the mathematical description of the relativistic physical quantities (Lorentz tensors)
(c) Defines the mathematical nature of the equations of the theory(Covariance Principle)

4.7.1 The Equation of Lorentz Isometry

Consider a RIO Σ who observes a light beam passing through the points A, B of real space. Let $\Delta\mathbf{r}$ be the relative position vector of the points as measured by Σ and Δt the time required according to the proper clock of Σ in order for the light to cover the distance AB. From the principle of the constancy of the speed of light we have

$$\Delta\mathbf{r}^2 - c^2\Delta t^2 = 0,$$

where c is the speed of light in vacuum.[6] For another RIO Σ' who is observing the same light beam (events) let the corresponding quantities be $\Delta\mathbf{r}'$ and $\Delta t'$. For Σ' we have again the relation

$$\Delta\mathbf{r}'^2 - c^2\Delta t'^2 = 0.$$

The last two equations imply

$$\Delta\mathbf{r}'^2 - c^2\Delta t'^2 = \Delta\mathbf{r}^2 - c^2\Delta t^2.$$

This equation relates the measurements of the components of events referring to light beam and it is a direct consequence of the principle of the constancy of c. The question which arises is: What will be the relation for events concerning mass points, e.g., an electron? The answer has been given – and perhaps this is his greatest contribution – by Einstein who stated the following principle and founded the Theory of Special Relativity:

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[6] Note that the four-vector AB^i is defined by the events: A: The light beam passes the point A and B: The light beam passes the point B. The four-vector AB^i is a null vector.

Definition 6 (Principle of Communication of Einstein) Let $(c\Delta t, \Delta\mathbf{r})$, $(c\Delta t', \Delta\mathbf{r}')$ be the components of the position four-vector AB^i (not necessarily null!) as measured by the RIOs Σ, Σ', respectively. Then the following equation holds:

$$\Delta\mathbf{r}'^2 - c^2\Delta t'^2 = \Delta\mathbf{r}^2 - c^2\Delta t^2, \tag{4.5}$$

which determines the communication of measurements for one RIO to the other.

The physical significance of (4.5) is that it determines the code of communication (i.e., the code for transferring information) between RIOs. We find the geometric significance of (4.5) if we write it in the LCF of the Σ, Σ', respectively. In these systems (and only there!) (4.5) takes the form

$$\begin{aligned}&(c\Delta t, \Delta x, \Delta y, \Delta z)diag(-1, 1, 1, 1)(c\Delta t, \Delta x, \Delta y, \Delta z)^t\\ &= (c\Delta t', \Delta x', \Delta y', \Delta z')diag(-1, 1, 1, 1)(c\Delta t', \Delta x', \Delta y', \Delta z')^t.\end{aligned} \tag{4.6}$$

The lhs of (4.6) (see also Sects. 1.3 and 3.5) consists of two parts:

- A. The 1×4 matrix $(c\Delta t, \Delta x, \Delta y, \Delta z)$.
- B. The 4×4 matrix $diag(-1, 1, 1, 1)$.

The first matrix concerns the chronometric measurements of the RIO and quantifies the first relativistic physical quantity, the *position four-vector* $x_i = (ct, \mathbf{r})$ whose components depend on the RIO. The second matrix is independent of the RIO.[7] The matrix $diag(-1, 1, 1, 1)$ can be considered as the canonical form of a metric in a four-dimensional real vector space. If this is done, then this metric is the Lorentz metric. If we set $\eta = diag(-1, 1, 1, 1)$ relation (4.6) is written as

$$(ct, \mathbf{r})^t(\eta - L^t\eta L)(ct, \mathbf{r}) = 0,$$

where L is the transformation matrix relating Σ, Σ' and defined by the requirement $(ct', \mathbf{r}')^t = (ct, \mathbf{r})^t L^t$ (t = transpose matrix). This relation must be satisfied for all position four-vectors, therefore we infer that the matrix L *is* defined by the relation

$$\eta = L^t\eta L. \tag{4.7}$$

Relation (4.7) is the isometry equation for the Lorentz metric in two LCFs. We infer that the coordinate transformations between RIOs using LCF are the Lorentz transformations we introduced in (1.31) of Sect. 1.3. Therefore, as is the case in Newtonian Physics where the Galileo Relativity Principle introduces the Galileo transformations, in Special Relativity the Einstein Relativity Principle introduces the Lorentz transformation as the transformations relating the components of four-vectors (and tensors in general) of two RIOs. These ideas are described pictorially in Fig. 4.7.

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[7] However, the form $diag(-1, 1, 1, 1)$ changes if frames different than LCF are used.

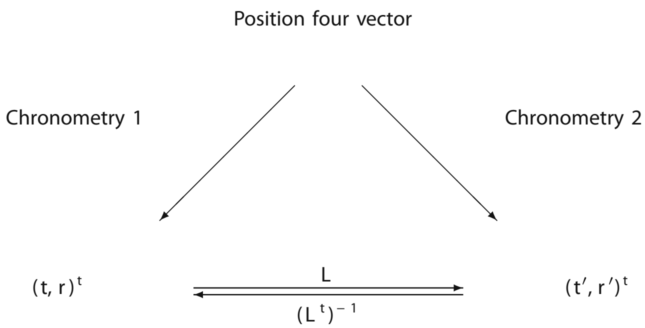

Fig. 4.7 Principle of Communication of Einstein

The general solution of (4.7) has been given in Sect. 1.6 and it is a matrix of the form $L(\boldsymbol{\beta}, E) = L(\boldsymbol{\beta})R(E)$. In this solution the matrix $L(\boldsymbol{\beta})$ is the pure relativistic part of the Lorentz transformation and it is given by (1.44) and (1.45). It is parameterized by the relative velocity $\boldsymbol{\beta}$ of the LCF whose space axes are parallel. The matrix $R(E)$ is a generalized Euclidean orthogonal transformation which depends on three parameters (e.g., Euler angles) and brings the axes of the LCF to be parallel.

4.8 The Lorentz Covariance Principle

In order to define the mathematical description of the relativistic physical quantities we must define the mathematical nature of the fundamental physical quantities of the theory. These quantities are – as in Newtonian theory – two: one four-vector and one invariant. In Special Relativity we also have two such quantities: the position four-vector and the invariant speed of light in vacuum. Based on this observation we define the first part of the Lorentz Covariance Principle as follows:

Lorentz Covariance Principle: Part I

The position four-vector and the Lorentz distance (equivalently the Lorentz metric) are relativistic physical quantities.

The position four-vector and the Lorentz distance are not sufficient for the development of Special Relativity in the same manner that the position vector and the time are not enough for the development of Newtonian Physics. Therefore we have to introduce more relativistic physical quantities and this is done by means of the Covariance Principle. In Newtonian Physics the position vector and the time defined the mathematical character of the Newtonian Physical quantities via the Galileo Principle of Relativity and the second part of the Principle of Covariance. Similarly in Special Relativity the position four-vector and the spacetime distance define the nature of the relativistic physical quantities via the Lorentz Covariance Principle Part II:

Lorentz Covariance Principle: Part II

The relativistic physical quantities are described with Lorentz tensors.

This means that the relativistic physical quantities are described with geometric objects which

(a) Are tensors of type (m, n) in Minkowski space, that is they have $4^n \times 4^m$ components $(m, n = 0, 1, 2, 3, \ldots)$, where m is the number of contravariant indices and n the number of covariant indices.
(b) If $T^{i'j'\cdots}_{a'b'\ldots}$ and $T^{ij\cdots}_{ab\ldots}$ are the components of a Lorentz tensor T wrt the RIOs Σ' and Σ, respectively, then

$$T^{i'j'\cdots}_{a'b'\ldots} = L^{i'}_{i} L^{j'}_{j} \ldots L^{a}_{a'} L^{b}_{b'} \ldots T^{ij\ldots}_{ab\ldots}, \tag{4.8}$$

where $L^{i'}_{i}$ is the Lorentz transformation which relates Σ and Σ' and $L^{i}_{i'}$ is its inverse.

4.8.1 Rules for Constructing Lorentz Tensors

The rules of Proportion 2 which were applied in order to define new Newtonian tensors from other Newtonian tensors also apply to Lorentz tensors. That is we have the following rules:

Rule 1:

$$d \begin{bmatrix} \text{Lorentz} \\ \text{Tensor} \\ \text{type } (r, s) \end{bmatrix} / d \begin{bmatrix} \text{Lorentz} \\ \\ \text{Invariant} \end{bmatrix} = \begin{bmatrix} \text{Lorentz} \\ \text{Tensor} \\ \text{type } (r, s) \end{bmatrix}. \tag{4.9}$$

Rule 2:

$$\begin{bmatrix} \text{Lorentz} \\ \\ \text{Invariant} \end{bmatrix} \begin{bmatrix} \text{Lorentz} \\ \text{Tensor} \\ \text{type } (r, s) \end{bmatrix} = \begin{bmatrix} \text{Lorentz} \\ \text{Tensor} \\ \text{type } (r, s) \end{bmatrix}. \tag{4.10}$$

In Newtonian Physics Rule 1 is used to define the kinematic quantities of the theory (velocity, acceleration, etc.) and Rule 2 is used for the definition of the dynamical quantities (momentum, force, etc.). In a similar manner in Special Relativity Rule 1 defines the four-velocity, the four-acceleration, and Rule 2 the four-momentum, the four-force, etc. These four-vectors are studied in the chapters to follow.

4.8.2 Potential Relativistic Physical Quantities

We come now to the following point: Does every Lorentz tensor represent a relativistic physical quantity? The answer to this question is the same with the one we gave in Newtonian Physics where we introduced the potential Newtonian physical quantities. These quantities were Euclidean tensors which *possibly* describe a Newtonian physical quantity. Based on that we consider that each Lorentz tensor is a potential relativistic physical quantity and in order to be a relativistic physical quantity it must satisfy the following criteria:

- In case it has a Newtonian limit for a characteristic observer (e.g., the proper observer) then that limit must correspond to a Newtonian physical quantity.
- In case it does not have a Newtonian limit (e.g., the speed of light) then it will obtain physical status only by means of a principle (e.g., as is done with the speed of light in vacuum).

In the chapters to follow we shall have the chance to see how these criteria are applied in practice.

4.9 Universal Speeds and the Lorentz Transformation

The purpose of the determination of the (proper) Lorentz transformation in Sect. 1.6.1 as the solution of the isometry equation $L^t \eta L = \eta$ was to show that the Lorentz transformation as such is of a pure mathematical nature. The physics of the Lorentz transformation comes from its kinematic interpretation.

In this section we shall derive the proper Lorentz transformation using the physical hypothesis that there exist in nature universal speeds, not necessarily one only. This approach is closer to the standard approach of the literature; however, it has a value because it produces at the same time the Galileo transformation and incorporates the kinematic interpretation of the transformation. Furthermore this derivation of the Lorentz transformation, although simple, is useful because it follows an axiomatic approach which familiarizes the reader with this important methodology of theoretical physics.[8]

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[8] In the literature one can find many derivations of the proper Lorentz transformation – mainly of the boosts – using a more or less axiomatic approach. Some of them are

1. D. Sardelis "Unified Derivation of the Galileo and the Lorentz transformation" *Eur. J. Phys.* **3**, 96–99, (1982).
2. A.R. Lee and T.Malotas "Lorentz Transformations from first Principles" *Am. J. Phys.* **43**, 434–437, (1975) and *Am. J. Phys.* **44**, 1000–1002, (1976).
3. V.Berzi and V. Gorini "Reciprocity Principle and the Lorentz Transformation" *J. Math. Phys.* **10**, 1518–1524, (1968).
4. C. Fahnline "A covariant four-dimensional expression for Lorentz transformations" *Am. J. Phys.* **50**, 818–821, (1982)

Consider two observers (not NIO or RIO but just observers who in some way produce coordinates for the physical phenomena in some four-dimensional space) $\Sigma,\ \Sigma'$ with coordinate systems (ct, x, y, z) and (ct', x', y', z'). We relate these coordinates with a number of mathematical assumptions which must satisfy the following criteria:

(a) Have a physical meaning or significance
(b) Are the least possible
(c) Lead to a unique result

Assumption I

There are linear transformations among the coordinates (ct, x) *and* (ct', x') *which are parameterized by a real parameter* V *(whose physical significance will be given below).*

This assumption implies that we can write

$$\begin{bmatrix} ct \\ x \end{bmatrix} = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{bmatrix} \begin{bmatrix} ct' \\ x' \end{bmatrix} + \begin{bmatrix} \gamma \\ \delta \end{bmatrix}, \tag{4.11}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma, \delta$ are well-defined (i.e., continuous and infinite differentiable) real functions of the real parameter V. We assume that the parameter V is such that the following assumption is satisfied:

Assumption II

$$\text{(a) If } t = t' = 0,\ \text{ then } x = x' = 0. \tag{4.12}$$

$$\text{(b) If } x' = 0 \text{ for all } t,\ \text{ then } x = Vt. \tag{4.13}$$

$$\text{(c) If } x = 0 \text{ for all } t',\ \text{ then } x' = -Vt'. \tag{4.14}$$

$$\text{(d) If } V = 0,\ \text{ then } x = x \text{ and } t = t'. \tag{4.15}$$

Let us examine the implications of Assumption II to geometry and kinematics. Condition (a) means that the transformation (4.11) is synchronous, that is, when the origins of Σ and Σ' coincide then the times t and t' are made to coincide. Condition (b) means that the plane (y', z') which is defined with the equation $x' = 0$ is always parallel to the plane (y, z) which is defined at each time coordinate t with the relation $x = Vt$. Similar remarks hold for condition (c) with the difference that now the plane (y, z) is kept parallel to the plane (y', z'). Finally, condition (d) implies that for the value $V = 0$ the two coordinate systems coincide and Σ is not differentiated from Σ'.

From (b) and (c) it follows that the parameter V has dimensions $[LT^{-1}]$, therefore its physical meaning is speed. We continue with the consequences of Assumption II on the coefficients $\alpha_1, \alpha_2, \beta_1, \beta_2$. Equation (4.12) implies $\gamma = \delta = 0$ hence the transformation is homogeneous. Equation (4.13) implies

$$\beta_1 = \frac{V\alpha_1}{c}.$$

Similarly (4.14) gives

$$\beta_1 = \frac{V\beta_2}{c}.$$

From these it follows that

$$\alpha_1 = \beta_2.$$

Therefore, the transformation equation (4.11) becomes

$$\begin{bmatrix} ct \\ x \end{bmatrix} = \begin{bmatrix} \beta_2 & \alpha_2 \\ \frac{V\beta_2}{c} & \beta_2 \end{bmatrix} \begin{bmatrix} ct' \\ x' \end{bmatrix}. \tag{4.16}$$

In this expression we have two unknowns – the functions $\alpha_2,\ \beta_2$ – thus we need two more assumptions before we determine the transformation.

We introduce the quantities $U = \frac{x}{t}$ and $U' = \frac{x'}{t'}$ where (ct, x) and (ct', x') are the coordinates of a moving particle in the frames Σ and Σ', respectively. The quantities U, U' represent the x- and x'-components of the velocity of the particle wrt Σ and Σ', respectively. From the transformation equation (4.16) it follows easily ($\beta_2 \neq 0$) that

$$U = \frac{x}{t} = \frac{U' + V}{1 + (\frac{\alpha_2}{\beta_2 c})U'}. \tag{4.17}$$

Equation (4.17) is a general relation which relates the velocities U, U' with the parameter V. In order to determine the ratio $\frac{\alpha_2}{\beta_2}$ we consider one more assumption:

Assumption III

There are at least two universal speeds, one with value infinity and the other with finite value c which are defined by the following requirements:
If $U \longrightarrow +\infty$ then $U' \longrightarrow +\infty$ (V finite).
If $U \longrightarrow c$ then $U' \longrightarrow c$.

The first speed gives ($\beta_2 \neq 0$)

$$\alpha_2 = 0 \tag{4.18}$$

and the second

$$\alpha_2 = \frac{V}{c}\beta_2. \tag{4.19}$$

Assumption III is not an assumption without content (that is, there do exist in nature such speeds) and this is what the experiments of Michelson–Morley have shown. Furthermore in Newtonian Physics we accept that the interactions are propagating with infinite speed (action at a distance). The value of the constant c is the speed of light in vacuum. The above do not exclude further universal speeds; however, since they have not been found yet we shall ignore them.

There is still one parameter to determine, therefore we need one more assumption. This assumption has to do with the isometry property of the transformation and has as follows:

Assumption IV

The determinant of the transformation (4.16) equals $+1$.

In order to quantify this assumption we compute the determinant D of the transformation. We find

$$D = \beta_2^2 - \left(\frac{V}{c}\right)\alpha_2\beta_2 = \beta_2\left(\beta_2 - \frac{V\alpha_2}{c}\right). \tag{4.20}$$

In the first case of infinite speed $\alpha_2 = 0$ and Assumption IV gives $\beta_2 = \pm 1$. The value $\beta_2 = -1$ is rejected because it leads to $t = -t'$ for all values of V and this contradicts (4.15). Hence $\beta_2 = 1$ and the transformation becomes

$$t = t' \qquad x = x' + Ut'. \tag{4.21}$$

These equations are the Galileo transformation of time and space coordinates of Newtonian Physics. We note that in this case (4.17) gives

$$U = U' + V, \tag{4.22}$$

which is the formula of composition of velocities of Newtonian Physics.

We come now to the case of finite speed in which $\alpha_2 \neq 0$. The assumption $D = 1$ and (4.20) in combination with (4.19) imply

$$\beta_2 = \pm\left(1 - \frac{V}{c^2}\right)^{-1/2} = \pm\gamma. \tag{4.23}$$

From (4.16) we find for $V = 0$, $\beta_2(0) = +1$, therefore we take the positive sign. In this case the transformation is

$$\begin{aligned} x' &= \gamma(x - Vt), \\ t' &= \gamma\left(t - \frac{xV}{c^2}\right). \end{aligned} \tag{4.24}$$

This is the boost along the common x, x'-axes with relative velocity V. Transformation (4.24) is symmetric. This can be seen if it is written as follows:

$$\begin{aligned} x' &= \gamma(x - \beta ct), \\ ct' &= \gamma(ct - \beta x). \end{aligned} \tag{4.25}$$

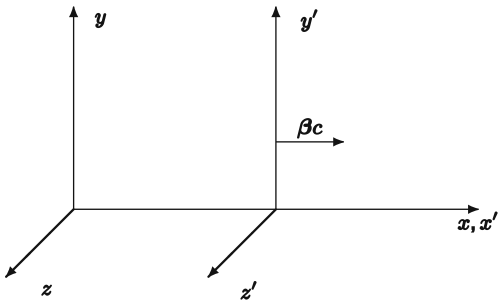


Fig. 4.8 Standard configuration of relative motion of two LCFs

We infer that the physical interpretation of the parameter V is that it corresponds to the relative speed of Σ, Σ' in the case the x, x'-axes are common and they move as in Fig. 4.8. This arrangement of motion shall be called *standard configuration* in order to economize space.

We examine condition (4.17) in order to determine the rule of composition of velocities for the case of the finite speed. Replacing in (4.24) (or otherwise) we compute

$$U = \frac{U' + V}{1 + \frac{U'V}{c^2}}. \tag{4.26}$$

This relation is the transformation of the three-velocities under the boost (4.25). It is easy to show that if $U = c$ then $U' = c$, as required by the universality of the speed c.[9] Also we note that the inverse transformation which expresses (ct, x) in terms of (ct', x') is the same as (4.24) with the difference that V is replaced with $-V$.

All the above assume a common spatial direction in the two LCFs Σ, Σ'. Along this direction the transformation of coordinates between Σ and Σ' has the characteristic that it is reversed if we take V in place of $-V$, that is, instead of considering Σ to move wrt Σ' we consider Σ' to move wrt Σ. This symmetry of relative motion between Σ, Σ' has been called *reciprocity of motion*. This direction is obviously related to the relative velocity of Σ and Σ', that is the $\mathbf{V}_{\Sigma\Sigma'}$ and $\mathbf{V}_{\Sigma'\Sigma}$. This leads us to the next assumption:

Assumption V

$$\mathbf{V}_{\Sigma'\Sigma} = -\mathbf{V}_{\Sigma\Sigma'}. \tag{4.27}$$

This assumption defines the common direction between the Σ and Σ'. Its geometric role is to "cut" the three-dimensional space into two-dimensional slices (y, z),

[9] Universal is a scalar physical quantity which has the *same* value in *all* (and the accelerated!) coordinate systems of the theory, whose is a physical quantity.

(y', z') (a foliation of the three-dimensional space!). Assumption V gives $\mathbf{V}$ the general physical significance as the relative velocity of Σ and Σ'.

With Assumption V we have assumed that the planes (y, z) and (y', z') remain parallel; however, we have not excluded that the axes y, z do not rotate as the plane (y, z) moves along the common axis x, x'. This requires one more assumption:

Assumption VI

Directions perpendicular to the characteristic common direction of Σ *and* Σ' *do not rotate.*

This implies that for the special type of motion we have considered

$$\begin{aligned} y &= y', \\ z &= z'. \end{aligned} \tag{4.28}$$

With (4.28) we have completed the derivation of the boost (and the Galileo transformation!) along the common direction x, x'.

Using Assumptions V and VI one is possible to produce the general Lorentz transformation for which no axis is coinciding with the characteristic common direction, the relative velocity is arbitrary, and the axes of the coordinate frames parallel. However, this has already been done in Chap. 1.

Chapter 5
The Physics of the Position Four-Vector

5.1 Introduction

The fundamental vector to all theories of motion is the position vector. In each such theory this vector is defined by means of a definite procedure, which reflects the way the theory incorporates the concepts of space and time. The position vector is a profound concept in Newtonian Physics due to the sensory relation of this theory with the concepts of space and time. However, the case is different with Special Relativity. The measurement of the position four-vector with light signals (chronometry) leads to a relation between the two concepts and necessitates their reconsideration. In the present chapter we consider the "relativistic" view of space and time and their relation (via a *definition*!) with the corresponding concepts of Newtonian Physics. This correspondence is necessary because we conceive the world via our Newtonian sensors. The results we find comprise the physics of the position four-vector and specify the relativistic kinematics at the level of everyday experience in the laboratory.

5.2 The Concepts of Space and Time in Special Relativity

In Newtonian Physics the measurement of the position vector is achieved by means of two procedures (=reading of scales) and three different and absolute (that is, the same for all Newtonian observers) physical systems: the mass gun, the (absolute) unit rigid rod, and the (absolute) cosmic clock. In Special Relativity the measurement of the position four-vector is done with *one procedure*, the chronometry as developed in Sect. 4.6.2, and two physical systems, the photongun and the proper clock, none of which is absolute (that is, common for all relativistic observers).

Unfortunately chronometry does not suffice for the study of physical problems in practice for the following reasons:

a. In the laboratory we still use the "absolute" rigid rod of Newtonian Physics to measure space distances. Furthermore our clocks are based (as a rule) on Newtonian physical systems, which we understand and use as being absolute.

M. Tsamparlis, *Special Relativity*, DOI 10.1007/978-3-642-03837-2_5,

The measurement of spatial and temporal differences involves the coordinates of two events whereas chronometry is concerned with the position vector of a single event.

Therefore standard practice imposes the question/demand:

How will one measure Newtonian spatial and Newtonian temporal differences in Special Relativity?

The answer to this question/demand will be given in the pages to follow. More specifically we shall define a new procedure, which we call the chronometry of two events which will associate the relative spacetime distance of two events, with a Newtonian spatial and temporal distance. The misunderstanding of this procedure has led to a type of problems in Special Relativity known as ***paradoxes***. The "paradoxicalness" of these problems is always due to the mistaken interpretation of the relativistic measurement of either the spatial distance or the temporal difference.

5.3 Measurement of Spatial and Temporal Distance in Special Relativity

Consider two events P, Q in spacetime with position four-vector[1] $x^i_P = (l, \mathbf{r})^t_\Sigma = (l', \mathbf{r}')^t_{\Sigma'}$ and $x^i_Q = (l + dl, \mathbf{r} + d\mathbf{r})^t_\Sigma = (l' + dl', \mathbf{r}' + d\mathbf{r}')^t_{\Sigma'}$ in the RIOs Σ and Σ', respectively. The four-vector PQ^i has components

$$(PQ)^i = (dl,\ d\mathbf{r})^t_\Sigma = (dl',\ d\mathbf{r}')^t_{\Sigma'}. \tag{5.1}$$

We collect this information in Table 5.1.

In order to find all allowable possibilities we consider the relations which connect the quantities $dl,\ d\mathbf{r}$ and $dl',\ d\mathbf{r}'$. These relations are the following two:

a. The invariance of the Lorentz length of the four-vector $(PQ)^i$:

Table 5.1 Table of coordinates of two events

	Σ	Σ'
$P:$	$(l,\ \mathbf{r})^t_\Sigma$	$(l',\ \mathbf{r}')^t_{\Sigma'}$
$Q:$	$(l + dl,\ \mathbf{r} + d\mathbf{r})^t_\Sigma$	$(l' + dl',\ \mathbf{r}' + d\mathbf{r}')^t_{\Sigma'}$
$PQ:$	$(dl,\ d\mathbf{r})^t_\Sigma$	$(dl',\ d\mathbf{r}')^t_{\Sigma'}$

[1] With $x^i_P = (l, \mathbf{r})^t_\Sigma$ we mean $x^i_P = \begin{pmatrix} l \\ \mathbf{r} \end{pmatrix}_\Sigma$. The sole reason for this writing is to economize space. Recall that according to our convention a contravariant vector has upper index and is represented by a column matrix and a covariant vector has lower index and it is represented by a row matrix.

$$-\,dl^2 + d\mathbf{r}^2 = -dl'^2 + d\mathbf{r}'^2. \tag{5.2}$$

b. The quantities dl, $d\mathbf{r}$ and dl', $d\mathbf{r}'$ are related by a (proper) Lorentz transformation according to relations (see (1.51), (1.52)):

$$d\mathbf{r} = d\mathbf{r}' + \left(\frac{\gamma}{\gamma+1}\boldsymbol{\beta}\cdot d\mathbf{r}' + dl'\right)\gamma\boldsymbol{\beta}, \tag{5.3}$$

$$dl = \gamma\left(dl' + \boldsymbol{\beta}\cdot d\mathbf{r}'\right). \tag{5.4}$$

From relations (5.2), (5.3), and (5.4) we construct Table 5.2, which shows all possible cases and indicates which are allowed and which are excluded.

Table 5.2 The relativistically allowable cases of measurement of spatial distance and temporal difference

	dl	$d\mathbf{r}$	dl'	$d\mathbf{r}'$	Remarks
(a)		All zero (1 case)			Trivial
(b)		One $\neq 0$ (4 cases)			Impossible due to (5.2)
(c)		Two $\neq 0$ (6 cases)			
	0	$\neq 0$	$\neq 0$	0	Impossible due to (5.2)
	$\neq 0$	0	0	$\neq 0$	–//–
	$\neq 0$	$\neq 0$	0	0	Impossible due to (5.3)
	0	0	$\neq 0$	$\neq 0$	–//–
	0	$\neq 0$	0	$\neq 0$	Impossible due to (5.3), (5.4)
	$\neq 0$	0	$\neq 0$	0	–//–
(d)		Three $\neq 0$ (4 cases)			
	$\neq 0$	0	$\neq 0$	$\neq 0$	Case 1
	0	$\neq 0$	$\neq 0$	$\neq 0$	Case 2
	$\neq 0$	$\neq 0$	0	$\neq 0$	Case 3
	$\neq 0$	$\neq 0$	$\neq 0$	0	Case 4
(e)		All $\neq 0$ (4 cases)			The general case

We note that, besides the most general case (e), it is possible to relate the quantities dl, $d\mathbf{r}$, dl', $d\mathbf{r}'$ in a relativistically consistent manner only in four cases, each case being characterized by the vanishing of one of the quantities $dl, d\mathbf{r}, dl', d\mathbf{r}'$. This result demands that we consider each of these cases as defining the relativistic measurement of the spatial and the temporal distance in the RIOs Σ and Σ'. In order to find the exact role for each case we consider the Newtonian measurement of spatial and temporal distance and note the following:

- The Newtonian measurement of spatial distance involves two points in space (i.e., $d\mathbf{r} \neq 0$ or $d\mathbf{r}' \neq 0$) and takes place at one time moment (i.e., $dl = 0$ or $dl' = 0$).
- The Newtonian measurement of temporal difference involves one point in space (i.e., $d\mathbf{r} =0$ or $d\mathbf{r}' = 0$) and two time moments (i.e., $dl = 0$ or $dl' = 0$).

Having as a guide the Newtonian method of measuring spatial distance and temporal difference we give the following definition for the measurement of the relativistic spatial distance and the measurement of the relativistic temporal difference.

Definition 7 Condition $d\mathbf{r} = 0$ (respectively, $d\mathbf{r}' = 0$) defines the relativistic measurement of temporal difference at one spatial point by the RIO Σ (respectively, Σ'). Condition $dl = 0$ (respectively, $dl' = 0$) defines the relativistic measurement of spatial distance at one time moment by the RIO Σ (respectively, Σ').

Definition 7 suffices in order

- To define a unique and consistent procedure for the measurement of the relativistic spatial distance and the relativistic temporal difference in Special Relativity and
- To define a well-defined correspondence between the relativistic measurement of the spatial distance and temporal difference with the corresponding Newtonian quantities.

We shall need the following result.

Example 8 Show that if two events coincide in a RIO Σ then they coincide in all RIOs.
Solution

Consider two events A, B say, which coincide in the RIO Σ. Then in Σ the four-vector $(AB^i)_\Sigma = 0$. Any other RIO is related to Σ via a Lorentz transformation. The Lorentz transformation is homogenous and therefore preserves the zero four-vector. This means that in all RIO $AB^i = 0$, that is, the events A, B coincide.

5.4 Relativistic Definition of Spatial and Temporal Distances

Consider the events P, Q in spacetime and the four-vector PQ^i they define. There are three possibilities: PQ^i timelike, spacelike, or null. In each case the measurement of the vector PQ^i has a different meaning. In accordance with what has been said in Sect. 5.3 we give the following definition:

Definition 8 If the four-vector PQ^i is timelike then the measurement of the distance of the events P, Q concerns temporal difference. If the four-vector PQ^i is spacelike the measurement of the distance of the events P, Q concerns spatial distance. Finally if PQ^i is null then the measurement of the distance of the events P, Q is not defined.

Definition 8 is compatible with Definition 7 because the vanishing of either dl or dl' implies that the position four-vector is spacelike whereas the vanishing of either $d\mathbf{r}$ or $d\mathbf{r}'$ means that the position four-vector is timelike. We conclude that in order to study the four allowable cases of Table 5.2 it is enough to study the cases that the position four-vector is timelike and spacelike.

5.5 Timelike Position Four-Vector – Measurement of Temporal Distance

Consider two events P, Q which define a timelike position four-vector PQ^i. Let Σ^+_{PQ} be the proper observer of the timelike four-vector PQ^i. The events P, Q define two hyperplanes in spacetime, which we call Σ^+_P and Σ^+_Q, respectively (see Fig. 5.1). Let Σ be the world line of another RIO whose world line makes (in spacetime) an angle ϕ with the world line of Σ^+_{PQ}. We are looking for a definition/procedure which will determine the time difference of the events P, Q for Σ.

In order to do that we recall that the measurement of time difference by a Newtonian observer requires that the events occur at the same spatial point. At this point one places a clock and reads the indications of the clock when the events occur, the difference of these indications being the required temporal difference of the events for the observer (and *all* other Newtonian observers!). We transfer these ideas in Special Relativity as follows.

We consider the two spatial hyperplanes Σ^+_P and Σ^+_Q of Σ^+_{PQ} at the points P, Q. These planes contain all spatial events of spacetime which are simultaneous for Σ^+_{PQ} with the events P, Q, respectively. The world line of Σ intersects the hyperplanes at the points P_1, Q_1, respectively. We *define* the temporal difference of the events P, Q for the RIO Σ to be the (Lorentz) distance of the events P_1, Q_1 (see Fig. 5.1).

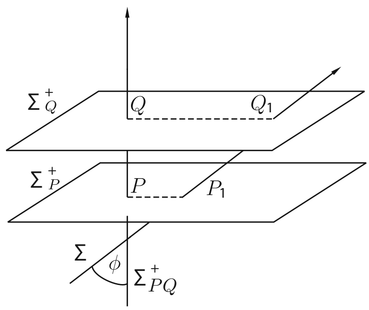
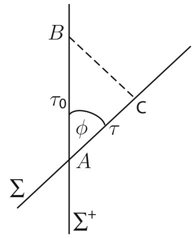


Fig. 5.1 Relativistic measurement of time difference of two events

Fig. 5.2 Hyperbolic triangle for the measurement of time difference



Let $(PQ)_{\Sigma^+_{PQ}} = c\tau_0$ be the distance (temporal difference) of the events P, Q as measured by the proper observer Σ^+_{PQ} with two readings of his clock (=Newtonian way of measuring time difference) and let $(P_1Q_1)_\Sigma = c\tau$ be the corresponding distance (temporal difference) measured relativistically by the procedure we defined above by Σ. The two quantities τ_0 and τ are related by the Lorentz transformation which connects the observers Σ^+_{PQ} and Σ. The components of the four-vector PQ^i in the LCFs Σ^+_{PQ} and Σ are $PQ^i = (c\tau_0, \mathbf{0})_{\Sigma^+_{PQ}}$ and $PQ^i = (c\tau, \mathbf{PQ})_\Sigma$, respectively. The Lorentz transformation for the zeroth component gives

$$\tau = \gamma\tau_0, \tag{5.5}$$

where $\gamma = \left(1-\beta^2\right)^{-1/2}$ and for the space component gives the trivial relation $\mathbf{v} = \frac{\mathbf{PQ}}{\tau}$. We remark that these relations can be obtained from the hyperbolic geometry of an orthogonal triangle if we make the following conventions (see Fig. 5.2):

(1) The side AB=time difference of the events $P,\ Q$ as measured by Σ^+_{PQ}.
(2) The side BC = space distance of the events $P,\ Q$ as measured byΣ.
(3) The angle ϕ is related to the rapidity (i.e., relative speed) of Σ^+_{PQ}, Σ by the relation

$$\cosh\phi = \gamma.$$

It is easy to see that with these conventions relation (5.5) is found from the (hyperbolic) orthogonal triangle ABC in the form $\tau = \tau_0\cosh\phi = \gamma\tau_0$.

Because $\gamma > 1$, $\tau > \tau_0$. Therefore the result of the measurement of the temporal difference of the events P, Q for two RIOs is different and the minimum value occurs for the proper observer Σ^+_{PQ}. In words (5.5) can be stated as follows[2]:

$$\begin{bmatrix} \text{Relativistically} \\ \text{measured} \\ \text{temporal difference} \\ \text{of the events } P,\, Q \\ \text{in the LCF of the RIO } \Sigma \end{bmatrix} = \gamma \begin{bmatrix} \text{Newtonian} \\ \text{measured} \\ \text{temporal difference} \\ \text{of the events } P,\, Q \\ \text{in the proper frame } \Sigma^+_{PQ} \end{bmatrix}. \tag{5.6}$$

[2] Spacetime measurement means: with the use of light signals (radar method). Newtonian measurement means: with direct reading of the observer's (not the absolute!) clock.

We note that the quantity $c\tau_0 = -(PQ)^2$ has the following properties:

(1) It is invariant, therefore the value τ_0 is the same for all RIOs.
(2) The value τ_0 is determined experimentally with a procedure similar to the Newtonian one (i.e., reading the clock indication) only by the proper observer Σ^+_{PQ}.

The inequality $\tau > \tau_0$ has been called *time dilation.* The name is not the most appropriate (and has created a lot of confusion) because the quantities τ and τ_0 refer to different measurements of different observers with different methods and concern temporal (i.e., coordinate) distance and not time difference.

We arrive now at the following question:

Is it possible for a Newtonian observer (as we are!) to measure relativistic temporal difference of two events in spacetime?

The answer is "yes" and it is given in the following example.

Example 9 A clock moves along the x-axis of a RIO Σ with constant velocity $\mathbf{u} = u\,\mathbf{i}$. In order for the observer in Σ to measure the time difference of the "moving" clock the observer decides to apply the following procedure:

The observer considers two positions P, Q along the x-axis which are at a distance $d_{(PQ)}$ apart and with its clock measures the time interval Δt required for the moving clock to pass through the point P (event A) and through the point Q (event B). Then the observer defines the time interval Δt^+ measured by the *moving* clock of the two events P, Q to be $\Delta t^+ = \Delta t/\gamma$. Is this method of measuring relativistic time difference with Newtonian observations compatible with the chronometric method proposed above?

Answer[3] using (a) the algebraic method and (b) the geometric method.

Solution

Let $\Sigma^+_{(AB)}$ be the proper frame of the moving clock, which has (unknown) speed u along the x-axis of the LCF Σ.

(a) **Algebraic solution**

We consider the events:

A: The moving clock passes through point P.
B: The moving clock passes through point Q.

Because the events A, B are points of the world line of a clock they define a timelike four-vector AB^i. We write the coordinates of the events A, B in the frames of Σ and Σ^+_{AB} in the following table:

The boost relating Σ^+_{AB}, Σ gives

$$d_{(AB)} = \gamma u \Delta t^+,$$
$$\Delta t = \gamma \Delta t^+.$$

[3] The algebraic and the geometric method of solving problems in Special Relativity involving spatial and temporal differences is explained below in Sect. 5.11.3.

	Σ	Σ^+_{AB}
$A:$	$(ct_A,\ x_A)$	$(ct^+_A,\ 0)$
$B:$	$(c(t_A+\Delta t),\ x_A+d_{AB})$	$(c(t^+_A+\Delta t^+),\ 0)$
$AB:$	$(c\Delta t,\ d_{AB})_\Sigma$	$(c\Delta t^+,\ 0)_{\Sigma^+_{AB}}$

Eliminating Δt^+ we find

$$u = \frac{d_{(AB)}}{\Delta t}, \tag{5.7}$$

which is compatible with the Newtonian measurement of the speed of the moving clock in Σ. The second relation gives

$$\Delta t^+ = \frac{\Delta t}{\gamma},$$

where γ is computed from the speed u which is measured with Newtonian methods (5.7). This relation is similar to (5.5) and leads us to the conclusion that with the proposed method observer Σ is able, with Newtonian means and methods, to determine relativistic temporal differences of a moving clock.

(b) **Geometric solution**

We consider Fig. 5.3:
Explanation of the figure
Events
A, B : Events of "moving" clock.
Σ: World line of observer Σ.
Σ^+: World line of the "moving" clock.
ϕ: Rapidity of the "moving" clock wrt observer Σ.

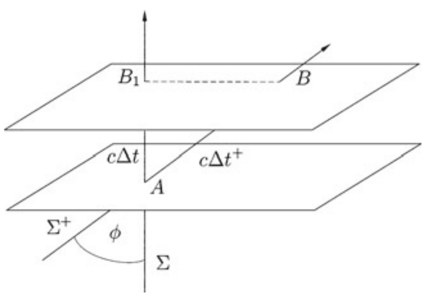


Fig. 5.3 Newtonian method of measurement of time differences

Data

(AB_1): Time interval of the events A, B measured relativistically by observer Σ, $(AB_1) = c\,\Delta t$.

(AB): Time interval of the events A, B measured by Newtonian methods by the proper observer of the moving clock, $(AB_1) = c\,\Delta t^+$.

(B_1B): Space distance of the events A, B as measured with Newtonian methods by Σ.

Requested

Relation between (AB), (AB_1).

From the triangle AB_1B we have

$$(AB_1) = (AB)\cosh\phi = \gamma\,(AB)$$

from which follows

$$\Delta t = \gamma\,\Delta t^+,$$

the same as before.

Again we note that

$$d_{AB} = (BB_1) = (AB)\sinh\phi = (AB)\beta\gamma = u\,\Delta t$$

as expected from the Newtonian measurement.

In Fig. 5.4 the measurement of the temporal difference AB of a clock from two RIOs Σ_1 and Σ_2 with speeds $v_1 < v_2$, respectively, wrt the clock (i.e., Σ^+) is shown. We note that $t_1 < t_2$, that is, the faster observer measures a larger time difference and as $v_2 \to c$ the $t_2 \to \infty$. It is important to note that the normals are on the world line of the observer Σ_1, Σ_2 and not on the world line of the "moving" clock.

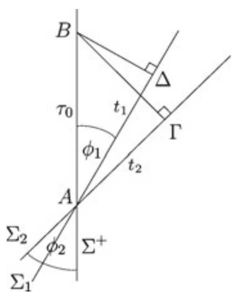


Fig. 5.4 Change of value of measured time difference for different speeds

5.6 Spacelike Position Four-Vector – Measurement of Spatial Distance

Consider two events P, Q which define a spacelike four-vector PQ^i and let Σ^- be a characteristic frame of PQ^i. In Σ^- the events P, Q occur at the same time and the four-vector PQ^i has components $(0, \mathbf{PQ}^-)$ where $\mathbf{PQ}^-$ is the vector giving the relative spatial position of the events P, Q in Σ^-. We consider a RIO Σ, which moves wrt Σ^- with parallel axes and relative speed $\boldsymbol{\beta}c$. In order to "measure" the relative spatial position of the events P, Q in Σ we must associate two corresponding events which are simultaneous in Σ. To achieve that we consider the hyperplane which is normal to the world line of the observer Σ at the point Q. This plane contains all events which are simultaneous with Q. The world line of Σ^-, which passes through the point P, intersects the hyperplane at a point P_1, hence this is the simultaneous event of Q for Σ. We define the spatial distance PQ_Σ of the events P, Q for observer Σ to be the spacetime distance of the events Q, P_1 (see Fig. 5.5).

To quantify the above we consider the (hyperbolic) orthogonal triangle QP_1P and take

$$P_1Q_\Sigma = \frac{PQ_{\Sigma^-}}{\cosh\phi} = \frac{PQ^-}{\gamma}. \tag{5.8}$$

In order to understand the physical meaning of (5.8) we note that PQ_{Σ^-} is the spatial distance of the events P, Q measured in the Newtonian way in Σ^- and PQ_Σ is (by definition!) the relativistic measurement of the same quantity in Σ. Then (5.8) can be understood as follows:

$$\begin{bmatrix}\text{Relativistic}\\ \text{Measurement of the}\\ \text{Spatial Distance}\\ \text{of the events } P, Q\\ \text{by the RIO } \Sigma\end{bmatrix} = \frac{1}{\gamma}\begin{bmatrix}\text{Newtonian}\\ \text{Measurement of the}\\ \text{Spatial Distance}\\ \text{of the events } P, Q\\ \text{by the RIO } \Sigma^-\end{bmatrix}. \tag{5.9}$$

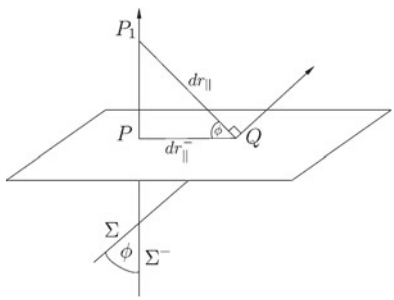


Fig. 5.5 Relativistic measurement of space distance

Because $\gamma > 1$ the $PQ_{\Sigma^-} > P_1 Q_\Sigma$, therefore PQ_{Σ^-} is the minimal value of the spatial distance of the events P, Q among all RIOs. This relation has been named length contraction where length means *Euclidean* length. The term "length" creates confusion because the concept "length" is absolute in Newtonian Physics and Euclidean geometry, therefore it appears to be absurd that the spatial distance of two events is different for relatively moving observers. However, this is not so, because (5.8) involves different quantities. In the rhs PQ_{Σ^-} is Newtonian length *measured* by the Newtonian method (of superposition) and by observer Σ^- whereas in the lhs the quantity $P_1 Q_\Sigma$ is the spatial (Euclidean) distance measured chronometrically by the observer Σ. The difference between these two lengths is due to the different method of measurement.

We arrive again at the question:

Is it possible for a Newtonian observer (as we are!) to measure the relativistic spatial difference of two events in spacetime?

The answer is affirmative and it is given in the following example.

Example 10 Consider a rod which slides along the x-axis of the LCF Σ with (known) constant velocity $\mathbf{u} = u\mathbf{i}$. In order for the observer Σ to measure experimentally the length of the rod it applies the following procedure: at first it measures the time period Δt required for the two ends of the rod to pass through a fixed point along the x-axis. Subsequently it multiplies with the speed of the rod in Σ and calculates the length L of the rod in Σ. Then Σ *defines* the "length" L^- of the rod in a characteristic RIO Σ^- to equal γL. Is the above procedure of measurement of spatial distance of events compatible (as far as the numeric results are concerned) with the relativistic? Answer using (a) the algebraic method and (b) the geometric method.

(a) **Algebraic Solution**

Let P, Q be the end points of the rod and x the coordinate of the observation point along the x-axis of the LCF of Σ. We consider the events P, Q to be the passage of the respective ends of the rod from the observation point. The coordinates of the events P, Q in the LCFs Σ and Σ^- are shown in the following table:

	Σ	Σ^-
P :	$(ct_A,\ x)$	$(ct_A^-,\ 0)$
Q :	$(c(t_A + \Delta t),\ x)$	$(c(t_A^- + \Delta t^-),\ L^-)$
PQ^i :	$(c\Delta t,\ 0)_\Sigma$	$(c\Delta t^-,\ L^-)_{\Sigma^-}$

where L^- is the length of the rod in Σ^-. From the boost relating Σ and Σ^- we obtain

$$L^- = \gamma\, u\, \Delta t. \tag{5.10}$$

All quantities in the rhs of (5.10) are measurable by the RIO Σ, therefore the length L^- can be computed in Σ only from the Newtonian measurement of the time difference Δt. Equation (5.10) can be written differently. The Newtonian distance L of the events P, Q in Σ is given by the relation

$$L = u\,\Delta t.$$

Replacing $u\,\Delta t$ in (5.10) we find

$$L = \frac{L^-}{\gamma}, \tag{5.11}$$

which is compatible with (5.8). We conclude that the above method of estimating the spatial distance of events in Σ using Newtonian methods is compatible with the relativistic one.

(b) **Geometric Solution**

We consider the spacetime diagram of Fig. 5.6:

Explanation of the figure
Events
A_1, B_1: Observation of the ends P, Q of the rod in Σ.
Σ: World line of observer.
A, B : World lines of the end points of the rod.
$\Sigma_{t_A},\ \Sigma_{t_B}$: Proper spaces of Σ at the proper moments $ct_A,\ ct_B$.
ϕ: rapidity of the rod in Σ.
Data
(B_1B_2): relativistically measured length of the rod in Σ, $(B_1B_2) = L$.
(B_2B_3): Newtonian measurement of the length of the rod in Σ^-, $(B_2B_3) = L^-$.
Requested
Relation between (B_1B_2) and (B_2B_3).

From the orthogonal triangle $B_1B_2B_3$ we have

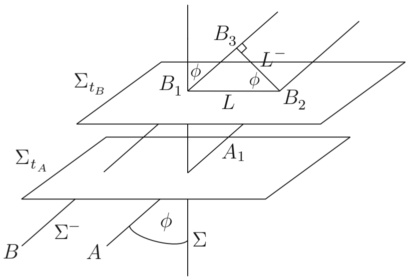


Fig. 5.6 Measurement of the length of a rod

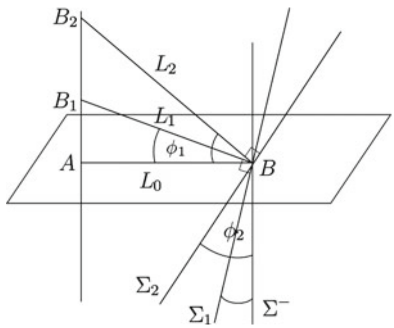


Fig. 5.7 Variation of the spatial distance of two events with velocity

$$(B_1 B_2) = \frac{(B_2 B_3)}{\cosh \phi}$$

from which follows $L = \frac{L^-}{\gamma}$ as before.

Figure 5.7 shows the measurement of the spatial distance of two events A, B by two RIOs Σ_1 and Σ_2 with speeds $v_1 < v_2$, respectively, wrt the characteristic observer Σ^- of the rod. We note that $L_2 < L_1$, that is the faster observer measures a smaller spatial distance and as $v_2 \to c$ the length $L_2 \to 0$. Furthermore we note that the normals are on the world line of the observers $\Sigma_1,\ \Sigma_2$ and not on the world line of the characteristic observer Σ^-.

5.7 The General Case

We have not yet covered the general case 5 of Table 5.2, in which all four components $dl,\ dl',\ d\mathbf{r},\ d\mathbf{r}'$ of the position four-vector in the LCF of the RIOs Σ and Σ' do not vanish. We do that in the following definition, which incorporates the considerations of Sects. 5.5, 5.6 and defines the correspondence between the Newtonian and the relativistic space and temporal distances.

Definition 9 (a) Let the events P, Q define a timelike position four-vector. The value of the Lorentz distance of the events P, Q (which is a relativistic invariant, therefore has the same value for all RIOs!) as measured in the proper frame Σ^+_{PQ} of the events P, Q coincides (numerically!) with the Newtonian time difference (that is with the time difference measured by the cosmic clock) of the events as measured by the Newtonian inertial observer who coincides with the RIO Σ^+_{PQ}.
(b) Let the events P, Q define a spacelike position four-vector. The value of the Lorentz distance of the events P, Q as measured in a characteristic frame Σ^- of the four-vector PQ^i coincides (numerically!) with the Newtonian spatial distance measured by a Newtonian inertial observer who coincides with the RIO Σ^-.

In the following the method of measurement of relativistic spatial and temporal distances is called *chronometry of a pair of events*.

Example 11 Two rods 1, 2 of length l_1 and $l_2 = nl_1$, respectively, move with constant speed along parallel directions and in the opposite sense. An observer Σ_1 at one end of the rod 1 measures that a time interval $\tau_1 = l_1/k$ $(k > 1)$ is required for the rod 2 to pass in front of him. Calculate

(a) The relative speed of the rods
(b) The time interval measured by the observer Σ_2 at one end of the rod 2, required for the rod 1 to pass in front of him

Solution

Let A, B be the events of observation of the ends of the rod l_2 as it passes in front of the observer Σ_1 positioned at one end of the rod l_1. We have the following table for the coordinates of these events in the LCF of the RIOs $\Sigma_1,\ \Sigma_2$:

	Σ_1	Σ_2
$A:$	$(0,\ 0)$	$(0,\ 0)$
$B:$	$(c\tau_1,\ 0)$	$(c\tau_2,\ l_2)$
$AB:$	$(c\tau_1,\ 0)_{\Sigma_1}$	$(c\tau_2,\ l_2)_{\Sigma_2}$

where τ_2 is the time interval required for the rod l_1 to pass from one end of the rod l_2. We have $\tau_2 = \frac{l_2}{c\beta}$ where $c\beta$ is the relative speed of the rods. The boost relating $\Sigma_1,\ \Sigma_2$ gives $\tau_2 = \gamma\tau_1$, hence

$$l_2 = c\gamma\beta\tau_1 \Rightarrow \beta\gamma = \frac{l_2}{c\tau_1} = \frac{nl_1}{cl_1/k} = \frac{nk}{c}.$$

Using the identity $\gamma^2 - 1 = \gamma^2\beta^2$ we find eventually

$$\gamma = \sqrt{1 + \frac{n^2k^2}{c^2}}, \quad \beta = \frac{nk}{\sqrt{c^2 + n^2k^2}}.$$

(b) The result does not change if we interchange the names of the rods $1 \to 2$. Therefore without any further calculations we write

$$\tau_2 = \frac{l_1}{c\beta\gamma} = \frac{l_1}{nk}.$$

5.8 The Reality of Length Contraction and Time Dilation

It is natural to state the question: Does the phenomenon of length contraction and time dilation exist? The answer to that question is crucial in order to avoid confusion especially with the various paradoxes which, as a rule, are concerned with these two phenomena.

Definition 9 relates the measurement of the relativistic spatial and temporal distance of two events, with the corresponding Newtonian measurement of these quantities. Verbally it can be stated as follows:

$$\begin{bmatrix} \text{Time difference } dl^{+} \\ \text{of events } P,\ Q \text{ } \textit{measured} \\ \text{relativistically in the} \\ \text{proper frame } \Sigma^{+}_{PQ} \end{bmatrix} \equiv \begin{bmatrix} \text{Newtonian measurement of the} \\ \text{time difference of the events } P,\ Q \text{ taken} \\ \text{by the Newtonian inertial observer} \\ \text{who coincides with } \Sigma^{+}_{PQ} \end{bmatrix},$$

$$\begin{bmatrix} \text{Spatial distance } d\mathbf{r}^{-}_{\parallel} \\ \text{of events } P,\ Q \text{ } \textit{measured} \\ \text{relativistically in the} \\ \text{characteristic frame} \\ \Sigma^{-}_{PQ} \end{bmatrix} \equiv \begin{bmatrix} \text{Newtonian measurement} \\ \text{of the spatial distance} \\ \text{of events } P,\ Q \text{ taken} \\ \text{by the Newtonian inertial observer} \\ \text{who coincides with } \Sigma^{-}_{PQ} \end{bmatrix}.$$

With this definition the relativistic measurements/observations of the spatial and temporal distance from the characteristic observer and the proper observer, respectively, attain a Newtonian "reality." However, this is true only for these observers! For the rest RIO there is no "Newtonian reality" – that is, comparison with a corresponding Newtonian physical quantity – and one must use the appropriate Lorentz transformation to estimate the value of spatial distance and time difference. Therefore relations (5.5) and (5.8) must be understood as follows:

$$\begin{bmatrix} \text{Chronometric } \textit{measurement} \\ \text{of time difference} \\ \text{of the events } P,\ Q \\ \text{by the RIO } \Sigma \end{bmatrix} = \gamma \begin{bmatrix} \text{Newtonian measurement} \\ \text{of time difference} \\ \text{of the events } P,\ Q \\ \text{in the proper frame } \Sigma^{+}_{PQ} \end{bmatrix}$$

and

$$\begin{bmatrix} \text{Chronometric } \textit{measurement} \\ \text{of spatial distance} \\ \text{of the events } P,\ Q \\ \text{by the RIO } \Sigma \end{bmatrix} = \gamma^{-1} \begin{bmatrix} \text{Newtonian measurement} \\ \text{of spatial distance} \\ \text{of the events } P,\ Q \\ \text{in the proper frame } \Sigma^{-}_{PQ} \end{bmatrix}.$$

Therefore the validation or not of the phenomena of length contraction and time dilation in nature does not concern the validity of Special Relativity as a theory (=logical structure) but the chronometry, that is, the proposed *method of measuring/estimating* Newtonian spatial distances and Newtonian time differences. Observation has confirmed that the above relations are true, and therefore validates the chronometry of the two events. In practice this means that if we produce in the laboratory a beam of particles at the point A and wish to focus them at the point B, then the Euclidean distance (AB) must be determined relativistically. That is the position of the point B depends on the energy (speed) of the particles of the beam, which is not the case with Newtonian Physics. In a similar manner we determine the lifetime of a given unstable particle. In the laboratory the experimentally measured lifetime of an unstable particle depends on the speed of the particle contrary to the Newtonian view that the lifetime of a particle is unique. According to our assumption the unique value of the Newtonian approach coincides with the relativistic measurement of the lifetime of the particle in its proper frame (photons excluded, they are stable anyway).

From the above we conclude that the phenomena of length contraction and time dilation are real and must be taken into consideration in our (obligatory!) Newtonian measurements in the laboratory and/or in space.

The dependence of the spatial distance and the time difference of two events from the velocity of the particles must not worry us in our everyday activities. Indeed, as we have already seen, the effects of relativistic kinematics are appreciable and show up at high (relative) speeds, far beyond the speeds of our sensory capabilities. Problems which refer to cars entering a garage while traveling with speeds, e.g., $0.9c$ concern "realities" limited at the level of student exercises and not further than that.

5.9 The Rigid Rod

The rigid rod is a purely Newtonian system which lies in the roots of Newtonian theory and expresses the absoluteness of three-dimensional space. The rigid rod does not exist in Special Relativity, in the sense that spatial distance is not a relativistic physical quantity. However, in practice and in the laboratory we do use rigid rods (e.g., the standard meters) to measure distances, therefore it is important and useful that we shall *define* the concept of rigid rod in Special Relativity. This "relativistic" rigid rod shall be associated via a correspondence principle with the Newtonian rigid rod, in the same way we did in Sect. 5.7 for the spatial distance and the time interval.

Definition 10 The (one-dimensional relativistic) *rigid rod* in Special Relativity is defined to be a (relativistic) physical system which in spacetime is represented by a bundle of timelike straight lines such that

(a) All lines are parallel (in the spacetime sense).
(b) All lie in the same timelike plane.

The system (relativistic) rigid rod is characterized by a single number (Lorentz invariant!), which we call *rest length* or *proper length* of the rigid rod. This number is specified by the following procedure.

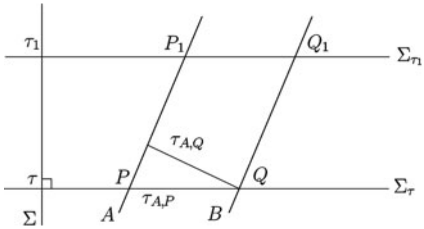


Fig. 5.8 Relativistic measurement of the length of a rigid rod

Each line in the bundle of the timelike straight lines defining the rigid rod can be associated with the world line of a RIO. If this is done then one can consider the (relativistic) rigid rod as a set of RIOs which have zero relative velocity.[4]

Let A, B be the limiting outmost world lines of the bundle of the timelike lines defining the rigid rod (see Fig. 5.8). Consider a RIO Σ and let Σ_τ be the proper space of Σ at proper time τ. This plane intersects the bundle of the lines defining the rigid rod at line PQ say. The points P, Q are the intersection of the outmost RIOs A, B of the rod with Σ_τ. The four-vector PQ^i is spacelike because it lies on the spacelike hyperplane Σ_τ. The length of the four-vector PQ^i is called the *length of the (relativistic) rigid rod* as measured by the observer Σ.

The length (PQ) is uniquely defined for each RIO Σ because of the following:

(1) It is defined geometrically by the intersection of two planes, each one uniquely defined by the rigid rod and the observer.
(2) The length of the rod is independent of the proper moment τ of Σ because Σ is a RIO and all proper spaces are parallel, therefore the length of the intersection with the two-dimensional plane defining the rigid rod is independent of the point at which we consider the proper plane.

From Fig. 5.8 we note two important properties concerning the length of a (relativistic) rigid rod:

- The length is not absolute, that is the same for all RIOs, and varies with the speed of the rod .
- The events P, Q are not simultaneous for the outmost observers A, B of the rod ($\tau_{A,P} \neq \tau_{A,Q}$ and similarly for B) while they are so for the observer Σ.

The RIOs for which the value of the distance (PQ) takes its maximum value are called *characteristic observers* of the rigid rod. Obviously the world lines of these observers are parallel to the world lines A, B which means that they have zero velocity relative to the rod. The characteristic observer whose world line has the

[4] These observers are called *comoving observers* of the rigid rod. With a similar token one studies the motion of a relativistic fluid.

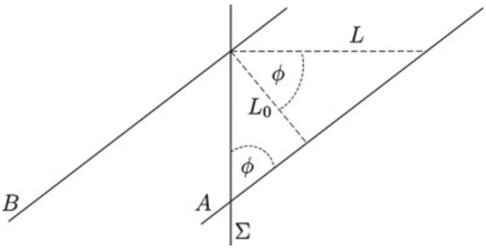


Fig. 5.9 Length of a (relativistic) rigid rod and Lorentz transformation

same spatial distance from the world lines A, B is called *rest observer*[5] of the rigid rod and denoted by Σ_0.

If the length of the rigid rod measured by a characteristic observer is L_0 and as measured by a RIO Σ, with rapidity ϕ wrt the characteristic observer, is L, then the following relation holds between the two lengths (see Fig. 5.9):

$$L = \frac{L_0}{\cosh \phi} = \frac{L_0}{\gamma}.$$

Because $\gamma > 1 \Rightarrow L < L_0$ from which it follows that the characteristic observer measures larger length for the rigid rod and in fact the maximal length (length contraction).

5.10 Optical Images in Special Relativity

The considerations of the previous sections were concerned with the temporal and the spatial components of the position four-vector and their relation with the corresponding concepts of time interval and space distance of Newtonian Physics. There is one more Newtonian physical quantity, the creation of optical images, which is related to the position vector but not to the measurement of the time difference and space distance because it does not involve the Lorentz transformation. The transfer of this quantity to Special Relativity is important although it applies only to extended luminous bodies, which as a rule do not have practical applications in Special Relativity.

In order to define the concept "appear" (= create optical image) in Special Relativity we generalize the corresponding concept of Newtonian Physics. In Newtonian Physics when we "see" a luminous object we receive at the eye (or the lens of a camera) simultaneously photons from the various parts of the body. Due to the finite speed of light and the different distances of the various points of the body

[5] The rest observer is the unique characteristic observer with the property that if it (recall that we do not use he/she) emits simultaneously two light signals toward the end points of the rod and these signals are reflected at these points, then it will receive back the two signals simultaneously and along antiparallel directions.

from the eye the photons which arrive simultaneously at the eye must be emitted at different times. If the luminous body moves slowly wrt the eye then we may assume that the photons are emitted simultaneously from all points of the body (Newtonian approximation). In this case we have a "faithful" depiction of the body. But for relativistic speeds this approximation does not hold and one should expect distortion of the optical image. Due to the above we define the creation of optical images in Special Relativity as follows.

Definition 11 The optical image of a set of luminous points at one space point and at one time moment of a RIO Σ is created from the points of emission, which are simultaneous in Σ.

Before we continue we need the following simple result.

Exercise 12 Let Σ and Σ' be two LCFs with parallel axes and relative speed $\boldsymbol{\beta}$. Consider a four-vector A^i which in the LCFs Σ and Σ' has components $A^i = \begin{pmatrix} A^0 \\ \mathbf{A} \end{pmatrix}_\Sigma$ and $A^i = \begin{pmatrix} A^{0'} \\ \mathbf{A}' \end{pmatrix}_{\Sigma'}$. Show that

$$A^0 = \frac{1}{\beta}\left(A_\parallel - \frac{1}{\gamma}A'_\parallel\right), \tag{5.12}$$

where $A_\parallel = \frac{1}{\beta}(\boldsymbol{A}\cdot\boldsymbol{\beta}),\ A'_\parallel = \frac{1}{\beta}(\boldsymbol{A}'\cdot\boldsymbol{\beta})$.

[Hint: Consider the Lorentz transformation which connects Σ, Σ':

$$\begin{aligned} A^{0'} &= \gamma(A^0 - \boldsymbol{\beta}\cdot\mathbf{A}), \\ \mathbf{A} &= \mathbf{A}' + \frac{\gamma-1}{\beta^2}(\boldsymbol{\beta}\cdot\mathbf{A})\boldsymbol{\beta} + \gamma A^{0'}\boldsymbol{\beta} \end{aligned}$$

and in the second equation (a) substitute $A^{0'}$ and (b) project parallel to $\boldsymbol{\beta}$.]

Suppose a photon is emitted from a point in space (event A) and the same photon is received at another point in space (event B). Let Σ be a RIO for which the events A, B have components $\begin{pmatrix} ct_A \\ \mathbf{r}_A \end{pmatrix}_\Sigma$, $\begin{pmatrix} ct_B \\ \mathbf{r}_B \end{pmatrix}_\Sigma$. The four-vector AB^i is a null vector (because it concerns the propagation of a photon), therefore

$$-c^2(t_B - t_A)^2 + (\mathbf{r}_B - \mathbf{r}_A)^2 = 0.$$

We write $R_{AB} =\mid \mathbf{r}_B - \mathbf{r}_A \mid$ and assuming that $t_B > t_A$ (both time moments in Σ!) we have

$$t_B = t_A + \frac{R_{AB}}{c}. \tag{5.13}$$

This relation connects in Σ (!) the distance covered by the photon after its emission in terms of the time interval after its emission. Let another RIO Σ' move wrt Σ with parallel axes and arbitrary constant velocity. Let $\begin{pmatrix} ct'_A \\ \mathbf{r}'_A \end{pmatrix}_{\Sigma'}$ be the components of the event A in Σ'. According to Exercise 12 for the zeroth component of the four-vector A^i we have

$$ct_A = \frac{1}{\beta}\left(r_{A\|} - \frac{1}{\gamma} r'_{A\|}\right), \tag{5.14}$$

where $r_{A\|} = \frac{1}{\beta}(\boldsymbol{r} \cdot \boldsymbol{\beta})$, $r'_{A\|} = \frac{1}{\beta}(\boldsymbol{r}' \cdot \boldsymbol{\beta})$. From (5.13), (5.14) we have the relation which connects the spatial position of the event A of emission in Σ' with the time and the spatial distance covered by the photon in Σ:

$$\begin{aligned} ct_B &= \frac{1}{\beta}\left(r_{A\|} - \frac{1}{\gamma} r'_{A\|}\right) + R_{AB} \Rightarrow \\ r'_{A\|} &= \gamma\left(r_{A\|} - \beta c t_B + \beta R_{AB}\right). \end{aligned} \tag{5.15}$$

This relation gives the distance of the image $r'_{A\|}$ in Σ' as a function of the distance $r_{A\|}$ in Σ.

Normal to the direction of the velocity $\boldsymbol{\beta}$ the Lorentz transformation has no effect, hence

$$\mathbf{r}_{A\perp} = \mathbf{r}'_{A\perp}. \tag{5.16}$$

Using relations (5.15) and (5.16) it is possible to compute the image in Σ if we know a luminous object in Σ'. Indeed let A, Γ be two points of a luminous body which are observed at the point B of Σ, the time moment t_0 (of Σ). Then for each point (5.15), (5.16) hold separately and the condition of the creation of the images of the points A, Γ in Σ is that the time in the rhs of the equations is common and equal to t_0. Subtracting (5.15) for the points A, Γ and eliminating the time t_0 we find the equation

$$r'_{A\|} - r'_{\Gamma\|} = \gamma(r_{A\|} - r_{\Gamma\|} + \beta(R_{AB} - R_{\Gamma B})), \tag{5.17}$$

which expresses the "parallel" spatial distance of the points A, Γ as it is seen by the observer Σ at the point B of Σ. We note that this expression is independent of the time of observation in Σ, hence the image parallel to the relative velocity remains the same; it is "frozen" as we say. In addition we note that for $c \to \infty$ the $r'_{A\|} - r'_{\Gamma\|} = r_{A\|} - r_{\Gamma\|}$, that is we have the absolute depiction of Newtonian Physics.

Without restricting generality one may consider the point B to be the origin of the LCF Σ in which case $R_A^2 = \mathbf{r}_A^2 = x_A^2 + y_A^2 + z_A^2$ and $R_\Gamma^2 = \mathbf{r}_\Gamma^2 = x_\Gamma^2 + y_\Gamma^2 + z_\Gamma^2$. If in addition we restrict our considerations to boosts, then $r_{\|A} = x_A$, $r'_{\|A} = x'_A$ and (5.15), (5.17) read

$$x'_A = \gamma(x_A - \beta c t_B + \beta R_A), \tag{5.18}$$

$$x'_\Gamma = \gamma(x_\Gamma - \beta c t_B + \beta R_\Gamma) \Rightarrow \tag{5.19}$$

$$x'_A - x'_\Gamma = \gamma[x_A - x_\Gamma + \beta(R_A - R_\Gamma)]. \tag{5.20}$$

This relation holds if Σ' moves wrt Σ with speed β. If the velocity of Σ' is $-\beta$ the relation remains the same with the change that the dashed quantities have to be interchanged with the unprimed, that is

$$x_A - x_\Gamma = \gamma\left[x'_A - x'_\Gamma + \beta(R'_A - R'_\Gamma)\right]. \tag{5.21}$$

Example 12 A rod AC of length l' is resting parallel to the x'-axis and in the plane $y'x'$ of the RIO Σ' while Σ' is moving in the standard configuration along the x-axis of the RIO Σ with velocity $\boldsymbol{\beta}$. Calculate the length l of the image of the rod on a film which is placed at the origin O of Σ.

Solution

Let

$$| x'_A - x'_C | = l', \; | x_A - x_C | = l.$$

From (5.20) we have

$$| x'_A - x'_C | = \gamma[| x_A - x_C | + \beta(R_A - R_C)]. \tag{5.22}$$

We consider two cases depending on whether the rod is resting along the x'- axis or not.

(A) The rod is resting along the x'-axis, hence $y' = y = 0$.

We consider two cases depending on whether the rod is approaching or moving away from the origin O of Σ. In the first case we have $x'_A - x'_C = -l' = -l_-, \; x_A - x_C = -l, \; R_A - R_C = -l$ and replacing in (5.20) we find

$$l_- = \sqrt{\frac{1+\beta}{1-\beta}} l' > l'. \tag{5.23}$$

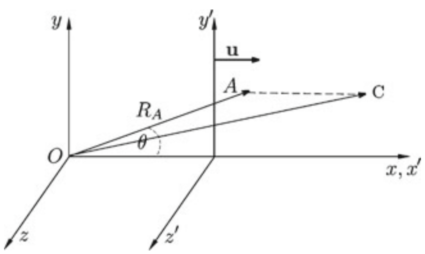
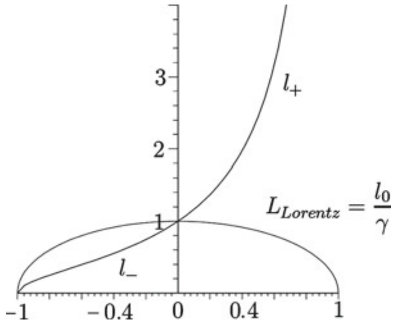


Fig. 5.10 Relativistic observation of an optical image

Fig. 5.11 Variation of the apparent length with velocity



When the rod is moving away we have $x'_A - x'_C = -l'$, $x_A - x_C = -l_+$, $R_A - R_C = -l'$ and (5.20) gives

$$l_+ = \sqrt{\frac{1-\beta}{1+\beta}}l' < l'. \tag{5.24}$$

Figure 5.11 shows the variation of the lengths l_+, l_- for various values of the velocity of the rod. In the same figure the standard Lorentz contraction (l'/γ) is also shown for comparison. We note that while the Lorentz contraction is independent of the direction of motion (hence symmetric about the value $\beta = 0$) the situation is different for the "apparent" length of the rod which behaves as a "wave" (because (5.23) and (5.24) express the Doppler effect).

(B) The rod is moving so that $y' = y = c_1 \neq 0$.

In this case (5.20) gives (see Fig. 5.10)

$$l = \frac{l'}{\gamma} - \beta(R_A - R_C). \tag{5.25}$$

If we set θ the angle of the position vector of the edge point A of the rod with the x-axis in Σ and $\theta - \Delta\theta$ for the other end B, relation (5.25) is written as follows:

$$l = \frac{1}{\gamma}\left[l' + \beta\gamma y_1\left(\frac{1}{\sin(\theta - \Delta\theta)} - \frac{1}{\sin\theta}\right)\right]. \tag{5.26}$$

When $\pi > \theta > \frac{\pi}{2}$ the rod approaches the origin O of Σ and when $\frac{\pi}{2} > \theta > 0$ the rod is moving away from the origin of Σ.

In order to get a feeling of the result we consider the limit $\Delta\theta \ll 0$ (equivalently $l \ll R_A$) in which case we can write (prove it!) for the difference $R_A - R_C = l\cos\theta$. In this case (5.25) becomes

$$l = \frac{l'}{\gamma(1+\beta\cos\theta)}. \tag{5.27}$$

We infer that in this case the observer sees the standard Lorentz contraction only if the observer looks along the direction of the y-axis (that is $\theta = \frac{\pi}{2}$).

Example 13 A right circular cylinder[6] of radius R and length L_0 is rotating with constant angular speed ω about its axis, which coincides with the x-axis of the LCF Σ. Let Σ' be another RIO which moves in the standard configuration along the common x, x'-axes with speed β.

(a) Prove that in Σ' the cylinder appears to be a right circular cylinder of radius R *and* length $\frac{L_0}{\gamma}$.
(b) Assume that along the surface of the cylinder have been marked some points (for example with the fixing of small flags whose height is negligible compared to the radius of the cylinder) which in the RIO Σ appear to be along a directrix of the cylinder. Prove that in Σ' these points do not appear to be along a directrix of the cylinder, but along a spiral which appears from one end of the cylinder and disappears at the other end of the cylinder.

Solution

(a) The equation of the surface of the (right circular) cylinder in Σ is $y^2+z^2 = R^2$. The boost relating Σ, Σ' gives $y' = y, z' = z$. Therefore in Σ' the equation of the surface is $y'^2 + z'^2 = R^2$, which is the surface of a right circular cylinder of radius R. For the remaining two coordinates (ct, x) and (ct', x') the boost gives

$$x' = \gamma(x - ut),$$
$$ct = \gamma(ct' + \beta x')$$

from which follows $x' = \frac{1}{\gamma}x - ut'$. Let A, B be the points of intersection of the base planes of the cylinder with the x-axis. The coordinates of these points in Σ are $0, L_0$. At the moment t' of Σ' these points have coordinates whose difference is $x'_B - x'_A = \frac{1}{\gamma}L_0$ which is the expected Lorentz contraction and at the same time the height of the cylinder in Σ'.

(b) The points of a directrix of the cylinder in Σ have coordinates

$$x, \quad y = R\cos\omega t, \quad z = R\sin\omega t$$

and velocity

$$\mathbf{v} = (0, -R\omega\sin\omega t, R\omega\cos\omega t).$$

[6] This problem has been proposed by the German physicist von Laue at the early steps of Special Relativity.

We compute the corresponding quantities in Σ'. The coordinates are

$$x' = \frac{1}{\gamma}x - ut',$$
$$y' = y = R\cos\omega t = R\cos\omega\gamma/c(ct' + \beta x') = R\cos\left(\frac{\omega}{\gamma}t' + \frac{\beta\omega}{c}x\right),$$
$$z' = z = R\sin\omega t = R\sin\omega\gamma/c(ct' + \beta x') = R\sin\left(\frac{\omega}{\gamma}t' + \frac{\beta\omega}{c}x\right).$$

In order to define how Σ' "sees" the marked points we consider t' =constant. From the equation of the transformation it follows that the marked points lie along a (right circular) helix of radius R and pace $\frac{2\pi c}{\beta\omega}$. The helix develops all along the length of the cylinder and it is "frozen" during the motion because both its radius and its pace are independent of t'. The effect of the rotation is that the helix appears in Σ' to emerge from the nearest end of the cylinder and disappear at the other end (see Fig. 5.12).

In order to calculate the velocity of the marked points of the cylinder in Σ' we take the derivative[7] of the coordinates of the points in the LCF Σ' with respect to t'. The answer (assuming x =constant) is

$$\mathbf{v}' = \left(-u, -\frac{R\omega}{\gamma}\sin\left(\frac{\omega}{\gamma}t' + \frac{\beta\omega}{c}x\right), \frac{R\omega}{\gamma}\cos\left(\frac{\omega}{\gamma}t' + \frac{\beta\omega}{c}x\right)\right).$$

This motion can be considered as a combination of two motions. A translational motion along the common x, x'- axes with speed $-\beta$ and a rotational motion about the same axis with angular speed $\omega' = \frac{\omega}{\gamma}$. (Question: What happens when the speed u of Σ' is such that $\omega = \gamma$?)

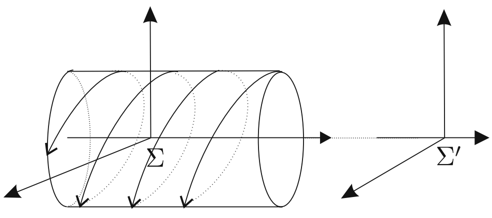


Fig. 5.12 The image of marked points in Σ'

[7] One can also use the relativistic rule of composition of velocities.

5.11 How to Solve Problems Involving Spatial and Temporal Distance

In this section we present a number of methods for the solution of problems involving the application of Lorentz transformation – mainly the boosts in problems involving spatial and temporal distance. These methods can be used for calculations in the laboratory and especially in the discussion of the paradoxes which are based in the misunderstanding of the relativistic spatial and temporal distance and their relation to the corresponding Euclidean quantities. In any case we strongly suggest that the material of this chapter should be studied thoroughly by those who are not experienced with Special Relativity.

5.11.1 A Brief Summary of the Lorentz Transformation

Before we proceed we collect the results we have found so far for the Lorentz transformation.

Consider two LCFs Σ and Σ' with parallel axes and relative velocity $\mathbf{u}$. The (proper) Lorentz transformation $L_{+\uparrow}$ which relates the physical quantities of Σ and Σ' is (see (1.47))

$$L_{+\uparrow} = \begin{bmatrix} \gamma & -\gamma\boldsymbol{\beta}^t \\ -\gamma\boldsymbol{\beta} & I + \frac{(\gamma-1)}{\beta^2}\boldsymbol{\beta}\boldsymbol{\beta}^t \end{bmatrix}. \tag{5.28}$$

For example for the position four-vector $(l,\ \mathbf{r})$ the transformation (5.28) gives

$$\begin{cases} \mathbf{r}' = \mathbf{r} + \left[\frac{\gamma-1}{\beta^2}(\boldsymbol{\beta}\cdot\mathbf{r}) - \gamma l\right]\boldsymbol{\beta}, \\ l' = \gamma\,(l - \boldsymbol{\beta}\cdot\mathbf{r}). \end{cases} \tag{5.29}$$

The vector form of the Lorentz transformations (5.28) and (5.29) holds only if the axes of Σ and Σ' are parallel (in the Euclidean sense) in the *same Euclidean space*. Relations (5.28) and (5.29) are not quantitative and, in general, can be used in qualitative analysis only. In case we want to make explicit calculations we have to employ a coordinate system and write these equations in coordinate form.

A special and important subclass of Lorentz transformations is the boosts. This is due to the fact (see (1.43)) that a general Lorentz transformation can be written as a product of two Euclidean rotations and one boost. Two LCFs are related with a boost of velocity $\boldsymbol{\beta}$ if and only if they move in the standard configuration, that is

(1) The axes x, x' are common and the axes y, y', z, z' are parallel.
(2) The speed u of Σ' relative to Σ is parallel to the common axis x.

(3) The clocks of Σ and Σ' have been synchronized so that when the origins coincide (that is $x = y = z = x' = y' = z' = 0$) the clocks are reset to zero (that is $t = t' = 0$).

The boost relating the coordinates (l, x, y, z) and (l', x', y', z') of an event P, say, in Σ and Σ' reads

$$\begin{aligned} x' &= \gamma(x - \beta l), \\ y' &= y, \\ z' &= z, \\ l' &= \gamma(l - \beta x). \end{aligned} \tag{5.30}$$

The inverse Lorentz transformation from Σ' to Σ is

$$\begin{aligned} x &= \gamma(x' + \beta l'), \\ y &= y', \\ z &= z', \\ l &= \gamma(l' + \beta x'), \end{aligned} \tag{5.31}$$

that is, the sign of β in (5.30) changes. The matrix form of the transformation is

$$L_{+}(\beta) = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{5.32}$$

This matrix is symmetric. This is not the case for the matrix representing the general Lorentz transformation (i.e., parallel axes and arbitrary relative velocity).

5.11.2 *Parallel and Normal Decomposition of Lorentz Transformation*

As we have seen the Lorentz transformation has no effect normal to the direction of the relative velocity $\mathbf{u}$ while parallel to the direction of $\boldsymbol{u}$ the action of the Lorentz transformation is a boost. This observation leads us to consider the following methodology of solving problems in Special Relativity. Consider a four-vector $A_i = (A_0, \mathbf{A})$ (not necessarily the position vector) and let $\mathbf{u}$ be the relative velocity of two LCFs Σ and Σ'. We decompose in Σ the vector $\mathbf{A}$ along and normal to $\boldsymbol{u}$:

$$\mathbf{A} = \mathbf{A}_{\parallel} + \mathbf{A}_{\perp}\ ,\ \mathbf{A}_{\parallel} = \frac{\mathbf{A}\cdot\boldsymbol{\beta}}{\beta^2}\boldsymbol{\beta}\ ,\ \mathbf{A}_{\perp} = \mathbf{A} - \mathbf{A}_{\parallel}$$

and write in a profound notation: $A_i = (A_0, \ \mathbf{A}_\parallel, \ \mathbf{A}_\perp)_\Sigma$. Suppose that in Σ' the four-vector A_i has components $A_i = (A'_0, \ \mathbf{A}'_\parallel, \ \mathbf{A}'_\perp)_{\Sigma'}$. Then the (proper) Lorentz transformation gives the relations

$$\begin{aligned} A'_0 &= \gamma(A_0 - \boldsymbol{\beta} \cdot \mathbf{A}_\parallel), \\ \mathbf{A}'_\parallel &= \gamma(\mathbf{A}_\parallel - \boldsymbol{\beta} A^0), \\ \mathbf{A}'_\perp &= \mathbf{A}_\perp. \end{aligned} \tag{5.33}$$

For the Euclidean length $\mathbf{A}'^2$ we have

$$\mathbf{A}'^2 = \mathbf{A}'^2_\parallel + \mathbf{A}'^2_\perp = \mathbf{A}^2_\perp + \gamma(\mathbf{A}_\parallel - \boldsymbol{\beta} A^0)^2. \tag{5.34}$$

The angle θ between the directions of $\mathbf{A}$ and $\mathbf{u}$ is given by

$$\tan\theta = \frac{|\mathbf{A}_\perp|}{|\mathbf{A}_\parallel|}. \tag{5.35}$$

The corresponding angle θ' of $\mathbf{A}'$ with the velocity $\mathbf{u}$ in Σ' is computed as follows:

$$\tan\theta' = \frac{|\mathbf{A}'_\perp|}{|\mathbf{A}'_\parallel|} = \frac{|\mathbf{A}_\perp|}{\gamma\left(|A_0\boldsymbol{\beta} - \mathbf{A}_\parallel|\right)} = \frac{|\mathbf{A}|\sin\theta}{\gamma(A_0\beta - |\mathbf{A}|\cos\theta)}. \tag{5.36}$$

We note that $\theta \neq \theta'$ for every three-vector $\mathbf{A}$. This result has many applications under the generic name *aberration*.

5.11.3 Methodologies of Solving Problems Involving Boosts

The difficult part in the solution of a problem in Special Relativity is the recognition of the events and their coordinate description in the appropriate LCF of the problem. As a rule in a relativistic problem there are involved

(a) Two events (or relativistic physical quantities) A, B (say),
(b) Two LCFs Σ and Σ' (say), and
(c) Data concerning some of the components of the events in one LCF and some data in the other. With the aid of the Lorentz transformation relating Σ and Σ' one is able to compute all the components of the involved quantities (usually four-vectors) A, B.

For the solution of problems in Special Relativity involving space distances and time distances one has to assign correctly who is the observer measuring the relevant quantity directly and who is the observer computing it via the appropriate Lorentz transformation. Concerning the question of measurement of space distances and time distances we have the following:

Case 1

Consider two events A, B, which define the timelike four-vector AB^i. As we have shown there exists a unique LCF Σ^+_{AB}, the proper frame of the events A, B, in which the Lorentz length of the four-vector equals the zeroth component $(AB)^+$. This number (by definition!) equals the time difference of the two events as measured by a Newtonian observer in Σ^+_{AB}. In every other LCF Σ, which has velocity $\boldsymbol{\beta}$ wrt to Σ^+_{AB}, the events A, B have (relativistic!) time difference

$$(AB) = \gamma\,(AB)^+. \tag{5.37}$$

The time difference (AB) *is not measured* by the Newtonian observer in Σ but either it is computed by the Lorentz transformation relating Σ and Σ^+_{AB}, or it is measured directly in Σ by chronometry. The value of the quantity (AB) varies with the frame Σ and holds $(AB) > (AB)^+$ (time dilation).

Geometrically, the measurement of the temporal difference is shown in Fig. 5.13.

Description of Diagram

Σ : World line of observer.
Σ^+ : World line of proper frame of the clock.
A : Newtonian observation of the indication A of the clock in Σ^+.
B : Newtonian observation of the indication B of the clock in Σ^+.
$(AB)^+$: Newtonian measurement of temporal length in Σ^+.
$(AB)_\Sigma$: Chronometric measurement of the temporal length (AB) by Σ.

Concerning the calculation, from the triangle ABC we have

$$(AC) = (AB)^+\,\cosh\phi = \gamma\,(AB)^+.$$

Case 2

Consider two events A, B, which define a spacelike four-vector AB^i and are studied in the LCF Σ. Let Σ^- be a characteristic LCF of the spacelike four-vector AB^i and let $(AB)^-$ be the Lorentz length of this four-vector. We have defined $(AB)^-$ to be the spatial distance of the events A, B as measured by a Newtonian observer at rest in Σ^-. If the (relativistic) spatial distance of the events A, B in Σ is (AB)

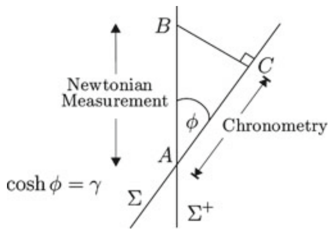


Fig. 5.13 Relativistic measurement of the temporal difference

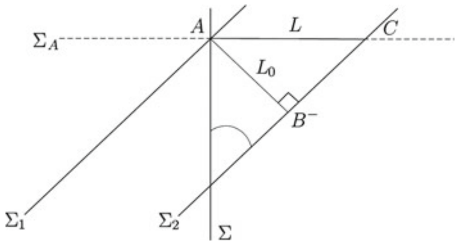


Fig. 5.14 Chronometric measurement of spatial distance

(measured chronometrically) and the relative speed of Σ, Σ^- is $\boldsymbol{\beta}$, Lorentz transformation gives the relation

$$(AB) = \frac{(AB)^-}{\gamma}. \tag{5.38}$$

The length $(AB)^-$ is an invariant whereas the length (AB) is not and depends on the observer Σ (the factor β). We have the obvious inequality $(AB) < (AB)^-$ (spatial distance contraction or, as it has been inappropriately established, length contraction).

The measurement of spatial distance is described geometrically in Fig. 5.14.

Description of Diagram

Σ :World line of observer.

Σ_1 : World line of end A of spatial distance.

Σ_2 : World line of end B of spatial distance.

Σ_A : Proper space of Σ at the point A.

C: Intersection of proper space Σ_A with world line Σ_2.

AB^- : Common (Lorentz) normal of Σ_1, Σ_2 at the event A.

$(AB)^-$: Newtonian distance (proper distance) L_0 of world lines in the characteristic LCF Σ^-_{AB}.

(AC): Spatial distance of world lines Σ_1 and Σ_2 as measured chronometrically by Σ.

Concerning the calculations, from the triangle ABC of Fig. 5.14 we find

$$(AB)^- = (AC)\cosh\phi = \gamma\,(AC).$$

Case 3

Consider two events A, B which define a null vector. A null four-vector does not have a characteristic LCF, therefore it makes no sense to measure (relativistic) spatial or temporal distance.

5.11.4 The Algebraic Method

This method involves the construction of a table containing the coordinates of the four-vectors in the LCFs Σ, Σ' and subsequently the application of the Lorentz transformation relating the data of the table. More precisely, for every pair of events A, B one constructs the following table of coordinates:

	Σ	Σ'
$A:$	(l_A, x_A, y_A, z_A)	(l'_A, x'_A, y'_A, z'_A)
$B:$	(l_B, x_B, y_B, z_B)	(l'_B, x'_B, y'_B, z'_B)
$AB^i:$	$(l_B - l_A, x_B - x_A, \ldots)_\Sigma$	$(l'_B - l'_A, x'_B - x'_A, \ldots)_{\Sigma'}$

The two events define the four-vector AB^i whose components in Σ and Σ' equal the difference of the components of the events A, B in Σ and Σ', respectively, and are related by the Lorentz transformation. In case Σ, Σ' move in the standard configuration with velocity β we have the following equations:

$$
\begin{aligned}
l'_B - l'_A &= \gamma\ [(l_B - l_A) - \beta(x_B - x_A)]\,,\\
x'_B - x'_A &= \gamma\ [(x_B - x_A) - \beta\,(l_B - l_A)]\,,\\
y'_B - y'_A &= y_B - y_A,\\
z'_B - z'_A &= z_B - z_A.
\end{aligned}
$$

Usually these equations suffice for the solution of the problem. It is important that before inserting the data in the table one clarifies which observer does the Newtonian and which the chronometric measurement of space and time distances. This method of solving relativistic problems is called the *algebraic method*.

Example 14 A rod AB moves in the plane xy of the LCF Σ in such a way that the end A slides along the x-axis with constant speed u whereas the rod makes at all times (in Σ!) with the x-axis a constant angle ϕ. Consider another LCF Σ' which moves relative to Σ in the standard configuration along the x-axis with speed v. Let $m = \tan\phi$ be the inclination of the rod in Σ. Calculate the corresponding quantity m' in Σ'.

First Solution

We consider the end points A, B of the rod and assume that the measurement of the length of the rod is done in Σ'. Furthermore we assume that at time $t = 0$ of Σ the point A is at the origin of Σ. These assumptions lead to the following table of coordinates of the events A, B:

	Σ	Σ'
$A:$	$(ct_A,\ ut_A,\ 0,\ 0)$	$(ct',\ x'_A,\ 0,\ 0)$
$B:$	$(ct_B,\ ut_B + l_x,\ l_y,\ 0)$	$(ct',\ x'_A + l'_x,\ l'_y,\ 0)$
$AB^i:$	$(c\,\Delta t,\ u\Delta t + l_x,\ l_y,\ 0)_\Sigma$	$(0,\ l'_x,\ l'_y,\ 0)_{\Sigma'}$

where $\Delta t = t_B - t_A$. The coordinates of the four-vector AB^i in Σ, Σ' are related to the boost along the x-axis with speed v:

$$\begin{aligned} l'_x &= \gamma\ (l_x + u\Delta t - v\Delta t)\,, \\ l'_y &= l_y, \\ c\,\Delta t &= \gamma\left(0 + \frac{v}{c}\,l'_x\right). \end{aligned}$$

Eliminating Δt from the third equation and replacing in the first we find

$$l'_x = \frac{1}{\gamma\,\left(1 - \frac{uv}{c^2}\right)}\,l_x.$$

The inclination m' of the rod in Σ' is

$$m' = \frac{l'_y}{l'_x} = \gamma\,\left(1 - \frac{uv}{c^2}\right)\frac{l_y}{l_x} = \gamma\,\left(1 - \frac{uv}{c^2}\right) m.$$

Second Solution

Let Σ^- be a characteristic frame of the rod AB. We assume measurement of the length of the rod in Σ^- and find the following table of coordinates of the events A, B:

	Σ	Σ^-
$A:$	$(ct_A,\ ut_A,\ 0,\ 0)$	$(ct^-_A,\ 0,\ 0,\ 0)$
$B:$	$(ct_B,\ ut_B + l_x,\ l_y,\ 0)$	$(ct^-_A,\ l^-_x,\ l^-_y,\ 0)$
$AB^i:$	$(c\,\Delta t,\ u\Delta t + l_x,\ l_y,\ 0)_\Sigma$	$(0,\ l^-_x,\ l^-_y,\ 0)_{\Sigma^-}$

The boost relating Σ and Σ^- gives

$$\begin{aligned} l^-_x &= \gamma_u\ (l_x + u\,\Delta t - u\,\Delta t) = \gamma_u\,l_x, \\ l^-_y &= l_y, \end{aligned}$$

hence

$$m^- = \frac{l_y^-}{l_x^-} = \frac{l_y}{\gamma_u\, l_x} = \frac{m}{\gamma_u}.$$

Because Σ is arbitrary we write without any further calculations for Σ':

$$m^- = \frac{m'}{\gamma_{u'}}.$$

It follows:

$$\frac{m'}{\gamma_{u'}} = \frac{m}{\gamma_u}.$$

But we know that the γ-factors are related as follows[8]

$$\gamma_{u'} = \gamma_u\, \gamma_v \left(1 - \frac{uv}{c^2}\right). \tag{5.39}$$

Replacing $\gamma_{u'}$ we find

$$m' = \gamma_v \left(1 - \frac{uv}{c^2}\right) m.$$

Third Solution (With Relative Velocities)

We assume (chronometric) measurement of the length of the rod in Σ and have the following table of coordinates of the events $A,\ B$:

	Σ	Σ^-
A :	$(ct_A,\ ut_A,\ 0,\ 0)$	$(ct'_A,\ x'_A,\ 0,\ 0)$
B :	$(ct_A,\ ut_A + l_x,\ l_y,\ 0)$	$(ct'_B,\ x'_B,\ l'_y,\ 0)$
AB^i :	$(0,\ l_x,\ l_y,\ 0)$	$(c\,\Delta t',\ x'_B - x'_A,\ l'_y,\ 0)$

[8] Proof of (5.39).
One method to prove (5.39) is to replace each γ in terms of the velocity and make direct calculations. This is awkward. The simple and recommended proof is to use the four-velocity vector, whose zeroth component is γc, and apply the Lorentz transformation for the four-velocity. Let us see this proof. The four-velocity of the event A in the LCFs Σ and Σ' has components

$$(\gamma_u c,\ \gamma_u u,\ 0,\ 0)_\Sigma,\quad (\gamma_{u'} c,\ \gamma_{u'} u',\ 0,\ 0)_{\Sigma'}.$$

These two expressions are related with a boost as above. For the zeroth component we have

$$\gamma_{u'} c = \gamma_v \left(\gamma_u c - \frac{v}{c}\,\gamma_u\, u\right) = \gamma_v\, \gamma_u\, c \left(1 - \frac{uv}{c^2}\right).$$

Check the boost for the space components. The result will be used in the next solution.

The boost relating Σ, Σ' gives

$$x'_B - x'_A = \gamma\, l_x,$$
$$\Delta t' = -\gamma\, \frac{v}{c^2}\, l_x.$$

But in Σ'

$$x'_B - x'_A = u'\,\Delta t' + l'_x$$

because the events A, B are not simultaneous in Σ'. From the relativistic rule of composing three-velocities (see end of Footnote 8) we have

$$u' = \frac{u - v}{1 - \frac{uv}{c^2}}.$$

On replacing,

$$x'_B - x'_A = l'_x - \frac{u - v}{1 - \frac{uv}{c^2}}\,\frac{\gamma v}{c^2} l_x = \gamma\, l_x.$$

It follows:

$$l'_x = \frac{l_x}{\left(1 - \frac{uv}{c^2}\right)\gamma}.$$

Replacing in the relation $m' = l'_y / l'_x$ one obtains the previous result.

Example 15 Two events A, B are simultaneous in the LCF Σ and have a space distance (in Σ!) c km along the x-axis. Compute the β-factor of another LCF Σ' moving in the standard way along the x-axis so that the event A has a time distance (i) 1/10 s, (ii) 1 s, (iii) 10 s from event B.

Algebraic Solution

The two events are simultaneous in Σ. Hence we have the following table of coordinates:

	Σ	Σ'
$A:$	$(ct,\ x_A)$	$(ct'_A,\ x'_A)$
$B:$	$(ct,\ x_A + d)$	$(ct'_B,\ x'_B)$
$AB^i:$	$(0,\ d)_\Sigma$	$(-c\Delta t', x'_B - x'_A)_{\Sigma'}$

where d is the distance of the events in Σ. The boost relating Σ, Σ' gives

$$-c\Delta t' = \gamma\,(0 - \beta\, d) = -\gamma\beta d \Rightarrow \beta\gamma = \frac{c\Delta t'}{d}.$$

Squaring and making use of the (useful!) identity $\gamma^2\,\beta^2 = \gamma^2 - 1$ we find

$$\gamma^2 = 1 + \frac{c^2\,\Delta t'^2}{d^2}.$$

Replacing γ in terms of β and setting $d = c$ we get eventually

$$\beta = \frac{c\,\Delta t'}{\left(c^2\,\Delta t'^2 + d^2\right)^{1/2}} = \frac{\Delta t'}{\sqrt{1 + \Delta t'^2}}.$$

For the given values of $\Delta t'$ we find the following:

(i) For $\Delta t' = 1/10\,\mathrm{s}$, $\beta = 0.0995$ or $v = 0.0995c$ ($\approx$ 10% c).
(ii) For $\Delta t' = 1\,\mathrm{s}$, $\beta = 0.707$ or $v = 0.707c$ ($\approx$ 71% c).
(iii) For $\Delta t' = 10\,\mathrm{s}$, $\beta = 0.995$ or $v = 0.995c$ ($\approx$ 99.5% c).

We observe that the more the speed of Σ' increases the more the event A delays (in Σ'!) wrt the event B. Indeed, solving the last expression for $\Delta t'$, we find $\Delta t' = \beta\gamma = \sqrt{\gamma^2 - 1}$.

5.11.5 The Geometric Method

For the solution of problems involving relativistic spatial and temporal distances we can use spacetime diagrams. This methodology, which is suitable mainly for simple problems, is called the *geometric method* and it is realized in the following steps:

(1) Events:
Draw the world lines of the observers involved and identify the events which concern the problem.
(2) Quantities:
Construct in the spacetime diagram the given space and time distances.
(3) Requested:
Construct in the spacetime diagram the requested space and/or time distances.
(4) Solution:
Use relations (5.30) and (5.31), which give (as a rule) direct solution to the problem.

As is the case with the algebraic method, when one draws the various quantities in the spacetime diagram, he/she must be aware of who measures with the Newtonian method and who with the chronometric.

Example 16 Two spaceships 1, 2, each of length l, pass each other moving parallel and in opposite directions. If an observer at one end of spaceship 1 measures that time T is required for the spaceship 2 to pass in front of him, find the relative speed u of the spaceships.

Algebraic Solution

We consider the events:

Event A: The "nose" of spaceship 2 passes in front of the observer.
Event A: The "tail" of spaceship 2 passes in front of the observer.

The table of coordinates of the events A, B in the LCF of the two spaceships is

	Spaceship 1	Spaceship 2
A :	$(ct_A,\ x)$	$(ct'_A,\ x'_A)$
B :	$(ct_B,\ x)$	$(ct'_A + \frac{cl}{u},\ x'_A + l)$
AB^i :	$(cT,\ 0)_1$	$(\frac{l}{\beta},\ l)_2$

l/u is the time required for spaceship 2 to pass in front of the nose of spaceship 1. The boost relating the components of the position four-vector AB^i gives ($\beta = u/c$)

$$\beta\gamma = \frac{l}{cT} \Rightarrow \beta = \frac{l}{\sqrt{c^2T^2 + l^2}}.$$

Geometric Solution

We consider the spacetime diagram of Fig. 5.15:

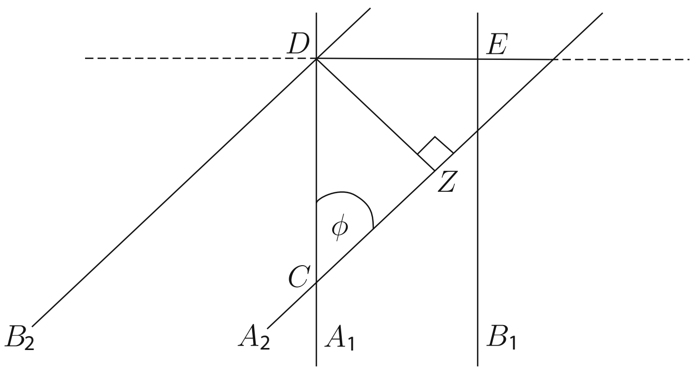


Fig. 5.15 Geometric measurement of the length of the spaceship

Description of Diagram

Events:

$A_1,\ B_1$: World lines of observers at the end points of spaceship 1.

$A_2,\ B_2$: World lines of observers at the end points of spaceship 2.

C: The nose of spaceship 2 passes in front of the nose A_1 of the spaceship 1.

D: The tail of spaceship 2 passes in front of the nose A_1 of the spaceship 1.

DZ: Proper space of observer B_2 at the event D.

Data:

CD: Time distance required for spaceship 2 to pass in front of the tip A_1 of spaceship 1, $(CD) = Tc$.

DE: Proper length of spaceship 1, $(DE) = l$.

DZ: Proper length of spaceship 2, $(DZ) = l$.

Requested:

Angle ϕ.

From the triangle CZD we have

$$(DC) = (DZ)/\sinh\phi = l/\beta\gamma \Rightarrow \beta\gamma = \frac{l}{cT} \Rightarrow \beta = \frac{l}{\sqrt{c^2T^2 + l^2}}.$$

Example 17 An astronaut on a cosmic platform sees a spaceship moving toward him with constant speed 0.6 c. Checking in the database of the known spaceships he finds that the spaceship could be the Enterprize. According to the file in the database the length of the Enterprize is 120 m and emits recognition signals every 80 s. The astronaut measures the length of the spaceship and finds 96 m. He measures the period of the emitted light signals and finds 100 s. Is it possible to conclude from these measurements that the approaching spaceship is the Enterprize?

Algebraic Solution

We consider the events:

A: Observation of one end of the unknown spaceship

B: Observation of other end of the unknown spaceship

Consider Σ to be the platform and let Σ' be the proper frame of the spaceship. For the (chronometric) measurement of the length of the approaching spaceship at the time t of Σ we have the following table of coordinates of the events $A,\ B$: The boost relating the components of the four-vector AB^i in Σ and Σ' gives

	Σ	Σ'
A :	$(ct,\ x_A,\ 0,\ 0)$	$(ct'_A,\ x'_A,\ 0,\ 0)$
B :	$(ct,\ x_A + \Delta x,\ 0,\ 0)$	$(ct'_A + c\Delta t',\ x'_A,\ 0,\ 0)$
AB^i :	$(0,\ \Delta x,\ 0,\ 0)_\Sigma$	$(c\,\Delta t',\ \Delta x',\ 0)_{\Sigma'}$

$$\Delta x' = \gamma\,(\Delta x - \beta \cdot 0) = \gamma\,\Delta x,$$

where $\gamma = (1 - 0.6)^{-1/2} = 1.25$. Replacing we find $\Delta x' = 1.25 \times 96\,\mathrm{m} = 120\,\mathrm{m}$, which coincides with the length of Enterprize given in the database.

We continue with the identification of the emission signal. Now the events A, B are the emission of successive light signals (from the same position of the spaceship). Let $\Delta t'$ be the time distance of the light signals in Σ' and Δt the corresponding difference in Σ. We have the following table of coordinates for the events A, B :

	Σ	Σ'
A :	$(ct_A,\ x,\ 0,\ 0)$	$(ct'_A,\ x'_A,\ 0,\ 0)$
B :	$(ct_A + c\,\Delta t,\ x + \Delta x,\ 0,\ 0)$	$(ct'_A + c\Delta t',\ x'_A,\ 0,\ 0)$
AB^i :	$(c\Delta t,\ \Delta x,\ 0,\ 0)_\Sigma$	$(c\,\Delta t',\ 0,\ 0,\ 0)_{\Sigma'}$

The boost relating the components of AB^i in Σ, Σ' gives

$$c\,\Delta t\ =\ \gamma\,(c\,\Delta t' + \beta\,0) = \gamma c \Delta t' = 1.25 \times 80\,\mathrm{s} = 100\,\mathrm{s}$$

and coincides with the period of signals given in the database. The astronaut concludes that as far as he can say the approaching spaceship is the Enterprize.

Geometric Solution

Consider the spacetime diagram of Fig. 5.16:

Description of Diagram

(a) Length measurement:

Events:

Σ: World line of cosmic platform.

$A_1,\ B_1$: World lines of the end points of the unknown spaceship.

$A,\ B$: Events of measurement of the space length of the unknown spaceship at the moment t_B of the platform.

Σ_A: Proper space of the platform at the event A.

Data:

(BE) : (Proper) length of Enterprize, $(BE) = 120\,\mathrm{m}$.

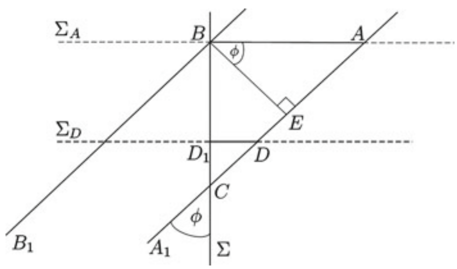


Fig. 5.16 Geometric measurement of spatial and temporal distance

(BA): Measured length of unknown spaceship, $(BA) = 96\,\text{m}$.
ϕ : $\tanh\phi = \beta = 0.6, \quad \cosh\phi = \gamma = 1.25$.
Requested:
Compatibility of the data.

From triangle BEA we have

$$(BE) = (BA)\cosh\phi = (BA)\gamma = 96 \times 1.25\,\text{m} = 120\,\text{m},$$

therefore the space lengths are compatible.

(b) Measurement of the time distance (period) of the recognition signal.
We consider again the spacetime diagram of Fig. 5.16:
Description of Diagram
Events:
Σ: World line of cosmic platform.
C, D: Events of emission of successive light signals.
C, D_1: Events of observation of successive light signals from the platform.
Σ_D: Proper space of the space platform Σ at the event D.
Data:
$(CD) = 80\,\text{s}$ (proper) time distance of successive light signals emitted from Enterprize.
$(CD_1) = 100\,s$, measured time distance of successive light signals in the space platform Σ.
ϕ : $\tanh\phi = \beta = 0.6$, $\cosh\phi = \gamma = 1.25$.
Requested:
Compatibility of the data.

From triangle CDD_1 we have

$$(CD_1) = (CD)\cosh\phi = \gamma(CD) = 1.25 \times 80\,\text{s} = 100\,\text{s},$$

which is compatible with the data of the database.

Chapter 6
Relativistic Kinematics

6.1 Introduction

In the previous chapters we developed the basic concepts of relativistic motion, that is, the time and spatial distance. The methodology we followed was the definition of these concepts, in agreement with the corresponding concepts of Newtonian theory, which is already a "relativistic" theory of motion.

In the present chapter we continue with the development of the kinematics of Special Relativity. We introduce the basic four-vectors of four-velocity and four-acceleration. We define the concept of relative four-vector and finally show how the Lorentz transformation defines the law of composition of three-velocities. The four-acceleration is discussed in a separate chapter, due to the "peculiarities" and its crucial role in both kinematics and dynamics of Special Relativity.

6.2 Relativistic Mass Point

Special Relativity is a geometric theory of physics. Therefore all its concepts must be defined geometrically before they are studied. The basic Physical object of kinematics is the "moving point," which in Special Relativity is described by a world line, that is, a spacetime curve whose position four-vector and the tangent four-vector at all its points is timelike or null. The reason we consider two types of curves is because in Special Relativity there are two types of particles with different kinematic (and dynamic) properties: the particles with mass and the photons. The particles with mass are called *Relativistic mass points* (ReMaP) and geometrize with a timelike world line, whereas the photons we correspond to null world lines. In this section we develop the kinematics of ReMaPs. The kinematics of photons are studied in another place.

A timelike world line can be affinely parameterized.[1] The affine parameter is called the *proper time* τ of the ReMaP.

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[1] Affine parameterization means that the length of the tangent vector $\frac{dx^i}{d\tau}$ has constant length along the world line. Proper time is the affine parameter (modulo a linear transformation) for which this constant equals $-c^2$.

M. Tsamparlis, *Special Relativity*, DOI 10.1007/978-3-642-03837-2_6,

In order to do physics we must associate with ReMaP relativistic Physical quantities. We assume that with each ReMaP there are associated two relativistic Physical quantities: the position four-vector x^i and the proper time τ. These quantities are sufficient to the kinematics of the ReMaP. Concerning the dynamics, one has to associate additional Physical quantities to a ReMaP.

Using the Physical quantities x^i, τ we define new ones along the world line, using the two basic rules of constructing new tensors from given ones (see Proposition 2): multiplication with an invariant and differentiation wrt an invariant. These new tensors are potential relativistic Physical quantities and become relativistic Physical quantities, either by means of a principle or by identifying with a Newtonian Physical quantity in a specific frame. From x^i, τ we define five new Lorentz tensors (two four-vectors and three invariants):

$$u^i = \frac{dx^i}{d\tau}, \tag{6.1}$$

$$a^i = \frac{d^2x^i}{d\tau^2}, \tag{6.2}$$

$$u^i u_i,\ a_i u^i,\ a^i a_i. \tag{6.3}$$

In order to study the new four-vectors we use special coordinate systems in which these vectors attain their reduced form. Because $x^i(\tau)$ is a timelike curve it admits a proper frame Σ^+ in which $x^i = \binom{c\tau}{\mathbf{0}}_{\Sigma^+}$. In this frame we compute

$$u^i = \frac{dx^i}{d\tau} = \begin{pmatrix} c \\ \mathbf{0} \end{pmatrix}_{\Sigma^+}$$

as well as

$$u^i u_i = -c^2.$$

From these two relations we infer the following:

(1) The timelike vectors x^i, u^i have common proper frame.
(2) The length of the four-vector u^i is the universal constant c.
(3) The four-vector u^i is determined in its proper frame by c only, hence it is *common* for all particles with mass.

Because u^i is determined only in terms of c, which (in Special Relativity) is a Physical quantity, we infer that u^i is a relativistic Physical quantity. This new relativistic quantity is called the *four-velocity* of the ReMaP at the point $x^i(\tau)$.

We continue with the rest of the quantities defined by (6.2) and (6.3). Relation $u^i u_i = -c^2 \Longrightarrow u^i a_i = 0$, which implies the following:

(1) The four-vector a^i is normal to u^i, therefore it is a spacelike four-vector.
(2) The invariant $u^i a_i$ is trivial.
(3) In the proper frame Σ^+ of the four-vectors x^i, u^i the four-vector a^i has components (why?)

$$a^i = \begin{pmatrix} 0 \\ \mathbf{a}^+ \end{pmatrix}_{\Sigma^+}. \tag{6.4}$$

We note that Σ^+ is the characteristic frame of the spacelike four-vector a^i. If we identify in Σ^+ the three-vector $\mathbf{a}^+$ with the Newtonian acceleration of the mass point in Σ^+, then the four-vector a^i becomes a relativistic Physical quantity. We call this new quantity *four-acceleration* of the world line at the point $x^i(\tau)$. Geometrically the four-acceleration accounts for the curvature of the world line at each of its points, therefore the vanishing of the four-acceleration (equivalently of the three-vector $\mathbf{a}^+$) means that the world line is a straight line in spacetime. This result is consistent with the requirement that the world line of a Relativistic inertial observer (RIO) is a straight line. Furthermore we see that with this correspondence the Newtonian inertial observers correspond to Relativistic inertial observers (RIO) which is important, because all observations are made by Newtonian observers (i.e., us).

Based on the above analysis, we give the following important *definition*.

Definition 12 Relativistic mass point (ReMaP) with timelike world line $A(\tau)$ is the set of all relativistic Physical quantities, which are defined at every point of $A(\tau)$ and have common proper frame or characteristic frame, depending on whether they are timelike or spacelike four-vectors.

According to this definition, x^i is the position four-vector, u^i is the four-velocity, and a^i is the four-acceleration of the ReMaP, whose world line is the curve $x^i(\tau)$.

It is to be noted that the above apply to particles with mass, not to photons (or particles with speed c), which are described by null position four-vectors. The position vectors of photons do not have a special type of frame in which they obtain a reduced form. The kinematics of photons is studied by means of other four-vectors.

Example 18 Calculate the components of the four-velocity u^i in the LCF, in which the particle has three-velocity $\mathbf{v}$.

Solution

Consider a ReMaP P with world line $x^i(\tau)$ and let Σ be an LCF, in which the position four-vector $x^i = \begin{pmatrix} ct \\ \mathbf{r} \end{pmatrix}_{\Sigma}$. In the proper frame Σ^+ of P we have $x^i = \begin{pmatrix} c\tau \\ \mathbf{0} \end{pmatrix}_{\Sigma^+}$. The two expressions are related with a Lorentz transformation (not necessarily a boost), hence

$$ct = \gamma c\tau \Rightarrow t = \gamma\tau. \tag{6.5}$$

The components of the four-velocity u^i of P in Σ are

$$u^i = \frac{dx^i}{d\tau} = \frac{dt}{d\tau}\frac{dx^i}{dt} = \gamma \begin{pmatrix} c \\ \mathbf{u} \end{pmatrix}_{\Sigma}, \tag{6.6}$$

where $\mathbf{u}$ is the three-velocity of the ReMaP in Σ. We find the same result if we apply the general Lorentz transformation (1.51), (1.52) to the four-vector $u^i = \begin{pmatrix} c \\ \mathbf{0} \end{pmatrix}_{\Sigma^+}$.

Exercise 13 Using the above expression for the coordinates of u^i in Σ, show that in Σ $u^i u_i = -c^2$, which reaffirms that the quantity $u^i u_i$ is an invariant.

In the following example we show how the four-velocity is computed for certain motions.

Example 19 The LCF Σ' is moving in the standard configuration with speed u wrt the LCF Σ. A particle moves in the x, y plane of Σ with constant speed v around a circular orbit of radius r centered at the origin of Σ. Calculate the four-velocity of the particle in Σ'. Calculate the γ'-factor of the particle in Σ'. Find the relation between the time t in Σ with the proper time τ of the particle. Write the equation of the trajectory of the particle in Σ in terms of the proper time τ.
First solution

The equation of the orbit of the particle in Σ is

$$x = r\cos\omega t = r\cos\left(\frac{vt}{r}\right), \quad y = r\sin\omega t = r\sin\left(\frac{vt}{r}\right).$$

Therefore the three-velocity of the particle in Σ is

$$v_x = -v\sin\left(\frac{vt}{r}\right), \quad v_y = v\cos\left(\frac{vt}{r}\right)$$

and the four-velocity

$$v_i = (\gamma c, \gamma v_x, \gamma v_y, 0)_\Sigma = \left(\gamma c, -\gamma v\sin\left(\frac{vt}{r}\right), \gamma v\cos\left(\frac{vt}{r}\right), 0\right)_\Sigma,$$

where $\gamma = 1/\sqrt{1-\frac{v^2}{c^2}}$.

In order to compute the four-velocity of the particle in Σ', we consider the boost relating Σ, Σ'. For the zeroth component we have

$$\gamma' c = \gamma_u\left(\gamma c - \frac{u}{c}\gamma v_x\right) = \gamma_u\gamma c\left(1 - \frac{u v_x}{c^2}\right) \Rightarrow \gamma' = \gamma_u\gamma Q, \tag{6.7}$$

where $Q \equiv (1 - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2})$ and $\gamma_u = 1/\sqrt{1-\frac{u^2}{c^2}}$. For the spatial components we compute

$$v_{x'} = \frac{v_x - u}{Q}, \tag{6.8}$$

$$v_{y'} = \frac{v_y}{\gamma_u Q}. \tag{6.9}$$

Hence the four-velocity of the particle in Σ' is

$$v^i_{\Sigma'} = \begin{pmatrix} \gamma' c \\ \gamma' v_{x'} \\ \gamma' v_{y'} \\ 0 \end{pmatrix}_{\Sigma'} = \begin{pmatrix} \gamma_u \gamma Q c \\ \gamma_u \gamma (v_x - u) \\ \gamma v_y \\ 0 \end{pmatrix}_{\Sigma'} = \begin{pmatrix} \gamma_u \gamma Q c \\ -\gamma_u \gamma (v \sin \frac{vt}{r} + u) \\ \gamma v \cos \frac{vt}{r} \\ 0 \end{pmatrix}_{\Sigma'} .$$

The proper time of the particle in Σ is $\tau = \frac{t}{\gamma}$, where t is the time in Σ. Therefore the orbit of the particle in Σ in terms of the proper time is

$$x = r \cos\left(\frac{v\gamma\tau}{r}\right), \qquad y = r \sin\left(\frac{v\gamma\tau}{r}\right).$$

[What do you have to say for the four-vector $(\frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, 0)_\Sigma$?]
Second solution

First we transform the orbit of the particle in Σ' and then we compute the four-velocity using derivation wrt time. The boost relating Σ, Σ' gives

$$ct' = \gamma_u \left(ct - \frac{u}{c}x\right), \; x' = \gamma_u (x - ut), \; y' = y.$$

From the first equation it follows that

$$dt' = \gamma_u \left(dt - \frac{u}{c^2} v_x dt\right) = \gamma_u Q dt \Rightarrow \frac{dt}{dt'} = \frac{1}{\gamma_u Q},$$

hence

$$v'_x = \frac{dx'}{dt'} = \frac{dx'}{dt}\frac{dt}{dt'} = \frac{v_x - u}{1 - \frac{uv_x}{c^2}} = \frac{-1}{Q}\left[v \cos\left(\frac{vt}{r}\right) + u\right],$$

$$v'_y = \frac{dy'}{dt'} = \frac{dy'}{dt}\frac{dt}{dt'} = \frac{v_y}{\gamma_u \left(1 - \frac{uv_x}{c^2}\right)} = \frac{1}{\gamma_u Q}\left[v \sin\left(\frac{vt}{r}\right)\right].$$

The rest of the solution is the same.

6.3 Relativistic Composition of Three-Vectors

A basic result of Newtonian Physics, with numerous applications, is the composition rule for Newtonian three-vectors. This rule relates the observations of *two* Newtonian inertial observers (NIO) and concerns *one* Newtonian Physical quantity. Let us recall how this rule is defined.

Consider two NIOs N_1 and N_2 and the Newtonian mass point P, which wrt to N_1, N_2 has, respectively,

- Position vector : $\mathbf{r}_{P_1}, \mathbf{r}_{P_2}$
- Velocity : $\mathbf{u}_{P_1}, \mathbf{u}_{P_2}$
- Acceleration : $\mathbf{a}_{P_1}, \mathbf{a}_{P_2}$

Then, if the NIO N_2 wrt the NIO N_1 has

- Position vector : $\mathbf{r}_{N_{21}}$
- Velocity : $\mathbf{u}_{N_{21}}$
- Acceleration: $\mathbf{a}_{N_{21}}$

Newtonian kinematics postulates the relations

$$\begin{aligned} \mathbf{r}_{P_2} &= \mathbf{r}_{P_1} + \mathbf{r}_{N_{21}}, \\ \mathbf{u}_{P_2} &= \mathbf{u}_{P_1} + \mathbf{u}_{N_{21}}, \\ \mathbf{a}_{P_2} &= \mathbf{a}_{P_1} + \mathbf{a}_{N_{21}}, \end{aligned} \tag{6.10}$$

which are called the *Newtonian composition rule* for the Newtonian position vector, velocity, and acceleration, respectively. This rule applies to any other Newtonian three-vector associated with the Newtonian point mass.

We note that relations (6.10) express the Galileo transformation for the position, velocity, and acceleration three-vectors, respectively. This observation is important, because it shows that the Newtonian composition rule for three-vectors is fundamental to geometry (it expresses the linearity of the Newtonian space) and to physics, because it expresses the Galileo Principle of Relativity. This explains why the breaking of this rule by the velocity of light necessitated the introduction of Special Relativity.

Using the above observation, we extend the three-vector (not anymore Newtonian three-vector!) composition rule in Special Relativity by a direct application of the Lorentz transformation. Working in a similar manner we consider *two* LCFs Σ_1 and Σ_2 and *one* ReMaP P, which relatively to Σ_1, Σ_2 has, respectively,

(1) Position four-vector : $x^i_{P_1}, x^i_{P_2}$
(2) Four-velocity : $u^i_{P_1}, u^i_{P_2}$
(3) Four-acceleration: $a^i_{P_1}, a^i_{P_2}$

If $L(1,2)$ is the Lorentz transformation relating Σ_1, Σ_2 then the following relations apply:

$$\begin{aligned} x^i_{P_2} &= L(1,2)x^i_{P_1}, \\ u^i_{P_2} &= L(1,2)u^i_{P_1}, \\ a^i_{P_2} &= L(1,2)a^i_{P_1}. \end{aligned} \tag{6.11}$$

These equations induce a relation among the three-vectors of the spatial parts of the respective four-vectors. We call this relation the *relativistic composition rule* for three-vectors. This rule is different from the Newtonian composition rule, because the first expresses the linearity of space and the Galileo Principle of Relativity, whereas the latter the linearity of Minkowski space and the Einstein Principle of Relativity.

Obviously, the detailed expression of the relativistic composition rule will depend on the four-vector concerned. Most useful (and important) is the relativistic composition rule for three-velocities, which we derive in the next example.

Example 20 a. Consider two LCFs Σ, Σ' with parallel axes and relative velocity $\mathbf{u}$. Let a ReMaP be P which in Σ, Σ' has velocities $\mathbf{v}$ and $\mathbf{v}'$, respectively. Prove that

$$\mathbf{v}' = \frac{1}{\gamma_u Q}\left\{\mathbf{v} + \left[\frac{\mathbf{u}\cdot\mathbf{v}}{u^2}(\gamma_u - 1) - \gamma_u\right]\mathbf{u}\right\}, \tag{6.12}$$

where $\gamma_{v'} = \gamma_u \gamma_v Q$ and $Q = 1 - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}$.

Replace $\frac{1}{u^2} = \frac{1}{c^2}\frac{1}{1-\frac{1}{\gamma_u^2}}$ to find the equivalent form:

$$\mathbf{v}' = \frac{1}{\gamma_u Q}\left\{\mathbf{v} + \left[\frac{\gamma_u}{1+\gamma_u}\frac{\mathbf{u}\cdot\mathbf{v}}{c^2} - 1\right]\gamma_u \mathbf{u}\right\}. \tag{6.13}$$

In the special case that the velocity $\mathbf{u}$ is parallel to the x-axis so that Σ, Σ' are related with a boost, show that (6.12) reduces to ($Q = 1 - \frac{uv_x}{c^2}$)

$$\begin{aligned} v_{x'} &= \frac{v_x - u}{Q},\\ v_{y'} &= \frac{v_y}{\gamma_u Q},\\ v_{z'} &= \frac{v_z}{\gamma_u Q}. \end{aligned} \tag{6.14}$$

b. Using the results of (a) prove that if the velocity of Σ' wrt Σ is $\mathbf{v}$, then the velocity of Σ wrt Σ' is $-\mathbf{v}$.

c. Consider three LCFs Σ_1, Σ_2, and Σ_3 such that Σ_2 is moving in the standard configuration wrt the LCF Σ_1 with speed u along the common axis x_1, x_2 and Σ_3 moves wrt Σ_2 in the standard configuration with speed v along the common axis y_2, y_3. Let $\mathbf{v}_{31}$ be the three-velocity of Σ_3 wrt Σ_1 and $\mathbf{v}_{13}$ the three-velocity of Σ_1 wrt Σ_3. Show that the angle θ_{31} of $\mathbf{v}_{31}$ with the x_1-axis is given by $\tan(\theta_{31}) = \frac{v}{u\gamma(u)}$ and the angle θ_{13} of $\mathbf{v}_{13}$ with the axis x_3 is given by $\tan(\theta_{13}) = \frac{v\gamma(v)}{u}$. Compute the difference $\Delta\theta = \theta_{13} - \theta_{31}$ in terms of the speeds u, v and discuss the result.

Solution

The four-velocity v^i of P in Σ, Σ', respectively, is

$$v^i = (\gamma_v c, \gamma_v \mathbf{v})^t_\Sigma = (\gamma_{v'} c, \gamma_{v'} \mathbf{v}')^t_{\Sigma'}.$$

These coordinates are related by the Lorentz transformations (1.51) and (1.52) relating Σ, Σ'. For the zeroth component we find ($Q = 1 - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}$)

$$\gamma_{v'} c = \gamma_u \left(\gamma_v c - \frac{\mathbf{u}\cdot\gamma_v\mathbf{v}}{c}\right) \Longrightarrow \frac{\gamma_v}{\gamma_{v'}} = \frac{1}{\gamma_u\left(1 - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right)} = \frac{1}{\gamma_u Q}. \tag{6.15}$$

For the spatial components we have

$$\begin{aligned}
\gamma_{\mathbf{v}'}\mathbf{v}' &= \gamma_{\mathbf{v}}\mathbf{v} + \left[(\gamma_{\mathbf{u}} - 1)\frac{\mathbf{u}\cdot\gamma_{\mathbf{v}}\mathbf{v}}{u^2} - \frac{\gamma_{\mathbf{u}}}{c}\gamma_{\mathbf{v}} c\right]\mathbf{u} \\
&= \gamma_{\mathbf{v}}\left\{\mathbf{v} + \left[(\gamma_{\mathbf{u}} - 1)\frac{\mathbf{u}\cdot\mathbf{v}}{u^2} - \gamma_{\mathbf{u}}\right]\mathbf{u}\right\}.
\end{aligned}$$

It follows:

$$\mathbf{v}' = \frac{\gamma_v}{\gamma_{v'}}\left\{\mathbf{v} + \left[(\gamma_u - 1)\frac{\mathbf{u}\cdot\mathbf{v}}{u^2} - \gamma_u\right]\mathbf{u}\right\}. \tag{6.16}$$

Replacing $\frac{\gamma_v}{\gamma_{v'}}$ from (6.15) we find the required result.
If in (6.16) we consider $\mathbf{u} = u\mathbf{i}$ we obtain (6.14).

Second solution

We consider first the boost for the position vector between Σ and Σ':

$$\begin{aligned}
t' &= \gamma_u\left(t - \frac{ux}{c^2}\right), \\
x' &= \gamma_u(x - ut), \\
y' &= y, \\
z' &= z.
\end{aligned}$$

It follows:

$$v'_x = \frac{dx'}{dt'} = \frac{dx'}{dt}\frac{dt}{dt'} = \gamma_u(v_x - u)\frac{dt}{dt'}.$$

From the transformation of the zeroth component we have

$$c\,dt' = \gamma_u\left(c\,dt - \frac{\mathbf{u}\cdot\mathbf{v}dt}{c^2}\right) \Rightarrow \frac{dt}{dt'} = \frac{1}{\gamma_u\left(1 - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right)} = \frac{1}{\gamma_u Q}. \tag{6.17}$$

Replacing we find

$$v'_x = \frac{v_x - u}{Q}.$$

Similarly, we calculate

$$v'_y = \frac{dy'}{dt'} = \frac{dy'}{dt}\frac{dt}{dt'} = \frac{v_y}{\gamma_u Q},$$
$$v'_z = \frac{v_z}{\gamma_u Q}.$$

In the general case we have

$$\mathbf{v}' = \frac{d\mathbf{r}'}{dt'} = \frac{d\mathbf{r}'}{dt}\frac{dt}{dt'} = \frac{dt}{dt'}\frac{d}{dt}\left\{\mathbf{r} + \left[(\gamma_u - 1)\frac{\mathbf{u}\cdot\mathbf{r}}{u^2} - \gamma_u t\right]\mathbf{u}\right\}$$
$$= \frac{dt}{dt'}\left\{\mathbf{v} + \left[(\gamma_u - 1)\frac{\mathbf{u}\cdot\mathbf{v}}{u^2} - \gamma_u\right]\mathbf{u}\right\}.$$

Replacing $\frac{dt}{dt'}$ from (6.17) (which holds for general motion with parallel axes), we obtain (6.12).

b. The velocity $\mathbf{v}$ of Σ wrt itself equals $\mathbf{0}$. Replacing in (6.16) $\mathbf{v} = \mathbf{0}$ we find $\mathbf{v}' = -\mathbf{u}$.

c. We consider Σ_3 to be the proper frame of P and apply (6.12). Let $\mathbf{v}_{31}$ be the relative velocity of Σ_3 wrt Σ_1 and $\mathbf{v}_{32} = (0, v, 0)$ wrt Σ_2. The velocity of Σ_1 wrt Σ_2 is $\mathbf{v}_{12} = (-u, 0, 0)$. Relation (6.12) gives

$$\mathbf{v}_{31} = \frac{dt_2}{dt_1}\left\{\mathbf{v}_{32} + \left[(\gamma_{12} - 1)\frac{\mathbf{v}_{12}\cdot\mathbf{v}_{32}}{v_{12}^2} - \gamma_{12}\right]\mathbf{v}_{12}\right\} \Rightarrow \mathbf{v}_{31} = \left(u, \frac{v}{\gamma_u}, 0\right),$$

where we have used that

$$\frac{dt_2}{dt_1} = \frac{1}{\gamma_{21}\left(1 - \frac{\mathbf{v}_{32}\cdot\mathbf{v}_{12}}{c^2}\right)} = \frac{1}{\gamma_u}.$$

Similarly, for $\mathbf{v}_{13}$ we have

$$\mathbf{v}_{13} = \frac{dt_2}{dt_3}\left\{\mathbf{v}_{12} + \left[(\gamma_{32} - 1)\frac{\mathbf{v}_{12}\cdot\mathbf{v}_{32}}{v_{32}^2} - \gamma_{32}\right]\mathbf{v}_{32}\right\} \Rightarrow \mathbf{v}_{13} = \left(-\frac{u}{\gamma_v}, -v, 0\right).$$

We note that the two three-velocities lie in the planes x_1y_1 and x_3y_3. Therefore, in order to give their direction, it is enough to give the angle they make with the axes x_1 and x_3, respectively. The angle of $\mathbf{v}_{31}$ with the x_1-axis is

$$\tan\theta_{31} = \frac{v}{u\gamma_u}$$

and similarly that of $\mathbf{v}_{13}$ with the x_3-axis is

$$\tan\theta_{13} = \frac{v\gamma_v}{u}.$$

In Newtonian Physics the following holds (because $\gamma_u = \gamma_v = 1$):

$$\theta_{31} = \theta_{13}.$$

In relativistic kinematics this does not hold and the two relative velocities are not collinear. They have a deviation $\Delta\theta = \theta_{31} - \theta_{13}$ as follows:

$$\tan\Delta\theta = \tan(\theta_{31} - \theta_{13}) = \frac{\tan\theta_{31} - \tan\theta_{13}}{1 + \tan\theta_{13}\tan\theta_{31}} = \frac{1 - \gamma_u\gamma_v}{u^2\gamma_u + v^2\gamma_v} uv.$$

Replacing the lengths $u^2,\ v^2$ in terms of the corresponding γ, we find the result

$$\tan\Delta\theta = -\frac{\gamma_u\gamma_v}{\gamma_u + \gamma_v}\frac{uv}{c^2}.$$

We note that in general $\Delta\theta \neq 0$. Kinematically this means that the Lorentz transformation does not preserve the three-directions. This does not bother us because three-directions are not covariant (as in the case of Newtonian Physics). Assuming speeds $v_i \ll c$, the $\gamma_i = 1 + O(\beta_u^2, \beta_v^2) \quad (i = u, v)$ and $\Delta\theta = -\tan^{-1}(\beta_u\beta_v/2) = -\beta_u\beta_v/2 + O(\beta_u^2, \beta_v^2)$. This result is related to the Thomas precession, which will be considered in Sect. 6.6.

In Sect. 4.5.2 we classified the relativistic particles in terms of the type of their world lines and their speed. More specifically, the particles with timelike world line and speed $u < c$ are named bradyons, the particles with null world line and speed $u = c$ luxons, and the (hypothetical) "particles" with spacelike world line and speed $u > c$ tachyons. Because the speed is not a relativistic quantity, we have to prove that this classification is covariant. That is, if a photon is a photon for one LCF then it must be a photon for all LCFs and not change to a mass particle for some LCFs. However, the frequency of the photon can change.

Example 21 A ReMaP P has three-velocity $\mathbf{u}$ and $\mathbf{u}'$ in the LCFs Σ and Σ', respectively. Assuming that Σ' is related to Σ with a boost along the common x, x'-axes with speed v, show that

(a) If $v < c$ then $u' < c \Rightarrow u < c$ and $u' > c \Rightarrow u > c$.
(b) Prove that $\frac{\gamma(u')}{\gamma(u)} = \gamma(v)\left(1 - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right)$.

Solution

(a) From the invariance of the length of the position vector we have

$$-c^2dt^2 + dx^2 + dy^2 + dz^2 = -c^2(dt')^2 + (dx')^2 + (dy')^2 + (dz')^2 \Rightarrow$$
$$dt^2(c^2 - u^2) = dt'^2(c^2 - u'^2). \tag{6.18}$$

The boost relating Σ and Σ' gives

$$dt' = \gamma(v)\left(dt - \frac{\beta}{c}dx\right) \Rightarrow dt' = \gamma(v)dt\left(1 - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right). \tag{6.19}$$

Replacing in (6.18) we find

$$(c^2 - u^2) = \gamma^2(v)\left(1 - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right)^2 (c^2 - u'^2) = \frac{1}{\alpha}(c^2 - u'^2),$$

where

$$\alpha \equiv \frac{1-\beta^2}{(1-\frac{\mathbf{u}\cdot\mathbf{v}}{c^2})^2} > 0.$$

Because α is always positive, we have that $u' < c \Rightarrow u < c$ and $u' > c \Rightarrow u > c$.

(b) Using (6.18) and (6.19) we find

$$\frac{\gamma^2(u')}{\gamma^2(u)} = \left(\frac{c^2-u^2}{c^2-u'^2}\right)\frac{dt'}{dt} = \gamma^2(v)(1 - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2})^2 \Rightarrow$$
$$\frac{\gamma(u')}{\gamma(u)} = \gamma(v)\left(1 - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right). \tag{6.20}$$

It is interesting to plot the speed of P in case P moves along the common x, x'-axes. If v_1, v_2 is the speed of P in the LCFs Σ_1, Σ_2, respectively, we have

$$v_2 = \frac{v_1 - u}{1 - \frac{u\cdot v_1}{c^2}}. \tag{6.21}$$

The plot of v_1 as a function of u for various values of v_2 is shown in Fig. 6.1.

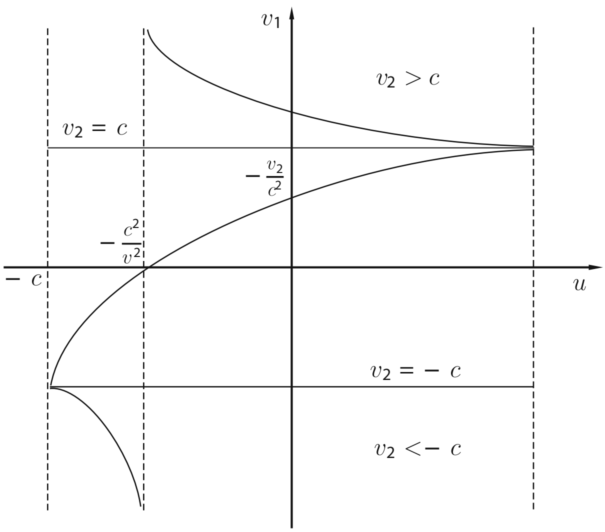

Fig. 6.1 Relativistic composition rule for three-velocities

We note that the plane (v_1, u) is divided into five distinct regions as follows:

$$v_1 < -c, \qquad v_1 = -c, \qquad -c < v_1 < c, \qquad v_1 = c, \qquad v_1 > c.$$

In each part we have values for the parameter v_2 so that v_1 and v_2 are always in the same region. For example, if the value of v_1 is in the region $-c < v_1 < c$, then in the same region is the value of v_2. This result shows that the classification of the particles according to their speed is covariant. More specifically, bradyons are always in the region $(-c, c)$, photons in the region $v_1 = v_2 = \pm c$, and "tachyons" in the region $(c, +\infty) \cup (-\infty, -c)$.

6.4 Relative Four-Vectors

Another important relation among velocity, acceleration, and Newtonian vectors in general is the relative velocity, acceleration, and vector, respectively. The concept of relative vector involves *one* observer and *two* mass points, contrary to the idea of the composition of vectors, which involves two observers and one mass point. Consequently, the concept of relative vector cannot involve the Galileo transformation, which also concerns two Newtonian observers. Let us recall the definition of relative vectors in Newtonian Physics.

Consider an NIO N wrt which two Newtonian mass points P, Q have, respectively,

- Position vectors: $\mathbf{r}_P, \mathbf{r}_Q$
- Velocities: $\mathbf{v}_P, \mathbf{v}_Q$
- Accelerations: $\mathbf{a}_P, \mathbf{a}_Q$

Then the relative position vector $\mathbf{PQ}$ of the Newtonian mass point Q wrt the Newtonian mass point P with reference to the observer N is defined to be the vector

$$\mathbf{PQ} = \mathbf{r}_{PQ} = \mathbf{r}_Q - \mathbf{r}_P.$$

For the velocity we have, respectively,

$$\mathbf{v}_{PQ} = \mathbf{v}_Q - \mathbf{v}_P \tag{6.22}$$

and for the acceleration

$$\mathbf{a}_{PQ} = \mathbf{a}_Q - \mathbf{a}_P.$$

This definition expresses the linearity of the space of Newtonian Physics and it is foreign to the rule of composition of Newtonian vectors, which is a consequence of the Galileo transformation (and involves two observers). The fact that the two definitions seem to "coincide" is due to the simplicity of the geometric structure of the linear space R^3 (something similar occurs with the real numbers, which practically share all properties).

The concept of relative vector is taken directly over to Special Relativity as follows. Consider an LCF Σ wrt which two relativistic mass points P, Q have, respectively,

(1) Position four-vector: x^i_P, x^i_Q
(2) Four-velocity : v^i_P, v^i_Q
(3) Four-acceleration : a^i_P, a^i_Q

Then the *relative position* four-vector of Q wrt P with reference to the observer Σ is defined to be the four-vector

$$x^i_{PQ} = x^i_Q - x^i_P.$$

In a similar manner, the relative four-velocity v^i_{PQ} and the relative four-acceleration a^i_{PQ} of Q wrt P with reference to the observer Σ are defined to be the four-vectors

$$v^i_{PQ} = v^i_Q - v^i_P, \tag{6.23}$$

$$a^i_{PQ} = a^i_Q - a^i_P. \tag{6.24}$$

The definition of the relative four-vector in Special Relativity expresses the linearity of the Minkowski space and, as is the case with Newtonian Physics, it is different from the composition rule of four-vectors.[2]

The most important of the relative vectors is the relative velocity, which is used in the study of colliding beams, in order to determine the maximum interaction energy. For photons, the concept of relative velocity and acceleration makes no sense because these four-vectors are not defined for photons.

Consider the LCF Σ with four-velocity u^i and a ReMaP P with four-velocity v^i. The inner product $u^i v_i$ is an invariant, therefore its value is the same in all LCFs. In the proper frame Σ^+ of Σ one has

$$u^i = \begin{pmatrix} c \\ \mathbf{0} \end{pmatrix}_{\Sigma^+}, \; v^i = \begin{pmatrix} \gamma(v)c \\ \gamma(v)\mathbf{v} \end{pmatrix}_{\Sigma^+}$$

from which it follows that

$$u^i v_i = -\gamma(v)c^2 \Rightarrow \gamma(v) = -\frac{1}{c^2} u^i v_i. \tag{6.25}$$

[2] The concept of relative four-vector cannot be extended to theories of physics formulated over a non-linear space, i.e., curved spaces (e.g., General Relativity). The reason is that the relative vector involves the difference of vectors defined at different points in space, therefore one has to "transport" one of the vectors at the point of application of the other, an operation which involves necessarily the curvature of the space.

Relation (6.25) expresses the γ-factor of a ReMaP wrt an observer in terms of the inner product of their four-velocities. Consider now two ReMaPs, the 1, 2 say, with four-velocities v_1^i, v_2^i and relative four-velocity $v_{21}^i = v_1^i - v_2^i$.
Then

$$v_{21}^i (v_{21})_i = (v_1^i - v_2^i)\,[(v_1)_i - (v_2)_i] = -2c^2 - 2v_1^i (v_2)_i.$$

Let $\mathbf{v}_{21}$ be the three-velocity of ReMaP 2 as measured in the proper frame of 1. Then according to (6.25)

$$\gamma(v_{21}) = -\frac{1}{c^2} v_1^i (v_2)_i, \tag{6.26}$$

hence

$$v_{21}^i (v_{21})_i = -2c^2 + 2c^2 \gamma(v_{21}) = 2c^2[\gamma(v_{21}) - 1]. \tag{6.27}$$

Relation (6.27) leads to two conclusions:

(1) $\gamma(v_{21}) > 1 \Rightarrow v_{21}^i (v_{21})_i > 0$, which means that the relative four-velocity is a spacelike vector, hence not a four-velocity.
(2) The speed of $\mathbf{v}_{12}$ is determined uniquely from the speeds of the two ReMaPs in Σ.

The second result is not obvious and it will be useful to prove it. Suppose that in Σ the four-velocities of 1, 2 are

$$v_1^i = \begin{pmatrix} \gamma(v_1)c \\ \gamma(v_1)\mathbf{v}_1 \end{pmatrix}_\Sigma, \; v_2^i = \begin{pmatrix} \gamma(v_2)c \\ \gamma(v_2)\mathbf{v}_2 \end{pmatrix}_\Sigma.$$

Then in Σ we have

$$v_1^i v_{2i} = -\gamma(v_1)\gamma(v_2)c^2 + \gamma(v_1)\gamma(v_2)(\mathbf{v}_1 \cdot \mathbf{v}_2) = -\gamma(v_1)\gamma(v_2)c^2 \left(1 - \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{c^2}\right).$$

From (6.27) we conclude

$$\gamma(v_{12}) = \gamma(v_{21}) = 1 - \frac{1}{2}\gamma(v_1)\gamma(v_2)Q, \tag{6.28}$$

where $Q = 1 - \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{c^2}$. Replacing the γ-factors in terms of $\beta = v/c$ we find the symmetric relation

$$\beta_{12}^2 = 1 - \frac{(1 - \boldsymbol{\beta}_1^2)(1 - \boldsymbol{\beta}_2^2)}{(1 - \boldsymbol{\beta}_1 \cdot \boldsymbol{\beta}_2)^2}. \tag{6.29}$$

We note that the right parts of (6.28) and (6.29) are symmetric wrt the speeds v_1, v_2. This implies that if $\mathbf{v}_{12}$ is the relative velocity of 1 wrt 2 and $\mathbf{v}_{21}$ is the relative velocity of 2 wrt 1 then

$$|\mathbf{v}_{12}| = |\mathbf{v}_{21}|\,, \tag{6.30}$$

that is, the relative speeds are equal (but as will be shown below, their velocities are different!). Furthermore, we note that the relative speed obtains its maximum value when the denominator in (6.29) takes its maximum value, that is, when $\boldsymbol{\beta}_1 \cdot \boldsymbol{\beta}_2 = -1$. This happens when 1, 2 move antiparallel in Σ.

Relation (6.29) can be written in a different form, useful in the solution of problems.

Exercise 14 Making use of the three-vector identity $\mathbf{a} \times \mathbf{b} = |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2$ show that (6.29) can be written as

$$\boldsymbol{\beta}_{12}^2 = \frac{(\boldsymbol{\beta}_1 - \boldsymbol{\beta}_2)^2 - (\boldsymbol{\beta}_1 \times \boldsymbol{\beta}_2)^2}{(1 - \boldsymbol{\beta}_1 \cdot \boldsymbol{\beta}_2)^2}. \tag{6.31}$$

Up to this point we have determined only the speeds, which are measured in the proper frames of the particles in terms of their velocities in Σ. The following question arises: Is it possible to determine completely the velocities $\mathbf{v}_{12}$, $\mathbf{v}_{21}$, which are measured in the proper frames of the particles 1 and 2 in terms of the velocities $\mathbf{v}_1, \mathbf{v}_2$ which are measured in Σ? The answer is "yes," provided that we work as we did for the composition of four-vectors. That is, we consider in Σ, Σ_2 the components of the four-velocity of particle 1:

$$v_1^i = \begin{pmatrix} \gamma(v_1)c \\ \gamma(v_1)\mathbf{v}_1 \end{pmatrix}_\Sigma, \quad v_1^i = \begin{pmatrix} \gamma(v_{12})c \\ \gamma(v_{12})\mathbf{v}_{12} \end{pmatrix}_{\Sigma_2}$$

and *demand* the two expressions of v_1^i to be related with a Lorentz transformation with velocity $\mathbf{v}_2$. The three-velocities $\mathbf{v}_{21}$, $\mathbf{v}_{12}$ defined in this way are called *relative three-velocity* of particle 2 wrt particle 1 and relative velocity of the particle 1 wrt particle 2, respectively. In Newtonian Physics we have

$$\mathbf{v}_{21} = -\mathbf{v}_{12} = \mathbf{v}_2 - \mathbf{v}_1.$$

However, in Special Relativity nothing is obvious and everything has to be calculated explicitly. Let us compute $\mathbf{v}_{12}$. Using the general result of (6.13) of Example 20, we write

$$\mathbf{v}_{12} = \frac{1}{\gamma(v_2)Q}\left\{\mathbf{v}_1 + \left[\frac{\gamma(v_2)}{\gamma(v_2)+1}\frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{c^2} - 1\right]\gamma(v_2)\mathbf{v}_2\right\}. \tag{6.32}$$

The term in the brackets is written as

$$\begin{aligned}
&\mathbf{v}_1 - \mathbf{v}_2 + \left\{1 + \left[\frac{\gamma^2(v_2)}{\gamma(v_2)+1}\frac{\mathbf{v}_1\cdot\mathbf{v}_2}{c^2} - \gamma(v_2)\right]\right\}\mathbf{v}_2 \\
&= \mathbf{v}_1 - \mathbf{v}_2 + \frac{1}{\gamma(v_2)+1}\left[1 - \gamma^2(v_2) + \gamma^2(v_2)\frac{\mathbf{v}_1\cdot\mathbf{v}_2}{c^2}\right]\mathbf{v}_2 \\
&= \mathbf{v}_1 - \mathbf{v}_2 + \frac{1}{\gamma(v_2)+1}\left[-\gamma^2(v_2)\frac{v_2^2}{c^2} + \gamma^2(v_2)\frac{\mathbf{v}_1\cdot\mathbf{v}_2}{c^2}\right]\mathbf{v}_2 \\
&= \mathbf{v}_1 - \mathbf{v}_2 - \frac{\gamma^2(v_2)\frac{v_2^2}{c^2}}{\gamma(v_2)+1}\left[1 - \frac{\mathbf{v}_1\cdot\mathbf{v}_2}{v_2^2}\right]\mathbf{v}_2 \\
&= \mathbf{v}_1 - \mathbf{v}_2 - [\gamma(v_2)-1]\left[1 - \frac{\mathbf{v}_1\cdot\mathbf{v}_2}{v_2^2}\right]\mathbf{v}_2,
\end{aligned}$$

where in the last two steps we have used the identity $\gamma^2 - 1 = \gamma^2\beta^2$. Finally

$$\mathbf{v}_{12} = \frac{1}{\gamma(v_2)\left(1 - \frac{\mathbf{v}_1\cdot\mathbf{v}_2}{c^2}\right)}\left\{\mathbf{v}_1 - \mathbf{v}_2 - [\gamma(v_2)-1]\left[1 - \frac{\mathbf{v}_1\cdot\mathbf{v}_2}{v_2^2}\right]\mathbf{v}_2\right\}. \tag{6.33}$$

We note that (6.33) is not symmetric in $\mathbf{v}_1, \mathbf{v}_2$. Indeed, we calculate

$$\mathbf{v}_{21} = \frac{1}{\gamma(v_1)\left(1 - \frac{\mathbf{v}_1\cdot\mathbf{v}_2}{c^2}\right)}\left\{\mathbf{v}_2 - \mathbf{v}_1 - [\gamma(v_1)-1]\left[1 - \frac{\mathbf{v}_1\cdot\mathbf{v}_2}{v_1^2}\right]\mathbf{v}_1\right\} \neq -\mathbf{v}_{12}. \tag{6.34}$$

This result is different from the expected Newtonian result $\mathbf{v}_{12} = -\mathbf{v}_{21}$. This must not bother us, because the vectors $\mathbf{v}_{12}, \mathbf{v}_{21}$ are three-vectors of different relativistic observers (the proper observers Σ_1 and Σ_2 of the particles 1, 2, respectively) and the kinematics of relativistic observers is different from that of Newtonian observers.[3]

The three-velocities $\mathbf{v}_{12}, \mathbf{v}_{21}$ differ only by a Euclidean rotation, therefore there exists an angle ψ and a direction $\mathbf{e}$ normal to the plane defined by the velocities $\mathbf{v}_{12}, \mathbf{v}_{21}$ such that the rotation of $\mathbf{v}_{12}$ about the direction $\hat{\mathbf{e}}$ for an angle ψ takes it over to $\mathbf{v}_{21}$. The angle ψ is known as *Wigner angle* and has wide use in the study

[3] There is a significant difference, which must be pointed out. The relative (relativistic) velocity $\mathbf{v}_{21}$ refers to the velocity of the particle 2 in the proper frame Σ_1 of particle 1 and therefore involves the relativistic measurement of velocity (not photon!) and the inequality $|\mathbf{v}_{12}| < c$ is expected to hold. Indeed, if we consider two particles, one moving along the positive x-axis with speed $c/2$ and the other moving along the negative x-axis with speed $c/2$, then using (6.31) we compute that the (relativistic) relative speed of the particles equals $4c/5 < c$, whereas the Newtonian relative speed is c.

of elementary particles.[4] In order to compute the Wigner angle we set $A \equiv v_{12}^2 = \frac{\gamma_{12}^2-1}{\gamma_{12}^2} = 1 - \frac{1}{\gamma_1^2\gamma_2^2 Q^2}$ where $\gamma_i = \gamma(v_i)$ $(i = 1, 2)$ and get

$$\begin{aligned}
\cos\psi &= \frac{\mathbf{v}_{12}\cdot\mathbf{v}_{21}}{v_{12}v_{21}} \\
&= -\frac{1}{A}\left[(\mathbf{v}_1-\mathbf{v}_2)^2 - (\gamma_2-1)\frac{(\mathbf{v}_1\cdot\mathbf{v}_2-\mathbf{v}_2^2)}{v_2^2} - (\gamma_1-1)\frac{(\mathbf{v}_1\cdot\mathbf{v}_2-\mathbf{v}_1^2)}{v_1^2}\right. \\
&\left. + (\gamma_1-1)(\gamma_2-1)\frac{(\mathbf{v}_1\cdot\mathbf{v}_2)(\mathbf{v}_1\cdot\mathbf{v}_2-\mathbf{v}_2^2)(\mathbf{v}_1\cdot\mathbf{v}_2-\mathbf{v}_2^2)}{v_1^2v_2^2}\right].
\end{aligned} \tag{6.35}$$

We write $\mathbf{v}_1\cdot\mathbf{v}_2 = v_1v_2\cos\phi$ and after a standard calculation we find

$$\cos\psi = 1 - \frac{2\sin^2\phi}{1+2\rho\cos\phi+\rho^2}, \tag{6.36}$$

where $\rho = \left[\frac{(\gamma_1+1)(\gamma_2+1)}{(\gamma_1-1)(\gamma_2-1)}\right]^{\frac{1}{2}}$ $(1 \le \rho \le \infty)$. Other equivalent forms of (6.36) are

$$\tan\frac{\psi}{2} = \frac{\sin\phi}{\rho+\cos\phi}, \tag{6.37}$$

$$\sin\psi = \frac{2\sin\phi(\rho+\cos\phi)}{1+2\rho\cos\phi+\rho^2}. \tag{6.38}$$

We note that the angle ψ

- Is in the region $[\frac{\pi}{2}, \pi]$
- Depends symmetrically on $\mathbf{v}_1, \mathbf{v}_2$

Exercise 15 Prove that for $\phi = \frac{\pi}{2}$ the angle ψ is given by the relation

$$\psi = \tan^{-1}\left[\frac{v_1v_2(\gamma_1\gamma_2-1)}{\gamma_1v_1^2+\gamma_2v_2^2}\right] = 2\cot^{-1}\rho.$$

Assume $v_1 = v_2$ and show that in the Newtonian limit the angle $\psi \to 0$ (is of order $O(\epsilon^2)$), while in the relativistic limit $\psi \to \frac{\pi}{2}$. Finally, show that for photons $\psi = \frac{\pi}{2}$.

Exercise 16 Working with the relative position, derive relation (6.33). Moreover, assume that the position four-vector of a particle 1 in the LCFs Σ and Σ_2 is, respectively,

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[4] The interested reader can find more on the Wigner angle in, e.g., A. Ben-Menahem "Wigner's rotation revisited" *Am. J. Phys.* **53**, pp. 62–66, (1983).

$$x_1^i = \begin{pmatrix} l_1 \\ \mathbf{r}_1 \end{pmatrix}_\Sigma , \begin{pmatrix} l_{12} \\ \mathbf{r}_{12} \end{pmatrix}_{\Sigma_2} .$$

Then consider the Lorentz transformation, which relates $\mathbf{r}_{12}$ with $\mathbf{r}_1$, and calculate the relative velocity from the relation $\mathbf{v}_{12} = \frac{d\mathbf{r}_{12}}{d\tau_2}$, where τ_2 is the proper time of Σ_2.

In the following example, we demonstrate the use of relative velocity in a "practical" problem.

Example 22 A mobile is sliding freely along the positive x-axis with constant speed ac $(0 < a < 1)$. A smaller mobile is moving freely on the first along the same direction with relative speed ac. A third even smaller mobile moves freely on the second in the same direction with relative speed ac and so on.

(1) Assuming that Newtonian kinematics applies, prove that the speed cv_{r+1} of the $r+1$ mobile wrt the *x-axis* is

$$v_{r+1} = v_r + a, \tag{6.39}$$

where $r = 1, 2, \ldots$. Also prove that in Newtonian kinematics the speed of the nth mobile is given by the relation

$$v_{n+1} = (n+1)a. \tag{6.40}$$

Conclude that the speeds v_n are members of an arithmetic series with step a. Finally, show that $\lim_{n\to\infty} v_n \to +\infty$.

(2) Repeat the same questions for relativistic kinematics and show that

$$v_{r+1} = \frac{v_r + a}{1 + av_r},$$

$$v_n = \frac{(1+a)^n - (1-a)^n}{(1+a)^n + (1-a)^n}, \tag{6.41}$$

$$\lim_{n\to\infty} v_n = 1.$$

Discuss the result.

Solution

The Newtonian part is obvious. Concerning the relativistic part, let cv_r be the speed of the rth mobile wrt the x-axis and ac the relative speed of the $r+1$ mobile wrt the mobile r. From the relativistic composition rule for three-velocities we have

$$cv_{r+1} = \frac{cv_r + ac}{1 + \frac{cv_r ac}{c^2}} = c\frac{v_r + a}{1 + v_r a} \Rightarrow v_{r+1} = \frac{v_r + a}{1 + v_r a}. \tag{6.42}$$

Consider the sequence $\zeta_r = \frac{1-v_r}{1+v_r}$, $r = 1, 2, \ldots$. We find

$$\zeta_{r+1} = \frac{1-v_{r+1}}{1+v_{r+1}} = \frac{1-\frac{v_r+a}{1+v_r a}}{1+\frac{v_r+a}{1+v_r a}} = \frac{1+av_r-v_r-a}{1+av_r+v_r+a}$$
$$= \frac{(1-a)(1-v_r)}{(1+a)(1+v_r)} = \frac{1-a}{1+a}\zeta_r.$$

This reduction formula gives

$$\zeta_r = \left(\frac{1-a}{1+a}\right)^{r-1}\zeta_1,$$

which implies that the terms ζ_r of the sequence are members of a geometric series with step $\frac{1-a}{1+a} \in (0,1)$. Obviously, this sequence converges to $\lim_{r\to\infty}\zeta_r = 0$, hence the velocity $\lim_{r\to\infty} cv_r = c$ as expected. The first term of the series is[5]

$$\zeta_1 = \frac{1-v_1}{1+v_1} = \frac{1-a}{1+a},$$

thus $\zeta_r = \left(\frac{1-a}{1+a}\right)^r$. From the relation defining ζ_r we have

$$\frac{1-v_r}{1+v_r} = \left(\frac{1-a}{1+a}\right)^r$$

from which follows:

$$v_r = \frac{(1+a)^r-(1-a)^r}{(1+a)^r+(1-a)^r}.$$

The rest are left as an exercise for the reader.

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[5] The solution of the problem is a particular case of the sequence $a_n = \frac{\kappa a_n+\lambda}{\mu a_n+\rho}$, whose terms a_n converge to a common value x. In this case, we set $x = \frac{\kappa x+\lambda}{\mu x+\rho}$ and compute the roots ρ_1, ρ_2 of the quadratic equation. Then the following recursion formula holds:

$$\frac{a_{n+1}-\rho_1}{a_{n+1}-\rho_2} = A\frac{a_n-\rho_1}{a_n-\rho_2},$$

where A is a constant. [Study the case $\rho_1 = \rho_2$.] In our example $a_n = v_r$ and $\kappa = 1, \lambda = a, \mu = 1$, $\rho = a$. The equation $x = \frac{x+a}{ax+1} \Rightarrow ax^2 + x = x + a \Rightarrow x = \pm 1$, hence the reduction relation is

$$\frac{v_{r+1}-1}{v_{r+1}+1} = A\frac{v_{r-1}}{v_{r+1}} \text{ or } \zeta_{r+1} = A\zeta_r.$$

In order to calculate A, we consider the terms $\zeta_2 = A\zeta_1$ and replacing we find $A = \frac{1-a}{1+a}$, etc.

6.5 The three-Velocity Space

In Special Relativity we introduce various spaces besides the standard spacetime. Such a space is the *three-velocity space*, which is used to study geometrically relative motion. The three-velocity of a particle is described in terms of three components, therefore in a three-dimensional space the three-velocity of a particle corresponds to a point. Following this observation, we consider a RIO Σ_0 and two particles, which in Σ_0 have velocities[6] $\mathbf{v}_{I0}$ $I = 1, 2$. Then the velocity of Σ_0 is the origin, O say, of the coordinates of the velocity space (the zero velocity $\mathbf{0}$) and the velocities $\mathbf{v}_{I0}$ $I = 1, 2$ are two other points, the $1, 2$ say, of that space. Joining the points $O, 1, 2$ with a "straight line" (note the quotation marks!) we obtain the triangle $O12$ (Fig. 6.2).

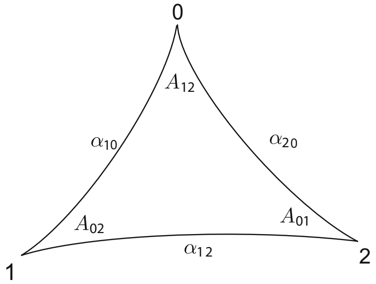

Fig. 6.2 The hyperbolic triangle

In order to give a kinematic interpretation of this triangle we relate the sides of the triangle with the speeds of the corresponding velocities. Thus we relate the side 01 to the length of the velocity $\mathbf{v}_{10}$ and the length of the side 12 to the length of the relative velocity $\mathbf{v}_{12}$ (in Σ_0!). Because the $\mathbf{v}_{IJ} = -\mathbf{v}_{JI}$ the side IJ is of the same length as the side JI with $I, J = 0, 1, 2$. In Newtonian kinematics due to the Newtonian composition rule of three-velocities (6.22) the velocity triangle is a typical triangle of Euclidean geometry, hence the velocity space is also a Euclidean space. In Special Relativity, due to the relativistic composition formula (6.23), this triangle is not Euclidean and the three-velocity space is not a linear space, neither is the Minkowski space.[7]

[6] In this section we follow the notation that two indices in a velocity indicate the first quantity with reference to the second. For example the velocity of Σ_1 wrt Σ_2 will be denoted as $\mathbf{v}_{12}$. Concerning the angles we follow the notation that the angle between the velocities $\mathbf{v}_{10}, \mathbf{v}_{20}$ of Σ_1 and Σ_2 in Σ_0 will be denoted by A_{12}.

[7] As we shall show in Sect. 15.4.3, when we study the covariant form of the Lorentz transformation, the three-velocity space is a three-dimensional Riemannian manifold of constant negative curvature whose metric is Lorentz covariant. Such spaces are known as Lobachevsky spaces.

In order to find more information about the three-velocity space, we study the velocity triangle. We consider a RIO Σ_0 and let the RIOs be Σ_1, Σ_2 with velocities $\mathbf{v}_{10}, \mathbf{v}_{20}$, respectively, wrt Σ_0. Then the relative velocity $\mathbf{v}_{21}$ of Σ_2 wrt Σ_1 (as measured by Σ_0!) is given by (6.34) and its γ-factor $\gamma_{21} = \gamma(v_{21})$ is given by (6.26) or equivalently by (6.28). Furthermore, according to the composition formula (6.33) the three three-velocities $\mathbf{v}_{10}, \mathbf{v}_{20}, \mathbf{v}_{21}$ are coplanar (in Σ_0). Let γ_{I0} be the γ-factor of $\mathbf{v}_{I0}$ and α_{I0} $I = 1, 2$ the corresponding rapidity. Then $\gamma_{I0} = \cosh \alpha_{I0}$, $\mathbf{v}_{I0} = \frac{1}{c} \tanh \alpha_{I0} \mathbf{e}_{I0}$, where $\mathbf{e}_{I0}$ is the unit along the direction of the three-velocity $\mathbf{v}_{I0}$ $I = 1, 2$. Let $\gamma_{21} = \cosh \alpha_{12}$ be the corresponding quantities for the relative velocity. Then (6.28) gives for γ_{21}

$$\begin{aligned} \cosh \alpha_{21} &= \cosh \alpha_{10} \cosh \alpha_{20} (1 - \tanh \alpha_{10} \tanh \alpha_{20} \mathbf{e}_{10} \cdot \mathbf{e}_{20}) \\ &= \cosh \alpha_{10} \cosh \alpha_{20} - \sinh \alpha_{10} \sinh \alpha_{20} \cos A_{12}, \end{aligned} \tag{6.43}$$

where A_{12} is the angle between the unit vectors $\mathbf{e}_{01}, \mathbf{e}_{02}$ along the three-velocities $\mathbf{v}_{10}, \mathbf{v}_{20}$ in Σ_0. Equation (6.43) is the cosine law of hyperbolic Euclidean geometry for a hyperbolic triangle of sides $\alpha_{10}, \alpha_{20}, \alpha_{21}$ and angles A_{12}, A_{01}, A_{02}. This geometry is identical to the Euclidean spherical trigonometry of a triangle with sides $i\alpha_{10}, i\alpha_{20}, i\alpha_{21}$ and angles A_{12}, A_{01}, A_{02} and we have considered the correspondence

$$\cos(i\alpha) = \cosh \alpha. \tag{6.44}$$

In order to make sure that we have a triangle in hyperbolic Euclidean geometry we must prove that the above result holds for all three vertices of the triangle. To do that we go to the definition of relative velocity and write

$$v^i_{21} = v^i_{20} - v^i_{10} \Rightarrow v^i_{20} = v^i_{21} + v^i_{10} = v^i_{10} - v^i_{12}.$$

But $\gamma_{21} = \gamma_{12}$, i.e., $\cosh \alpha_{12} = \cosh \alpha_{21}$. It is an easy exercise to show that

$$\cosh \alpha_{20} = \cosh \alpha_{12} \cosh \alpha_{10} \cosh \alpha_{12} - \sinh \alpha_{10} \sinh \alpha_{12} \cos A_{02}, \tag{6.45}$$

where A_{02} is the angle between the velocities $\mathbf{v}_{12}$ and $\mathbf{v}_{02}$ in Σ_2. Similarly, we prove the corresponding relation for the quantity $\cosh \alpha_{10}$. These three relations establish that the three-velocity triangle is a hyperbolic triangle with sides $\alpha_{10}, \alpha_{20}, \alpha_{21}$ and corresponding angles A_{12}, A_{01}, A_{02} in a three-dimensional hyperbolic Euclidean space. Note that the sides of this triangle are the rapidity and not the speed of the three-velocities.

A crucial question is if the relative velocity is compatible with the Lorentz transformation. This is not obvious, although it is suggested by the result we have just derived. To prove that this is the case, we note that the three-velocity of Σ_0 in Σ_1 is $-\mathbf{v}_{10}$ and in Σ_2 is $-\mathbf{v}_{20}$ whereas the three-velocity of Σ_2 wrt Σ_1 is $\mathbf{v}_{21}$. Therefore the definition of the relative velocity we gave will be compatible with the Lorentz transformation if the components of the four-velocity of Σ_0 in Σ_1 and Σ_2 are related by a proper Lorentz transformation with velocity $\mathbf{v}_{21}$.

We choose the coordinate frame in Σ_0 so that the x-axis is along the direction of $\mathbf{v}_{21}$. According to the composition formula (6.33) the three three-velocities $\mathbf{v}_{10}, \mathbf{v}_{20}, \mathbf{v}_{21}$ are coplanar (in Σ_0). We choose the y-axis to be normal to the x-axis in the plane of the three-velocities. We recall that the angle between $\mathbf{v}_{21}$ and $\mathbf{v}_{01}$ is A_{01}, hence between $\mathbf{v}_{21}$ and $-\mathbf{v}_{10}$ (the three-velocity of Σ_0 in Σ) is $\pi - A_{10}$ and similarly the angle between $\mathbf{v}_{21}$ and $-\mathbf{v}_{20}$ is $\pi - A_{02}$. We write for the four-velocities of Σ_0 in Σ_1 and Σ_2 :

$$v_{01}^i = \begin{pmatrix} c\cosh\alpha_{10} \\ -c\sinh\alpha_{10}\cos A_{01} \\ c\sinh\alpha_{10}\sin A_{01} \\ 0 \end{pmatrix}_{\Sigma_1}, v_{02}^i = \begin{pmatrix} c\cosh\alpha_{20} \\ -c\sinh\alpha_{20}\cos A_{02} \\ c\sinh\alpha_{20}\sin A_{02} \\ 0 \end{pmatrix}_{\Sigma_2}.$$

The matrix representing the Lorentz transformation in the coordinate system we have chosen is

$$L(\mathbf{v}_{21}) = \begin{pmatrix} \cosh\alpha_{21} & \sinh\alpha_{21} & 0 & 0 \\ \sinh\alpha_{21} & \cosh\alpha_{21} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

therefore we must have

$$[v_{20}^i]_{\Sigma_2} = [L(\mathbf{v}_{21})]\,[v_{10}^i]_{\Sigma_1}$$

or

$$\begin{aligned} \cosh\alpha_{20} &= \cosh\alpha_{10}\cosh\alpha_{21} - \sinh\alpha_{10}\sinh\alpha_{21}\cos A_{02}, \\ \sinh\alpha_{20}\cos A_{02} &= \sinh\alpha_{21}\cosh\alpha_{10} - \cosh\alpha_{21}\sinh\alpha_{10}\cos A_{01}, \\ \sinh\alpha_{20}\sin A_{02} &= \sinh\alpha_{10}\sin A_{01}. \end{aligned} \tag{6.46}$$

The first equation is identical with (6.45) while the rest two are identities of Euclidean spherical trigonometry if we use imaginary sides $i\alpha_1, i\alpha_2, i\alpha_{21}$. For example, the last relation is the "sine law." Equivalently, they are the relations of hyperbolic trigonometry, that is the trigonometry of the hyperbolic sphere. We conclude that the definition of relative motion is compatible with the Lorentz transformation.

In plane Euclidean trigonometry the angles of a triangle add to π. In spherical and hyperbolic trigonometry this does not hold and one defines the spherical/hyperbolic defect ε by the formula

$$\varepsilon = \pi - (A_{01} + A_{02} + A_{12}). \tag{6.47}$$

In spherical trigonometry ε is negative and its absolute value equals the area of the triangle (on the unit sphere). In hyperbolic trigonometry ε is positive. An expression of ε in terms of the three sides and an angle is the following[8]:

$$\sin\frac{\varepsilon}{2} = \left[\frac{(\cosh\alpha_2 - 1)\,(\cosh\alpha_{12} - 1)}{2(\cosh\alpha_1 + 1)}\right]^{1/2} \sin A_{01} \tag{6.48}$$

and cyclically for each vertex. It is important to see that the velocity triangle contains all the information concerning the relativistic relative three-velocity, that is, gives both the speeds and the relative directions of the velocities.

6.6 Thomas Precession

The Thomas precession is a purely relativistic kinematic phenomenon, which is due to the properties of the Lorentz transformation, and more specifically, to the non-covariant character of Euclidean parallelism. The first to note this property of the Lorentz transformation was L. Thomas[9] who applied it in the study of the emission spectra of certain atoms. According to the simple model of the atom at that time, the electrons were considered as negatively charged spheres, which were rotating around the axis of their spin like gyroscopes and also around the positive-charged nucleus. According to the Newtonian theory, the two rotations do not interact and after a complete rotation about the nucleus the axis of spin returns to its original position. However, the non-covariant character of Newtonian parallelism under the action of Lorentz transformation leads to an interaction of these two rotations with the effect that after a complete revolution around the nucleus the axis of spin makes an angle with its initial direction. This type of rotational motion is called precession. The special case we consider here has been called *Thomas precession*.

In order to study the Thomas precession we consider three LCFs Σ_1, Σ_2, Σ_3 such that Σ_1, Σ_2 and Σ_2, Σ_3 have parallel axes and non-collinear relative velocities. Let the Lorentz transformations, which relate the pairs Σ_1, Σ_2 and Σ_2, Σ_3, be L_{12} and L_{23}, respectively. The composite Lorentz transformation $L_{23}L_{12}$ relating Σ_1, Σ_3 is also a Lorentz transformation, because Lorentz transformations form a group. However, as we shall show, this transformation is of a different type than the L_{12}, L_{23} and corresponds to a Lorentz transformation in which the space axes of the transformed Σ_1 are not parallel (in the Euclidean sense) with the space axes of the original Σ_1. This rotation of the space axes is a purely kinematic relativistic phenomenon, without a Newtonian analogue, which we call *Thomas phenomenon*.

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[8] See for example B.P. (1929) Peirce A short table of integrals ginn, *Boston* formulae 631 and 632 pp. and for a theoretical treatment *A. Ungar Foundations of Physics* **28**, 1283–1321, (1998).

[9] L.H. Thomas The kinematics of an electron with an axis, *Philosophical Magazine* **3**, 1–22 (1927).

We have seen a first approach to Thomas phenomenon in Example 20, where we studied the composition of boosts along different directions. However, as we shall see below, the real power of the Thomas phenomenon is in accelerated motions.

Consider two LCFs Σ_1, Σ_2 with parallel space axes and let $\mathbf{u}$ be the velocity of Σ_2 wrt Σ_1. The Lorentz transformation $L_{12}(\mathbf{u})$ which relates Σ_1, Σ_2 is given by the general relations (1.51), (1.52), or, equivalently in the form of a matrix, from (1.47):

$$L^i_{(12)j}(\mathbf{u}) = \begin{pmatrix} \gamma & -\gamma\beta_\mu \\ -\gamma\beta^\mu & \delta^\mu_\nu + \frac{\gamma^2}{1+\gamma}\beta^\mu\beta_\nu \end{pmatrix}, \tag{6.49}$$

where $c\boldsymbol{\beta}^\mu = \mathbf{u} = \begin{pmatrix} u^1 \\ u^2 \\ u^3 \end{pmatrix}_{\Sigma_1}$ and $c\boldsymbol{\beta}_\mu = (u_1, u_2, u_3)_{\Sigma_1}$. The inverse Lorentz transformation is $L^i_{(21)j}(-\mathbf{u}) = \left[L^i_{(12)j}\right]^{-1}$.

We consider a third LCF Σ_3, which has space axes parallel to those of Σ_2 and velocity $\mathbf{v}$. The Lorentz transformation $L^i_{(23)j}(\mathbf{v})$ relating Σ_2 and Σ_3 is again of the general form (1.51) and (1.52). Because the Lorentz transformations form a group, there must exist a third Lorentz transformation $L^i_{(13)j}(\mathbf{w})$ relating Σ_1, Σ_3 with velocity $\mathbf{w}$, say, defined by the relation

$$L^i_{(13)j}(\mathbf{w}) = L^i_{(12)k}(\mathbf{u})L^k_{(23)j}(\mathbf{v}). \tag{6.50}$$

In order to compute the velocity $\mathbf{w}$, we note that Σ_2 has velocity $-\mathbf{u}$ wrt Σ_1 and Σ_3 has velocity $\mathbf{v}$ wrt Σ_2. Therefore $\mathbf{w}$, which is the velocity of Σ_3 wrt Σ_1, is given from relation (6.12) of Example 20, which is

$$\mathbf{w} = \frac{1}{\gamma_u Q}\left\{\mathbf{v} + \mathbf{u}\left[\frac{\mathbf{u}\cdot\mathbf{v}}{u^2}(\gamma_u - 1) + \gamma_u\right]\right\}. \tag{6.51}$$

Now we examine if the composite Lorentz transformation $L^i_{(13)j}(\mathbf{w})$ is of the general form (1.51) and (1.52) as it is the case with the transformations $L^i_{(12)j}(\mathbf{u})$, $L^i_{(23)j}(\mathbf{v})$. For that, we examine if the space axes of Σ_3 are parallel (in the Euclidean sense!) to those of Σ_1.

We consider an arbitrary spacetime point, whose position four-vector in the LCFs Σ_1, Σ_2, Σ_3 has components

$$\begin{pmatrix} l_1 \\ \mathbf{r}_1 \end{pmatrix}_{\Sigma_1} \overset{\mathbf{u}}{\Longrightarrow} \begin{pmatrix} l_2 \\ \mathbf{r}_2 \end{pmatrix}_{\Sigma_2} \overset{\mathbf{v}}{\Longrightarrow} \begin{pmatrix} l_3 \\ \mathbf{r}_3 \end{pmatrix}_{\Sigma_3}. \tag{6.52}$$

Then relations (1.51) and (1.52) give
$L_{12}(\mathbf{u})$:

$$l_2 = \gamma_u \left(l_1 - \frac{\mathbf{u}\cdot\mathbf{r}_1}{c} \right), \tag{6.53}$$

$$\mathbf{r}_2 = \mathbf{r}_1 - \gamma_u \frac{\mathbf{u} l_1}{c} + (\gamma_u - 1)\frac{\mathbf{u}\cdot\mathbf{r}_1}{u^2}\mathbf{u}. \tag{6.54}$$

$L_{23}(\mathbf{v})$:

$$l_3 = \gamma_v \left(l_2 - \frac{\mathbf{v}\cdot\mathbf{r}_2}{c} \right), \tag{6.55}$$

$$\mathbf{r}_3 = \mathbf{r}_2 - \gamma_v \frac{\mathbf{v}}{c} l_2 + (\gamma_v - 1)\frac{\mathbf{v}\cdot\mathbf{r}_2}{v^2}\mathbf{v}. \tag{6.56}$$

From (6.53), (6.54), and (6.55) we have

$$\begin{aligned} l_3 &= \gamma_v \left[\gamma_u \left(l_1 - \frac{\mathbf{u}\cdot\mathbf{r}_1}{c} \right) - \frac{\mathbf{v}}{c} \cdot \left(\mathbf{r}_1 - \gamma_u \frac{\mathbf{u}}{c} l_1 + (\gamma_u - 1)\frac{\mathbf{u}\cdot\mathbf{r}_1}{u^2}\mathbf{u} \right) \right] \\ &= \gamma_u \gamma_v \left(1 + \frac{\mathbf{u}\cdot\mathbf{v}}{c^2} \right) \left[l_1 - \frac{B}{c\gamma_u \left(1 + \frac{\mathbf{u}\cdot\mathbf{v}}{c^2} \right)} \right], \end{aligned} \tag{6.57}$$

where

$$\begin{aligned} B &= \mathbf{v}\cdot\mathbf{r}_1 + (\gamma_u - 1)\frac{1}{u^2}(\mathbf{u}\cdot\mathbf{r}_1)(\mathbf{u}\cdot\mathbf{v}) + \gamma_u \mathbf{u}\cdot\mathbf{r}_1 \\ &= \mathbf{r}_1 \cdot \left[\mathbf{v} + \gamma_u \mathbf{u} + (\gamma_u - 1)\frac{\mathbf{u}\cdot\mathbf{v}}{u^2}\mathbf{u} \right] \\ &= \gamma_u \left(1 + \frac{\mathbf{u}\cdot\mathbf{v}}{c^2} \right) (\mathbf{r}_1 \cdot \mathbf{w}). \end{aligned}$$

Replacing B in (6.57) we find

$$l_3 = \gamma_u \gamma_v \left(1 + \frac{\mathbf{u}\cdot\mathbf{v}}{c^2} \right) \left[l_1 - \frac{\mathbf{w}\cdot\mathbf{r}_1}{c} \right]. \tag{6.58}$$

But we know that (see (6.7))

$$\gamma_w = \gamma_u \gamma_v \left(1 + \frac{\mathbf{u}\cdot\mathbf{v}}{c^2} \right),$$

therefore

$$l_3 = \gamma_w \left(l_1 - \frac{\mathbf{w}\cdot\mathbf{r}_1}{c} \right). \tag{6.59}$$

We conclude that, as far as the zeroth component of the position four-vector is concerned, the composite Lorentz transformation $L_{13}(\mathbf{w})$ behaves as a Lorentz transformation with parallel axes and velocity $\mathbf{w}$. This means that whatever differences of $L_{13}(\mathbf{w})$ concern the spatial part $\mathbf{r}$ of position four-vector and more specifically either the length or the direction of $\mathbf{r}$, or both. We treat each case separately.

Let $r_3(\mathbf{w})$ be the spatial part of the position four-vector after the action of $L^i_{13j}(\mathbf{w})$ (on the $\begin{pmatrix} l_1 \\ \mathbf{r}_1 \end{pmatrix}_{\Sigma_1}$) and let $l_3(\mathbf{w})$ be the temporal component. The invariance of the Lorentz length of the four-vector implies

$$l_3^2(\mathbf{w}) - \mathbf{r}_3^2(\mathbf{w}) = l_3^2 - \mathbf{r}_3^2.$$

From (6.59) we have $l_3(\mathbf{w}) = l_3$, hence

$$\mathbf{r}_3^2(\mathbf{w}) = \mathbf{r}_3^2. \tag{6.60}$$

We conclude that the two three-vectors $\mathbf{r}_3(\mathbf{w})$, $\mathbf{r}_3$ of Σ_3 differ only in their direction (a Euclidean rotation), therefore they must be related with an orthogonal Euclidean transformation A, say

$$\mathbf{r}_3(\mathbf{w}) = A\mathbf{r}_3, \tag{6.61}$$

where $A^t A = I_3$. This Euclidean transformation is the essence of the Thomas phenomenon. In order to compute the matrix A, we consider the infinitesimal form of the Euclidean transformation. We write $\mathbf{v} = \delta\mathbf{u}$ and have in second-order approximation $O(\delta u^2)$

$$\mathbf{r}_3(\mathbf{w}) = \mathbf{r}_3 + d\mathbf{\Omega} \times \mathbf{r}_3(\mathbf{w}) + O(\delta u^2)$$

or

$$\mathbf{r}_3(\mathbf{w}) - d\mathbf{\Omega} \times \mathbf{r}_3(\mathbf{w}) = \mathbf{r}_3 + O(\delta u^2). \tag{6.62}$$

The three-vector $d\mathbf{\Omega}$ of Σ_3 represents the rotation angle $d\phi(\mathbf{u})$, which corresponds to the Euclidean matrix A. Because the matrix A *is the same* for all four-vectors in Minkowski space, we take the four-vector x^i to be timelike and furthermore we take Σ_1 to be its proper frame. Then $\mathbf{r}_1 = 0$ and $l_1^2 = -x^i x_i$. From (6.53) and (6.54) we find in Σ_2 for this four-vector

$$\begin{aligned} l_2 &= \gamma_u l_1, \\ \mathbf{r}_2 &= -\gamma_u \frac{\mathbf{u}}{c} l_1. \end{aligned} \tag{6.63}$$

The three-vector $d\boldsymbol{\Omega}$ is an axial three-vector,[10] which depends on the three-vectors $\mathbf{u}$, $\delta\mathbf{u}$ and vanishes when $\mathbf{u}$, $\delta\mathbf{u}$ are parallel. Indeed, we observe that in case $\mathbf{u}//\mathbf{v}$ the composite velocity $\mathbf{w}//\mathbf{u}$ and the spatial axes of Σ_1, Σ_2, Σ_3 are parallel. This means that $d\boldsymbol{\Omega}$ has the general form

$$d\boldsymbol{\Omega} = \alpha\mathbf{u} \times \delta\mathbf{u}, \tag{6.64}$$

where α is a function, which has to be determined. In this general form, (6.56) combined with (6.63) and (6.53) gives

$$\begin{aligned}\mathbf{r}_3 &= \mathbf{r}_2 - \gamma_{\delta u}\frac{\delta\mathbf{u}}{c}l_2 + O(\delta\mathbf{u}^2)\\ &= -\frac{\gamma_u}{c}(\mathbf{u}+\delta\mathbf{u})l_1 + O(\delta\mathbf{u}^2),\end{aligned} \tag{6.65}$$

where in our approximation $\gamma_{\delta u} = 1$.

Concerning $\mathbf{r}_3(\mathbf{w})$, we have from (1.52) for velocity $\mathbf{w}$ and $\mathbf{r}_1 = 0$ ($Q = 1 + \frac{\mathbf{u}\cdot\delta\mathbf{u}}{c^2}$),

$$\mathbf{r}_3(\mathbf{w}) = -\gamma_w\frac{\mathbf{w}}{c}l_1.$$

Equation (6.51) gives for $\mathbf{w}$ if we set $\mathbf{v} = \delta\mathbf{u}$ and apply the composition of the γs, i.e., $\gamma_w = \gamma_u\gamma_{\delta u}Q$:

$$\begin{aligned}\mathbf{r}_3(\mathbf{w}) &= -\gamma_u\gamma_{\delta u}Q\frac{1}{\gamma_u Qc}\left[\delta\mathbf{u} + \gamma_u\mathbf{u} + (\gamma_u - 1)\frac{\mathbf{u}\cdot\delta\mathbf{u}}{u^2}\mathbf{u}\right]l_1\\ &= -\frac{1}{c}\left[\delta\mathbf{u} + \gamma_u\mathbf{u} + (\gamma_u - 1)\frac{\mathbf{u}\cdot\delta\mathbf{u}}{u^2}\mathbf{u}\right]l_1.\end{aligned} \tag{6.66}$$

The term[11]

$$\begin{aligned}d\boldsymbol{\Omega} \times \mathbf{r}_3(\mathbf{w}) &= \alpha(\mathbf{u}\times\delta\mathbf{u}) \times \left(-\frac{1}{c}\right)\left\{\delta\mathbf{u} + \left[\gamma_u + (\gamma_u - 1)\frac{\mathbf{u}\cdot\delta\mathbf{u}}{u^2}\right]\mathbf{u}\right\}l_1\\ &= -\frac{\alpha}{c}\left[\gamma_u u^2\delta\mathbf{u} + \gamma_u(\mathbf{u}\cdot\delta\mathbf{u})\mathbf{u} + O(\delta u^2)\right]l_1.\end{aligned}$$

Replacing in (6.62) we find

$$\begin{aligned}(-1 + \alpha\gamma_u u^2 + \gamma_u)\delta\mathbf{u} + \left[-\gamma_u + (\gamma_u - 1)\frac{\mathbf{u}\cdot\delta\mathbf{u}}{u^2} + \alpha\gamma_u(\mathbf{u}\cdot\delta\mathbf{u}) + \gamma_u\right]\mathbf{u} = 0 \Rightarrow\\ (-1 + \alpha\gamma_u u^2 + \gamma_u)\left(\delta\mathbf{u} + \frac{\mathbf{u}\cdot\delta\mathbf{u}}{u^2}\mathbf{u}\right) = 0.\end{aligned}$$

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[10] These vectors are one-forms but this is not crucial for our considerations here.

[11] Apply the identity $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\,\mathbf{B} - (\mathbf{B}\cdot\mathbf{A})\mathbf{C}$.

This relation holds for all $\mathbf{u}$, $\delta\mathbf{u}$, therefore

$$\alpha = \frac{1-\gamma_u}{\gamma_u u^2}. \tag{6.67}$$

Replacing α in (6.64) we find

$$d\boldsymbol{\Omega} = -\frac{\gamma_u - 1}{\gamma_u}\frac{\mathbf{u}\times\delta\mathbf{u}}{u^2}. \tag{6.68}$$

This formula gives $d\boldsymbol{\Omega}$ in terms of $\delta\mathbf{u}$, which is velocity relative to Σ_2. As it turns out, in applications we need the velocity $\delta\mathbf{v}_1$ relative to Σ_1. In order to calculate $\delta\mathbf{v}_1$, we apply the general relation (6.12) for the following velocities:

$$\begin{aligned}
\mathbf{u} &\to -\mathbf{u} && (\Sigma_2 \to \Sigma_1),\\
\mathbf{v} &\to \delta\mathbf{u} && (\Sigma_2 \to \Sigma_3),\\
\mathbf{v}' &\to \delta\mathbf{v}_1 && (\Sigma_1 \to \Sigma_3)
\end{aligned}$$

and have

$$\begin{aligned}
\delta\mathbf{v}_1 &= \frac{1}{\gamma_u Q}\left[\delta\mathbf{u}+\gamma_u\mathbf{u} + (\gamma_u - 1)\frac{\mathbf{u}\cdot\delta\mathbf{u}}{u^2}\mathbf{u}\right] \Rightarrow\\
\mathbf{u}\times\delta\mathbf{v}_1 &= \frac{1}{\gamma_u Q}\mathbf{u}\times\delta\mathbf{u}.
\end{aligned}$$

Then (6.68) becomes

$$d\boldsymbol{\Omega} = -\frac{\gamma_u - 1}{u^2}Q\mathbf{u}\times\delta\mathbf{v}_1. \tag{6.69}$$

In case the velocities $\mathbf{u}$, $\delta\mathbf{u} \ll c$ the $Q \simeq 1 + O(\delta u^2)$ and $d\boldsymbol{\Omega}$ reduces to

$$d\boldsymbol{\Omega} = -\frac{\gamma_u - 1}{u^2}(\mathbf{u}\times\delta\mathbf{v}_1). \tag{6.70}$$

The rotation $d\boldsymbol{\Omega}$ in Σ_1 takes place with angular velocity $\boldsymbol{\omega}_T$, which is defined by the relation

$$\boldsymbol{\omega}_T = \frac{d\boldsymbol{\Omega}}{dt_1} = -\frac{\gamma_u - 1}{u^2}\mathbf{u}\times\mathbf{a}, \tag{6.71}$$

where $\mathbf{a} = \frac{d\mathbf{v}_1}{dt_1}$ is the three-acceleration of the origin of Σ_3 the moment the velocity of Σ_3 wrt Σ_2 is $\delta\mathbf{u}$.

Although the Thomas precession has a direct geometric explanation, in general it is considered as abstruse. This is due to the fact that it is a purely relativistic phenomenon incompatible with the Euclidean concept of parallelism, which is a strongly empirical concept.

In order to give a Physical meaning to the Thomas precession, we consider the simple atom model (by now obsolete!), which is a nucleus with electrons rotating around it with constant angular velocity. We assume that the LCF Σ_1 is the proper frame of the nucleus and let τ be the proper time of a rotating electron. Then, at each point along the trajectory of the electron, the position three-vector $\mathbf{r}_1(\tau)$ and the three-velocity are determined by the proper time τ. The relation between the proper time τ and the time t_1 in Σ (the nucleus) is

$$t_1 = \gamma(u)\tau.$$

Consider the neighboring points along the trajectory of the electron with position vectors $\mathbf{r}_1(\tau)$ and $\mathbf{r}_3(\tau + \phi\tau)$. At the point $\mathbf{r}_1(\tau)$ we consider the LCF Σ_2 with space axes parallel to the axes of Σ_1 and at the point $\mathbf{r}_3(\tau + \phi\tau)$ we consider the LCF Σ_3 with space axes parallel to the axes of Σ_2. Due to the Thomas phenomenon, the space axes of Σ_3 will appear to rotate in Σ_1 with angular velocity ω_T. The angle of rotation in time dt_1 is $\omega_T dt_1$. Because the velocity is normal to the acceleration $\mathbf{a}$, $a = \frac{u^2}{\rho}$, where ρ is the radius of the atom in Σ_1 (proper frame of the nucleus), we have

$$\boldsymbol{\omega}_T = -\frac{\gamma_u - 1}{u^2} u \frac{u^2}{\rho}\mathbf{n} = -(\gamma_u - 1)\frac{u}{\rho}\mathbf{n}, \tag{6.72}$$

where $\mathbf{n}$ is the unit normal to the plane $\mathbf{u} \times \mathbf{a}$ (in which lies the orbit of the electron). Obviously, due to the small value of the quantity $\gamma_u - 1$ for usual speeds, the angular velocity $\boldsymbol{\omega}_T$ of precession is very small. If we set $\omega = \frac{u}{\rho}$, where ω is the angular speed of the electron around the nucleus, we find that in Σ_1

$$\omega_T = -(\gamma_u - 1)\omega\mathbf{n}. \tag{6.73}$$

Taking the coordinates so that the z-axis is along the direction of $\mathbf{n}$, we have that the $x - y$ plane is rotating counter-clockwise with angular velocity $\boldsymbol{\omega}_T = (\gamma_u - 1)\omega\mathbf{z}$. For every complete rotation of the electron, the precession angle is $2\pi(\gamma_u - 1)$ rad.

Corollary 1 *Assuming that the period or rotation of the electron (in Σ_1) is T, prove that the period of the Thomas precession is $T_T = \frac{T}{\gamma_u - 1}$, where u is the (constant) speed of rotation of the electron. Show that for usual speeds $T_T \gg T$.*

Chapter 7
Four-Acceleration

7.1 Introduction

Although the study of accelerated motion is necessary in Special Relativity, as a rule, in standard textbooks little attention is payed to this subject. Perhaps this is due to the difficulty of the comprehension of "acceleration" in spacetime and its involved "behavior" under the Lorentz transformation. Indeed, with the exemption of the proper frame, the zeroth component of the acceleration four-vector enters in the spatial part creating confusion. Furthermore the Lorentz transformation of the acceleration four-vector does not reveal a clear kinematic role for the temporal and the spatial parts. However, the extensive study of four-acceleration is necessary, because it completes our understanding of relativistic kinematics, relates kinematics with the dynamics and finally it takes Special Relativity over to General Relativity in a natural way. In addition the four-acceleration finds application in many physical phenomena, such as, the radiation of an accelerated charge, the annihilation of anti-proton, the resonances of strange particles.

The structure of this chapter is as follows. In the first sections we study the general properties of four-acceleration and its behavior under the Lorentz transformation, especially under boosts. We prove a number of general results which are applied to one-dimensional accelerated motion and in particular to hyperbolic motion, which is a covariant relativistic motion. We consider two types of hyperbolic motion: the case of a rigid rod and the case of a "fluid," the later being characterized by the fact that the relative velocity between its parts (particles) does not vanish.

The most important concept in the study of accelerated motion is the synchronization of the clocks of the inertial and the accelerated observer. As we have seen the clocks of two inertial observers are synchronized with chronometry (Einstein synchronization), whereas we have not defined a synchronization procedure between an inertial and an accelerated observer. The definition of a synchronization for accelerated observers is necessary if we want to compare the kinematics of such observers. We emphasize that there is not a global synchronization for all accelerated observes and one defines a synchronization per case.

We shall also discuss the extension of the Lorentz transformation between an inertial and an accelerated observer and shall introduce the concept of the

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"generalized" Lorentz transformation. The generalized Lorentz transformation will lead us to examine the incorporation of gravity within Special Relativity. The result will be negative as it will be shown by means of three thought experiments, the gravitational redshift, the gravitational time dilatation, and finally the curvature of spacetime, which leads directly to General Relativity.

7.2 The Four-Acceleration

Consider a relativistic mass particle (ReMaP) P (not a photon!) with position four-vector x^i and four-velocity $u^i = \frac{dx^i}{d\tau}$. The four-acceleration a^i of P is defined as follows [see (6.2)]:

$$a^i = \frac{du^i}{d\tau}, \tag{7.1}$$

and it is a spacelike vector normal to the four-velocity vector (because $u^i a_i = 0$). Consider an arbitrary RIO Σ, say, in which the four-vectors $x^i$and u^i have components $x^i = \begin{pmatrix} l \\ \mathbf{r} \end{pmatrix}_\Sigma$ and $u^i = \begin{pmatrix} c\gamma \\ \gamma\mathbf{v} \end{pmatrix}_\Sigma$, respectively, where $\gamma = (1-\beta^2)^{-1/2}$ and $l = ct$. Then in Σ the components of the four-acceleration are

$$a^i = \frac{dt}{d\tau}\frac{du^i}{dt} = \gamma \begin{pmatrix} c\dot{\gamma} \\ \mathbf{v}\dot{\gamma} + \gamma\dot{\mathbf{v}} \end{pmatrix}_\Sigma, \tag{7.2}$$

where a dot over a symbol indicates derivative wrt t, e.g., $\dot{\gamma} = \frac{d\gamma}{dt}$. We set $\dot{\mathbf{v}} = \mathbf{a}$, the three-acceleration of P in Σ and (7.2) is written as follows:

$$a^i = \gamma \begin{pmatrix} c\dot{\gamma} \\ \mathbf{v}\dot{\gamma} + \gamma\mathbf{a} \end{pmatrix}_\Sigma. \tag{7.3}$$

One computes the quantity $\dot{\gamma}$ most conveniently by means of the orthogonality condition $u^i a_i = 0$. Indeed we have

$$(\mathbf{v}\dot{\gamma} + \gamma\mathbf{a}) \cdot \mathbf{v} - c^2\dot{\gamma} = 0 \Rightarrow \dot{\gamma} = \frac{1}{c^2}\gamma^3(\mathbf{v}\cdot\mathbf{a}). \tag{7.4}$$

Replacing in (7.3) we obtain the final expression

$$a^i = \begin{pmatrix} ca^0 \\ a^0\mathbf{v} + \gamma^2\mathbf{a} \end{pmatrix}_\Sigma, \tag{7.5}$$

where

$$a^0 = \gamma\dot{\gamma} = \frac{1}{c^2}\gamma^4(\mathbf{v}\cdot\mathbf{a}) = \frac{\mathbf{v}\cdot\mathbf{a}}{(c^2-v^2)^2}c^2. \tag{7.6}$$

We note that both the temporal part and the spatial part of the four-vector a^i depend on both $\mathbf{v}$ and $\mathbf{a}$. Therefore it is not obvious why we should consider the four-vector a^i as the "generalization" of acceleration of Newtonian Physics in Special Relativity. In addition we note that the temporal part a^0 enters the spatial part. These observations indicate that we should be careful and search for proper answers.

We recall (see Sect. 6.2) that the proper frame Σ^+ (say) of P is also a characteristic frame of the four-vector a^i and that in Σ^+ a^i has the form

$$a^i = \begin{pmatrix} 0 \\ \mathbf{a}^+ \end{pmatrix}_{\Sigma^+}, \tag{7.7}$$

where $\mathbf{a}^+$ is the three-acceleration of P in its proper frame. But how can the statement "*I am accelerating in my own proper frame*" be conceived kinematically? The answer is as follows.

We consider two relativistic mass particles, the P, Q say, which at some moment t in the frame of a RIO Σ have relative velocity $\mathbf{v}_{PQ}(t) = 0$ and relative position vector $\mathbf{r}_{PQ}(t)$. Furthermore we consider that P has constant velocity in Σ whereas Q is accelerating in Σ. Then the proper frame Σ_P^+ of P is a RIO (related to Σ with a Lorentz transformation) whereas the proper frame Σ_Q^+ of Q is not a RIO and it is not related with a Lorentz transformation either with Σ or with Σ_P^+. This is expected because the Lorentz transformation relates two RIO and not a RIO with an accelerated observer. There are two options to continue

- Either find a way to interpret the accelerated motion in terms of inertial motions or
- Extend by means of some new definitions and principles the Lorentz transformation to relate a RIO with an accelerated observer. In this generalization one must demand that in the limit of vanishing acceleration the transformation will reduce to the standard Lorentz transformation.

In this section we consider the first option and leave the second for subsequent sections.

The moment $t+dt$ of Σ the relative position of P, Q in Σ is $\mathbf{r}_{PQ}(t+dt)$ different from $\mathbf{r}_{PQ}(t)$ and to be specific

$$\mathbf{r}_{PQ}(t+dt) = \mathbf{r}_{PQ}(t) + \mathbf{v}_{PQ}(t+dt)dt,$$

where the relative velocity $\mathbf{v}_{PQ}(t+dt)$ of P, Q the moment $t+dt$ of Σ is

$$\mathbf{v}_{PQ}(t+dt) = \mathbf{a}_{PQ}(t)dt,$$

where $\mathbf{a}_{PQ}(t)$ is the relative acceleration of Q in Σ_P^+. Because Σ_P^+ is a RIO $\mathbf{a}_{PQ}(t)$ equals the acceleration of Q in Σ, therefore it is different from zero. From this analysis we infer the following:

(1) At the moment t of Σ the proper frames Σ_P^+ and Σ_Q^+ coincide and Σ_P^+ is a RIO. We call Σ_P^+ the *Local Relativistic Inertial Observer (LRIO)* of Q the moment t of Σ. The moment $t + dt$ of Σ, the Σ_P^+, and Σ_Q^+ do not coincide and there is another LRIO for Q whose velocity in Σ is $\mathbf{v}_{PQ}(t + dt)$. This means that we can interpret kinematically the acceleration $\mathbf{a}_{PQ}(t)$ in Σ as a continuous change of LRIO. We conclude that the relativistic accelerated motion can be understood as a *continuous sequence* of (relativistic) inertial motions of varying velocity, or, equivalently as a *continuous sequence* of Lorentz transformations, parameterized by the proper time of Q.

(2) Geometrically the above interpretation can be understood as follows. The world line of an accelerated observer is a timelike differentiable curve in Minkowski space. This curve can be approximated by a great number of small straight line segments, as it is done, for example, in the computation of the length of a curve. Each such segment can be seen as a portion of a straight line, which subsequently is identified with the world line of a RIO. In the limit the world line of an accelerated observer is approximated by the continuous sequence of its tangents, parameterized by the proper time of the accelerated observer

By means of the above approach we achieve two goals as follows:

- We explain the accelerated motion in terms of inertial motions.
- It is possible to apply the Lorentz transformation along the world line of the accelerated observer provided that at each event we change the LRIO or the transformation.

Exercise 17 Show that if the three-acceleration of a ReMaP vanishes in a RIO, then it vanishes for all RIO. This implies that the accelerated motion is covariant in Special Relativity. That is, if a relativistic observer is accelerating wrt a RIO then accelerates wrt any other RIO. This result differentiates the inertial motion from the accelerated motion and in fact it is the expression of First Newton's Law in Special Relativity.

As it is the case with all relativistic physical quantities, the four-acceleration must be associated with a Newtonian physical quantity or must be postulated as a pure relativistic physical quantity. This is done as follows. In the proper frame Σ^+ of the position vector of an accelerated ReMaP (=relativistic mass point) P the four-acceleration has the reduced form $(0, \mathbf{a}^+)_{\Sigma^+}$, i.e., it is specified completely by a three-vector $\mathbf{a}^+$. We postulate that the three-acceleration which is measured by the LRIO Σ_τ of Σ^+ at the proper time moment τ of P coincides with the Newtonian acceleration of P as measured by Σ_τ. We assume that this is the acceleration that "feels" or measures (for example, with a gravitometer) the proper observer of P. We note the following result, which is consistent with this identification.

Exercise 18 Show that $\mathbf{a}^+ = \mathbf{0}$ if and only if $a^i = 0$ (see Exercise 17)

We note that

$$a^i a_i = (\mathbf{a}^+)^2 = a^2 \tag{7.8}$$

that is $(\mathbf{a}^+)^2$ is an invariant. This means that if a RIO measures the quantity $(\mathbf{a}^+)^2$ then this number is the same for all other RIO. However, the vector $\mathbf{a}^+$ itself is differentiated in a complex way, and this is what makes the study of four-acceleration difficult.

Example 23 Calculate the Lorentz length a^2 of the four-acceleration a^i in a RIO Σ in which the velocity of a ReMaP is $\mathbf{v}$ and its three-acceleration $\mathbf{a}$. Show that if in Σ the three-acceleration $\mathbf{a}$ is parallel to the three-velocity $\mathbf{v}$ then

$$a^2 = \left(\frac{d(\gamma\boldsymbol{\beta})}{dt}\right)^2_\Sigma, \tag{7.9}$$

whereas if it is normal to the three-velocity then

$$a^2 = \gamma^4\dot{\boldsymbol{\beta}}^2 \quad \text{where} \quad \boldsymbol{\beta} = \mathbf{v}/c.$$

[Hint: Use the identity $\dot{\gamma} = \boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}\gamma^3 = \frac{\mathbf{v}\cdot\dot{\mathbf{v}}\gamma^3}{c^2}$ (see (7.4))].

Solution

In Σ $a^i = \gamma\begin{pmatrix} c\dot{\gamma} \\ \mathbf{v}\dot{\gamma} + \gamma\mathbf{a}\end{pmatrix}_\Sigma$ (see (7.3)). Hence

$$\begin{aligned}
a^2 = a^i a_i &= c^2\gamma^2\left(-\dot{\gamma}^2 + (\dot{\gamma}\boldsymbol{\beta} + \gamma\dot{\boldsymbol{\beta}})^2\right) = \\
&= c^2\gamma^2\left(-\dot{\gamma}^2 + \dot{\gamma}^2\boldsymbol{\beta}^2 + \gamma^2\dot{\boldsymbol{\beta}}^2 + 2\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}\gamma\dot{\gamma}\right) = \\
&= c^2\gamma^2\left(-\dot{\gamma}^2(1-\boldsymbol{\beta}^2) + \gamma^2\dot{\boldsymbol{\beta}}^2 + 2\beta\dot{\beta}\cos\phi\gamma\dot{\gamma}\right) = \\
&= c^2\gamma^2\left(-\frac{\dot{\gamma}^2}{\gamma^2} + \gamma^2\dot{\boldsymbol{\beta}}^2 + 2\beta\dot{\beta}\cos\phi\gamma\dot{\gamma}\right).
\end{aligned}$$

From the hint we have $\dot{\gamma} = \beta\dot{\beta}\cos\phi\gamma^3$. Replacing we find[1]

$$\begin{gathered}
a^2 = c^2\gamma^2\left(-\beta^2\dot{\beta}^2\cos^2\phi\gamma^4 + \gamma^2\dot{\beta}^2 + 2\gamma^4\beta^2\dot{\beta}^2\cos^2\phi\right) \Rightarrow \\
a^2 = c^2\gamma^4\dot{\beta}^2\left(1 + \gamma^2\beta^2\cos^2\phi\right).
\end{gathered} \tag{7.10}$$

[1] Show that this result can be written in the equivalent form

$$\mathbf{a}^{+2} = [\mathbf{a}^2 - (\mathbf{a}\times\boldsymbol{\beta})^2\gamma^6].$$

This is the general case. In case $\boldsymbol{\beta} \parallel \dot{\boldsymbol{\beta}}$ (that is $\mathbf{v} \parallel \mathbf{a}$) $\cos\phi = 1$ and the above formula reduces to

$$a^2 = c^2\gamma^4\dot{\boldsymbol{\beta}}^2\left(1+\gamma^2\beta^2\right) = c^2\gamma^6\dot{\boldsymbol{\beta}}^2 = \gamma^6\mathbf{a}^2c^2.$$

This can be written differently. We have

$$(\boldsymbol{\beta}\gamma)^{\cdot} = \dot{\boldsymbol{\beta}}\gamma + \boldsymbol{\beta}\dot{\gamma} = \dot{\boldsymbol{\beta}}\gamma + \boldsymbol{\beta}^2\dot{\boldsymbol{\beta}}\gamma^3 = \dot{\boldsymbol{\beta}}\gamma(1+\beta^2\gamma^2) = \dot{\boldsymbol{\beta}}\gamma\gamma^2 = \dot{\boldsymbol{\beta}}\gamma^3$$

hence

$$a^2 = c^2\left[(\boldsymbol{\beta}\gamma)^{\cdot}\right]^2 = \left[(\gamma\mathbf{v})^{\cdot}\right]^2. \tag{7.11}$$

In case $\boldsymbol{\beta} \perp \dot{\boldsymbol{\beta}}$, $\cos\phi = 0$ and we get

$$a^2 = c^2\gamma^4\dot{\boldsymbol{\beta}}^2 = \gamma^4\mathbf{a}^2. \tag{7.12}$$

It becomes clear that the concept of four-acceleration is much more complicated and involved than that of the four-velocity. This should be expected because, unlike velocity, acceleration relates kinematics with dynamics. We continue with some useful examples.

Example 24 A ReMaP P in the RIO Σ has three-acceleration $\mathbf{a}$ and three-velocity $\mathbf{v}$. Calculate the three-acceleration of P in the RIO Σ' which moves wrt Σ in the standard configuration with speed u along the common axis x, x'.
Solution

The four-acceleration in Σ is given by

$$a^i = \begin{pmatrix} ca^0 \\ a^0\mathbf{v} + \gamma_v^2\mathbf{a} \end{pmatrix}_\Sigma ,$$

where $a^0 = \gamma_v\dot{\gamma}_v = \gamma_v^4c^{-2}(\mathbf{v}\cdot\mathbf{a})$. Suppose that in Σ' P has velocity $\mathbf{v}'$ and acceleration $\mathbf{a}'$, so that

$$a^i = \begin{pmatrix} ca^{0'} \\ a^{0'}\mathbf{v}' + \gamma_{v'}^2\mathbf{a}' \end{pmatrix}_{\Sigma'} ,$$

where $a^{0'} = \gamma_{v'}\dot{\gamma}_{v'} = \gamma_{v'}^4c^{-2}(\mathbf{v}'\cdot\mathbf{a}')$.
The two expressions are related with a boost

$$\begin{aligned}
ca^{0'} &= \gamma_u \left[ca^0 - \frac{u}{c}(a^0 v^x + \gamma_v^2 a^x) \right] \\
a^{0'} v^{x'} + \gamma_{v'}^2 a^{x'} &= \gamma_u (a^0 v^x + \gamma_v^2 a^x - \frac{u}{c} a^0 c) \\
a^{0'} v^{y'} + \gamma_{v'}^2 a^{y'} &= a^0 v^y + \gamma_v^2 a^y \\
a^{0'} v^{z'} + \gamma_{v'}^2 a^{z'} &= a^0 v^z + \gamma_v^2 a^z .
\end{aligned}$$

The first equation gives (we write $v^x = v_x$ and $a^x = a_x$ why?)

$$a^{0'} = \gamma_u a^0 \left(1 - \frac{u v^x}{c^2} \right) - \frac{u}{c^2} \gamma_u \gamma_v^2 a^x . \tag{7.13}$$

Replacing in the second follows

$$\gamma_u a^0 \left(1 - \frac{u v^x}{c^2} \right) v_{x'} - \frac{u}{c^2} v_{x'} \gamma_u \gamma_v^2 a_x + \gamma_{v'}^2 a_{x'} =$$

$$= \gamma_u (a^0 v_x + \gamma_v^2 a_x - u a^0) \Rightarrow$$

$$\gamma_{v'}^2 a_{x'} = \gamma_u \gamma_v^2 \left(1 + \frac{u v_{x'}}{c^2} \right) a_x + a^0 \gamma_u \left[v_x - u - v_{x'} \left(1 - \frac{u v^x}{c^2} \right) \right] .$$

But from the composition rule of three-velocities we have $v_{x'} = \frac{v_x - u}{1 - \frac{u v_x}{c^2}}$ therefore the second term in the rhs vanishes. Hence

$$a_{x'} = \frac{\gamma_u \gamma_v^2}{\gamma_{v'}^2} \left(1 + \frac{u v_{x'}}{c^2} \right) a_x . \tag{7.14}$$

But from (6.7) we have

$$\gamma_v = \gamma_u \gamma_{v'} \left(1 + \frac{u v_{x'}}{c^2} \right) \Rightarrow \frac{\gamma_v}{\gamma_{v'}} = \gamma_u \left(1 + \frac{u v_{x'}}{c^2} \right) . \tag{7.15}$$

Replacing in (7.14) we find

$$a_{x'} = \frac{\gamma_v^3}{\gamma_{v'}^3} a_x .$$

This relation can be written in the form

$$\gamma_{v'}^3 a_{x'} = \gamma_v^3 a_x , \tag{7.16}$$

which indicates that the quantity $\gamma_v^3 a_x$ is an invariant under boosts. This is a very useful result in applications.

Equation (7.14) can be written differently. From (7.15) follows

$$\gamma_{v'} = \gamma_u \gamma_v \left(1 - \frac{u v_x}{c^2}\right) \Rightarrow \frac{\gamma_v}{\gamma_{v'}} = \frac{1}{\gamma_u \left(1 - \frac{u v_x}{c^2}\right)}$$

and (7.14) becomes

$$a_{x'} = \frac{1}{\gamma_u^3 \left(1 - \frac{u v_x}{c^2}\right)^3} a_x. \tag{7.17}$$

Working similarly we show that

$$a_{y'} = \frac{1}{\gamma_u^2 \left(1 - \frac{u v_x}{c^2}\right)^2} \left(a_y + \frac{\frac{u v_y}{c^2}}{1 - \frac{u v_x}{c^2}} a_x \right), \tag{7.18}$$

$$a_{z'} = \frac{1}{\gamma_u^2 \left(1 - \frac{u v_x}{c^2}\right)^2} \left(a_z + \frac{\frac{u v_z}{c^2}}{1 - \frac{u v_x}{c^2}} a_x \right). \tag{7.19}$$

It is worth discussing the kinematic implications of the transformation relations (7.17), (7.18), and (7.19). They imply that the three-acceleration $\mathbf{a}'$ in Σ' depends on both the three-acceleration $\mathbf{a}$ *and* the three-velocity $\mathbf{v}$ of P in Σ. Therefore, in general, a motion along the x-axis with constant acceleration $\mathbf{a} = a_x \mathbf{i}$ does not imply that in Σ' $\mathbf{a}'_y, \mathbf{a}'_z$ vanish unless $v_y = v_z = 0$! This means that in Σ' the motion is in general three-dimensional and furthermore it is not with uniform acceleration.

Let us examine the important case of planar motion. It is easy to show that if the motion in Σ is accelerated in the y–z plane (in general, a plane normal to the direction of the relative velocity of Σ, Σ') then the motion in Σ' is also planar. In all other cases the motion in Σ' is not planar. This means that the concept of planar motion in Special Relativity is not covariant, hence it has no physical meaning in that theory. An important motion of this type of great interest in Physics is the central motion, which we examine in the following exercise.

Exercise 19 A RIO Σ' is moving wrt the RIO Σ in the standard configuration with speed u along the common x, x'-axis. Show that a central motion in the plane y, z of Σ is also a central motion in the plane y', z' of Σ'.

For a general relative velocity $\mathbf{u}$ of Σ, Σ' relations (7.17), (7.18), and (7.19) read as follows:

$$\mathbf{a}' = \frac{1}{\gamma_u^2 \left(1 - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right)^3} \left[\left(1 - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right)\mathbf{a} + \frac{\mathbf{u}\cdot\mathbf{a}}{c^2}\mathbf{v} + \frac{\mathbf{u}\cdot\mathbf{a}}{u^2}\left(\frac{1}{\gamma_u} - 1\right)\mathbf{u} \right], \tag{7.20}$$

where $\mathbf{a}, \mathbf{a}'$ is the three-acceleration of the ReMaP P in Σ, Σ', respectively, and $\mathbf{v}$ is the velocity of the ReMaP in Σ. From this relation we find that the parallel and the normal components of $\mathbf{a}'$ wrt $\mathbf{u}$ are

$$\mathbf{a}'_{\|} = \frac{1}{\gamma_u^3 \left(1 - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right)^3} \mathbf{a}_{\|}, \tag{7.21}$$

$$\mathbf{a}'_{\perp} = \frac{1}{\gamma_u^2 \left(1 - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right)^3} \left[\mathbf{a}_{\perp} - \frac{1}{c^2}\mathbf{u} \times (\mathbf{a} \times \mathbf{v})\right] \tag{7.22}$$

$$= \frac{1}{\gamma_u^2 \left(1 - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right)^2} \left[\mathbf{a}_{\perp} + \frac{u v_{\perp}}{c^2 \left(1 - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right)} \mathbf{a}_{\|}\right]. \tag{7.23}$$

The proof of (7.20), (7.21), (7.22), and (7.23) is tedious but standard.

Exercise 20 Consider two LCF Σ, Σ' with parallel axes and relative velocity $\mathbf{u}$. A ReMaP P in Σ, Σ' has three-velocity $\mathbf{v}$, $\mathbf{v}'$ and three-acceleration $\mathbf{a}$, $\mathbf{a}'$ respectively. Show that the proper three-acceleration of P satisfies the relation

$$\mathbf{a}^{+} = \gamma_v^3 \mathbf{a}_{\|} + \gamma_v^2 \mathbf{a}_{\perp} = \gamma_v'^3 \mathbf{a}'_{\|} + \gamma_v'^2 \mathbf{a}'_{\perp}, \tag{7.24}$$

where the parallel and the normal analysis refers to the direction of the relative velocity $\mathbf{u}$ of Σ, Σ'. Using this result prove that

(1) A ReMaP moves along a straight line in both Σ and Σ' if and only if it is moving along the direction of the relative velocity $\mathbf{u}$ of Σ, Σ'. If this is the case then the accelerations satisfy the relation

$$\gamma_+^3 \mathbf{a}_{\|} = \gamma_+'^3 \mathbf{a}'_{\|} = \mathbf{a}^{+}. \tag{7.25}$$

(2) A ReMaP moves in a planar motion in both RIO Σ, Σ' if, and only if, this plane is normal to the relative velocity $\mathbf{u}$ of Σ, Σ'. Furthermore in that case the accelerations satisfy the relation

$$\gamma_v^2 \mathbf{a}_{\perp} = \gamma_v'^2 \mathbf{a}'_{\perp} = \mathbf{a}^{+}. \tag{7.26}$$

[Hint: Assume that one of the RIO, the Σ say, coincides with the proper frame of P. Then $\mathbf{u} = \mathbf{v}$, $\mathbf{v}' = \mathbf{0}$ and equations (7.21) and (7.23) give $\mathbf{a}_{\|} = \gamma_v^{-3} \mathbf{a}_{\|}^{+}$, $\mathbf{a}_{\perp} = \gamma_v^{-2} \mathbf{a}_{\perp}^{+}$].

7.3 Calculating Accelerated Motions

When we are given an accelerated motion in a RIO Σ' and wish to describe the motion in another RIO Σ which is related to Σ' with the Lorentz transformation $L(\Sigma', \Sigma)$ there are two methods to work:

- First method: Use the Lorentz transformation to calculate the acceleration in Σ and then solve the differential equation $\frac{d^2x^i}{d\tau^2} = a^i$ in Σ. Because the transformation of the four-acceleration involves the three-velocity we have to Lorentz transfer and the four-velocity.
- Second method: Solve the quadratic equation $\frac{d^2x^i}{d\tau^2} = a^i$ in Σ', that is compute the three-orbit $\mathbf{r}'(t')$ and then use the Lorentz transformation to find the orbit in Σ.

As a rule the second method is simpler because we choose Σ' so that the equations of motion are simpler.

Example 25 The LCF Σ and Σ' move in the standard configuration with speed u along the common axis x, x'. A ReMaP P departs from rest the time moment $t' = 0$ (of Σ') and moves along the x'-axis with constant acceleration (in Σ'!) $a = \frac{c}{\tau}$, where τ is the life time of P (a constant). If $u = c/\sqrt{2}$, show that the motion in Σ is described by the equation

$$x^2 - 2\sqrt{2}c(t+\tau)x + 2c^2t(t+c) = 0.$$

For $t \ll \tau$ show that the accepted solution is $x = \frac{ct}{\sqrt{2}}$. Comment on the result.
Solution

The position four-vector of P in Σ and Σ' is

$$x^i = \begin{pmatrix} ct \\ \mathbf{x} \end{pmatrix}_\Sigma = \begin{pmatrix} ct' \\ \mathbf{x}' \end{pmatrix}_{\Sigma'}.$$

It is easy to calculate that in Σ' the trajectory of P is described by the equation

$$x' = \frac{1}{2}\frac{c}{\tau}t'^2. \tag{7.27}$$

To calculate the equation of motion of P in Σ we apply the boost along the x, x'-axis with speed u:

$$x = \gamma(u)\left(\frac{1}{2}\frac{c}{\tau}t'^2 + ut'\right),$$
$$ct' = \gamma(u)\left(ct - \frac{u}{c}x\right).$$

Replacing in (7.27) we find

$$x = \gamma(u)\left\{\frac{1}{2}\frac{c}{\tau}\left[\gamma(u)\left(t - \frac{u}{c^2}x\right)\right]^2 + u\left[\gamma(u)\left(t - \frac{u}{c^2}x\right)\right]\right\}.$$

For $u = \frac{c}{\sqrt{2}}$ we compute $\gamma(u) = \sqrt{2}$. It follows that the motion of P in Σ is described by the equation

$$x^2 - 2\sqrt{2}c(t+\tau)x + 2c^2 t(t+\tau) = 0.$$

The last relation can be written as

$$x^2 - 2\sqrt{2}c\tau\left(1+\frac{t}{\tau}\right)x + 2c^2\tau t\left(1+\frac{t}{\tau}\right) = 0.$$

If $\frac{t}{\tau} \ll 1$ this becomes

$$x^2 - 2\sqrt{2}c\tau x + 2c^2\tau t = 0 \Longrightarrow x = \sqrt{2}\tau c \pm \sqrt{2c^2\tau^2 - 2c^2\tau t}$$

therefore

$$x = \sqrt{2}\tau c\left(1 \pm \sqrt{1-\frac{t}{\tau}}\right) \approx \sqrt{2}c\tau\left[1 \pm \left(1-\frac{t}{2\tau}\right)\right] = \begin{cases} \frac{ct}{\sqrt{2}} \\ \text{or} \\ 2\sqrt{2}c\tau\left(1-\frac{t}{4\tau}\right). \end{cases}$$

The first solution is selected if the initial condition is $x(0) = 0$ and the second solution if the initial condition is $x(0) = 2\sqrt{2}c\tau$. According to Lorentz transformation when $t = t' = 0$ the $x = x' = 0$, which implies that the accepted solution is $x = \frac{ct}{\sqrt{2}}$.

Example 26 (Hyperbolic and Harmonic Oscillator) A ReMaP P is moving with constant speed u along the x'-axis of a LCF Σ'. When P passes through the origin of Σ'', it begins to accelerate with acceleration $a' = k^2 x'$ where k is a positive constant. Let Σ be another LCF which is related to Σ' with a boost along the x-axis with velocity factor β. Compute the position, the velocity, and the acceleration of P in Σ. Repeat the calculations for $a' = -k^2 x'$.

Solution

The equation of motion of P in Σ' is $\frac{d^2x'}{dt'^2} = k^2 x'$ with the initial condition $x'(0) = 0, \dot{x}(0) = u$. Therefore in Σ' the orbit is

$$x'(t') = \frac{u}{k}\sinh kt'. \tag{7.28}$$

In order to find the orbit in Σ we apply the boost relating Σ, Σ'. We have

$$t' = \gamma\left(t - \frac{\beta}{c}x\right), \tag{7.29}$$

$$x = \gamma\,(x' + \beta ct'). \tag{7.30}$$

From (7.28) and (7.29) follows

$$x' = \frac{u}{k}\sinh\left[k\gamma\left(t - \frac{\beta}{c}x\right)\right]. \tag{7.31}$$

Replacing x', t' from (7.29) and (7.31) in (7.30) we compute the required orbit:

$$x = \gamma \left\{ \frac{u}{k} \sinh \left[k\gamma \left(t - \frac{\beta}{c} x \right) \right] + \beta c \gamma \left(t - \frac{\beta}{c} x \right) \right\} \Longrightarrow$$

$$x = \frac{\gamma u}{k} \sinh \left[k\gamma \left(t - \frac{\beta}{c} x \right) \right] + \beta c \gamma^2 t - \beta^2 \gamma^2 x \Longrightarrow$$

$$(1 + \beta^2\gamma^2)x - \beta c \gamma^2 t = \frac{\gamma u}{k} \sinh \left[k\gamma \left(t - \frac{\beta}{c} x \right) \right].$$

But $(1 + \beta^2\gamma^2) = \gamma^2$ hence

$$x - \beta c t = \frac{u}{k\gamma} \sinh \left[k\gamma \left(t - \frac{\beta}{c} x \right) \right]. \tag{7.32}$$

Differentiating (7.32) wrt t we find

$$v - \beta c = \frac{u}{k\gamma} k\gamma \left(1 - \frac{\beta}{c} v \right) \cosh \left[k\gamma \left(t - \frac{\beta}{c} x \right) \right] \Longrightarrow$$

$$v - \beta c = u \left(1 - \frac{\beta}{c} v \right) \cosh \phi \text{ where } \phi = k\gamma \left(t - \frac{\beta}{c} x \right).$$

Solving in terms of v we find the speed of P in Σ:

$$v = \frac{u \cosh \phi + \beta c}{1 + \frac{u\beta}{c} \cosh \phi} \tag{7.33}$$

and by differentiation the acceleration:

$$a = \frac{ku}{\gamma} \frac{\sinh \phi \left(1 - \frac{\beta}{c} v \right)}{\left(1 + \frac{u\beta}{c} \cosh \phi \right)^2}. \tag{7.34}$$

Working similarly for the acceleration $a' = -k^2 x'$ (harmonic oscillator in Σ' not in Σ!) we find

$$x'(t') = \frac{u}{k} \sin kt' \tag{7.35}$$

and

$$x = \frac{u}{\gamma k} \sin \phi + \beta c t, \quad v = \frac{u \cos \phi + \beta c}{1 + \frac{u\beta}{c} \cos \phi}, \quad a = -\frac{ku}{\gamma} \frac{\sin \phi \left(1 - \frac{\beta}{c} v \right)}{\left(1 + \frac{u\beta}{c} \cos \phi \right)^2}. \tag{7.36}$$

7.4 Hyperbolic Motion of a Relativistic Mass Particle

In the last section we have shown that the motion of a ReMaP along the straight line defined by the direction of the proper acceleration $\mathbf{a}^+$ is covariant only in the LCF whose velocity is parallel to $\mathbf{a}^+$. Although this type of motion is very special, it is useful because it gives us the possibility to study the kinematics of four-acceleration in rather simple problems. More interesting is the case $\mathbf{a}^+ =$ constant. This type of motion we name *hyperbolic motion* and discuss it in Example 27.

Example 27 A ReMaP P moves along the x-axis of the LCF Σ with velocity u and constant proper acceleration a^+. Let τ be the proper time of P. Ignore the superfluous coordinates y, z.

(1) Calculate the four-acceleration of P in Σ.
(2) Calculate the three-acceleration $\mathbf{a} = \frac{d\mathbf{v}}{dt}$ of P in Σ.
(3) If ψ is the rapidity of P in Σ show that $\frac{d\psi}{d\tau} = \frac{a^+}{c}$. Calculate the four acceleration in terms of the rapidity ψ.
(4) Give the component of the four-velocity in Σ for a general proper acceleration $a^+(\tau)$.
(5) In case $a^+ =$ constant *compute* the four-velocity in Σ assuming the initial condition: For $\tau = 0, t = 0, x = 0$ (departure from the origin of Σ) and $u = 0$ (departure from rest in Σ). In addition show that in Σ the following statements are valid:

   – The β-factor of P in Σ is $\beta = \tanh \frac{a^+\tau}{c}$.
   – The γ-factor of P in Σ is $\gamma = \cosh \frac{a^+\tau}{c}$.
   – The position of P in Σ in terms of the proper time τ is

$$x(\tau) = \frac{c^2}{a^+}\left(\cosh \frac{a^+\tau}{c} - 1\right).$$

   – The time of P in Σ in terms of the proper time τ is

$$t(\tau) = \frac{c}{a^+} \sinh \frac{a^+\tau}{c}.$$

   – The position of P in terms of the time t in Σ is

$$x(t) - x_0 = \frac{c^2}{a^+}\left(\sqrt{1 + \frac{a^{+2}t^2}{c^2}} - 1\right).$$

   It is given that $\dot{\gamma} = \boldsymbol{\beta}\dot{\boldsymbol{\beta}}\gamma^3$ (see (7.4)).

Solution

(1) In the proper frame $\Sigma(\tau)$ of P at the proper time τ, the four-acceleration is

$$a^i = \begin{pmatrix} 0 \\ a^+ \end{pmatrix}_{\Sigma(\tau)},$$

where for clarity we have ignored the y, z coordinates. Suppose that in Σ the four-acceleration is

$$a^i = \begin{pmatrix} a^0 \\ a^1 \end{pmatrix}_{\Sigma}.$$

The boost relating Σ, $\Sigma(\tau)$ gives

$$a^1 = \gamma a^+, \ a^0 = \beta\gamma a^+ \tag{7.37}$$

hence

$$a^i = \begin{pmatrix} \beta\gamma a^+ \\ \gamma a^+ \end{pmatrix}_{\Sigma}. \tag{7.38}$$

Second solution
The orthogonality relation $u^i a_i = 0$ between the four-velocity and the four-acceleration gives

$$a^0 u^0 - a^1 u^1 = 0 \Longrightarrow \begin{pmatrix} a^0 = \lambda u^1 \\ a^1 = \lambda u^0 \end{pmatrix} \tag{7.39}$$

($u^0 > 0, \lambda > 0$ assuming motion toward the positive x-axis). The Lorentz length of the four-acceleration is a^+ hence

$$(a^+)^2 = (a^1)^2 - (a^0)^2 = -\lambda^2[(u^1)^2 - (u^0)^2] = \lambda^2 c^2 \Longrightarrow \lambda = \frac{a^+}{c}.$$

From these relations and the components of the four-velocity $u^1 = \gamma u$, $u^0 = \gamma c$ we obtain again (7.37) and (7.38).

(2) We know that in Σ the four-acceleration a^i has components (see (7.5))

$$a^i = \begin{pmatrix} c a_0 \\ a_0 u + \gamma^2 a \end{pmatrix}_{\Sigma}.$$

Comparing with (7.38) we find the equations

$$\left\{ \begin{array}{l} \beta\gamma a^+ = c a_0 \\ \gamma a^+ = a_0 u + \gamma^2 a \end{array} \right\} \Longrightarrow a = \frac{1}{\gamma^3} a^+. \tag{7.40}$$

Second solution to the second question
As we have shown in Example 25 (see (7.16)) the quantity $a_x \gamma^3$ is an invariant. In the proper frame $\gamma^+ = 1$ hence $a_x \gamma^3 = a^+$, etc.

(3) The rapidity of P in Σ is defined by the relation $\gamma = \cosh\psi$. Differentiating wrt t we find $\dot{\gamma} = \sinh\psi\dot{\psi}$. Replacing $\dot{\gamma}$ from $\dot{\gamma} = \beta\dot{\beta}\gamma^3$ and $\sinh\psi = \beta\gamma$ we find

$$\frac{1}{c^2}ua\gamma^3 = \frac{u}{c}\gamma\dot{\psi} \Longrightarrow \dot{\psi} = \frac{a\gamma^2}{c} \Longrightarrow \frac{d\tau}{dt}\frac{d\psi}{d\tau} = \frac{a}{c}\gamma^2 \Longrightarrow$$
$$\frac{d\psi}{d\tau} = \frac{a}{c}\gamma^3.$$

But from question 2. we have $a\gamma^3 = a^+$, hence

$$\frac{d\psi}{d\tau} = \frac{a^+}{c} \Longrightarrow \psi = \frac{a^+\tau}{c}. \tag{7.41}$$

We infer the (important) result that the quantity $d\psi$ is an invariant. Concerning the expression of the four-acceleration in terms of the rapidity we have[2]

$$a^0 = \beta\gamma a^+ = a^+\sinh\psi = a^+\sinh\frac{a^+\tau}{c}, \tag{7.42}$$

$$a^1 = \gamma a^+ = a^+\cosh\frac{a^+\tau}{c}. \tag{7.43}$$

(Notice the difference between a_1 and a^+).

(4) Calculation of the four-velocity
From (7.39) we find

$$u^0 = \frac{1}{\lambda}a^1 = \frac{c}{a^+}a^+\cosh\frac{a^+\tau}{c} = c\cosh\frac{a^+\tau}{c},$$
$$u^1 = \frac{1}{\lambda}a^0 = \frac{c}{a^+}a^+\sinh\frac{a^+\tau}{c} = c\sinh\frac{a^+\tau}{c}.$$

Hence the four-velocity of P in Σ is

$$u^i = \begin{pmatrix} c\cosh\frac{a^+\tau}{c} \\ c\sinh\frac{a^+\tau}{c} \end{pmatrix}_{\Sigma}. \tag{7.44}$$

Another way to calculate the components of the four-velocity is the following. From the definition of the four-velocity and relations (7.42) and (7.43) we have

$$u^0 = \int a^0 d\tau = \int a^+\sinh\frac{a^+\tau}{c}d\tau = c\cosh\frac{a^+\tau}{c} + A,$$

[2] We consider a^+ =constant, however, the working remains the same if $a^+(\tau)$.

where A is a constant. Similarly we compute $u^1 = c \sinh \frac{a^+\tau}{c} + B$ where B is a constant. Applying the constraint $u^i u_i = -c^2$ and the initial conditions we compute $A = B = 0$ and the previous result follows.

In order to calculate the motion of P in Σ in terms of the proper time τ we consider the definition of u^0, u^1 and write

$$\frac{dt}{d\tau} = \frac{u^0}{c} = \cosh \frac{a^+\tau}{c}, \tag{7.45}$$
$$\frac{dx}{d\tau} = u^1 = c \sinh \frac{a^+\tau}{c}. \tag{7.46}$$

Integrating we find

$$t = \frac{c}{a^+} \sinh \left(\frac{a^+\tau}{c} \right) + A_1,$$
$$x = \frac{c^2}{a^+} \cosh \frac{a^+\tau}{c} + B_1,$$

where A_1, B_1 are constants. Assuming the initial condition $t(0) = 0,\ x(0) = x_0$ we compute $A_1 = 0,\ B_1 = -\frac{c^2}{a^+} + x_0$. Finally

$$\begin{aligned} t &= \tfrac{c}{a^+} \sinh \left(\tfrac{a^+\tau}{c} \right), \\ x - x_0 &= \tfrac{c^2}{a^+} (\cosh \tfrac{a^+\tau}{c} - 1). \end{aligned} \tag{7.47}$$

In order to calculate the motion of P in Σ in terms of the time t in Σ we use the transformation of the time component

$$\tau = \frac{t}{\gamma} = \frac{t}{\cosh \frac{a^+\tau}{c}} = \frac{t}{\sqrt{1 + \sinh^2 \frac{a^+\tau}{c}}} = \frac{t}{\sqrt{1 + \left(\frac{a^+t}{c} \right)^2}}. \tag{7.48}$$

Similarly for the position we have from (7.46)

$$x - x_0 = \frac{c^2}{a^+} \left(\sqrt{1 + \sinh^2 \frac{a^+\tau}{c}} - 1 \right) = \frac{c^2}{a^+} \left(\sqrt{1 + \left(\frac{a^+t}{c} \right)^2} - 1 \right). \tag{7.49}$$

7.4.1 Geometric Representation of Hyperbolic Motion

It is useful to discuss the geometric representation of hyperbolic motion in the Euclidean plane (ct, x). From (7.47) we have for the world line of P

$$\left(x - x_0 + \frac{c^2}{(a)^+}\right)^2 - c^2t^2 = q^2, \tag{7.50}$$

where $q = \frac{c^2}{a^+}$. Equation (7.50) is the equation of the world-line of the ReMaP P. The graphic representation of (7.50) on the Euclidean plane $x - ct$ is a hyperbola (hence the name of this type of motion) with asymptotes $x + \frac{c^2}{(a)^+} - x_0 = \pm ct$. The asymptotes are null spacetime curves which can be considered as the world lines of photons emitted from the point $x = -\frac{c^2}{(a)^+} + x_0$ at the moment $t = 0$ of Σ. The motion is a "rotation" (along a hyperbolic circle) of the line connecting the point P with the pivot point $x = -\frac{c^2}{(a)^+} + x_0$. The "radius" of this rotation is $q = \frac{c^2}{a^+}$. The above are represented in Fig. 7.1.

Exercise 21 Based on the above kinematic interpretation of the asymptotes $x + \frac{c^2}{(a)^+} = \pm ct$ show that if a ReMaP P which rests at the origin $x = 0$ of Σ departs at the moment $t = 0$ in Σ along the x-axis with constant proper acceleration a^+ while at the same moment ($t = 0$) of Σ a photon is emitted from the point $-\frac{c^2}{a^+}$ of the x-axis toward P, the photon will never reach P.

The geometric representation of the world line becomes more explanatory if we introduce the new variable $X = x + \frac{c^2}{(a)^+}$. Then the equation of the world line reads (see Fig. 7.2)

$$(X - x_0)^2 - c^2t^2 = q^2. \tag{7.51}$$

In the new coordinates the pivot point is located at the value $X = x_0$ and the parameter q is the radius of hyperbolic rotation (see Fig. 7.2).

If we represent the world line in the complex plane (X, ict) we obtain a circle of radius q (Fig. 7.3).

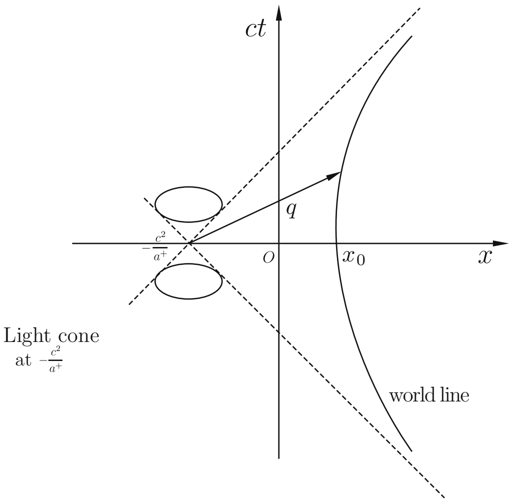


Fig. 7.1 Geometric representation of hyperbolic motion

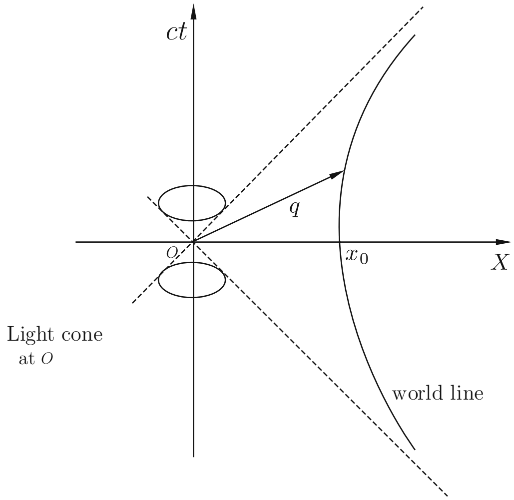


Fig. 7.2 Geometric representation of hyperbolic motion in coordinates (X, q)

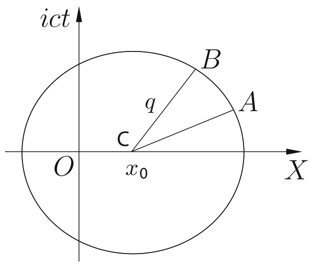


Fig. 7.3 Geometric representation of hyperbolic motion in the complex plane

Exercise 22 Show that in the complex plane the asymptotes $X - x_0 = \pm ct$ pass through the point $(x_0, 0i)$ and the world line becomes a circle of radius q. If $\phi(A, B)$ is the angle between two rays CA, CB in the complex plane, then

$$arc(AB) = q\phi(A, B). \tag{7.52}$$

The angle ϕ (A, B) equals the rapidity between the LRIO with world lines tangent to the events A, B.

Exercise 23 Consider the expressions ($\psi \in R, q$ =constant)

$$X - x_0 = q \cosh \psi, \tag{7.53}$$

$$ct = q \sinh \psi \tag{7.54}$$

and show that they define a parametric expression of the world line $(X - x_0)^2 - c^2t^2 = q^2$ with parameter ψ. Also show that the Lorentz length ds along the world line is given by the expression

$$ds = qd\psi \tag{7.55}$$

and it is reduced to (7.52).

Exercise 24 Show that the parameter ψ of Exercise 23 satisfies the relations

$$\psi = \tanh^{-1}\beta, \tag{7.56}$$
$$\gamma = \cosh\psi, \tag{7.57}$$
$$\frac{d\psi}{d\tau} = \frac{a^+}{c}, \tag{7.58}$$
$$\psi = \frac{a^+\tau}{c}, \tag{7.59}$$

where τ is the proper time of P and the initial condition is $\tau = 0$ when $\psi = 0$.

The hyperbolic motion is a covariant motion, in the sense that if a ReMaP P moves in a LCF with hyperbolic motion then its motion in any other LCF is also hyperbolic. Indeed we observe that for hyperbolic motion in Σ

$$\begin{aligned} a^i &= \begin{pmatrix} a^0 \\ a^1 \end{pmatrix} = \begin{pmatrix} a^+ \sinh \frac{a^+}{c} \\ \\ a^+ \cosh \frac{a^+}{c} \end{pmatrix}_\Sigma = a^+ \begin{pmatrix} \sinh \frac{a^+\tau}{c} \\ \\ \cosh \frac{a^+\tau}{c} \end{pmatrix}_\Sigma \\ &= a^+ \begin{pmatrix} \frac{a^+}{c} t \\ (x - x_0 + \frac{c^2}{a^+})\frac{a^+\tau}{c^2} \end{pmatrix}_\Sigma = \frac{(a^+)^2}{c^2} x^i, \end{aligned}$$

where x^i is the position four-vector in the plane (ct, x). That is, we have

$$a^i = \frac{(a^+)^2}{c^2} x^i \tag{7.60}$$

or

$$a^i = \frac{a^+}{q} x^i. \tag{7.61}$$

Furthermore from the orthogonality condition $a^i u_i = 0$ we find

$$u^i x_i = 0. \tag{7.62}$$

Relations (7.60), (7.61), and (7.62) constitute the complete covariant characterization of hyperbolic motion. They may be considered as defining the uniform rotational motion in Special Relativity, because (7.62) shows that the position four-vector (the "radius" of rotation of the cyclic motion) is normal to the tangent of the spacetime orbit, which is the four-velocity u^i.

7.5 Synchronization

The concept of synchronization is a key concept in the understanding of Special Relativity. However, it appears that there does not exist a clear exposition of it in the literature. In the case of RIO the synchronization of the clocks is done by means of light signals in the well-known way (Einstein synchronization) but little is said for the synchronization of accelerated relativistic observers. In the subsequent sections we shall deal with this concept, always within the limits set by the level of this book.

In Special (and General) Relativity there are two "times," the coordinate time and the proper time. The first corresponds to the zeroth coordinate of a specific event measured by a relativistic observer (inertial or not!) by the method of chronometry. The second is the indication of the personal clock read (no measurement procedure is used!) by the same observer. The two concepts are also different mathematically. The first is a coordinate and the second is a Lorentz invariant. The Lorentz transformation relates the coordinate time, not the proper time.

Suppose we have a set of observers describing the various events in spacetime. Because in Special Relativity there is no universal (i.e., absolute) time, they are not able to relate their kinematical observations, they can only exchange their measurements by the appropriate Lorentz transformation – and that only in the case they are RIO – and the Lorentz transformation by itself does not produce information. For an intrinsic description of kinematics we must define a correspondence between the proper clocks of the relativistic observers (inertial or not!). Every such correspondence we call a ***synchronization.*** For a synchronization to be "satisfactory" we demand that it complies with the following requirements:

- Must be independent of the coordinate system employed by the observers
- In case the observers are RIO must be symmetric (i.e., observer independent in order to preserve the equivalence of the observers) and Lorentz covariant
- Must define a 1:1 correspondence between the world lines of the observers (that is at each proper moment of one observer must correspond one proper moment of the other and the opposite)

In Newtonian Physics the synchronization is unique (i.e., absolute) and it is defined by the identification of the "proper time" of all Newtonian observers (inertial or not) with the absolute clock. This is the reason why we do not consider explicitly the concept of synchronization in Newtonian Physics.

It is evident that there are infinite ways to define a synchronization, the most important and with a physical significance being those which are closest to the concept of Newtonian time. In the following we consider first the standard synchronization between two RIOs (Einstein synchronization) and subsequently a synchronization between a RIO and an accelerated observer.

7.5.1 Einstein Synchronization

It is natural to expect that the most useful and natural synchronization between RIO must be defined in terms of light signals. This synchronization is the one considered

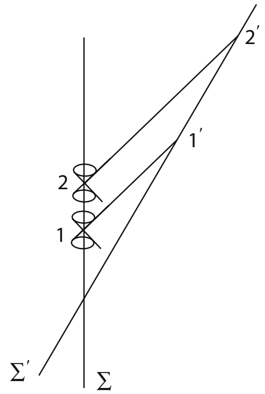


Fig. 7.4 Einstein synchronization of RIO

initially by Einstein and for this reason it is known as *Einstein synchronization*. This synchronization is defined as follows. Consider the world lines of two RIO Σ, Σ', which are straight lines. At the event 1 along the world line of one of the RIO, the Σ say, consider the light cone with apex at that event. This cone is unique and furthermore intersects the world line of the RIO Σ' at the point $1'$ say. This procedure can be done at every point along the world line of Σ and it is easy to show that it defines a synchronization between the RIO Σ, Σ' (see Fig. 7.4). If we consider a number of equidistant points $1, 2, 3, \ldots$ along the world line of Σ then with the Einstein synchronization the corresponding points $1', 2', 3', \ldots$ along the world line of Σ' are also equidistant due to the constancy of the velocity of light and the constancy of the relative velocity of the observers. This implies that if τ, τ' is the proper time of Σ, Σ' respectively then the Einstein synchronization is expressed analytically with the relation[3]

$$\tau = \tau'. \tag{7.63}$$

7.6 Rigid Motion of Many Relativistic Mass Points

To see the necessity of rigid motion in Special Relativity let us start with the following plausible situation. Suppose a spaceship moves with high speed in space in a long journey. We expect

(a) that during its motion the spaceship will accelerate and decelerate as required during the course of traveling
(b) the spaceship will not tear apart during the course of the motion and in fact neither the astronauts nor the various instruments will change shape or become shorter or longer, i.e., in general, the spaceship and all its contents will be "rigid" in the Newtonian sense

[3] The coordinate time of events are related by the Lorentz transformation, i.e., $ct' = \gamma(ct - \mathbf{r} \cdot \boldsymbol{\beta})$.

Obviously this contradicts Special Relativity due to length contraction. On the other hand it is a necessity, otherwise there is no point for us (the Newtonians!) to develop space traveling! The first to realize this situation was Max Born, a German physicist, who shortly after the introduction of Special Relativity (i.e., in 1910) asked if the untenable concept of "rigid body" can be generalized in Special Relativity to the concept of "rigid motion," that is a relativistic motion in which the spatial distances of a cluster of ReMaPs remains constant during the motion. It turned out that such motions are possible (under special conditions, not in general (!)) and have been called *Born rigid motions*. In the following we discuss *Born rigidity* for various types of motion. We start with the simple hyperbolic motion, continue with an arbitrary one-dimensional motion, and end up with the rotational motion.

However this is not the end of the story. Indeed how one keeps time in a spaceship? Due to the time dilatation effect two clocks set at identical readings at different points along the spaceship, as the motion occurs their indications will be different. Which is correct and which is false? Which clock the astronauts should believe? The situation can be dealt with an "equivalence" of clocks and this is what the synchronization is all about. We will define a number of synchronizations showing that this concept is conventional and indeed it is the transfer of the concept of absolute Newtonian time in Special Relativity.

7.7 Rigid Motion and Hyperbolic Motion

Consider two ReMaPs $1, 2$ which depart from rest at positions x_{01}, x_{02} and move along the x-axis with constant proper accelerations a_1^+ and a_2^+, respectively. According to the results of Sect. 7.4 the world lines of the ReMaPs are given by the equations

$$\left(x - x_{0i} + \frac{c^2}{a_i^+}\right)^2 - c^2t^2 = q_i^2 \qquad i = 1, 2,$$

where (ct, x) are coordinates in Σ and $q_i = \frac{c^2}{a_i^+}$. Figure 7.5 shows the world lines of the particles in the Euclidean plane (ct, x). We consider a point A_1 along the world line of ReMaP 1 and extend O_1A_1 until it intersects the world line of ReMaP 2 at the point A_1''. The length A_1A_1'' is the distance of the two ReMaPs as measured by ReMaP 1. If we draw the horizontal line from A_1'' to the axis ct we define the point A_1' of the world line 1. The length $A_1'A_1''$ is the distance of the two ReMaPs at the moment ct_1 of Σ. As ReMaP 1 "moves" along its world line the distance between the two ReMaPs changes and eventually becomes infinite when the projection of the "radius" of ReMaP 1 does not intersect the world line of ReMaP 2, e.g., $B_1B_1'' > A_1A_1''$. Note that the line O_1A_1 is the x-axis of the ReMaP 1.

Consider now the case $a_1^+ = a_2^+ = a^+$ and study the new situation (see Fig. 7.6). In this case the O_1, O_2 coincide and the distance of the ReMaP $1, 2$ according to 1 equals $A_1'' - A_1 = q_2 - q_1 = x_{02} - x_{01} =$ constant whereas the distance of the

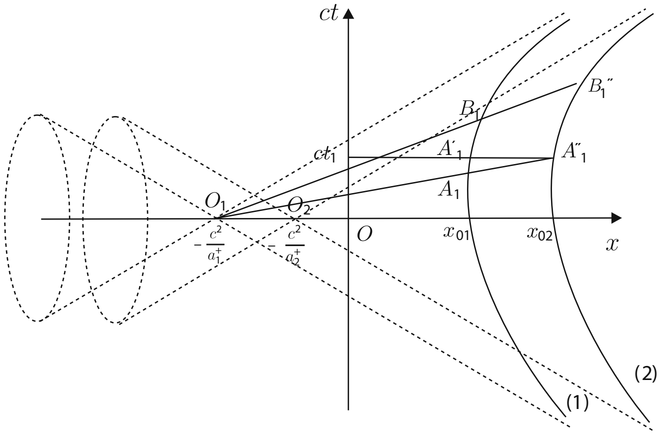


Fig. 7.5 General non-rigid motion

ReMaP in Σ, i.e., $A_1'' - A_1'$, changes as the point A_1 moves along the world line 1. Assume now that the two ReMaPs 1, 2 are the end points of a "rod." Can this rod be rigid? The answer is "no" because the rigid rod assumes invariance of its length under the Galileo group. The next question is: Can this rod "appear" to be rigid to the ReMaP 1 (and of course 2)? The answer is "yes" provided that $a_1^+ = a_2^+ = a^+$. To connect the result with the previous considerations we conclude that the astronauts in the spaceship will have the impression that the spaceship is like a rigid Newtonian structure provided all parts of the ship have the same proper acceleration (the result holds for one-dimensional motion!). We also note that the rapidity is the same for all parts of the rod therefore they have the same speed which is an additional confirmation that the rod moves as a coherent body. This type of motion is rigid motion in Special Relativity called *Born rigid motion* and the distance $A_1 A_1''$ we call the *proper length* of the (relativistic) rod 1, 2. Concerning the motion of the points of the rod we say that the rod is *Born accelerated* along the x-axis of the RIO Σ. Note that in Σ the spatial dimension of the spacecraft changes!

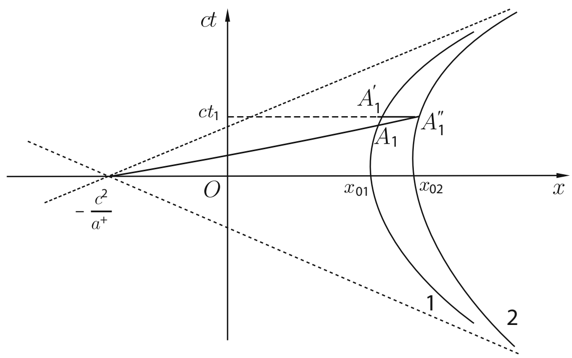


Fig. 7.6 Born rigid motion

The next step is to define a synchronization of the proper times among the points of the rod. This synchronization will define the "proper" time of the rod and will correlate the kinematics of the various parts of the rod. It is possible to define two different synchronizations for 1, 2.

7.7.1 The Synchronization of LRIO

Let A_1 be an arbitrary event along the world line of ReMaP 1 at the proper moment τ_1 and let A_1'' be the corresponding event on the world line of 2 (see Fig. 7.7), which is defined by the extension of O_1A_1. From (7.59) we have

$$\psi_{A_1} = \frac{a_1^+}{c}\tau_1 = \frac{a_2^+}{c}\tau_2 \Rightarrow a_1^+\tau_1 = a_2^+\tau_2. \tag{7.64}$$

Relation (7.64) defines a diffeomorphism between the world lines of 1, 2 hence a synchronization of the proper times of 1, 2. This synchronization we call the *synchronization of the LRIO*. The name is due to the fact that the line OA_1A_1'' defines the x'-axis of the LRIO of the events A_1, A_1''. In terms of proper times the synchronization is expressed by the relation

$$\tau_2 = \frac{a_1^+}{a_2^+}\tau_1 \tag{7.65}$$

and, in terms of proper time intervals

$$\delta\tau_2 = \frac{a_1^+}{a_2^+}\delta\tau_1. \tag{7.66}$$

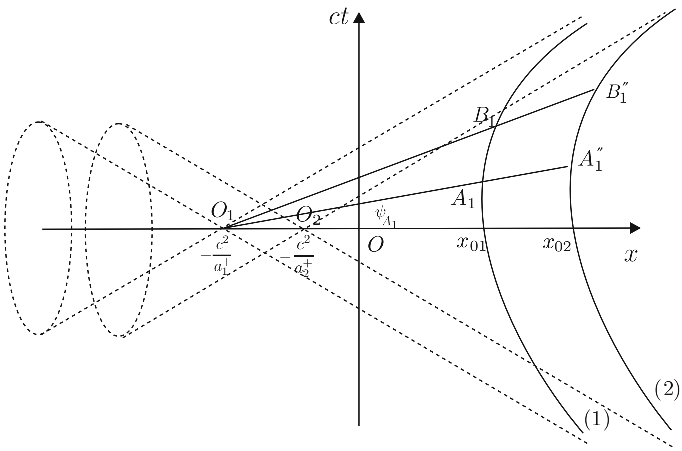


Fig. 7.7 Synchronization of LRIO

Obviously $\delta\tau_2 > \delta\tau_1$ since, as we have assumed $a_2^+ < a_1^+$. Relation (7.66) means that the rate of the proper clock of 1 is slower than the rate of the proper clock of 2, the difference being measured by the quotient of the proper accelerations a_1^+, a_2^+ of $1, 2$. This difference of proper clocks we call *acceleration time dilatation*. It is apparent that this dilatation has nothing to do with the Lorentz time dilatation, which is based on the Einstein synchronization and assumes inertial motion of the clocks $1, 2$.

In order to express geometrically the acceleration time dilatation we write condition (7.66) in terms of spacetime angles and arcs. From (7.59) and (7.66) we have

$$\delta\psi_{A_1''B_1''} = \delta\psi_{A_1B_1} \tag{7.67}$$

hence

$$\frac{s_{A_1'B_1'}}{s_{A_1B_1}} = \frac{\sqrt{\frac{c}{a_2^2} + x_{02}}}{\sqrt{\frac{c}{a_1^2} + x_{01}}}. \tag{7.68}$$

The synchronization of the LRIO is not the only one possible, but it has physical significance, because it is related directly with the observation of the spatial distance by the LRIO of the rod. We note that even if we have Born rigid motion, in which case the ReMaPs have the common proper acceleration a^+, the rates of time for the two ReMaPs are different due to the terms x_{01}, x_{02}. Hence it is not possible to have a "common" time for the two ReMaPs when the synchronization of the LRIO is used.

7.7.2 Synchronization of Chronometry

Consider two ReMaPs $1, 2$ which move with Born rigidity. Consider the events A, B along the world line of the ReMaP 1 (see Fig. 7.8) and assume that two light signals are send toward the ReMaP 2 and reach the world line of 2 at the points A', B', respectively. The lines AA', BB' are parallel to the surface of the light cones with vertex at the events A, B, respectively. Obviously this procedure defines a diffeomorphism between the world lines of $1, 2$ therefore a synchronization which is expressed with the following relation:

$$\tau_{A'} = \tau_A. \tag{7.69}$$

This synchronization we call the *synchronization of chronometry* and it coincides with the synchronization of the LRIO for this type of motion (see Fig. 7.9).

Let us compare the arcs s_{AB} and $s_{A'B'}$ along the world lines of 1 and 2. These are the rates of the proper clocks of $1, 2$. Assuming that the points A, B are "near" we write

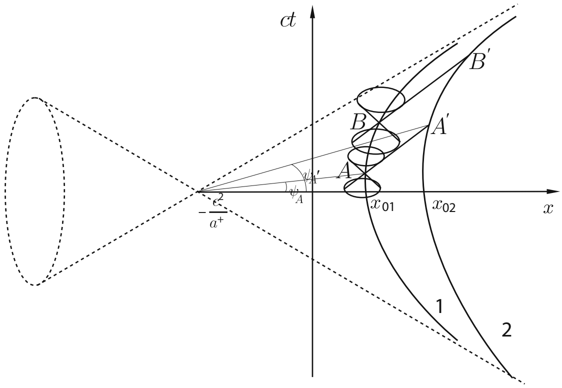


Fig. 7.8 Synchronization of chronometry

$$s_{A'B'} = q_2(\psi_{B'} - \psi_{A'}), \tag{7.70}$$

$$s_{AB} = q_1(\psi_B - \psi_A), \tag{7.71}$$

where ψ_i is the rapidity of the event i ($i = A, B, A', B'$). Using (7.59) and (7.64) we find

$$\frac{s_{A'B'}}{\tau_{B'} - \tau_{A'}} = \frac{s_{AB}}{\tau_B - \tau_A}, \tag{7.72}$$

that is, the quotient $\frac{s}{\Delta\tau}$ is invariant, justifying the name for this synchronization. We note the difference between the synchronization of the LRIO and that of chronometry. Which one we should follow is a matter of choice, convenience, and above all physical reality. For example, if we design the cardan of the proper clocks where the handles of the clocks move to be as the hyperbolic world lines and the rate of the clocks is the same, then if the indication of the clocks is the same at one moment it will stay the same at all future moments.

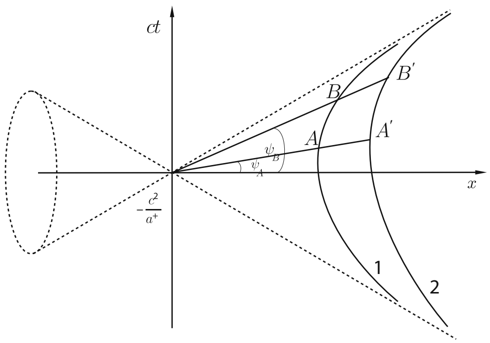


Fig. 7.9 Synchronization of LRIO for Born rigid motion

7.7.3 The Kinematics in the LCF Σ

The kinematics of a set of ReMaP in Σ does not depend on the synchronization chosen to relate the proper clocks of the particles. Indeed in Σ observation is done chronometrically therefore the quantities involved are only the coordinate ones. For example, let us examine the velocities and the accelerations as observed in Σ of the ReMaP 1, 2, which move hyperbolically with the same proper acceleration a. When the positions of 1, 2 are observed simultaneously in Σ the ReMaP 1 is at the event A_1' and the ReMaP 2 at the event A_1'' (see Fig. 7.6). The events A_1, A_1' do not have the same rapidity hence the observed velocities and accelerations of 1, 2 in Σ are different. More specifically we have

- Velocity of 1 as measured in Σ: $u_1 = c\tanh\psi_1$.
- Acceleration of 1 as measured in Σ: $a_1 = a^+\cosh\psi_1$.
- Velocity of 2 as measured in Σ: $u_2 = c\tanh\psi_2$.
- Acceleration of 2 as measured in Σ: $a_2 = a^+\cosh\psi_2$.

Using the results of Example 27 we write these quantities in terms of the proper times of 1,2

$$u_1 = c\tanh\left(\frac{a^+}{c}\tau_1\right) \qquad a_1 = a^+\cosh\left(\frac{a^+}{c}\tau_1\right)$$
$$u_2 = c\tanh\left(\frac{a^+}{c}\tau_2\right) \qquad a_2 = a^+\cosh\left(\frac{a^+}{c}\tau_2\right).$$

We infer that Σ measures the relative velocity

$$u_{21} = u_2 - u_1 = c\left[\tanh\left(\frac{a^+}{c}\tau_2\right) - \tanh\left(\frac{a^+}{c}\tau_1\right)\right] \tag{7.73}$$

and the relative acceleration

$$a_{21} = a_2 - a_1 = a^+\left[\cosh\left(\frac{a^+}{c}\tau_2\right) - \cosh\left(\frac{a^+}{c}\tau_1\right)\right]. \tag{7.74}$$

These equations can be written as follows:

$$u_{21} = \frac{c\tanh\left[\frac{a^+}{c}(\tau_2 - \tau_1)\right]}{1 - \tanh\left(\frac{a^+}{c}\tau_2\right)\tanh\left(\frac{a^+}{c}\tau_1\right)}, \tag{7.75}$$
$$a_{21} = 2a^+\sinh\left[\frac{a^+}{2c}(\tau_2 - \tau_1)\right]\sinh\left[\frac{a^+}{2c}(\tau_2 + \tau_1)\right]. \tag{7.76}$$

Example 28 A rod of length l is resting along the x-axis of the LCF Σ. At the time moment $t = 0$ of Σ the rod (that is all its points) starts to accelerate with proper acceleration $\mathbf{a}^+ = a^+\hat{\mathbf{x}}$. Assuming that the (Euclidean) proper length of the rod

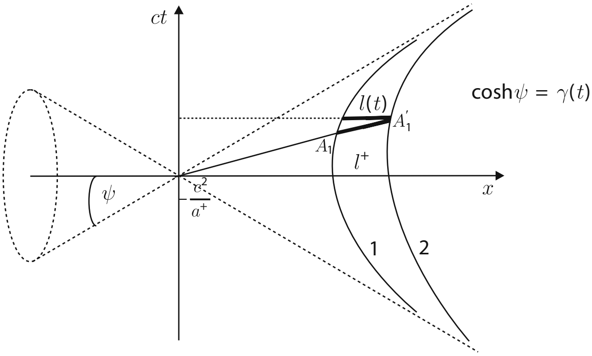


Fig. 7.10 Measurement of length of a Born accelerated rod

does not change during the motion (rigid body motion) calculate the length of the rod in Σ at the time moment t (of Σ!).[4]
Solution

We apply the boost, relating the proper observer of the rod (LRIO!) Σ^+ with the RIO in Σ, to the four-vector defined by the end points A, B of the rod. In Σ^+ we have $(AB)^+ = l^+, \delta t^+ = t_A^+ - t_B^+ \neq 0$ and in Σ $(AB) = l, \delta t_2 = t_{A,\Sigma} - t_{B,\Sigma} = 0$ because the ends A, B are observed simultaneously in Σ. The boost gives $l = \frac{l^+}{\gamma}$ where $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$ is a function of t (in Σ!). Because l^+ is assumed to be constant we differentiate this relation and get $dl = -\frac{l^+}{\gamma^2} d\gamma$. Integrating in the region $[0, t]$ and making use of the initial conditions $v(0) = 0$ and $l(0) = l^+$ it follows

$$l(t) = l^+ + l^+ \left(\frac{1}{\gamma(t)} - \frac{1}{\gamma(0)}\right) = \frac{l^+}{\gamma(t)}.$$

We note that at every moment this implies the standard Lorentz contraction with a γ-factor depending on the time t in Σ. Obviously as $t \to \infty$ the $\gamma(t) \to \infty$ hence $l \to 0$ (see Fig. 7.10).

Example 29 A rod is resting along the x'-axis of a RIO Σ' when it starts moving (all its points!) with constant acceleration a parallel to the y'-axis. At the same time (in Σ) the LCF Σ' starts moving in the standard configuration wrt Σ along the common x, x'-axis with speed v. Find the equation of motion of the rod in Σ. Comment on the result. Examine the case that the rod is not accelerating, that is $a = 0$.
Solution

In Σ' the equation of motion of an arbitrary point of the rod is

$$y' = \frac{1}{2} a t'^2, \quad x' = \text{constant}, \quad z' = 0.$$

[4] The assumption that the proper length of the rod does not change during acceleration is independent of the assumption $a^+ =$ constant. We shall discuss more on this matter elsewhere.

The boost relating Σ', Σ gives

$$x = \gamma(v)(x' + \beta ct'),$$
$$y = y' = \frac{1}{2}at'^2, \qquad z = z' = 0,$$
$$ct' = \gamma(v)\left(ct - \frac{v}{c}x\right).$$

From the first of these relations we find

$$ct' = \frac{1}{\beta}\left(-x' + \frac{x}{\gamma(v)}\right).$$

Using this, we eliminate t' from the other two equations and get

$$y = \frac{1}{2}a\frac{1}{c^2}\frac{1}{\beta^2}\left(-x' + \frac{x}{\gamma(v)}\right)^2 = \frac{a}{2c^2(\gamma^2(v)-1)}(x - \gamma(v)x')^2.$$

This equation shows that in the plane $x - y$ of Σ the rod appears to be a parabola (see Fig. 7.11).

This "distortion" of the shape of the rod in Σ is due to the fact that the rod is "seen" in Σ, that is, the events defined by the points of the rod are simultaneous in Σ' and not in Σ. As a result the points that are further away appear "later" in Σ, hence the distortion of the shape of the rod in Σ.

We consider now the case $a = 0$, that is, v =constant. Then we have

$$y = y' = ut',$$
$$ct' = \frac{c}{v}\left(\frac{x}{\gamma(v)} - x'\right),$$

hence

$$y = \frac{u}{v}\left(\frac{x}{\gamma(v)} - x'\right).$$

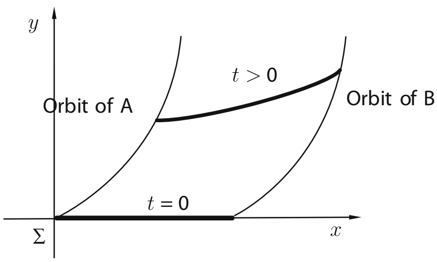
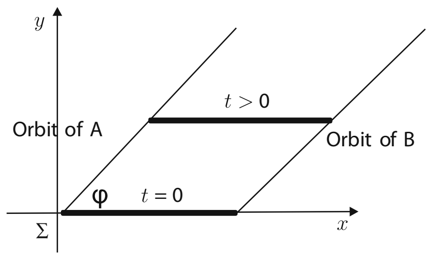


Fig. 7.11 The accelerated rigid rod along the y-axis

Fig. 7.12 Rigid rod moving inertially along y-axis



The end point A has $x'(A) = 0$ therefore $y(A) = \frac{u}{v}\frac{x(A)}{\gamma(v)}$. Similarly for the end point B of the rod we have $x'(B) = L$ hence $y(B) = \frac{u}{v}(\frac{x(B)}{\gamma(v)} - L)$. In the plane $x - y$ the curves $(y(A), x)$, $(y(B), x)$ are parallel straight lines with slope (see Fig. 7.12)

$$\tan\phi = \frac{dy}{dx} = \frac{u}{v\gamma(v)}.$$

We conclude that the rod remains normal to the y'-axis and its end points A, B are simultaneous in Σ'. This condition gives

$$\frac{x(B) - x(A)}{\gamma(v)} = x'(B) - x'(A) = L \Longrightarrow y(A) = y(B),$$

i.e., the rod is moving normal to the y-axis of Σ. The length of the rod in Σ is

$$x(B) - x(A) = \gamma(v)L.$$

This result does not conflict with the Lorentz contraction, because in Σ we do not measure the length of the rod – that is the spatial distance of the events A, B (because $t(A) \neq t(B)$) – hence $x(B) - x(A)$ is the coordinate length of the rod in Σ.

7.7.4 The Case of the Gravitational Field

In order to obtain a feeling of the kinematic results obtained in the previous section we consider an one-dimensional rocket of proper length l_0, which in some RIO Σ starts from rest and moves rigidly. Due to the type of motion the observer inside the rocket will notice no change in the spatial distances inside the rocket. What one can say about the (proper) clocks within the rocket?

If we consider the synchronization of chronometry then if the clocks at each point inside the rocket are set at the same indication and their handles are moving along the world line of the clock (if the clocks are digital then their reading must change accordingly), that is, they will always show the same indication, because this synchronization is independent of the position within the rocket. If the same clocks are synchronized with the synchronization of LRIO then, if they start with the same reading then according to (7.68) after a while they will have different readings depending on their position within the rocket. Obviously the synchronization of the LRIO could create confusion for the crew inside the rocket.

Due to the rigidity of the motion there is a common LRIO for all points of the rocket. If this observer uses his clock to time the events into the rocket then he will make use of the synchronization of the LRIO. This synchronization is more close to the Newtonian concept of absolute time and perhaps it will be the "natural" one to be used by a *single* traveler. However, if there are more than one travelers in the rocket who set their clocks at the same reading and move into the rocket, after a while when they meet again they will find that their clocks have different readings! It becomes apparent how crucial is the synchronization procedure (the time keeping of more than one clocks) and furthermore how conventional it is.

Let us assume that the observer chooses the synchronization of the LRIO and examine what happens with the light signals inside the rocket.

Consider two identical oscillators placed at a fixed distance l_0^+ within the rocket and let τ_1, τ_2 be their periods according to the observer in the rocket. Due to either synchronization the period of each oscillator depends on its position within the rocket. We stipulate that the number of complete oscillations of 1 and 2 is the same, because this has to do only with the internal function of the oscillators and they are assumed to be identical. Then according to the time keeping of the observer in the rocket, for n periods oscillator 1 requires a duration $n\tau_1$ and oscillator 2 a duration $n\tau_2$. From (7.68) we have

$$\frac{\nu_1 - \nu_2}{\nu_2} = \frac{\sqrt{\frac{c}{a_2^2} + x_{02}} - \sqrt{\frac{c}{a_1^2} + x_{01}}}{\sqrt{\frac{c}{a_1^2} + x_{01}}}. \tag{7.77}$$

This frequency shift $\Delta\nu = \nu_1 - \nu_2$ is not due to the relative motion of the oscillators, but to the existence of the acceleration and the considered synchronization of the clocks in the rocket. This change of frequency we call *acceleration redshift*.

One practical application of this result is when the rocket falls freely in a weak gravitational field. In this case the acceleration can be considered as constant in the dimensions of the rocket (rigid body motion) and (7.77) means that inside the rocket the frequency of the oscillators changes with the position. To show that this is the case, it is enough to send one photon from one end of the rocket toward the other and observe if the photon changes color or not. As we shall discuss later this occurs and this phenomenon is known as *gravitational redshift*. It is one of the phenomena, which indicated the Equivalence Principle of General Relativity, the later being stated as follows:

> The local Physics in a gravitating field can be described equivalently with the Physics in a properly accelerated coordinate system in a space free of gravitational field.

7.8 General One-Dimensional Rigid Motion

In this section we generalize[5] the study of rigid motion by considering two ReMaP A, B which are moving arbitrarily along the x-axis of a RIO Σ their position being described by the functions $x_A(t)$, $x_B(t)$, respectively. We shall say that A, B undergo rigid motion if in the *proper frame* of A the distance of B from A is constant and equal to L_0 during all motion.

We note that the requirement of rigid motion does not refer to the RIO Σ, hence in Σ the two particles can have different velocities and accelerations.

Let us formulate the conditions of rigid motion in terms of the kinematic variables.

At the time moment t in Σ let $\beta_A(t)$, $\gamma_A(t)$ be the kinematic factors of the proper frame $\Sigma_A(t)$ of A at that moment. Because we measure length in $\Sigma_A(t)$ the events must be simultaneous in that frame, hence not simultaneous in Σ. We have the following table of coordinates:

	Σ	$\Sigma_A(t)$
$A:$	$(ct,\ x_A(t))$	$(c\bar{t}_A,\ 0)$
$B:$	$(ct_B,\ x_B(t_B))$	$(c\bar{t}_A,\ L_0)$
$BA^i:$	$(ct_B - ct, x_B(t_B) - x_A(t))_\Sigma$	$(0,\ L_0)_{\Sigma_A(t)}$

The boost relating Σ, $\Sigma_A(t)$ gives

$$ct_B = ct - \gamma_A(t)\beta_A(t)L_0, \tag{7.78}$$

$$x_B(t_B) = x_A(t) - \gamma_A(t)\, L_0. \tag{7.79}$$

Because we want the time $t_B > 0$ we demand the restriction

$$L_0 < \frac{ct}{\gamma_A(t)\beta_A(t)}, \tag{7.80}$$

which sets an upper bound for the value of the distance L_0. Eliminating t_B we find in Σ the following equation for the motion of B:

$$x_B\left(t - \frac{\gamma_A(t)\beta_A(t)L_0}{c}\right) = x_A(t) - \gamma_A(t)\, L_0. \tag{7.81}$$

---

[5] See D. Kim and Sang Gyo Jo "Rigidity in Special Relativity" *J. Phys. A: Math Gen.* **37**, 4369, (2004).

We compute

$$\frac{dx_B}{dt_B}\frac{dt_B}{dt} = \frac{dx_A(t)}{dt} - \frac{d\gamma_A(t)}{dt}L_0. \tag{7.82}$$

But from (7.4) we have $\frac{d\gamma_A(t)}{dt} = \gamma_A^3(t)\beta_A(t)\frac{d\beta_A(t)}{dt}$. Hence the rhs gives

$$\frac{dx_A(t)}{dt} - \gamma_A^3(t)\beta_A(t)\frac{d\beta_A(t)}{dt}L_0 = \frac{dx_A(t)}{dt}\left(1 - \frac{\gamma_A^3(t)\frac{d\beta_A(t)}{dt}L_0}{c}\right).$$

Using (7.78) we prove easily that $\frac{dt_B}{dt} = 1 - \frac{\gamma_A^3(t)\frac{d\beta_A(t)}{dt}L_0}{c}$ hence imposing the second restriction

$$1 - \frac{\gamma_A^3(t)\frac{d\beta_A(t)}{dt}L_0}{c} \neq 0 \tag{7.83}$$

(7.82) gives

$$\frac{dx_B(t_B)}{dt_B} = \frac{dx_A(t)}{dt}, \tag{7.84}$$

which implies that the velocity of B in Σ is the same with the velocity of A in Σ. We infer that the proper frame A is also the proper frame of B, therefore it is enough to consider the rigid motion in the proper frame of one of the particles, both particles being equivalent in that respect.

This equivalence can be seen formally as follows. Equation (7.84) implies $\beta_A(t) = \beta_B(t_B)$, $\gamma_A(t) = \gamma_B(t_B)$ hence if we replace $ct = ct_B + \gamma_A(t)\beta_A(t)L_0 = ct_B + \gamma_B(t_B)\beta_B(t_B)L_0$ in (7.79) we find

$$x_A\left(t_B + \frac{\gamma_B(t_B)\beta_B(t_B)L_0}{c}\right) = x_B(t_B) + \gamma_B(t_B)\,L_0. \tag{7.85}$$

The above results are general and hold for any motion of A, B along the x-axis of Σ.

7.8.1 The Case of Hyperbolic Motion

Let us apply the above results in the special case A executes hyperbolic motion with proper acceleration a_A^+. As it has been shown in (7.49) in this case

$$x_A(t) - x_{0,A} = \frac{c^2}{a_A^+}\left(\sqrt{1 + \left(\frac{a_A^+ t}{c}\right)^2} - 1\right). \tag{7.86}$$

For this type of motion we also have (see also (7.45) and (7.46))

$$\gamma_A(t)\beta_A(t) = \sinh \frac{a_A^+ \tau}{c} = \frac{a_A^+ t}{c}, \tag{7.87}$$

$$\gamma_A(t) = \cosh \frac{a_A^+ \tau}{c} = \frac{a_A^+}{c^2}\left(x_A(t) - x_A(0)\right) + 1. \tag{7.88}$$

We examine conditions (7.80) and (7.83). The first gives

$$ct - \gamma_A(t)\beta_A(t)L_0 > 0 \Rightarrow ct - \frac{a_A^+ t}{c} L_0 > 0 \Rightarrow L_0 < \frac{c^2}{a_A^+}. \tag{7.89}$$

Concerning the second condition we note that $a_A \gamma_A^3(t) = a_A^+ \Rightarrow c \frac{d\beta_A(t)}{dt} \gamma_A^3(t) = a_A^+$. Therefore the second condition reads

$$1 - \frac{a_A^+ L_0}{c^2} \neq 0$$

and is trivially satisfied due to (7.89).

Concerning the motion of B we have from (7.78)

$$t_B = \left(1 - \frac{a_A^+ L_0}{c^2}\right) t \tag{7.90}$$

and from (7.79) and (7.88)

$$x_B\left(t_B\right) = x_A(t)\left(1 - \frac{a_A^+ L_0}{c^2}\right) - L_0 + x_A(0)\frac{a_A^+ L_0}{c^2}. \tag{7.91}$$

In order to compute $x_B\left(t_B\right)$ in terms of the time t_B we use (7.90) to replace t in the rhs of (7.91). Then

$$x_B\left(t_B\right) = x_A\left(\frac{t_B}{1 - \frac{a_A^+ L_0}{c^2}}\right)\left(1 - \frac{a_A^+ L_0}{c^2}\right) - L_0 + x_A(0)\frac{a_A^+ L_0}{c^2}.$$

From (7.86) we find

$$x_A\left(\frac{t_B}{1 - \frac{a_A^+ L_0}{c^2}}\right) = \frac{c^2}{a_A^+}\left(\sqrt{1 + \left(\frac{a_A^+}{c}\right)^2 \frac{t_B^2}{\left(1 - \frac{a_A^+ L_0}{c^2}\right)^2}} - 1\right) + x_A(0)$$

hence

$$x_B(t_B) = \frac{c^2}{a_A^+}\left(\sqrt{1+\left(\frac{a_A^+}{c}\right)^2 \frac{t_B^2}{\left(1-\frac{a_A^+ L_0}{c^2}\right)^2}} - 1\right)\left(1-\frac{a_A^+ L_0}{c^2}\right)$$
$$- L_0 + x_A(0)\frac{a_A^+ L_0}{c^2} + x_A(0)\left(1-\frac{a_A^+ L_0}{c^2}\right)$$
$$= \frac{c^2}{a_A^+}\left(\sqrt{\left(1-\frac{a_A^+ L_0}{c^2}\right)^2 + \frac{a_A^{+2} t_B^2}{c^2}} - 1 + \frac{a_A^+ L_0}{c^2}\right) - L_0 + x_A(0).$$

We rewrite this expression as

$$x_B(t_B) = \frac{c^2}{\frac{a_A^+}{1-a_A^+ L_0/c^2}}\left(\sqrt{1+\left(\frac{a_A^+}{1-a_A^+ L_0/c^2}\right)^2 \frac{t_B^2}{c^2}} - 1\right) - L_0 + x_{0,A} \tag{7.92}$$

from which we infer that the particle B is also executing hyperbolic motion with proper acceleration

$$a_B^+ = \frac{a_A^+}{1-a_A^+ L_0/c^2}. \tag{7.93}$$

Note the dependence of proper acceleration on the relative distance of the particles A, B.

7.9 Rotational Rigid Motion

We generalize the previous considerations to rotational rigid motion. Consider two ReMaP A, O such that O is resting at the origin of the coordinates of a LCF Σ and A rotates in the $x-y$ plane of Σ in a circular orbit of radius a. At each moment the velocity of A is normal to the radius in Σ, hence there is no Lorentz contraction for the LRIO at that point. This means that our assumption that the radius a remains constant is not incompatible with the Lorentz contraction. We shall refer to this type of motion as *rigid rotation*.

We consider now a third ReMaP B which is rotating rigidly around O in Σ at a radius $b > a$. We want to define the rigid motion of the pair A, B. For this we consider a time moment t_0 in Σ and assume that at this moment A makes an angle $\theta_A(t_0)$ with x-axis while B makes the angle $\theta_B(t_0)$. Let $\Sigma_A(t_0)$ be the LRIO of A the moment t_0. The velocity of this frame wrt Σ is $a\dot{\theta}_A(t_0)\hat{\mathbf{y}}$ where $\hat{\mathbf{y}}$ is the unit vector along the y-axis. We have the following table of coordinates between the RIO Σ and the LRIO $\Sigma_A(t_0)$ at the moment t of Σ

	Σ	$\Sigma_A(t_0)$
$A:$	$(ct_0,\ a,\ 0)$	$(c\bar{t}_A, 0, 0$
$B:$	$(ct_B,\ b\cos\Phi_B,\ b\sin\Phi_B)$	$(c\bar{t}_B, \bar{x}_B(t_B), \bar{y}_B(t_B))$
$BA^i:$	$\begin{pmatrix} ct_B - ct_0 \\ b\cos\Phi_B - a \\ b\sin\Phi_B \end{pmatrix}_\Sigma$	$\begin{pmatrix} c\bar{t}_B - c\bar{t}_A \\ \bar{x}_B(t_B) \\ \bar{y}_B(t_B) \end{pmatrix}_{\Sigma_A(t_0)}$

The boost along the y-axis gives

$$\begin{aligned} c\bar{t}_B - c\bar{t}_A &= \gamma_A(t_0)\,(ct_B - ct_0 - \beta_A(t_0) b \sin\Phi_B)\,, \\ \bar{x}_B(t) &= b\cos\Phi_B - a, \\ \bar{y}_B(t) &= \gamma_A(t_0)\,[b\sin\Phi_B - \beta_A(t_0)(ct_B - ct_0)]\,, \end{aligned} \tag{7.94}$$

where $\sin\Phi_A = \sin(\theta_A(t) - \theta_A(t_0))$, $\cos\Phi_A = \cos(\theta_A(t) - \theta_A(t_0))$ and $\sin\Phi_B = \sin(\theta_B(t) - \theta_A(t_0))$, $\cos\Phi_B = \cos(\theta_B(t) - \theta_A(t_0))$.

We define the rigid motion of the pair A, B by the requirement their distance in the LRIO $\Sigma_A(t_0)$, which at time t_0 equals

$$a^2 + b^2 - 2ab\cos\phi_{AB},$$

where ϕ_{AB} is the angle between OA, OB at the moment t_0 of Σ to remain constant. To formulate this requirement we note that the measure of the spatial distance of two events in a LRIO requires that the events are simultaneous in that LRIO, therefore

$$c\bar{t}_B - c\bar{t}_A = 0.$$

Then (7.94) implies

$$ct_B - ct_0 = \beta_A(t_0) b \sin\Phi_B \tag{7.95}$$

from which we have

$$\sin\Phi_B = \frac{ct_B - ct_0}{b\beta_A(t_0)}. \tag{7.96}$$

The requirement of rigidity gives

$$(\bar{x}_B(t_A))^2 + (\bar{y}_B(t_A))^2 = a^2 + b^2 - 2\cos\phi_{AB}.$$

Replacing we find

$$(b\cos\Phi_B - a)^2 + [\gamma_A(t_0)\,(b\sin\Phi_B - \beta_A(t_0)(ct_B - ct_0))]^2 = a^2 + b^2 - 2\cos\phi_{AB}.$$

The first term

$$
\begin{aligned}
(b\cos\Phi_B - a)^2 &= b^2\cos^2\Phi_B - 2ab\cos\Phi_B + a^2\\
&= b^2\left[1-\left(\frac{c(t_B-t_0)}{\beta_A(t_0)b}\right)^2\right] - 2ab\cos\Phi_B + a^2\\
&= a^2+b^2-\frac{c^2(t_B-t_0)^2}{\beta_A^2(t_0)} - 2ab\cos\Phi_B,
\end{aligned}
$$

where in the second step we have used (7.95).

The second term gives

$$
\begin{aligned}
[\gamma_A(t_0)\,(b\sin\Phi_B-\beta_A(t_0)(ct_B-ct_0))]^2 &= \gamma_A^2(t_0)\left[\frac{ct_B-ct_0}{\beta_A(t_0)}-\beta_A(t_0)(ct_B-ct_0)\right]^2\\
&= \gamma_A^2(t_0)\frac{c^2(t_B-t_0)^2}{\beta_A^2(t_0)}(1-\beta_A^2(t_0))^2\\
&= \frac{c^2(t_B-t_0)^2}{\beta_A^2(t_0)}.
\end{aligned}
$$

Replacing in the rigidity condition we find

$$
a^2+b^2-\frac{c^2(t_B-t_0)^2}{\beta_A^2(t_0)}-2ab\cos\Phi_B+\frac{c^2(t_B-t_0)^2}{\beta_A^2(t_0)\gamma^2(t_0)} = a^2+b^2-2ab\cos\phi_{AB} \Rightarrow
$$

$$
\frac{c^2}{2ab}(t_B-t_0)^2+\cos\Phi_B = \cos\phi_{AB}. \tag{7.97}
$$

Conditions (7.95) and (7.97) are the two equations quantifying the rotational rigid motion. The first equation gives the time t_B in Σ of the event B so that when the event A occurs at time t_0 in Σ both events occurring at the same moment in the LRIO $\Sigma_A(t_0)$. Given $\theta_A(t)$ the second equation determines the angle $\theta_B(t)$ of B so that the spatial distance of A, B will be constant for the LRIO $\Sigma_A(t_0)$ of A at the time moment t_0. Note that for this motion the angle ϕ_{AB} is constant and equals the angle AOB at the moment t_0.

From the rigidity relation we get by differentiating wrt t_B

$$
\begin{aligned}
\frac{c^2}{2ab}2(t_B-t_0)-\frac{d\Phi_B}{dt_B}\sin\Phi_B &= 0 \Rightarrow\\
\frac{c^2}{ab}(t_B-t_0)-\frac{d\Phi_B}{dt_B}\frac{ct_B-ct_0}{\beta_A(t_0)b} &= 0 \Rightarrow\\
\frac{d\theta(t_B)}{dt_B} &= \frac{\beta_A(t_0)}{a}c \Rightarrow\\
\frac{d\theta(t_B)}{dt_B} &= \frac{d\theta_A(t_0)}{dt_0},
\end{aligned} \tag{7.98}
$$

that is, the angular velocity of B in Σ at the moment t_B equals the angular velocity of A at the time moment t_0 in Σ. This means that the instantaneous rest frame for B at the moment t_B coincides with the instantaneous rest frame of A at the moment t_0, hence what we have said for the second automatically applies for the first. We conclude that the definition of rigidity we gave is independent of which particle we consider as a reference particle. We also note that if $\frac{d\theta_A(t_0)}{dt_0}$ is a monotonically increasing function of time, then so is $\frac{d\theta(t_B)}{dt_B}$.

A special case occurs when the two particles are on the same radius at the times t_0 and t_B, respectively. In this case ϕ_{AB} is either 0 or π and for any given $\theta_A(t_A)$ the

$$\theta_B(t_B) = \theta_A(t_A) + \phi_{AB}. \tag{7.99}$$

In order to write the rigidity conditions with reference to the particle B we interchange $A \leftrightarrow B$ and $a \leftrightarrow b$. Let $\bar{t}_A$ be the time moment of Σ for which the event corresponding to particle A and the event corresponding to particle B are simultaneous at the LRIO of B at the moment t_B. Then the rotational rigidity conditions for particle B read

$$c\bar{t}_A - ct_B = \beta_B(t_B) a \sin[\theta_A(\bar{t}_A) - \theta_B(t_B)], \tag{7.100}$$

$$\frac{c^2}{2ab}(\bar{t}_A - t_B)^2 + \cos[\theta_A(\bar{t}_A) - \theta_B(t_B)] = \cos\phi_{AB}. \tag{7.101}$$

7.9.1 The Transitive Property of the Rigid Rotational Motion

We consider four ReMaP A, B, D, O which are moving so that A, B are in rigid rotation about O, and B, D are also in rigid rotation about O. We define the transitive property of the rigid rotational motion by the requirement that the ReMaPs A, D execute rigid rotational motion about O. Let us examine if this kinematic requirement can be satisfied and under what conditions.

The fact that the particles A, B are in rigid rotation about O implies that for given $t_A, \theta_A(t_A)$ of the particle A there are $t_B, \theta_B(t_B)$ (time in the RIO Σ with origin at O) satisfying the following two equations:

$$\frac{d\theta(t_B)}{dt_B} = \frac{d\theta_A(t_A)}{dt_A}, \tag{7.102}$$

$$ct_B - ct_A = \beta_A(t_A) b \sin[\theta_B(t_B) - \theta_A(t_A)]. \tag{7.103}$$

Similarly for the pair B, D we have the equations

$$\frac{d\theta(t_D)}{dt_D} = \frac{d\theta_B(t_B)}{dt_B}, \tag{7.104}$$

$$ct_D - ct_B = \beta_B(t_B) d \sin[\theta_D(t_D) - \theta_B(t_B)]. \tag{7.105}$$

The transitive property will be satisfied if the following two conditions are satisfied:

$$\frac{d\theta(\tilde{t}_D)}{d\tilde{t}_D} = \frac{d\theta_A(t_A)}{dt_A}, \tag{7.106}$$

$$c\tilde{t}_D - ct_A = \beta_A(t_A) d \sin[\theta_D(\tilde{t}_D) - \theta_A(t_A)]. \tag{7.107}$$

From (7.102) and (7.104) follows

$$\frac{d\theta(t_D)}{dt_D} = \frac{d\theta_A(t_A)}{dt_A}.$$

Because $\theta(\tilde{t}_D)$ is uniquely defined in terms of $\theta_A(t_A)$ and also $\frac{d\theta(t_D)}{dt_D}$ is a monotonic function, we infer that the first condition (7.106) is satisfied provided $\tilde{t}_D = t_D$. It remains the second condition to be checked. From condition (7.107) setting $\tilde{t}_D = t_D$ we have

$$ct_D - ct_A = \beta_A(t_A) d \sin[\theta_D(t_D) - \theta_A(t_A)]. \tag{7.108}$$

Subtracting (7.105) and (7.106) we obtain

$$ct_D - ct_A = \beta_B(t_B) d \sin[\theta_D(t_D) - \theta_B(t_B)] - \beta_A(t_A) b \sin[\theta_B(t_B) - \theta_A(t_A)].$$

The last two relations and (7.103) imply

$$\begin{aligned}\beta_B(t_B) d \sin[\theta_D(t_D) - \theta_B(t_B)] - \beta_A(t_A) b \sin[\theta_D(t_D) - \theta_A(t_A)] \\ = \beta_A(t_A) d \sin[\theta_B(t_B) - \theta_A(t_A)]\end{aligned}$$

or replacing $\beta_B(t_B) = \frac{b}{c}\frac{d\theta_B(t_B)}{dt_B}$, $\beta_A(t_A) = \frac{a}{c}\frac{d\theta_A(t_A)}{dt_A} = \frac{a}{c}\frac{d\theta_B(t_B)}{dt_B}$:

$$bd \sin[\theta_D(t_D) - \theta_B(t_B)] - ab \sin[\theta_B(t_B) - \theta_A(t_A)] = ad \sin[\theta_D(t_D) - \theta_A(t_A)]. \tag{7.109}$$

This condition is satisfied automatically in Newtonian Physics but not in Special Relativity, in which it serves as an additional condition/constraint. This condition restricts severely the possible cases in Special Relativity. Indeed (7.109) has the following solutions:

Solution 1

$\theta_D(t_D) = \theta_B(t_B) = \theta_A(t_A)$ and any values of the constants a, b, d. This solution forces the three points along the same radius on the plane of rotation, therefore the rigid rotation of a rigid rod normal to the pivot at one of its ends.

Solution 2

$\theta_B(t_B) = \theta_A(t_A) \neq \theta_D(t_D), d \neq 0$. In this case condition (7.109) reduces to $a = b$ that is the points A, B coincide hence we are forced back to the two points rigid rotational motion.

We conclude that the only possibility for rigid rotational motion in Special Relativity is the rotation of a rigid rod about one of its points. This implies that if we have a disk of radius R then the disk can rotate rigidly if the points in each radius stay at all times in the same radius, i.e., they have the same angular velocity.

7.10 The Rotating Disk

We come to the final case of our considerations that is the rigid motion of a rotating disk. This is a complex problem which appears to be still unsolved in the current literature, something to be expected because it requires so many conventions that the choice of a unique approach is not feasible. In any case it is an interesting problem and in this section we shall follow the rather standard and widely accepted – but not unique! – approach of Grøn.

7.10.1 The Kinematics of Relativistic Observers

Before we enter into the discussion of the rotating disk problem we have to advance a little our knowledge of relativistic kinematics. Let us see why. The kinematics of Special Relativity developed in the previous sections was concerned mainly with the relativistic mass point (ReMaP). This kinematics has been extended with the Born's rigid motion to cover the rigid body of Newtonian Physics. However the problem in Special (and General) Relativity is that one does not have solids, therefore one has to *define* the relativistic body, and then develop a kinematics appropriate for this type of body. This will become apparent from the discussion of the rotating disk.

Kinematics is a comparative study which requires two observers. In Special Relativity there are two different cases to consider:

(a) Kinematics between two relativistic inertial observers (RIO)
(b) Kinematics between a RIO and an accelerating observer.

In the first case the kinematics is determined by the Lorentz transformation relating the observers. That is, the choice of the second observer is equivalent to the choice of the appropriate Lorentz transformation (via the $\boldsymbol{\beta}$-factor). In the second case the choice of the accelerating observer is at will because the Lorentz transformation does not relate a RIO with an accelerating observer, therefore in this case the kinematics is *defined*. This is done by the *definition* of a metric (of Lorentzian character) for the accelerated observer by means of two actions:

(a) The definition of a coordinate transformation which relates the coordinates of the RIO with those of the accelerated observer.
(b) By assuming that this transformation leaves the form of the metric the same $(ds^2 = ds'^2)$, that is, the new metric is found from the Minkowski metric (of the RIO) if one replaces the RIO coordinates in terms of the coordinates of the

accelerated observer. Using this new metric one determines the relation of the time and the spatial intervals between the two observers.

One may consider the proposed coordinate transformation as “a generalized Lorentz transformation.” However, this transformation is very different from the standard Lorentz transformation. Indeed the proposed transformation is as follows:

(a) It is specific to the two observers involved
(b) It is not an isometry of Minkowski space, that is, it is not represented with a Lorentz matrix[6]
(c) It does not necessarily generates a (Lie) group, hence it is possible that one cannot define a covariance property[7]

We note that the metric $d\tilde{s}^2$ is still flat because with a coordinate transformation it is not possible to create new tensors.[8]

Having a metric for the relativistic observer it is possible to define coordinate time and the space intervals. This is important and it is done as explained below.[9]

7.10.2 Chronometry and the Spatial Line Element

The coordinatization in spacetime by a relativistic observer (inertial or not) is done by means of chronometry. Let us recall briefly this procedure. The relativistic observer is equipped with a photongun (a torch) and a clock. In order to determine the coordinates of an event the observer emits a light beam toward the point in space where the event takes place and in the direction $\mathbf{e}$ such that the beam is reflected at a “mirror,” placed at the point of space and returns along the *same* direction back to the observer. The observer notes the reading of his clock for the event of emittance t_e and the event of reception t_r. He also reads from the scale of the photongun the direction (angles) of the beam in his coordinate frame, Σ say. Subsequently the observer defines the following coordinates for the event in Σ

[6] For those who have a knowledge of geometry, it is not generated by a Killing vector of the Minkowski metric.

[7] For example the standard Lorentz transformations do form a group and the tensors which transform covariantly under this group are the Lorentz tensors. These tensors describe covariantly (within Special Relativity) the physical quantities of the theory.

[8] Flat means the curvature tensor vanishes. This means that the coordinate functions are defined everywhere in the space. Under a coordinate transformation it is possible to have new quantities, which are not tensors, (e.g., the connection coefficients Γ^a_{bc}), such as the curvature tensor. The reason for this is that the curvature tensor is covariant wrt the group of the general coordinate transformations (manifold mapping group) which includes whatever coordinate transformations we introduce between the coordinates of the RIO and the coordinates of the accelerated observer.

[9] See also *Ref: Landau and Lipsitz. The classical theory of fields 5th Edition p. 234.*

$$\begin{pmatrix} \frac{t_e+t_r}{2}c \\ \frac{t_e-t_r}{2}c\mathbf{e} \end{pmatrix}_\Sigma . \qquad (7.110)$$

Using the chronometric coordinatization one can determine geometrically if a given relativistic observer Σ is a RIO or an accelerating observer. This is done as follows (see Fig. 7.13). Let P be an arbitrary spacetime event whose coordinates have being determined by Σ. Draw from P the normal plane to the world line of the observer. Let t_P be the intersection of that plane with the world line of the observer. If $t_P - t_e = t_r - t_P$ for all spacetime events P then the observer Σ is a RIO, otherwise it is an accelerating observer.

Let us assume now that the metric of the relativistic observer – defined in the way explained above – in his coordinate frame, Σ say, has the form

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu + 2g_{0\mu}dx^0 dx^\mu + g_{00}(dx^0)^2. \qquad (7.111)$$

The emitted and the received light rays by the observer performing the chronometric measurement of the coordinates of an event P belong to the light cone of the point P. The equation of this cone is

$$g_{\mu\nu}dx^\mu dx^\nu + 2g_{0\mu}dx^0 dx^\mu + g_{00}(dx^0)^2 = 0, \qquad (7.112)$$

where dx^0 is to be taken as the time of traveling of the light beam to and from P and dx^μ are the coordinates defining the space axes of the coordinate system of the observer. Considering (7.112) as a quadratic equation in dx^0 and solving in terms of dx^0 we find

$$dx^0_\pm = \frac{1}{g_{00}}\left[-g_{0\mu}dx^\mu \pm \sqrt{|(g_{0\mu}g_{0\nu} - g_{\mu\nu}g_{00})|dx^\mu dx^\nu}\right]. \qquad (7.113)$$

These values of dx^0 concern the bidirectional propagation of light, one value for the "go" and the other for the "return" trip of the light ray. The difference of the two

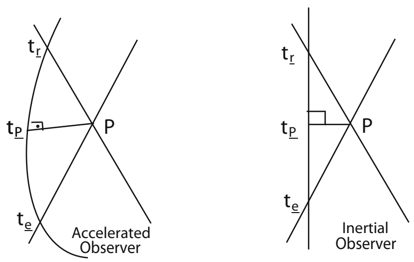


Fig. 7.13 The coordination of an accelerating and an inertial observer

roots gives the total traveling (coordinate) time of the beam. We find

$$dx^0_{CD} = dx^0_+ - dx^0_- = \frac{2}{g_{00}}\sqrt{|(g_{0\mu}g_{0\nu} - g_{\mu\nu}g_{00})|dx^\mu dx^\nu}, \tag{7.114}$$

where (C) is the event of emittance and (D) the event of reception.

The proper time between the events C, D is obtained if we set $dx^\mu = 0$ in (7.111)

$$ds^2 = -c^2 d\tau^2 = g_{00}(dx^0_{AB})^2 \tag{7.115}$$

from which follows

$$d\tau = \frac{1}{c}\sqrt{|g_{00}|}dx^0. \tag{7.116}$$

This equation relates the proper time interval of the relativistic observer (that is, the time period measured by its proper clock) with the coordinate time determined by the chronometric procedure.

Concerning the spatial distance of the event P we have

$$dl_P = \frac{1}{2}dx^0_{CD} = \sqrt{\left|\frac{g_{0\mu}g_{0\nu}}{g_{00}} - g_{\mu\nu}\right| dx^\mu dx^\nu}. \tag{7.117}$$

This leads us to consider the three-dimensional line element

$$dl^2 = \left|\frac{g_{0\mu}g_{0\nu}}{g_{00}} - g_{\mu\nu}\right| dx^\mu dx^\nu \tag{7.118}$$

as a positive definite line element, which defines the spatial metric of the relativistic observer[10] Σ.

Relations (7.116) and (7.118) describe the kinematics of a relativistic (inertial or accelerating) observer. The quantities $d\tau, dl_\mu$ are the units of time and spatial distance determined from the accelerating observer and dx^0, dx^μ the coordinate values of these quantities determined by the chronometric coordinatization. These relations provide us all the necessary equipment to discuss the rigid motion of the rotating disk.

(1) In the derivation we have considered $dx^0_+ > 0$, $dx^0_- < 0$ but this is not necessary; dx^0_+, dx^0_- can have any sign. The fact that in this case the value of $x^0(D)$ at the moment of arrival of the light signal in D might be less than the value $x^0(C)$ at the moment of departure from C contains no contradiction since the rate of clocks at different points in space are not assumed to be synchronized in any way.

[10] Not of spacetime! Each relativistic observer has its own spatial geometry!

(2) If we wish to find the spatial length dl_μ along the coordinate x^μ say, we simply write

$$dl_\mu = \sqrt{\left|\frac{g_{0\mu}g_{0\mu}}{g_{00}} - g_{\mu\mu}\right|}dx^\mu \tag{7.119}$$

In this equation dx^μ is the coordinate spatial length, that is the spatial length determined by the chronometric coordinatization. The above equation means that the spatial distance of the point P along the coordinate line x^μ attributed by the observer is $\sqrt{\left|\frac{g_{0\mu}g_{0\mu}}{g_{00}} - g_{\mu\mu}\right|}$ times the coordinate length (i.e., coordinate unit) along that coordinate line.

(3) If the observer is inertial then the metric has components $diag(-1, 1, 1, 1)$ and we find $d\tau = t,\ dl^2 = \delta_{\mu\nu}dx^\mu dx^\nu$, which are the expected answers.

7.10.3 The Rotating Disk

A circular disk of radius R lying on the plane x, y of a RIO Σ rotates with constant angular velocity ω about the z-axis. The center of the disk is assumed to be at the point $(-R, 0, 0)$. This description, although perfectly alright in Newtonian Physics, is meaningless in Special Relativity if we do not define(!) precisely for Σ what we mean by a rotating disk. Indeed in Newtonian Physics the disk is a solid, therefore we do not expect any change in the disk when it is resting and when it is set in rotation. However, in Special (and General) Relativity there do not exist solid bodies and one has to define the moving body.

There is not a unique or "the correct" definition of the rotating disk and one can define it as one wishes. Then the theory produces the consequences of the specific definition and the "experiment" tests the validity or not of the results. This means that all approaches/assumptions concerning the rotating disk are acceptable until experiment will select "the one," if it exists. It is to be noted that the outcome does not prove or disprove Special Relativity, but concerns only the specific assumptions, which have been made in the definition of the moving body. If the experiment verifies these assumptions then Special Relativity is extended to apply to this type of moving body and all is well done. If not, then the assumptions are abandoned but Special Relativity stays!

In the following we will consider a "standard" approach due to Grøn.[11] For further information on the problem of the rotating disk and other approaches the reader is referred to the work of R. Klauber.[12]

Following the previous analysis we consider three types of observers. The RIO Σ for whom the rotating disk is defined (this is the reference RIO), a Locally

[11] Grøn "Relativistic description of a rotating disk" *Am. J. Physics* **43**, 869–876, (1975).

[12] R. Klauber "Relativistic rotation: A comparison of theories" gr-qc /0604118 16 December 2006.

Relativistic Inertial Observer (LRIO) defined by an accelerating point at the rim of the rotating disk and the accelerated observer rotating together with the rotating disk (say the "proper" observer of the disk).

The kinematics between the two RIO is determined by the Lorentz transformation corresponding to the velocity of the second RIO. The kinematics between the standard RIO and the rotating observer requires

(a) The definition of the rotating disk for the accelerated, i.e., (rotating) observer and through this.
(b) The definition of the coordinate transformation, relating the coordinate systems of the two observers. This transformation determines a metric for the rotating observer via relations (7.115) and (7.117) and determines the kinematics of the accelerated observer in terms of the kinematics of the RIO.

7.10.4 Definition of the Rotating Disk for a RIO

When the disk rotates

(a) It preserves its shape (that is it remains as a plane circular disk).
(b) The disk rotates with constant angular velocity about the fixed direction of z-axis.
(c) There is no relative rotation of the parts of the disk. Kinematically this means that the points on a radius remain on the radius during the rotation of the disk. As we have seen in Sect. 7.9 this is necessary if the transitivity property of the rigid rotational motion is to be satisfied.
(d) The radius R of the disk when at rest and after it is made to rotate does not change.[13]

It is obvious that under the above assumptions for Σ, the disk is assumed to rotate as a solid (Born conditions are satisfied). Due to the above assumptions the angle a between two radii of the disk before and during the motion remains constant. This implies for the arc length across the rotating disk

$$\frac{ds}{r} = d\theta \quad 0 < r \leq R. \tag{7.120}$$

[13] These assumptions can be stated in a more advanced language. Here we state without further comment the geometric significance of each of the four assumptions in terms of the fluid of observers defined by the disk:

(a) The fluid defined by the disk has zero shear $\sigma_{ab} = 0$
(b) $\omega_{ab;c} = 0$
(c) $\dot{u}^a = 0$
(d) $\theta = 0$. The Born rigidity conditions $\sigma_{ab} = 0, \theta = 0$ are satisfied.

The angular velocity of the disk for the observer Σ is defined as follows:

$$\omega = \frac{d\theta}{dt}, \tag{7.121}$$

where t is time in Σ. Replacing in (7.120) we find

$$\frac{ds/dt}{r} = \omega.$$

But $ds/dt = v$ is the speed of rotation (at distance r form the axis of rotation), therefore

$$v = \omega r. \tag{7.122}$$

The condition

$$v < c \Rightarrow \omega R < c \Rightarrow \omega < \frac{c}{R}$$

restricts the values of the angular velocity ω.

The above exhaust the assumptions defining the rotating disk for the RIO Σ.

7.10.5 The Locally Relativistic Inertial Observer (LRIO)

We consider the disk at rest and divide its rim in a large number n (say) of line elements of length $2\pi R/n$ each. Then at one of these line elements, the k say, we consider the comoving LRIO Σ_k whose velocity (in Σ!) is $\mathbf{v} = \omega R \mathbf{j}$, where $\mathbf{j}$ is the unit along the y-axis (see Fig. 7.14). We consider a point P at the rim of the disk and write the components of the position four-vector, the four-velocity, and the four-acceleration of P in Σ[14]:

$$x_P^i = \begin{pmatrix} t_p \\ -R(1-\cos\theta_P) \\ R\sin\theta_P \\ 0 \end{pmatrix}_\Sigma, \qquad u_P^i = \gamma \frac{dx_P^i}{dt} = \begin{pmatrix} \gamma \\ -\beta\gamma\sin\theta_P \\ \beta\gamma\cos\theta_P \\ 0 \end{pmatrix}_\Sigma,$$

$$a_P^i = \gamma \frac{du_P^i}{dt} = \begin{pmatrix} \gamma\dot{\gamma} \\ -\beta\gamma(\dot{\gamma}\sin\theta_P + \omega\gamma\cos\theta_P) \\ \beta\gamma(\dot{\gamma}\cos\theta_P - \omega\gamma\sin\theta_P) \\ 0 \end{pmatrix}_\Sigma.$$

Assume that in Σ_k

[14] t is time in Σ. We set $c = 1$.

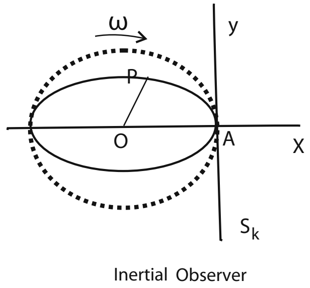


Fig. 7.14 Definition of coordinates for the LRIO Σ_k

$$x^i_{k,P} = \begin{pmatrix} t_{k,p} \\ x_{k,P} \\ y_{k,P} \\ 0 \end{pmatrix}_{\Sigma_k}, u^i_{k,P} = \begin{pmatrix} \gamma_{k,p} \\ \gamma_{k,p} v_{k,x} \\ \gamma_{k,p} v_{k,y} \\ 0 \end{pmatrix}_{\Sigma_k}, \tag{7.123}$$

where $\gamma_{k,p}$ is the γ-factor of the point P in Σ_k. Σ_k is related to Σ with a boost along the y-axis with $\beta = \omega R/c$. For the four-position vector the boost gives

$$t_p = \gamma(t_{k,p} + \beta y_{k,P}), \tag{7.124}$$

$$x_{k,P} = -R(1 - \cos\theta_P), \tag{7.125}$$

$$y_{k,P} = \gamma(R\sin\theta_P - \beta t_p). \tag{7.126}$$

Replacing t_p from (7.124) in (7.126) we get

$$y_{k,P} = \gamma\left[R\sin\theta_P - \beta\gamma(t_{k,p} + \beta y_{k,P})\right] \Rightarrow$$

$$y_{k,P} = R\left(\frac{1}{\gamma}\sin\theta_P - \omega t_{k,p}\right). \tag{7.127}$$

Replacing $y_{k,P}$ from (7.127) in (7.124) follows that

$$t_p = \frac{1}{\gamma} t_{k,p} + \beta R \sin\theta_P, \tag{7.128}$$

which relates the time of Σ, Σ_k for the point P along the rim of the disk. (Obviously all the above go through if we assume instead of the rim a point P at a distance $r < R$ from the center of the rotating disk).

From the above relations we compute

$$\left(1 + \frac{x_{k,P}}{R}\right)^2 + \left(\omega\gamma t_{k,p} + \frac{\gamma y_{k,P}}{R}\right)^2 = 1. \tag{7.129}$$

It follows that for the observer Σ_k at time $t_{k,p} = 0$ the points at the rim of the disk (that is, the shape of the disk) are on an ellipse centered at the point $(-R, 0)$

whose minor semi-axis is R/γ (along the y-axis) and its major semi-axis is R (along the x-axis) (see Fig. 7.14).

Let us assume now the initial condition

At time $t_p = 0$ the point P is at the point A of Σ_k.

This implies

$$t_{k,A} = -\beta R \sin\theta_A \Leftrightarrow \omega t_{k,A} = -\beta^2 \sin\theta_A. \tag{7.130}$$

The point A arrives at the point P after time t_P, therefore the angle

$$\theta_P = \theta_A + \omega t_p. \tag{7.131}$$

Then

$$\begin{aligned} x_{k,P} &= -R\left(1-\cos(\theta_A + \omega t_p)\right), \\ y_{k,P} &= R\left(\frac{1}{\gamma}\sin(\theta_A + \omega t_p) - \omega t_{k,p}\right). \end{aligned} \tag{7.132}$$

We infer that for the observer Σ_k the point P describes a cycloid. The same trajectory (i.e., a cycloid) describes the point P for the RIO Σ but with equations (see Fig. 7.15):

$$\begin{aligned} x_P &= -R\left(1-\cos(\theta_A + \omega t_p)\right), \\ y_P &= R\left(\sin(\theta_A + \omega t_p) - \omega t_P\right). \end{aligned} \tag{7.133}$$

Equations (7.133) can be written in terms of the coordinates in Σ_k if we substitute t_p from (7.124)

$$\begin{aligned} x_P &= -R\left[1-\cos(\theta_A + \omega\gamma(t_{k,p} + \beta y_{k,P}))\right], \\ y_P &= R\left[\sin(\theta_A + \omega\gamma(t_{k,p} + \beta y_{k,P})) - \omega t_{k,p}\right]. \end{aligned}$$

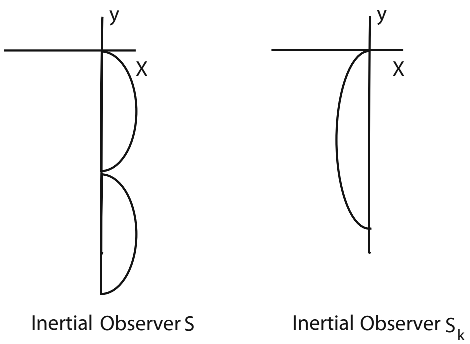


Fig. 7.15 The cycloid for the RIO Σ and the rotating observer Σ_k

In (7.131) the angle θ_A is measured in Σ. To compute it in Σ_k we use the transformation formulae

$$\sin\theta_A = \frac{\sin\theta_{k,A}}{\sqrt{1-\beta^2\cos^2\theta_{k,A}}}, \qquad \cos\theta_A = \frac{\cos\theta_{k,A}}{\gamma\sqrt{1-\beta^2\cos^2\theta_{k,A}}} \tag{7.134}$$

7.10.5.1 Transformation of Angles

Let $\theta_{k,P}$ the angle between the x-axis and the direction of the radius of the disk from the center of the disc to the point P. Then we have for the center of the disk

In $\Sigma : x_{center} = -R,\ y_{center} = 0$.

In Σ_k the boost gives: $x_{k,center} = x_{center} = -R,\ \ y_{k,center} = -\beta\gamma t_{k,p}$.

We have then for $\theta_{k,P}$

$$\begin{aligned}\tan\theta_{k,P} &= \frac{y_{k,P}-y_{k,A}}{x_{k,P}-x_{k,A}} = \frac{R\left(\frac{1}{\gamma}\sin\theta_P - \omega t_{k,p}\right)-0}{-R\left(1-\cos\theta_P\right)+R} \\ &= \frac{1}{\gamma}\tan\left[\omega\gamma(\beta y_k + t_{k,p})\right] + \omega(\gamma-1)t_{k,p}\end{aligned} \tag{7.135}$$

Grøn seems to miss the term $\omega(\gamma-1)t_{k,p}$. See relation (17) of his paper. Note that $\theta_P = \omega t_P$.

7.10.5.2 The Four-Velocities

We compute first the relative velocity of the point P in Σ_k. The four-velocity of Σ_k in Σ is ($c=1$)

$$u_k^i = \begin{pmatrix}\gamma\\0\\\gamma\beta\\0\end{pmatrix}_\Sigma .$$

Therefore the relative four-velocity of the point P in Σ_k is

$$u_{k,P}^i = u_P^i - u_k^i = \begin{pmatrix}\gamma\\-\beta\gamma\sin\theta_P\\\beta\gamma\cos\theta_P\\0\end{pmatrix}_\Sigma - \begin{pmatrix}\gamma\\0\\\gamma\beta\\0\end{pmatrix}_\Sigma = \begin{pmatrix}0\\-\beta\gamma\sin\theta_P\\-\beta\gamma(1-\cos\theta_P)\\0\end{pmatrix}_\Sigma .$$

The invariant $u_P^i u_{ki} = -\gamma_{k,P}$ where $\gamma_{k,P}$ is the γ-factor of P in Σ_k. Replacing we find

$$\gamma_{k,P} = \gamma^2(1-\beta^2\cos\theta_P). \tag{7.136}$$

This is an invariant therefore its value is the same for all RIO, including Σ!

We note that the point P' of the disk opposite to the point P (i.e., $\theta_P = \pi$) has γ-factor

$$\gamma_{k,P'} = \gamma^2(1+\beta^2) = 2\gamma^2 - 1.$$

We consider now the boost relating Σ, Σ_k applied to the four-velocity vector of the point P. We have the components $\begin{pmatrix} \gamma_{k,p} \\ \gamma_{k,p} v_{k,x} \\ \gamma_{k,p} v_{k,y} \\ 0 \end{pmatrix}_{\Sigma_k}$ and $\begin{pmatrix} \gamma \\ -\beta\gamma \sin\theta_P \\ \beta\gamma\cos\theta_P \\ 0 \end{pmatrix}_{\Sigma}$.

For the zeroth component we find

$$\gamma_{k,p} = \gamma(\gamma - \beta\beta\gamma\cos\theta_P) = \gamma^2(1-\beta^2\cos\theta_P)$$

which coincides with the previous result (7.136). For the other two components we get

$$\begin{aligned} \gamma_{k,p} v_{k,x} &= -\beta\gamma\sin\theta_P, \\ \gamma_{k,p} v_{k,y} &= \gamma(\beta\gamma\cos\theta_P - \beta\gamma) = \beta\gamma^2(\cos\theta_P - 1). \end{aligned}$$

or, replacing $\gamma_{k,p}$

$$\begin{aligned} v_{k,x} &= -\frac{\beta\gamma\sin\theta_P}{\gamma^2(1-\beta^2\cos\theta_P)} = -\frac{\beta\sin\theta_P}{\gamma(1-\beta^2\cos\theta_P)}, \\ v_{k,y} &= \frac{\beta\gamma^2(\cos\theta_P - 1)}{\gamma^2(1-\beta^2\cos\theta_P)} = \frac{\beta(\cos\theta_P - 1)}{1-\beta^2\cos\theta_P}. \end{aligned}$$

If we replace further ($\theta_A = 0$)

$$\theta_P = \omega\gamma(t_{k,p} + \beta y_{k,P}),$$

we compute the velocity of the point P in the RIO Σ_k.

Concerning the direction of motion we have

$$\begin{aligned} \tan\theta_{k,P} = \frac{v_{k,y}}{v_{k,x}} &\Rightarrow v_{k,y} = v_{k,x}\tan\theta_P = -\frac{\beta\sin\theta_P}{\gamma(1-\beta^2\cos\theta_P)}\tan\theta_P \\ &= -\frac{\beta\frac{\tan\theta_P}{\sqrt{1+\tan^2\theta_P}}}{\gamma\left(1-\beta^2\frac{1}{\sqrt{1+\tan^2\theta_P}}\right)}\tan\theta_P = -\frac{\beta\tan^2\theta_P}{\gamma\left(\sqrt{1+\tan^2\theta_P}-\beta^2\right)}. \end{aligned}$$

7.10.6 The Accelerated Observer

Let $\tilde{\Sigma}$ be the accelerated (rotating) observer who rotates with the disk. For this observer we must define a coordinate transformation, which will relate its coordinates with those of the RIO Σ. The definition considered by Grøn (we emphasize that this is not the sole choice!) consists of the following assumptions:

(a) The disk remains a plane circular disk as it rotates. This means that the rotating observer can use the polar coordinates $(\tilde{r}, \tilde{\theta}, \tilde{z})$ where $(\tilde{r}, \tilde{\theta})$ are the polar and the angular coordinate in the rotating plane $(\tilde{x}, \tilde{y})$, which rotates wrt the plane (x, y).

(b) The radius of the rotating disk is the same with the radius of the disk at rest, that is R. This implies for the transformation of the radial coordinate:

$$\tilde{r} = r.$$

(c) The clocks of observer $\tilde{\Sigma}$ are synchronized with the clocks of Σ with spherical optical waves which are emitted from the center of rotation of the disk. This implies that the coordinate time in the two frames is related as follows:

$$\tilde{t} = t$$

(d) For the accelerated observer the rotation is "rigid" in the sense that points on a radius remain on that radius during the rotation. This implies that the angles on the disk for the accelerated observer are related to the angles on the disk for the inertial observer Σ by the formula

$$\tilde{\theta} = \theta - \omega t.$$

(e) For the remaining coordinate z normal to the plane of the disk we assume that there is no change and write

$$\tilde{z} = z.$$

In conclusion the coordinate system $(\tilde{t}, \tilde{r}, \tilde{\theta}, \tilde{z})$ of the rotating observer is related to the coordinate system (t, r, θ, z) of the RIO Σ by the transformation equations

$$\tilde{t} = t, \;\; \tilde{r} = r, \;\; \tilde{\theta} = \theta - \omega t, \;\; \tilde{z} = z. \tag{7.137}$$

Having the coordinate transformation we compute the metric for the accelerated observer by requiring that $d\tilde{s}^2 = ds^2$. We compute

$$
\begin{aligned}
d\tilde{s}^2 &= -dt^2 + dr^2 + r^2 d\theta^2 + dz^2 \\
&= -d\tilde{t}^2 + d\tilde{r}^2 + \tilde{r}^2(d\tilde{\theta} + \omega d\tilde{t})^2 + d\tilde{z}^2 \Rightarrow \\
d\tilde{s}^2 &= -(1-\omega^2\tilde{r}^2)d\tilde{t}^2 + d\tilde{r}^2 + \tilde{r}^2 d\tilde{\theta}^2 + 2\omega\tilde{r}^2 d\tilde{\theta} d\tilde{t}.
\end{aligned}
\tag{7.138}
$$

This gives the following metric components in the rotating coordinate system

$$
[g_{ab}] = \begin{bmatrix} -(1-\omega^2\tilde{r}^2) & 0 & \omega\tilde{r}^2 & 0 \\ * & 1 & 0 & 0 \\ * & * & \tilde{r}^2 & 0 \\ * & * & * & 1 \end{bmatrix}. \tag{7.139}
$$

Using the components of the metric and relations (7.115) and (7.117) one is able to compare the kinematics of the accelerated observed $\tilde{\Sigma}$ with that of the RIO Σ.

7.10.6.1 Time Intervals

Let $\tilde{\tau}$ be the proper time of the accelerating observer. Then (7.116) gives

$$
d\tilde{\tau} = \sqrt{|g_{00}|}dt = \sqrt{1-\omega^2\tilde{r}^2/c^2}dt \quad (\omega\tilde{r} < c). \tag{7.140}
$$

The same result we compute if we set $d\tilde{r} = d\tilde{\theta} = 0$ in the expression of the metric and then use $d\tilde{s}^2 = -c^2 d\tilde{\tau}^2$.

Kinematically this means that all proper clocks of the rotating observer go slower when compared to the clock of the RIO[15] Σ and with a rate increasing with the position $\tilde{r}$ according to (7.140). Thus events at different distances from the center of the disk, measured as simultaneous on the coordinate clocks, are not simultaneous for Σ.

It is to be noted that the clocks in $\tilde{\Sigma}$ do not agree with the clocks in Σ_k ($d\tilde{\tau} \neq dt_k$). This is due to the synchronization procedure followed between the clocks of Σ, $\tilde{\Sigma}$ (spherical light waves from the center of rotation at fixed intervals) and that between the clocks of Σ, Σ_k (Einstein synchronization).

7.10.6.2 Spatial Geometry

The positive definite line element, which defines the intrinsic spatial geometry of the accelerating observer is

$$
d\sigma^2 = h_{\mu\nu}dx^\mu dx^\nu = |g_{\mu\nu} - g_{0\mu}g_{o\nu}/g_{00}|dx^\mu dx^\nu. \tag{7.141}
$$

---

[15] This is another view of the twin paradox. The first tween is aging with the clock of Σ and the traveling twin with the proper clock of the rotating disk. Furthermore if we have two twins positioned at different distances from the center of the rotating disk the twin nearer to the center of rotation center ages quicker than the twin further apart.

From the components of the metric for the rotating observer we compute (see (7.139))

$$d\sigma^2 = d\tilde{r}^2 + \frac{\tilde{r}^2}{1-\omega^2\tilde{r}^2} d\tilde{\theta}^2. \tag{7.142}$$

It follows that the line element along the radial coordinate is (set $d\tilde{\theta} = 0$)

$$d\sigma_r = d\tilde{r} \tag{7.143}$$

and the tangential line element along the $\tilde{\theta}$-direction is (set $d\tilde{r} = 0$)

$$d\sigma_\theta = \frac{\tilde{r}}{\sqrt{1-\omega^2\tilde{r}^2}} d\tilde{\theta}. \tag{7.144}$$

The fact that the proper spatial length depends on $\tilde{r}$ shows that the disk cannot pass from rest to rotation in such a way that both the radial and the tangential line elements remain unchanged.

In order to compute the length of the periphery of the disk in $\tilde{\Sigma}$ we take in (7.144) $d\tilde{\theta} = 2\pi$ and $\tilde{r} = R$ and get

$$\tilde{S} = \frac{2\pi R}{\sqrt{1-\omega^2 R^2}} < S, \tag{7.145}$$

where S is the length of the periphery of the disk for the RIO Σ. The difference is due to the fact that the unit of spatial length for the observer $\tilde{\Sigma}$ changes along the periphery of the disk.

7.10.6.3 The Velocity of Light for the Accelerating Observer $\tilde{\Sigma}$

From (7.141) we have that the line element along the coordinate direction l is found by setting $\mu = \nu = l$

$$d\sigma_l = \sqrt{(g_{ll} - g_{0l}g_{0l}/g_{00})} dx^l.$$

The velocity of a particle in $\tilde{\Sigma}$ along the direction l is (we do not take $c = 1$ in order to show the differentiation from the value c)

$$c_l = \frac{d\sigma_l}{dx^0} c = \sqrt{(g_{ll} - g_{0l}g_{0l}/g_{00})} \frac{dx^l}{dx^0} c. \tag{7.146}$$

The line element between two points with coordinates (x^0, x^l) and (x^0+dx^0, x^l+dx^l) (which are related with velocity $\frac{dx^l}{dx^0}$) is

$$ds^2 = g_{ll}(dx^l)^2 + 2g_{l0}dx^l dx^0 + g_{00}(dx^0)^2.$$

If these points are along the trajectory of a light ray, then $ds^2 = 0$ and we obtain

$$g_{ll}(dx^l)^2 + 2g_{l0}dx^l dx^0 + g_{00}(dx^0)^2 = 0.$$

Dividing with $(dx^0)^2$ we find

$$\frac{dx^l}{dx^0} = \frac{g_{00}}{\sqrt{|g_{l4}^2 - g_{ll}g_{00}|} + g_{l4}}.$$

Substituting in (7.146) we obtain the velocity of light in the direction l

$$c_l = \frac{d\sigma_l}{dx^0}c = \sqrt{|g_{ll} - g_{0l}g_{0l}/g_{00}|}\frac{g_{00}c}{\sqrt{|g_{l0}^2 - g_{ll}g_{00}|} + g_{l0}} = \frac{\sqrt{|g_{00}|}}{\frac{g_{l0}}{\sqrt{|g_{l0}^2 - g_{ll}g_{00}|}} + 1}c. \tag{7.147}$$

Let us see what we get for the two characteristic coordinate directions. Along the radial direction $\tilde{r}$ we have $g_{0r} = 0$, $g_{00} = -(1 - \omega^2\tilde{r}^2)$, $g_{rr} = 1$ therefore

$$c_r = \sqrt{1 - \omega^2\tilde{r}^2}c \ < \ c.$$

For the tangential velocity of light we have $g_{0\theta} = \omega\tilde{r}^2$, $g_{00} = -(1 - \omega^2\tilde{r}^2)$, $g_{\theta\theta} = \tilde{r}^2$. Hence

$$c_\theta = \frac{\sqrt{1 - \omega^2\tilde{r}^2}}{\frac{\omega\tilde{r}^2}{\sqrt{|\omega\tilde{r}^2 - \tilde{r}^2[-(1-\omega^2\tilde{r}^2)]}} + 1}c = \frac{\sqrt{1 - \omega^2\tilde{r}^2}}{1 + \omega\tilde{r}}c = \sqrt{\frac{1 - \omega\tilde{r}}{1 + \omega\tilde{r}}}c \ < \ c.$$

Exercise 25 Compute the spatial length of the velocity of light that is the quantity $c_r^2 + \tilde{r}^2 c_\theta^2$ and compare it with c.

The reason that the velocity of light is locally different from c is due to the fact that the clocks of the rotating disk are synchronized with light signals from the center of the disk, whereas the clocks of the RIO Σ are synchronized by the Einstein convention (i.e., the Lorentzian kinematics).

Exercise 26 Show that the time Δt_r required by a light ray to reach from the center of the disk to the periphery of a circle of radius $\tilde{r}_1$ is given by

$$\Delta t_r = \frac{1}{c}\int_0^{\tilde{r}_1} \frac{d\tilde{r}}{\sqrt{1 - \omega^2\tilde{r}^2/c^2}} = \frac{1}{\omega}\arcsin\frac{\omega\tilde{r}_1{}^2}{c}. \tag{7.148}$$

7.11 The Generalization of Lorentz Transformation and the Accelerated Observers

The Lorentz transformation contains all the mathematical structure of Special Relativity. Indeed being linear relates the characteristic observers of the theory (the RIO) and, by means of the Lorentz group, defines the Lorentz tensors, which are the mathematical objects describing the physical quantities of the theory. Therefore within the framework setup by the Lorentz transformation it is possible to study all problems involving relativistic inertial motions. However, in practice the rule is accelerated motions whereas inertial motions are the exception. Therefore it is natural to ask

> Is it possible to study accelerated motions by means of a (generalized) Lorentz transformation?

One answer to this question has been given in Sect. 7.2 by the introduction of the Locally Relativistic Inertial Observers (LRIO). In that approach one approximates the world line of an accelerated observer (which of course is not a straight line) with a continuous sequence of straight lines each line being the tangent at each and every point of the world line. This continuous sequence of straight lines is parameterized by the proper time τ (say) of the accelerated observer and can be considered equivalently as a continuous sequence of Lorentz transformations whose parameter $\boldsymbol{\beta}(\tau)$ is again a function of the proper time τ. This approach has been used in the study of the Thomas phenomenon.

In the last sections we considered a different approach, that is, we induced a metric for the accelerated observer using the metric of the inertial observer $\eta = diag(-1, 1, 1, 1)$ and the transformation relating the coordinates of the two observers. Then we were able to calculate temporal and spatial distances for the accelerated observer. In the present section we wish to take (7.116) and (7.119) one step further and answer the question

> Is it possible to determine a transformation which relates a RIO with a given accelerated observer in the same way the Lorentz transformation relates a RIO with a RIO?

As we shall show this is possible, and the resulting transformation – which is not universal as the Lorentz transformation, but depends on the particular accelerated observer – we call *generalized Lorentz transformation*. We shall consider only simple configurations, because in more general situations one has to go directly to General Relativity. This "generalization" of the Lorentz transformation will take us to the limits of Special Relativity and furthermore indicates the direction one should take in order to extend the theory. Indeed, as will be shown, the extension – generalization of Special Relativity is not of a simple academic interest but it is a necessity emerging from physical reality. That is to say, there are relativistic phenomena which cannot be answered within the scenario of Special Relativity, because they are not due to relative motion.

7.11.1 The Generalized Lorentz Transformation

We seek a generalization of Lorentz transformation which will meet the following demands:

(1) The assumptions which will be made shall be minimal
(2) In the limit, when the acceleration vanishes the "generalized" Lorentz transformation will reduce to the standard Lorentz transformation

We recall that the standard Lorentz transformation has the following basic futures:

(1) It is an isometry, that is, preserves the Lorentz length $ds^2 = ds'^2$.
(2) It is linear
(3) Preserves the canonical form of the Lorentz metric $diag(-1, 1, 1, 1)$.

The generalization we are looking for must defy some of these assumptions. But which one(s)?

The assumption of isometry is essential because it concerns fundamental physical principles of the theory. For example, the particles are classified into photons and particles with mass according to the (Lorentz) length of the momentum or position four-vector. Therefore a transformation which will not preserve the length of four-vectors it is possible to change the nature of a particle. This would imply that, by means of a transformation, a photon could become a particle with mass and conversely, a situation which cannot be accepted.

There remain the other two assumptions. The property of linearity must be defied, because the world-line of an accelerated observer is not a straight line and a linear transformation cannot relate a straight line with a non-straight line. Concerning the last property this must also be abandoned because the Lorentz metric for an accelerated observer cannot have the canonical form.

Based on the above remarks we demand that the generalized Lorentz transformation must satisfy the following requirements:

- Must be an isometry, that is $ds^2 = ds'^2$
- Will not be linear except when acceleration vanishes
- Will apply in general only locally, that is, around every point of the world line of the (specific) accelerated observer and not globally (i.e., in the whole of Minkowski space) as it is the case with the standard Lorentz transformation. Furthermore in general the set of all "generalized" Lorentz transformations associated with an accelerated motion cannot be a subgroup of the Lorentz group and need not form a group

Consider a RIO Σ with coordinates (l, x, y, z) and let (l', x', y', z') be the coordinates of an accelerated observer E. The distance of two "adjacent" points in

Minkowski space for Σ is[16]

$$ds^2 = -dl^2 + dx^2 + dy^2 + dz^2 \tag{7.149}$$

while for E let us assume that it has the general form[17]

$$ds'^2 = -u_0^2 dl'^2 + u_1^2 dx'^2 + u_2^2 dy'^2 + u_3^2 dz'^2, \tag{7.150}$$

where u_0, u_1, u_2, u_3 are real, smooth functions of the coordinates (l', x', y', z'). The demand of isometry gives

$$-dl^2 + dx^2 + dy^2 + dz^2 = -u_0^2 dl'^2 + u_1^2 dx'^2 + u_2^2 dy'^2 + u_3^2 dz'^2. \tag{7.151}$$

In order to simplify the mathematics we consider motion along the x-direction only and set $y = y'$, $z = z'$. Then the transformation we are looking for is of the form

$$x = x(l', x'), l = l(l', x'), \tag{7.152}$$

where the functions are such that the transformation will be non-singular, that is, it will be reversible. The condition for this is that the Jacobian shall be non-zero. We compute

$$\begin{aligned} dx &= \frac{\partial x}{\partial l'} dl' + \frac{\partial x}{\partial x'} dx' = a(l', x')dl' + b(l', x')dx' \\ dl &= \frac{\partial l}{\partial l'} dl' + \frac{\partial l}{\partial x'} dx' = c(l', x')dl' + d(l', x')dx' \end{aligned}$$

from which follows

$$\begin{aligned} ds^2 = -dl^2 + dx^2 &= -(cdl' + ddx')^2 + (adl' + bdx')^2 \\ &= (-c^2 + a^2)dl'^2 + (b^2 - d^2)dx'^2 + 2(ab - cd)dl'dx'. \end{aligned}$$

But

$$ds'^2 = -u_0^2 dl'^2 + u_1^2 dx'^2$$

hence isometry implies

---

[16] The concept "adjacent" is outside the scope of this book and requires more advanced mathematics (topology). We simply mention that "adjacent" is modulated according to the curvature (= measure of acceleration) at each point of the world line of the accelerated observer.

[17] This form is general, because it can be shown that every (non-singular) metric can be put in this form by an appropriate choice of the coordinate system. Note that the quantities u_0, u_1, u_2, u_3 are functions of the coordinates and not constants.

$$c^2 - a^2 = u_0^2, \tag{7.153}$$
$$b^2 - d^2 = u_1^2, \tag{7.154}$$
$$ab - cd = 0. \tag{7.155}$$

Because the functions u_0, u_1 are real we demand $| c |>| a |, | b |>| d |$. From the continuity of the transformation we also have the conditions

$$\frac{\partial^2 x}{\partial l' \partial x'} = \frac{\partial^2 x}{\partial x' \partial l'} \Longrightarrow a_{,x'} = b_{,l'}\,, \tag{7.156}$$
$$\frac{\partial^2 l}{\partial l' \partial x'} = \frac{\partial^2 l}{\partial x' \partial l'} \Longrightarrow c_{,x'} = d_{,l'}\,, \tag{7.157}$$

where "," indicates partial derivative wrt the value that follows. Relations (7.154), (7.155), (7.156), (7.157), and (7.158) are the relations among the coefficients a, b, c, d of the required transformation and the components u_0, u_1 of the metric.

We note that if we demand the transformation to be linear the coefficients a, b, c, d must be constants. In this case from (7.154), (7.155), and (7.156) follows that the coefficients u_0, u_1 are also constants, hence if we consider the change of units $x' \Rightarrow u_1 x', l' \Rightarrow u_0 l'$ the metric takes its canonical form $diag(-1, 1, 1, 1)$ and we return to the standard Lorentz transformation. We infer that in order to have a generalized Lorentz transformation the coefficients $a,\ b,\ c,\ d$ or, equivalently, the functions $u_0(l', x'), u_1(l', x')$ must be non-constants.

Obviously the general solution of the system of equations (7.154), (7.155), (7.156), (7.157), and (7.158) is difficult to find – and perhaps it is of no interest – therefore we are looking for special solutions, which have a profound kinematic and/or geometric significance.

7.11.2 The Special Case $u_0(l', x') = u_1(l', x') = u(x')$

We consider the accelerated (one-dimensional) motion which is defined by the conditions

$$u_0(l', x') = u_1(l', x') = u(x'), \tag{7.158}$$

where $u(x')$ is a real smooth function. We note that for this choice the metric for the accelerated observer is conformal[18] to the canonical form of the Lorentz metric $diag(-1, 1, 1, 1)$:

$$ds'^2 = u^2(x')(-dl^2 + dx^2).$$

[18] Two metrics $g_1(x^i),\ g_2(x^i)$ are said to be *conformal* if there exists a smooth function $\phi(x^i)$ such that $g_2(x^i) = \phi(x^i) g_1(x^i)$. The conformally related metrics have important applications in physics and especially in electromagnetism and General Relativity. A metric which is conformal to the Lorentz metric is called *conformally flat*.

In this particular case relations (7.154)–(7.158) read

$$c^2 - a^2 = u^2, \tag{7.159}$$
$$b^2 - d^2 = u^2, \tag{7.160}$$
$$ab - cd = 0, \tag{7.161}$$
$$a_{,x'} = b_{,l'}\,, \tag{7.162}$$
$$c_{,x'} = d_{,l'} \tag{7.163}$$

with the restriction $\mid c \mid > \mid a \mid, \mid b \mid > \mid d \mid$. From (7.160) and (7.161) follows

$$c^2 - a^2 = b^2 - d^2 \Longrightarrow a^2 + b^2 = c^2 + d^2$$

and from (7.162)

$$ab = cd.$$

Adding and subtracting we find

$$\mid a + b \mid = \mid c + d \mid, \mid a - b \mid = \mid c - d \mid .$$

These relations are equivalent to the following four systems of algebraic equations

$a + b = c + d$ and $a - b = c - d \Rightarrow a = c, b = d$ not accepted
$a + b = c + d$ and $a - b = -c + d \Rightarrow a = d, b = c$ accepted
$a + b = -c - d$ and $a - b = c - d \Rightarrow a = -d, b = -c$ accepted
$a + b = -c - d$ and $a - b = -c + d \Rightarrow a = -c, b = -d$ not accepted.

We conclude that the solution is

$$a = \epsilon_1 d, \qquad b = \epsilon_1 c, \tag{7.164}$$

where $\epsilon_1 = \pm 1$.

We continue with relations (7.163) and (7.164). We differentiate (7.161) wrt l' and find (note $u(x')$!)

$$bb_{,l'} - dd_{,l'} = 0.$$

From (7.163) and (7.164) we have

$$ca_{,x'} - ac_{,x'} = cb_{l'} - ad_{l'} = \frac{c}{b} bb_{l'} - \frac{a}{d} dd_{l'} = \frac{1}{b}(cd - ab)\, d_{,l} = 0$$

from which follows

$$\left(\frac{a}{c}\right)_{,x'} = 0 \Rightarrow a = \Phi(l')c, \tag{7.165}$$

where $\Phi(l')$ is a smooth function. Replacing a in (7.160) we find ($\epsilon_2 = \pm 1$)

$$c = \frac{\epsilon_2 u}{\sqrt{1-\Phi^2}}, \mid \Phi \mid < 1.$$

and from (7.165) follows

$$d = \epsilon_1 \Phi(l')c.$$

Finally the solution is

$$\begin{aligned} a &= \epsilon_1 d = \frac{\epsilon_1 \epsilon_2 u \Phi}{\sqrt{1-\Phi^2}}, \\ b &= \epsilon_1 c = \frac{\epsilon_1 \epsilon_2 u}{\sqrt{1-\Phi^2}}. \end{aligned} \tag{7.166}$$

This solution is determined in terms of the functions $u(x')$, $\Phi(l')$ ($\mid \Phi \mid < 1$), which have to be determined. In order to do that we consider (7.163) from which follows

$$a_{,x'} = b_{,l'} \Rightarrow \frac{\Phi}{\sqrt{1-\Phi^2}}\, u_{,x'} = \frac{u\Phi\Phi_{,l'}}{\left(1-\Phi^2\right)^{3/2}} \Rightarrow \frac{u_{,x'}}{u} = \frac{\Phi_{,l'}}{1-\Phi^2}.$$

The lhs is a function of x' and the rhs is a function of l'. Therefore each side of the equation must equal a constant p, say

$$\frac{1}{u}\frac{du}{dx'} = p, \tag{7.167}$$

$$\frac{1}{1-\Phi^2}\frac{d\Phi}{dl'} = p. \tag{7.168}$$

Integration of the first gives

$$u(x') = Ae^{px'}, \qquad A = \text{constant} \tag{7.169}$$

and integration of the second

$$\frac{1+\Phi}{1-\Phi} = e^{2pl'+2B} \Rightarrow \Phi(l') = \tanh(pl'+B), \qquad B = \text{constant}. \tag{7.170}$$

Eventually we have the solution

$$a(l', x') = \epsilon_1 d(l', x') = \epsilon_1\epsilon_2 A e^{px'} \frac{\tanh(pl' + B)}{\sqrt{1 - \tanh^2(pl' + B)}} = \epsilon_1\epsilon_2 A e^{px'} \sinh(pl' + B)$$

$$b(l', x') = \epsilon_1 c(l', x') = \epsilon_1\epsilon_2 A e^{px'} \cosh(pl' + B).$$

In order to compute the constant B we assume the initial condition: the motion starts from rest. Then $\frac{dx'}{dl'}(0) = \frac{dx}{dl}(0) = 0$ and follows

$$0 = \frac{dx}{dl}_{|0} = \frac{a(0)dl' + b(0)dx'}{c(0)dl' + d(0)dx'}_{|0} = \frac{a(0) + b(0)\frac{dx'}{dl'}(0)}{c(0) + d(0)\frac{dx'}{dl'}(0)} \Rightarrow a(0) = 0 \Rightarrow B = 0.$$

This implies the final expression for the coordinates of the metric

$$a(l', x') = \epsilon_1 d(l', x') = \epsilon_1\epsilon_2 A e^{px'} \sinh pl', \tag{7.171}$$

$$b(l', x') = \epsilon_1 c(l', x') = \epsilon_1\epsilon_2 A e^{px'} \cosh pl'. \tag{7.172}$$

Having the expression of the coordinates of the metric we are in a position to determine the "generalized" Lorentz transformation for this particular case we discuss (not for every accelerated motion!). We find

$$x = \int a dl' + b dx' = \epsilon_1\epsilon_2 \frac{A}{p} \int d[e^{px'} \cosh pl'] = \epsilon_1\epsilon_2 \frac{A}{p} e^{px'} \cosh pl' + B_1 \tag{7.173}$$

and similarly

$$l = \epsilon_1\epsilon_2 \frac{A}{p} e^{px'} \sinh pl' + B_2. \tag{7.174}$$

We determine the constants B_1, B_2 by demanding the initial conditions: $l = l' = 0, x' = 0, x = k$, that is, the motion starts from the point $x = k$ of Σ and with the clocks synchronized at the origin of the coordinates. We compute easily $B_1 = -\epsilon_1\epsilon_2 \frac{A}{p} + k$, $B_2 = 0$, hence

$$l = \epsilon_1\epsilon_2 \frac{A}{p} e^{px'} \sinh pl', \tag{7.175}$$

$$x = \epsilon_1\epsilon_2 \frac{A}{p} (e^{px'} \cosh pl' - 1) + k. \tag{7.176}$$

It remains to compute the constant A which concerns initial conditions for u. We require that when $x' = 0$ then $dx' = 0$, that is, at the origin of E the coordinate $l' = \tau$ where τ is the proper time of the accelerated observer. This demand restricts the motion of the origin of the accelerated frame to be the hyperbolic motion we studied in Sect. 7.4.

For $dx' = 0$ we find ($c = 1$)

$$ds'^2 = -u_0^2(0)dl'^2.$$

But we also have that $ds^2 = -d\tau^2$ so that from the isometry condition $ds^2 = ds'^2$ follows

$$-d\tau^2 = -u_0^2(0)d\tau^2 \Rightarrow u_0^2(0) = 1 \Rightarrow A = \epsilon_3, \tag{7.177}$$

where $\epsilon_3 = \pm 1$. We conclude that the generalized Lorentz transformation for this transformation of accelerated motion is

$$l = \epsilon_1\epsilon_2\epsilon_3 \frac{1}{p} e^{px'} \sinh pl', \qquad x = \epsilon_1\epsilon_2\epsilon_3 \frac{1}{p}\left(e^{px'} \cosh pl' - 1\right) + k. \tag{7.178}$$

Indeed we note that when $x' = 0$ relations (7.179) read

$$l = \epsilon_1\epsilon_2\epsilon_3 \frac{1}{p} \sinh pl' \qquad x = \epsilon_1\epsilon_2\epsilon_3 \frac{1}{p}\left(\cosh pl' - 1\right) \tag{7.179}$$

which are identical with the corresponding relations of hyperbolic motion if we set $l' = \tau$ and $\epsilon_1\epsilon_2\epsilon_3 = 1$.

It is interesting to compute the inverse transformation $x' = x'(x, l),\ l' = l'(x, l)$. This is achieved as follows. We define the constant $\bar{p} = \frac{p}{\epsilon_1\epsilon_2\epsilon_3}$ and use (7.179) to find

$$\frac{l\bar{p}}{(x-k)\bar{p}+1} = \tanh pl' \Rightarrow l' = \frac{1}{p}\tanh^{-1}\left[\frac{l\bar{p}}{(x-k)\bar{p}+1}\right]. \tag{7.180}$$

Replacing in the first of (7.179) we get

$$e^{px'} = \frac{\bar{p}l}{\sinh pl'} \Rightarrow x' = \frac{1}{p}\ln\frac{\bar{p}l}{\sinh\tanh^{-1}\frac{l\bar{p}}{(x-k)\bar{p}+1}}. \tag{7.181}$$

Let us examine the equations of transformation in the limit $px' \ll 0$, that is, very close to the origin of the accelerated frame. In this case $e^{px'} = \sum_{n=0}^{\infty} \frac{(px')^n}{n!} \cong 1 + px' \cong 1$ and the equations of the transformation (7.180) are simplified as follows:

$$l = \frac{1}{\bar{p}}\sinh pl', \tag{7.182}$$

$$x = \frac{1}{\bar{p}}(\cosh pl' - 1) + k. \tag{7.183}$$

We infer that very close to the origin of the accelerated frame the equations of transformation tend to the equations of hyperbolic motion, a result which was expected.

7.11.2.1 Conclusion

It is possible to find a non-linear isometry which relates the coordinates (l, x) of a RIO with the coordinates of an accelerated observer E (l', x'). Therefore we arrive at the important conclusion that

> The accelerated motion can be "absorbed" into the generalization of the metric, that is the Geometry.

Although we proved this for the simple case we considered above, there is no reason why we should not assume the validity of this conclusion in general. This conclusion is of fundamental importance for the following reason. According to the Principle of Equivalence the massive bodies (and, as we shall see, the photons) suffer the same acceleration in a gravitational field independently of their internal structure. Consider a RIO Σ in which there exists a gravitational field whose strength is $\mathbf{g}$. Then according to the Equivalence Principle all particles with mass (and the photons), when moving freely in Σ, attain acceleration $\mathbf{g}$. According to the generalization we have considered it is possible to alter the metric $diag(-1, 1, 1, 1)$ to a new metric, which will be determined from the form of the gravitational field. Therefore

> The gravitational field is "absorbed" into Geometry (i.e. the metric) of spacetime (not Minkowski space anymore) and in the new metric the effect of the gravity does not exist.

This type of motion is called *free fall*. It exists in Newtonian Physics. However, it is not attributed to the change of the background geometry of (absolute) space and one simply ignores the gravitational field as a force in Newton's Second Law. The geometrization of the gravitational field started with Special Relativity and finally found its natural place in General Relativity, which is also a geometric theory of physics.

7.11.3 Equation of Motion in a Gravitational Field

An important application of the generalized Lorentz transformation, which takes us very close to General Relativity, is the Principle of Least Action for accelerated motions.

Let A, B two events along the world line of a ReMaP P. If P moves inertially then the world line of P is a straight line in Minkowski space and it is defined by the requirement that the integral

$$\int_A^B \sqrt{|ds^2|}$$

is extremum wrt the metric $\eta = diag(-1, 1, 1, 1)$. This requirement is equivalent to the demand that the variation of the action integral vanishes (Principle of Least Action)

$$\delta \int_A^B \sqrt{|ds^2|} = 0.$$

Because for an accelerated motion we have assumed that the metric remains the same, that is $ds^2 = ds'^2$, whereas the components change we stipulate that the world line of an accelerated relativistic mass point is given by the condition

$$\delta \int_A^B \sqrt{|ds'^2|} = 0, \tag{7.184}$$

that is, it is a geodesic (curve of extremum length defined by the points A, B) wrt the metric g_{ij}. Therefore the straight lines of an inertial motion become geodesics of a proper Riemannian metric of Lorentzian character. In case the acceleration is due to gravitational forces only, (7.184) defines the world line of a relativistic particle in free fall in the gravitational field.

We conclude that the effect of accelerated motion in Special Relativity is twofold:

- "Changes" the Lorentz metric from its canonical form $\eta = diag(-1, 1, 1, 1)$ to another general form g_{ij}. The theory does not say how this new metric is to be defined and one if free to do it as one considers best. General Relativity gives field equations for the determination of this metric.
- The world line of a relativistic particle in free motion is a (timelike) straight line in Minkowski space when the gravitational field vanishes and (timelike) geodesic of the Riemannian space with metric g_{ij} when there exists a gravitational field.

The above conclusions lead us to the following generalization of the Equivalence Principle of Newtonian gravity in General Relativity (as we shall see Special Relativity cannot accommodate the gravitational field):

> Accelerated motions which are caused by the gravitational field only (free fall) take place along geodesics of the metric which corresponds to the particular gravitational field.[19]

7.12 The Limits of Special Relativity

In Sect. 7.11 we showed that accelerated motions, and consequently the gravitational field, can be absorbed in the geometry of spacetime without introducing new principles but the Principle of Equivalence, which is borrowed from Newtonian theory. Therefore it is logical to expect that with this differentiation–generalization Special Relativity can become a theory of gravity. The first attempts toward this direction

[19] This metric is computed (partially) from Einstein field equations.

were done by Einstein himself (and others followed). However, it became clear that Special Relativity cannot accommodate the gravitational field. However, it offers many of its elements for such a theory to be constructed. This road led eventually to the Theory of General Relativity. In the following we present some thought experiments which justify these remarks. The common characteristic of these "experiments" is that they consider photons interacting with the gravitational field and show that this interaction leads to new phenomena which cannot be explained within the framework of Special Relativity and require a new relativistic theory of gravitation.

7.12.1 Experiment 1: The Gravitational Redshift

Consider a positron and one electron which rest within a gravitational field at the potential level V in some RIO Σ. At some moment the particles are left to fall freely in the gravitational field. At the potential level $V+\Delta V$ ($\Delta V<0$) the two particles annihilate producing two photons of frequency ν (why the two photons must have the same frequency?). Subsequently the photons are reflected elastically on a large mirror and return to the potential level V where they interact creating again a pair positron – electron which are left to fall again and so on. This thought experiment was suggested by Einstein.[20]

We discuss this thought experiment with the sole restriction that it does not give rise to a perpetual mobile, that is, the Second Principle of Thermodynamics is not violated. Because the particles have equal masses, at the potential level $V+\Delta V$ they will have equal kinetic energies $\frac{1}{2}mv^2=m\mid\Delta V\mid$. At the first event of annihilation of the particles conservation of energy implies for the frequency of the photons (in the following we assume $c=1$)

$$2h\nu=-2m\Delta V+2m. \tag{7.185}$$

Let $\overline{\nu}$ the frequency of the photons when they reach the potential level V. If the photons do not interact with the gravitational field their frequency at the potential level V will again be ν, hence the produced pair of particles must have non-zero kinetic energy, which contradicts the Second Principle of Thermodynamics. Therefore the photons *must interact* with the gravitational field and more specifically they must loose energy (i.e., they must be redshifted) as they propagate to higher potential levels. In order to compute the amount of redshift we assume that the pair of electron – positron at the potential level V is produced at rest, therefore

$$2h\overline{\nu}=2m. \tag{7.186}$$

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[20] See A. Einstein "Uber den Einfluss der Schwerkraft auf die Ausbreitung des Lichtes" Ann. Phys. 898–908, (1911). Translation of this paper can be found in Lorentz H. A, Einstein A, Minkowski H and Weyl H "The Principle of Relativity: A Collection of Original Memoirs" Mathewen, London, (1923). Paperback reprint Dover (1952) New York.

From relations (7.186) and (7.187) follows

$$\frac{\overline{\nu} - \nu}{\overline{\nu}} = \Delta V < 0. \tag{7.187}$$

The redshift of an electromagnetic wave is measured with the quantity $z = \frac{\lambda - \overline{\lambda}}{\overline{\lambda}}$ or, in terms of the frequency $z = \frac{\Delta\nu}{\overline{\nu}}$. It follows that

$$z = \Delta V. \tag{7.188}$$

The phenomenon of variation of the frequency of a photon as it propagates in a non-homogeneous gravitational field we call *gravitational redshift*. This phenomenon is neither Newtonian nor it can be explained within the framework of Special Relativity. Indeed the gravitational redshift has served as one of the first experimental facts toward the justification of General Relativity. It has been verified by astronomical observations (with the shift of the line D_1 of the spectrum of Na in the solar spectrum[21] as well as with relevant observations in the laboratory.

Concerning the latter the best available measurements have been done by Pound and Rebka[22] and Pound and Snider.[23] In these experiments γ radiation of 14.4 KeV produced from the radioactive source ^{57}Co, was made to propagate opposite to the gravitational field into a tube of length 22.5 m filled up with helium, which was placed along the tower of Harvard in the campus of the University of Harvard. At the reception of the photons it was placed an absorber enriched with ^{57}Fe, which was connected with a proportional counter. If the photons do not interact with the gravitational field the energy of the photons is proportional to the square of the speed u of the emitter (this velocity can change as we shall see below in this book when we study the relativistic reactions). If we consider various speeds of the emitter then the distribution of the number of photons in terms of speed must be a Gaussian symmetric around the value $u = 0$.

In the experiment, the distribution of photons for two ranges of the velocity was measured and the existence of asymmetry was examined. The first range was the velocities $V_D \pm u$ and the second range the velocities $-V_D \pm u$. The asymmetry *was observed* and the calculations agreed with the value given by the gravitational redshift (that is $z = -mgh$). Therefore the gravitational redshift is indeed a physical phenomenon, which must satisfy every theory of gravity. One of these theories is General Relativity.

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[21] See Brault J. W. "Gravitational redshift of solar lines" in *Bull Amer Phys Soc* **8**, 28, (1963).

[22] See Pound R. V. and Rebka G. A. "Apparent weight of photons" *Phys Rev* **4**, 337–341, (1960).

[23] See Pound R. V. and Snider J. L. "Effect of gravity on gamma radiation" *Phys Rev* **B140**, 788–803, (1965).

7.12.2 Experiment 2: The Gravitational Time Dilation

The fundamental conclusion of the gravitational redshift is that the photons interact with the gravitational field. If one considers the photon as an oscillator, hence as a clock, this implies that identical proper clocks placed at rest at different places within a non-homogeneous gravitational field have different rates! This is not conceivable within the framework of Special Relativity. To show the validity of this assertion we consider a second thought experiment proposed by Schild.[24]

A source of monochromatic electromagnetic radiation is placed at the potential level $V + \Delta V$ of a gravitational field and a receiver not moving wrt the source, at the potential level V. If the electromagnetic wave (photon) does not interact with the gravitational field the frequency of the wave at the emitter and the receiver must be the same. However, as we have shown the electromagnetic field does interact with the gravitational field, therefore the two frequencies must be different, i.e., $\nu \neq \overline{\nu}$. Suppose that the frequency of the wave at the emitter is ν and the frequency at the receiver is $\overline{\nu}$. Let τ be the proper time at the emitter and $\overline{\tau}$ the proper time at the receiver. Assume that the emitter sends photons for a period τ and assume that these photons are received at the receiver during a period $\overline{\tau}$. Because the number of photons (oscillations) must be the same at the emitter and the receiver, we must have $\nu\tau = \overline{\nu}\,\overline{\tau}$. But $\nu \neq \overline{\nu}$ therefore $\tau \neq \overline{\tau}$. However, according to Special Relativity the emitter and the receiver neither move relative to each other, nor within the gravitational field, therefore the indications of the proper clocks at the emitter and the receiver once identical (synchronized with the Einstein synchronization) should stay so, i.e., $\tau = \overline{\tau}$. We conclude that

(1) Special Relativity cannot be used in the study of gravitational phenomena except in the cases of very small regions and for weak gravitational fields, which can be treated practically as homogeneous, where the gravitational redshift can be neglected.
(2) The rate of a clock depends on the strength of the gravitational field at the point where the clock is situated. This implies that the proper time of relativistic observers which rest at different potential levels in a gravitational field is not the same. More specifically the rate of the clock of the observer at the lower potential level is higher than the rate of the clock at the higher potential level. This results in a time dilation effect between observers at different potential levels. This new phenomenon we call *gravitational time dilation*. In order to compute the amount of the gravitational time dilation we consider the relation $\nu\tau = \overline{\nu}\,\overline{\tau}$ and obtain

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[24] 1. Schild A. Lectures on General Relativity Theory, in “Relativity Theory and Astrophysics I: Relativity and Cosmology” p. 27 in Vol. 8 of the series Lectures in Applied Mathematics. *Am. Math. Soc.* (1967).

2. Schild A. in “Proceedings International School of Physics Enrico Fermi” Academic Press, NY, pp. 69–115, (1963).

$$\frac{\overline{\nu} - \nu}{\overline{\nu}} = \frac{\overline{\tau} - \tau}{\overline{\tau}}. \tag{7.189}$$

Replacing the lhs from (7.188) which gives the gravitational redshift we find

$$\frac{\overline{\tau} - \tau}{\overline{\tau}} = \Delta V. \tag{7.190}$$

From (7.190) it is apparent that the rate of a clock depends on the strength of the gravitational field at the position of the clock.

7.12.3 Experiment 3: The Curvature of Spacetime

The phenomenon of the gravitational time dilation prohibits the use of Special Relativity but also does indicates the course one should take to "extend" this theory. This direction is the one we have already chosen in the generalization of the Lorentz transformation. Indeed the generalization of Lorentz transformation through the coordinates of a RIO and an accelerating observer leads directly to the generalization of the Newtonian Principle of Equivalence to incorporate all physical systems (including photons which do not exist – in the sense that they are not Newtonian physical systems – in Newtonian theory). The following thought experiment has been proposed[25] by Schild.

Consider a spherical mass M which creates in the surrounding space a gravitational field and study the propagation of photons between an emitter on the surface of the sphere and a receiver (of negligible mass so that it has no effect on the gravitational field of M) near the surface of M. Let us assume that spacetime around the mass is still the Minkowski space. Let A be the event of emission of a photon of frequency ν from the emitter the proper moment τ_A of the emitter and B the event of reception of the photon at the receiver, the proper time τ_B of the receiver. Because the emitter and the receiver do not move wrt the sphere their world lines are parallel straight lines in Minkowski space. Due to the gravitational redshift the frequency $\overline{\nu}$ of the photon which reaches the receiver is $\overline{\nu} \neq \nu$. Suppose we repeat the experiment the proper time τ'_A of the emitter and send a photon (event A') which is obtained by the receiver (event B') the proper time τ'_B of the receiver. Because everything is static the frequency of the second received photon must also be $\overline{\nu}$. This implies that the distance $AB = A'B'$ and due to the parallelism of the world lines we infer that $ABB'A'$ is a parallelogram in Minkowski space. But then $AA' = BB'$ which contradicts the gravitational time dilation!

The solution to this conflict is to assume that in the presence of gravity spacetime is not anymore the Minkowski space but a more general metric space, which in the small vicinity of any of its points (where the gravitational field can be considered homogeneous) can be approximated by Minkowski space. This new space has curvature and it is the substratum of General Relativity.

[25] See reference in previous footnote.

Chapter 8
Paradoxes

8.1 Introduction

As has been remarked repeatedly in the previous chapters, the theory of Special Relativity is not based on direct sensory experience, as it is the case with Newtonian Physics. This leads to situations which contradict the "common sense"[1] in the sense that the theory gives different results depending on the observer describing a given phenomenon. This contradicts the fundamental hypothesis of physics that "reality" is observer independent. If this was true, then that would be a real problem for the theory.

As it is expected, after the introduction of Special Relativity and indeed after its early success, many people (who react to the new) posed several "hypothetical" experiments with the purpose to prove the confrontation of the "new" theory with the common sense, hence its invalidation. Most of these suggestions involve the length contraction and the time dilation, because both (Euclidean) space and (Newtonian) time are fundamental to our perception of the world. All these "experiments" with one name are referred to as "paradoxes."

All paradoxes concern Newtonian Physical situations ("common sense"), which are transferred over to Special Relativity without checking if this is possible and in what way. All paradoxes can be explained if one considers the problem from a true relativistic point of view. We believe that whatever paradoxes will be proposed they will be explained by Special Relativity. Since if Special Relativity had even a small "bag" it is rather unreal that this has not yet shown after the everyday routine application of the theory in the laboratory for more than 100 years.

However, paradoxes are useful because they help us understand better the theory and it is possible that they reveal facets of the theory that we have not noticed yet. A similar case is the contribution of Einstein to quantum theory. Indeed Einstein

[1] Einstein in order to show the insufficiency of the "common sense" in various projections of the human mind has remarked that "*Common sense is the aggregate of prejudices acquired by the age of eighteen.*" However, many years before him Heraclitus has remarked that "*Του λογου δεοντος ξυνου ζωουσιν οι πολλοι ως ιδιαν εχοντες φρονησιν*" which means *"Although common sense is common, most people consider it as if it is their own."*

M. Tsamparlis, *Special Relativity*, DOI 10.1007/978-3-642-03837-2_8,

was very sceptical about quantum theory ("The God does not play dice") and proposed many paradoxes and arguments in order to disprove that theory. The search for answers to these arguments greatly helped quantum mechanics, to the degree that later these arguments were considered to be Einstein's contribution in the development of quantum mechanics.

One of the most well-known paradoxes, for which many papers and arguments have been put forward, is the tween paradox according to which two twins in relative non-uniform motion age differently. Today, this paradox should be considered as rather trivial and shall not be considered here.[2] Instead in the following we shall discuss more advanced and relatively unknown paradoxes.

8.2 Various Paradoxes

In this section we discuss a number of paradoxes which concern the length contraction, the time dilation, and other Newtonian Physical quantities.

Example 30 In Fig. 8.1 the H-shaped slider $CDEZ$ slides with constant velocity βc along the ends of the circuit Σ. We assume that the current can flow only along the sides CE and DZ and that the distance of the points CD equals the distance of the points AB when the slider $CDEZ$ rests in the circuit Σ. Let Σ' be the proper observer of the slider. According to Σ as the slider moves the distance of the points C, D is shortened due to length contraction; therefore the lamp will be off for a period of time T_{off} (say). According to the proper observer Σ' of the slider for the same reason the distance of the points AB is shorter than that of the points CD, therefore the lamp will be always on. Prove that the two views, however radical, can be explained and hence do not lead to a paradox.
Solution

Before we consider the solution of the paradox[3] we note that when we close a switch the current develops in the conductor with speed $\beta_s c$ where $\beta_s c$ is the speed

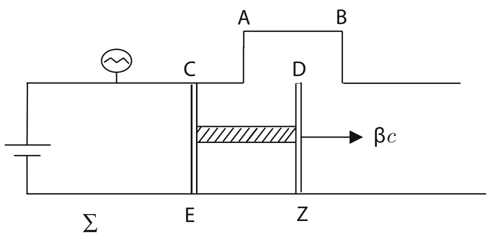


Fig. 8.1 The paradox of the lamp

[2] For literature relevant to this paradox see for example D. Greenberger "The reality of the twin paradox effect" *Am. J. Phys.* **40**, 750–754, (1972).

[3] For a different approach see G. Sastry "Is length contraction really paradoxical?" *Am. J. Phys.* **55**, 943–46, (1987).

of propagation of the electromagnetic field in the conductor. This implies that for an interruption of the development of a current in a conductor there is a "dead" time. We use this effect in order to explain the paradox.

The circuit is open when the following events take place:

(a) The point C leaves contact A.
(b) The electric pulse which is created when the point B comes in contact with point D travels along the part AB of the circuit.

Assuming that the lamp is at the end A of the conductor we have for each of the observers the following time intervals concerning the events (a) and (b):

Observer Σ

The time period that the circuit stays "off" due to dissociation of the points A, C (Lorentz contraction, event a) equals

$$\frac{l_0 - \frac{l_0}{\gamma}}{\beta c}.$$

The time required by the current to develop in the circuit after point D closes the circuit at the point B (event b) is

$$\frac{AB}{\beta_s c} = \frac{l_0}{\beta_s c}.$$

Therefore for Σ the time period for which the lamp is off equals

$$T_{\text{off}} = \frac{l_0 - \frac{l_0}{\gamma}}{\beta c} + \frac{l_0}{\beta_s c} = \frac{l_o}{c}\left(\frac{1}{\beta} + \frac{1}{\beta_s}\right) - \frac{l_0}{\beta\gamma c}. \tag{8.1}$$

Observer Σ'

Assume that the observer Σ' counts time zero when the end D coincides with the point B. The moment that the end point C disassociates from the end point A is[4] (event a)

$$t_1' = \frac{\text{Difference of length } AB \text{ in } \Sigma' \text{ and in } \Sigma}{\beta c} = \frac{l_0 - \frac{l_0}{\gamma}}{\beta c}.$$

[4] We note that $t_2' > t_1'$, therefore it is possible that for observer Σ' the circuit remains open and therefore the lamp will be turned off. The proof that $t_2' > t_1'$ is as follows. It is enough to show that

$$\frac{\gamma l_0}{c\beta_s} + \frac{\gamma\beta l_0}{c} > \frac{l_0}{\beta c} - \frac{l_0}{\gamma\beta c} \Leftrightarrow \frac{\gamma}{\beta_s} > \frac{1}{\beta} - \frac{1}{\gamma\beta} - \gamma\beta = -\frac{\gamma}{1+\gamma},$$

which is true.

Assuming that the speed of the pulse for observer Σ' is $\beta_s' c$ the distance AB, which for Σ' has length $\frac{l_0}{\gamma}$ and speed $-\beta c$, is covered in the interval (event b)

$$t_2' = \frac{l_0}{\gamma(\beta_s' - \beta)c}.$$

$(\beta_s' - \beta)c$ is the speed of propagation of the pulse in the "moving" circuit AB. The time period for which the circuit stays off for Σ' is

$$T'_{\text{off}} = t_2' - t_1' = \frac{l_0}{\gamma(\beta_s' - \beta)c} - \frac{l_0 - (l_0/\gamma)}{\beta c}.$$

From the relativistic rule of composition of three-velocities we have

$$\beta_s' = \frac{\beta_s + \beta}{1 + \beta_s \beta} \Rightarrow \beta_s' - \beta = \frac{\beta}{\gamma^2(1 + \beta_s \beta)}.$$

Replacing we calculate T'_{off}:

$$T'_{\text{off}} = \frac{\gamma l_0}{\beta_s c} + \gamma \frac{l_0 \beta}{c} - \frac{l_0}{\beta c} + \frac{l_0}{\gamma \beta c} = \gamma T_{\text{off}}.$$

This relation proves that if the lamp turns off for Σ then it does so for Σ' for time periods related by the time dilation formula

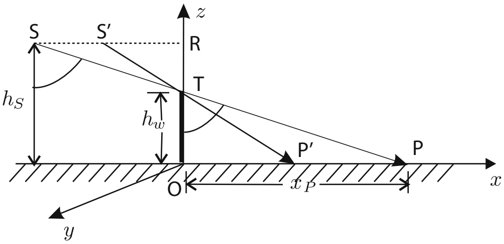
$$T'_{\text{off}} = \gamma T_{\text{off}}.$$

Example 31 A source of monochromatic light is moving in the LCF Σ with constant velocity u in the plane $x{-}z$ parallel to the x-axis at a height h_{s} along the z-axis. A wall of negligible thickness and of height h_{w} ($h_{\text{w}} < h_{\text{s}}$) is placed parallel to the plane $y{-}z$ and in front of the light source. Due to the presence of the wall a shadow is created in the rear part of the wall along the x-axis (see Fig. 8.2). From Newtonian Physics we expect that

(a) The shadow will be continuous.
(b) As the source approaches the wall the length of the shadow on the x-axis will diminish and will vanish when the source is exactly over the wall (that is at the z-axis according to Fig. 8.2).
(c) First the most distant points from the wall will be lighted.

Show that in Special Relativity the behavior of the shadow is different than the Newtonian one, if the velocity u is such that $\frac{u}{c} > 1 - \frac{h_w}{h_s}$. Specifically, show that the

Fig. 8.2 The shadow paradox



shadow vanishes before the light source reaches the wall. Calculate the distance of the light source from the wall when the shadow vanishes.[5]

Solution

First we discuss the phenomenon qualitative. We consider a point S along the orbit of the light source and let P be the point along the x-axis which is lighted from the light ray emitted from the source at the point S and greases the wall at the point T (see Fig. 8.2). The shadow at the point P disappears when this ray reaches the x-axis. Since light propagates with constant finite velocity, when this light ray reaches the point P, the light source will emit a new light ray from another point S'. This light ray will reach another point P' at which the shadow will disappear (see Fig. 8.2). Depending on the velocity of the source it is possible that the time required for the light ray to cover the distance SP is larger than the sum of the time intervals required by the source to cover the distance SS' and the light ray emitted from the point S' to cover the distance $S'P'$. If this is the case the shadow disappears first at the point P' and later at the point P! This behavior is different than the expected Newtonian behavior and it is due to the finite speed of light.

Let us come now to the quantitative discussion. Let Σ be the rest frame of the wall and Σ' the rest frame of the light source. Let t_S be the moment of Σ at which the light source is at the point S and t_P the moment at which the light ray reaches the point P, which we assume to be situated at a distance x_P from the wall. From the geometry of Fig. 8.2 we have

$$x_P = h_{\mathrm{w}} \tan\phi, \tag{8.2}$$

$$(SP) = \frac{h_{\mathrm{S}}}{\cos\phi} = k\sqrt{h_{\mathrm{w}}^2 + x_P^2}, \tag{8.3}$$

where we have set $k = \frac{h_{\mathrm{w}}}{h_{\mathrm{s}}} < 1$. We note that when the light source is infinitely far from the wall the angle $\phi = \frac{\pi}{2}$ and $x_P \to \infty$ hence the shadow covers all the parts of the x-axis at the other side of the wall.

[5] For a different approach see H.-M. Lai "Extraordinary shadow appearance due to fast moving light" *Am. J. Phys.* **43**, 818–820, (1975).

The moment of Σ at which the light ray reaches the x-axis, destroying the shadow at the point P, is

$$t_P = t_S + \frac{(SP)}{c} = t_\text{S} + \frac{h_\text{S}}{c\cos\phi}. \tag{8.4}$$

The speed at which the shadow disappears in Σ is given by the speed v_P of the P in Σ. In time dt_S (of Σ) the source moves the distance udt_S and the point P the distance $dx_P = v_P dt_S$. From (8.2) we obtain

$$dx_P = -\frac{h_w}{\cos^2\phi} d\phi.$$

But from Fig. 8.2 we have

$$(SR) = (h_\text{s} - h_\text{w})\tan\phi \Longrightarrow d\phi = \frac{u\cos^2\phi}{h_\text{s} - h_\text{w}} dt_S,$$

where we have replaced $d(SR) = -udt_S$. The last two relations imply

$$v_P = \frac{dx_P}{dt_\text{S}} = -\frac{h_w}{h_\text{s} - h_\text{w}} u = -\frac{u}{k-1} \qquad (k > 1). \tag{8.5}$$

We note that the velocity v_P has always direction toward the wall and has a constant speed depending (as expected) on the speed u of the source. Furthermore, we note that the value of the speed v_P can be as great as we wish depending on the relation of the heights h_s, h_w. This result does not conflict the upper limit set by the speed of light in vacuum, because the point P is not a material point but an image (has no energy).

We consider now a new position S' of the light source after the point S and let P' be the point where the light ray emitted by the source at the point S' and greasing the wall at the point T hits the x-axis. The time moment $t_{P'}$ at which the shadow at the point P' is destroyed is

$$\begin{aligned} t_{P'} &= t_S + \frac{(SS')}{u} + \frac{(S'P')}{c} \\ &= t_S + \frac{(SR) - (S'R)}{u} + \frac{h_S}{c\cos\phi'} \\ &= t_S + \frac{h_s - h_w}{u}(\tan\phi - \tan\phi') + \frac{h_S}{c\cos\phi'}. \end{aligned}$$

We look now for a position S' such that the light rays from the points $S,\ S'$ reach the x-axis at the same moment destroying the shadow at the respective points $P,\ P'$.

To find this position we equate the times $t_P,\ t_{P'}$ and find

$$t_S + \frac{h_S}{c\cos\phi} = t_S + \frac{h_S - h_w}{u}(\tan\phi - \tan\phi') + \frac{h_S}{c\cos\phi'} \Rightarrow$$
$$\sin\frac{\phi+\phi'}{2} = \frac{k-1}{\beta}\cos\frac{\phi-\phi'}{2}. \tag{8.6}$$

Obviously there are values of β', ϕ' for which this condition is satisfied, therefore the point P', which is closer to the wall, is lighted before the point P! In these cases the shadow in the rear of the wall is not destroyed from the motion of the point P but from the lightening of the point P'.

The condition for the complete disappearance of the shadow is $\phi' = 0$. When this is the case the angle ϕ_0 is given by the relation

$$\tan\frac{\phi_0}{2} = \frac{k-1}{\beta}.$$

In order for this value to be acceptable it must satisfy the condition $\phi_0 < \frac{\pi}{2}$, that is $\tan\frac{\phi_0}{2} < 1$. This condition gives for β:

$$\beta > k - 1,$$

which is possible because the rhs is < 1. We conclude that for speeds of the source $\beta > k - 1$ the shadow disappears, although the source has not reached the wall yet. This behavior contradicts the Newtonian one. For smaller speeds the Newtonian and the relativistic behaviors do not differ qualitatively.

We have still to calculate the position S of the source for which the shadow disappears completely. From the previous calculations we have

$$(SR) = (h_S - h_w)\tan\phi_0 = (h_S - h_w)\frac{2\tan\frac{\phi_0}{2}}{1-\tan^2\frac{\phi_0}{2}} = -\frac{2h_S\beta(k-1)^2}{\beta^2-(k-1)^2}.$$

We note that as $\beta \to 0$ the $(SR) \to 0$, that is, we have the expected Newtonian result.

The critical distance $x_{P,0}$ at which the shadow disappears along the x-axis is given by

$$x_{P,0} = h_w\tan\phi_0 = \frac{2h_w\beta(k-1)}{\beta^2-(k-1)^2}.$$

Before that point we have two light rays destroying the shadow. Which of these will destroy first and how will this be done? To answer this question we calculate the velocity $v_{P'} = \frac{dx_{P'}}{dt_S}$ with which the image point P' moves along the x-axis. We have

$$\begin{aligned}v_{P'} &= \frac{dx_{P'}}{dt_S} = \frac{d}{dt_S}(h_\mathrm{w} \tan\phi') = -\frac{h_\mathrm{w}}{\cos^2\phi}\frac{d\phi}{dt_S}\frac{\cos^2\phi}{\cos^2\phi'}\frac{d\phi'}{d\phi} \\ &= v_P \frac{\cos^2\phi}{\cos^2\phi'}\frac{d\phi'}{d\phi}.\end{aligned}$$

Because $\frac{\pi}{2} > \phi > \phi'$, $\frac{\cos^2\phi}{\cos^2\phi'} < 1$. Also from (8.6) one can show that $-1 < \frac{d\phi'}{d\phi} < 0$.

We note that the velocity $v_{P'}$ has direction opposite to that of v_P and smaller magnitude. This is expected as the point P' is closer to the wall hence it must move opposite to the point P in order to destroy the shadow in the interval $x_P - x_{P'}$.

Finally, we note that the length of the shadow along the x-axis equals

$$x_P - x_{P'} = h_\mathrm{w}(\tan\phi - \tan\phi') = h_\mathrm{w}\frac{\sin(\phi - \phi')}{\cos\phi\cos\phi'} > 0.$$

This is always positive, therefore the shadow is constantly diminishing.

Example 32 In the LCF Σ a right angle lever BAC with equal lengths $AB = BC = L$ is resting under the influence of a pair of forces $\boldsymbol{f}$ as shown in Fig. 8.3. At some moment, the lever starts sliding along the x-axis with constant speed u while remaining under the action of the same couple of forces in its rest frame. Obviously, the proper observer, Σ' say, of the lever will observe no change (because nothing changes concerning the couple of forces). However, the observer Σ will "see" the angle to rotate, because due to length contraction of the side AB a net moment will apply to the lever. Explain the above paradox.

It is given that the transformation of the four-force under a boost along the x-axis is

$$\begin{aligned}f'_x &= \frac{1}{(1 - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2})}(f_x - \frac{u}{c^2}\mathbf{f}\cdot\mathbf{v}), \\ f'_y &= \frac{1}{\gamma(u)(1 - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2})}f_y, \\ f'_z &= \frac{1}{\gamma(u)(1 - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2})}f_z.\end{aligned}$$

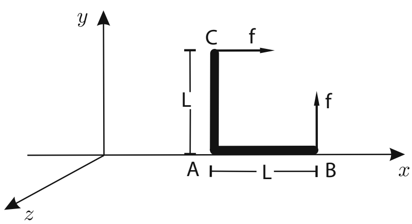


Fig. 8.3 The paradox of the L-shaped object

Solution

In the proper frame Σ' of the rod the velocity vanishes, therefore the inverse boost from Σ to Σ' of the components of the four-force gives

$$\begin{aligned} f_x &= f'_x, \\ f_y &= f'_y/\gamma(u), \\ f_z &= f'_z/\gamma(u). \end{aligned}$$

The forces which act on the lever angle in the LCF Σ' are $\mathbf{f}'_B = f\mathbf{j}$ and $\mathbf{f}'_C = f\mathbf{i}$. These forces in Σ transform to

$$\begin{aligned} \mathbf{f}_B &= (0, f/\gamma(u), 0) = \frac{f}{\gamma(u)}\mathbf{j}, \\ \mathbf{f}_C &= (f, 0, 0) = f\mathbf{i}. \end{aligned}$$

We take moments in Σ wrt the point A of the lever angle:

$$f_B(AB)_\Sigma - f_C(AC)_\Sigma = \frac{1}{\gamma(u)} f \frac{1}{\gamma(u)} L - fL = -\beta^2 fL < 0.$$

We note that the total moment about the point A in Σ does not vanish, hence the lever angle will rotate about A for the one observer (the Σ) and will not for the proper observer Σ'. The source of this paradoxical behavior is due to the assumption that the angle is a solid body. In Relativity (both Special and General) there are no solid bodies, because their existence relies in the Euclidean metric of the three-dimensional space.[6]

Example 33 An equilateral triangle of side a slides with speed u along the x-axis of the LCF Σ. Compute the perimeter of the triangle when it slides

(a) Along one of its heights
(b) Along one of its sides

Comment on the results in the limit $\beta \to 0$ (Newtonian limit) and $\beta \to 1$ (relativistic limit).

Solution

(a) We consider the events A, B, C to be the position of the tips of the triangle at each moment. We know the coordinates of these events in the proper frame Σ'

[6] To find more on this paradox which has a long history in Special Relativity the reader can check the following articles:

1. J. C. Nockerson and R. McAdory "Right-angle lever paradox" *Am. J. Phys.* **43**, 615, (1975).
2. D. Jensen "The paradox of the L-shaped object" *Am. J. Phys.* **57**, 553, (1989).

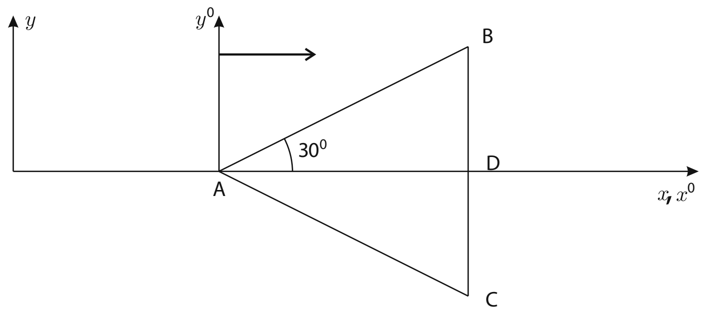


Fig. 8.4 Motion of an equilateral triangle

say of the triangle and we compute them in the frame Σ using the appropriate boost. With the help of Fig. 8.4 we obtain the following table of coordinates:

	Σ	Σ'
$A:$	$(ct_A, x_A, 0, 0)_\Sigma$	$(ct'_A, 0, 0, 0)_{\Sigma'}$
$B:$	$(ct_B, x_B, y_B, 0)_\Sigma$	$(ct'_A, \frac{a\sqrt{3}}{2}, \frac{a}{2}, 0)_{\Sigma'}$
$C:$	$(ct_C, x_C, y_C, 0)_\Sigma$	$(ct'_A, \frac{a\sqrt{3}}{2}, -\frac{a}{2}, 0)_{\Sigma'}$
BA^i	$(c\Delta t_{BA}, x_B - x_A, y_B, 0)_\Sigma$	$(0, \frac{a\sqrt{3}}{2}, \frac{a}{2}, 0)_{\Sigma'}$
CA^i	$(c\Delta t_{CA}, x_C - x_A, y_C, 0)_\Sigma$	$(0, \frac{a\sqrt{3}}{2}, -\frac{a}{2}, 0)_{\Sigma'}$
CB^i	$(c\Delta t_{CB}, x_C - x_B, y_C - y_C, 0)_\Sigma$	$(0, 0, -a, 0)_{\Sigma'}$

$x_B - x_A = x_C - x_A = (AD)_\Sigma = a' \cos\phi$ where a' is the side $(AB)_\Sigma = (AC)_\Sigma$ in Σ and ϕ is the angle $\widehat{BAD}$ in Σ. From the triangle ADB we obtain $\sin\phi = \frac{(BD)_\Sigma}{(AB)_\Sigma} = \frac{a}{2a'}$, hence

$$x_B - x_A = a'\sqrt{1 - \frac{a^2}{4a'^2}} = \sqrt{a'^2 - \frac{a^2}{4}}.$$

The boost gives $x_B - x_A = \frac{1}{\gamma}\frac{a\sqrt{3}}{2}$. Replacing $x_B - x_A$ we find

$$a' = \frac{\sqrt{1 + \frac{3}{\gamma^2}}}{2}a.$$

Obviously in the LCF Σ the triangle ABC is isosceles with perimeter

$$T = 2a' + a = \left[1 + \sqrt{4 - 3\beta^2}\right]a.$$

Second Solution

Since the side (BC) is normal to the relative velocity it will stay normal and with the same length, that is $(BC)_\Sigma = a$. The points B, C are symmetric about the x-axis which coincides with the direction of the relative velocity, hence they will stay symmetric in Σ. This implies that the point D is the middle point of $(BC)_\Sigma$ and in Σ. The side $(AD)_\Sigma$ due to length contraction in Σ has length $\frac{a\sqrt{3}}{2\gamma}$. Therefore

$$(AB)_\Sigma = \sqrt{(BD)^2_{\Sigma'} + (AD)^2_{\Sigma'}} = \sqrt{\frac{a^2}{4} + \frac{3a^2}{4\gamma^2}} = \frac{a}{2}\sqrt{4 - 3\beta^2}$$

from which the perimeter follows easily.

(b) This case is described in Fig. 8.5. We give only the "practical" solution and leave the coordinate solution for the reader. We consider the height AD and for obvious reasons we write

$$(BD)_{\Sigma'} = (BD)_\Sigma = \frac{a\sqrt{3}}{2},$$
$$(AD)_\Sigma = \frac{1}{\gamma}(AD)_{\Sigma'} = \frac{a}{2\gamma}.$$

Therefore

$$(BC)_\Sigma = (AB)_\Sigma = \sqrt{(AD)^2_\Sigma + (BD)^2_\Sigma} = \frac{a}{2}\sqrt{4 - \beta^2}.$$

The perimeter of the triangle ABC for Σ is

$$T = 2(AB)_\Sigma + (AC)_\Sigma = a\left[\sqrt{1-\beta^2} + \sqrt{4-\beta^2}\right].$$

Concerning the Newtonian and the relativistic limits of the perimeter we see that in both cases the Newtonian limit ($\beta \to 0$) of the perimeter of the triangle equals $3a$, that is, the triangle behaves as a solid body and its angles and the lengths of its sides

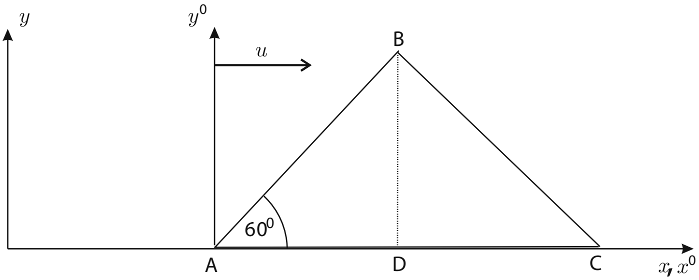


Fig. 8.5 Motion of an isosceles triangle

do not change. In the relativistic limit ($\beta \to 1$) in the first case, the perimeter of the triangle becomes $2a$, the angle $\widehat{BAC} \to 180^\circ$ and the triangle degenerates to a straight line. In the second case the perimeter equals $a\sqrt{3}$ and the angle $\widehat{ABC} \to 0^\circ$, that is the triangle degenerates again to a straight line of length twice the height BD.

We see that the perimeter of a triangle in Σ depends on the way the triangle moves in Σ. This shows clearly that there are no rigid bodies (in the Newtonian sense!) in Relativity.

Chapter 9
Mass – Four-Momentum

9.1 Introduction

In the previous sections we considered the kinematics of Special Relativity, which concerns the study of the four vectors of position, velocity, and acceleration. The major result of this study was the geometric description/definition of the Relativistic Mass Particle (ReMaP) as a set of four-vectors, which at each point along the world line of the particle have common proper frame or characteristic frame if they are timelike or spacelike, respectively.

With the three four-vectors of four-position, four-velocity, and four-acceleration one is able to study the geometric properties of the world line of the ReMaP but not the individual characteristics of the ReMaP and its interaction with the environment. For example, it is not possible to say if a given world line concerns an electron moving in an electromagnetic field or a ReMaP which is mechanically accelerated. Furthermore it is not possible to predict the world line of ReMaP moving under the action of a given dynamical field. In conclusion kinematics studies the world line only as a geometric result independently of the cause and the internal structure of the system.

The situation is the same with Newtonian Physics in which the three corresponding quantities of Newtonian kinematics do not suffice for the study of Newtonian motion. As we do in Newtonian Physics we introduce in Special Relativity new relativistic physical quantities, the Dynamical Relativistic Physical Quantities, which characterize the world lines with more information. The set of all these quantities together with the "laws" which govern them comprises the field of *Relativistic Dynamics*.

We recall that the relativistic physical quantities are Lorentz tensors which in the proper frame of the ReMaP must satisfy the following two conditions:

- They must attain their reduced or canonical form
- Either their components must be directly related to corresponding Newtonian physical quantities or their physical meaning must be defined by a relativistic principle.

M. Tsamparlis, *Special Relativity*, DOI 10.1007/978-3-642-03837-2_9,

These two conditions must be satisfied by all dynamic physical quantities introduced in the following.

The simplest tensors are the invariants. Furthermore from the invariants we are able to construct new tensors using the rules of Proposition 2, that is, either by multiplying or by differentiating wrt an invariant.

The first invariant we introduce is the (relativistic) mass and, using this, the four-vectors of four-momentum, four-force, etc.

We emphasize that the photons, and in general the particles with speed c are not ReMaP, therefore they do not have a proper frame. This implies that the definition of the dynamical quantities for these particles, hence their dynamics, is different than the dynamics of ReMaP and must be treated accordingly.

9.2 The (Relativistic) Mass

Before we define the (Lorentz) invariant dynamical physical quantity (relativistic) mass we refer some general comments which concern all invariant dynamical physical quantities.

- Any (Lorentz) invariant is a potential relativistic physical quantity and it is characterized by the fact that it has the *same (arithmetic) value* in all LCF. Therefore it suffices to know the value of an invariant in one LCF.
- In order a potentially invariant physical quantity to become a relativistic physical quantity its value in the proper frame of the ReMaP must coincide with a corresponding Newtonian physical quantity. If there is no such a quantity, then its relativistic physical role will be defined by means of a principle (for example, the invariant c).
- Every invariant relativistic physical quantity of a ReMaP, say the A^{0+}, defines in the proper frame Σ^+ of the ReMaP a new potential relativistic physical quantity by means of the timelike four-vector $(A^{0+}, \mathbf{0})_{\Sigma^+}$. One such four-vector we have already defined in kinematics by the invariant c, and it is the four-velocity whose components in Σ^+ are $(c, \mathbf{0})_{\Sigma^+}$.
- In addition to the four-vector $(A^{0+}, \mathbf{0})_{\Sigma^+}$ the invariant defines more dynamical relativistic physical quantities by means of the rules 1, 2 of Proposition 2.

The first invariant quantity we introduce is the (relativistic) mass m of the ReMaP. In order to make the mass a relativistic physical quantity we stipulate that in the proper frame of the ReMaP its value will be identical with the value of the Newtonian mass of the ReMaP in that frame.[1]

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[1] Usually in the literature the relativistic mass of a ReMaP is referred as rest mass and it is written as m_0. This is due to the fact that its value is defined in the proper frame of the ReMaP where the ReMaP is at rest. However, it is important to become clear that the relativistic mass is an invariant hence its value is the same in all LCF and does not change "with the velocity" as it is erroneously claimed. Therefore the term "rest mass" is misleading and should be abandoned.

The mass of a ReMaP need not be a constant. For example, in the case of a relativistic rocket the mass of the rocket changes along the world line, i.e., $m = m(\tau)$ where τ is the proper time of the rocket. In this case every value $m(\tau)$ is a (Lorentz) invariant and the mass of the rocket is a continuous sequence of relativistic invariants parameterized by the proper time of the rocket. In this sense we consider the rate of change $dm/d\tau$ of the proper mass of the rocket and whatever consequences this has.

As we have already remarked the luxons (photons) do not have proper frame therefore for those we cannot define the relativistic physical quantity mass as we did for ReMaP. However, we can define a "mass" which will be common for all luxons as a limiting case of the (relativistic) mass of the ReMaP. Indeed we note that the mass of a ReMaP is not possible to equal zero because no Newtonian particle has mass zero. However, the mass can approach zero as closely as one wishes, hence zero is the minimum (lowest limit) of the mass of the ReMaP. Furthermore the Lorentz transformation being homogeneous preserves zero. Therefore we *define* the relativistic mass of luxons to be zero. As we shall see this choice is compatible with the dynamic physical quantities of photons, to be considered further on.

9.3 The Four-Momentum of a ReMaP

Consider a ReMaP P of mass m and four-velocity u^i. By means of Rule 1 of Proposition 2 we define the potential relativistic physical quantity

$$p^i = m\,u^i. \tag{9.1}$$

p^i is a timelike four-vector with length

$$p^2 = m^2\,u^i u_i = -m^2\,c^2 < 0. \tag{9.2}$$

In the proper frame Σ^+ of P the reduced form of the four-velocity is $(c, 0)_{\Sigma^+}$ hence the four-momentum $p_i = (mc, 0)_{\Sigma+}$. The zeroth component mc of p^i in the proper frame has physical meaning (by definition, not by Newtonian analogue!) because both m and c are relativistic physical quantities. Therefore the four-vector p^i is a relativistic physical quantity. In Special Relativity the four-vector of (linear) momentum is as important as the linear momentum vector in Newtonian Physics. As it will be shown in the following, its components encounter both for the energy and the three-momentum of the ReMaP in a LCF.

To find the components of p^i we recall that in Σ the four-velocity u^i has components $u^i = \gamma \begin{pmatrix} c \\ \mathbf{u} \end{pmatrix}_\Sigma$ hence the four-momentum

$$p^i = \gamma \begin{pmatrix} mc \\ m\mathbf{u} \end{pmatrix}_\Sigma. \tag{9.3}$$

In order to reveal the physical meaning of the components of the four-momentum in the LCF Σ, we consider the Taylor expansion of γ around the value 1 and compare the result with known Newtonian quantities. For the spatial components we have

$$m\gamma \boldsymbol{u} = m\boldsymbol{u} + O(u^2/c^2)\boldsymbol{u}. \tag{9.4}$$

$m\boldsymbol{u}$ is the "Newtonian" linear momentum of the ReMaP P in Σ. Therefore it is logical to assume that the quantity

$$\boldsymbol{p} = m\,\gamma \boldsymbol{u} \tag{9.5}$$

is the (linear) three-momentum of P in Σ.

Working in a similar fashion we find for the zeroth component of the four-momentum

$$m\gamma c^2 = mc^2 + \frac{1}{2}mu^2 + O(u^4/c^2)\,. \tag{9.6}$$

The quantity $\frac{1}{2}mu^2$ is the Newtonian kinetic energy of P in Σ, therefore the terms $m\gamma c^2$ and mc^2 must be related to energy quantities. The term mc^2 involves only the ReMaP P and it is independent of its motion. This term we identify with the *internal energy* of the ReMaP P and the term $m\gamma c^2$ with the total energy E of P in Σ

$$E = m\gamma c^2. \tag{9.7}$$

With these identifications the four-momentum of P in Σ has the following representation:

$$p^i = \begin{pmatrix} E/c \\ \mathbf{p} \end{pmatrix}_\Sigma . \tag{9.8}$$

Using (9.7) and (9.8) and replacing in (9.2) we obtain the following fundamental formula which relates E, $\boldsymbol{p}$, m:

$$E = \sqrt{\boldsymbol{p}^2c^2 + m^2c^4}. \tag{9.9}$$

This relation is possible to be displayed on the Euclidean plane by means of an orthogonal triangle (see Fig. 9.1). T is the kinetic energy of P in Σ.

Exercise 27 Prove that the angle ϕ in Fig. 9.1 is given by the relation

$$\sin\phi = \frac{pc}{E} = \beta.$$

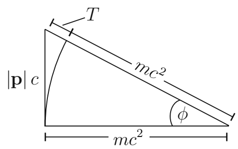

Fig. 9.1 Geometric representation of the relation between energy and three-momentum

Determine the relation of ϕ with the rapidity of the ReMaP. Using this result represent geometrically the boost in terms of the four-momentum. [Hint: Use the invariants of the the Lorentz transformation.]

The triangle of Fig. 9.1 allows one to use ϕ and distinguish between the non-relativistic ($\beta \to 0$) and the ultra relativistic ($\beta \to 1$) motions (see Fig. 9.2).

Indeed we note that for low velocities (in Σ!) the major part of the (total) energy E is due to the mass mc^2 while for high velocities the energy is mainly due to the momentum pc, that is, the kinetic energy. This observation is important in practice, because it indicates which terms could be ignored without affecting significantly the final result.

The energy is the zeroth component of the four-momentum, therefore its value depends on the LCF Σ where the motion is considered. The following mistake is often made. Because the quantity $\frac{E}{c^2} = m\gamma$ has dimensions of mass, some people set $M = m\gamma$ and assume that M is the "mass" of the ReMaP. Then they conclude that "the mass depends on the velocity." This is absurd because M is not a relativistic physical quantity (it is not an invariant) hence it has no physical meaning in Special Relativity. Simply it is another name for the energy E of the ReMaP and varies from frame to frame according to the Lorentz transformation of the zeroth component of the four-momentum.

Example 34 (a) Show that if the momentum of a ReMaP is larger than the mass by two orders of magnitude, then the energy equals the measure of the three-momentum to the order of 10^{-4}. Conclude that in these cases practically $E = pc$. This is the ultra relativistic case.

(b) Repeat the calculation assuming that the mass of the ReMaP is two orders of magnitude larger of the (length) of three-momentum and show that this case corresponds to the non-relativistic case, therefore we may take $E = mc^2$.

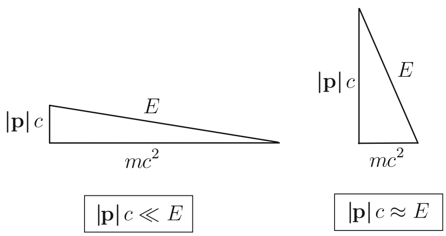

Fig. 9.2 Relativistic and Newtonian limit of the energy–three-momentum relation

Solution
(a) From (9.9) we find

$$E^2 = p^2c^2 + m^2c^4 \approx p^2c^2\left(1 + \frac{m^2c^4}{p^2c^2}\right).$$

Assuming that the momentum is two orders of magnitude larger than the mass term mc we have

$$E = pc\sqrt{1 + 10^{-4}} \approx pc\left(1 + \frac{1}{2}10^{-4}\right).$$

Therefore the energy equals the momentum to the order 10^{-4}. It follows that $\sin\phi = O(10^{-4})$ (ultra relativistic limit).

(b) Working similarly we find

$$E = mc^2\left[1 + \frac{pc}{m^2c^4} + O\left(\frac{pc}{m^2c^4}\right)^2\right] \approx mc^2(1 + 10^{-4}).$$

From (9.9) we conclude that the energy of a ReMaP P in a LCF Σ varies with the three-velocity of P in Σ. This is also the case with Newtonian Physics. The difference with Special Relativity is that for relativistic speeds ($\beta > 0.8$) the change of energy with the velocity is much higher and tends to infinity as $\beta \to 1$ (see Fig. 9.3). This implies that infinite kinetic energy is required for a ReMaP to be accelerated to the velocity c, which is consistent with the relativistic requirement that c is the highest possible speed and furthermore either a particle is and stays for ever a ReMaP or it is and stays for ever a photon!

From (9.5) and (9.7) we have that in a LCF Σ

$$\boldsymbol{\beta} = \frac{\boldsymbol{p}c}{E}. \tag{9.10}$$

In spite of its simplicity relation (9.10) is very useful because it computes directly the β-factor if the three-momentum and the energy of the particle are known. Another useful relation we find if we differentiate (9.9) is

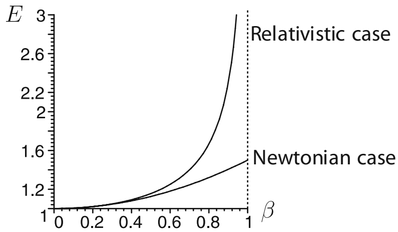

Fig. 9.3 Change of energy with the speed

$$E dE = \boldsymbol{p} d\boldsymbol{p} c = |\boldsymbol{p}| d|\boldsymbol{p}| c \text{ (why?)}$$

from which follows

$$\beta = \frac{dE}{d|\boldsymbol{p}|c}. \tag{9.11}$$

Exercise 28 Let Σ, Σ' be two LCF which are related with a boost in the standard configuration with velocity factor β. Let P be a ReMaP with mass m which in Σ, Σ' has linear momentum $\boldsymbol{p}$, $\boldsymbol{p}'$ and energies E, E', respectively. Prove the relations

$$E' = \gamma(E - \beta p_x c), \tag{9.12}$$

$$p'_x = \gamma(p_x - \beta E/c), \tag{9.13}$$

$$p'_y = p_y, \quad p'_z = p_z\,. \tag{9.14}$$

Example 35 P, Σ, Σ' as in Exercise 28. Prove the relations

$$\left(p'_x - \sqrt{\frac{1+\beta}{1-\beta}}p_x\right)\left(p'_x - \sqrt{\frac{1-\beta}{1+\beta}}p_x\right) = \beta^2\gamma^2(p_y^2 + p_z^2 + m^2c^2),$$

$$\left(E' - \sqrt{\frac{1+\beta}{1-\beta}}E\right)\left(E' - \sqrt{\frac{1-\beta}{1+\beta}}E\right) = \beta^2\gamma^2c^2(p_y^2 + p_z^2 + m^2c^2).$$

Plot these relations when $p_y^2 + p_z^2 =$ constant. What happens when P is a photon?
Solution

The four-momentum of P in Σ and Σ' is $p_i = (E/c, \boldsymbol{p})_\Sigma$ and $p_i = (E'/c, \boldsymbol{p}')_{\Sigma'}$, respectively. The two expressions of the four-momentum are related by a boost along the common x, x' axis. Relation (9.13) gives

$$(p'_x - \gamma p_x)^2 = \gamma^2\beta^2\frac{E^2}{c^2} = \gamma^2\beta^2(p_x^2 + a^2),$$

where $a^2 = p_y^2 + p_z^2 + m^2c^2$. This relation can be written as

$$(p'_x - \gamma p_x)^2 - (\gamma\beta p_x)^2 = \gamma^2\beta^2a^2$$

or

$$[p'_x - (\gamma + \gamma\beta)p_x][p'_x - (\gamma - \gamma\beta)p_x] = \gamma^2\beta^2a^2.$$

But

$$\gamma \pm \beta\gamma = \gamma(1 \pm \beta) = \sqrt{\frac{1 \pm \beta}{1 \mp \beta}}.$$

Replacing we find

$$\left(p'_x - \sqrt{\frac{1+\beta}{1-\beta}}p_x\right)\left(p'_x - \sqrt{\frac{1-\beta}{1+\beta}}p_x\right) = \beta^2\gamma^2 a^2. \tag{9.15}$$

Working similarly for the energy we show

$$\left(E' - \sqrt{\frac{1+\beta}{1-\beta}}E\right)\left(E' - \sqrt{\frac{1-\beta}{1+\beta}}E\right) = \beta^2\gamma^2 a^2. \tag{9.16}$$

If $a^2 = $ constant these equations describe hyperbolae with asymptotes the straight lines

$$p'_x = \sqrt{\frac{1+\beta}{1-\beta}}p'_x \qquad p'_x = \sqrt{\frac{1-\beta}{1+\beta}}p_x$$

and, respectively,

$$E' = \sqrt{\frac{1+\beta}{1-\beta}} \qquad E' = \sqrt{\frac{1-\beta}{1+\beta}}$$

The plotting is left to the reader. In case P is a photon relations (9.15) and (9.16) become identical and furthermore the mass $m = 0$.

Exercise 29 The LCF Σ and Σ' are moving with parallel axes and relative velocity $\mathbf{u}$. A ReMaP P has four-momentum $(E/c, \mathbf{p})_\Sigma$ and $(E'/c, \mathbf{p}')_{\Sigma'}$ in Σ, Σ', respectively.

(a) Show that the components of the four-momentum in Σ and Σ' are related as follows:

$$E' = \gamma_u \left(E - \mathbf{u}\cdot\mathbf{p}\right), \tag{9.17}$$

$$\mathbf{p}' = \mathbf{p} + \mathbf{u}\left[\frac{\mathbf{u}\cdot\mathbf{p}}{u^2}(\gamma_u - 1) - \gamma_u \frac{E}{c}\right]. \tag{9.18}$$

Show that in the case of a boost these relations reduce to (9.12) – (9.14).

(b) Show that the angles θ, θ' of the three-momenta $\mathbf{p}$, $\mathbf{p}'$ with the relative velocity $\mathbf{u}$ of Σ and Σ' are related as follows:

$$\cot\theta' = \left(\cot\theta - \frac{\beta}{B'}\frac{1}{\sin\theta}\right), \tag{9.19}$$

where B' is the speed factor of P in Σ'. Relation (9.19) is called the *particle aberration formula*. It is used in the focussing of beams of particles.

Exercise 30 Let $\boldsymbol{p}$, E be the three-momentum and the energy of a ReMaP of mass m in the LCF Σ. Prove the relations

$$d\boldsymbol{p} = \gamma dm\boldsymbol{u} + md(\gamma \boldsymbol{u}), \tag{9.20}$$
$$dE = mc^2 d\gamma + \gamma c^2 dm. \tag{9.21}$$

Deduce that the change of three-momentum or the energy of a ReMaP in Σ is due to either a change of γ (the speed) or a change of mass or both.

Exercise 31 Assumptions as in Exercise 30. Prove the relations

$$E = m\gamma c^2 \qquad p^2 = \beta^2\gamma^2 m^2 c^2.$$

Deduce that in order to compute the energy and the measure of the three-momentum of a ReMaP in Σ (not a photon!) it is enough to know the γ-factor of the ReMaP in Σ.

Note: For the photons we have $pc = E = h\nu$ where h is the Plank's constant and ν is the frequency of the photon.

Example 36 A ReMaP of mass m and four-velocity u^i has four-momentum $p^i = mu^i$. Prove that the energy and the length of the three-momentum of the ReMaP in a LCF Σ in which the ReMaP has velocity factor γ is given by the relations

$$E = -\gamma(p^i u_i), \tag{9.22}$$
$$\boldsymbol{p}^2 = (p^i p_i) + (\gamma^2/c^2)(p^i u_i)^2. \tag{9.23}$$

Solution

We note the relations $p^i u_i = -mc^2$ and $p^i p_i = -m^2c^2$. Then

$$E = m\gamma c^2 = -\gamma(p^i u_i),$$
$$\mathbf{p}^2 = \frac{E^2}{c^2} - m^2c^2 = \left(\frac{\gamma^2}{c^2}\right)(p^i u_i)^2 + (p^i p_i).$$

Example 37 A beam of electrons of average energy E is produced in a linear accelerator. The focussing of the beam is assumed to be perfect and also that all electrons have the same energy with deviation $\alpha\%$. This means that the speeds of the electrons are parallel and their energy is between the values $E_+ = (1+\alpha)E$ and $E_- = (1-\alpha)E$. Calculate the maximum relative speed of the electrons in the beam if $\alpha = 0.1,\ E = 1\,\text{GeV},\ m = 0.5\,\text{MeV/c}^2$. Does the result depend on the value of the average energy E?

Solution

Let $u^i_+,\ u^i_-$ be the four-velocities of the electrons in the beam with energies $E_+ = (1+\alpha)E$ and $E_- = (1-\alpha)E$, respectively. The relative velocity of the electrons is the four-vector $u^i_\pm = u^i_+ - u^i_-$ and has length

$$u^i_{\pm}u_{\pm i} = (u^i_+ - u^i_-)(u_{+i} - u_{-i}) = -2c^2 - 2u^i_+ u_{-i}. \tag{9.24}$$

We calculate the inner product $u^i_+ u_{-i}$ in the proper frame of u^i_-. In that frame $u^i_- = \begin{pmatrix} c \\ \mathbf{0} \end{pmatrix}_-$, $u^i_+ = \begin{pmatrix} \gamma_{\pm} c \\ \gamma_{\pm}\mathbf{v}_{\pm} \end{pmatrix}_-$ where $\gamma_{\pm}$ is the γ-factor of the relative velocity $\mathbf{v}_{\pm}$. The product $u^i_+ u_{-i} = -\gamma_{\pm}c^2$. Replacing in (9.24) we find

$$u^i_{\pm}u_{\pm i} = 2(\gamma_{\pm} - 1)c^2. \tag{9.25}$$

We note that the length of the four-vector $u^i_{\pm}$ is positive hence $u^i_{\pm}$ is not a four-velocity!

We calculate $\gamma_{\pm}$ in the laboratory where the four-velocities u^i_+, u^i_- are given. From the relations $u^i_+ = \frac{p^i_+}{m}$, $u^i_- = \frac{p^i_-}{m}$ and the relation $u^i_+ u_{-i} = -\gamma_{\pm}c^2$ we find

$$\gamma_{\pm} = -\frac{1}{m^2c^2} p^i_+ p_{-i}.$$

We also calculate the inner product in the laboratory. We have the components

$$p^i_+ = \begin{pmatrix} E_+/c \\ p\hat{\mathbf{e}}_+ \end{pmatrix}_L, \; p^i_- = \begin{pmatrix} E_-/c \\ p\hat{\mathbf{e}}_- \end{pmatrix}_L$$

from which follows

$$\gamma_{\pm} = -\frac{1}{m^2c^4}(-E_+E_- + p_+p_-c^2). \tag{9.26}$$

The length

$$\begin{aligned} p_+c &= \sqrt{E_+^2 - m^2c^4} = E_+\sqrt{1 - \frac{m^2c^4}{E_+^2}} \\ &\approx E_+\left[1 - \frac{1}{2}\left(\frac{mc^2}{E_+}\right)^2 + O\left(\left(\frac{mc^2}{E_+}\right)^4\right)\right] \end{aligned}$$

and similarly

$$p_-c = \sqrt{E_-^2 - m^2c^4} \approx E_-\left[1 - \frac{1}{2}\left(\frac{mc^2}{E_-}\right)^2 + O\left(\left(\frac{mc^2}{E_-}\right)^4\right)\right].$$

Therefore the product

$$p_+p_-c^2 \approx E_+E_-\left[1 - \frac{1}{2}\left(\frac{mc^2}{E_+}\right)^2 - \frac{1}{2}\left(\frac{mc^2}{E_-}\right)^2 + O\left(\left(\frac{mc^2}{E_-}\right)^4, \left(\frac{mc^2}{E_-}\right)^4\right)\right].$$

Replacing in (9.26) we find

$$\gamma_{\pm} \approx -\frac{1}{m^2c^4}\left[-E_+E_- + E_+E_-\left[1 - \frac{1}{2}\left(\frac{mc^2}{E_+}\right)^2 - \frac{1}{2}\left(\frac{mc^2}{E_-}\right)^2\right]\right]$$
$$= \frac{1}{2}\left(\frac{E_-}{E_+} + \frac{E_+}{E_-}\right).$$

The $\beta_{\pm}$-factor which corresponds to the $\gamma_{\pm}$-factor is given by the relation

$$\beta_{\pm} = \sqrt{1 - \frac{1}{\gamma_{\pm}^2}} = \frac{E_+^2 - E_-^2}{E_+^2 + E_-^2}.$$

In terms of the deviation α of the energy we have

$$\beta_{\pm} = \frac{(1+\alpha)^2 - (1-\alpha)^2}{(1+\alpha)^2 + (1-\alpha)^2} = \frac{4\alpha}{(1+\alpha)^2 + (1-\alpha)^2}.$$

We note that in our approximation (which is reasonable) $\beta_{\pm}$ is independent of the average energy E and depends only on the deviation α.

9.4 The Four-Momentum of Photons (Luxons)

Luxons are particles (photons and probably other particles) whose speed in all LCF equals c. These particles do not have a proper frame. They play a double fundamental role in Special Relativity. They are the natural "bullets" which are used by the relativistic observers in order to coordinate the events in spacetime (chronometry) and also they are the carriers of information among these observers. Beyond that, the photons are fundamental particles in the constituents of nuclear reactions. Therefore the study of their kinematics and especially their dynamics is a must. However, due to their characteristics they have "peculiarities" which must be taken into account. Let us start with the geometric considerations by assuming that the only luxons are the photons.

The world lines of photons are null straight lines in Minkowski space M^4. If x^i is the position vector of a photon, then x^i is a null four-vector and in any LCF Σ can be written in the form

$$x^i = A(t, \mathbf{r})\begin{pmatrix} 1 \\ \hat{\mathbf{e}} \end{pmatrix}_{\Sigma}, \tag{9.27}$$

where $A(t, \boldsymbol{r})$ is a scalar function (not invariant!) depending on the coordinates $t, \mathbf{r}$ of the photon in Σ and $\hat{\mathbf{e}}$ is the unit three-vector in the direction of propagation of the photon in Σ. For example, a photon which propagates along the three-direction

$(1, 1, 1)^t$ in Σ and the moment $t = 0$ of Σ passes through the origin has position vector

$$x^i = ct\left(1, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)^t_\Sigma. \tag{9.28}$$

The four-velocity of a photon cannot be defined by the relation $dx^i/d\tau$ because photons do not have proper time. The same holds for the four-acceleration, which also has the additional constraint that the speed in all LCF equals c. We also have a problem with the four-momentum because one cannot consider mass with the standard connotation for the photons, except as a minimum of the mass of the ReMaP.

In order to circumvent these difficulties and be consistent with what we have done already, we continue using the process of the limit. We start with the four-momentum.

Because we have assumed the “mass” of photons to equal 0 the four-momentum $p^i p_i = -m^2c^2 = 0$ must also be a zero four-vector. This means that in any LCF Σ p^i can be written in the form

$$p_i = A(t, \mathbf{r})(1, \mathbf{e})_\Sigma, \tag{9.29}$$

where $A(t, \boldsymbol{r})$ is a scalar (but not an invariant!) quantity and the three-vector $\mathbf{e}$ defines the direction of propagation of the photon in the three-space of Σ. From Newtonian Physics we know that a photon of frequency ν, which propagates along the direction $\hat{\mathbf{e}}$, has energy E and momentum $\mathbf{p}$ given by

$$E = h\nu, \tag{9.30}$$

$$\mathbf{p} = \frac{h\nu}{c}\hat{\mathbf{e}} = \frac{E}{c}\hat{\mathbf{e}}. \tag{9.31}$$

Based on the above analysis we *define* the four-momentum of a photon in a LCF Σ in which the photon has energy $E(= h\nu)$ and three-momentum $\mathbf{p}(= \frac{h\nu}{c}\hat{\mathbf{e}})$ with the relation

$$p^i = \begin{pmatrix} E/c \\ (E/c)\hat{\mathbf{e}} \end{pmatrix}_\Sigma. \tag{9.32}$$

This definition connects the two natures of the photon, that is, the wave and the particle nature.

Example 38 In the LCF Σ a photon has energy E and three-momentum $\mathbf{p}$. Calculate the energy E' and the direction of propagation $\mathbf{e}'$ of the photon in an LCF Σ', which moves relative to Σ with velocity $\mathbf{u}$. Express the angle θ' between $\mathbf{e}'$ and $\mathbf{u}$ in terms of the angle θ between $\mathbf{e}$ and $\mathbf{u}$. Comment on the result.

Solution

Suppose the four-momentum p^i of the photon in Σ and Σ' has representation $p_i = (E/c, \mathbf{p})_\Sigma$ and $p_i = (E'/c, \mathbf{p}')_{\Sigma'}$, respectively. The Lorentz transformation parallel and normal to the relative velocity $\mathbf{u}$ gives (recall that $\mathbf{p}_{\|} = \frac{\mathbf{p}\cdot\mathbf{u}}{u^2}$):

$$E'/c = \gamma(u)(E/c - \boldsymbol{\beta} \cdot \mathbf{p}_{\|}), \tag{9.33}$$

$$\frac{E'}{c}\hat{\mathbf{e}}'_{\perp} = \mathbf{p}_{\perp}, \tag{9.34}$$

$$\frac{E'}{c}\hat{\mathbf{e}}'_{\|} = \gamma(u)\left(\mathbf{p}_{\|} - \boldsymbol{\beta}\frac{E}{c}\right). \tag{9.35}$$

From (9.33) we have for the energy E' in Σ'

$$E' = \gamma(u)(E - \mathbf{p}\cdot\mathbf{u}). \tag{9.36}$$

The other two equations give

$$\begin{aligned}(E'/c)\hat{\mathbf{e}}' &= \mathbf{p}_{\perp} + \left[\gamma(u)\frac{\mathbf{p}\cdot\mathbf{u}}{u^2} - \gamma(u)\frac{E}{c^2}\right]\mathbf{u} \\ &= \mathbf{p}\left[(\gamma(u) - 1)\frac{\mathbf{p}\cdot\mathbf{u}}{u^2} - \gamma(u)\frac{E}{c^2}\right]\mathbf{u}\end{aligned}$$

from which follows

$$\hat{\mathbf{e}}' = \frac{c}{\gamma(u)(E - \mathbf{p}\cdot\mathbf{u})}\left\{\mathbf{p} + \left[(\gamma(u) - 1)\frac{\mathbf{p}\cdot\mathbf{u}}{u^2} - \gamma(u)\frac{E}{c^2}\right]\mathbf{u}\right\}. \tag{9.37}$$

Relation (9.37) is known as *light aberration formula*. [Prove (9.37) by direct application of the Lorentz transformation].

Concerning the angle θ we consider $\theta \neq k\pi/2,\ k = 0, 1, 2, 3$ and $|\mathbf{p}c| = E$ (because we have photons) hence $|\mathbf{p}_{\|}| = \frac{E}{c}\cos\theta$, $|\mathbf{p}_{\perp}| = \frac{E}{c}\sin\theta$. It follows

$$\begin{aligned}\cot\theta' = \frac{|\mathbf{e}'_{\|}|}{|\mathbf{e}'_{\perp}|} &= \gamma(u)\frac{(E/c)\cos\theta - (E/c)\beta}{(E/c)\sin\theta} \\ &= \gamma(u)\frac{\cos\theta - \beta}{\sin\theta} = \gamma(u)\left(\cot\theta - \frac{\beta}{\sin\theta}\right).\end{aligned} \tag{9.38}$$

[Compare this relation with (9.19). What are their similarities and differences?]

We consider next the special cases $\theta = k\pi/2,\ k = 0, 1, 2, 3$. When $\theta = 0$ from (9.34) we have $\mathbf{p}_{\perp} = 0 \Longrightarrow \hat{\mathbf{e}}'_{\perp} = 0$ hence $\theta' = 0$, that is, the directions of propagation of the photon on Σ and Σ' coincide. The same result we find if we work directly with (9.37). Indeed we have ($\mathbf{u} = u\hat{\mathbf{e}}$)

$$\hat{\mathbf{e}}' = \frac{c}{\gamma(u)(E - \beta E)}\left[\frac{E}{c}\hat{\mathbf{e}} + \left[(\gamma(u) - 1)\frac{\beta E}{u^2} - \gamma(u)\frac{E}{c^2}\right]\hat{\mathbf{e}}\right] = \hat{\mathbf{e}}.$$

When $\theta = \frac{\pi}{2}$ then $\mathbf{p}\cdot\mathbf{u} = 0$ and (9.37) gives

$$\hat{\mathbf{e}}' = \frac{1}{\gamma}(\hat{\mathbf{e}}_{\perp} - \beta\gamma\hat{\mathbf{e}}_{\parallel}).$$

The $\cot\theta' = -\beta\gamma < 0$ hence $\theta' \epsilon \ (\frac{\pi}{2}, \pi)$.

Working similarly we show that when $\theta = \pi$ the $\theta' = \pi$ and when $\theta = 3\pi/2$ the $\theta' \epsilon \ (-\pi/2, \pi/2)$.

9.5 The Four-Momentum of Particles

In the previous sections we defined the four-momentum of the ReMaP and the photons in a different way. The necessity for this approach was due to the different character of this four-vector for each class of particles, that is, timelike for the ReMaP and null for the photons. However, in nuclear reactions both types of particles are involved therefore it would be useful to have a unified approach. This leads us to the introduction of *particle four-vector* which is a four-vector whose (Lorentz) length is either negative or zero. This approach gives us the possibility to give a dynamic definition for the elementary particles.

Definition 13 Particle in the Theory of Special Relativity is any physical system whose four-momentum is a particle four-vector.

This definition allows us to use all the results on particle four-vectors in the study of relativistic collisions. However, it is to be noted that the results of that section apply to all particle four-vectors not only to the four-momentum.

9.6 The System of Natural Units

In Newtonian Physics the physical quantities spatial length (L), time (T) and mass (M) are absolute (i.e., Euclidean invariants) therefore it is logical to develop a system of units whose fundamental elements are the [L,T,M]. This system is the International System of Units (SI units) which is an evolution (in 1960) of the well-known MKSA system with the addition of the units of Kelvin, Candela, and (later on) the mole.

In the Theory of Special Relativity the quantities T, L are not relativistic physical quantities hence the SI, although it can still be used, is not anymore inherent in that theory.

In order to find a new system of units suitable for Special Relativity we consider the new fundamental invariant relativistic quantities. The first such quantity is the value c, therefore in the "relativistic" system the universal constant c will equal the *pure* number 1. The new system we call the *System of Natural Units (NU)*.

In this system the unit of spatial length 1 m is related to the unit of time 1 s with the relation

$$1\,\mathrm{s} = 3 \times 10^8\,\mathrm{m}. \tag{9.39}$$

The independent dimensions in the System of Natural Units are (among other) the [L, M].

In order to convert the value of a quantity from the SI system in the system of Natural Units we apply the following simple rule:

Multiply the value of the physical quantity in the SI units with the quantity $c = [c]m/s$ where $[c] = 3 \times\ 10^8$ in the same power as the time T appears in the units of the physical quantity in the SI system.

Let us see some applications of this rule.

Spatial length (L in SI)

The dimensions are the same in both systems of units, i.e., 1 m. Indeed, according to the rule

$$1\,\mathrm{m\ in\ SI} = [c]^0\,(\mathrm{m/s})^0\,\mathrm{m} = 1\,\mathrm{m\,in\ NU}.$$

Time (T in SI)

The unit of time in NU is the spatial length m. Indeed according to the rule

$$1\,\mathrm{s\ in\ SI} = [c]^1\,(\mathrm{m/s})^1\,\mathrm{s\ in\ NU} = [c]\,\mathrm{m}$$

which is in agreement with (9.39).

Velocity(L/T in SI)

$$1\,\mathrm{m/s\ in\ SI} = [c]^{-1}\,(\mathrm{m/s})^{-1}\,(\mathrm{m/s}) = [c]^{-1}\mathrm{in\ NU}.$$

We note that in NU the velocity is a pure number, as expected.

Acceleration (L/T^2 in SI)

$$1\,\mathrm{m/s^2\ in\ SI} = [c]^{-2}\,(\mathrm{m/s})^{-2}\,\mathrm{m/s^2} = [c]^{-2}\,\mathrm{m}^{-1}\mathrm{in\ NU}.$$

Force (ML/T^2 in SI)

$$1\mathrm{N\ in\ SI} = 1\,\mathrm{kg\,m/s^2} = [c]^{-2}(\mathrm{m/s})^{-2}\mathrm{kg\,m/s^2} = [c]^{-2}\mathrm{kg\,m}^{-1}\ \mathrm{in\ NU}.$$

Exercise 32 Show the validity of the following transformations between the values of the physical quantities in the SI and the NU system.

Energy (Nm)

$$1\mathrm{J\ in\ SI} = [c]^{-2}\,\mathrm{kg\ in\ NU}.$$

(Note that the dimensions of energy in NU is mass (kg) which is compatible with the relation $E = mc^2$ and allows us to measure energy in kg and conversely mass in kJ.)
Pressure

$$1\,\mathrm{N/m^2}\text{ in SI} = [c]^{-2}\,\mathrm{kg/m^3}\text{ in NU}$$

Energy density

$$1\,\mathrm{J/m^3}\text{ in SI} = [c]^{-2}\,\mathrm{kg/m^3}\text{ in NU}$$

Mass density

$$1\,\mathrm{kg/m^3}\text{ in SI} = 1\,\mathrm{kg/m^3}\text{ in NU}$$

Momentum

$$1\,\mathrm{kg\,m/s^1}\text{ in SI} = [c]^{-1}\,\mathrm{kg}\text{ in NU}$$

(Note that in the NU system the dimensions of energy are identical with those of linear momentum).

Plank's constant:

$$\begin{aligned} h = 6.63\times10^{-34}\,\mathrm{J\,s}\text{ in SI} &= 6.63\times10^{-34}[c]^{-2}\mathrm{kg}\times[c]\,\mathrm{m}\\ &= 2.21\times10^{-42}\,\mathrm{kg\,m}\text{ in NU.}\end{aligned}$$

Except the introduction of the NU system, there is also the necessity to adjust the relativistic units of measurement to the relativistic reality. Indeed as it is nonsense to measure the speed of a car in mm/days, similarly it makes no sense to measure the energy of an elementary particle in J. The basic physical quantity in the study of elementary particles (at the level we are dealing with) is the energy. Since in NU the units of energy are identical with the units of mass (because $c = 1$ and $E = mc^2$), we measure the mass of elementary particles in units of energy. For reasons we mentioned above these units are not the erg (C.G.S.) or the Joule (S.I.) but the eV and its multiples. By definition 1 eV is the kinetic energy acquired by an electron which starts from rest and moves in empty space between two points whose potential difference is 1 V. Let us calculate how much energy is in 1 eV.

The charge of the electron equals 1.6021×10^{-19} Cb, therefore,

$$1\,\mathrm{eV} = 1.6021\times10^{-19}\,\mathrm{J} = 1.602\times10^{-12}\,\mathrm{erg}.$$

Inverting this relation we find

$$1\,\mathrm{J} = 6.24\times10^{18}\,\mathrm{eV}.$$

The multiples of eV (which are common to all physical quantities) are the MeV $= 10^6$ eV, GeV $= 10^9$ eV $= 10^3$ MeV, and the TeV $= 10^{12}$ eV $= 10^3$ GeV $= 10^6$ MeV. Obviously

$$\begin{aligned}1\,\text{MeV} &= 1.602 \times 10^{-13}\,\text{J} = 1.602 \times 10^{-6}\,\text{erg}\\ 1\,\text{J} &= 6.24 \times 10^{12}\,\text{MeV}.\end{aligned}$$

As we mentioned in the NU the units of energy (e.g., MeV) have the same dimension with the units of mass. In Exercise 32 we have noted that the dimensions of linear momentum in the SI system follow from the units of energy in the NU if one multiplies with the factor $[c]^{-1}$. Similarly the units of mass in the SI system are found from the units of energy in the NU system by the multiplication with the factor $[c]^{-2}$. For this reason the units of three-momentum are written as MeV/c and the units of mass as MeV/c^2. Let us see some practical calculations.

Example 39 The mass of the electron in SI system equals 9.1×10^{-28} g. What is the mass of the electron in MeV/c^2 (that is in the NU system)?
Solution
We have

$$\begin{aligned}&\text{Mass of electron in NU } (\text{MeV}/\text{c}^2) =\\ &\text{Mass of electron in grams} \times [c]^2 = (9.1 \times 10^{-31}\,\text{Kg}) \times (3 \times 10^8\,\text{m/s})^2\\ &= 8.19 \times 10^{-14}\,\text{J} = \frac{8.19 \times 10^{-14}}{1.602 \times 10^{-19}}\text{eV} = 0.51\,\text{MeV}.\end{aligned}$$

Example 40 The mass of the proton in the NU system equals 938 MeV/c^2. Calculate the mass of the proton in grams (that is in SI units).
Solution
We have

$$\begin{aligned}&\left[\text{Mass of the proton in grams}\right] =\\ &\left[\text{Mass of the proton in 938MeV/c}^2\right]/c^2 =\\ &\left[(938 \times 1.602 \times 10^{-6})\text{erg}\right]/(3 \times 10^{10}\text{cm/s})^2 = 1.67 \times 10^{-24}\text{g}.\end{aligned}$$

Exercise 33(a) Show that if the mass of a ReMaP equals m g, then its mass in NU equals $5.618\,\text{m} \times 10^9\,\text{MeV/c}^2$.
(b) Show that if the mass of a ReMaP in NU equals E MeV/c^2, then its mass in grams is $m = \frac{E}{5.618} \times 10^{-9}$ g.
(c) A particle has linear momentum 15 GeV/c. Show that in NU its momentum equals 8.01×10^{-18} Kg m/s^1.

Example 41 In the LCF Σ two electrons have equal kinetic energies 1 MeV and are moving in opposite directions. Compute

(a) The speed of each electron in Σ
(b) The relative velocity of the electrons

(It is given that the mass of the electron in NU is $m = 0.51\,\mathrm{MeV/c^2}$).
Solution

(a) If T is the kinetic energy of the electron then the total energy $E = T + mc^2$. But $E = m\gamma c^2$. Therefore

$$\gamma = \frac{T + mc^2}{mc^2} = 1 + \frac{T}{mc^2} \Longrightarrow \gamma = 2.96 \text{ and } \beta = 0.94.$$

(b) The relative velocity of the electrons equals the velocity of one of them in the rest frame of the other. The relativistic rule for spatial velocities gives

$$|v_{\mathrm{rel}}| = \frac{u + u}{1 + uu/c^2} = \frac{2\beta}{1 + \beta^2}\, c,$$

where $\beta = u/c$ is the β-factor of each electron in Σ. Replacing, we find $|v_{\mathrm{rel}}| = 0.998c$.

Example 42 Calculate to the order of the third decimal digit the β-factor of a pion of momentum 10.0 GeV/c. It is given that the mass of the pion is $1.40\,\mathrm{GeV/c^2}$.
Solution

It is given that $|pc| = 10.0\,\mathrm{GeV}$ and $mc^2 = 1.40\,\mathrm{GeV}$. Then

$$\beta = \frac{|pc|}{E} = \frac{|pc|}{m\gamma c^2} \Rightarrow \beta\gamma = \frac{|pc|}{mc^2} = 7.14$$

therefore

$$\gamma = \sqrt{1 + (\beta\gamma)^2} = 7.21,$$
$$\beta = \sqrt{1 - \frac{1}{\gamma^2}} = 0.990.$$

Chapter 10
Relativistic Reactions

10.1 Introduction

According to the Newtonian point of view, matter consists of absolute units, the particles, which were created once, and ever since their number and their identity are preserved. Today we know that this point of view is not valid. In nature, nuclear reactions occur at all times so that particles are created and destroyed in such a way that their number and identity are constantly changing. A dramatic example of such a change is the explosion of the nuclear bomb and on a bigger scale the "burning" of the mass of the sun. A world in which both the number and the identity of the particles are not constant but change, either spontaneously or by external causes, is compatible with the point of view of Special Relativity. Because, according to that theory,

1. A system is characterized only by the values of the various relativistic physical quantities (mass, charge, spin, etc.) defining the system.
2. A system is described by means of states, each state being an aggregation of a different (in general) number and type of particles and fields binding them.
3. The state of the system can change to another state either spontaneously or by external causes. We say that each change of state is a *state transition* of the system. The physical processes which cause phase transitions of particle systems are called *particle reactions* or *scattering*.
4. The state transitions of a system take place so that all the physical quantities (mass, charge, spin, etc.), which define the system, are conserved.

For example according to the relativistic view the reaction

$$n \to p + e^- + \bar{\nu}_\beta$$

describes the transition of a physical system, which in one state appears as a neutron and in the other as the aggregate of a proton, an electron, and a neutrino. What is the same in the two "appearances" of the system are the total four-momentum, the total charge, the total spin, etc. As a second example, the states electron and $\gamma + \gamma$ are not possible for a system, because the charge is not conserved. This implies

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that the Newtonian distinction of systems, in simple and complex according to the number of particles they involve, does not apply. Indeed, as we have seen in the above example, the same system in one state appears as a single particle (the n) and in another as the aggregate of three particles (the $p,\ e^-,\ \bar{\nu}_\beta$). Furthermore, it is possible that each particle is by itself a system and in some states can appear as more particles and so on.

If in a reaction (or scattering) of particles the number and the identity of particles are preserved, we call the reaction *elastic*. In all other cases, the reaction is called *inelastic*.

In practice, particle reactions are achieved by the collision of beams of fast moving particles either with targets at rest in the laboratory or with other traveling beams. These beams of particles (the mother particles) are created in special man-made machines (particle colliders, linear accelerators, synchrotrons, etc.) or in interstellar space, e.g., by magnetic fields of strongly magnetized stellar bodies. The particles which emerge from a reaction (the daughter particles) are observed with special machines, which measure their distribution in space (number density and energy density in a specified direction and a small solid angle) in some LCFs. For a long time the experimental measurements involved photographing and counting the particle trajectories (bubble chamber), but today there are more accurate and powerful electronic methods employing computers and photosensitive materials (scintillator, etc.). In every case, what we observe in a particle reaction is a set of orbits, whose study provides us with information concerning the values of the physical quantities of the system in the LCF we are working. The experimentally measured quantities we observe are

1. The identity of the particles (partially)
2. The distribution of the space velocities of the particles (i.e., the number of trajectories per given direction and unit solid angle)
3. The distribution of energies and three-momenta (curvatures and other geometric elements of the orbits in given electromagnetic fields)

The "metamorphosis" of a physical system from one state to another can be studied in two stages. The first stage considers the initial and the final states and it is concerned only with the conservation of the physical quantities characterizing the system. The second stage studies the dynamics, that is the mechanism, which brings the initial state to the final state. In the following, we shall be concerned with the first study, sometimes called the kinetics of the reaction. The second approach requires the methods of quantum physics and it is outside the scope of this book.

10.2 Representation of Particle Reactions

The spacetime representation of a particle reaction is shown in Fig. 10.1. This representation simply shows the particles which react and the particles which are produced. The dark circle represents the mechanism of the reaction (the "dynamics"

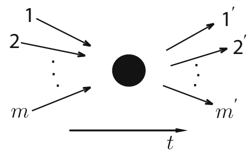

Fig. 10.1 Spacetime representation of a reaction

of the reaction) and, as we remarked above, concerns the study of the reaction by means of quantum field theory.

The study of the kinetics of a particle reaction is also done in two stages, as it is done with conventional chemistry. In the first stage, the reaction is studied stoichiometrically, that is, we consider a small number of individual reacting particles and write

$$\begin{array}{cccccccc} A & + & B & + \cdots \rightarrow & A' & + & B' & + \cdots \\ 1 & & 2 & + \cdots & 1' & & 2' & + \cdots \end{array}$$

This defines the initial and the final states of the system. Then one considers the conservation equations of the physical quantities of the system in one or more states and draws conclusions with the purpose to explain or foresee the experimental data. In the second stage, the reaction is studied qualitatively, that is, one considers the distributions of particles (not individual particles) and subsequently studies the conservation of the various physical quantities of the system. In the following, we shall be concerned with the stoichiometric study only, because the quantitative study requires special knowledge beyond the scope of this book.

10.3 Relativistic Reactions

In Special Relativity the basic physical quantity, which characterizes a system of particles is the four-momentum. The four-momentum of a relativistic system in a certain state is the vector sum of the four-momenta of all components comprising the system. If the state of the system consists only of free particles, then the four-momentum of the system equals the sum of four-momenta of all individual particles. If the state of the system contains fields of interaction among the particles, then the four-momentum of the system contains the four-momenta of the particles plus the four-momenta of the fields of interaction. For example, the four-momentum of an atom includes the four-momenta of the nucleus, and the electrons plus the momentum of the electromagnetic field coupling the nucleus and the electrons. Furthermore, the nucleus itself is a system whose four-momentum is the sum of the momenta of the nucleons which make it up and the hadronic and the electromagnetic fields which couple them. Moreover, each nucleon is a relativistic system and so on. We note that there is an endless process, which does not allow for a clear-cut distinction between particle momentum and field momentum. Therefore, what is experimentally measured is the total momentum of an experimentally identifiable system.

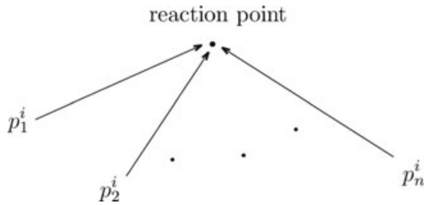

Fig. 10.2 Schematic representation of the reacting (mother) momenta

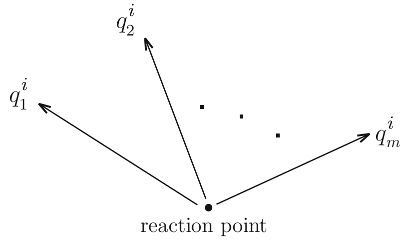

Fig. 10.3 Schematic representation of the produced (daughter) momenta

Consider a set of particles $P_1,\ P_2, \ldots, P_n$ with corresponding four-momenta p_1, $p_2, \ldots, p_n$. We say that these particles react or collide if their world lines have a common point as shown in Fig. 10.2.

The reaction is a common event for all particles involved. As a result of the reaction, suppose that new particles $Q_1, Q_2, \ldots, Q_m$ are produced with four-momentum $q_1,\ q_2, \ldots, q_m$ respectively. The new particles are in general different both in identity $(P_{(\alpha)} \neq Q_{(\beta)})$ and in number (see Fig. 10.3).

The world lines of the daughter particles have again a common spacetime event, which is the same as the event of reaction/collision of the mother particles. Therefore, at the spacetime event of the reaction we have the following sets of timelike or null four-vectors:

$$p_1,\ p_2, \ldots, p_n \text{ and } q_1,\ q_2, \ldots, q_m.$$

These vectors are elements of Minkowski space, therefore they must comply with the geometry of that space. In Chap. 1 we stated Proposition 4, which concerns the sum of particle four-vectors and the triangle inequality in Minkowski space. Let us see the effect of each of these geometric results on the four-momenta.

10.3.1 The Sum of Particle Four-Vectors

According to Proposition 4, the sum of a finite number of future/past-directed timelike or null four-vectors (particle four-vectors) is a future/past-directed timelike four-vector except, if and only if, all vectors are null and parallel, in which case the sum is a null vector parallel to the individual vectors. Let us see the significance of

Fig. 10.4 Relativistic reaction

Particles $\xrightarrow{\text{Reaction}}$ Particles

this result in the case of spacetime collisions. The second part assures that in Special Relativity the light beams exist and are geometrically sound quantities. Indeed, a light beam is defined by a bunch of parallel null straight world lines and according to Proposition 4, the sum of such an aggregate of lines is again a parallel null line corresponding to the beam. Furthermore, it is easy to show that if two photons (or null straight world lines) are parallel in Minkowski space, then their directions of propagation in Euclidean three-space are also parallel. Therefore, a light beam in Newtonian space is also a light beam in spacetime. This is an important conclusion otherwise we would not be able to coordinate spacetime by means of light rays (i.e., to do chronometry in Special Relativity) working within our Newtonian environment.

Concerning the first part of the proposition we have that – excluding the case that all four-momenta are future/past-directed null and parallel – the sums $P^i = \sum_{B=1}^{n} p_B^i$ and $Q^i = \sum_{\alpha=1}^{m} q_\alpha^i$ are timelike four-vectors. This geometric result, when transferred to the relativistic reactions, means that they must be of the general form (Fig. 10.4)

Certainly, geometry does not (and should not!) say which particles are produced when certain particles are involved in a relativistic reaction. This is the work of physics and its laws. However, what it does say is that in a relativistic reaction particles produce particles and only particles. This is not a self-evident fact and shows that our belief of how a relativistic reaction occurs is justified by the mathematical model we employ.

Because four-momentum is a relativistic physical quantity, we demand that during a relativistic reaction the total four-momentum is conserved, that is, the following equation holds:

$$P^i = Q^i. \tag{10.1}$$

The understanding of this law is different from the usual point of view of the corresponding law of conservation of three-momentum (and energy) of Newtonian Physics, the difference being in the absolute character of the particles of the latter. Indeed, in Special Relativity it is assumed that during the reaction, the reacting particles form a new unstable (or virtual) particle with four-momentum P^i which subsequently, with some kind of esoteric mechanism, brakes up into the products of the reaction. This virtual particle is called *center of momentum particle* and is indicated as shown in Fig. 10.5. Such an assumption is clearly incompatible with the Newtonian point of view.

The law of conservation of four-momentum is a pillar of contemporary physics and appears to hold exactly without exception. For example, an apparent violation in radioactive beta decay back in the 1930s led Pauli to the speculative hypothesis that in that reaction there is also an undetected neutral particle of small proper mass,

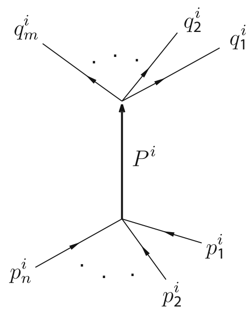

Fig. 10.5 The center of momentum particle

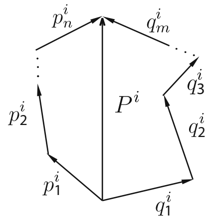

Fig. 10.6 The closed polygon of linear four-momenta

which was later named "neutrino." This speculation was established[1] 23 years later, beyond any doubt, by its detection in a nuclear reaction.

Geometrically the law of conservation of four-momentum in spacetime is represented as in Newtonian three-space, that is, with two closed polygons with one common side (the four-vector P^i) the rest of the sides being the four-momenta $p_1, p_2, \ldots, p_n$ and $q_1, q_2, \ldots, q_m$ as shown in Fig. 10.6.

Algebraically, this law is described by the equation

$$p_1 + p_2 + \cdots + p_n = q_1 + q_2 + \cdots + q_m. \tag{10.2}$$

10.3.2 The Relativistic Triangle Inequality

Let us consider now the second geometric result of Proposition 4 concerning the relativistic triangle inequality. Let OA^i, OB^i, and AB^i be the position four-vectors of three spacetime points O, A, B in the interior of the (future) light cone. Then for the spacetime triangle (OAB) it is true that the Lorentz lengths (not the Euclidean lengths!) satisfy the relation

(Lorentz length of OB) > (Lorentz length of OA) + (Lorentz length of AB).

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[1] C. L. Cowan Jr and F. Reines Phys. Rev. **92**, 830, (1953); F. Reines and C. L. Cowan Jr Phys. Rev. **113**, 273, (1959).

This relation is opposite to the one of Euclidean geometry and can be extended easily to any spacetime polygon, whose sides are timelike and/or null four-vectors. The physical meaning of this geometric result is the following:

Consider a closed spacetime polygon with sides the four-momentum vectors p^i_A $(A = 1, 2, \ldots, n)$ and let P^i be their sum. The length of each side equals $m_A c$, where m_A is the mass of the particle A with four-momentum p^i_A $(A = 1, 2, \ldots, n)$. The geometric result implies that the mass of the center of mass particle is more than (and in extreme cases equal to) the sum of the masses of the individual reacting particles. Because the mass is directly related to energy, this result can be understood as follows.

A relativistic system in every state consists of a set of particles and has total energy E, which includes the masses of the constituting particles, their kinetic energies, and the energy of the dynamical fields among the particles:

$$\begin{aligned}[\text{Mass of the system}] =&[\text{Masses of particles}] + [\text{Kinetic energy of particles}]\\ &+ [\text{Energy of dynamical fields}].\end{aligned}$$

At every moment, these three types of energy are in dynamic equilibrium so that it is possible, e.g., to reduce the mass of the individual particles while increasing the kinetic energy in order to keep the total mass of the system constant. This phenomenon happens in nuclear reactors (e.g., in the sun), where a part of the mass of the system is transformed into kinetic energy of the products which, accordingly, is dissipated as radiation and heat.

10.4 Working with Four-Momenta

Having given the basic definitions and notions concerning the four-momentum we continue with practical examples, which show how one works in practice with four-momenta.

Example 43 Show that there are no relativistic reactions, whose product will be a single photon. Also show that the photon does not decay.

Solution

Assume that particles 1, 2 with masses m_1, m_2, respectively, interact and produce a photon according to the reaction $1 + 2 \rightarrow \gamma$.

Conservation of four-momentum gives

$$p^i_1 + p^i_2 = p^i_\gamma.$$

Squaring and using the relation $p^i p_i = -m^2c^4$ we find

$$-m_1^2c^2 - m_2^2c^2 + 2p^i_1 p_{2i} = 0. \tag{10.3}$$

Relation (10.3) contains invariant quantities, therefore its value can be computed in any LCF. We choose the proper system Σ_1 of particle 1, in which $\mathbf{p}_1 = 0$ and let $p_2^i = \begin{pmatrix} E_2'/c \\ \mathbf{p}_2' \end{pmatrix}_{\Sigma_1}$. Then relation (10.3) gives

$$(m_1 + m_2)^2 c^2 + 2m_1(E_2' - m_2 c^2) = 0.$$

In this relation all terms are ≥ 0, therefore it vanishes only if each term vanishes. The vanishing of the first term gives $m_1 = m_2 = 0$, which means that the two reacting particles must be photons.

Conservation of energy in Σ gives

$$E_1 + E_2 = E_3,$$
$$\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_3.$$

Squaring the second and using relation $\mathbf{p}_i^2 = E_i^2$ $(i = 1,\ 2,\ 3)$ for the reacting photons, we find that one of the energies $E_1,\ E_2$ must vanish which is impossible. We conclude that a single photon cannot be the result of a relativistic reaction.

Considering the reaction in the equivalent form

$$(-\gamma) \to (-1) + (-2),$$

we conclude that a photon cannot decay.

The following example shows the connection of the relativistic law of conservation of four-momentum with the conservation of energy and three-momentum of Newtonian Physics.

Example 44 a. Show that if the time component of a four-vector vanishes in *all* LCFs, then the four-vector is the zero four-vector. (This result is known as the Theorem of the zeroth component).

b. Show that in the Theory of Special Relativity the conservation of energy implies the conservation of three-momentum and conversely. This result shows that the Newtonian laws of conservation of energy and three-momentum are contained as a set in the conservation law of four-momentum.

Solution

a. Consider the four-vector A^i, which in the LCFs Σ and Σ' has components $A^i = \begin{pmatrix} 0 \\ \mathbf{A} \end{pmatrix}_{\Sigma}$ and $A^i = \begin{pmatrix} 0 \\ \mathbf{A}' \end{pmatrix}_{\Sigma'}$, respectively. If the relative velocity of $\Sigma,\ \Sigma'$ is $\boldsymbol{\beta}$, then the Lorentz transformation gives

$$0 = \gamma(0 - \boldsymbol{\beta} \cdot \mathbf{A}),$$
$$\mathbf{A}' = \gamma(\mathbf{A} - \boldsymbol{\beta} \cdot 0)$$

from which follows

$$\boldsymbol{\beta} \cdot \mathbf{A} = 0, \qquad \mathbf{A}' = \gamma \mathbf{A}.$$

Because $\boldsymbol{\beta}$ is arbitrary, the first relation gives $\mathbf{A} = 0$ and then from the second relation $\mathbf{A}' = 0$ follows. In both cases we have $A^i = 0$.

b. Let P^i be the four-momentum of a physical system, which undergoes an interaction. Let ΔP^i be the change in the four-momentum of the system during this interaction and let an arbitrary LCF be Σ, in which the four-vector ΔP^i has components $\Delta P^i = \begin{pmatrix} \frac{\Delta E}{c} \\ \Delta \mathbf{p} \end{pmatrix}_\Sigma$. If the energy is conserved during the interaction $\Delta E = 0$ in all LCFs then by the Theorem of the zeroth component $\Delta P^i = 0 \Rightarrow \Delta \mathbf{p} = 0$, that is the three-momentum of the physical system is also conserved during the interaction. Working in a similar manner we show that $\Delta \mathbf{p} = 0 \Rightarrow \Delta E = 0$.

10.5 Special Coordinate Frames in the Study of Relativistic Collisions

The quantitative study of a relativistic reaction involves calculations with components, therefore it requires the decomposition of the four-vectors involved in some LCF. In practice, we use the following three LCFs:

- The proper frame of the center of momentum particle. This frame is called the *center of momentum* (CM) frame and denoted as Σ^* or Σ^{CM}.
- The proper frames of the reacting and the product particles (excluding of course the photons). We call these LCFs *target frame* of the relevant particle.
- The laboratory frame, which as a rule coincides with one of the target frames but in the case of colliding beams it can be the proper frame of the measuring apparatus. The laboratory frame is denoted with (L).

It is useful to compute the $\mathbf{B}$, Γ-factors of the CM frame in the L frame. To do that we decompose the four-vector P^i in both systems:

$$P^i = \begin{pmatrix} Mc \\ \mathbf{0} \end{pmatrix}_{CM} = \begin{pmatrix} E^L/c \\ \mathbf{P}^L \end{pmatrix}_L .$$

From the relation $E^L = M\Gamma c^2$, or from the time component of the Lorentz transformation, we have

$$\Gamma = \frac{E^L}{Mc^2}. \tag{10.4}$$

The space component of the Lorentz transformation gives

$$\mathbf{B} = \frac{\mathbf{P}^L c}{E^L}. \tag{10.5}$$

In words the above relations are [see also (9.10)]

$$\mathbf{B} = \frac{\text{Total momentum of system of particles in } L}{\text{Total energy of system of particles in } L}, \tag{10.6}$$

$$\Gamma = \frac{\text{Total energy of system of particles in } L}{\text{Mass of CM Particle}}. \tag{10.7}$$

In order to compute the quantities $\mathbf{B}$, Γ in terms of the energies and the three-momenta of the reacting particles, we decompose in the lab frame (or in any other frame we desire) the four-momenta of the particles $p_A^i = \begin{pmatrix} E_A^L/c \\ \mathbf{p}_A^L \end{pmatrix}_L$, $A = 1, 2, \ldots, n$, and have the relations

$$E^L = \sum_A E_A^L, \qquad \mathbf{P}^L = \sum_A \mathbf{p}_A^L. \tag{10.8}$$

We recall that the four-momentum of a photon is $p_A^i = \frac{E}{c}\begin{pmatrix} 1 \\ \hat{\mathbf{e}} \end{pmatrix}_\Sigma$, where $\hat{\mathbf{e}}$ is the direction of propagation of the photon in the LCF Σ and E is the energy of the photon in Σ.

10.6 The Generic Reaction $A + B \to C$

Let us consider the reaction

$$1 + 2 + \cdots \to 1' + 2' + \cdots .$$

According to our previous considerations this reaction expresses two different appearances of the *same* physical system. Furthermore, the particles themselves can be complex physical systems consisting of more particles. This point of view allows us to consider the reaction

$$A + B \to C,$$

where A, B are hypothetical (virtual) "particles" which consist of some particles, $1, 2, \ldots$ and C is the particle $1' + 2' + \cdots$. With this approach we can think of any relativistic reaction as a sequence of reactions of the form $A + B \to C$. We remark that the reaction (decay) $A \to B + C$ is the same with $A + B \to C$ because it can be written as $A + (-B) \to C$. We conclude that the reaction

$$A + B \to C$$

is the *generic reaction* in terms of which all reactions can be studied. In the following, we study the generic reaction and apply the results to specific problems.

In order to facilitate the writing of the results and show their cyclical property, we introduce the following notation. Let the physical quantity be A, which refers to the particle N in the LCF Σ. We shall write

$${}^{N}_{\Sigma}A.$$

For example, the energy of particle 1 in the proper frame Σ_2 of particle 2 is written as ${}^{1}_{\Sigma_2}E$. If the LCF Σ is obvious it shall be omitted. For the CM frame Σ^* we shall write ${}^{1}_{\Sigma^*}E$ or ${}^{1}E^*$.

10.6.1 The Physics of the Generic Reaction

Consider an arbitrary LCF Σ, in which the four-momenta of particles A, B, C are decomposed as follows:

$$\begin{pmatrix} {}^{A}_{\Sigma}E/c \\ {}^{A}_{\Sigma}\mathbf{p} \end{pmatrix}, \quad \begin{pmatrix} {}^{B}_{\Sigma}E/c \\ {}^{B}_{\Sigma}\mathbf{p} \end{pmatrix}, \quad \begin{pmatrix} {}^{C}_{\Sigma}E/c \\ {}^{C}_{\Sigma}\mathbf{p} \end{pmatrix}.$$

Conservation four-momentum gives

$$\begin{pmatrix} {}^{A}_{\Sigma}E/c \\ {}^{A}_{\Sigma}\mathbf{p} \end{pmatrix} + \begin{pmatrix} {}^{B}_{\Sigma}E/c \\ {}^{B}_{\Sigma}\mathbf{p} \end{pmatrix} = \begin{pmatrix} {}^{C}_{\Sigma}E/c \\ {}^{C}_{\Sigma}\mathbf{p} \end{pmatrix} \tag{10.9}$$

or in terms of components

$${}^{A}_{\Sigma}E + {}^{B}_{\Sigma}E = {}^{C}_{\Sigma}E, \tag{10.10}$$

$${}^{A}_{\Sigma}\mathbf{p} + {}^{B}_{\Sigma}\mathbf{p} = {}^{C}_{\Sigma}\mathbf{p}. \tag{10.11}$$

To these equations we must add the corresponding equations, which define the masses:

$${}^{I}_{\Sigma}E = \sqrt{{}^{I}_{\Sigma}\mathbf{p}^2c^2 + m_I^2c^4} \qquad I = A,\ B,\ C. \tag{10.12}$$

We end up with a system of seven equations in 15 quantities. Usually, the masses are assumed to be given, therefore the unknown quantities reduce to five. As we show below, the masses suffice in order to fix the energies and the lengths of three-momenta, therefore the remaining five unknowns involve the directions of the three-momenta.

In order to compute the energies in terms of the masses we square (10.9) and evaluate the Lorentz product of the four-momenta in the proper frame of one of the involved particles, as we did in Example 43. We have

$$-m_A^2c^2 - m_B^2c^2 + 2\left(-{}^{A}_{\Sigma_A}E\;{}^{B}_{\Sigma_A}E/c^2 + {}^{A}_{\Sigma_A}\mathbf{p}\cdot{}^{B}_{\Sigma_A}\mathbf{p}\right) = -m_C^2c^2. \tag{10.13}$$

In the proper frame Σ_A of particle A we have ${}^{A}_{\Sigma_A}\mathbf{p}_A = 0$, ${}^{A}_{\Sigma_A}E = m_Ac^2$ and let $p_B^i = \begin{pmatrix} {}^{B}_{A}E/c \\ {}^{B}_{A}\mathbf{p} \end{pmatrix}_{\Sigma_A}$. Then (10.13) gives

$${}^{B}_{\Sigma_A}E = \frac{m_C^2 - m_A^2 - m_B^2}{2m_A}\,c^2. \tag{10.14}$$

In order to compute the energy ${}^{A}_{\Sigma_B}E$, we simply interchange in (10.14) $A,\ B$ and find

$${}^{A}_{\Sigma_B}E = \frac{m_C^2 - m_B^2 - m_A^2}{2m_B}\,c^2. \tag{10.15}$$

Finally, in order to compute the energy ${}^{A}_{\Sigma_C}E$, we write the reaction as

$$A + (-C) \to (-B)$$

and apply (10.14) with the following change of letters:

$$A \leftrightarrow A,\ B \leftrightarrow -C,\ C \leftrightarrow -B,$$

namely,

$$p_A \leftrightarrow p_A,\ p_B \leftrightarrow -p_C,\ p_C \leftrightarrow -p_B.$$

We write the result leaving the squares of the four-momenta (the masses) the same and changing the sign of the energy:

$${}^{C}_{\Sigma_A}E = \frac{-m_B^2 + m_A^2 + m_C^2}{2m_A}\,c^2. \tag{10.16}$$

We note that the relations giving the energies are cyclical and independent of the nature of the particles, therefore there must exist a purely geometric method for the calculation of the energies. This observation leads us to look for a geometric description of the reaction as a collision of four-vectors, a point of view which we shall develop in Sect. 16.3.

Concerning the computation of the length of the three-momenta, we have

$$|{}^{B}_{\Sigma_A}\,\mathbf{p}\,|^2 c^2 = {}^{B}_{\Sigma_A}E^2 - m_B^2 c^4 = c^4 \left[\frac{m_C^2 - m_A^2 - m_B^2}{2m_A}\right]^2 - m_B^2 c^4$$

$$= \frac{c^4}{4m_A^2}\,(m_A + m_B + m_C)\,(m_A + m_B - m_C)$$

$$(m_A - m_B + m_C)\,(m_A - m_B - m_C)\,.$$

It follows:

$$|{}^{B}_{\Sigma_A}\,\mathbf{p}\,| = \frac{c}{2m_A}\lambda(m_A, m_B, m_C), \tag{10.17}$$

where the function λ is defined by the relation

$$\lambda(m_A, m_B, m_C) = \sqrt{(m_A + m_B + m_C)\,(m_A + m_B - m_C)}$$
$$\sqrt{(m_A - m_B + m_C)\,(m_A - m_B - m_C)} > 0 \tag{10.18}$$

and has dimensions $[M^2\,L^0\,T^0]$. The function $\lambda(m_A,\ m_B,\ m_C)$ is symmetric in all its arguments, therefore the only difference between the length of the three-momenta of the particles is the mass in the denominator of relation (10.17). The function λ is characteristic in the study of the relativistic reactions/collisions and it is called the *triangle function*.

Exercise 34 In general, the λ function for three variables is defined as follows:

$$\lambda(x, y, z) = \sqrt{\left(x^2 + y^2 - z^2\right)^2 - 4x^2 y^2}. \tag{10.19}$$

Show that the function $\lambda(x, y, z)$ has the following algebraic and geometric properties.

A. Algebraic

1.

$$\lambda(x, y, z) = \lambda(x, y, z) = \lambda(z, y, x) = \lambda(y, z, x). \tag{10.20}$$

$$\lambda(x, y, z) = \left[x - (\sqrt{y} + \sqrt{z})^2\right]^{\frac{1}{2}} \left[x - (\sqrt{y} - \sqrt{z})^2\right]^{\frac{1}{2}}. \tag{10.21}$$

$$\lambda(x, y, y) = \sqrt{x^2(x^2 - 4y^2)}. \tag{10.22}$$

$$\lambda(x, y, 0) = \|x - y\|. \tag{10.23}$$

$$\lambda(x, y, z) = \sqrt{(z - x - y)(z - x + y)(z + x - y)(z - x + y)}. \tag{10.24}$$

2. $\lambda(x, y, z) \geq 0$ if $z \geq x + y$ (relativistic case) and $\lambda(x, y, z) < 0$ if $z < x + y$ (Newtonian case).

B. Geometric

The quantity $\frac{1}{4}\sqrt{-\lambda(x, y, z)}$ equals the area of a Euclidean triangle with sides x, y, z. The negative sign is needed because in the Euclidean case $z < x + y$ and the function $\lambda(x, y, z) < 0$. It is this property that gave $\lambda(x, y, z)$ its name. $\lambda(x, y, z)$ is the formula of Heron[2] for the area of a triangle in terms of its sides (see Example 79).

One could forward the question: How much information is contained in the above equations? To answer this question we consider an arbitrary LCF Σ, which is not the proper frame of any of the particles A, B, C. Then relation (10.13) allows us to compute the (Euclidean) inner products of the three-momenta of the particles in Σ (note that these products are not Lorentz invariant, therefore they depend on the LCF Σ). The inner products give the (Euclidean) angles among the three-momenta in Σ, hence the triangle of the three-momenta is fully determined in Σ. In case Σ coincides with the proper frame of any of the particles A, B, C, then the triangle of the three-momenta degenerates into a straight line segment. We conclude that assuming that the masses of the particles are given, we have the following information in an arbitrary LCF Σ:

1. Complete knowledge of the triangle (or straight line segment) of the three-momenta
2. Complete knowledge of the energies of the particles

Therefore the remaining five parameters concern the positioning of the triangle of the three-momenta in the three-space of Σ (more precisely in the momentum space of Σ). This positioning requires three parameters for fixing one of the vertices of the triangle and two parameters (two angles) for the determination of the orientation of the plane of the triangle.

A different way to study the effect of the masses on the "internal" structure of the system of particles is via the geometric representation of the relation $E^2 = |\mathbf{p}|^2 c^2 + m^2c^4$. In the proper space of one of the particles it is easy to check the validity of the triangles of Fig. 10.7.

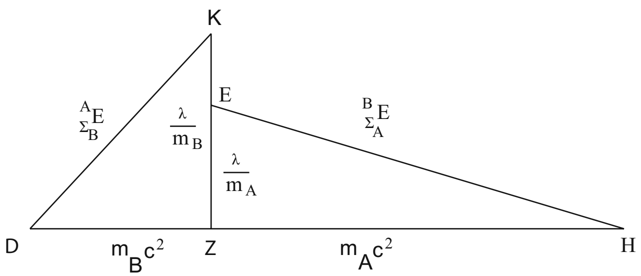

Fig. 10.7 Representation of the reaction $A + B \to C$

[2] See http://mathpages.com/home/kmath196.htm.

We note that the masses $m_A,\ m_B$ fix the point Z of the projection of the vertex K of the Euclidean triangles (KZH) and (KZD), the height in each case being fixed by the function $\lambda(m_A,\ m_B,\ m_C)$ and the masses $m_A,\ m_B$.

Consequently, the three masses fix the figure completely and what remains is its positioning in the three-dimensional Euclidean space of the particle C.

The following example shows the application of the above discussion in practice.

Example 45 Show that in the decay $1 \to 2 + 3$ the factors $\beta_2,\ \beta_3$ in the CM are constant, depending only on the masses of the particles and the energy in the CM.

Solution

Relation (10.17) gives the length of the three-momenta of particles $2,\ 3$:

$$\mid \mathbf{p}_2^* \mid = \mid \mathbf{p}_3^* \mid = \frac{\lambda(m_1, m_2, m_3)}{2M} c,$$

where M is the mass of the CM particle. The energies of the particles 2, 3 in the CM frame, according to (10.16), are

$$E_2^* = \frac{M^2 + m_2^2 - m_3^2}{2M} c^2, \qquad E_3^* = \frac{M^2 - m_2^2 + m_3^2}{2M} c^2.$$

From (10.6) the factor β_2^* of particle 2 in the CM frame is

$$\beta_2^* = \frac{\mid \mathbf{p}_2^* \mid c}{E_2^*} = \frac{\lambda(m_1, m_2, m_3)}{M^2 + m_2^2 - m_3^2}. \tag{10.25}$$

We note that β_2^* depends only on the masses of the particles and the mass of the CM particle, which in the CM frame equals the energy of the particles in that frame. We work similarly for particle 3.

10.6.2 Threshold of a Reaction

An interesting limiting case occurs when the triangle of the three-momenta degenerates to a straight line segment. This happens when the height of the triangle vanishes, that is, when the function $\lambda(m_A,\ m_B,\ m_C) = 0$. In this case, all the three particles $A,\ B,\ C$ are at rest in the proper frame of C (which is also the CM frame) and we say that this limiting case is the *threshold* of the reaction. Conservation of energy in the proper frame of C gives

$$\,^A_{\Sigma_C}E + \,^B_{\Sigma_C}E = m_C c^2 \Rightarrow m_C \geq m_A + m_B,$$

where equality holds only at the threshold of the reaction.[3] This result is the physical explanation of the triangle inequality we mentioned in Sect. 10.3 or, equivalently, the geometric interpretation of the function λ, as the area of a Euclidean triangle with sides m_A, m_B, m_c. In practice, this means that in case the two particles collide totally inelastically, so that after collision they become one particle, their kinetic energy (in the proper frame of the daughter particle) transforms into the mass of the new particle. Conversely, when a particle decays, a part of its mass becomes kinetic energy of the produced particles. At the threshold of a reaction, there is no transformation of mass into kinetic energy.

Exercise 35 Show that the necessary and sufficient condition for the particles A and B of non-zero mass to react at the threshold is

$$\frac{p_A^i}{m_A} = \frac{p_B^i}{m_B} \Leftrightarrow u_A^i = u_B^i,$$

where u^i is the four-velocity of the particles.

Exercise 36 Consider the case $m_A = m_B = m$ and show that in this case the triangle of the reaction is isosceles with height (= length of three-momentum) $\mid \mathbf{p} \mid = \frac{c}{2}\sqrt{m_C^2 - 4m^2}$ and common side (= energy) $E = \frac{m_c}{2}c^2$. Deduce that at the threshold of the reaction, the mass $m = \frac{m_c}{2}$ and the triangle degenerates to a straight line segment.

In case one of the particles A, B is a photon, only one of the triangles (KZH), (KZD) of Fig. 10.7 survives. We infer that it is not possible that both particles are photons, because in that case there does not exist a triangle (see also Example 43).

Exercise 37 Assume that one of the particles, the A say, is a photon and that $m_B \neq 0$. Show that in this case, the energies of A, B in the proper frame of C are the following:

$$\underset{\Sigma_C}{\overset{A}{}} E = \frac{m_C^2 - m_B^2}{2m_C}, \qquad \underset{\Sigma_C}{\overset{B}{}} E = \frac{m_C^2 + m_B^2}{2m_C}. \tag{10.26}$$

What can you say about the lengths of the three-momenta without any further calculations?

Consider that the mass $m_B \to 0$ and show that the mass cannot take the value zero, because then the sum of the angles of the triangle of reaction becomes greater than 2π. Conclude that it is not possible for one particle to decay to two photons or two photons to react and produce a *single* particle (decay in *two* particles is possible!).

[3] The vanishing of the function $\lambda(m_A, m_B, m_C)$ implies the condition $\left(m_C^2 + m_A^2 - m_B^2\right)^2 - 4m_C^2 m_B^2 = 0$, from which follows $m_C = m_B \pm m_A$. But the function λ is symmetric in all its arguments, therefore the result must not change if we interchange the masses m_A, m_B. This property selects only the case $m_C = m_B + m_A$.

The *threshold energy* of a reaction is the energy of the bullet particle in the CM frame at the threshold of the reaction. The threshold energy is different for different bullet particles. Let us see how the threshold energy is computed in practice.

Example 46 Consider the reaction $\pi^- + p \to K^0 + \Lambda^0$, where the proton rests in the laboratory.

(a) Compute the threshold energy of the pion.
(b) In an experiment, in which the pions have three-momentum 2.50 GeV/c, it is observed that the Λ^0 particles have momentum 0.60 GeV/c and they emerge at an angle of 45° with the direction of motion of the pions. Compute the γ^*-factor of the CM frame.
(c) Compute the three-momentum of the kaons K^0 in the laboratory frame and in the CM frame.
It is given that $m_{\pi^-} = 140\,\mathrm{MeV}/c^2$, $m_p = 938\,\mathrm{MeV}/c^2$, $m_{K^0} = 498\,\mathrm{MeV}/c^2$, $m_{\Lambda^0} = 1116\,\mathrm{MeV}/c^2$.

Solution

(a) The reaction is

$$\begin{array}{cccc} \pi^- + & p \to & K^0 + & \Lambda^0 \\ 1 & 2 & 3 & 4 \end{array}.$$

Conservation of four-momentum gives

$$p_1^i + p_2^i = p_3^i + p_4^i.$$

At the threshold of the reaction at the CM frame the total three-momentum of the daughter particles vanishes. Therefore

$$p_3^i + p_4^i = \begin{pmatrix} (m_3 + m_4)c \\ \mathbf{0} \end{pmatrix}_{CM}.$$

Squaring the conservation equation we find

$$-m_1^2c^2 - m_2^2c^2 + 2p_1^i p_{2i} = -(m_3 + m_4)^2c^2. \tag{10.27}$$

The term $p_1^i p_{2i}$ is invariant, therefore it can be computed in any LCF. We choose the lab frame, where p is at rest, and assume that at the threshold we have the components

$$p_1^i = \begin{pmatrix} E_{1,th}/c \\ \mathbf{p}_{th} \end{pmatrix}_L, \quad p_2^i = \begin{pmatrix} m_2c \\ \mathbf{0} \end{pmatrix}_L.$$

Replacing in (10.27) and solving for $E_{1,\,th}$ we find

$$E_{1,\,th} = \frac{(m_3 + m_4)^2 - m_1^2 - m_2^2}{2m_2}\,c^2.$$

A second solution, which is based in the previous considerations, is the following. The energy of particle 1 in the lab frame is (see (10.14))

$${}_L^1E = \frac{M^2 - m_1^2 - m_2^2}{2m_2}\,c^2,$$

where M is the mass of the CM particle. At the threshold of the reaction

$$M = m_3 + m_4$$

from which follows

$${}_L^1E_{th} = \frac{(m_3 + m_4)^2 - m_1^2 - m_2^2}{2m_2}c^2.$$

(b) In the experiment, the four-momentum of the CM particle in the lab frame is

$$P_{KO}^i = p_1^i + p_2^i = \begin{pmatrix} E/c \\ \mathbf{P} \end{pmatrix}_L,$$

where

$$E = E_1 + E_2 = \sqrt{\mathbf{p}_1^2c^2 + m_1^2c^4} + m_2c^2 = 3.44\,\text{GeV},$$
$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 = 2.50\,\mathbf{i}\,\text{GeV}/c.$$

The β^*-factor of the CM frame in the lab frame is given by

$$\beta^* = \frac{|\,\mathbf{P}\,|\,c}{E} = 0.727$$

from which we compute $\gamma^* = 1.456$.

(c) Conservation of three-momentum in the lab frame gives

$$\mathbf{p}_3 = \mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_4 = 2.50\,\mathbf{i} - 0.60(\cos 45\,\mathbf{i} + \sin 45\,\mathbf{j}) = (2.08\,\mathbf{i}{-}0.42\,\mathbf{j})\,\text{GeV}/c$$

from which we compute $|\,\mathbf{p}_3\,| = 2.12\,\text{GeV}/c$ and

$$\tan\theta_3 = \frac{-0.42}{2.08} = -0.2019 \Rightarrow \theta_3 = -11^0\,24'\,57''.$$

The energy of K^0 in the lab is

$$E_3 = \sqrt{\mathbf{p}_3^2 c^2 + m_3^2 c^4} \approx 2.18\,\text{GeV}/c.$$

In order to compute the three-momentum of the kaons in the CM frame we use the Lorentz transformation, which connects the lab and the CM frames. We have

$$\begin{aligned} p_{3x}^* &= \gamma^*(p_{3x} - \beta^* E_3/c), \\ p_{3y}^* &= p_{3y}, \\ E_3^*/c &= \gamma^*(E_3/c - \beta^* p_{3x}). \end{aligned}$$

Replacing we compute

$$\mathbf{p}_3^* = (0.721\,\mathbf{i} - 0.42\,\mathbf{j})\,\text{GeV}/c.$$

It follows that the length $|\,\mathbf{p}_3^*\,| = 0.834\,\text{GeV}/c$ and the angle

$$\tan\theta_3^* = \frac{-0.42}{0.721} = -0.5825 \Rightarrow \theta_3^* = -30^0\ 13'\ 19''.$$

Example 47 A particle A of mass $m_1 \neq 0$, which in the lab frame has energy E_1, decays into two particles B, C with masses $m_2,\ m_3$, respectively.

a. Show that $m_1 \geq m_2 + m_3$, where the equal sign holds at the threshold of the reaction.
b. Calculate the range of energies of particle A (in the lab) and the conditions that the masses must satisfy in order for the reaction to be possible.

Solution

a. The reaction is

$$\begin{array}{ccccc} A & \to & B & + & C \\ 1 & & 2 & & 3 \end{array}.$$

Conservation of four-momentum $p_1^i = p_2^i + p_3^i$ in the proper frame of A, Σ_A gives

$$m_1 \begin{pmatrix} c \\ 0 \end{pmatrix}_{\Sigma_A} = m_2\gamma_{21} \begin{pmatrix} c \\ \mathbf{v}_2 \end{pmatrix}_{\Sigma_A} + m_3\gamma_{31} \begin{pmatrix} c \\ \mathbf{v}_3 \end{pmatrix}_{\Sigma_A},$$

where $\gamma_{21},\ \gamma_{31}$ are the γ-factors of particles 2, 3 in Σ_A. From the zeroth component we have

$$m_1 = m_2\gamma_{21} + m_3\gamma_{31} \geq m_2 + m_3.$$

The "=" holds when $\gamma_{21} = \gamma_{31} = 1$, that is at the threshold of the reaction.

b. Square the conservation equation to get

$$-m_2^2c^2 = -m_1^2c^2 - m_3^2c^2 - 2p_1^i p_{3i}. \tag{10.28}$$

We compute the product $p_1^i p_{3i}$ in the lab frame. We have

$$p_1^i \cdot p_{3i} = \begin{pmatrix} E_1/c \\ \mathbf{p}_1 \end{pmatrix}_L \cdot \begin{pmatrix} E_3/c \\ \mathbf{p}_3 \end{pmatrix}_L = -\frac{E_1E_3}{c^2} + p_1 p_3 \cos\theta,$$

where $p_1 = |\mathbf{p}_1|,\ p_3 = |\mathbf{p}_3|$ and θ is the angle between $\mathbf{p}_1,\ \mathbf{p}_3$ in the lab. Replacing in (10.28) we find

$$\begin{aligned} E_1E_3 - c^2 p_1 p_3 \cos\theta &= \frac{1}{2}(m_1^2 + m_3^2 - m_2^2)c^4 \Rightarrow \\ cp_1 \cos\theta \sqrt{E_3^2 - m_3^2c^4} &= -Xc^2 + E_1E_3, \end{aligned} \tag{10.29}$$

where $X = \frac{1}{2}(m_1^2 + m_3^2 - m_2^2)c^2 > 0$. Squaring we find

$$\begin{aligned} c^2p_1^2\cos^2\theta(E_3^2 - m_3^2c^4) &= X^2c^4 + E_1^2E_3^2 - 2Xc^2E_1E_3 \Rightarrow \\ (E_1^2 - c^2p_1^2\cos^2\theta)E_3^2 - 2Xc^2E_1E_3 &+ X^2c^4 + c^6p_1^2m_3^2\cos^2\theta = 0. \end{aligned}$$

This is a quadratic equation in E_3, therefore it obtains the extreme values when the discriminant $\Delta = 0$. We compute

$$\begin{aligned} \Delta &= (Xc^2E_1)^2 - (E_1^2 - c^2p_1^2\cos^2\theta)(X^2c^4 + c^6p_1^2m_3^2\cos^2\theta) \\ &= X^2E_1^2c^4 - [E_1^2X^2c^4 + (c^6E_1^2p_1^2m_3^2 - c^6p_1^2X^2)\cos^2\theta - c^8p_1^4m_3^2\cos^4\theta] \end{aligned}$$

and find the equation

$$\left[c^2p_1^2m_3^2\cos^2\theta - (E_1^2m_3^2 - X^2)\right]\cos^2\theta = 0\ .$$

The roots of this equation are

$$\cos\theta = 0, \quad \cos^2\theta = \frac{E_1^2m_3^2 - X^2}{m_3^2p_1^2c^2}.$$

We examine each root separately.

When the first root $\cos\theta = 0$ is replaced in the original equation (10.29) it gives $E_1E_3 = Xc^2$, hence E_1E_3 is an invariant.

The second root gives

$$\cos^2\theta = \frac{E_1^2}{p_1^2c^2} - \frac{X^2}{m_3^2p_1^2c^2} = 1 + \frac{m_1^2c^2}{p_1^2} - \frac{X^2}{m_3^2p_1^2c^2} \Rightarrow$$
$$\sin^2\theta = \frac{X^2}{m_3^2p_1^2c^2} - \frac{m_1^2c^2}{p_1^2}.$$

Therefore, we have the condition

$$0 \le \frac{X^2}{m_3^2p_1^2c^2} - \frac{m_1^2c^2}{p_1^2} \le 1 \Leftrightarrow 0 \le \frac{X^2}{m_3^2} - m_1^2c^4 \le p_1^2c^2 \Leftrightarrow$$
$$0 \le \frac{X^2}{m_3^2} - m_1^2c^4 \le E_1^2 - m_1^2c^4.$$

The left-hand side gives the following condition for the energy:

$$X^2 - m_1^2m_3^2c^4 \ge 0 \Leftrightarrow$$
$$(X - m_1m_3c^2)\,(X + m_1m_3c^2) \ge 0 \Leftrightarrow$$
$$((m_1 - m_3)^2 - m_2^2)\,((m_1 + m_3)^2 - m_2^2) \ge 0\,.$$

Because $m_1 \ge m_2$ and $m_1 \ge m_3$, this inequality has the solution

$$m_1 \ge m_2$$

(as well as the condition $m_1 \ge m_2 + m_3$, which we already know). The other condition for the energy of the mother particle is

$$E_1^2 \ge \frac{X^2}{m_3^2} \Rightarrow E_1 \ge \frac{|X|}{m_3} = \frac{1}{2}\frac{|m_1^2 + m_3^2 - m_2^2|}{m_3}c^2.$$

Because in the proper frame of the mother particle, the daughter particles in general have energies $E_2 \ge m_2c^2,\ E_3 \ge m_3c^2$, it must be true that $m_1 \ge m_2 + m_3$, which we already know.
(Examine the case that one of the daughter particles is a photon.)

Example 48 In the generic reaction $A + B \to C$ define the invariants

$$s_{AB} = \frac{1}{c^2}(p_A^i + p_B^i)^2, \qquad t_{AB} = \frac{1}{c^2}(p_A^i - p_B^i)^2. \tag{10.30}$$

The first of these invariants gives the mass of the CM particle and the second the three-momentum transfer (equivalently the percentage of energy of the system, which is transformed into kinetic energy of the CM particle). Show that the quantities $s_{AB},\ t_{AB}$ attain their extreme values simultaneously and that this is happening at the threshold of the reaction. Also show that in that case the mass of the CM particle has its minimum value $m_A + m_B$.

Solution

We note that the sum $s_{AB} + t_{AB} = \frac{1}{c^2}\left[(p^i_A)^2 + (p^i_B)^2\right] = -m_A^2 - m_B^2$= constant. Therefore, when s_{AB} is maximum, t_{AB} becomes minimum and conversely. Because s_{AB} is an invariant, its value is the same in all LCFs. We compute it in the proper frame of the particle B. In that frame, the inner product $p^i_A p_{Bi} = -{}^A_B E m_B$, therefore $s_{AB} = -m_A^2 - m_B^2 - 2\,{}^A_B E m_B$. The maximum value of the right-hand side occurs when ${}^A_B E = m_A c^2$, namely at the threshold of the reaction. For this energy, the value $s_{AB,\,\max} = -(m_A + m_B)^2$ and the corresponding minimum value of the quantity t_{AB} at that energy is $t_{AB,\,\min} = (m_A - m_B)^2$. The last question is left to the reader.

10.7 Transformation of Angles

In the last section we have shown that, given the masses, the remaining unknown elements of the reaction are (a) the angles between the three-momenta of the reacting particles and (b) the orientation of the plane of the triangle of the three-momenta in the LCF, in which the reaction is studied. Because in practice the angular data in general are given in different LCF, it is necessary to study the transformation of angles under Lorentz transformation. This study will be done between the CM frame (Σ^*) and the lab frame (Σ^L), because these are the frames mostly used in practice.

Exercise 38 Consider a particle with four-momentum p^i, which in the LCF Σ^* and Σ^L has components

$$p^i = \begin{pmatrix} E^L/c \\ \mathbf{p}^L \end{pmatrix}_{\Sigma^L}, \begin{pmatrix} E^*/c \\ \mathbf{p}^* \end{pmatrix}_{\Sigma^*}.$$

Assume that the velocity factor of Σ^* in Σ^L is $\boldsymbol{\beta}$ and show that between the three-vectors $\mathbf{p}^{\mathrm{L}}$ and $\mathbf{p}^*$ the following relation holds:

$$\mathbf{p}^L = \mathbf{p}^* + (\gamma - 1)\frac{\mathbf{p}^* \cdot \boldsymbol{\beta}}{\beta^2}\boldsymbol{\beta} + \gamma E^* \boldsymbol{\beta}. \tag{10.31}$$

Furthermore, show that the factors $\mathbf{B}^L, \mathbf{B}^*$ of the particle in the frames Σ^L and Σ^*, respectively, are given by the relations

$$\mathbf{B}^L = \frac{c\mathbf{p}^L}{E^L}, \qquad \mathbf{B}^* = \frac{c\mathbf{p}^*}{E^*}. \tag{10.32}$$

Using relations (10.31), (10.32) we are able to compute the components of the four-momentum in Σ^L when they are known in Σ^* and conversely.

Without loss of generality (why?) we direct the axes of Σ^* and Σ^L in such a way that they are related by a boost along the z-axis with velocity factor β. Then the

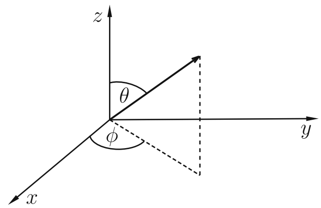

Fig. 10.8 Spherical coordinates in momentum space

Lorentz transformation gives

$$\begin{aligned} E^*/c &= \gamma(E^L/c - \beta p_z^L), \\ p_x^* &= p_x^{\mathrm{L}}, \\ p_y^* &= p_y^{\mathrm{L}}, \\ p_z^* &= \gamma(p_z^{\mathrm{L}} - \beta E^{\mathrm{L}}/c). \end{aligned} \tag{10.33}$$

We consider in each of the frames Σ^*, Σ^L spherical coordinates (see Fig. 10.8) and have the relations

$$\begin{aligned} p_x^L &= P^L \sin\theta^L \cos\phi^L, \; p_x^* = P^* \sin\theta^* \cos\phi^*, \\ p_y^L &= P^L \sin\theta^L \sin\phi^L, \; p_y^* = P^* \sin\theta^* \sin\phi^*, \\ p_z^L &= P^L \cos\theta^L, \qquad\qquad p_z^* = P^* \cos\theta^*. \end{aligned}$$

In the new coordinates the equations of the transformation (10.33) are written as follows:

$$E^*/c = \gamma(E^L/c - \beta P^L \cos\theta^L), \tag{10.34}$$

$$P^* \sin\theta^* \cos\phi^* = P^L \sin\theta^L \cos\phi^L, \tag{10.35}$$

$$P^* \sin\theta^* \sin\phi^* = P^L \sin\theta^L \sin\phi^L, \tag{10.36}$$

$$P^* \cos\theta^* = \gamma(P^L \cos\theta^L - \beta E^L/c). \tag{10.37}$$

Dividing (10.35) by (10.36) we find

$$\tan\phi^L = \tan\phi^*. \tag{10.38}$$

Because $0 < \phi^{\mathrm{L}}, \phi^* < \pi$ it follows:

$$\phi^L = \phi^*. \tag{10.39}$$

Then the equations of the transformation are written as

$$E^*/c = \gamma(E^L/c - \beta P^L \cos\theta^L), \tag{10.40}$$

$$P^* \sin\theta^* = P^L \sin\theta^L, \tag{10.41}$$

$$\phi^* = \phi^L, \tag{10.42}$$

$$P^* \cos\theta^* = \gamma(P^L \cos\theta^L - \beta E^L/c). \tag{10.43}$$

In order to derive the transformation between the angles θ^L, θ^* we divide relations (10.43), (10.41) and find (compare with (9.38))

$$\cot\theta^* = \gamma\left(\cot\theta^L - \frac{r^L}{\sin\theta^L}\right), \tag{10.44}$$

where the quantity r^L is the quotient of the β-factor of Σ^L and Σ^*, and B^L the β-factor of the particle in the lab frame:

$$r^L = \frac{\beta}{B^L}. \tag{10.45}$$

We have the following result illustrated below.

Exercise 39 Show that when $r^L > 1$, the angle θ^L has a maximum, which occurs for the value

$$\theta^L_{\max} = \cos^{-1}\left(-\frac{1}{r^L}\right) \tag{10.46}$$

and equals

$$\sin\theta^L_{\max} = \frac{B^*\Gamma^*}{B^L\Gamma^L}, \tag{10.47}$$

where $\Gamma^* = 1/\sqrt{1 - B^{*2}}$ is the γ-factor of the particle in Σ^*.

Exercise 40 Using the inverse Lorentz transformation, show that the following relation holds:

$$\cot\theta^L = \gamma\left(\cot\theta^* + \frac{r^*}{\sin\theta^*}\right), \tag{10.48}$$

where the quantity r^* is defined as follows:

$$r^* = \frac{\beta}{B^*}. \tag{10.49}$$

Show that when $r^* > 1$, the angle θ^* has a maximum for the value

$$\theta^*_{\max} = \cos^{-1}\left(-\frac{1}{r^*}\right) \tag{10.50}$$

and this maximum equals

$$\sin\theta^*_{\max} = \frac{B^L\Gamma^L}{B^*\Gamma^*}. \tag{10.51}$$

The length $P^L = \mid \mathbf{p}^L \mid$ of the three-momentum of the particle remains to be computed in the laboratory frame Σ^L in terms of the components of the four-momentum in the CM frame Σ^*. From the transformation of energy – see (10.40) – we have

$$\gamma E^L = E^* + \gamma\beta c P^L \cos\theta^L.$$

Replacing this in $E^L = \sqrt{(\mathbf{P}^L)^2c^2 + m^2c^4}$ and squaring we find

$$\gamma^2(1-\beta^2\cos^2\theta^L)(cP^L)^2 - 2\beta\gamma E^*\cos\theta^L cP^L + \gamma^2m^2c^4 - (E^*)^2 = 0. \tag{10.52}$$

The term

$$1-\beta^2\cos^2\theta^L = 1-\beta^2(1-\sin^2\theta^L) = \frac{1}{\gamma^2}(1+\beta^2\gamma^2\sin^2\theta^L)$$

and similarly the term

$$\gamma^2m^2c^4 - (E^*)^2 = \gamma^2m^2c^4 - (cP^*)^2 - m^2c^4 = -(cP^*)^2 + \beta^2\gamma^2m^2c^4.$$

Replacing in (10.52) we find a quadratic equation in terms of P^L:

$$(1+\beta^2\gamma^2\sin^2\theta^L)(cP^L)^2 - 2\beta\gamma E^*\cos\theta^L cP^L - (cP^*)^2 + \beta^2\gamma^2m^2c^4 = 0, \tag{10.53}$$

whose solution is

$$cP^L = \frac{1}{1+\beta^2\gamma^2\sin^2\theta^L}X, \tag{10.54}$$

where

$$X = \beta\gamma E^*\cos\theta^L \pm \sqrt{(\beta\gamma E^*\cos\theta^L)^2 - (1+\beta^2\gamma^2\sin^2\theta^{\mathrm{L}})(\beta^2\gamma^2m^2c^4 - (cP^*)^2)}. \tag{10.55}$$

The term in square root can be written as

$$\gamma^2\left[(cP^*)^2 - \beta^2\gamma^2m^2c^4\sin^2\theta^{\mathrm{L}}\right].$$

Therefore, finally, we have

$$cP^{\mathrm{L}} = \frac{1}{1+\beta^2\gamma^2\sin^2\theta^L}\left[\beta\gamma E^*\cos\theta^L \pm \gamma\sqrt{(cP^*)^2 - \beta^2\gamma^2 m^2 c^4 \sin^2\theta^{\mathrm{L}}}\,\right]. \tag{10.56}$$

From relation (10.48) one computes $\sin\theta^L$, $\cos\theta^L$ in terms of the trigonometric functions of θ^*, therefore relation (10.56) gives the three-momentum of the particle in the lab frame Σ^L, if the four-momentum is known in the CM frame Σ^*.

10.7.1 Radiative Transitions

One important application of the results of the last section occurs, when one of the produced particles is a photon. These reactions are of the general form[4]

$$A \to B + \gamma,$$

where A, B are some sets of particles or a single particle. Such reactions are the excitation and de-excitation of atoms, various decay reactions of particles (e.g., $\Sigma^0 \to \Lambda + \gamma$, $\pi^+ \to \mu^+ + \nu_\mu$), etc. The study of these reactions is done in the CM frame, which coincides with the proper frame of the mother particle, or in the lab frame, in which the mother particle A has three-momentum $\mathbf{p}$. The three-dimensional schematic representation of the reaction is shown in Fig. 10.9.

Without making any further calculations, we use relations (10.14), (10.15), (10.16), and (10.17) and write the energies of the particle B and the photon in the CM frame:

$${}^{\gamma}_{A}E = \frac{m_A^2 - m_B^2}{2m_A}c^2, \qquad {}^{B}_{A}E = \frac{m_A^2 + m_B^2}{2m_A}c^2, \tag{10.57}$$

$${}^{\gamma}_{A}P = {}^{B}_{A}P = \frac{c}{2m_A}\lambda(m_A, m_B, m_\gamma) = {}^{\gamma}_{A}E/c = \frac{m_A^2 - m_B^2}{2m_A}c. \tag{10.58}$$

From the above relations it is clear that for the reaction to be possible, the mass of the mother particle A must be greater than the mass of the daughter particle B. The difference of the masses $\Delta m = m_A - m_B$ is called *proper mass loss* and equals the kinetic energy of the products in the CM frame. Indeed, we calculate for particle B

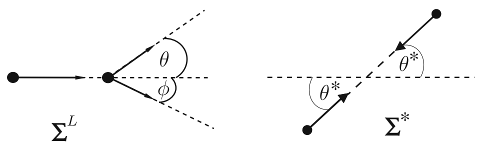

Fig. 10.9 Decay in the lab frame and in the CM frame

[4] $m_A \neq 0$, see Example 43.

$$ {}^{B}_{A}T = {}^{B}_{A}E - m_B c^2 = \frac{c^2}{2m_A}(m_A - m_B)^2 $$

and for the photon

$$ {}^{\gamma}_{A}T = {}^{\gamma}_{A}E = \frac{m_A^2 - m_B^2}{2m_A} c^2. $$

Therefore, the total kinetic energy ${}_{A}T_{\text{total}}$ of the products in the CM frame is

$$ {}_{A}T_{\text{total}} = {}^{B}_{A}T + {}^{\gamma}_{A}T = \frac{c^2}{2m_A}\left[(m_A - m_B)^2 + m_A^2 - m_B^2\right] = (m_A - m_B)c^2. \quad (10.59) $$

Finally, concerning the factor ${}^{B}_{A}\beta$ of the particle B in the CM frame we have

$$ {}^{B}_{A}\beta = \frac{{}^{B}_{A}Pc}{{}^{B}_{A}E} = \frac{m_A^2 - m_B^2}{m_A^2 + m_B^2}. \quad (10.60) $$

In order to compute the corresponding quantities in the laboratory frame we apply the boost relating the two frames with velocity factor ${}^{A}_{L}\beta = \beta_A$ of the particle A in the lab frame. For particle B we have

$$ {}^{B}_{L}E = \gamma_A\left({}^{B}_{A}E + \beta_A\,{}^{B}_{A}Pc\cos\theta^*\right). $$

But ${}^{B}_{A}Pc = {}^{\gamma}_{A}Pc = {}^{\gamma}_{A}E$, therefore

$$ {}^{B}_{\mathrm{L}}E = \gamma_A\left({}^{B}_{A}E + \beta_A\,{}^{\gamma}_{A}E\cos\theta^*\right). \quad (10.61) $$

For the photon,

$$ {}^{\gamma}_{\mathrm{L}}E = \gamma_A\left({}^{\gamma}_{A}E - \beta c\,{}^{\gamma}_{A}P\cos\theta^*\right) = \gamma_A\,{}^{\gamma}_{A}E\left(1 - \beta\cos\theta^*\right). \quad (10.62) $$

Using the energy ${}^{B}_{\mathrm{L}}E$ we compute the three-momentum

$$ {}^{B}_{L}Pc = \sqrt{{}^{B}_{L}E^2 - m_B^2 c^4} $$

while for the photon we have

$$ {}^{\gamma}_{L}Pc = {}^{\gamma}_{L}E. $$

The angular quantities of the daughter particles remains to be computed. In order to compute the total angle $\theta+\phi$ in the lab frame, we square the conservation relation:

$$ m_A^2 c^2 = m_B^2 c^2 - 2\left(-{}^{B}_{L}E\,{}^{\gamma}_{L}E + {}^{B}_{L}P\,{}^{\gamma}_{L}Pc^2\cos(\theta+\phi)\right) $$

and solve for $\cos(\theta+\phi)$. The result is

$$\cos(\theta+\phi)=\left[{}^{B}_{L}E\,{}^{\gamma}_{L}E-\frac{m_A^2-m_B^2}{2}c^2\right]\left({}^{B}_{L}P\,{}^{\gamma}_{L}Pc^2\right)^{-1}.$$

This relation calculates the angle $\theta+\phi$ in terms of the energies and the three-momenta of the daughter particles in the lab frame.

In order to compute the angles in the lab frame, we apply relations (10.48) and (10.49). For particle B we have

$$\cot\theta^L=\gamma_A\left(\cot\theta^*+\frac{r_B^*}{\sin\theta^*}\right),$$

where $r^*=\beta_A/{}^{B}_{A}\beta$. Similarly, for the photon we find

$$\cot\phi^{\mathrm{L}}=\gamma_A\left(\cot\phi^*+\frac{r_\gamma^*}{\sin\phi^*}\right)=\gamma_A\left(\cot\phi^*-\frac{\beta_A}{\sin\phi^*}\right).$$

The maximum angle between the direction of emission of the photon and the direction of motion of the mother particle in the lab frame is given by the relation (10.51):

$$\sin\phi^L_{\max}=\frac{{}^{\gamma}_{A}P}{{}^{\gamma}_{L}P}=\frac{{}^{\gamma}_{A}E}{{}^{\gamma}_{L}E}=\frac{1}{\gamma_A\ (1-\beta\cos\theta^*)}.$$

Example 49 An excited atom of mass m^* makes a transition from the excited state to the ground state of mass m by the emission of a photon. Subsequently, this photon collides with a similar atom, which is in the ground state. Determine the condition on the energy of the second particle in the ground state in order to make a transition to the excited state.

Solution

The reactions are $1\to\gamma+2$ and $3+\gamma\to4$ where 1, 4 are the atoms of mass m^* in the excited state and 2, 3 are the atoms of mass m in the ground state. If we denote by $p_I,\ I=1,2,3,4$, the four-momenta of the corresponding particles, we have the conservation equations

$$p_1^i=p_\gamma^i+p_2^i,\qquad p_\gamma^i+p_3^i=p_4^i.$$

The energy of the photon in the proper frame of atom 1 is (see (10.16) or write the reaction in the form $p_1^i-p_\gamma^i=p_2^i$ and square)

$$E_\gamma=\frac{m^{*2}-m^2}{2m^*}\,c^2. \tag{10.63}$$

Squaring the second conservation equation and using (10.63) we find

$$2p_{\gamma}^{i} p_{3i} = (m^2 - m^{*2})c^2 = -2m^* E_{\gamma}. \tag{10.64}$$

Assume that in the proper frame Σ_1 of particle 1 the four-momenta $p_{\gamma}^{i} = \frac{E_{\gamma}}{c}\begin{pmatrix}1\\ \hat{\mathbf{e}}\end{pmatrix}_{\Sigma_1}$, $p_3^i = \begin{pmatrix}E_3/c\\ \mathbf{p}_3\end{pmatrix}_{\Sigma_1}$, where $\hat{\mathbf{e}}$ is the direction of propagation of the photon in Σ_1. Then (10.64) gives

$$\hat{\mathbf{e}} \cdot \mathbf{p}_3 = \frac{1}{c}(E_3 - m^* c^2). \tag{10.65}$$

Let $\mathbf{p}_{3\perp}$ be the component of $\mathbf{p}_3$ normal to $\hat{\mathbf{e}}$ and $\mathbf{p}_{3\parallel} = \frac{1}{c}(E_3 - m^* c^2)$ the component parallel to the direction $\hat{\mathbf{e}}$. We have

$$E_3^2 - \mathbf{p}_{3\perp}^2 c^2 - \mathbf{p}_{3\parallel}^2 c^2 - m^2 c^2 = 0 \Rightarrow$$
$$E_3^2 - \mathbf{p}_{3\parallel}^2 c^2 - m^2 c^2 \geqslant 0.$$

Replacing $\mathbf{p}_{3\parallel}$ we find the required condition:

$$E_3 \geqslant \frac{m^{*2} + m^2}{2m^*} c^2.$$

We conclude that the transition is possible only if the atom 3 has a non-vanishing speed in the proper frame of the initial excited atom 1. To find the γ-factor of this motion we write $E_3 = m\gamma_3 c^2$ and have

$$\gamma_3 \geqslant \frac{m^{*2} + m^2}{2mm^*}.$$

The corresponding β-factor is computed to be

$$\beta > \frac{(m^* - m)(m^* + m)}{m^{*2} + m^2}.$$

To obtain a physical estimate of the result, we consider the excitation energy $\Delta E = (m^* - m)c^2$ and have

$$\beta > \frac{(\Delta E/c^2)(m^* + m)}{m^{*2} + m^2}.$$

If the excitation energy $\Delta E \ll mc^2$, we can approximate $m^* \simeq m$ and the above relation reduces to

$$\beta > \frac{\Delta E}{mc^2} = \frac{h\nu}{mc^2},$$

where ν is the frequency of the emitted photon.

10.7.2 Reactions With Two-Photon Final State

The reactions with two-photon final state[5] are important in practice and include the positron–electron annihilation $e^+ + e^- \rightarrow \gamma + \gamma$, the two-photon decay of the pion $\pi^0 \rightarrow \gamma + \gamma$, which is used in experimental particle research to recognize photons from π^0 or η^0 decay against a background of uncorrected photons. In the case of two-photon final state, the results of Sect. 10.7.1 do not apply and we have to consider this case as a new one.

We consider the reaction

$$\begin{matrix} A & \rightarrow & \gamma & + & \gamma \\ 1 & & 2 & & 3 \end{matrix},$$

where A is the CM particle. For example, we consider the reaction $e^+ + e^- \rightarrow \gamma + \gamma$ in the following two stages:

$$e^+ + e^- \rightarrow A \rightarrow \gamma + \gamma$$

and assume that in the lab frame the CM particle A describes the system of the particles e^+, e^-. This consideration is not obligatory but, as a rule, it is the one recommended, because it leads to the result with the use of the general results on the generic reaction $A + B \rightarrow C$.

We consider the electron to be at rest in the lab frame when a positron of energy E_1 (in the lab frame) interacts producing two photons. Let us assume that in the CM frame the direction of propagation of photons creates an angle θ^* with the common direction of e^+, e^- (see Fig. 10.9). First we compute the various parameters of the CM particle and then the angle between the directions of the photons in the lab frame.

We consider the first part of the reaction (i.e., $e^+ + e^- \rightarrow A$) and we have for the mass M of the CM particle

$$p_1^i + p_{2i} = p_A^i \Rightarrow p_1^i p_2^i = \frac{1}{2}\left(-M^2 + 2m^2\right)c^2.$$

[5] Reactions with products $\nu + \gamma$ and $\nu + \nu$ have not been observed and seem to be forbidden by a quantal conservation law.

Because the electron rests in the lab we have $p_1^i p_{2i} = -E_1 m$. Replacing we find

$$M = m\sqrt{2(1+\gamma_1)},$$

where $\gamma_1 = \frac{E_1}{mc^2}$ is the γ-factor of the positron in the lab frame.

Next we compute the β^*-factor of the CM particle in the lab frame. We have

$$\beta^* = \frac{\mid \mathbf{p}_1 + \mathbf{p}_2 \mid c}{E_1 + mc^2} = \frac{m\beta_1\gamma_1 c^2}{m\gamma_1 c^2 + mc^2} = \frac{\beta_1\gamma_1}{1+\gamma_1} = \sqrt{\frac{\gamma_1 - 1}{\gamma_1 + 1}}.$$

Also

$$\gamma^* = 1/\sqrt{1-\beta^{*2}} = \sqrt{\frac{\gamma_1 + 1}{2}}.$$

The energies E_3^*, E_4^* of the photons in the CM frame are equal (to show this simply consider the projection of the three-momentum of the photons normal to the direction of motion of the mother particle in the CM frame). Let $E_3^* = E_4^* = E^*$ be the common value of the energy. To compute E^*, we consider relation (10.14) for the right part of the reaction (i.e., $A \to \gamma + \gamma$) and write

$$E^* = E_3^* = \frac{M^2 - 0 - 0}{2M}c^2 = \frac{Mc^2}{2} = mc^2\sqrt{\frac{\gamma_1 + 1}{2}}.$$

To calculate the energies of the photons in the lab frame we consider the x-axis along the direction of motion of the positron and use the boost from the CM frame to the lab frame. We find

$$\frac{E_3}{c} = \gamma^*\left(\frac{E_3^*}{c} + \beta^* p_{3,x}\right) = \gamma^*\frac{E^*}{c}\left(1 + \beta^*\cos\theta^*\right).$$

Similarly, for the other photon we compute

$$\frac{E_4}{c} = \gamma^*\frac{E^*}{c}\left(1 - \beta^*\cos\theta^*\right).$$

We note that the ratio of the energies is

$$\frac{E^*}{E_3} = \frac{1}{\gamma^*(1+\beta^*\cos\theta^*)}, \tag{10.66}$$

$$\frac{E^*}{E_4} = \frac{1}{\gamma^*(1-\beta^*\cos\theta^*)}. \tag{10.67}$$

The angle θ^L between the directions of the photons in the lab frame remains to be computed. For this, we write the four-momenta p_3^i, p_4^i in the lab frame and compute the inner product $p_3^i\, p_{4i}$. We have

$$p_3^i = \frac{E_3}{c}\begin{pmatrix}1\\ \mathbf{e}_3\end{pmatrix}_{\mathrm{L}}, \qquad p_4^i = \frac{E_4}{c}\begin{pmatrix}1\\ \mathbf{e}_4\end{pmatrix}_{\mathrm{L}},$$
$$p_3^i p_{4i} = \frac{E_3E_4}{c^2}\left(-1+\cos\theta^{\mathrm{L}}\right).$$

In the CM frame we have

$$p_3^i p_{4i} = \frac{E^{*2}}{c^2}\left(-1+\cos\pi\right) = -\frac{2E^{*2}}{c^2}.$$

Equating the two results and solving for θ^L we get

$$-2\sin^2\frac{\theta^L}{2} = -\frac{2E^{*2}}{E_3E_4} \Rightarrow \sin\frac{\theta^L}{2} = \sqrt{\frac{E^{*2}}{E_3E_4}}.$$

We replace the ratios $\frac{E^*}{E_3}$, $\frac{E^*}{E_4}$ from (10.66) and (10.67) and have the final answer:

$$\sin\frac{\theta^L}{2} = \frac{1}{\gamma^*\sqrt{\left(1-\beta^{*2}\cos^2\theta^*\right)}}. \tag{10.68}$$

We note that when $\theta^* = \pi/2$ (that is, in the CM frame the directions of the photons are perpendicular to the direction of motion of the electron and the positron)

$$\sin\frac{\theta^L}{2} = \frac{1}{\gamma^*} = \sqrt{\frac{2}{1+\gamma_1}}. \tag{10.69}$$

In this special case, we also have $E_3 = E_4$ (why?) so that the photons are emitted in the lab symmetrically to the direction of motion of the positrons and with equal frequency (color). Finally for the value $\gamma_1 = 3$, the angle $\theta^L = \pi/2$ and the photons in the lab frame are propagating at 45° to the direction of motion of the positrons in the lab frame.

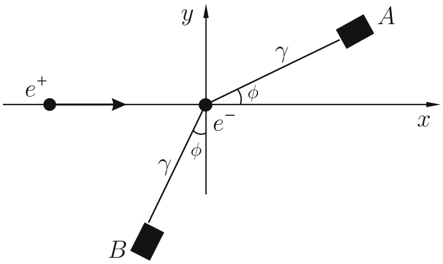

Fig. 10.10 Annihilation e^+–e^-

Example 50 (Isotropy and constancy of c) A beam of positrons of energy E hits a target of electrons, which is resting in the lab. From the scattering, photons are produced, which are detected with two counters A, B placed in the plane x–y at equal distances from the target and making an angle ϕ with the axes x and y, respectively (see Fig. 10.10).

1. Calculate the energy E as a function of the angle ϕ (and the mass m of the electron). What happens when the detectors A, B are placed at an angle $\phi = 45°$?
2. Consider the pair e^+, e^- as a source of photons and show how it is possible to demonstrate that the speed of light is isotropic and independent of the speed of the emitter.

Solution

1. The reaction is

$$\begin{array}{ccccc} e^+ & + e^- & \to & \gamma & + \gamma \\ 1 & +2 & & 3 & 4 \end{array}.$$

In the lab frame (= proper frame of the electron) the energy of particle 1 is

$$ {}^1_L E = E_1 = \frac{M^2 - 2m^2}{2m} c^2, \tag{10.70}$$

where M is the mass of the CM particle. Squaring the conservation equation of four-momentum, we find the mass M in terms of the scalar product $p_3^i \cdot p_4^i$:

$$ - M^2 c^2 = 2 p_3^i \cdot p_4^i. \tag{10.71}$$

Conservation of three-momentum along the y-axis gives

$$E_3 \sin\phi = E_4 \cos\phi \Rightarrow E_4 = E_3 \tan\phi. \tag{10.72}$$

Conservation of energy in the lab frame gives

$$E_1 + mc^2 = E_3 + E_4 \Rightarrow E_3 = \frac{E_1 + mc^2}{1 + \tan\phi}. \tag{10.73}$$

In order to compute M we decompose p_3^i, p_4^i in the lab frame:

$$p_3^i = (E_3/c) \begin{pmatrix} 1 \\ \cos\phi \\ \sin\phi \\ 0 \end{pmatrix}_L, \quad p_4^i = (E_4/c) \begin{pmatrix} 1 \\ -\sin\phi \\ -\cos\phi \\ 0 \end{pmatrix}_L$$

and replace the unknown energies E_3, E_4 from relations (10.72), (10.73). The result is

$$p_3^i \cdot p_4^i = -\frac{E_3^2}{c^2}\tan\phi(1+\sin 2\phi) = -\frac{1}{2c^2}\sin 2\phi\left(E_1 + mc^2\right)^2.$$

We replace the inner product $p_3^i \cdot p_4^i$ in (10.71) and find the mass:

$$M^2 = \sin 2\phi\frac{\left(E_1 + mc^2\right)^2}{c^4}.$$

Therefore, the energy of the photons in the lab frame in terms of the angle ϕ is

$$E_1 = \frac{2-\sin 2\phi}{\sin 2\phi}mc^2. \tag{10.74}$$

The factor $\gamma(\phi)$ of the positrons in the lab frame is

$$\gamma(\phi) = \frac{E_1}{mc^2} = \frac{2-\sin 2\phi}{\sin 2\phi}.$$

For $\phi = 45°$ we find $\gamma(45°) = 1$, that is, the positron reacts at rest in the lab frame (threshold of the reaction) and the energy has its minimal value $E_1 = mc^2$. [Exercise: Compute the derivative $\frac{dE_1}{d\phi}$ and verify that the minimum value of the energy E_1 occurs for the value $\phi = 45°$.]

From (10.72), (10.73) we compute $E_3(45°) = E_4(45°) = mc^2$, which means that the emitted photons have the same frequency, which is determined by the mass of the electron only. Kinematically the angle ϕ measures the β-factor of the positron in the lab frame and the value 45° corresponds to the threshold of the reaction.

2. We consider the pair e^+, e^- as a source of photons (photon emitter), whose velocity is the velocity of the CM particle. Then we have the following situation:

- For the various values of the angle ϕ the detectors A, B detect photons with energies E_3, E_4, which are fixed uniquely by the value of the angle ϕ and the energy E_1 of the positrons in the lab frame.
- The speed of the photon emitter in the lab is determined by the energy E_1.
- The detectors A, B are at equal distances from the photon emitter, therefore, if the speed of light is constant, the photons will be detected simultaneously at the detectors A, B (for all angles ϕ).

From the above, we conclude that in order to prove the independence of the speed of light from the velocity of the photon emitter, it is enough to measure the time difference between the photons received at the detectors.

This leads to the following experimental procedure. For a given energy E_1 of the positrons, we measure the frequency of events registered by the detectors and

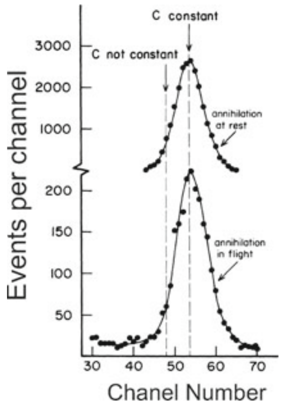

Fig. 10.11 Sandeh's experimental curves

determine the time difference for each event. If the speed of light is independent of the speed of the photon emitter, then most events for each energy E_1 (and angle ϕ) will occur at the value $\Delta t = 0$ (simultaneous events). The value $\phi = 45°$ has no special significance, besides the fact that it corresponds to the minimum value of the energy E_1, therefore it is meaningless to continue the measurements for angles larger than $45°$.

This experiment has been realized by Sandeh[6] for various values of the positron energy in the lab. The experimental measurements (see Fig. 10.11) did indeed show that most events occur for simultaneous events ($\Delta t = 0$). If we rotate the whole measuring apparatus around the z-axis and repeat the experimentation, then we find the same result. This proves the isotropy of the speed of light.

10.7.3 Elastic Collisions – Scattering

We say that a reaction or particle collision is elastic if the number and the identity of the reacting particles are preserved. The most well-known elastic scattering is the Compton scattering, in which a beam of X-rays is scattered on a source of free electrons according to the reaction

$$e^- + \gamma \to \gamma + e^-.$$

Other well-known reactions of elastic scattering are

- Rutherford scattering:

$$a + \text{nucleus} \to a + \text{nucleus at the same state.}$$

[6] D. Sandeh Phys. Rev. Lett. **10**, 271, (April 1983).

- Proton–proton scattering without the production of other particles:

$$p + p \to p + p.$$

In this reaction it is not possible to relate a given mother proton with a specific daughter proton, that is which initial proton corresponds to which final. We can only say that two protons are reacting and two protons are emerging from the reaction.

- Elastic collision of pion with proton:

$$\pi^- + p \to \pi^- + p.$$

The physical quantities we wish to find in an elastic collision are the energy and the angle of scattering of the daughter particles in the laboratory, in terms of the energy of the bullet beam of particles. In an elastic scattering, as a rule, the lab frame coincides with the proper frame of the target particle. However, in the study of elastic collisions in space, the lab frame is the frame of distant stars. Such collisions are the elastic scattering of particles of cosmic radiation of very high energy on thermal photons, which are radiated from stellar objects. During these collisions it is possible that these particles lose most of their energy, whereas the scattered photons increase, respectively, their energy with the result of increase in their frequency from the visible to γ-ray frequencies. Similar scattering phenomena occur in accelerators with the head-on collision of laser beams with a beam of accelerating electrons.

In order to study the elastic scattering, we write the reaction in two stages with the intermediate CM particle (Fig. 10.12):

$$\begin{array}{ccccccccc} A & + & B & \to & P & \to & A & + & B \\ 1 & & 2 & & & & 3 & & 4 \end{array}$$

and assume that the lab frame coincides with the proper frame of particle 2. We compute first the quantities of the CM particle P. For the mass M we have

$$ {}_2^1E = E_1 = \frac{M^2 - m_1^2 - m_2^2}{2m_2}c^2 \Rightarrow M = \sqrt{m_1^2 + m_2^2 + 2m_2 E_1/c^2}.$$

The β^*-factor of P in the lab frame is

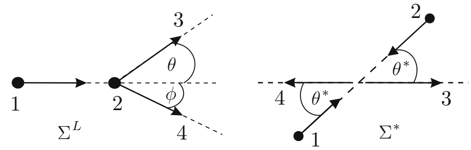

Fig. 10.12 Elastic collision in the lab frame and the CM frame

$$\boldsymbol{\beta}^* = \frac{\sqrt{E_1^2 - m_1^2 c^4}}{E_1 + m_2 c^2}\,\mathbf{i}$$

and the γ^*-factor

$$\gamma^* = \frac{E_1 + m_2 c^2}{M c^2}.$$

Next, we compute the energy and the scattering angle of the daughter particle 3 of mass m_1. From the conservation of four-momentum we have

$$\left(p_4^i\right)^2 = \left(p_1^i + p_2^i - p_3^i\right)^2. \tag{10.75}$$

In the lab frame, the decomposition of the four-vectors is

$$p_1^i = \begin{pmatrix} E_1/c \\ \mathbf{p}_1 \end{pmatrix}_L, \quad p_2^i = \begin{pmatrix} m_2 c \\ \mathbf{0} \end{pmatrix}_L, \quad p_3^i = \begin{pmatrix} E_3/c \\ \mathbf{p}_3 \end{pmatrix}_L, \quad p_4^i = \begin{pmatrix} E_4/c \\ \mathbf{p}_4 \end{pmatrix}_L.$$

We note that the angle θ_{13} is given by the expression

$$\mathbf{p}_1 \cdot \mathbf{p}_3 = P_1 P_3 \cos\theta_{13},$$

where $P_1,\ P_3$ are the lengths of the three-momenta of the particles $1,\ 3$ in the lab frame. Replacing in Eq. (10.75) and solving for cos θ_{13} we find

$$\cos\ \theta_{13} = \frac{-2m_1^2 c^4 + (E_3 - E_1)\, m_2 c^2 + E_1 E_3}{P_1 P_3 c^2}.$$

The unknown quantities in this equation are the quantities $P_1,\ P_3$. The length P_1 is computed from the relation

$$P_1 c = \sqrt{E_1^2 - m_1^2 c^4}$$

while for the calculation of P_3 we need to know the energy E_3 of particle 3 in the lab frame. This is computed as follows:

The length P_3^* of the three-momentum and the energy E_3^* of particle 3 in the CM frame are given by the relations

$$P_3^* = |{}_P^3\ \mathbf{p}\ | = \frac{1}{2M}\sqrt{\lambda(M, m_1, m_2)}, \qquad E_3^* = \sqrt{P_3^{*2} c^2 + m_1^2 c^4}.$$

The boost relating the lab and the CM frame gives for the energy E_3 of particle 3 in the lab: $E_3 = {}_L^3\ E = \gamma^* \left(E_3^* + \beta^* c P_3^* \cos\theta_{13}^*\right)$, where θ_{13}^* is the scattering angle in the CM frame and $\beta^*,\ \gamma^*$ are the velocity factors of the CM frame in the lab.

A different method to compute the scattering angle in the lab frame is by the use of the transformation equation (10.48) for the angles or, equivalently, to work as follows.

We decompose the three-momentum of particle 3 along and normal to the direction of motion of particle 1 in the lab frame. Then the boost between the lab frame and the CM frame gives

$$\cot\theta_{13} = \frac{{}^3_L\mathbf{p}_\parallel}{{}^3_L\mathbf{p}_\perp} = \frac{\Gamma_3^*\left(P_3^*\cos\theta_{13}^* + \beta^* E_3^*/c\right)}{P_3^*\sin\theta_{13}^*} = \Gamma_3^*\left(\cot\theta_{13}^* + \frac{\beta^*}{B_3^*\sin\theta_{13}^*}\right),$$

where $B_3^*,\ \Gamma_3^*$ are the $\beta,\ \gamma$-factors of particle 3 in the CM frame and are computed as follows:

$$\mathbf{B}_3^* = \frac{P_3^*}{E_3^*}\left(\cos\theta_{13}^*\mathbf{i} + \sin\theta_{13}^*\mathbf{j}\right), \qquad \Gamma_3^* = \frac{E_3^*}{m_1}.$$

Exercise 41 Consider two particles of non-zero mass, which collide elastically, and show that in the CM frame the speeds of particles remain unchanged after the collision.

Example 51 (Compton scattering) A photon is scattered elastically on an electron, which rests in the laboratory. Calculate the energy of the scattered particles in the laboratory as functions of the energy of the incident photon and the scattering angle of the photon in the lab frame (Fig. 10.13).

Solution

The reaction is

$$\underset{1}{\gamma} + \underset{2}{e^-} \rightarrow \underset{3}{\gamma} + \underset{4}{e^-}.$$

Conservation of four-momentum gives $p_4^i = p_1^i + p_2^i - p_3^i$, which after squaring becomes

$$p_1^i p_{2i} - p_1^i p_{3i} - p_2^i p_{3i} = 0.$$

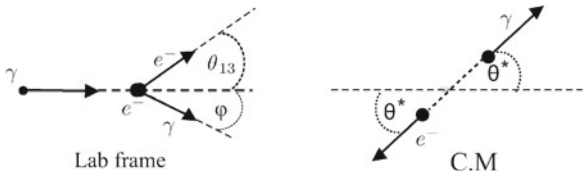

Fig. 10.13 Compton scattering in the lab frame and the CM frame

The decomposition of the four-momenta in the lab frame is

$$p_1^i = \frac{E_1}{c}\begin{pmatrix}1\\ \mathbf{i}\end{pmatrix}_L, \quad p_2^i = \begin{pmatrix}mc\\ \mathbf{0}\end{pmatrix}_L, \quad p_3^i = \frac{E_3}{c}\begin{pmatrix}1\\ \hat{\mathbf{e}}_3\end{pmatrix}_L, \quad p_4^i = \begin{pmatrix}E_4/c\\ \mathbf{p}_4\end{pmatrix}_L.$$

We compute the inner products in the last equation and solving in terms of E_3, we find

$$E_3 = \frac{mc^2}{E_1(1-\cos\theta_{13}) + mc^2} E_1,$$

where θ_{13} is the scattering angle. This relation gives the energy E_3 in terms of the required quantities E_1, θ_{13}. We introduce the quantity $\mathcal{E} = mc^2/E_1$ and have the dimensionless ratio

$$\frac{E_3}{E_1} = \frac{\mathcal{E}}{1+\mathcal{E}-\cos\theta_{13}} = \frac{\mathcal{E}}{\mathcal{E}+2\sin^2\frac{1}{2}\theta_{13}}. \tag{10.76}$$

For the energy E_4 of the scattered electron in the lab frame we have

$$E_4 = E_1 + mc^2 - E_3 = \frac{\left(E_1+mc^2\right)\left[E_1(1-\cos\theta_{13}) + mc^2\right] - mc^2 E_1}{E_1(1-\cos\theta_{13}) + mc^2}.$$

We note that the ratio

$$\frac{E_4}{E_3} = -1 + \frac{(\mathcal{E}+1)(1+\mathcal{E}-\cos\theta_{13})}{\mathcal{E}}. \tag{10.77}$$

The angle θ_{34} between the scattered particles remains to be computed in terms of the quantities E_1, θ_{13} or $\mathcal{E}$, θ_{13}. We consider the equation of conservation of four-momentum in the form

$$p_1^i + p_2^i = p_3^i + p_4^i$$

and squaring we get

$$p_1^i p_{2i} = p_3^i p_{4i}.$$

We calculate the inner products in the lab and find

$$-E_1 m = -\frac{E_3 E_4}{c^2} + \frac{E_3}{c} P_4 \cos\theta_{34}.$$

We solve this equation in terms of $\cos\theta_{34}$ and we write the result in terms of the ratios $\frac{E_3}{E_4}$ and $\frac{E_3}{E_1}$:

$$\cos\theta_{34} = \frac{E_3 E_4 - E_1 mc^2}{E_3\sqrt{E_4^2 - m^2c^4}} = \frac{E_4/E_3 - (E_1/E_3)^2\mathcal{E}}{\sqrt{\frac{E_4^2}{E_3^2} - \frac{\mathcal{E}^2}{E_3^2/E_1^2}}}.$$

Replacing $\frac{E_3}{E_4}$, $\frac{E_3}{E_1}$ from (10.76) and (10.77) we calculate the angle θ_{34} in terms of the quantities $\mathcal{E}$, θ_{13}.

A different way to compute the angle θ_{34} is to compute the recoil angle ϕ of the electron. This is done as follows.

First we note that the three-momenta of the involved particles are coplanar (why?). Then, conservation of three-momentum along and normal to the direction of the incident photon gives the equations

$$|\mathbf{p}_3|\sin\theta_{13} = |\mathbf{p}_4|\sin\phi,$$
$$|\mathbf{p}_1| = |\mathbf{p}_3|\cos\theta_{13} + |\mathbf{p}_4|\cos\phi.$$

Dividing these equations we find

$$\tan\phi = \frac{|\mathbf{p}_3|\sin\theta_{13}}{|\mathbf{p}_1| - |\mathbf{p}_3|\cos\theta_{13}} = \frac{\frac{E_3}{E_1}\sin\theta_{13}}{1 - \frac{E_3}{E_1}\cos\theta_{13}},$$

where we have used the fact that $|\mathbf{p}_1| = \frac{E_1}{c}$, $|\mathbf{p}_3| = \frac{E_3}{c}$. Replacing $\frac{E_3}{E_1}$ from (10.76) and applying standard trigonometry we find (Fig. 10.13)

$$\tan\phi = \frac{\cot\frac{1}{2}\theta_{13}}{1 + \frac{1+\mathcal{E}}{\mathcal{E}}}.$$

We note that $\tan\phi$ is always positive, therefore the electron is always thrown forward.

Finally, let us compute the electron's recoil speed. Conservation of energy gives

$$E_4 = E_1 + mc^2 - E_3 = \frac{mc^2}{\mathcal{E}} + mc^2 - \frac{\mathcal{E}}{\mathcal{E} + 2\sin^2\frac{1}{2}\theta_{13}}\frac{mc^2}{\mathcal{E}}$$
$$= \frac{1 + \frac{2(1+\mathcal{E})}{\mathcal{E}^2}\sin^2\frac{1}{2}\theta_{13}}{1 + \frac{2}{\mathcal{E}}\sin^2\frac{1}{2}\theta_{13}}mc^2.$$

But $E_4 = mc^2\gamma_4$, where γ_4 is the γ-factor of the recoil electron in the lab frame. Therefore

$$\gamma_4 = \frac{1 + \frac{2}{\mathcal{E}}\left(1 + \frac{1}{\mathcal{E}}\right)\sin^2\frac{1}{2}\theta_{13}}{1 + \frac{2}{\mathcal{E}}\sin^2\frac{1}{2}\theta_{13}}. \tag{10.78}$$

From this the β-factor is computed as

$$\beta_4 = \frac{\frac{2}{\mathcal{E}} \sin \frac{1}{2}\theta_{13}\sqrt{1 + \frac{2\mathcal{E}+1}{\mathcal{E}^2} \sin^2 \frac{1}{2}\theta_{13}}}{1 + \frac{2(1+\mathcal{E})}{\mathcal{E}^2} \sin^2 \frac{1}{2}\theta_{13}}.$$

Exercise 42 Determine the conditions for which the photon is scattered opposite to its initial direction (back scattering). Note that in this case $\theta_{13} = \theta_{34} = \pi$. Similarly study the forward scattering for which $\theta_{13} = \theta_{34} = 0$. Show that in the forward scattering the energy of the scattered photon is maximum and equal to E_1, whereas for the backward scattering it has the minimum value $\frac{\mathcal{E}}{2+\mathcal{E}} E_1$. Also show that the velocity of the recoil electron ranges from zero for the forward scattering to the maximum value $\beta_{4,\ \max} = \frac{2(1+\mathcal{E})}{E^2+2\mathcal{E}+2}$.

Example 52 A particle of mass m and kinetic energy T in the laboratory is scattered elastically on another similar particle which rests in the laboratory. Calculate the kinetic energy of the scattered particle in terms of the kinetic energy of the bullet particle and the scattering angle. Determine the maximal value of the scattering angle.

Solution

The reaction is

$$\begin{array}{ccccccc} A & + & B & \rightarrow & A & + & B \\ 1 & & 2 & & 3 & & 4 \end{array}.$$

Conservation of four-momentum gives

$$p_4^i = p_1^i + p_2^i - p_3^i$$

and upon squaring

$$m^2c^2 = p_1^i p_{2i} - p_1^i p_{3i} - p_2^i p_{3i}.$$

We calculate the inner products in the lab frame and find the equation

$$\begin{aligned} m^2c^2 &= -E_1 m - \mathbf{p}_1 \cdot \mathbf{p}_3 + \frac{E_3}{c}\left(\frac{E_1}{c} + mc\right) \Longrightarrow \\ \mathbf{p}_1 \cdot \mathbf{p}_3 &= (E_3 - mc^2)(E_1 + mc^2)/c^2. \end{aligned}$$

But

$$c^2\mathbf{p}_1 \cdot \mathbf{p}_3 = p_1 p_3 c^2 \cos\theta_{13} = \sqrt{E_1^2 - m^2c^4}\sqrt{E_3^2 - m^2c^4} \cos\theta_{13}.$$

Replacing in the last relation, we find

$$\begin{aligned} \left(E_1^2 - m^2c^4\right)\left(E_3^2 - m^2c^4\right) \cos^2\theta_{13} &= \left(E_1 + mc^2\right)^2 \left(E_3 - mc^2\right)^2 \Rightarrow \\ \left(E_1 - mc^2\right)\left(E_3 + mc^2\right) \cos^2\theta_{13} &= \left(E_3 - mc^2\right)\left(E_1 + mc^2\right). \end{aligned}$$

The kinetic energies of the initial and the scattered particles in the lab frame are

$$T_1 = E_1 - mc^2, \qquad T_3 = E_3 - mc^2.$$

Replacing in the last relation, we find

$$T_3 = \frac{2mc^2 T_1 \cos^2\theta_{13}}{2mc^2 + T_1 \sin^2\theta_{13}}. \tag{10.79}$$

We note that the maximal value of the scattering angle θ_{13} occurs when $T_1 = T_3$, that is, when the kinetic energy of the bullet particle is transferred to the scattered particle. Setting $T_1 = T_3$ in (10.79) we find

$$\left(T_1 + 2mc^2\right)\sin^2\theta_{13} = 0 \Rightarrow \theta_{13} = 0^\circ.$$

Chapter 11
Four-Force

11.1 Introduction

In the previous chapters we considered four-vectors, which describe the evolution of a relativistic system in spacetime without taking into account the environment, which modulates motion. In the present chapter, we develop the dynamics of Special Relativity by introducing the four-vector of four-force. However, the real power of relativistic dynamics is via the introduction of the Lagrangian and the Hamiltonian formalism, which we shall also consider briefly. In what follows, we present a number of solved problems, which will familiarize the reader with simple applications of relativistic dynamics.

11.2 The Four-Force

Consider a ReMaP P with four-velocity u^i and mass $m(\tau)$, where τ is the proper time of P. We define the four-vector (= potential relativistic physical quantity)

$$F^i = \frac{d}{d\tau}(m(\tau)u^i) = \frac{dp^i}{d\tau}, \tag{11.1}$$

where $p^i = m(\tau)u^i$ is the four-momentum of P. The four-vector F^i we name *four-force* acting on P. Because P has only mass, the only reaction it can have is the change of the mass $\frac{dm}{d\tau}$. The remaining dynamical fields (electromagnetic field, forces due to other mechanical systems, e.g., springs) demand a "charge," therefore they have an effect on the four-acceleration only. This is the reason that gravity has a different role from the rest of the dynamical fields, i.e., interacts with the mass.[1] This becomes clear if we write the four-force as follows:

[1] Recall that gravity interacts with the gravitational mass, whereas the mass which enters Newton' s Second Law is the inertial mass. However, due to Eötwos experiments we identify these two masses (Equivalence Principle) in Newtonian Gravity. The same Principle is assumed to hold in General Relativity.

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$$F^i = \frac{dm}{d\tau}u^i + ma^i. \tag{11.2}$$

The first term is parallel to the four-velocity and the second normal to it. Each part we associate with a different Newtonian physical quantity. Indeed, as it will be shown, the timelike part concerns the total variation of the energy of P due to the action of an inertial three-force $\mathbf{f}$ (that is a force which changes the kinematics of P) and the spacelike part is defined in terms of $\mathbf{f}$.

Before we continue with the study of the four-force, we must prove that it corresponds to a relativistic physical quantity and it is not simply a four-vector without physical significance. To do this, we consider the proper frame Σ^+ of P and from (11.2) we obtain

$$F^i = \frac{dm}{d\tau}\begin{pmatrix} c \\ \mathbf{0} \end{pmatrix}_{\Sigma^+} + m\begin{pmatrix} 0 \\ \mathbf{a}^+ \end{pmatrix}_{\Sigma^+} = \begin{pmatrix} c\frac{dm}{d\tau} \\ m\mathbf{a}^+ \end{pmatrix}_{\Sigma^+}. \tag{11.3}$$

If in Σ^+ we identify the invariant $\frac{dm}{d\tau}$ with the Newtonian physical quantity $(\frac{dm}{dt})_{\Sigma^+}$ and the space part $m\mathbf{a}^+$ with the Newtonian three-force $\mathbf{f}_{\Sigma^+}$ then the components of the four-vector F^i attain physical significance, therefore this four-vector represents a relativistic physical quantity.

In order to understand the physical meaning of the two parts of the four-force in an arbitrary frame Σ, we consider the decomposition of F^i in Σ. If the four-momentum p^i in Σ is $\begin{pmatrix} E/c \\ \mathbf{p} \end{pmatrix}_{\Sigma}$, we have

$$F^i = \frac{dp^i}{d\tau} = \frac{dt}{d\tau}\frac{d}{dt}\begin{pmatrix} E/c \\ \mathbf{p} \end{pmatrix}_{\Sigma} = \gamma\begin{pmatrix} \dot{E}/c \\ \mathbf{f} \end{pmatrix}_{\Sigma}, \tag{11.4}$$

where $\mathbf{f} = \frac{d\mathbf{p}}{dt}$ is the three-force in Σ and $\dot{E} = \frac{dE}{dt}$. The quantity $\dot{E}$ is possible to be computed in two ways. Either from the relation

$$E = \sqrt{\mathbf{p}^2c^2 + m^2c^4} \Rightarrow \dot{E} = \frac{1}{2E}\left(2\mathbf{f}\cdot\mathbf{p}c^2 + 2m\dot{m}c^4\right) = \mathbf{f}\cdot\mathbf{u} + \frac{c^2}{\gamma^2}\frac{dm}{d\tau} \tag{11.5}$$

or by using the invariant F^iu_i. In this case we have

$$F^iu_i = \left(\frac{dm}{d\tau}u^i + ma^i\right)u_i = -\frac{dm}{d\tau}c^2. \tag{11.6}$$

In Σ the value of the invariant is

$$F^iu_i = \gamma\begin{pmatrix} \frac{1}{c}\dot{E} \\ \mathbf{f} \end{pmatrix}\cdot\gamma\begin{pmatrix} c \\ \mathbf{u} \end{pmatrix} = \gamma^2(-\dot{E} + \mathbf{f}\cdot\mathbf{u}). \tag{11.7}$$

Equating the two expressions and solving in terms of $\dot{E}$ we obtain again

$$\dot{E} = \frac{c^2}{\gamma^2}\frac{dm}{d\tau} + \mathbf{f}\cdot\mathbf{u}. \tag{11.8}$$

Verbally, the above equation can be stated as follows:

$$\begin{bmatrix}\text{Change of the}\\ \text{total energy of } P\\ \text{in } \Sigma\end{bmatrix} = \frac{c^2}{\gamma^2}\begin{bmatrix}\text{Change of}\\ \text{the internal energy}\\ \text{(mass) of } P\\ \text{in its proper frame}\end{bmatrix} + \begin{bmatrix}\text{Rate of production}\\ \text{of work in } \Sigma \text{ by}\\ \text{the external forces}\\ \text{acting on } P \text{ in } \Sigma.\end{bmatrix} \tag{11.9}$$

Equation (11.9) is the conservation of energy in Special Relativity. The new element is the relation of the (proper) mass of P with the work produced by the external force $\mathbf{f}$ acting on P in Σ.

From (11.2) follows that one is possible to classify the four-forces in a covariant manner using the vanishing or not of the quantity $F^i u_i = -\frac{dm}{d\tau}c^2$. The four-forces F^i, for which $\frac{dm}{d\tau} = 0$, we call *pure* or *inertial* four-forces and are the ones that create motion in three-dimensional space and correspond to the Newtonian forces. The second type of four-forces is defined by the requirement $a^i = 0$ and corresponds to four-forces which do not produce motion in the three-dimensional space. These four-forces we call *thermal*. The general expression of a thermal four-force is

$$F^i = \frac{dm}{d\tau}u^i. \tag{11.10}$$

To understand this type of force consider a ReMaP P, whose four-velocity is u^i and in its proper frame it is heated (e.g., by means of the flame of a candle) or is loosing mass as it is the case, e.g., with a rocket. Then $\frac{dm}{d\tau} \neq 0$ and on P acts a thermal force F^i. The three-momentum of P varies as follows:

$$d\mathbf{p} = dm\mathbf{u}$$

which implies that on P acts the three-force

$$\mathbf{f} = \frac{dm}{d\tau}\mathbf{u}. \tag{11.11}$$

The change of the energy $\dot{E}$ of P in Σ is given by (11.8). For a pure force we compute

$$\dot{E} = \mathbf{f}\cdot\mathbf{u}, \tag{11.12}$$

that is, we recover the corresponding expression of Newtonian Physics.

For a thermal force

$$\dot{E} = \frac{c^2}{\gamma^2}\frac{dm}{d\tau} + \mathbf{f}\cdot\mathbf{u} = \frac{c^2}{\gamma^2}\frac{dm}{d\tau} + \frac{dm}{d\tau}\mathbf{u}^2 = \frac{dm}{d\tau}c^2,$$

that is, the rate of change of energy is independent of Σ. In this case, the energy

$$E(t) = \int \frac{dm}{d\tau}c^2 dt + \text{constant}. \tag{11.13}$$

Let us discuss an example with a thermal four-force.

Example 53 A black (i.e., absolutely absorbing) surface is resting in the laboratory. At what rate photons of wavelength λ cm must fall normally to the surface in order to exert a three-force F N/m^2 normal to the surface?
Numerical application: $\lambda = 5\times 10^{-7}$ m, $F = 10^{-5}$ N, $h = 6.63\times 10^{-34}$ Js, $S = 0.01$ m^2.
First Solution

Let J be the number of photons that fall on the surface S per second and square meter. Because the surface is black, all photons are absorbed. This implies that per s and m^2 on the surface the photons transfer momentum

$$J\cdot\frac{h\nu}{c} = J\frac{h}{\lambda}.$$

The force F due to change of the three-momentum is

$$F = J\frac{h}{\lambda}S \Rightarrow J = \frac{F\lambda}{hS}.$$

Numerical application:

$$J = \frac{10^{-5}\text{N}\times 5\times 10^{-7}\text{m}}{6.63\times 10^{-34}\text{J s}\times 0.01\ \text{m}^2} = 7.54\times 10^{23}\text{photons}.$$

Second Solution [with the use of (11.13)].

Per unit of time (s) and surface (m^2) on the surface fall J photons of energy $E = h\nu = \frac{hc}{\lambda}$. Therefore, the increase of the mass of the surface per s and m^2 is

$$\frac{dm}{dt} = J\frac{hc}{\lambda c^2} = J\frac{h}{\lambda c}.$$

The thermal force, which is produced from this increase of proper mass equals

$$\mathbf{f} = \frac{dm}{d\tau}\mathbf{u} = Jc\frac{hS}{\lambda c}\hat{\mathbf{z}} = J\frac{hS}{\lambda}\hat{\mathbf{z}},$$

where $\hat{\mathbf{z}}$ is the unit normal to the surface and parallel to the direction of the falling photons.

In the following we shall consider pure forces only, because these are the ones that interest this book. This implies that in the sequel the following relations are assumed to hold:

$$F^i = m\,a^i \tag{11.14}$$

$$F^i = \gamma \begin{pmatrix} \frac{1}{c}\mathbf{f}\cdot\mathbf{u} \\ \mathbf{f} \end{pmatrix}_\Sigma \tag{11.15}$$

$$\dot{E} = \mathbf{f}\cdot\mathbf{u} = m\dot{\gamma}c^2. \tag{11.16}$$

If we replace in (11.14) F^i and a^i in terms of their components in an arbitrary LCF Σ, we find (11.16) and the equation

$$\mathbf{f} = m\dot{\gamma}\mathbf{u} + m\gamma\mathbf{a} \tag{11.17}$$

which defines the three-force $\mathbf{f}$ in terms of the three-acceleration $\mathbf{a}$. This expression can be simplified significantly. Indeed, from (7.4) we have $\dot{\gamma} = \frac{1}{c^2}\gamma^3(\mathbf{v}\cdot\mathbf{a})$ therefore replacing in (11.17) we find

$$\begin{aligned} \mathbf{f} &= m\gamma\left(\frac{1}{c^2}\gamma^2(\mathbf{v}\cdot\mathbf{a})\mathbf{v} + \mathbf{a}\right) = m\gamma\left(\frac{1}{c^2}\gamma^2 v a_\parallel v\mathbf{e}_\parallel + \mathbf{a}_\parallel + \mathbf{a}_\perp\right) \\ &= m\gamma\left[(\gamma^2\beta^2+1)\mathbf{a}_\parallel + \mathbf{a}_\perp\right] = m\gamma(\gamma^2\mathbf{a}_\parallel + \mathbf{a}_\perp). \end{aligned} \tag{11.18}$$

Exercise 43

(a) Show that (11.17) is compatible with (11.16).
[Hint: Make use of (11.8)].
(b) Show that (11.17) can be written as $\mathbf{f} = \frac{d\mathbf{p}}{dt}$, where $\mathbf{p} = m\gamma\mathbf{u}$, ($m$ = constant) and t is time in Σ. This result is used in the solution of problems, when the force $\mathbf{f}$ is given.

From (11.17) we note that, although the form of the four-force $F^i = ma^i$ has the form of Newton's Second Law, the corresponding three-force is related to three-acceleration in a different manner due to the presence of the extra term $m\dot{\gamma}\mathbf{u}$, which is an "apparent" force related to the three-velocity of P. This term never vanishes except if $\dot{\gamma} = 0$, in which case (prove this result) $a^i = 0$, therefore $\mathbf{f} = \mathbf{0}$. Furthermore, the three three-vectors $\mathbf{f}$, $\mathbf{u}$, $\mathbf{a}$ are coplanar, but the direction of $\mathbf{f}$ is different from that of $\mathbf{a}$.

This fact has led people to consider for the relativistic inertial three-forces two "masses" with reference to the direction of three-velocity. The parallel mass ($m_\parallel$) and the perpendicular mass ($m_\perp$.) The calculation of these masses is as follows:

Projecting (11.17) along $\mathbf{u}$ we find

$$\begin{aligned} f_{\|} = \mathbf{f} \cdot \mathbf{u} &= m\dot{\gamma} u^2 + m\gamma \mathbf{a} \cdot \mathbf{u} \\ &= m\gamma \left(\frac{u^2\gamma^2}{c^2} + 1 \right) \mathbf{a} \cdot \mathbf{u} \\ &= m\gamma^3 \mathbf{a} \cdot \mathbf{u} \end{aligned} \tag{11.19}$$

therefore

$$m_{\|} = m\gamma^3. \tag{11.20}$$

Working similarly we find for $m_{\perp}$

$$m_{\perp} = m\gamma. \tag{11.21}$$

A useful observation concerning the one-dimensional problems is the following. In these cases the three-force is collinear to the three-velocity, therefore from (11.19) we have for any type of force the condition

$$f = m\gamma^3 \frac{dv}{dt} = m\gamma^3 v \frac{dv}{dx} = \frac{1}{2} mc^2 \gamma^3 \frac{d\beta^2}{dx} = \frac{1}{2} mc^2 \gamma^3 \frac{d}{dx} \left(1 - \frac{1}{\gamma^2} \right) = mc^2 \frac{d\gamma}{dx}$$

hence

$$\int_1^2 f(x) dx = mc^2 (\gamma_2 - \gamma_1), \tag{11.22}$$

that is, for a one-dimensional motion under the action of any type of (smooth) force $f(x)$ the speed is given by the first integral (11.22).

Exercise 44 (*Conservation of mechanical energy*) Consider two events 1, 2 along the world line c_P of a ReMaP P. By making use of (11.16) show that for a pure force in an arbitrary LCF Σ

$$E_2 - E_1 = \int_{c_1}^{c_2} \mathbf{f} \cdot \mathbf{v} dt, \tag{11.23}$$

where $\mathbf{v}$ is the three-velocity of P in Σ. Making use of the relation $E = mc^2 + T$, where T is the kinetic energy of P in Σ, show that the change of kinetic energy in Σ is given by the expression

$$T_2 - T_1 = \int_{c_1}^{c_2} \mathbf{f} \cdot \mathbf{v} dt. \tag{11.24}$$

Conclude that in case m = constant, then in any LCF Σ the following Law of Conservation of Energy holds

$$\begin{bmatrix} \text{Change of the} \\ \text{Kinetic energy of } P \text{ in } \Sigma \\ \text{between the events} \\ 1,2 \text{ along its worldline} \end{bmatrix} = \begin{bmatrix} \text{Work done by} \\ \text{the external inertial} \\ \text{three-forces on } P \text{ in } \Sigma \text{ between} \\ \text{the events } 1,2 \end{bmatrix}. \tag{11.25}$$

Prove that in the case of one-dimensional motion relation (11.22) follows as a special case of (11.24).

Exercise 45 A ReMaP P is moving in the LCF Σ under the action of the three-force $\mathbf{f}$. Let Σ' be another LCF, whose axes are parallel to those of Σ and its velocity is $\mathbf{u}$. Prove that the three-force $\mathbf{f}$ and the rate of change of energy of P in Σ' are given by the following relations:

$$\mathbf{f}' = \frac{1}{\gamma_u Q}\left\{\mathbf{f} + \left[(\gamma_u - 1)\frac{\mathbf{u}\cdot\mathbf{f}}{u^2} - \frac{1}{c^2}\gamma_u\,(\mathbf{f}\cdot\mathbf{v})\right]\mathbf{u}\right\}, \tag{11.26}$$

$$\frac{dE'}{dt'} = \frac{1}{Q}\left(\frac{dE}{dt} - \mathbf{u}\cdot\mathbf{f}\right), \tag{11.27}$$

where $Q = \left(1 - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right)$ and $\mathbf{v}$ is the velocity of P in Σ.

In the special case Σ, Σ' are related with a boost along the direction of the common x-axis then the above relations read

$$f_x' = \frac{1}{Q}\left\{f_x - \frac{u}{c^2}(\mathbf{f}\cdot\mathbf{v})u\right\}, \tag{11.28}$$

$$f_y' = \frac{f_y}{\gamma_u Q}, \tag{11.29}$$

$$f_z' = \frac{f_z}{\gamma_u Q}. \tag{11.30}$$

[Hint: Apply the general Lorentz transformation (1.51) and (1.52) to the four-force].

A direct application of Exercise 45 is the following example:

Example 54 Let Σ and Σ' be two LCF with parallel axes and relative velocity $\mathbf{u}$. A ReMaP P is moving in Σ under the action of the three-force $\mathbf{f}$, which is always normal to the direction of the three-velocity of P in Σ.

(a) Show that the total energy of P in Σ is a constant of motion.
(b) Show that the energy E' of P in Σ' is not in general a constant of motion.
(c) Apply the result of (a) in order to show that the total energy of an electric charge, which moves in a LCF Σ under the action of a magnetic field only is a constant of motion.

Solution

(a) Let $\mathbf{v}$ be the velocity of P in Σ. From (11.16) we have in Σ

$$\dot{E} = \mathbf{f} \cdot \mathbf{v} = 0 \Rightarrow E = \text{constant in } \Sigma.$$

(b) Equation (11.27) gives

$$\frac{dE'}{dt'} = \frac{1}{Q}(-\mathbf{u} \cdot \mathbf{f}) \neq 0 \quad (\text{except if} \quad \mathbf{u} \cdot \mathbf{f} = 0).$$

(c) When an electric charge q moves in a LCF Σ, in which there is a magnetic field $\mathbf{B}$ only, the force on the charge is the Lorentz force $\mathbf{f} = kq\mathbf{v} \times \mathbf{B}$ (k is a constant depending on the units). This force is always normal to the velocity of the charge, therefore according to (a) the energy of the charge is a constant of motion in Σ. In Σ' the electromagnetic field will have in general and electric field[2] hence in Σ' the Lorentz force on the charge is $\mathbf{f}' = q(\mathbf{E}' + \mathbf{v}' \times \mathbf{B}')$, which is not normal to the velocity of the charge in Σ'. Therefore (in general), the total energy of the charge is not a constant of motion in Σ'. (Does this result contradicts (b)?)

In the following examples we show how one studies the motion of a ReMaP, which moves under the action of a given three-force.

Example 55 A particle P of mass m is moving along the x-axis of the LCF Σ under the action of the three-force ($a > 0$)

$$f = \frac{2mc^2 a}{(a-x)^2},$$

where a is a positive constant. Assuming that the motion starts from rest at the origin of Σ, prove that P is at the position x ($x < a$) along the x-axis of Σ after a time interval

$$t = \frac{1}{3c}\sqrt{\frac{x}{a}}(x + 3a).$$

[Hint: It is given that

$$\int \frac{xdx}{\sqrt{ax+b}} = \frac{2(ax-2b)}{3a^2}\sqrt{ax+b} + C \quad].$$

[2] This will be shown in Chap. 13.

Solution

The motion of P is one-dimensional, therefore (11.22) applies,[3] which gives in Σ taking into consideration the initial condition

$$\int_0^x f(x)dx = mc^2(\gamma - 1). \tag{11.31}$$

Replacing $f(x)$ in the lhs we compute

$$\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right)^2 = \left(\frac{a+x}{a-x}\right)^2 \Rightarrow v = 2c\frac{\sqrt{ax}}{a+x}.$$

Concerning the calculation of $x(t)$ we have

$$\frac{dx}{dt} = 2c\frac{\sqrt{ax}}{a+x} \Rightarrow \int_0^x \frac{a+x}{\sqrt{ax}}dx = 2c\int_0^t dt \Rightarrow$$
$$t = \frac{1}{3c}\sqrt{\frac{x}{a}}(x+3a).$$

Example 56 A particle of mass m is moving in the xy plane of the LCF Σ under the action of the constant force $\mathbf{F} = f\mathbf{j}$. If at $t = 0$ the particle starts to move from the origin with three-momentum $\mathbf{p}(0) = p_0\mathbf{i}$, show that the equation of motion of the particle in Σ is

$$y = \frac{E_0}{f}\left(-1 + \cosh\frac{fx}{cp_0}\right),$$

where $E_0 = \sqrt{(p_0^2c^2 + m^2c^4)}$.

Compute the Newtonian limit. How this result is related to the projectile motion of Newtonian Physics in a constant gravitational field?

Solution

The equation of motion of the particle in Σ is

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}$$

from which follows

$$\frac{dp_x}{dt} = 0, \qquad \frac{dp_y}{dt} = f, \qquad \frac{dp_z}{dt} = 0.$$

[3] The same result we find if we consider the conservation of energy given by relation (11.24).

The first and the last equation give

$$p_x(t) = p_x(0) = p_0$$
$$p_z(t) = p_z(0) = 0.$$

The second

$$p_y = ft + p_0(y) = ft.$$

Hence, the three-momentum is

$$\mathbf{p} = (p_0,\ ft,\ 0).$$

The three-velocity of the particle in Σ is

$$\begin{aligned}\mathbf{u} = \frac{\mathbf{p}}{m\gamma} = \frac{c^2}{E}\mathbf{p} &= \frac{c^2}{c\sqrt{m^2c^2 + p^2}}\mathbf{p}\\ &= \frac{c^2}{c\sqrt{m^2c^2 + p_0^2 + f^2t^2}}(p_0,\ ft,\ 0)\\ &= \frac{c^2}{\sqrt{E_0^2 + f^2t^2c^2}}(p_0,\ ft,\ 0),\end{aligned}$$

where E_0 is the total energy of the particle at $t = 0$. From this relation follows

$$\frac{d\mathbf{r}}{dt} = \frac{c^2}{\sqrt{E_0^2 + f^2t^2c^2}}(p_0,\ ft,\ 0) \qquad \Rightarrow$$
$$\mathbf{r} - \mathbf{r}_0 = ((x - x_0),\ (y - y_0),\ (z - z_0)) = \int_0^t \frac{c^2}{\sqrt{E_0^2 + f^2t^2c^2}}(p_0,\ ft,\ 0)dt.$$

Using the initial condition $\mathbf{r}(0) = 0$ we find

$$\begin{aligned}x(t) &= \int_0^t \frac{p_0c^2}{\sqrt{E_0^2 + f^2t^2c^2}}dt = \frac{p_0c}{f}\sinh^{-1}\frac{cft}{E_0}\\ y(t) &= \int_0^t \frac{ftc^2}{\sqrt{E_0^2 + f^2t^2c^2}}dt = \frac{E_0}{f}\left[-1 + \sqrt{1 + \left(\frac{cft}{E_0}\right)^2}\right]\\ z(t) &= 0.\end{aligned}$$

The orbit of the particle is in the plane xy. Eliminating t we find the equation of the orbit

$$y = \frac{E_0}{f}\left(-1 + \cosh\frac{fx}{p_0 c}\right).$$

In order to find the Newtonian limit, we expand cosh around the point $x = 0$ and find

$$y = \frac{E_0}{f}\left[-1 + 1 + \frac{f^2x^2}{2p_0^2c^2} + O(x^4)\right] = \frac{E_0 f}{2p_0^2c^2}x^2 + \cdots .$$

Ignoring x^4 terms, we obtain the result

$$y = \frac{E_0 f}{2p_0^2c^2}x^2 = \frac{m\gamma_0 c^2 f}{2m^2\gamma_0^2c^2u_0^2}x^2 = \frac{f}{2m\gamma_0 u_0^2}x^2,$$

which coincides with the orbit of a Newtonian projectile in a constant force field $\mathbf{f} = f\mathbf{j}$ with initial velocity $u_0 \ll c$.

Example 57 (Hyperbolic motion) A particle of mass m is moving along the x-axis of the LCF Σ under the action of the constant three-force $\mathbf{f} = f\mathbf{i}$ (f = constant in Σ).

(1) Assume that the particle starts its motion from rest at the origin of Σ and show that its position $x(t)$ is given by the expression

$$x(t) = \frac{c}{k}(\sqrt{1 + k^2t^2} - 1).$$

Display the quantities $\frac{k}{c}v(t)$ and $\frac{k}{c}x(t)$ in terms of t, where $v(t)$ is the velocity of the particle.

(2) If τ is the proper time of the particle show that

$$x(\tau) = \frac{c}{k}(\cosh k\tau - 1) \quad , \qquad t = \frac{1}{k}\sinh k\tau.$$

Making use of the above results, show that the length of the position four-vector $x^2 - c^2t^2$ is a constant of motion (hyperbolic motion, see Sect. 7.4).

Solution

(1) The equation of motion of the particle in Σ is

$$\frac{f}{m} = \frac{d}{dt}\frac{v}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

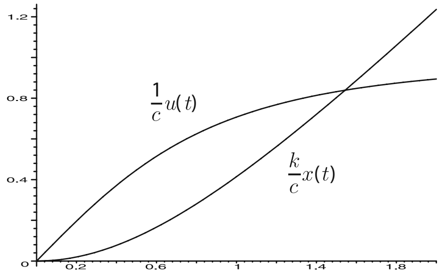


Fig. 11.1 Velocity and speed in one-dimensional hyperbolic motion

Integrating and taking into consideration the initial condition $v(0) = 0$ we find

$$v(t) = \frac{f}{m}\frac{t}{\sqrt{1 + (\frac{f}{mc}t)^2}}.$$

We introduce the constant k (with dimensions $[T^{-1}]$) $k = \frac{f}{mc}$ and write

$$v(t) = \frac{ckt}{\sqrt{1 + k^2t^2}}.$$

Integrating again and using the initial condition $x(0) = 0$ follows

$$x(t) = \frac{c}{k}(\sqrt{1 + k^2t^2} - 1). \tag{11.32}$$

The graph of the quantities $\frac{1}{c}v(kt)$, $\frac{k}{c}x(t)$ in terms of t is shown in Fig. 11.1.

(2) Let τ be the proper time of the particle. Then

$$d\tau = \frac{1}{\gamma}dt = \frac{dt}{\sqrt{1 + k^2t^2}} \Rightarrow t = \frac{1}{k}\sinh k\tau.$$

Replacing t in $x(t)$ we find

$$x(\tau) = \frac{c}{k}(\cosh k\tau - 1). \tag{11.33}$$

The last part is obvious.

In the next example we show that the three-force is not necessarily collinear with the three-acceleration, a characteristic which differentiates drastically the relativistic dynamics from the Newtonian. In general, the accelerated motions (for large and persistent accelerations) give rise to paradoxes, which are not easy to understand or explain and must be considered with great care.

Example 58 A particle of mass m moves in the xy plane of a LCF Σ under the action of the three-force $\mathbf{f} = f_x\mathbf{i} + f_y\mathbf{j}$. If $\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j}$ is the three-acceleration of the particle in Σ, show that

$$f_x = m\frac{[a_x(1-\beta_y^2)+a_y\beta_x\beta_y]}{(1-\beta_x^2-\beta_y^2)^{3/2}}, \tag{11.34}$$

$$f_y = m\frac{[a_y(1-\beta_x^2)+a_x\beta_x\beta_y]}{(1-\beta_x^2-\beta_y^2)^{3/2}}. \tag{11.35}$$

These relations show that the three-force acting on the particle and the resulting three-acceleration are not collinear (both in Σ!)

Using these relations prove that in order $a_x = 0$, the three-force must have component along the x-axis. Compute this component and give a physical interpretation of this "paradox."

In order to study the relation between the components of the three-force and the three-acceleration introduce the quantity $R = \frac{f_x}{f_y}$ and show that

$$\frac{a_y}{a_x} = \frac{k_1 - Rk_2}{Rk_3 - k_2}, \tag{11.36}$$

where $k_1 \equiv 1 - \frac{u_y^2}{c^2}$, $k_2 \equiv \frac{u_x u_y}{c^2}$, $k_3 \equiv 1 - \frac{u_x^2}{c^2}$. Consider the special case $\beta_x(t) = \beta_y(t)$ and show that

$$k_1 = k_3 = 1 - k_2.$$

Display $\frac{a_y}{a_x}$ as a function of k_2 for the values $R \in \{1.0, 1.1, 2.0, 10.0, \infty\}$. Comment on the result.[4]
[Hint: $\dot{\gamma} = \frac{\mathbf{u}\cdot\mathbf{a}}{c^2}\gamma^3$].
Solution

Because the motion is in the plane xy the component $u_z = 0$. We compute

$$\begin{aligned}
f_x &= \frac{d}{dt}(m\gamma u_x) = mu_x\frac{d\gamma}{dt} + m\gamma a_x = \frac{1}{c^2}mu_x\gamma^3(u_x a_x + u_y a_y) + m\gamma a_x \\
&= m\beta_x\gamma^3\left(\beta_x a_x + \beta_y a_y + \frac{1}{\gamma^2}\frac{a_x}{\beta_x}\right) \\
&= m\beta_x\gamma^3\left(\beta_x a_x + \beta_y a_y + \frac{1-\beta_x^2-\beta_y^2}{\beta_x}a_x\right), \\
f_x &= m\gamma^3\left[(1-\beta_y^2)a_x + \beta_x\beta_y a_y\right].
\end{aligned}$$

[4] The reader can find more information on this interesting approach in the following references:

1. G. W. Ficken "A relativity paradox: The negative acceleration component" *Am. J. Phys.* **44**, 1136–1137, (1976).
2. P. F. Gonzalez Diaz "Some additional results on the directional relationship between forces and acceleration in Special Relativity" *IL NUOVO CIMENTO* **51B**, 104–116, (1979).

Similarly we show the result for f_y.

We set $a_x = 0$ in the last relations and we get

$$f_x = m\gamma^3 a_y \beta_x \beta_y \qquad \text{and} \qquad f_y = m\gamma^3 a_y (1 - \beta_x^2)$$

from which follows

$$f_x = f_y \frac{\beta_x \beta_y}{1 - \beta_x^2}.$$

The component f_x of the three-force along the x-axis when the three-acceleration vanishes is explained as follows. Condition $a_x = 0$ implies $u_x =$ constant. However, the time derivative of the momentum $p_x = m\gamma(u_x^2 + u_y^2)u_x$ does not vanish because $\dot{\gamma} \neq 0$, therefore the component $f_x \neq 0$.

We define $R = \frac{f_x}{f_y}$ and replacing in the expressions which give f_x, f_y we find

$$R = \frac{a_x k_1 + a_y k_2}{a_x k_2 + a_y k_3} = \frac{k_1 + k_2 \frac{a_y}{a_x}}{k_2 + k_3 \frac{a_y}{a_x}} \Rightarrow$$
$$\frac{a_y}{a_x} = \frac{k_1 - Rk_2}{Rk_3 - k_2},$$

where $k_1 \equiv 1 - \beta_y^2$, $k_2 \equiv \beta_x \beta_y$, $k_3 \equiv 1 - \beta_x^2$.

In case $\beta_x(t) = \beta_y(t)$ follows easily that $k_1 = k_3 = 1 - k_2$, $\beta_x^2 = \frac{1}{2}\beta^2$.

Furthermore, $\gamma = \frac{1}{\sqrt{-1+2k_1}}$ hence

$$\frac{a_y}{a_x} = \frac{1 - (1 + R)k_2}{R - (1 + R)k_2}. \tag{11.37}$$

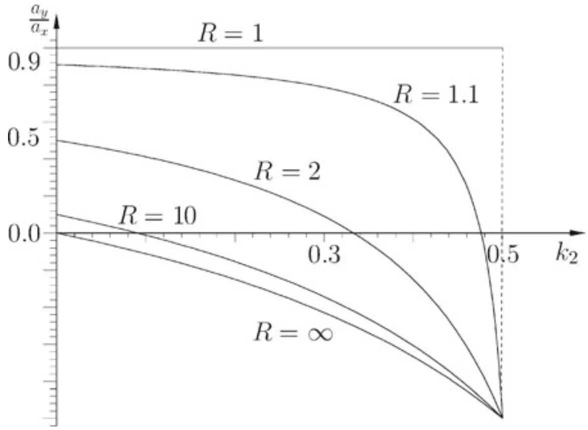
The graph of $\frac{a_y}{a_x}(k_2)$ is shown in Fig. 11.2.

Since $k_2 = \frac{1}{2}\beta^2$ the deviation from the Newtonian result appears when the speed u has large values. For small speeds $k_2 \longrightarrow 0$ and $\frac{a_y}{a_x} \longrightarrow \frac{1}{R} = \frac{f_y}{f_x}$, which is the expected Newtonian expression. It is to be noted that when $f_y = 0$ and $f_x \neq 0$, that is, when there is force only along the x-axis the $R = \infty$ and from the graph follows that $\frac{a_y}{a_x} < 0$ hence the quantity a_x measures retardation. In this case (11.35) gives

$$a_x = -\frac{k_3}{k_2} a_y = \left(1 - \frac{1}{k_2}\right) a_y < 0,$$
$$f_x = m\gamma^3 \frac{k_2^2 - k_1 k_3}{k_2} a_y = m\gamma^3 \left(2 - \frac{1}{k_2}\right) a_y > 0.$$

When $f_x = 0$ then (11.34) gives $a_x = -\frac{k_2}{1-k_2} a_y$, which means that $a_x < 0$ and $a_x < a_y$ because $0 \leq k_2 < 1/2$ as it can be seen from Fig. 11.2. From this relation follows

Fig. 11.2 $\frac{a_y}{a_x}$ when $v_x(t) = v_y(t)$



$$f_y = m\gamma^3(1 - k_2)\left(1 - \frac{k_2}{1 - k_2^2}\right) a_y.$$

We note that as $k_2 \longrightarrow 1/2$ the $f_y \longrightarrow \infty$ because $\gamma \longrightarrow \infty$.

If the component f_x has small value, e.g., $f_x = f_y/10$, then from Fig. 11.2 follows that for this value of R the quotient $\frac{a_y}{a_x} > 0$. The vanishing of $\frac{a_y}{a_x}$ occurs when $k_2 \simeq 0.1$ which implies that the component f_x of the three-force "finishes" at the value $k_2 \simeq 0.1$. Finally, when $R = 1$ from (11.37) follows $\frac{a_y}{a_x} = \frac{1-2k_2}{1-2k_2} = 1$, that is, the quotient $\frac{a_y}{a_x}$ is independent of k_2. This is a limiting case.

Exercise 46

(a) Using the identity of the (Euclidean) three-dimensional vector calculus

$$\mathbf{A}\times(\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$$

show that

$$\mathbf{A} = \frac{1}{B^2}(\mathbf{A} \cdot \mathbf{B})\mathbf{B} - \frac{\mathbf{B}\times(\mathbf{B} \times \mathbf{A})}{B^2}. \tag{11.38}$$

This relation decomposes the vector $\mathbf{A}$ normal and parallel to the vector $\mathbf{B}$.

(b) Take $\mathbf{B}$ to be the unit vector $\hat{\mathbf{e}}_\parallel$ along the direction of the velocity and show the relation

$$\mathbf{f}' = \frac{1}{\gamma_u Q}\mathbf{f}_\perp + \left[f_\parallel - \frac{\mathbf{u}\times(\mathbf{v} \times \mathbf{u})}{Quc^2} \cdot \mathbf{f}\right]\hat{\mathbf{e}}_\parallel. \tag{11.39}$$

Subsequently, show that when $\mathbf{f}_\perp \neq 0$ the $\mathbf{f}'$ can be written as a Lorentz force as follows[5]

$$\mathbf{f}' = \frac{f_\perp \gamma_u}{Q} \left[\hat{\mathbf{e}}_\perp + \frac{1}{c^2} \mathbf{V} \times \mathbf{h} \right], \tag{11.40}$$

where $\mathbf{h} = \mathbf{u} \times \hat{\mathbf{e}}_\perp$ is the "magnetic field" and $\mathbf{V} = \mathbf{u} + \frac{c^2 R}{f_\perp \gamma_u} \hat{\mathbf{e}}_\perp$ is the "corrected" velocity in the LCF Σ and $R = Q \left[\mathbf{f}_\| - \frac{\mathbf{u}\times(\mathbf{v}\times\mathbf{u})}{Quc^2} \cdot \mathbf{f} \right]$.
[Hint: Use (11.26)].

11.3 Inertial Four-Force and Four-Potential

In order to develop the dynamics of Special Relativity along the analytic formalism of Lagrange and Hamilton, we have to introduce potential functions for four-forces. It is obvious that we shall consider only inertial four-forces because analytic dynamics concerns inertial motion and not, e.g., thermal forces.

The first type of potential functions we look for are the (Lorentz) invariants.[6] Consider a ReMaP with four-velocity u^i which is moving under the action of the inertial four-force F^i. Suppose $\phi(l,\ \mathbf{r})$ $(l = ct)$ is an invariant such that

$$F_i = -q\phi_{,i}, \tag{11.41}$$

where q is a constant. The spacelike character of F^i implies the constraint

$$F_i u^i = -q\phi_{,i} u^i = 0$$

for all four-velocities u^i. The geometric meaning of this constraint is that the "equipotential surfaces" $\phi =$ constant are (Lorentz) orthogonal to the four-velocity u^i, hence they are spacelike hypersurfaces. Since the four-velocity is arbitrary, this implies $\phi_{,i} = 0$ (why?). Clearly this is impossible, therefore it is not possible to consider an invariant four-potential.

Our next choice is a vector four-potential $\Phi_i(l,\ \mathbf{r})$ say. In this case, we assume that the four-potential is related with the four-force F^i as follows

$$F_i = \frac{q}{c}(\Phi_{i,j} - \Phi_{j,i})u^j. \tag{11.42}$$

[5] A special case of this result can be found in D. Bedford and P. Krumm Am. J. Phys **53**, 889, (1985).

[6] The potential is a relativistic physical quantity, therefore must be expressed in terms of a Lorentz tensor.

We note that the constraint $F_i u^i = 0$ is satisfied identically for all u^i, therefore it is possible that we shall consider such a four-potential. In the more general case, that is, when there does not exist a four-potential, it is still possible to write the (inertial) four-force in the form $F_j = \Omega_{ij} u^i$, where Ω_{ij} is an antisymmetric tensor. Such four-forces are due to various dynamical fields.

Now we come to the question: Suppose one is given a three-force **f**, which in the LCF Σ is conservative (that is, it can be expressed as the derivative of a scalar three-potential – not Lorentz invariant in general). How one will compute a four-potential for the four-force defined by this three-force in Σ and subsequently in all other LCF via the appropriate Lorentz transformation? Furthermore the transformed three-force in the other LCF is conservative and, if it is, does there exist a transformation law between the two three-potentials?

In order to answer these questions we note that the Lorentz transformation of the three-force does not necessarily preserve the conservative character of the force, that is, in general the transformed three-force will not be conservative. However, if we manage to define for the three-force **f** a four-potential in Σ then this will give a four-potential in any other LCF via the Lorentz transformation of the first. This is shown clearly by the following example.

Example 59 Show that a three-force which is central in the LCF Σ is not (in general) central in another LCF Σ'. Because the central forces are conservative conclude that the property of a three-force to be conservative is not (in general) Lorentz covariant.
Solution

Let $\mathbf{f}(\mathbf{r}) = f(r)\mathbf{r}$ be a central force in the LCF Σ where $\mathbf{r} = r\hat{\mathbf{r}}$ is the position vector in Σ. As we know from Newtonian Mechanics, this force is conservative with potential function $U(\mathbf{r}) = -\int \mathbf{f} \cdot d\mathbf{r} = -\int f(r)dr$. Let P be a ReMaP which in Σ has velocity $\mathbf{v}$, position vector $\mathbf{r}$ while moves under the action of the three-force $\mathbf{f}(\mathbf{r})$. Then the four-force on P in Σ is

$$F^i = \gamma \begin{pmatrix} \mathbf{f} \cdot \mathbf{v}/c \\ \mathbf{f} \end{pmatrix}_\Sigma .$$

Let Σ' be another LCF in which P has velocity $\mathbf{v}'$ and position vector $\mathbf{r}'$ while is moving under the action of the same three-force, which in Σ' we denote by $\mathbf{f}'$. In order to compute $\mathbf{f}'$, we consider the Lorentz transformation of the four-vector F^i from Σ to Σ' and have (($Q = 1 - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}$), see Exercise 45)

$$\begin{aligned} \mathbf{f}' &= \frac{1}{\gamma_u Q}\left\{ f(r)\mathbf{r} + \left[(\gamma_u - 1)\frac{\mathbf{u}\cdot\mathbf{r}}{u^2} f(r) - \frac{\gamma_u}{c^2} f(r)(\mathbf{r}\cdot\mathbf{v}) \right] \mathbf{u} \right\} \\ &= \frac{f(r)}{\gamma_u Q}\left\{ \mathbf{r} + \left[(\gamma_u - 1)\frac{\mathbf{u}\cdot\mathbf{r}}{u^2} - \frac{\gamma_u}{c}(ct) \right] \mathbf{u} + \frac{\gamma_u}{c}(ct)\mathbf{u} - \frac{\gamma_u}{c^2}(\mathbf{r}\cdot\mathbf{v})\mathbf{u} \right\}. \end{aligned}$$

The first two terms in brackets equal $\mathbf{r}'$ and the quantity $ct - \frac{\mathbf{r}\cdot\mathbf{v}}{c} = \frac{1}{\gamma_v} d\tau$, where $d\tau$ is the proper time of P. Finally, we find

$$\mathbf{f}' = \frac{f(r)}{\gamma_u Q}\left(\mathbf{r}' + \frac{\gamma_u}{\gamma_v c} d\tau \mathbf{u}\right). \tag{11.43}$$

Obviously the three-force $\mathbf{f}'$ is not central (due to the term $\frac{f(r)d\tau}{\gamma_v Qc}\mathbf{u}$) in Σ' hence, in general, $\mathbf{f}'$ is not conservative.

Exercise 47 In Example 59 calculate $\nabla' \times \mathbf{f}'$ and find the conditions (if they exist) for which the three-force $\mathbf{f}'$ is conservative.

11.3.1 The Vector Four-Potential

Consider an LCF Σ in which there exists a conservative three-force field $\mathbf{f}$ with potential[7] $\phi(l,\ \mathbf{r})$ ($l = ct$) so that $\mathbf{f} = -q\nabla\phi$. We *define* the four-potential of the four-force which is defined by the three-force $\mathbf{f}$ with the timelike four-vector Φ_i, which in Σ has components

$$\Phi_i = (\phi, 0,\ 0,\ 0) \tag{11.44}$$

and require that the inertial four-force F_i on an arbitrary ReMaP P which in Σ moves under the action of the three-force $\mathbf{f}$ is given by

$$F_i = \frac{q}{c}(\Phi_{i,j} - \Phi_{j,i})v^i. \tag{11.45}$$

Before we continue, we have to check that this definition of the four-potential is compatible with the decomposition of the four-force in Σ, that is, that the definition (11.45) leads to the expression $F^i = \gamma \begin{pmatrix} \frac{1}{c}\mathbf{f}\cdot\mathbf{v} \\ \mathbf{f} \end{pmatrix}_\Sigma$.[8]

In order to show this, we write $\mathbf{f} = -\nabla\phi$ and assume that in Σ $v^i = \gamma \begin{pmatrix} c \\ \mathbf{v} \end{pmatrix}_\Sigma$. We compute the components of the four-force defined by $\mathbf{f}$ by considering the definition (11.45). For the temporal component we have[9]

$$F_0 = \frac{q}{c}(\Phi_{0,j} - \Phi_{j.0})v^j = \frac{q}{c}(\Phi_{0,\mu} - \Phi_{\mu,0})\gamma v^\mu = \frac{q}{c}\Phi_{0,\mu}\gamma v^\mu \tag{11.46}$$

$$= \frac{q}{c}\gamma\nabla\phi\cdot\mathbf{v} = -\frac{1}{c}\gamma\mathbf{f}\cdot\mathbf{v}. \tag{11.47}$$

---

[7] The scalar potential $\phi(\mathbf{r},\ t)$ corresponds to the three-force $\mathbf{f}$ not the four-force F^i defined by $\mathbf{f}$! As we have shown, there does not exist an invariant four-potential.

[8] One could work in the opposite direction, that is consider the decomposition of the four-force in Σ and derive the definition (11.45).

[9] Care! $F_0 = -F^0$.

Similarly, for the spatial components we have

$$F_\mu = \frac{q}{c}(\Phi_{\mu,\ j} - \Phi_{j,\ \mu})v^j = -\frac{q}{c}\Phi_{0,\ \mu}v^0 = -\frac{q}{c}\phi_{,\ \mu}\gamma c = \gamma \mathbf{f}$$

which completes the proof.

Concerning the physical meaning of the four-potential, we note that it is a four-vector which is determined by the motion of P, therefore must be expressible in terms of the invariants associated with the ReMaP P. The only such quantity (in the absence of dynamical fields) is the mass of P. Therefore, there must exist a relation between the mass m of P, which is moving under the action of the inertial four-force, and the four-potential of the four-force. Obviously, this relation must be sought in the conservation of energy.

We consider the condition $F^i v_i = 0$ which gives in Σ

$$\begin{gathered}\dot{E} = \mathbf{f} \cdot \mathbf{v} \Rightarrow \\ dE = \mathbf{f} \cdot d\mathbf{r} = -q\nabla\phi \cdot d\mathbf{r} = -qd\phi\end{gathered}$$

hence in Σ

$$E + q\phi = \text{constant}.$$

But $E = T + mc^2$ where T is the kinetic energy of P in Σ. Hence

$$T + q\phi = \text{constant} - mc^2.$$

The lhs is the same with the corresponding Newtonian expression whereas the rhs differs by the term mc^2, which indicates the different dynamics of the two theories. Indeed, the common appearance of this term with the quantities T and $q\phi$ indicates that the mass m of P can change by the kinetic energy and/or the potential energy of the dynamical field, which modulates the motion. Therefore, if $E_1 = M_1 c^2$ and $E_2 = M_2 c^2$ ($M_i = \gamma m_i$ where $i = 1, 2$) are the energies of P at the events 1, 2 of its world line and ϕ_1, ϕ_2 the corresponding values of the potential of the three-force in Σ, then

$$(M_2 - M_1)c^2 = q(\phi_2 - \phi_1),$$

that is, the energy of the field changes the "inertial" mass of P and conversely.

11.4 The Lagrangian Formalism for Inertial Four-Forces

The Lagrangian and the Hamiltonian formulations of dynamics of Special Relativity are equally important as it is in Newtonian Physics. However, it presents "peculiarities" and it is more obscure due to the hyperbolic geometry of spacetime and the possibility of null four-vectors. In the following, we shall discuss only the rudiments of these formalisms and advise the interested reader to consult books whose subject is the relativistic field theory.

The main role of Lagrange equations is to produce the equations of motion of a ReMaP moving under the action of an inertial dynamical field. The Lagrange equations involve a scalar (not invariant!) function, the Lagrangian and the equations of motion (for conservative dynamical fields) have the well-known form

$$\frac{d}{d\theta}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = 0, \tag{11.48}$$

where a dot over a symbol indicates derivation along the parameter θ. The function L cannot be arbitrary. Indeed, the equations of motion (11.48) must be covariant wrt the Lorentz group (see Covariance Principle in Sect. 4.8) therefore, the quantity $Ld\theta$ must be a (Lorentz) invariant. This requirement restricts L depending on the type of the parameter θ. A significant difference from Newtonian Physics is that θ need not be the time but it can be any (increasing and smooth) parameter along the world line of the ReMaP. From the variational calculus it is known that (11.48) follows from the stationary integral

$$\delta \int L d\theta = 0. \tag{11.49}$$

In case θ is the time in a LCF Σ then the integral $\int Ldt$ is the *action integral* and the Lagrangian function L is called the relativistic Lagrangian of the ReMaP in Σ.

We start our study by computing the Lagrangian of a free relativistic particle of mass m and four-velocity u^i. In Newtonian Physics the Lagrangian $L_{0,\,N}$ for a free Newtonian particle is

$$L_{0,\,N} = 2T = mv^2$$

and the Principle of Least Action[10] is

$$\delta \int L_{0,\,N} dt = 0,$$

where t is time along the orbit of the moving particle in space. This leads us to consider[11] in Special Relativity as the Lagrangian of the free particle the Lagrangian $L_{0,\,M}$:

[10] It is not always least. The correct is to say stationary action. But this is what prevailed in the literature and we shall follow it.

[11] As a matter of fact this choice is unique, because m and the proper time are the only invariants associated with a free particle.

$$L_{0,\ M} = m(u^i u_i) = -mc^2 \tag{11.50}$$

and write for the (relativistic) Principle of Least Action

$$\delta \int_1^2 L_{0,\ M} d\tau = 0, \tag{11.51}$$

where 1, 2 are events along the world line of the particle and τ is the proper time of the particle. Equation (11.51) does not apply to photons, because photons do not have proper time. However, the formalism can be extended to include the photons, by considering a different parameter along the null world line of a photon. In order for the function $L_{0,\ M}$ to be accepted must lead to the equation of motion of a free particle

$$\frac{dp^i}{d\tau} = 0. \tag{11.52}$$

As we assume the four-force to be inertial ($F^i u_i = 0$) the mass m of the particle is constant, hence (11.52) is written

$$\frac{du^i}{d\tau} = 0 \Rightarrow \frac{d^2 x^i}{d\tau^2} = 0 \tag{11.53}$$

and has solution

$$x^i = u^i \tau + B^i. \tag{11.54}$$

This is consistent with the assumption that the world line of a free particle is a timelike straight line.

Let us examine the Lagrangian we considered in more detail. Replacing $L_{0,M}$ in (11.51) we find

$$-mc^2 \delta \int_1^2 d\tau = 0 \Longrightarrow \delta \int_1^2 d\tau = 0. \tag{11.55}$$

Equation (11.55) is the equation of geodesics, which in Minkowski space are indeed the straight lines (11.54). By definition, the world lines of photons are null straight lines in Minkowski space, therefore, the formalism we developed is ample and consistent with the assumptions we have made so far.

It remains to prove that the variation (11.51) leads to the equation of motion (11.52) when $L d\theta = L_{0,\ M} d\tau$ and θ is a smooth function of τ such that the derivative $\frac{d\theta}{d\tau}$ between two events 1, 2 of the world line of the particle does not change sign. Without restriction of generality we assume $\frac{d\theta}{d\tau} > 0$.

The necessity of consideration of the parameter θ comes from the necessity to cover the case of photons, which do not have proper time as well as the need to compute the equations of motion in an LCF other than the proper frame of the particle. For example, if we consider $\theta = t$ where t is time in Σ, then the equations which follow from the variation of the action integral are the equations of motion in Σ. With the introduction of the parameter θ the Lagrangian becomes

$$\begin{aligned} L_{0M} &= mu^i u_i = -mc\sqrt{-u^i u_i} = -mc\sqrt{-\eta_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}} \\ &= -mc\sqrt{-\eta_{ij}\frac{dx^i}{d\theta}\frac{dx^j}{d\theta}}\frac{d\theta}{d\tau} \end{aligned}$$

hence

$$L_{0M}d\tau = -mc\sqrt{-\eta_{ij}\dot{x}^i\dot{x}^j}d\theta, \tag{11.56}$$

where $\dot{x}_i = \frac{dx^i}{d\theta}$. The Principle of Least Action reads for the parameter θ:

$$\delta\int \Lambda d\theta = 0, \tag{11.57}$$

where

$$\Lambda = -mc\sqrt{-\eta_{ij}\dot{x}^i\dot{x}^j}. \tag{11.58}$$

In the following, we shall show that indeed the Lagrangian Λ leads to the correct equations of motion.

Consider an inertial four-force field F^i which is described by the four-vector potential Φ_i according to the relation

$$F_i = k(\Phi_{i,j} - \Phi_{j,i})\dot{x}^j\frac{d\theta}{d\tau}, \tag{11.59}$$

where $k = \frac{q}{c}$ is a constant. As we have said, the equation of motion of a ReMaP P of mass m and four-velocity u^i in the four-force field F^i is

$$\frac{dp^i}{d\tau} = F^i, \tag{11.60}$$

where $p^i = mu^i$ and τ is the proper time of P. This relation in terms of the parameter θ is written as follows:

$$\begin{aligned} \frac{dp^i}{d\theta}\frac{d\theta}{d\tau} &= F_i = k(\Phi_{i,j} - \Phi_{j,i})\dot{x}^j\frac{d\theta}{d\tau} \Longrightarrow \\ \frac{dp^i}{d\theta} &= k(\Phi_{i,j} - \Phi_{j,i})\dot{x}^j. \end{aligned} \tag{11.61}$$

Equation (11.61) must follow from Lagrange equations for the Lagrangian Λ, that is, from the equation

$$\frac{d}{d\theta}\frac{\partial \Lambda}{\partial \dot{x}^i} - \frac{\partial \Lambda}{\partial x^i} = 0. \tag{11.62}$$

The Lagrangian Λ besides the term $-mc\sqrt{-\eta_{ij}\dot{x}^i\dot{x}^j}$ which is the Lagrangian of the free particle, must have a second term which will take into account the potential field F^i. In analogy with Newtonian Physics we *define* the relativistic Lagrangian as follows:

$$\Lambda = -mc\sqrt{-\eta_{ij}\dot{x}^i\dot{x}^j} - k\Phi_i\dot{x}^i \tag{11.63}$$

and compute the Lagrange equation (11.62). We set for convenience $A = \sqrt{-\eta_{ij}\dot{x}^i\dot{x}^j}$ and get

$$\frac{\partial \Lambda}{\partial \dot{x}^i} = -mc\frac{\partial A}{\partial \dot{x}^i} - k\Phi_i$$

But

$$\frac{\partial A}{\partial \dot{x}^i} = -\frac{1}{A}\eta_{ij}\dot{x}^j.$$

hence

$$\frac{\partial \Lambda}{\partial \dot{x}^i} = \frac{mc}{A}\eta_{ij}\dot{x}^j - k\Phi_i.$$

The derivative

$$\frac{d}{d\theta}\frac{\partial \Lambda}{\partial \dot{x}^i} = -\frac{mc\dot{A}}{A^2}\eta_{ij}\dot{x}^j - \frac{mc}{A}\eta_{ij}\ddot{x}^j - k\dot{\Phi}_i = -\frac{mc}{A^2}\eta_{ij}[\dot{A}\dot{x}^j - A\ddot{x}^j] - k\dot{\Phi}_i.$$

The term

$$\dot{\Phi}^i = \frac{d\Phi^i}{d\theta} = \frac{d\Phi^i}{dx^j}\dot{x}^j = \Phi^i_{,j}\dot{x}^j$$

so that

$$\frac{d}{d\theta}\frac{\partial \Lambda}{\partial \dot{x}^i} = -\frac{mc}{A^2}\eta_{ij}\left(\dot{A}\dot{x}^j - A\ddot{x}^j\right) - k\Phi_{i,j}\dot{x}^j.$$

We also compute

$$\frac{\partial \Lambda}{\partial x^i} = -k\Phi_{j,i}\dot{x}^j.$$

Replacing in (11.62) we find

$$-\frac{mc}{A^2}\eta_{ij}\left(\dot{A}\dot{x}^j - A\ddot{x}^j\right) - k\Phi_{i,j}\dot{x}^j + k\Phi_{j,i}\dot{x}^j = 0$$

or

$$\frac{mc}{A^2}\eta_{ij}\left(A\ddot{x}^j - \dot{A}\dot{x}^j\right) = k\left(-\Phi_{j,i} + \Phi_{i,j}\right)\dot{x}^j \tag{11.64}$$

In terms of the four-force (11.64) is written as follows:

$$\frac{m}{A^2}\eta_{ij}\left(A\ddot{x}^j - \dot{A}\dot{x}^j\right) = F_i. \tag{11.65}$$

Let us consider various cases for the parameter θ. The choice $\theta = \tau$ gives $A = c$ and (11.65) reads as

$$m\eta_{ij}\frac{d^2x^j}{d\tau^2} = F_i \tag{11.66}$$

which is the correct equation of motion. For a free particle $F_i = 0$ and (11.65) becomes $\frac{d^2x^j}{d\tau^2} = 0$, which is the expected result.

In order to compute the equation of motion of the ReMaP in the LCF Σ in which the time is t we set $\theta = t$ and the Lagrangian Λ reads as

$$\begin{aligned}
\Lambda_\Sigma &= -mc\sqrt{-\eta_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}} - k\Phi_i\frac{dx^i}{dt} \\
&= -mc\sqrt{-\eta_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}}\frac{d\tau}{dt} - k\Phi_i\frac{dx^i}{d\tau}\frac{d\tau}{dt} \\
&= -mc\cdot c\frac{1}{\gamma} - \frac{k}{\gamma}\Phi_i u^i \Longrightarrow \\
\Lambda_\Sigma &= -\frac{mc^2}{\gamma} - \frac{k}{\gamma}\Phi_i u^i.
\end{aligned} \tag{11.67}$$

We write the Lagrangian Λ_Σ in terms of the components of the four-velocity in Σ. If we assume in Σ

$$\Phi_i = \left(\phi, \mathbf{w}\right)_\Sigma, \qquad u^i = \begin{pmatrix}\gamma c\\ \gamma\mathbf{v}\end{pmatrix}_\Sigma.$$

then

$$\Phi_i u^i = -\phi\gamma c + \gamma\mathbf{w}\cdot\mathbf{v} = \gamma(-\phi c + \mathbf{w}\cdot\mathbf{v}).$$

Replacing in (11.67) we find

$$\Lambda_\Sigma = -\frac{mc^2}{\gamma} - k(-\phi c + \mathbf{w} \cdot \mathbf{v}). \tag{11.68}$$

Exercise 48

(a) Let $v^\rho = \frac{dx^\rho}{dt}$ be the three-velocity of a ReMaP P in a LCF Σ. Prove the identity

$$\frac{\partial}{\partial v^\rho}\frac{1}{\gamma} = -\frac{1}{c^2}\gamma v^\rho \tag{11.69}$$

[Hint: $\frac{\partial}{\partial v^\rho}v^2 = 2v^\rho$].

(b) Show that the three-momentum p^ρ of P in Σ can be written as

$$p^\rho = \frac{\partial}{\partial v^\rho}\left(-\frac{mc^2}{\gamma}\right), \tag{11.70}$$

where $m =$ constant.
[Hint: $p^\rho = m\gamma v^\rho$].

(c) Prove the relations

$$\frac{dp^\rho}{dt} = \frac{d}{dt}\left(\frac{\partial L_{0,\Sigma}}{\partial v^\rho}\right) - \frac{\partial L_{0,\Sigma}}{\partial x^\rho},$$
$$\frac{dp^0}{dt} = \frac{dE}{d(ct)} = \frac{\partial L_{0,\Sigma}}{\partial(ct)},$$

where $L_{0,\ \Sigma} = -\frac{mc^2}{\gamma}$ and $v^\rho = \frac{dx^\rho}{dt}$. Finally show that

$$\frac{dp^i}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L_{0,\Sigma}}{\partial x^i},$$

where $\dot{x}^i = \frac{dx^i}{dt}$. Conclude that the Lagrangian of a free particle in Σ is the function $L_{0,\ \Sigma} = -\frac{mc^2}{\gamma}$.

Exercise 49

(a) Let θ be a parameter along the world line of a ReMaP P such that $\frac{d\theta}{d\tau} > 0$ where τ is the proper time of P. Show that the Lagrangian L_θ which corresponds to the parameter θ is related to the Lagrangian L_τ which corresponds to the parameter τ, with the relation

$$L_\tau = L_\theta \frac{d\theta}{d\tau} \tag{11.71}$$

[Hint: $\delta \int L_\theta d\theta = \delta \int L_\tau d\tau$].

(b) Consider $\theta = t$ where t is the time in a LCF Σ and show that

$$L_{\Sigma^+} = \gamma L_{t,\ \Sigma}, \tag{11.72}$$

where L_{Σ^+} is the Lagrangian in the proper frame of P and $L_{t,\ \Sigma}$ the Lagrangian in Σ.

(c) Let Σ, Σ' be two LCF with parallel axes and relative velocity $\mathbf{u}$. Prove that the Lagrangians of an inertial conservative four-vector field in $\Sigma,\ \Sigma'$ are related as follows:

$$L_\Sigma \gamma_\Sigma = L_{\Sigma'} \gamma_{\Sigma'}, \tag{11.73}$$

where $\gamma_\Sigma = \frac{1}{\sqrt{1-\beta^2}}$, $\gamma_{\Sigma'} = \frac{1}{\sqrt{1-(\beta')^2}}$, and β, β' are the speeds of P in $\Sigma,\ \Sigma'$, respectively.

Making use of the relation $\gamma_{\Sigma'} = \gamma_u \gamma_\Sigma Q$, where $Q = 1 - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}$, prove that

$$L_{\Sigma'}(q', \frac{dq'}{dt'}, t') = \frac{1}{\gamma_u Q} L_\Sigma(q, \frac{dq}{dt}, t). \tag{11.74}$$

This relation gives the transformation law of the Lagrangian function under a Lorentz transformation.

11.5 Motion in a Central Potential

As an application of the Lagrange formalism in Special Relativity we consider the motion of a ReMaP P of mass $m \neq 0$ (not a photon!) in the LCF Σ, in which there exists a central potential given by the function $U(r)$. Recall (see Example 59) that the motion of P in another LCF which moves wrt Σ is not in general central. Therefore, whatever results we derive in this section are valid in Σ only. If one wishes to find these results in another LCF, the Σ' say, then the results must be transformed by the Lorentz transformation relating Σ and Σ'.

According to the previous considerations the four-potential defined by the three-potential $U(r)$ in Σ is the timelike four-vector $(U(r),\ 0,\ 0,\ 0)$. The motion in Σ is taking place on a plane as in Newtonian Mechanics. Indeed, in Σ the three-force on P is $\mathbf{F} = -\frac{dV}{dr}\hat{\mathbf{e}}_r$, therefore the angular momentum $\mathbf{L}$ of P wrt the origin of Σ is constant:

$$\frac{d\mathbf{L}}{dt} = \mathbf{r} \times \mathbf{F} = 0.$$

In addition $\mathbf{L}\cdot\mathbf{r} = 0$, that is, the position vector $\mathbf{r}$ is always normal to the constant vector $\mathbf{L}$ consequently the orbit of P in Σ is on a plane.

The Lagrangian of P equals the sum of the Lagrangian $L_0(\tau) = -mc^2$, where τ is the proper time of P and the potential function $U(r)$. In order to express L_0 in terms of the parameter t the time in Σ, we use the relation

$$L_0(\tau)d\tau = L_0(t)dt$$

from which follows

$$L_0(t) = -\frac{1}{\gamma}mc^2. \tag{11.75}$$

Consequently the Lagrangian of P in Σ is[12]

$$L_\Sigma(r,\ t) = -\frac{m}{\gamma}c^2 - U(r). \tag{11.76}$$

We consider polar coordinates $r,\ \theta$ so that the speed $v^2 = \dot{r}^2 + r^2\dot{\theta}^2$ and the final form of the Lagrangian reads

$$L_\Sigma(r\ , t) = -mc^2\sqrt{1 - \frac{\dot{r}^2 + r^2\dot{\theta}^2}{c^2}} - U(r). \tag{11.77}$$

The angle θ is a cyclic coordinate, therefore, the generalized momentum $P_\theta \equiv L_z$ is a constant of motion. We compute

$$L_z = \frac{\partial L}{\partial \dot{\theta}} = m\gamma r^2\dot{\theta}. \tag{11.78}$$

The Lagrange equation for the r-coordinate is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0$$

from which follows

$$\frac{d}{dt}(\gamma\dot{r}) - \gamma r\dot{\theta}^2 + \frac{1}{m}\frac{dU}{dr} = 0. \tag{11.79}$$

Because the Lagrangian is independent of time (and there are no non-holonomic constraints) the energy is a constant of motion or, equivalently a first integral of (11.79). This implies $E + U = C$ where C is a constant. Solving in terms of γ we find

[12] This Lagrangian follows directly from (11.67) if we change $L_\Sigma \to \Lambda_\Sigma$.

$$m\gamma c^2 + U(r) = C \Rightarrow \gamma = \frac{C - U}{mc^2}. \tag{11.80}$$

The speed of the particle can be written as

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2 = \left(\frac{dr}{d\theta}\right)^2 \dot{\theta}^2 + r^2\dot{\theta}^2 = \left[\left(\frac{dr}{d\theta}\right)^2 + r^2\right] \frac{L_z^2}{m^2\gamma^2 r^4}$$

from which follows

$$v^2\gamma^2 = \frac{L_z^2}{m^2}\left[\left(\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 + \frac{1}{r^2}\right]. \tag{11.81}$$

Replacing v^2, $v^2\gamma^2$ in the identity $1 + \gamma^2\beta^2 = \gamma^2$, we get the equation of the trajectory of the particle in Σ:

$$1 + \frac{L_z^2}{m^2c^2}\left[\left(\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 + \frac{1}{r^2}\right] = \left(\frac{C - U}{mc^2}\right)^2. \tag{11.82}$$

We introduce the new variable $u = \frac{1}{r}$ and (11.82) is written as follows:

$$1 + \frac{L_z^2}{m^2c^2}\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] = \left(\frac{C - U}{mc^2}\right)^2. \tag{11.83}$$

Differentiating this relation wrt u we find

$$\frac{2L_z^2}{m^2c^2}\frac{du}{d\theta}\left[\frac{d^2u}{d\theta^2} + u\right] = -\frac{2}{mc^2}\frac{du}{d\theta}\frac{dU}{du}\frac{C - U}{mc^2}.$$

We assume $\frac{du}{d\theta} \neq 0$ (that is the points at which the function r has extreme values are excluded) and we have the final form of the equation of motion of the particle in Σ for a general central potential $U(r)$ ($U(r)$ is given in Σ!)

$$\frac{d^2u}{d\theta^2} + u = -\frac{1}{L_z^2}\frac{dU}{du}\frac{C - U}{c^2}. \tag{11.84}$$

Of historical interest is the study of the motion of an electron around the nucleus. In this case, the potential function in the proper frame of the nucleus is assumed to have the form $U(r) = -\frac{Ze^2}{r}$ where Z is the atomic number of the nucleus. For this potential function, the equation of the orbit (11.84) reads

$$\frac{d^2u}{d\theta^2} + \left(1 - \frac{Z^2e^4}{c^2L_z^2}\right)u = \frac{Ze^2C}{c^2L_z^2}. \tag{11.85}$$

If we introduce the new constant $\lambda^2 = 1 - \frac{Z^2e^4}{c^2L_z^2}$ this equation is written as

$$\frac{d^2u}{d\theta^2} + \lambda^2 u = \frac{Ze^2C}{c^2L_z^2}. \tag{11.86}$$

This equation describes a forced harmonic oscillation with proper frequency λ and driving force $\frac{Ze^2C}{c^2L_z^2}$. The solution is

$$u = A\cos\lambda\theta + B\sin\lambda\theta + \frac{1}{\lambda^2}\frac{Ze^2C}{c^2L_z^2}, \tag{11.87}$$

where the constants A, B are determined from the initial conditions. We assume that $\theta = 0$ when the electron is at the perihelion (see Fig. 11.3) and have

$$\left(\frac{dr}{d\theta}\right)_{\theta=0} = u^2\left(\frac{du}{d\theta}\right)_{\theta=0} = 0 \Rightarrow \left(\frac{du}{d\theta}\right)_{\theta=0} = 0.$$

Then (11.87) gives $B = 0$ and the general solution reads

$$\frac{1}{r} = A\cos\lambda\theta + \frac{1}{\lambda^2}\frac{Ze^2C}{c^2L_z^2}. \tag{11.88}$$

We infer that in the proper frame of the nucleus Σ, the orbit of the electron is an ellipse with one focal point at the origin of Σ and eccentricity $\frac{Ac^2L_z^2}{Ze^2c}$. The constant $\frac{1}{A}$ is the distance of the directrix from the focal point (see Fig. 11.3).

The Newtonian limit of the orbit occurs when $c \to \infty$ that is, $\lambda = 1$. This orbit is the well-known circular orbit assumed by Bohr in his model of the atom. For finite values of c, $\lambda \neq 1$ and the orbit is precessing between two enveloping circles, which are defined by the maximum and the minimum values of r, as shown in Fig. 11.4. The precession is due to the fact that after a change of θ by 2π the factor $\lambda\theta$ does not change 2π except iff $\lambda = 1$, which is the Newtonian orbit. After a complete rotation the ellipsis "closes" by an angle $2\pi + \Delta\theta$ where per complete rotation $\Delta\theta$ is defined by the relation

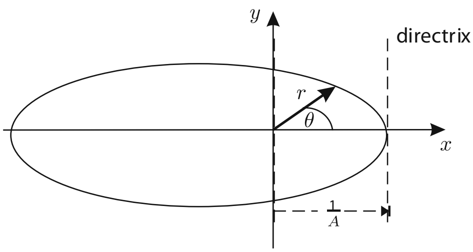


Fig. 11.3 Specifying the coordinates r, θ

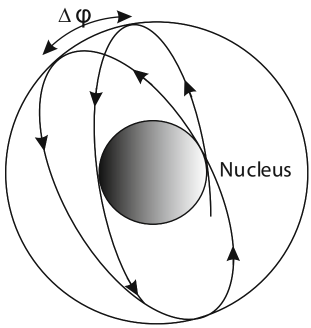


Fig. 11.4 Precession of the orbit

$$\lambda(2\pi + \Delta\theta) = 2\pi \Longrightarrow \Delta\theta = 2\pi\frac{1-\lambda}{\lambda}.$$

Because $1 - \lambda$ deviates from 1 in terms of the order $\frac{1}{c^2}$ its value is close to 1, hence the deviation $\Delta\theta$ per rotation is very small. Kinematically this means that the the orbit precesses along the enveloping circles with very small angular velocity.

Sommerfeld used the precession of the relativistic orbit in the old Quantum Theory in order to determine the quantum conditions for the energy levels and the probability for the transition between stable orbits of the hydrogen atom. As it is well known, in the old Quantum Theory for every periodicity of a stable classical orbit, there corresponds a quantum number. For example, Bohr had used the periodicity of the Newtonian orbit in order to introduce the quantum number (= quantize) of angular momentum and, based on that, to compute the energy levels of the hydrogen atom. By using the relativistic orbit, Sommerfeld calculated in addition the fine energy levels of the energy spectrum of the hydrogen atom.

If we assume that the angular momentum L_z is quantized in multiples of $\hbar = \frac{h}{2\pi}$ then we find that the minimum value of $L_z = \hbar$ gives the minimum value of λ, which is

$$\lambda^2_{\text{min}} = 1 - \left(\frac{Ze^2}{\hbar c}\right)^2 = 1 - Z^2\alpha^2,$$

where

$$\alpha = \frac{e^2}{\hbar c} = 7,2972 \times 10^{-3} = \frac{1}{134,04}$$

is a universal constant known as the *fine structure constant*.

The constant α determines the strength of the electromagnetic interaction among the fundamental particles by relating the quantum of the electric charge (e) with the quantum of the angular momentum $\hbar$ and the speed of light c. The deviation of λ_{min} from 1 is of the order of $\left(\frac{Z}{134}\right)^2$. For H_2 ($Z = 1$) hence the quotient $\frac{1}{134} = 5.6 \times 10^{-5}$, which can be observed in spectroscopy The fine lines in the spectrum of hydrogen were observed and that was one of the most accurate tests of relativistic dynamics.

However, when the dynamics of Special Relativity is applied to the gravitational field fails to justify the observational results.

Let us consider the gravitational field of the sun $U(r) = -\frac{GM}{r}$. Then all previous relations hold if one replaces Ze^2 with GMm where m is the mass of the particle which moves under the action of the gravitational field. If we compute from these relations the precession of Mercury we find that it precesses $7''$ per century, while the measured value (with accuracy $5''$) is $43''$ per century. Therefore, when dealing with the gravitational field we should apply the dynamics of General and not of Special Relativity.[13]

11.6 Motion of a Rocket

The motion of a rocket is a motion in which the mass changes. In the study of this motion, the rocket is considered to be a ReMaP with proper time τ, four-velocity u^i, four-acceleration a^i whose mass is changing at a rate $dm/d\tau$, e.g., by the emission of exhaust gases. The physical variables are the ones which can be measured by the observer inside the rocket (the proper frame). These are

- The rate of reduction of (proper) mass $dm/d\tau$ (an invariant) and
- The relative speed of emitted gases w'.

In the following, we assume one-dimensional motion so that the velocity of the exhausted gases is antiparallel to the (constant) direction of the velocity of the rocket. To determine the equation of motion of the rocket we consider the change of the four-momentum between the proper time moments τ and $\tau + d\tau$. Suppose that the proper moment τ the mass of the rocket is $m(\tau)$, its four-velocity is $u^i(\tau)$ and that the proper moment $\tau + d\tau$ these quantities are (assuming a continuous variation) $m(\tau + d\tau) = m(\tau) - dm$ and $u^i(\tau + d\tau) = u^i(\tau) + du^i$.

Let dm' the mass of the gases (we do not have conservation of mass!) and let w^i their four-velocity. Conservation of four-momentum of the system between the proper time moments τ and $\tau + d\tau$ gives

$$mu^i = (m - dm)(u^i + du^i) + dm'w^i \Rightarrow mdu^i - dmu^i + dmw^i = 0, \quad (11.89)$$

where we have ignored the term $dmdu^i$.

This equation is covariant, that is, it is valid in all LCF. We write it in the LCF Σ^+ of the IRIO of the rocket and compute it in any other LCF using the appropriate Lorentz transformation. The $u^i u_i = -c^2 \Rightarrow u^i du_i = 0$, that is, du^i is a spacelike four-vector.

[13] For a different view of this topic see T. E. Philips Jr., "Mercury precession according to Special Relativity" *Am. J. Phys.* **54**, 245–247, (1986).

Let $\mathbf{e}$ be the unit vector along the direction of motion of the rocket. Then for the four-vectors involved we have the following decomposition:

$$u^i = \begin{pmatrix} c \\ \mathbf{0} \end{pmatrix}_{\Sigma^+}, \qquad du^i = \begin{pmatrix} 0 \\ du'\mathbf{e} \end{pmatrix}_{\Sigma^+}, \qquad w^i = \begin{pmatrix} \gamma(w')c \\ -\gamma(w')w'\mathbf{e} \end{pmatrix}_{\Sigma^+},$$

where du' is the change in the four-velocity of the rocket during the period τ and $\tau + d\tau$. Replacing in (11.89) we find the following two equations:

$$dm' = \frac{1}{\gamma(w')} dm, \tag{11.90}$$

$$m\frac{du'}{dm} = w'. \tag{11.91}$$

Equation (11.90) gives the mass of the exhaust gases in terms of the diminution of the mass of the rocket and (11.91) is the equation of motion of the rocket in its proper frame Σ^+[14].

We determine the equation of motion of the rocket in a LCF Σ (the Earth say) in which the rocket at the moments $t,\ t + dt$ has velocities u and $u + du$, respectively. To do that we write the decomposition of the four-velocity of the rocket in Σ and Σ^+:

$$u^i(t + dt) = \begin{pmatrix} \gamma(du')c \\ \gamma(du')du'\hat{\mathbf{e}} \end{pmatrix}_{\Sigma^+}, \quad u^i(t + dt) = \begin{pmatrix} \gamma(u + du)c \\ \gamma(u + du)(u + du)\hat{\mathbf{e}} \end{pmatrix}_{\Sigma}$$

and apply the boost with velocity u:

$$\gamma(du') = \gamma(u)\left(\gamma(u + du) - \frac{u}{c^2}\gamma(u + du)(u + du)\right),$$
$$\gamma(du')du' = \gamma(u)\left(\gamma(u + du)(u + du) - u\gamma(u + du)\right) = \gamma(u)\gamma(u + du)du.$$

Replacing $\gamma(du')$ from the first and solving the second in terms of du' we find

$$du' = \gamma^2(u)du. \tag{11.92}$$

Replacing du' in (11.91) we find the equation of motion of the rocket in Σ to be

$$m\gamma^2(u)\frac{du}{dm} = w'. \tag{11.93}$$

[14] Equation (11.90) might give the impression that it is wrong because it relates two invariants, the masses dm and dm' with the scalar factor $\gamma(w')$! However, this view is not correct and it is due to Newtonian ideas. Indeed, the observer inside the rocket does not "know" (that is, cannot measure) the mass of the emitted gases, therefore the quantity dm' refers to the energy of the gases of mass dm and velocity w' wrt the rocket.

Equation (11.93) makes possible for the observer in the rocket, to be able to determine his motion in Σ by looking at the indication (a) of the instrument which measures dm/dt, that is, the rate of consumption of fuel and (b) a second instrument which measures the relative velocity of the exhausted gases w' wrt the rocket. Indeed, if these two quantities are known then by integrating (11.93) the traveler is able to compute the speed and the position of the rocket in Σ. If we consider an approximation of the order $O(\beta^2)$ then (11.93) reduces to the corresponding equation of Newtonian Physics, as it is expected.

In order to solve the equation of motion (11.93) we have to make certain assumptions, which have a sound physical basis. We assume $w' =$ constant, which means that the engines of the rocket work steadily. Furthermore we consider the initial conditions $m(0) = M,\ u(0) = u_0$. Then (11.93) gives

$$\int_{u_0}^{u} \frac{du}{1-\frac{u^2}{c^2}} = \int_{M}^{m} w' \frac{dm}{m}$$

from which follows

$$\frac{c}{2}\left[\ln\left(\frac{1+\frac{u}{c}}{1-\frac{u}{c}}\right) - \ln\left(\frac{1+\frac{u_0}{c}}{1-\frac{u_0}{c}}\right)\right] = -\ln\left(\frac{m}{M}\right)^{w'}.$$

Solving for $u(t)$ we find

$$u(t) = \frac{A-\left(\frac{m}{M}\right)^B}{A+\left(\frac{m}{M}\right)^B}\,c, \tag{11.94}$$

where $A = \frac{1+\frac{u_0}{c}}{1-\frac{u_0}{c}}$ and $B = \frac{2w'}{c}$. For initial speeds $u_0 \ll c$ the term u_0/c can be ignored ($A = 1$) and the solution (11.94) is written as

$$u(t) = \frac{1-\left(\frac{m}{M}\right)^B}{1+\left(\frac{m}{M}\right)^B}\,c. \tag{11.95}$$

We note that always $u < c$ and $u \to c$ only when the quotient $m/M \to 0$. Furthermore, the rate of change of the mass dm/dt does not enter the calculations. For $u_0/c \ll 1$ (and also for $m/M = 1$) we recover again the well-known Newtonian result. In general, the Newtonian behavior appears when the parameter $\left(\frac{m}{M}\right)^B \ll 1$. This happens at two phases of the motion:

- At the beginning of the motion, when $m/M = 1$ (and small velocities)
- When $2w'/c \ll 1$ approximately for all the duration of motion.

Practically, we have Newtonian behavior for $w'/c = 0.1$. In Fig. 11.5 we display the quotient u/c as a function of the ratio m/M and for the values $A = 1$ and $B = 0.1$ (Newtonian limit) and $B = 0.4$ (relativistic limit).

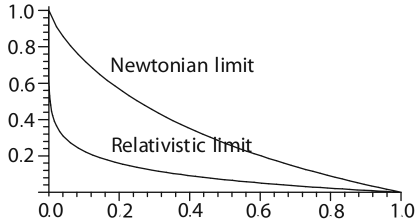


Fig. 11.5 Newtonian ($B = 0.1$) and relativistic ($B = 0.4$) motion of the rocket

[Question: What one can say for photonic fuel, that is fuel for which the exhausted gases are photons? What is the mass dm' in that case?].

Example 60 A spacecraft departs from the Earth and moves along a constant direction toward the center of galaxy, which is a distance of 30,000 ly as measured from the Earth. The engines of the spacecraft work steadily and create a constant proper acceleration g for the first half of the trip and a constant retardation g for the return trip.

(1) How long will the trip last according to the proper clock of the spacecraft?
(2) What distance has the spacecraft covered as estimated by the crew in the spacecraft?
(3) What fraction of the initial mass of the spacecraft will be used if we assume that the engines of the spacecraft transform the mass of the spacecraft into photons (photonic fuel) with an efficiency of 100% and the radiation is emitted in the opposite direction to the direction of the velocity of the spacecraft? Assume $c = 1$.

Solution

(1) Assume that the x'-axis joins the Earth with the center of galaxy (this is possible because the three-dimensional space in Special Relativity is flat, i.e., has zero curvature). Then $y = z = 0$ during all the duration of the trip. For the four-velocity and the four-acceleration of the rocket in the frame Σ of the Earth we have the decomposition

$$u^i = \begin{pmatrix} u^0 \\ u^1 \end{pmatrix}_\Sigma, a^i = \begin{pmatrix} a^0 \\ a^1 \end{pmatrix}_\Sigma.$$

The orthogonality relation is

$$u^i a_i = 0 \Rightarrow -a^0 u^0 + a^1 u^1 = 0 \Rightarrow a^0 = \lambda u^1, a^1 = \lambda u^0,$$

where $\lambda \neq 0$ is a constant. It is given that $a^i a_i = g^2$. Because the length of a four-vector is invariant we compute it in Σ and find

$$-(a^0)^2 + (a^1)^2 = g^2 \Rightarrow \lambda = \pm g,$$

where $\lambda = +g$ is for the first part of the trip and $\lambda = -g$ for the return trip back to Earth. Finally, for the departure trip we have the equations of motion

$$\frac{du^0}{d\tau} = gu^1,$$
$$\frac{du^1}{d\tau} = gu^0.$$

Differentiating the first wrt τ and replacing $\frac{du^1}{d\tau}$ from the second we find

$$\frac{d^2u^0}{d\tau^2} = g\frac{du^1}{d\tau} = g^2u^0 \Rightarrow \frac{d^2u^0}{d\tau^2} - g^2u^0 = 0.$$

The solution of this equation is

$$u^0 = A \sinh g\tau + B \cosh g\tau,$$

where $A,\ B$ are constants. In a similar manner we compute

$$u^1 = A' \sinh g\tau + B' \cosh g\tau.$$

The initial conditions are

$$\tau = t = 0,\ u^1(0) = 0,\ \frac{du^1}{dt}(0) = g, u^0(0) = 1$$

from which follows

$$u^1(\tau) = \sinh g\tau, \qquad u^0(\tau) = \cosh g\tau. \tag{11.96}$$

Concerning the position we have from (11.96)

$$\frac{dt}{d\tau} = u^0(\tau) = \cosh g\tau,$$
$$\frac{dx}{d\tau} = \frac{dx}{dt}\frac{dt}{d\tau} = \frac{1}{\gamma}u^1(\tau)\gamma = \sinh g\tau.$$

Assuming the initial conditions $\tau = t = x = 0$ we end up with

$$t(\tau) = \frac{1}{g}\sinh g\tau, \qquad x(\tau) = \frac{1}{g}(\cosh g\tau - 1)$$

which is (as it should be expected!) the expression of hyperbolic motion (see Example 27).

In order to get an idea of the result let us make some calculations. To do that we have to express g in units of distance and time. We note that $1\,\text{ly} =$

lightyear $\approx 9.5\times10^{15}$ m and 1 year $\approx 3.15\times10^{17}$ s. Moreover we have assumed $c = 1 \Rightarrow 1\,s = 3 \times 10^8$ m. Using these, we compute in units of distance

$$g = 10\,\mathrm{m/s}^2 = 10 \times \frac{1}{(3 \times 10^8)^2}\,\mathrm{m}^{-1}$$
$$= 10 \times \frac{1}{(3 \times 10^8)^2} \times (9.5 \times 10^{15})\,\mathrm{ly}^{-1} \approx 1\,(\mathrm{ly})^{-1}.$$

Similarly in units of time we have

$$g = 10\,\mathrm{m/s}^2 = 10 \times \frac{1}{3 \times 10^8}\,\mathrm{s}^{-1} = 10\frac{1}{3 \times 10^8} \times (3\times10^7)(\mathrm{years})^{-1} \approx 1\,(\mathrm{years})^{-1}.$$

Using these units we find that for the first part of the trip, that is, $x = 15,000$ ly, the time

$$\tau = \frac{1}{g}\cosh^{-1}(gx + 1) \Rightarrow \tau = [(1\text{ year})^{-1}]^{-1}\cosh^{-1}[(\mathrm{ly})^{-1}(15000\text{ ly}) + 1]$$
$$= \cosh^{-1}(15001)\text{ years} = 10.31\text{ years}.$$

Obviously for the whole trip it is required twice this time period, that is

$$\tau_{\mathrm{total}} = 2\,\tau = 20.62\text{ years}.$$

The time period between the events of departure and arrival of the spacecraft for the observer on Earth is

$$t = \frac{1}{g}\sinh g\tau = \sinh 20.62 = 4.5 \times 10^8\text{ years}.$$

Observe the difference between t and τ and appreciate the well-known twin paradox.

(2) In order to compute the distance covered by the spacecraft from the Earth, as estimated by the crew at the spacecraft, we consider coordinates $(cT,\ X)$ in the proper frame of the spacecraft and apply the boost with velocity u^1/γ where u^1 is the x-component of the four-velocity. We find

$$X = \gamma\left(x - \frac{u^1}{\gamma}t\right), \tag{11.97}$$

$$T = \gamma\left(t - \frac{u^1}{\gamma}x\right). \tag{11.98}$$

But

$$\gamma = u^0 = \cosh g\tau.$$

therefore (11.97) and (11.98) give

$$X = \frac{1}{g}(1 - \cosh g\tau) = -x.$$

This result is important because it proves that the distance of the spacecraft from the Earth is the same as estimated either by the observer at the Earth or by the crew in the spacecraft.

(3) Let M_0 be the initial mass of the spacecraft and $M(\tau)$ the mass after proper time τ. Then the energy $E(\tau) = M(\tau)\gamma(\tau) = M(\tau)u^0$ and the three-momentum $P(\tau) = M(\tau)\gamma(\tau)u = M(\tau)u^1$. The proper moment $\tau + d\tau$ the change of the energy and the momentum of the spacecraft, dE and dP, respectively, equal the corresponding quantities dE_{photons} and dP_{photons} of the emitted photons, respectively. But for photons we have that $dE_{\text{photons}} = dP_{\text{photons}}$ hence we have the following equation of motion:

$$-dP = dE \Rightarrow d(E + P) = 0 \Rightarrow E + P = \text{constant}.$$

From the initial conditions $E(0) = M_0,\ P(0) = 0$ we compute that the value of the constant is M_0, therefore we have finally

$$M(\tau)u^0 + M(\tau)u^1 = M_0 \Rightarrow M(\tau) = \frac{M_0}{u^0 + u^1}.$$

We replace $u^0,\ u^1$ from (11.96) and get

$$M(\tau) = M_0 e^{-g\tau}.$$

For proper time $\tau_{\text{total}} = 20.62$ years we find that the percentage of the mass which has been used equals

$$\Delta M = \left(1 - \frac{M(\tau)}{M_0}\right) \times 100 \ \approx (1 - e^{-20.26}) = 99.999\%.$$

Example 61 A particle of mass m and velocity u moves along the x-axis of the LCF Σ.

(1) Show that the three-momentum and the energy of the particle are given by the relations $P = mc \sinh \chi,\ E = mc^2 \cosh \chi$ where χ is the rapidity of the particle in Σ. Deduce that the speed and the rapidity of the particle in Σ are related as follows $u(t) = c \tanh \chi$.

(2) Consider a second LCF Σ' which moves wrt Σ in the standard configuration along the x-axis with speed v. Show that in Σ' the three-momentum P', E', and the energy E' of the particle are given by the following formulae:

$$P' = mc \sinh(\chi + \psi), \qquad E' = mc^2 \cosh(\chi + \psi),$$

where $\tanh\psi = v/c$. Explain the geometric meaning of this result. Assume that the speed u changes to $u + du$. Prove that

$$du\gamma^2(u) = cd\chi.$$

If du' is the corresponding change of the velocity in Σ' show that

$$du'^2/(u'^2(u) = cd\chi.$$

(3) Application.
A rocket moves freely by exhausting gases with a constant rate $\mu = dM/d\tau$ and velocity $-w'\hat{\mathbf{e}}$, where $\hat{\mathbf{e}}$ is the (constant) direction of motion of the rocket. If the rocket starts from a space platform A with initial mass M_0 and moves along the x-axis calculate the velocity $u(t)$ of the rocket wrt the platform when its mass is $M(\tau)$.

Solution
(2) The speed u and the rapidity χ of a particle in a LCF Σ are related as follows $u = c\tanh\chi$. Differentiating we find

$$du = \frac{1}{\cosh^2\chi}cd\chi = \frac{1}{\gamma^2(u)}cd\chi \Rightarrow \gamma^2(u)du = cd\chi. \tag{11.99}$$

From the relativistic rule of composing velocities we have that the rapidity χ' of the particle in Σ' is

$$\chi' = \chi + \psi,$$

where ψ is the rapidity of Σ' wrt Σ ($v = c\tanh\psi$). If v' is the velocity of the particle in Σ', then applying relation (11.99) for the velocity u' we find

$$du' = \frac{1}{\cosh^2\chi'}cd\chi' = \gamma^2(u')du' = cd\chi = \gamma^2(u)du \tag{11.100}$$

because $d\psi = 0$. From (11.99) and (11.100) we infer

$$\gamma^2(u')du' = \gamma^2(u)du.$$

which means that the quantity $\gamma^2(u)du$ is an invariant (but depends on the time t in Σ).
Application
The equation of motion of the rocket in Σ is (see (11.93)):

$$m\gamma^2(u)\frac{du}{dm} = w',$$

where w' is the speed of the exhausted gases wrt the rocket. In order to write this equation in terms of the rapidity χ we use (11.99) and find

$$cd\chi = w'\frac{dm}{m}.$$

The solution of this equation is

$$\chi - \chi_0 = \ln\left(\frac{M}{M_0}\right)^{\frac{w'}{c}}.$$

From the initial conditions we have $\chi_0 = \tanh^{-1} 0 = 0$, hence

$$\chi = \ln\left(\frac{M_0}{M}\right)^{w'/c}. \tag{11.101}$$

From the relation which gives the rapidity we find $e^{\chi} = \sqrt{\frac{c+u}{c-u}}$. Replacing in (11.101)

$$u(t) = \frac{1-(M/M_0)^{2\beta'}}{1+(M/M_0)^{2\beta'}}\,c,$$

where $\beta' = \frac{w'}{c}$.

11.7 The Frenet–Serret Frame in Minkowski Space

In classical vector calculus the Frenet–Serret equations define at each point along a smooth curve an orthonormal basis – or a "moving frame" – which is called the *Frenet–Serret frame*. The usefulness of the Frenet–Serret frame is that it characterizes and is characterized uniquely by the curve. In this section, we define the Frenet–Serret frame along a curve in Minkowski space and demonstrate its application in the definition of a generic four-force vector. It is apparent that in Minkowski space the equations defining the Frenet–Serret frame along a given curve must be four and the orthonormal tetrad defining the frame will involve Lorentz orthogonal vectors.

The first difference between the Euclidean and the Lorentzian Frenet–Serret frame is that in the first case one has one type of curves whereas the second one has three types of curves, the timelike, the spacelike, and the null.[15] Furthermore, the null curves are degenerate in the sense that their tangent vector is also normal to

[15] Besides these curves there are infinite other curves in Minkowski space which, however, do not interest us in relativity.

the curve. Therefore, the Frenet–Serret frame applies only to non-null (i.e., timelike and spacelike) curves.

Let $c^i(r)$ be a smooth non-null curve in Minkowski space and let r be an affine parameter[16] so that the tangent vector to the curve is $A^i = dc^i/dr$:

$$A^i A_i = \varepsilon(A), \tag{11.102}$$

$\varepsilon(A)$ being the sign of A^i, that is, $\varepsilon(A) = +1/-1$ if A^i is spacelike/timelike, respectively.

In the following, we shall denote the covariant differentiation along A^i with a dot, e.g.,

$$\dot{T}^{i_1\ldots i_r}_{j_1\ldots j_s} = T^{i_1\ldots i_r}_{j_1\ldots j_s,k} A^k. \tag{11.103}$$

Because A^i is unit $\dot{A}^i A_i = 0$.

Let B^i be the unit vector along $\dot{A}^i$ so that

$$\dot{A}^i = bB^i \qquad (b > 0). \tag{11.104}$$

Then

$$B^i A_i = 0, \qquad B^i B_i = \varepsilon(B). \tag{11.105}$$

We consider the derivative $\dot{B}^i$ and decompose it parallel and normal to the plane defined by A^i, B^i. We write

$$\dot{B}^i = aA^i + cC^i, \tag{11.106}$$

where a, c are coefficients and C^i is a unit vector normal to both A^i and B^i and so that the quantity $\eta_{ijk} A^i B^j C^k > 0$ (this defines the positive orientation of the frame). Then we have

$$C^i C_i = \varepsilon(C) \tag{11.107}$$

and

$$A^i C_i = B^i C_i = 0. \tag{11.108}$$

In order to compute the coefficients a, c we differentiate (11.105) and use (11.104) to get

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[16] Recall that a parameter along a non-null parameterized curve is called affine if the tangent vector along the curve (for this parametrization) has unit length, i.e., ± 1. If r is an affine parameter then $ar + b$ where $a, b \epsilon R$ is also an affine parameter.

$$\dot{B}^i A_i = -B^i \dot{A}_i = -b\varepsilon(B).$$

But from (11.106) we have

$$\dot{B}^i A_i = a\varepsilon(A) \Rightarrow a = -\varepsilon(A)\varepsilon(B)b. \tag{11.109}$$

Replacing in (11.106) we obtain finally

$$B^i = -\varepsilon(A)\varepsilon(B)bA^i + cC^i. \tag{11.110}$$

We apply the same procedure to the unit vector $\dot{C}^i$ and write

$$\dot{C}^i = \beta A^i + \gamma B^i + dD^i, \tag{11.111}$$

where D^i is the unit normal to the trihedral defined by the four-vectors A^i, B^i, C^i, i.e.,

$$D^i D_i = \varepsilon(D) \tag{11.112}$$

and

$$A^i D_i = B^i D_i = C^i D_i = 0. \tag{11.113}$$

In order to compute the coefficients β, γ we contract (11.111) with A^i, B^i and use (11.108) to find

$$\beta\varepsilon(A) = \dot{C}^i A_i = -C^i \dot{A}_i = -C^i bB_i = 0$$

and

$$\gamma\varepsilon(B) = \dot{C}^i B_i = -C^i \dot{B}_i = -\varepsilon(C)c.$$

Hence

$$\beta = 0, \qquad \gamma = -\varepsilon(B)\varepsilon(C)c$$

and (11.111) is written as follows:

$$\dot{C}^i = -\varepsilon(B)\varepsilon(C)cB^i + dD^i. \tag{11.114}$$

We decompose similarly $\dot{D}^i = \delta A^i + \epsilon B^i + nC^i$ and making use of (11.113) we find $\delta = \epsilon = 0$ and $n = -d\varepsilon(C)\varepsilon(D)$, so that[17]

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[17] The four-vector D^i is determined uniquely from the "outer product" of the three four-vectors A^i, B^i, C^i, i.e., $D^i = \eta^i_{jkl} A^j B^k C^l$ where $\eta^i_{jkl} = \sqrt{-g}\epsilon^i_{jkl}$ and ϵ^{ijkl} is the Levi-Civita symbol

$$\dot{D}^i = -\varepsilon(C)\varepsilon(D)dC^i. \tag{11.115}$$

From the above analysis we reach at the following conclusion:

Proposition 7 *At every point along a smooth non-null curve in Minkowski space, affinely parameterized with the parameter $r =$ arc length it is possible to construct an orthonormal tetrad A^i, B^i, C^i, D^i such that the four-vector A^i is tangent to the curve and the rest three spacelike mutually perpendicular four-vectors B^i, C^i, D^i to follow from the solution of the system of differential equations*

$$\dot{A}^i = bB^i, \tag{11.116}$$
$$\dot{B}^i = -\varepsilon(A)\varepsilon(B)bA^i + cC^i, \tag{11.117}$$
$$\dot{C}^i = -\varepsilon(B)\varepsilon(C)cB^i + dD^i, \tag{11.118}$$
$$\dot{D}^i = -\varepsilon(C)\varepsilon(D)dC^i, \tag{11.119}$$

where a dot over a symbol indicates differentiation wrt r.

The four-vectors $B^i,\ C^i,\ D^i$ we call the *first, second, and third normal*, respectively, of the curve c^i and the parameters $b,\ c,\ d$ the first, second and third curvature of c^i. The geometric significance of the Frenet–Serret curvatures is that they define uniquely the curve c^i (for given initial conditions).[18] If we denote the curvatures $b,\ c,\ d$ of the curve by $\kappa_1, \kappa_2, \kappa_3$, respectively, the Serret–Frenet relations are written in the following matrix notation:

$$\begin{pmatrix} \dot{A}^i \\ \dot{B}^i \\ \dot{C}^i \\ \dot{D}^i \end{pmatrix} = S \begin{pmatrix} A^i \\ B^i \\ C^i \\ D^i \end{pmatrix}, \tag{11.120}$$

where the 4×4 matrix

$$S = \begin{pmatrix} 0 & \kappa_1 & 0 & 0 \\ -\varepsilon(A)\varepsilon(B)\kappa_1 & 0 & \kappa_2 & 0 \\ 0 & -\varepsilon(B)\varepsilon(C)\kappa_2 & 0 & \kappa_3 \\ 0 & 0 & -\varepsilon(C)\varepsilon(D)\kappa_3 & 0 \end{pmatrix}. \tag{11.121}$$

($\epsilon_{ijkl} = +1$ if $ijkl$ is an even permutation of 0123 and $\epsilon_{ijkl} = -1$ otherwise). This implies that the parameter d is possible to be expressed in terms of the parameters $a,\ b,\ c$ and their derivatives. Therefore, in Minkowski space we have three four-vectors and three parameters associated uniquely with a given smooth curve and the associated Frenet–Serret frame. The orientation of this frame is positive or negative according the sign of the determinant of the 4×4 matrix defined by the four-vectors $A^i,\ B^i,\ C^i,\ D^i$ or equivalently from the sign of the quantity $\eta_{jkls}A^jB^kC^lD^s$.

[18] We note that the same relations hold in the case of General Relativity with the difference that the partial derivative is replaced with the covariant Riemannian derivative. Moreover if we set $\varepsilon(A) = 1,\ D^i = 0$ we recover the Frenet–Serret formulae of the Euclidean three-dimensional geometry.

In case the curve c^i is timelike $\varepsilon(A) = -1, \varepsilon(B) = \varepsilon(C) = \varepsilon(D) = +1$ and (11.116), (11.117), (11.118) and (11.119) become

$$A^i = \kappa_1 B^i, \tag{11.122}$$
$$\dot{B}^i = \kappa_1 A^i + \kappa_2 C^i, \tag{11.123}$$
$$\dot{C}^i = -\kappa_2 B^i + \kappa_3 D^i, \tag{11.124}$$
$$\dot{D}^i = -\kappa_3 C^i. \tag{11.125}$$

Consequently the matrix S:

$$S = \begin{pmatrix} 0 & \kappa_1 & 0 & 0 \\ \kappa_1 & 0 & \kappa_2 & 0 \\ 0 & -\kappa_2 & 0 & \kappa_3 \\ 0 & 0 & -\kappa_3 & 0 \end{pmatrix}. \tag{11.126}$$

In case c^i is a spacelike curve then $\varepsilon(A) = 1, \varepsilon(B) = -1$ (say) $\varepsilon(C) = \varepsilon(D) = 1$ and relations (11.116), (11.117), (11.118) and (11.119) are

$$A^i = \kappa_1 B^i, \tag{11.127}$$
$$\dot{B}^i = \kappa_1 A^i + \kappa_2 C^i, \tag{11.128}$$
$$\dot{C}^i = \kappa_2 B^i + \kappa_3 D^i, \tag{11.129}$$
$$\dot{D}^i = -\kappa_3 C^i \tag{11.130}$$

Exercise 50 Prove that the Frenet–Serret coefficients can be written in covariant form as follows:

$$\kappa_1 = \dot{A}^i B_i \qquad \kappa_2 = \dot{B}^i C_i \qquad \kappa_3 = \dot{C}^i D_i. \tag{11.131}$$

Deduce that the curvatures $b, \ c, \ d$ are invariants – being inner products of four-vectors – and characterize a curve in an intrinsic manner (that is, independent of the choice of coordinate system). As we have already remarked, it is this result which makes the Frenet–Serret formalism a useful geometric tool in the study of world lines.

Exercise 51 The straight line in Minkowski space is defined by the condition A^i =constant. Use the Frenet–Serret equations (11.116), (11.117), (11.118) and (11.119) to show that for a straight line the curvatures $\kappa_1 = \kappa_2 = \kappa_3 = 0$. Conversely using the values $\kappa_1 = \kappa_2 = \kappa_3 = 0$ in the same equations prove that the curve they describe is a straight line. This result makes possible the definition of the straight line in a algebraic and covariant manner.

In order to show how one computes the Frenet–Serret basis along a world line (i.e., a timelike curve) we consider a ReMaP P, which starts to move from the origin of an LCF Σ along the x-axis with constant proper acceleration $a > 0$ (hyperbolic motion). As we have shown (see equation (7.47)) the world line of P is ($c = 1$)

$$x = \frac{1}{a}(\cosh(a\tau) - 1) \qquad y = 0, \qquad z = 0,$$

where τ is the proper time of P (related to the time t of Σ with the relation $t = \frac{1}{a}(\sinh(a\tau))$. The four-vector A^i is the four-velocity u^i with components $u^i = (u^0, u^1, 0, 0)^t_\Sigma$ where $u^0 = \cosh(a\tau), u^1 = \sinh(a\tau)$. The four-vector $\dot{A}^i$ is the four-acceleration of P, which in Σ is given by $a^i = (a^0, a^1, 0, 0)^t_\Sigma$ where $a^0 = a\sinh(a\tau) = au^1, a^1 = a\cosh(a\tau) = au^0$. The length of the four-acceleration is a, hence the unit vector along the direction of a^i, which is also the first normal to the curve, is $B^i = (u^1, u^0, 0, 0)_\Sigma$. In order to compute the second normal to the curve we differentiate B^i and find

$$\dot{B}^i = (\dot{u}^1, \dot{u}^0, 0, 0)^t = (au^0, au^1, 0, 0)^t = au^i.$$

From (11.123) follows $\kappa_2 = 0$. For this value of κ_2 (11.124) and (11.125) are independent of the remaining two and are satisfied by infinitely many pairs of four-vectors C^i, D^i. This is due to the fact the the motion takes place in the plane $x - ct$, which can be embedded in infinitely many ways into the four-dimensional Minkowski space.

11.7.1 The Physical Basis

The Frenet–Serret frame has a special physical significance in the case of timelike curves. Indeed in this case the vector A^i is the four-velocity of the ReMaP P, the first normal is the direction of the four-acceleration while the first curvature κ_1 is the measure of the four-acceleration. Therefore, the Frenet–Serret frame covers the basic physical quantities of kinematics. There still remains a part which contains higher than the first derivatives of the four-velocity. This extra part must be related to the dynamics of motion, that is, the external four-force which modulates the motion of P.

It is speculated that the four-force depends on the higher derivatives of the four-velocity when a charge accelerates. Indeed, it is assumed that an accelerating charge radiates an electromagnetic field which exerts a force back on the charge. This force is not a Lorentz force. Up to now, many formulae for this type of force have been proposed but it appears that "the force" has not yet been found. Although we do not know the actual expression of this type of force, nonetheless it is possible by the use of the Frenet–Serret frame, to give the *generic* expression for this four-force in terms of the "physical" kinematic quantities $u^a, \dot{u}^a, \ddot{u}^a, \dddot{u}^a$. That is, we give a parameterized geometric expression which contains all possible four-forces one could conceive. The role of physics is to select the appropriate values of the parameters!

In the following, we assume that the generic four-force is a pure four-force, that is a spacelike four-vector normal to the four-velocity. Summarizing we have to solve the following problem:

> Compute the generic form of a spacelike four-vector along a smooth non-null curve – which is not a straight line – in a basis which consists of the unit tangent to the curve and its higher derivatives, assuming that they do not vanish. We call this new (not in general orthonormal) basis the *physical basis* along the curve.

This means that we are looking – if it exists! – for an expression of the form

$$r^i = f_i^0(x^i, u^i, \dot{u}^i, ...)u^i + f_i^1(x^i, u^i, \dot{u}^i, ...)\dot{u}^i + \cdots , \tag{11.132}$$

where r^i is an arbitrary four-vector (null or non-null) defined along the curve.

Let c^i be a non-null, smooth, affinely parameterized curve in Minkowski space with unit tangent vector A^i. From the Frenet–Serret relations we have

$$A^i, \quad \dot{A}^i = bB^i, \tag{11.133}$$

where

$$A^i A_i = \varepsilon(A), \quad B^i A_i = 0, \qquad B^i B_i = \varepsilon(B).$$

Differentiating twice $\dot{A}^i$ along the curve we find[19]

$$\ddot{A}^i = -\varepsilon(A)\varepsilon(B)b^2 A^i + \dot{b}B^i + bcC^i. \tag{11.134}$$

$$\begin{aligned} \dddot{A}^i &= [-3\varepsilon(A)\varepsilon(B)\dot{b}b]A^i + [\ddot{b} - \varepsilon(A)\varepsilon(B)b^3 - \varepsilon(B)\varepsilon(C)bc^2]B^i \\ &\quad + [2c\dot{b} + b\dot{c}]C^i + bcdD^i. \end{aligned} \tag{11.135}$$

We conclude that in general the tangent and the Frenet–Serret bases are related as follows:

$$\begin{bmatrix} A^i \\ \dot{A}^i \\ \ddot{A}^i \\ \dddot{A}^i \end{bmatrix} = R \begin{bmatrix} A^i \\ B^i \\ C^i \\ D^i \end{bmatrix}, \tag{11.136}$$

where the 4×4 matrix

[19] The three four-vectors $A^i, \dot{A}^i, \ddot{A}^i$ are sufficient (provided they are independent) in order to compute the fourth by the formula $T^i = \eta^i_{jkl} A^j \dot{A}^k \ddot{A}^l$. This means that, in general, the physical basis requires only up to the second derivative of the vector A^i along the curve.

$$R = \left[\begin{array}{c|c|c|c} 1 & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ -\varepsilon(A)\varepsilon(B)b^2 & \dot{b} & bc & 0 \\ -3\varepsilon(A)\varepsilon(B)b\dot{b} & \ddot{b} - \varepsilon(A)\varepsilon(B)b^3 - \varepsilon(B)\varepsilon(C)bc^2 & b\dot{c} + 2c\dot{b} & bcd \end{array} \right].$$

The matrix R defines a change of bases (not necessarily change of coordinates[20]) if its determinant does not vanish. We compute

$$\det(R) = b^3c^2d \tag{11.137}$$

from which we infer that R defines a change of basis provided $bcd \neq 0$. Kinematically this means that the physical basis is possible only for non-planar accelerated motions (see Exercise 51). In case one of the curvatures $b,\ c,\ d$ vanishes we can consider higher derivatives of A^i until we obtain a basis along the curve.

In the following we assume $bcd \neq 0$ and express the four-vectors of the Frenet–Serret tetrahedron in terms of the four-vectors of the physical basis using the matrix R. We have

$$\begin{bmatrix} A^i \\ B^i \\ C^i \\ D^i \end{bmatrix} = R^{-1} \begin{bmatrix} A^i \\ \dot{A}^i \\ \ddot{A}^i \\ \dddot{A}^i \end{bmatrix}. \tag{11.138}$$

We compute

$$R^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{b} & 0 & 0 \\ \frac{\varepsilon(A)\varepsilon(B)b}{c} & -\frac{\dot{b}}{b^2c} & \frac{1}{bc} & 0 \\ \Lambda & M & N & \Xi \end{bmatrix}, \tag{11.139}$$

where

$$\Lambda = (R^{-1})_{41} = -\frac{\varepsilon(A)\varepsilon(B)(\dot{c}b - \dot{b}c)}{c^2d}, \tag{11.140}$$

$$M = (R^{-1})_{42} = \frac{\varepsilon(A)\varepsilon(B)cb^4 + \varepsilon(B)\varepsilon(C)c^3b^2 - b(\ddot{b}c - \dot{b}\dot{c}) + 2c\dot{b}^2}{c^2b^3d}, \tag{11.141}$$

[20] A basis in a linear space does not necessarily follow from a system of coordinate functions, that is, the basis vectors cannot be written as tangent vectors to a system of coordinate lines. A criterion that a set of vectors is generated by a coordinate system is that the vectors commute. If this is the case, we call the basis *holonomic*, otherwise *unholonomic*.

$$N = (R^{-1})_{43} = \frac{-\dot{c}b - 2\dot{b}c}{b^2c^2d}, \tag{11.142}$$

$$\Xi = (R^{-1})_{44} = \frac{1}{bcd}. \tag{11.143}$$

Exercise 52 Verify that the matrix R^{-1} is the inverse of the matrix R. Then show that

$$\begin{aligned}
A^i &= u^i, \\
B^i &= \frac{1}{b}\dot{u}^i, \\
C^i &= \varepsilon(A)\varepsilon(B)\frac{b}{c}u^i - \frac{\dot{b}}{b^2c}\dot{u}^i + \frac{1}{bc}\ddot{u}^i, \\
D^i &= \Lambda u^i + M\dot{u}^i + N\ddot{u}^i + \Xi\dddot{u}^i.
\end{aligned}$$

Consider now an arbitrary four-vector which in the Frenet–Serret basis has the following analysis:

$$r^i = aA^i + \beta B^i + \gamma C^i + \delta D^i. \tag{11.144}$$

Replacing A^i, B^i, C^i, D^i in terms of the vectors of the physical basis we find

$$\begin{aligned}
r^i = &\left[\alpha + \gamma\frac{\varepsilon(A)\varepsilon(B)b}{c} - \delta\frac{\varepsilon(A)\varepsilon(B)(\dot{c}b - \dot{b}c)}{c^2d}\right]A^i + \\
&+ \left[\beta\frac{1}{b} - \gamma\frac{\dot{b}}{b^2c} + \delta\frac{\varepsilon(A)\varepsilon(B)cb^4 + \varepsilon(B)\varepsilon(C)c^3b^2 - b(\ddot{b}c - \dot{b}\dot{c}) + 2c\dot{b}^2}{c^3b^2d}\right]\dot{A}^i + \\
&+ \left[\gamma\frac{1}{bc} - \delta\frac{\dot{c}b + 2\dot{b}c}{b^2c^2d}\right]\ddot{A}^i + \delta\frac{1}{bcd}\dddot{A}^i.
\end{aligned} \tag{11.145}$$

Equation (11.145) is the answer to our problem, that is, gives the generic form of an *arbitrary* four-vector in terms of the tangent vector to the curve and its derivatives. If we denote the curvatures b, c, d as $\kappa_1,\ \kappa_2,\ \kappa_3$ then expression (11.145) reads

$$\begin{aligned}
r^i = &\left[\alpha + \gamma\frac{\varepsilon(A)\varepsilon(B)\kappa_1}{\kappa_2} - \delta\frac{\varepsilon(A)\varepsilon(B)(\dot{\kappa}_2\kappa_1 - \dot{\kappa}_1\kappa_2)}{\kappa_2^2\kappa_3}\right]A^i + \\
+ &\left[\beta\frac{1}{\kappa_1} - \gamma\frac{\dot{\kappa}_1}{\kappa_1^2\kappa_2} + \delta\frac{\varepsilon(A)\varepsilon(B)\kappa_2{\kappa_1}^4 + \varepsilon(B)\varepsilon(C)\kappa_2^3\kappa_1^2 + \kappa_1(\dot{\kappa}_2\dot{\kappa}_1 - \ddot{\kappa}_1\kappa_2) + 2\kappa_2\dot{\kappa}_1^2}{\kappa_2^3\kappa_1^2\kappa_3}\right]\dot{A}^i + \\
&+ \left[\gamma\frac{1}{\kappa_1\kappa_2} - \delta\frac{\dot{\kappa}_2\kappa_1 + 2\dot{\kappa}_1\kappa_2}{{\kappa_1}^2{\kappa_2}^2\kappa_3}\right]\ddot{A}^i + \delta\frac{1}{\kappa_1\kappa_2\kappa_3}\dddot{A}^i
\end{aligned} \tag{11.146}$$

11.7.2 The Generic Inertial Four-Force

We are now ready to compute the generic expression of a pure four-force which modulates the motion of a ReMaP P. Let c^i be the world line of P which we assume to be such that $\kappa_1\kappa_2\kappa_3 \neq 0$ and let u^i be the four-velocity of P. The physical frame along c^i consists of the four-vectors

$$A^i = u^i,\ \dot{A}^i = \dot{u}^i, \qquad \ddot{A}^i = \ddot{u}^i, \qquad \dddot{A}^i = \dddot{u}^i.$$

A pure four-force F^i on P is defined by the condition $F^i u_i = 0$. The general expression (11.146) reads for $\varepsilon(A) = -1$, $\varepsilon(B) = \varepsilon(C) = +1$

$$\begin{aligned} r^i &= \left[\alpha - \gamma\frac{\kappa_1}{\kappa_2} + \delta\frac{\dot{\kappa}_2\kappa_1 - \dot{\kappa}_1\kappa_2}{\kappa_2^2\kappa_3}\right]u^i \\ &+ \left[\beta\frac{1}{\kappa_1} - \gamma\frac{\dot{\kappa}_1}{\kappa_1^2\kappa_2} + \delta\frac{-\kappa_2{\kappa_1}^4 + \kappa_2^3\kappa_1^2 + \kappa_1(\dot{\kappa}_2\dot{\kappa}_1 - \ddot{\kappa}_1\kappa_2) + 2\kappa_2\dot{\kappa}_1^2}{\kappa_2^3\kappa_1^2\kappa_3}\right]\dot{u}^i \\ &+ \left[\gamma\frac{1}{\kappa_1\kappa_2} - \delta\frac{\dot{\kappa}_2\kappa_1 + 2\dot{\kappa}_1\kappa_2}{{\kappa_1}^2{\kappa_2}^2\kappa_3}\right]\ddot{u}^i \\ &+ \delta\frac{1}{\kappa_1\kappa_2\kappa_3}\dddot{u}^i. \end{aligned} \tag{11.147}$$

Condition $r^i u_i = 0$ implies $\alpha = 0$ (due to (11.144)), therefore, the generic expression of a pure (or inertial) four-force acting on P (and in general of a spacelike vector normal to the four-velocity u^i!) is

$$\begin{aligned} F^i &= \left[-\gamma\frac{\kappa_1}{\kappa_2} + \delta\frac{\dot{\kappa}_2\kappa_1 - \dot{\kappa}_1\kappa_2}{\kappa_2^2\kappa_3}\right]u^i \\ &+ \left[\beta\frac{1}{\kappa_1} - \gamma\frac{\dot{\kappa}_1}{\kappa_1^2\kappa_2} + \delta\frac{-\kappa_2{\kappa_1}^4 + \kappa_2^3\kappa_1^2 + \kappa_1(\dot{\kappa}_2\dot{\kappa}_1 - \ddot{\kappa}_1\kappa_2) + 2\kappa_2\dot{\kappa}_1^2}{\kappa_2^3\kappa_1^2\kappa_3}\right]\dot{u}^i \\ &+ \left[\gamma\frac{1}{\kappa_1\kappa_2} - \delta\frac{\dot{\kappa}_2\kappa_1 + 2\dot{\kappa}_1\kappa_2}{{\kappa_1}^2{\kappa_2}^2\kappa_3}\right]\ddot{u}^i + \delta\frac{1}{\kappa_1\kappa_2\kappa_3}\dddot{u}^i. \end{aligned} \tag{11.148}$$

Note that this expression contains u^i although $F^i u_i = 0$! This is due to the fact that the basis $\{u^i,\ \dot{u}^i,\ \ddot{u}^i,\ \dddot{u}^i\}$ is not orthonormal. For example

$$u^i\ddot{u}_i = (u^i\dot{u}_i)^{\cdot} - \dot{u}^i\dot{u}_i = -\dot{u}^2 = -b^2 \neq 0. \tag{11.149}$$

Exercise 53 Prove the relations

$$\dddot{u}^i u_i = -{\kappa_1}^2, \tag{11.150}$$

$$\ddot{u}^i\dot{u}_i = \kappa_1\dot{\kappa}_1, \tag{11.151}$$

$$\dddot{u}^i u_i = -3\kappa_1\dot{\kappa}_1. \tag{11.152}$$

[Hint: Use (11.136) to write $\ddot{u}^i = b^2 A^i + \dot{b}B^i + bcC^i$]

During the many years of Special Relativity there have been proposed many types of pure four-forces. Each of these satisfies a physical "need" and follows from some physical considerations or inspirations. All the proposed four-forces *must* follow from the generic expression (11.148) for an appropriate set of values of the parameters β, γ, δ, otherwise they are exempted mathematically. In the following, we shall examine a few types of pure four-forces proposed over the years and will decide on their *geometric* acceptance.

11.7.2.1 Newtonian Type Four-Force

This force has the general expression $F^i = m\dot{u}^i$ and is the generalization of Newton's Second Law in Special Relativity. To prove that this type of force is acceptable, we have to prove that it follows from the generic expression (11.148) for a special set of values of the parameters β, γ, δ. For this, we examine if the following system of equations has a unique solution:

$$-\gamma\frac{\kappa_1}{\kappa_2} + \delta\frac{\dot{\kappa}_2\kappa_1 + 2\dot{\kappa}_1\kappa_2}{\kappa_2^2\kappa_3} = 0,$$

$$\beta\frac{1}{\kappa_1} - \gamma\frac{\dot{\kappa}_1}{\kappa_1^2\kappa_2} + \delta\frac{-\kappa_2{\kappa_1}^4 + \kappa_2^3\kappa_1^2 + \kappa_1(\dot{\kappa}_2\dot{\kappa}_1 - \ddot{\kappa}_1\kappa_2) + 2\kappa_2\dot{\kappa}_1^2}{\kappa_2^3\kappa_1^2\kappa_3} = m,$$

$$\gamma\frac{1}{\kappa_1\kappa_2} - \delta\frac{\dot{\kappa}_2\kappa_1 + 2\dot{\kappa}_1\kappa_2}{{\kappa_1}^2{\kappa_2}^2\kappa_3} = 0,$$

$$\delta = 0.$$

The solution of the system is $\gamma = \delta = 0$, $\beta = \kappa_1 m$ where κ_1 is the first curvature (= the measure of the four-acceleration) of the world line. We conclude that the proposed Newtonian four-force is geometrically acceptable. This does not mean of course that it is also physically acceptable. Only experiment can establish this.

11.7.2.2 The Lorentz–Dirac Four-Force

Dirac suggested that the four-force on a particle of mass m and charge q, which accelerates under the action of a three-force is given by the following formula (Lorentz–Dirac force):

$$F^i = m\dot{u}^i - \frac{2}{3}q^2[\ddot{u}^i - (\dot{u}^i\dot{u}_i)u^i]. \tag{11.153}$$

The second term is assumed to describe the self interaction of the charge with its own radiation field.

We examine if this type of force is geometrically acceptable. For that we check if it is normal to the four-velocity. We find

$$F^i u_i = -\frac{2}{3}q^2[\ddot{u}^i u_i - (\dot{u}^j \dot{u}_j)(u^i u_i)] = -\frac{2}{3}q^2(-\dot{u}^i \dot{u}_i + \dot{u}^i \dot{u}_i) = 0$$

therefore, F^i is a pure four-force. We continue with the computation of the parameters β, γ, δ. We note that $\dot{u}^i \dot{u}_i = k_1^2$, hence (11.153) can be written as

$$F^i = m\dot{u}^i + \frac{2}{3}q^2{\kappa_1}^2 u^i - \frac{2}{3}q^2\ddot{u}^i. \tag{11.154}$$

Comparing (11.154) with the generic expression (11.148) we find the following algebraic system of four-equations in the three unknowns β, γ, δ

$$\begin{aligned}
-\gamma\frac{\kappa_1}{\kappa_2} + \delta\frac{\dot{\kappa}_2\kappa_1 + 2\dot{\kappa}_1\kappa_2}{\kappa_2^2 d} &= \frac{2}{3}\kappa_1^2 q^2,\\
\beta\frac{1}{\kappa_1} - \gamma\frac{\dot{\kappa}_1}{\kappa_1^2 c} + \delta\frac{-\kappa_2\kappa_1^4 + \kappa_2^3\kappa_1^2 + \kappa_1(\dot{\kappa}_2\dot{\kappa}_1 - \ddot{\kappa}_1\kappa_2) + 2\kappa_2\dot{\kappa}_1^2}{\kappa_1^2 c^3 d} &= m,\\
\gamma\frac{1}{\kappa_1\kappa_2} - \delta\frac{\dot{\kappa}_2\kappa_1 + 2\dot{\kappa}_1\kappa_2}{\kappa_1^2\kappa_2^2 d} &= -\frac{2}{3}q^2,\\
\delta &= 0.
\end{aligned}$$

The solution of this system is

$$\begin{aligned}
\beta &= m\kappa_1 - \frac{2}{3}\dot{\kappa}_1 q^2,\\
\gamma &= -\frac{2}{3}\kappa_1\kappa_2 q^2,\\
\delta &= 0.
\end{aligned} \tag{11.155}$$

We conclude that the Lorentz–Dirac four-force is geometrically acceptable and, furthermore in the Frenet–Serret basis can be written as

$$F^i = \left(m - -\frac{2}{3}q^2\frac{\dot{\kappa}_1}{\kappa_1}\right)\kappa_1 B^i - \frac{2}{3}\kappa_1\kappa_2 q^2 C^i. \tag{11.156}$$

Note that the case $F^i = 0$ is impossible, because

$$\kappa_1\kappa_2 \neq 0.$$

11.7.2.3 Bonnor Four-Force

Bonnor[21] assumed that the four-force on an accelerating charged particle depends (i) on the u^i, $\dot{u}^i$ and (ii) the amount of radiated energy is measured by the change $\dot{m}$ of the mass of the particle. Then he gave the expression

$$F^i = m\dot{u}^i + \left[\dot{m} - \frac{2}{3}q^2(\dot{u}^i\dot{u}_i)\right]u^i. \tag{11.157}$$

As we have seen $\dot{u}_i\dot{u}^i = \kappa_1{}^2$ hence this can be written as

$$F^i = m\dot{u}^i + \left(\dot{m} - \frac{2}{3}\kappa_1^2 q^2\right)u^i. \tag{11.158}$$

We demand the condition $F^i u_i = 0$ (pure four-force) and find from (11.158)

$$F^i u_i = -\left(\dot{m} - \frac{2}{3}\kappa_1{}^2 q^2\right) = 0 \Leftrightarrow \dot{m} = \frac{2}{3}\kappa_1{}^2 q^2. \tag{11.159}$$

Replacing this back in (11.157) follows $F^i = m\dot{u}^i$, that is, the Newtonian type of force but with varying mass. We do not expect that the Newtonian type of force will be possible to describe the self force on the charge, therefore, it appears that the Bonnor four-force does not serve its purpose. Indeed, it was soon abandoned.

The conclusion is

When geometry is used properly becomes a great tool to make good physics!

[21] See W. B. Bonnor "A new equation of motion for a radiating charged particle" *Proc. R. Soc. Lond.* A, **337**, 591–598, (1974). Also A. Schild "On the radiation emitted by an accelerated point charge" *J. Math. Analysis. Appl.* **1**, 127–131, (1960).

Chapter 12
Irreducible Decompositions

12.1 Decompositions

In order to understand the concept of decomposition it is best to start with a well-known example. Consider a two-index tensor T_{ab} and write it in terms of the symmetric and antisymmetric parts as follows:

$$T_{ab} = \frac{1}{2}(T_{ab} + T_{ba}) + \frac{1}{2}(T_{ab} - T_{ba}) = T_{(ab)} + T_{[ab]}. \tag{12.1}$$

This breaking (=decomposition) of the tensor into two parts has the advantage that under a change of coordinates, each part transforms independently of the other. Hence, one is possible to study the behavior of the tensor under coordinate transformations by studying the behavior of each part independently, thus facilitating the study. Since (12.1) is an identity, no information is lost. Behind this obvious remark there is a general conclusion as follows. The coordinate transformations of a space (i.e., all differentiable coordinate transformations) form a group known as the manifold mapping group (MMG). As we have said, a group of transformations defines a type of tensors, i.e., the geometric objects which transform covariantly under this group. The tensors defined by the MMG are the most general ones defined on a space. This is so because the MMG requires no other structure on the space but the transformations themselves, which are inherent in the definition of the space. The decomposition of the tensor T_{ab} we considered in symmetric and antisymmetric parts is covariant wrt the MMG group in the sense that (a) each part is a tensor of the same type as T_{ab} and (b) each part transforms independently under the action of the MMG (i.e., under coordinate transformations).

What has been said for the MMG applies to any other group of transformations. In particular, let us consider the Lorentz group which is the group of Special Relativity. The tensors of this group are the Lorentz tensors. A covariant decomposition of a Lorentz tensor in irreducible parts is the decomposition (via an identity) of this tensor in a sum of Lorentz tensors each with fewer components, which transform independently under the action of the Lorentz group. In the present chapter, we discuss the irreducible decomposition of vectors and tensors wrt a general vector and wrt a pair of vectors. We apply the results in Minkowski space and produce the

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well-known and important 1+3 and 1+1+2 decompositions. These decompositions are usually hidden behind the discussions in Special Relativity; however, they have to be considered explicitly when one considers the energy momentum tensor, the relativistic fluids, and other relevant material.

12.1.1 Writing a Tensor of Valence (0,2) as a Matrix

The calculations involving tensors with two indices are simplified significantly if we write them as square matrices. In order to do that, we must follow a definite convention which will secure the validity of the results. For the vectors in a linear space V^3 (we use V^3 for economy of space, the same apply to any finite n) we have made the convention to represent the contravariant vectors with $n \times 1$ (or column) matrices and the covariant vectors with $1 \times n$ (or row) matrices. For the tensors with two covariant indices we make the following convention:

The first index counts rows and the second counts columns.

According to this convention we write

$$T_{ij} = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}.$$

A special case of tensors with two indices are the products of two vectors $A^\mu B^\nu$. According to the definition of tensor product, the components of the tensor $A^\mu B^\nu$(in the coordinate system in which the components of the vectors A^μ and B^ν are given) are found by multiplying each component of A^μ with all the components of B^ν and setting the result as a row in the resulting matrix. For example, if we are given

$$A^\mu = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, B^\nu = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

then the tensor product

$$A^\mu \otimes B^\nu = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to show that the matrix which is defined by the product $A^\nu \otimes B^\mu$ is the transpose of the matrix corresponding to the product $B^\mu \otimes A^\nu$.

Another case which we meet frequently in practice is the computation of the components of the tensor $T_{ij} A^j$ if we are given the components of the tensors T_{ij}, A^j

(in the same coordinate system!). According to the convention above, this is found as follows:

$$T_{ij}A^j = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} \begin{pmatrix} A^1 \\ A^2 \\ A^3 \end{pmatrix} = \begin{pmatrix} T_{11}A^1 + T_{12}A^2 + T_{13}A^3 \\ T_{21}A^1 + T_{22}A^2 + T_{23}A^3 \\ T_{31}A^1 + T_{32}A^2 + T_{33}A^3 \end{pmatrix}$$

and

$$\begin{aligned} T_{ij}A^i &= \begin{pmatrix} A^1 \\ A^2 \\ A^3 \end{pmatrix}^t \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} \\ &= \left(A^1, A^2, A^3 \right) \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} \\ &= \begin{pmatrix} A^1T_{11} + A^2T_{21} + A^3T_{31}, \; A^1T_{12} + A^2T_{22} + A^3T_{32}, \\ A^1T_{13} + A^2T_{23} + A^3T_{33} \end{pmatrix}. \end{aligned}$$

In the following, we shall apply extensively the above conventions. However, it is advised that before one accepts a result as final, one should check it at least partially using the standard analytical method of components. Of course the best is to do the calculations by means of an algebraic computer program.

12.2 The Irreducible Decomposition wrt a Non-null Vector

With every non-null vector we associate a unique projection operator which projects in a plane normal to the vector. This operator can be defined in any metrical space (positive definite or not) and especially in the Euclidean space. In this section, we shall study this projective tensor and we shall define a covariant decomposition of any tensor in irreducible parts. We call this decomposition the $1 + (n - 1)$ decomposition and it is used extensively in relativistic Physics (however, not in Newtonian Physics).

In order to emphasize the concept and the generality of the projection operator in the following we shall consider initially a Euclidean space of dimension $n(\geq 2)$ and subsequently we shall consider the 1+3 decomposition in Minkowski space.

12.2.1 Decomposition in a Euclidean Space E^n

12.2.1.1 Decomposition of a Euclidean Vector

Consider a Euclidean space of dimension $n(\geq 2)$ endowed with a metric $g_{E\mu\nu}$. In an ECF the components of the metric are $\delta_{\mu\nu}$. Let A^μ be a vector in space whose length $\mathbf{A}^2 = g_{E\mu\nu} A^\mu A^\nu > 0$. We define the projective tensor $h_{\mu\nu}$ associated with the vector A^μ as follows:

$$h_{\mu\nu}(A) = g_{E\mu\nu} - \frac{1}{\mathbf{A}^2} A_\mu A_\nu. \tag{12.2}$$

It is easy to prove the following identities:

$$h_{\mu\nu}(A) = h_{\nu\mu}(A) \qquad \text{(symmetric tensor)}, \tag{12.3}$$

$$h_{\mu\nu}(A)\delta^{\mu\nu} = h^\mu_\mu(A) = n - 1 \qquad \text{(trace)}, \tag{12.4}$$

$$h_{\mu\nu}(A)A^\nu = 0 \qquad \text{(projects normal to the vector } A^\mu\text{)}. \tag{12.5}$$

Exercise 54 Show that the components of the projection tensor $h_{\mu\nu}(A)$ in an ECF Π and for an arbitrary vector A^μ are $diag(1 - \frac{1}{\mathbf{A}^2}(A^1)^2, \ldots, 1 - \frac{1}{\mathbf{A}^2}(A^n)^2)$. Furthermore show that if A^μ is unit (i.e., $\mathbf{A}^2 = 1$) then $h_{\mu\nu}(A) = \delta_{\mu\nu} - A_\mu A_\nu$.

Using $h_{\mu\nu}(A)$ we can decompose any other vector B^μ along and normal to the vector A^μ as follows:

$$B^\mu = \delta^\mu_\nu B^\nu = \left(h^\mu_\nu + \frac{1}{\mathbf{A}^2} A^\mu A_\nu \right) B^\nu = h^\mu_\nu B^\nu + \frac{1}{\mathbf{A}^2}(A_\nu B^\nu) A^\mu. \tag{12.6}$$

We call the vector $\mathbf{B}_\perp \equiv h^\mu_\nu B^\nu$ the *normal component* of B^μ wrt A^μ and the vector $\frac{1}{\mathbf{A}^2}(A_\nu B^\nu)A^\mu$ the *parallel component* of B^μ wrt A^μ. In the following, we assume that we work in an ECF, therefore the components of the metric are $\delta_{\mu\nu}$.

Example 62 Decompose the vector $B^\nu = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ normal and parallel to the vector $A^\mu = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$.

Solution

We compute $|\mathbf{A}|^2 = 5$. Hence

$$h(A)_{\mu\nu} = \delta_{\mu\nu} - \frac{1}{\mathbf{A}^2} A_\mu A_\nu = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{1}{5} \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 4 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

(Check that $h(A)_{\mu\nu} A^\nu = 0$!) Then for the normal and parallel parts of B^μ we have

$$B^\mu_\perp = h(A)\frac{\mu}{\nu} B^\nu = \frac{1}{5} \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}, \quad B^\mu_\parallel = B^\mu - B^\mu_\perp = \frac{1}{5} \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix}.$$

(Check that $B^\mu = B^\mu_\perp + B^\mu_\parallel$).

12.2.1.2 Decomposition of a Euclidean Second-Order Tensor

The 1+3 decomposition of an arbitrary tensor $T_{\mu\nu}$ in a Euclidean space of dimension n wrt the vector A^{μ} of (Euclidean) length $\mathbf{A}^2$ is done by means of the following identity:

$$\begin{aligned}
T_{\mu\nu} = \delta_{\mu}{}^{\alpha}\delta_{\nu}{}^{\beta}T_{\alpha\beta} &= \left(h_{\mu}{}^{\alpha} + \frac{1}{\mathbf{A}^2}A_{\mu}A^{\alpha}\right)\left(h_{\nu}{}^{\beta} + \frac{1}{\mathbf{A}^2}A_{\nu}A^{\beta}\right)T_{\alpha\beta} \\
&= \left[\frac{1}{\mathbf{A}^4}A^{\alpha}A^{\beta}A_{\mu}A_{\nu} + \frac{1}{\mathbf{A}^2}h_{\mu}{}^{\alpha}A^{\beta}A_{\nu} + \frac{1}{\mathbf{A}^2}h_{\nu}{}^{\beta}A^{\alpha}A_{\mu} + h_{\mu}{}^{\alpha}h_{\nu}{}^{\beta}\right]T_{\alpha\beta} \\
&= \frac{1}{\mathbf{A}^4}\left(T_{\alpha\beta}A^{\alpha}A^{\beta}\right)A_{\mu}A_{\nu} \qquad (12.7) \\
&\quad + \frac{1}{\mathbf{A}^2}h_{\mu}{}^{\alpha}A^{\beta}T_{\alpha\beta}A_{\nu} + \frac{1}{\mathbf{A}^2}h_{\nu}{}^{\beta}A^{\alpha}T_{\alpha\beta}A_{\mu} + h_{\mu}{}^{\alpha}h_{\nu}{}^{\beta}T_{\alpha\beta}.
\end{aligned}$$

We note that the irreducible parts of the tensor $T_{\mu\nu}$ are the following tensors:

- An invariant: $\frac{1}{\mathbf{A}^4}T_{\alpha\beta}A^{\alpha}A^{\beta}$
- Two vectors normal to A^{μ}: $\frac{1}{\mathbf{A}^2}h_{\mu}{}^{\alpha}A^{\beta}T_{\alpha\beta}$ and $\frac{1}{\mathbf{A}^2}h_{\nu}{}^{\beta}A^{\alpha}T_{\alpha\beta}$
- A tensor with two indices but with components only in the space which is normal to the vector A^{μ}: $h_{\mu}{}^{\alpha}h_{\nu}{}^{\beta}T_{\alpha\beta}$

The above result is written conveniently in the form of the following block matrix:

$$\begin{pmatrix} \frac{1}{\mathbf{A}^4}T_{\alpha\beta}A^{\alpha}A^{\beta} & \frac{1}{\mathbf{A}^2}h_{\mu}{}^{\alpha}A^{\beta}T_{\alpha\beta} \\ \frac{1}{\mathbf{A}^2}h_{\nu}{}^{\beta}A^{\alpha}T_{\alpha\beta} & h_{\mu}{}^{\alpha}h_{\nu}{}^{\beta}T_{\alpha\beta} \end{pmatrix}, \qquad (12.8)$$

where the blocks (1,2) and (2,1) are matrices of order $1 \times n$ and $n \times 1$, respectively, and the block (2,2) is a square matrix of dimension $(n-1)\times(n-1)$. The $1+(n-1)$ decomposition (12.8) is used extensively in the study of the energy momentum tensor and the kinematics of Special Relativity. Strange enough it appears that it is not used in Newtonian Physics!

Example 63 Decompose the tensor of order $(0,2)$ $T_{\mu\nu} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ wrt the Euclidean vector $A^{\mu} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$.

Solution

In Example 62 we computed the projection operator

$$h(A)_{\mu\nu} = \frac{1}{5}\begin{pmatrix} 4 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

We are ready to compute the irreducible parts of the decomposition as given in (12.7). Since the purpose of the example is to present practices of calculation we shall follow the intermediate steps and show how they are computed using matrices. We denote by $[h]$, $[T]$, $[A]$ the matrices which correspond to the tensors $h_{\alpha\beta}$, $T_{\alpha\beta}$, A^{α} and give the answer as a product of matrices. We emphasize that the calculation is done in a Euclidean space. In a Lorentz space, we have to take into account the signs which can be positive or negative. We also note that the components of the metric are $\delta_{\mu\nu}$, therefore when we raise or lower indices we observe no change in the coordinates. For example, we have $[h_{\alpha\beta}] = [h^{\beta}_{.\alpha}] = [h^{\alpha}_{.\beta}]$. This is not the case with the Lorentz metric.

For the invariant part we have

$$\begin{aligned}\frac{1}{\mathbf{A}^4}(T_{\alpha\beta}A^{\alpha}A^{\beta}) &= \frac{1}{\mathbf{A}^4}[A]^t[T][A] \\ &= \frac{1}{25}\begin{pmatrix} 1\,2\,0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \frac{11}{25}.\end{aligned}$$

The second irreducible part is a 3×1 matrix. We recall that the lower indices (covariant) count rows and the upper indices (contravariant) count columns. We have

$$\begin{aligned}\frac{1}{\mathbf{A}^2}h^{\mu\alpha}A^{\beta}T_{\alpha\beta} &= \frac{1}{\mathbf{A}^2}(h^{\mu 1}A^{\beta}T_{1\beta} + h^{\mu 2}A^{\beta}T_{2\beta} + h^{\mu 3}A^{\beta}T_{3\beta}) \\ &= \frac{1}{\mathbf{A}^2}\begin{pmatrix} h_{\mu 1} & h_{\mu 2} & h_{\mu 3} \end{pmatrix}\begin{pmatrix} A^{\beta}T_{1\beta} \\ A^{\beta}T_{2\beta} \\ A^{\beta}T_{3\beta} \end{pmatrix} \\ &= \frac{1}{\mathbf{A}^2}[h][T][A] \\ &= \frac{1}{25}\begin{pmatrix} 4 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \frac{1}{25}\begin{pmatrix} -6 \\ 3 \\ 0 \end{pmatrix}.\end{aligned}$$

Similarly, for the 1×3 irreducible part we have

$$\begin{aligned}\frac{1}{\mathbf{A}^2}h_{\mu}{}^{\alpha}A^{\beta}T_{\beta\alpha} &= \frac{1}{\mathbf{A}^2}(h_{\mu}{}^{1}A^{\beta}T_{\beta 1} + h_{\mu}{}^{2}A^{\beta}T_{\beta 2} + h_{\mu}{}^{3}A^{\beta}T_{\beta 3}) \\ &= \frac{1}{\mathbf{A}^2}\begin{pmatrix} A^{\beta}T_{\beta 1} & A^{\beta}T_{\beta 2} & A^{\beta}T_{\beta 3} \end{pmatrix}\begin{pmatrix} h_{\mu}{}^{1} \\ h_{\mu}{}^{2} \\ h_{\mu}{}^{3} \end{pmatrix} = \frac{1}{\mathbf{A}^2}[A]^t[T][h]\end{aligned}$$

$$
= \frac{1}{25}\begin{pmatrix} 1 & 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}
$$

$$
= \frac{1}{25}\begin{pmatrix} 4 & -2 & 15 \end{pmatrix}.
$$

Finally, for the 3×3 irreducible part we compute

$$
h^{\alpha}_{\mu} h^{\beta}_{\nu} T_{\alpha\beta} = [h][T][h] = \frac{1}{25}\begin{pmatrix} 4 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}
$$

$$
= \frac{1}{25}\begin{pmatrix} 16 & -8 & 10 \\ -8 & 4 & -5 \\ 0 & 0 & 25 \end{pmatrix}.
$$

It would be a good exercise to check that the above results are correct. To do this one must add the computed irreducible parts and get the original tensor. Let us do this. We have

$$
A_{\mu} A_{\nu} = \begin{pmatrix} 1 & 2 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix},
$$

$$
\left(\frac{1}{\mathbf{A}^2} h_{\mu}{}^{\alpha} A^{\beta} T_{\alpha\beta}\right) A_{\nu} = \frac{1}{25}\begin{pmatrix} -6 & 3 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 2 & 0 \end{pmatrix} = \frac{1}{25}\begin{pmatrix} -6 & -12 & 0 \\ 3 & 6 & 0 \\ 0 & 0 & 0 \end{pmatrix},
$$

$$
\left(\frac{1}{\mathbf{A}^2} h_{\mu}{}^{\alpha} A^{\beta} T_{\beta\alpha}\right) A_{\nu} = \frac{1}{25}\begin{pmatrix} 1 & 2 & 0 \end{pmatrix} \otimes \begin{pmatrix} 4 & -2 & 15 \end{pmatrix} = \frac{1}{25}\begin{pmatrix} 4 & -2 & 15 \\ 8 & -4 & 30 \\ 0 & 0 & 0 \end{pmatrix}.
$$

It is easy to show that the sum of these three matrices and the matrix

$$
h_{\mu}{}^{\alpha} h_{\nu}{}^{\beta} T_{\alpha\beta} = \frac{1}{25}\begin{pmatrix} 16 & -8 & 10 \\ -8 & 4 & -5 \\ 0 & 0 & 25 \end{pmatrix}
$$

give indeed the original tensor $T_{\mu\nu}$.

12.2.2 1 + 3 Decomposition in Minkowski Space

The main difference between the decomposition of a tensor in Minkowski space and in Euclidean space is that in the former when we change an index from covariant (lower index) to contravariant (upper index) in the computations we have to change the sign of the zeroth component. Furthermore, in Minkowski space we have vectors with zero length for which the projection operator is not defined. Finally, for the timelike vectors $A^i A_i < 0$, therefore we have to replace $A^i A_i$ with $\operatorname{sign}(A)A^2$ where $A^2 > 0$ $(A \in R)$ and $\operatorname{sign}(A) = -1, +1$ for timelike and spacelike four-vectors, respectively.

The first application of the 1+3 decomposition in Minkowski space is in the kinematics. For this reason, we consider the 1+3 decomposition wrt the four-velocity, that is a unit timelike vector[1] u^a $(u^i u_i = -1)$. In this case, relations (12.6), (12.7) we computed for the Euclidean metric apply and taking into account that $\operatorname{sign}(u) < 0$ we write[2]

- Projection operator:

$$h(u)_{ab} = \eta_{ab} + u_a u_b. \tag{12.9}$$

- 1+3 decomposition of a four-vector:

$$w^a = -(w_b u^b)u^a + h(u)^a{}_b u^b. \tag{12.10}$$

Decomposition of a tensor of type (0, 2):

$$T_{ab} = \left(T_{cd} u^c u^d\right) u_a u_b - h(u)_a{}^c u^d T_{cd} u_b - h(u)_b{}^d u^c T_{cd} u_a + h(u)_a{}^c h(u)_b{}^d T_{cd}. \tag{12.11}$$

We note that these formulae coincide with those of the last section when we replaced $\mathbf{A}^2$ with -1. However, the calculation of the components is another story. This will become clear from the following example which we advise the reader to follow through step by step.

Example 64 1+3 decompose the four-vector

$$w^i = \begin{pmatrix} \sqrt{3} \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

and the Lorentz tensor

$$T_{ab} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

---

[1] We consider $c = 1$. Otherwise we have $u^a u_a = -c^2$.

[2] Relations (12.10), (12.11) can be proved directly by using the identity $T_{ab} = \eta_a{}^c \eta_b{}^d T_{cd}$ and then replace $\eta_{ab} = h(u)_{ab} + u_a u_b$.

wrt the four-vector $u^a = \begin{pmatrix} \sqrt{3} \\ 1 \\ 1 \\ 0 \end{pmatrix}$.

It is assumed that the components of all tensors are in the same LCF system.
Solution

Since we work in an LCF system the Lorentz metric has components $\eta = diag(-1, 1, 1, 1)$. We compute[3]

$$u_a \otimes u_b = \begin{pmatrix} 3 & -\sqrt{3} & -\sqrt{3} & 0 \\ -\sqrt{3} & 1 & 1 & 0 \\ -\sqrt{3} & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$u^a u_a = -1 \qquad \text{(Unit timelike four-vector).}$$

$$h(u)_{ab} = \begin{pmatrix} 2 & -\sqrt{3} & -\sqrt{3} & 0 \\ -\sqrt{3} & 2 & 1 & 0 \\ -\sqrt{3} & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Concerning the raising and the lowering of the indices we have

$$h(u)_a{}^b = \eta^{bc} h(u)_{ac} = \begin{pmatrix} -2 & -\sqrt{3} & -\sqrt{3} & 0 \\ \sqrt{3} & 2 & 1 & 0 \\ \sqrt{3} & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$h(u)^a{}_b = \eta^{ac} h(u)_{cb} = \begin{pmatrix} -2 & \sqrt{3} & \sqrt{3} & 0 \\ -\sqrt{3} & 2 & 1 & 0 \\ -\sqrt{3} & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note that $[h(u)_a{}^b] = [h(u)^b{}_a]^t$. We are ready to apply the 1+3 decomposition. For the four-vector w^a we have

$$w^a_{||} = w^a u_a = 0,$$

$$w^a_{\perp} = h(u)^a{}_b w^b = \begin{pmatrix} \sqrt{3} \\ 2 \\ 1 \\ 0 \end{pmatrix}.$$

[3] Recall: First index row, second index column!

We verify that $w^a = w^a_{||} + w^a_{\perp} = \begin{pmatrix} \sqrt{3} \\ 2 \\ 1 \\ 0 \end{pmatrix}$.

Similarly, for the tensor T_{ab} we compute

$$T_{ab}u^a u^b = 7,$$

$$h(u)_a{}^c T_{cd} u^d = [h(u)_a{}^c][T_{cd}][u^d]$$

$$= \begin{pmatrix} -2 & -\sqrt{3} & -\sqrt{3} & 0 \\ \sqrt{3} & 2 & 1 & 0 \\ \sqrt{3} & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{3} \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$= -\left(6\sqrt{3}, -8, -10, 0\right).$$

$$h(u)^d{}_a T_{cd} u^c = [u^c]^t [T_{cd}][h(u)^d{}_a]$$

$$= \left(\sqrt{3}, 1, 1, 0\right) \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -2 & \sqrt{3} & \sqrt{3} & 0 \\ -\sqrt{3} & 2 & 1 & 0 \\ -\sqrt{3} & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= -\left(6\sqrt{3},\ -9,\ -9,\ -\sqrt{3} - 2\right).$$

$$h(u)_a{}^c h(u)_b{}^d T_{cd} = [h(u)_a{}^c][T_{cd}][h(u)_b{}^d]$$

$$= \begin{bmatrix} -2 & -\sqrt{3} & -\sqrt{3} & 0 \\ \sqrt{3} & 2 & 1 & 0 \\ \sqrt{3} & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} -2 & \sqrt{3} & \sqrt{3} & 0 \\ -\sqrt{3} & 2 & 1 & 0 \\ -\sqrt{3} & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & -8\sqrt{3} & -8\sqrt{3} & -2\sqrt{3} - 2 \\ -7\sqrt{3} & 11 & 10 & \sqrt{3} + 3 \\ -9\sqrt{3} & 13 & 14 & \sqrt{3} + 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

It is left as an exercise for the reader to verify that the above decomposition is correct.

In case the four-vector A^a is not unit, the projection tensor becomes

$$h(A)_{ab} = \eta_{ab} - \frac{\epsilon(A)}{\mathbf{A}^2} A_a A_b \qquad (\mathbf{A} > 0) \tag{12.12}$$

and the formulae which give the 1+3 decomposition change accordingly.

For example, the 1+3 decomposition of a four-vector B^a (null, timelike, or spacelike) wrt the vector A^a is

$$B^a = \delta^a_b B^b = \eta^a_b B^b = \left(h^a_b(A) + \frac{\varepsilon(A)}{\mathbf{A}^2} A^a A_b \right) B^b = \frac{\varepsilon(A)}{\mathbf{A}^2} (A_b B^b) A^a + h^a_b(A) B^b. \tag{12.13}$$

For a general tensor T_{ab} of order (0,2) working in a similar manner we find[4]

$$T_{ab} = \frac{1}{\mathbf{A}^4} \left(T_{cd} A^c A^d \right) A_a A_b + \frac{\varepsilon(A)}{\mathbf{A}^2} h(A)_a{}^c T_{cd} A^d A_b - \frac{\varepsilon(A)}{\mathbf{A}^2} h(A)_b{}^d T_{cd} A^c A_a \\ + h(A)_a{}^c h(A)_b{}^d T_{cd}. \tag{12.14}$$

We emphasize that relations (12.13) and (12.14) are mathematical identities, not new equations!

Example 65 Decompose the four-vector $B^a = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix}$ and the tensor $T_{ab} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ wrt the four-vector $A^a = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix}$. All components are assumed to be in the same LCF.

Solution

We compute $A^a A_a = -4$, therefore the four-vector A^a is timelike with measure $A = 2$. The tensor product

[4] The proof is as follows:

$$T_{ab} = \delta^c_a \delta^d_b T_{cd} = \left(h^c_a + \frac{\varepsilon(A)}{A^2} A^c A_a \right) \left(h^d_b + \frac{\varepsilon(A)}{A^2} A^d A_b \right) T_{cd}$$
$$= \frac{1}{A^4} \left(T_{cd} A^c A^d \right) A_a A_b + \frac{\varepsilon(A)}{A^2} \left(h_a{}^c A^d A_b T_{cd} + h_b{}^d A^c A_a T_{cd} \right) + h^c_a h^d_b T_{cd}.$$

$$A^a \otimes A^b = \begin{pmatrix} 9 & 6 & 3 & 0 \\ 6 & 4 & 2 & 0 \\ 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and the projection operator

$$h(A)_{ab} = \begin{pmatrix} \frac{5}{4} & \frac{-3}{2} & \frac{-3}{4} & 0 \\ \frac{-3}{2} & 2 & \frac{1}{2} & 0 \\ \frac{-3}{4} & \frac{1}{2} & \frac{5}{4} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

For the four-vector B^a we have

$$B^a A_a = -4,$$

$$h(A)^a_b B^b = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

therefore $B^a = A^a + h(A)^a_b B^b$.

For the tensor T_{ab} we compute

$$\frac{1}{A^4} T_{ab} A^a A^b = \frac{17}{16},$$

$$\frac{1}{A^2} h(A)^c_a T_{cb} A^b = \left(\frac{39}{16}, -\frac{21}{8}, -\frac{33}{16}, 0 \right),$$

$$\frac{1}{A^2} h(u)^b_a T_{cb} A^c = \left(\frac{39}{16}, -\frac{23}{8}, -\frac{25}{16}, -\frac{3}{2} \right),$$

$$h(A)^c_a h(A)^d_b T_{cd} = \begin{pmatrix} \frac{97}{16} & \frac{-57}{8} & \frac{-63}{16} & \frac{-7}{2} \\ \frac{-51}{8} & \frac{31}{4} & \frac{29}{8} & 4 \\ \frac{-87}{16} & \frac{47}{8} & \frac{73}{16} & \frac{5}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Verify the above result using relation (12.14).

12.3 1+1+2 Decomposition wrt a Pair of Timelike Vectors

In Sect. 12.2.1 we considered the $1+3$ decomposition wrt a non-null vector. However, practice has shown that we have to consider in Minkowski space the decomposition wrt a pair of non-null non-collinear four-vectors.[5] Obviously, we have the following three possibilities (timelike, timelike), (timelike, spacelike), (spacelike, spacelike). In the present section, we discuss the first case.

Let A^a be a timelike four-vector and B^a a non-null four-vector with lengths A, B, respectively. We are looking for a symmetric tensor p_{ab} of order (0, 2) ($p_{ab} = p_{ba}$) which will project normal to both four-vectors. The general form of this tensor is

$$p_{ab}(A, B) = \eta_{ab} + a_1 A_a A_b + a_2 B_a B_b + a_3(A_a B_b + B_a A_b), \tag{12.15}$$

where $a_1,\ a_2,\ a_3$ are coefficients which must be determined. It will be convenient to introduce the invariant

$$\gamma = -\eta_{ab} A^a B^b = -A_a B^a. \tag{12.16}$$

We shall determine the coefficients $a_1,\ a_2,\ a_3$ from the requirement

$$p_{ab}(A, B)A^b = p_{ab}(A, B)B^b = 0. \tag{12.17}$$

Requirement $p_{ab}(A, B)A^b = 0$ gives the equations

$$\begin{aligned} 1 - a_1 A^2 &= a_3\gamma, \\ a_2\gamma &= -a_3 A^2 \end{aligned} \tag{12.18}$$

and requirement $p_{ab}(A, B)B^b = 0$,

$$\begin{aligned} 1 + \epsilon(B)a_2 B^2 &= a_3\gamma, \\ a_1\gamma &= a_3\epsilon(B)B^2. \end{aligned} \tag{12.19}$$

The solution of the system of the four equations is

$$a_1 = \frac{\epsilon(B)B^2}{\gamma^2 + \epsilon(B)A^2B^2},\ a_2 = -\frac{A^2}{\gamma^2 + \epsilon(B)A^2B^2},\ a_3 = \frac{\gamma}{\gamma^2 + \epsilon(B)A^2B^2}.$$

[5] Because Minkowski space is flat, it is possible to transport a four-vector from one point to another along any path. This implies that the four-vectors need not have a common point of application. Simply they must define a two-plane. We shall use this observation in the derivation of the covariant Lorentz transformation.

Therefore, the projection operator $p_{ab}(A, B)$ is defined as follows:

$$p_{ab}(A, B) = \eta_{ab} + \frac{\epsilon(B)B^2}{\gamma^2 + \epsilon(B)A^2B^2}A_aA_b - \frac{A^2}{\gamma^2 + \epsilon(B)A^2B^2}B_aB_b + \frac{\gamma}{\gamma^2 + \epsilon(B)A^2B^2}(A_aB_b + B_aA_b). \tag{12.20}$$

In case the four-vectors A^a, B^a are unit and B^a is timelike, we write $A^a = u^a$, $B^a = v^a$ where $u^a u_a = v^a v_a = -1$ and relation (12.20) becomes

$$p_{ab}(u, v) = \eta_{ab} - \frac{1}{\gamma^2 - 1}\left[u_a u_b + v_a v_b - \gamma(u_a v_b + v_a u_b)\right]. \tag{12.21}$$

Exercise 55 Show that the tensor $p_{ab}(A, B)$ satisfies the requirement (12.17). Furthermore, show that the trance

$$p^a_a(A, B) = 2 \tag{12.22}$$

and that[6]

$$p^c_a(A, B)h_{cb} = p_{ab}(A, B). \tag{12.23}$$

Exercise 56 Let A^a be timelike four-vector and B^a a non-null four-vector. Consider an arbitrary four-vector C^a and decompose it wrt the four-vectors A^a, B^a as follows:

$$C_a = a_4 A_a + a_5 B_a + p_{ab}(A, B)C^b, \tag{12.24}$$

where a_4, a_5 are coefficients to be determined.[7] Contracting with A^a, B^a show that

$$a_4 = \frac{-\epsilon(B)B^2(CA) - \gamma(CB)}{\gamma^2 + \epsilon(B)A^2B^2}, \qquad a_5 = \frac{A^2(CB) - \gamma(CA)}{\gamma^2 + \epsilon(B)A^2B^2},$$

where $(CA) = C^a A_a$, $(CB) = C^a B_a$. Infer that the 1+1+2 decomposition of the four-vector C^a wrt the pair of four-vectors A^a, B^a is given by the identity

$$C_a = \frac{-\epsilon(B)B^2(CA) - \gamma(CB)}{\gamma^2 + \epsilon(B)A^2B^2}A_a + \frac{A^2(CB) - \gamma(CA)}{\gamma^2 + \epsilon(B)A^2B^2}B_a + p_{ab}(A, B)C^b. \tag{12.25}$$

Finally, in case the four-vectors A^a, B^a are the unit four-vectors u^a, v^a show that (12.25) reduces to

[6] The proof is easy: $p^c_a(A, B)h^b_c = p^c_a(A, B)\left(\delta^b_c + A_c A^b\right) = p^b_a(A, B)$.

[7] You can find the result by writing $C_a = \eta_{ab}C^b$ and using (12.20) to replace η_{ab}.

$$C_a = \frac{(Cu) - \gamma(Cv)}{\gamma^2 - 1} u_a + \frac{(Cv) - \gamma(Cu)}{\gamma^2 - 1} v_a + p_{ab}(u, v) C^b. \tag{12.26}$$

In example 66, we derive the standard decomposition of a timelike four-vector wrt a pair of another two timelike four-vectors. It would be an instructive exercise to derive the same result using the 1+1+2 decomposition.

Example 66 Consider the timelike four-vector p^i, of length $p^2 = -M^2$. Determine two timelike four-vectors $p_1^i,\ p_2^i$ such that

$$p^i = p_1^i + p_2^i \tag{12.27}$$

assuming that the lengths

$$p_1^2 = -m_1^2, \qquad p_2^2 = -m_2^2, \tag{12.28}$$

where $M, m_1 > m_2 > 0$ are given and that the following conditions are satisfied[8]:

$$p_1^i p_i < 0\ ,\ \ p_2^i p_i < 0. \tag{12.29}$$

Solution

Let p_1^i, p_2^i be the required four-vectors. We consider the parallel projection

$$p_{A\|}^i = \frac{p_A^i p_i}{-p^2}\, p^i$$

and the normal projection

$$p_{A,\perp}^i = p_A^i - p_{A\|}^i$$

of the vectors p_A^i $(A = 1, 2)$ wrt the four-vector p^i. Then (12.27) projected parallel and normal to p^i gives

$$p^i = p_{1\|}^i + p_{2\|}^i, \tag{12.30}$$

$$0 = p_{1\perp}^i + p_{2\perp}^i. \tag{12.31}$$

Let $\hat{n}^i$ be a unit four-vector normal to the four-vector p^i. We define the invariants λ, μ in terms of p_1^i with the relations

$$p_{1\perp}^i = \mu \hat{n}^i, \qquad p_{1\|}^i = \lambda p^i\ .$$

[8] The last requirement means that all three four-vectors have the same sign of their zero component, that is, they point in the same part of the light cone.

From (12.30), (12.31) we have

$$p_{2\perp}^i = -\mu\hat{n}^i, \qquad p_{2\parallel}^i = (1-\lambda)p^i.$$

We compute the lengths of p_1^i, p_2^i in terms of λ, μ. We find

$$-\lambda^2 M^2 + \mu^2 = -m_1^2, \tag{12.32}$$
$$-(1-\lambda)^2 M^2 + \mu^2 = -m_2^2. \tag{12.33}$$

The solution of the system of equations (12.32), (12.33) is

$$\lambda = \frac{M^2 + m_1^2 - m_2^2}{2M^2} = \frac{1}{M}\,{}^{1}_{M}E, \tag{12.34}$$
$$\mu^2 = \frac{[M^2 - (m_1+m_2)^2][M^2 - (m_1-m_2)^2]}{2M} = \frac{1}{4M^2}\lambda^2(M, m_1, m_2), \tag{12.35}$$

where ${}^{2}_{M}E$ is the energy of "particle" p_2^i in the proper frame of p^i and μ is the length of the three-momentum of 1 in the same frame. We note that we recover the results of the reaction $p^i \to p_1^i + p_1^i$. The invariants λ, μ satisfy certain restrictions. Indeed, from the inequalities (12.29) and because $\mu^2 \geq 0$, one has[9]

$$M^2 + m_1^2 - m_2^2 > 0,$$
$$[(M-m_1)^2 - m_2^2][(M+m_1)^2 - m_2^2] \geq 0.$$

The first inequality gives

$$M \geq m_1 + m_2$$

and the second inequality is trivially satisfied.

Therefore, the condition is

$$M \geq m_1 + m_2 \tag{12.36}$$

as expected.

Conversely, we note that if $M > m_1 + m_2$ and $\hat{n}^i$ is a unit normal four-vector to p^i, then the equations give a solution to the problem. Therefore, condition (12.36) is sufficient for the solution of the problem and the general solution is given from (12.34), (12.35). We note that the general solution is determined completely in terms of the data of the problem modulo the spacelike unit four-vector $\hat{n}^i$. Therefore we have ∞^2 solutions.

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[9] The first two conditions mean that the energy is positive. The third is the restriction that the measure of the three-momentum is non-negative.

Exercise 57 Let A^a be a unit timelike four-vector ($A^a A_a = -1$) and B^a be a unit spacelike unit four-vector ($B^a B_a = 1$) and let $\phi = A^a B_a$ be their inner product. Define the quantity $\Delta = 1+\phi^2$ and prove that the tensor of order (0, 2) $p_{ab}(A, B) = \eta_{ab} + \frac{1}{\Delta}[A_a A_b - B_a B_b - \phi(A_a B_b + A_b B_a)]$ has the following properties

1. Is symmetric.
2. Projects normal to both A^a, B^a, that is

$$p_{ab}(A, B)A^b = p_{ab}(A, B)B^b = 0. \tag{12.37}$$

3. Its trace

$$p^a_a(A, B) = 2. \tag{12.38}$$

4.

$$p_{ab}(A, B)h^b_c(A) = p_{ac}(A, B). \tag{12.39}$$

$$p^c_a(A, B)h^b_c = p^b_a(A, B). \tag{12.40}$$

We call the projection tensor $p_{ab}(A, B)$ the *screen projection operator* associated with the four-vectors A, B. This tensor is used in the study of spacelike congruences, e.g., the field lines of the electric filed.

Chapter 13
The Electromagnetic Field

13.1 Introduction

The Theory of Special Relativity (and consequently the Theory of General Relativity) would have never been discovered if Maxwell had not formulated the theory of electrodynamics.

Before Maxwell, the electric and the magnetic fields were considered to be independent physical fields and as such had been studied for many years by a number of pioneer physicists who discovered many laws concerning the physics of these fields. Maxwell was the first to foresee the common origin of these fields and introduced the electromagnetic field as the underlying physical entity. Subsequently he stated the basic equations governing the evolution of the electromagnetic field and reproduced all the then known physical laws concerning the electric and the magnetic fields.

However, Maxwell's field equations had a stoning property: They were not covariant under the classical Galileo transformations,[1] that is, their form was dependent on the particular Newtonian inertial frame they were written!

The non-covariance of Maxwell equations wrt the Galileo group of transformations was a serious defect. Indeed according to the covariance principle the physical quantities as well as the physical laws of a theory of physics must be covariant under the fundamental group of transformations of the theory. Therefore the non-covariance of the electromagnetic field wrt the group of Galileo transformations meant that the electromagnetic field was not a Newtonian physical quantity! But then the electric field and the magnetic field are not Newtonian physical quantities, a fact that one could hardly understand and accept easily.

Furthermore it was found that these equations were covariant under another more general group of transformations, which today we call the Lorentz group. But at the time there was not a theory of physics covariant wrt the Lorentz group, therefore the situation appeared to be impossible!

In addition to this theoretical – but important – aspect came a number of experiments involving light, which could not be explained in terms of Newtonian Physics.

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[1] That is, three-space rotation and translation or equivalently rigid body motion.

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These experiments were performed mainly by Michelson and Morley and were indicating that the speed of light in vacuum was independent of the speed of the source and the speed of the receiver and in fact was a universal constant.

As it is the case with such revolutionary and "out of the current line" situations, experts reacted with "wisdom" and tried to find "generalizations" or "hidden" fields in the well-established Newtonian Theory, which could explain the peculiar behavior of light. They invented a new "substance" to replace the absolute character of space and time, the ether, whose properties were postulated to be just enough to explain the new experimental facts. However, at that very moment the Theory of Special Relativity had been born and it was a matter of time who will present it first.

Again as it is the case with great ideas the development was slow and gradual. An eminent physicist of the time, H. Lorentz (see Footnote 1, Sect. 4.1), who was working on the effects of the electromagnetic field formulated a theory for the electron and introduced (in an artificial way) for the first time the Lorentz contraction. Another great theoretician H. Poincaré discussed the theoretical aspects of relative motion and practically stated what we call today the Einstein Relativity Principle. However, the birth of the new theory required a young and very special man, not yet established, and therefore able to think outside the current trend. That was Albert Einstein, who starting from the invariance of Maxwell equations derived the Lorentz transformations and rewrote Maxwell equations in their intrinsic four-dimensional form. Then taking a step further he postulated the Einstein Principle of Relativity (see Footnote 1, Sect. 4.1) claiming that in addition to the electromagnetic field there are many more physical fields which are covariant wrt the Lorentz transformation. These fields are the physical quantities of a new theory of physics and together with the laws which govern their behavior constitute what we call today the Theory of Special Relativity.[2]

In the following sections of this chapter we discuss the relativistic form of Maxwell equations and we solve some well-known problems indicating the application of the four-dimensional formalism in standard electrodynamics.

13.2 Maxwell Equations in Newtonian Physics

The theory of the electromagnetic field developed by Maxwell is a macroscopic theory, that is, it does not enter into the structure of matter as we understand it today. According to that theory the electromagnetic field in an arbitrary medium is described by means of four-vector fields: the electric field **E**, the magnetic field **H**, the electric induction **D**, and the magnetic induction **B**.

[2] Sometimes it is stated erroneously that this principle concerns all physical phenomena – see physical quantities. It does not. All Newtonian physical quantities do not obey Einstein's Relativity Principle whereas they do obey the Galileo Relativity Principle. For this reason a Newtonian physical quantity (velocity for example) is not a relativistic physical quantity, etc. Every theory of physics has a separate domain of application which comprises a "subset" of physical phenomena. See Chap. 2 for details.

As we have remarked in the introduction Maxwell's equations are not covariant under the Galileo group, therefore their form depends on the Newtonian inertial system, K say, we write them. If in K there are all four fields **E,H,D,B** then Maxwell equations for the electromagnetic field (in K only!) in SI units[3] are the following:

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0}, \tag{13.1}$$

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{j}, \tag{13.2}$$

$$\nabla \mathbf{j} + \frac{\partial \rho}{\partial t} = 0. \tag{13.3}$$

In these equations $\mathbf{j}$ is the electric current density in K, which is defined by the relation[4] $\mathbf{j} = \rho \mathbf{v}$, where ρ, $\mathbf{v}$ is the density and the velocity of the electric charge in the frame K. Equation (13.3) expresses the conservation of electric current and it is called the *continuity equation* (for electric charge).

In empty space these equations are known with various names. The first is known as Faraday's Law and the second in the static case (that is when $\frac{\partial \mathbf{D}}{\partial t} = \mathbf{0}$) as Ampére's Law.

An immediate consequence of (13.1)[5] is

$$\nabla \cdot \mathbf{B} = 0. \tag{13.4}$$

This equation implies that there is no magnetic charge and therefore magnetic current.

From (13.2) and (13.3) it follows:

$$\nabla \cdot \mathbf{D} = \rho. \tag{13.5}$$

Equation (13.5) (or one of its versions) is known as *Gauss Law*.

Equations (13.1), (13.2), (13.4), and (13.5) constitute Maxwell equations for a general electromagnetic field in a general medium. In practice we consider special cases depending on the particular physical problem. In these cases Maxwell equations take a simplified form. A standard method to obtain such simplified forms is by considering relations among the fields **D**, **B** and **E**, **H** known as *constitutive*

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[3] Electromagnetism is an old subject with applications in the most diverse areas of science and engineering. As a result there is a number of units in terms of which Maxwell equations have been written. In this book we shall use the SI system. In Sect. 13.16 we show how one writes these equations in other systems and especially in the Gauss system.

[4] The current $\mathbf{j}$ is more general than this but for the time being the conduction current suffices.

[5] To be precise (13.1) implies that $\nabla \cdot \mathbf{B} =$ constant in K. The requirement that the value of this constant is 0 is an extra assumption whose discussion is outside the scope of this book. For this reason this equation is considered as an extra independent equation compatible with the rest of Maxwell equations.

relations. The first such simplifying assumption we do is that the medium is homogeneous and isotropic (that is invariant under the Galileo group). For such mediums we assume that the constitutive relations are

$$\mathbf{D} = \varepsilon \mathbf{E}, \qquad \mathbf{B} = \mu \mathbf{H}. \tag{13.6}$$

The quantities ε *dielectric constant* and μ *magnetic permeability* are characteristic quantities of a medium and satisfy the relation

$$\varepsilon\mu = 1/u^2, \tag{13.7}$$

where u is the speed of the electromagnetic field in the medium.[6] Because the electromagnetic field propagates in empty space we must consider the empty space as a medium. In Special Relativity (but not in General Relativity!) empty three-space is considered to be a homogeneous and isotropic medium with dielectric constant ε_0 and magnetic permeability μ_0.[7] Because in empty space the electromagnetic field propagates with speed c (13.7) implies the relation

$$\varepsilon_0\mu_0 = 1/c^2. \tag{13.9}$$

We note that (13.9) relates the constancy of the speed of light with the electromagnetic properties of empty space.

For a homogeneous but not isotropic medium the fields **D** and **H** are related with the fields **E** and **B** by the relations

$$\mathbf{D} = \varepsilon \mathbf{E} + \mathbf{P}, \qquad \mathbf{B} = \mu(\mathbf{H} + \mathbf{M}), \tag{13.10}$$

where we have introduced two new vector fields, the *polarization vector* **P** and the *magnetization vector* **M** to count for the anisotropy. Obviously in empty space $\mathbf{P} = \mathbf{M} = \mathbf{0}$.

The first question to pose in the relativistic study of the electromagnetic field is if there exists a "generalized" potential in K (which cannot be a scalar) which, with proper derivations, produces both the electric and the magnetic fields in K. We address this question in the next section.

[6] The quantities ε, μ are scalar only for homogeneous and isotropic media. For anisotropic and non-homogeneous materials these quantities are described by second-order symmetric tensors.

[7] The values assigned to these constants are the following:

$$\varepsilon_0 = 8.85 \times 10^{-12}\,\text{Farad/m}, \quad \mu_0 = 1.26 \times 10^{-6}\,\text{Henry/m}. \tag{13.8}$$

13.3 The Electromagnetic Potential

Equation (13.4) implies that (provided certain mathematical assumptions are fulfilled, which we assume to be the case) there exists a differentiable vector field $\mathbf{A}(\mathbf{r}, t)$ such that

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{13.11}$$

Replacing in (13.1) we find

$$\nabla \times \left[\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right] = \mathbf{0}. \tag{13.12}$$

In the derivation of (13.12) we have used the fact that the operators $\frac{\partial}{\partial t}$ and ∇ commute[8]:

$$\frac{\partial}{\partial t}\nabla = \nabla\frac{\partial}{\partial t}.$$

From a well-known proposition of classical vector calculus, (13.12) implies that there exists a scalar function $\phi(\mathbf{r}, t)$ such that

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla\phi,$$

hence

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}. \tag{13.13}$$

The functions ϕ and $\mathbf{A}$ are the potentials for the electric and the magnetic fields, because they produce these fields by proper derivations.

Up to this point the functions ϕ and $\mathbf{A}$ are general. To determine them we are using the remaining two Maxwell equations. In order to make things simpler (without restricting seriously the generality, since the same hold in the case of a homogeneous and isotropic medium) we consider Maxwell equations in empty space[9] and write the constitutive equations:

$$\mathbf{D} = \varepsilon_0 \mathbf{E}, \qquad \mathbf{B} = \mu_0 \mathbf{H}.$$

If we replace $\mathbf{E}$ and $\mathbf{B}$ in terms of ϕ and $\mathbf{A}$ in (13.2) and (13.5) it follows:

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[8] The operators ∇ and $\frac{d}{dt}$ do not commute, because the operator $\frac{d}{dt}$ includes variations in space.

[9] In order to write Maxwell equations for a homogeneous and isotropic medium it is enough to replace c^2 with the product $\varepsilon\mu$ or with the speed u^2 of the electromagnetic field in the medium.

$$\nabla^2\phi + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{\rho}{\varepsilon_0}, \tag{13.14}$$

$$\nabla \times (\nabla \times \mathbf{A}) + \frac{1}{c^2}\frac{\partial}{\partial t}\left(\nabla\phi + \frac{\partial}{\partial t}\mathbf{A}\right) = \mu_0 \mathbf{j}. \tag{13.15}$$

Equation (13.14) does not simplify further. Equation (13.15) can be written differently. Using the identity

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

and the fact that $\nabla \frac{\partial \phi}{\partial t} = \frac{\partial}{\partial t}\nabla\phi$ after a simple calculation we find the equation

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla\left[\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t}\right] = -\mu_0 \mathbf{j}. \tag{13.16}$$

Equations (13.14) and (13.16) are the dynamical equations determining the potentials ϕ and $\mathbf{A}$.

We note that these equations do not determine the potentials $\phi, \mathbf{A}$ completely. Indeed if we replace $\mathbf{A}$ with $\mathbf{A}' = \mathbf{A} + \nabla\Lambda$ where $\Lambda(\mathbf{r}, t)$ is an arbitrary (but differentiable) function, then

$$\nabla \times \mathbf{A}' = \nabla \times \mathbf{A} = \mathbf{B}$$

because $\nabla \times (\nabla\Lambda) = \mathbf{0}$ for every function $\Lambda(\mathbf{r}, t)$.

Therefore the transformation $\mathbf{A}' = \mathbf{A} + \nabla\Lambda$ leaves the magnetic field invariant. However, this is not so for the electric field, which under the transformation $\mathbf{A}' = \mathbf{A} + \nabla\Lambda$ transforms as follows:

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} = -\nabla\phi - \frac{\partial \mathbf{A}'}{\partial t} + \nabla\left(\frac{\partial \Lambda}{\partial t}\right) \Longrightarrow$$
$$\mathbf{E} = -\nabla\left(\phi - \frac{\partial \Lambda}{\partial t}\right) - \frac{\partial \mathbf{A}'}{\partial t}.$$

In order that the electric field remain invariant we have to consider the following transformation of the scalar field ϕ:

$$\phi \longrightarrow \phi' = \phi - \frac{\partial \Lambda}{\partial t}.$$

The transformation of the potentials, which leave the dynamical fields $\mathbf{E}, \mathbf{B}$ invariant, is called a *gauge transformation*. This type of transformations is very important in physics.

The indeterminacy in the determination of the potentials gives us the freedom to define the function Λ by means of a scalar relation among the potentials $\phi, \mathbf{A}$ and select the pair of potentials which is most convenient to our purposes. The basic criterion for this relation is that it must be tensorial, that is covariant under the fundamental group of the theory. Otherwise the relation will be frame dependent. It is evident that in the Theory of Special Relativity this condition must be covariant under the Lorentz group, whereas in Newtonian Physics under the Galileo group. A second criterion, however of less importance, is that the condition which will be considered must lead to potential functions which simplify (13.14) and (13.16).

In the Theory of Special Relativity the condition which fulfils both requirements is the *Lorentz gauge*, which for empty space is defined as follows:

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0. \tag{13.17}$$

We note that with the Lorentz gauge the field equations (in empty space!) become

$$\nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\varepsilon_0} \quad \text{or} \quad \Box \phi = -\frac{\rho}{\varepsilon_0}, \tag{13.18}$$

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{j} \quad \text{or} \quad \Box A = -\mu_0 \mathbf{j}, \tag{13.19}$$

where

$$\Box = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \tag{13.20}$$

is the D'Alembert operator.[10] As we have seen in Exercise 11 D'Alembert's operator is Lorentz covariant. Therefore with the use of the Lorentz gauge the equations of the electromagnetic potentials become similar and are written in Lorentz covariant form in terms of the D'Alembert operator. This observation leads us in a natural way to the four-potential and enables us to write Maxwell equations in four-dimensional formalism.

Another aspect of the Lorentz gauge is that it implies the equation of continuity. Indeed from (13.18) and (13.19) we have

$$\nabla(\Box \mathbf{A}) = -\mu_0 \nabla \cdot \mathbf{j}\,, \quad \frac{\partial}{\partial t}(\Box \phi) = -\frac{1}{\varepsilon_0}\frac{\partial \rho}{\partial t} \Longrightarrow$$

$$\Box\left(\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t}\right) = -\mu_0\left(\nabla \cdot \mathbf{j} + \frac{\partial \rho}{\partial t}\right),$$

which proves this assertion.

[10] It is interesting to read about the life and the work of D'Alembert. The interested reader can visit the web site http://www-history.mcs.st-andrews.ac.uk/Biographies/D'Alembert.html

Because the dynamical equations for the potentials are covariant wrt the gauge transformation of the fields, we demand that the Lorentz gauge also will be invariant under the gauge transformation. This condition determines the gauge function $\Lambda(\mathbf{r}, t)$. Indeed the condition $\nabla \cdot \mathbf{A}' + \frac{1}{c^2}\frac{\partial \phi'}{\partial t} = 0$ gives

$$\begin{aligned}
0 &= \nabla \cdot \mathbf{A}' + \frac{1}{c^2}\frac{\partial \phi'}{\partial t} = \nabla \cdot (\mathbf{A} + \nabla\Lambda) + \frac{1}{c^2}\frac{\partial}{\partial t}\left(\phi - \frac{\partial \Lambda}{\partial t}\right)\\
&= \nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} + \boldsymbol{\nabla}^2\Lambda - \frac{1}{c^2}\frac{\partial^2 \Lambda}{\partial t^2}\\
&= \boldsymbol{\nabla}^2\Lambda - \frac{1}{c^2}\frac{\partial^2 \Lambda}{\partial t^2} = \Box\Lambda.
\end{aligned}$$

We conclude that the function Λ must be a solution of the wave equation in empty space.

There are gauge conditions for the potentials $\phi, \mathbf{A}$ other than the Lorentz condition. However, they are not Lorentz covariant and/or they do not imply the continuity equation for the charge. For example in Newtonian Physics the gauge condition for the electromagnetic potentials is the Coulomb gauge, which is defined by the relation

$$\nabla \cdot \mathbf{A} = 0. \tag{13.21}$$

The name of this gauge is due to the fact that by its use the field equation (13.18) becomes

$$\boldsymbol{\nabla}^2\phi = -\frac{\rho}{\varepsilon_0}, \tag{13.22}$$

which is the well-known Poisson equation whose solution is

$$\phi(\mathbf{r}, t) = \int\limits_{\Omega} \frac{\rho(\mathbf{r}', t)}{|\mathbf{r}' - \mathbf{r}|} d^3\mathbf{r}', \tag{13.23}$$

where $\rho(\mathbf{r}', t)$ is the charge density (in the Newtonian inertial system K where Maxwell equations are considered). This relation means that the scalar potential is the instantaneous Coulomb potential due to the charge density $\rho(\mathbf{r}, t)$ distributed in the space Ω (and measured in K).

Collecting the above results we state that Maxwell equations in empty space in the Lorentz gauge are as follows:

a. For the fields:

$$\mathbf{B} = \nabla \times \mathbf{A}, \tag{13.24}$$

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}. \tag{13.25}$$

b. For the potentials:

$$\Box\phi = -\frac{\rho}{\varepsilon_0}, \tag{13.26}$$

$$\Box \boldsymbol{A} = -\mu_0 \mathbf{j}. \tag{13.27}$$

c. Equation of continuity:

$$\nabla \cdot \mathbf{j} + \frac{\partial \rho}{\partial t} = 0. \tag{13.28}$$

d. Lorentz gauge:

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0. \tag{13.29}$$

The main result which follows from the above equations is that if we solve the wave equation for scalar and vector waves, then we have solved completely Maxwell equations and we have determined the electromagnetic field.

The difference between the Lorentz gauge and the Coulomb gauge is that the first is covariant wrt the Lorentz group whereas the second wrt the Euclidean group. This becomes clear from the following example.

Example 67 a. In the LCF Σ consider the four-quantity $\begin{pmatrix} -\frac{1}{c}\frac{\partial}{\partial t} \\ \boldsymbol{\nabla} \end{pmatrix}_\Sigma$ and show that it defines a four-vector. Show that the length of this four-vector is the D'Alembert operator $\Box = \boldsymbol{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$ and conclude that the D'Alembert operator is covariant wrt the Lorentz group.

b. Show that in an LCF Σ' moving with parallel axes wrt the LCF Σ with velocity $\mathbf{u}$, the ∇ transforms as follows:

$$\boldsymbol{\nabla}' = \boldsymbol{\nabla} + \left\{(\gamma - 1)\frac{\mathbf{u}\cdot\boldsymbol{\nabla}}{\boldsymbol{u}^2} + \frac{\gamma}{c^2}\frac{\partial}{\partial t}\right\}\mathbf{u}. \tag{13.30}$$

Solution

a. We consider an LCF Σ' moving in the standard configuration relative to the LCF Σ with parameter β. The Lorentz transformation relating Σ' and Σ is

$$x = \gamma(x' + \beta ct'), \quad ct = \gamma(ct' + \beta x'), \quad y = y', \quad z = z'.$$

We have then

$$\begin{aligned}
\frac{\partial}{\partial x'} &= \frac{\partial}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial}{\partial y}\frac{\partial y}{\partial x'} + \frac{\partial}{\partial z}\frac{\partial z}{\partial x'} + \frac{\partial}{\partial (ct)}\frac{\partial (ct)}{\partial x'} = \gamma\frac{\partial}{\partial x} + \beta\gamma\frac{\partial}{\partial (ct)},\\
\frac{\partial}{\partial y'} &= \frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial z'} = \frac{\partial}{\partial z},\\
\frac{\partial}{\partial (ct')} &= \frac{\partial}{\partial x}\frac{\partial x}{\partial (ct')} + \frac{\partial}{\partial y}\frac{\partial y}{\partial (ct')} + \frac{\partial}{\partial z}\frac{\partial z}{\partial (ct')} + \frac{\partial}{\partial (ct)}\frac{\partial (ct)}{\partial (ct')} = \gamma\frac{\partial}{\partial (ct)} + \beta\gamma\frac{\partial}{\partial x}.
\end{aligned}$$

Therefore

$$\begin{pmatrix} -\frac{\partial}{c\partial t'} \\ \frac{\partial}{\partial x'} \\ \frac{\partial}{\partial y'} \\ \frac{\partial}{\partial z'} \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{\partial}{c\partial t} \\ \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}, \tag{13.31}$$

where in the rhs the multiplication is matrix multiplication. This equation means that the quantity $\nabla_i \equiv \left(\frac{\partial}{\partial (ct)}, \nabla\right)$ (note the lowering of the index and the consequent change in the sign of the zeroth component![11]) is a four-vector.
The length of this four-vector is

$$\eta_{ij}\nabla^i\nabla^j = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \equiv \Box.$$

This quantity is invariant, that is

$$\nabla'^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2} = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}. \tag{13.32}$$

b. We consider the LCF Σ' whose axes are parallel to the ones of the LCF Σ and moves with velocity $\mathbf{u}$. Then the general Lorentz transformation (1.52) which relates Σ and Σ' gives for the three-vector ∇

[11] We recall at this point the rule for signs of the components when we lower and raise the index of a four-vector. To the contravariant four-vector A^i we correspond the covariant four-vector A_i with the relation $A_i = \eta_{ij}A^j$ where η_{ij} is the Lorentz metric. In case we have Lorentz orthonormal frames (which we assume to be the rule) the Lorentz metric has its canonical form $\eta_{ij} = diag(-1, 1, 1, 1)$. Therefore in such frames (and only there!) the four-vector $A^i = \begin{pmatrix} B \\ \mathbf{A} \end{pmatrix}_\Sigma$ corresponds to the four-vector $A_i = (-B, \mathbf{A})_\Sigma$, that is, the sign of the zeroth component changes and the matrix from column becomes row. If the frames are not Lorentz orthonormal then the above simple rule does not apply and one has to (a) compute the components of the Lorentz metric and then (b) multiply the resulting 4×4 matrix with the column (or row) matrix defined by the components of the four-vector in the same frame.

$$\nabla' = \nabla + \left\{(\gamma - 1)\frac{\mathbf{u} \cdot \nabla}{u^2} + \frac{1}{c^2}\gamma\frac{\partial}{\partial t}\right\}\mathbf{u}.$$

13.4 The Equation of Continuity

The equations of continuity appear in all theories of physics and express the conservation of a current. In the Theory of Special Relativity the equations of continuity have a special significance which we show in this section.

In an LCF Σ we consider a Euclidean vector field $\mathbf{A}$ and a Euclidean invariant B which satisfy an equation of continuity, that is

$$\nabla \cdot \mathbf{A} + \frac{\partial B}{\partial t} = 0. \tag{13.33}$$

This relation can be written as follows:

$$\begin{pmatrix} -\frac{\partial}{c\partial t} \\ \nabla \end{pmatrix} \cdot \begin{pmatrix} Bc \\ \mathbf{A} \end{pmatrix} = 0, \tag{13.34}$$

where the dot indicates Lorentz inner product. This equation attains a physical meaning in Special Relativity only when it is covariant under the action of Lorentz transformations, which we assume to be the case. What does this assumption imply on the four-quantity $\begin{pmatrix} Bc \\ \mathbf{A} \end{pmatrix}_\Sigma$?

If the rhs was not zero then by the quotient theorem the four-quantity $A^i \equiv \begin{pmatrix} Bc \\ \mathbf{A} \end{pmatrix}_\Sigma$ should be a four-vector, because the operator $\left(\frac{\partial}{c\partial t}, \ \nabla\right)_\Sigma$ is a four-vector. But the rhs vanishes and the quotient theorem does not apply because zero is covariant wrt any homogeneous transformation (the Lorentz transformation included!).

However, although the equation of continuity does not determine the Lorentz covariance of the four-quantity $\begin{pmatrix} Bc \\ \mathbf{A} \end{pmatrix}_\Sigma$ it does say that the assumption A^i four-vector is acceptable. In this case the equation of continuity is written as $A^i{}_{,i} = 0$; it is Lorentz covariant and has a physical meaning in Special Relativity. The physical meaning we give it is that the four-vector A^i represents a "physical current" and the equation of continuity expresses the lack of charge for this current or, equivalently, the lack of sinks and sources for this physical field.

The consideration A^i four-vector is not the only possible way and of course it need not be compatible with other transformations[12] which the Euclidean quantities $\mathbf{A}$, Bc may obey.

[12] Indeed let us assume that the four-quantity $(Bc, \mathbf{A})_\Sigma$ transforms from an LCF Σ to the LCF Σ' according to the rule

In electromagnetism the equation of continuity is the following:

$$\nabla \cdot \mathbf{j} + \frac{\partial \rho}{\partial t} = 0,$$

where $\mathbf{j} = \rho \mathbf{u}$ is the conduction current in an LCF Σ, ρ the density of the charge in Σ, and $\mathbf{u}$ the velocity of the charge in Σ. This equation gives in Σ the four-quantity

$$j^i = \begin{pmatrix} \rho c \\ \mathbf{j} \end{pmatrix}_\Sigma = \begin{pmatrix} \rho c \\ \rho \mathbf{u} \end{pmatrix}_\Sigma = \frac{\rho}{\gamma} \gamma \begin{pmatrix} c \\ \mathbf{u} \end{pmatrix}_\Sigma = \frac{\rho}{\gamma} u^i, \tag{13.35}$$

where u^i is the four-velocity of the charge. Therefore the requirement j^i to be a four-vector implies that the quantity $\rho_0 = \frac{\rho}{\gamma}$ must be Lorentz invariant. The invariant ρ_0 is called *proper density* of the charge and we assume that it measures the density of the charge in the rest frame of the charge. The $\rho = \rho_0 \gamma$ is the density of the charge in the LCF Σ. We note that $\rho > \rho_0$, that is, we have "charge dilation." We call the four-vector j^i *conduction four-current* and we define it covariantly as follows:

$$j^i = \rho_0 u^i. \tag{13.36}$$

The equation of continuity in electromagnetism reads

$$j^i_{,i} = 0. \tag{13.37}$$

We assume the electric charge to be Lorentz invariant and define it in the LCF Σ with the relation

$$dQ = \rho dV, \tag{13.38}$$

where dV is an elementary volume at the point of three-space of Σ where the charge density is ρ. In the proper frame of the charge, $dQ = \rho_0 dV_0$ from which follows the well-known transformation of three-volume:

$$dV = \frac{1}{\gamma} dV_0. \tag{13.39}$$

$$\begin{pmatrix} B'c \\ \mathbf{A}' \end{pmatrix}_{\Sigma'} = \kappa(\upsilon) \begin{pmatrix} Bc \\ \mathbf{A} \end{pmatrix}_\Sigma,$$

where $\kappa(\upsilon)$ is a function of relative velocity $\mathbf{v}$ of the frames Σ, Σ'. Then relation (13.34) is preserved in the new frame. Indeed

$$A^i \longrightarrow A^{i'} = \kappa(\upsilon) A^i,$$

therefore

$$A^{i'}_{\ ,i'} = Q(\upsilon)(\kappa(\upsilon) A^i)_{,i} = Q(\upsilon)\kappa(\upsilon) A^i_{\ ,i},$$

where $Q(\upsilon)$ is some function of the relative velocity resulting from the Lorentz transformation. This condition is not Lorentz covariant except if and only if $Q(\upsilon)\kappa(\upsilon) = 1$.

Example 68 a. Show that the elementary three-volume dV under a Lorentz transformation with parameter γ transforms as follows: $dV = \frac{1}{\gamma} dV^{+}$ where dV^{+} is the three-volume in the proper frame.

b. Let A be an invariant physical quantity (e.g., mass, charge) with density ρ_A. Show that under the same Lorentz transformation the density transforms with the relation $\rho_A = \gamma \rho_A^{+}$ where ρ_A^{+} is the density in the proper frame of the volume.

c. Suppose the quantity A of question (b) satisfies an equation of continuity, that is, in an arbitrary LCF Σ:

$$\nabla \cdot \mathbf{j} + \frac{\partial \rho_A}{\partial t} = 0,$$

where $\mathbf{j}$ is a current which is related to A and ρ_A is the density of A in Σ. This equation is written as

$$\begin{pmatrix} \frac{\partial}{\partial ct} \\ \nabla \end{pmatrix} \begin{pmatrix} c\rho_A \\ \mathbf{j}_A \end{pmatrix} = 0.$$

Assume that the four-quantity $J^a_A = \begin{pmatrix} c\rho_A \\ \mathbf{j}_A \end{pmatrix}$ is a four-vector and find a covariant relation among J^a and the four-velocity u^a of the elementary volume dV in Σ. Finally show that in this case the equation of continuity of the quantity A is written covariantly as $J^a{}_{,a} = 0$.

Solution

a. We consider the elementary surface dS normal to the three-velocity $\mathbf{u}$ of the elementary volume in the LCF Σ and we have $dV = dSdl$. Under the action of the Lorentz transformation $dS = dS^{+}$ and $dl = \frac{1}{\gamma} dl^{+}$ from which follows $dV = \frac{1}{\gamma} dV^{+}$.
b. Because A is invariant we have

$$\rho_A dV = \rho_A^{+} dV^{+} \Longrightarrow \rho_A = \gamma \rho_A^{+}.$$

c. Considering that the quantity $\begin{pmatrix} c\rho_A \\ \mathbf{j}_A \end{pmatrix}$ is a four-vector we have

$$\begin{pmatrix} c\rho_A \\ \mathbf{j}_A \end{pmatrix} = \rho_A \begin{pmatrix} c \\ \mathbf{j}_A/\rho_A \end{pmatrix} = \rho_A^{+} \begin{pmatrix} \gamma c \\ \gamma \mathbf{j}_A/\rho_A \end{pmatrix}.$$

The only four-vector whose zeroth component equals γc in the arbitrary LCF Σ is the four-velocity u^a of the elementary volume dV. It follows that $\mathbf{j}_A = \rho_A \mathbf{u}$ where $\mathbf{u}$ is the three-velocity of the elementary volume dV in Σ. We infer that the four-current of the quantity A is defined covariantly with the relation $J^a_A = \rho_A^{+} u^a$ where u^a is the four-velocity of every elementary volume dV of A. In terms of the four-current the equation of continuity is written as the Lorentz product:

$$0 = \frac{\partial \rho}{\partial ct} + \nabla \cdot \mathbf{j}_A = \frac{\partial J_A^0}{\partial x^0} + \nabla \cdot J_A^\mu = J_{A,a}^a.$$

Example 69 a. A current I flows through a straight conductor of infinite length. Calculate the magnetic field in a distance r from the conductor.

b. A straight conductor of infinite length is charged homogeneously with a positive charge of density ρ. Calculate the electric field at a distance r from the conductor.

c. In the LCF Σ the current I flows through a straight conductor of infinite length while a charge q moves parallel to the direction of the conductor with constant speed u at a distance r from the conductor. Study the motion of the charge in the proper frame of the conductor and the proper frame of the charge.

(It is given that $\varepsilon_0 = 8.85418 \times 10^{-12}\ \frac{\mathrm{Cb}^2}{\mathrm{N\,m}^2}$ (the useful quantity is $\frac{1}{4\pi\varepsilon_0} = 9.0 \times 10^9 \frac{\mathrm{N\,m}^2}{\mathrm{Cb}^2}$) and $\mu_0 = 4\pi \times 10^{-7}\ \frac{\mathrm{Weber}}{\mathrm{A\,m}}$.)

Solution

a. According to Ampére's Law the magnetic field $\mathbf{B}$ which is created from a current I is given (in the SI system) by the relation

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I, \tag{13.40}$$

where the integral is computed over a smooth closed curve in the space of the magnetic field and $d\mathbf{l}$ is the element of length along this curve (see Fig. 13.1).

In the case of a straight conductor of infinite length, due to symmetry, the magnetic field at a point P in space must have a direction normal to the direction of the conductor and a strength depending on the distance r of the P from the conductor. In order to compute the magnetic field at the point P we consider the plane through P which is normal to the conductor and in that plane we consider a circle centered at the conductor with radius r. Then $\mathbf{B} = B\hat{\mathbf{e}}_\phi$ and $d\mathbf{l} = dl\hat{\mathbf{e}}_\phi$, hence

$$\oint \mathbf{B} \cdot d\mathbf{l} = \oint Brd\phi = 2\pi r B.$$

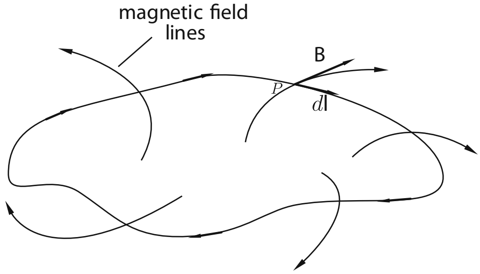

Fig. 13.1 Ampére's Law

Ampére's Law gives

$$\mathbf{B} = \frac{\mu_0 I}{2\pi r}\widehat{\mathbf{e}}_\phi,$$

where $(\widehat{\mathbf{e}}_r, \widehat{\mathbf{e}}_\phi, \widehat{\mathbf{e}}_\theta)$ is the right-handed system of spherical coordinates.

b. If a charge, Q say, is enclosed in the interior of a smooth closed surface S, then the electric field $\mathbf{E}$ on the surface is given by Gauss Law, which in the SI system of units is as follows:

$$\oint \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\varepsilon_0} Q,$$

where $d\mathbf{S}$ is the elementary surface at a point P of S and $\mathbf{E}$ is the value of the electric field at the point P.

In the case of a straight conductor of infinite length with charge density ρ we consider a cylindrical surface with axis the conductor, radius r, and length $2l$ and have

$$\oint \mathbf{E} \cdot d\mathbf{S} = \int_{-l}^{l} E 2\pi r dl.$$

The $\mathbf{E} \parallel d\mathbf{S}$ due to symmetry, that is, $\mathbf{E} = E\widehat{\mathbf{e}_r}$, $\mathbf{S} = dS\widehat{\mathbf{e}_r}$. Gauss Law gives

$$E = \frac{\rho}{2\pi r \varepsilon_0},$$

where $\rho = \frac{Q}{l}$ is the linear density of charge. The direction of $\mathbf{E}$ depends on the sign of the charge density.

c. In the proper frame of the conductor, Σ say, the current flowing through the conductor creates a magnetic field. There does not exist an electric field because the metallic conductor is electrically neutral. Indeed the conductor consists of positive ions of charge density ρ_+, which have velocity zero and free electrons of density $\rho_- = \rho_+$ which move with small velocity v, opposite to the direction of the current. In the frame Σ we consider a charge q which moves with velocity u parallel to the direction of the current in the conductor, therefore it suffers the Lorentz force

$$\mathbf{F} = q\mathbf{u} \times \mathbf{B}.$$

As we saw in (a) the magnetic field $\mathbf{B} = \frac{\mu_0 I}{2\pi r}\widehat{\mathbf{e}}_\phi$, therefore

$$\mathbf{F} = qu\frac{\mu_0 I}{2\pi r}\widehat{\mathbf{e}}_\theta \times \widehat{\mathbf{e}}_\phi = -\frac{qu\mu_0 I}{2\pi r}\widehat{\mathbf{e}}_r,$$

where $\hat{\mathbf{e}}_r$ is the unit normal with direction from the conductor to the charge. We note that the charge is attracted from the conductor.[13]

We consider now the motion in the proper frame of the charge, Σ' say. In that frame the densities of the positive ions ρ'_+ and the free electrons ρ'_- are not equal and they must be calculated from the Lorentz transformation relating Σ, Σ'. Suppose that in Σ' the four-current of the positive ions j^i_+ and that of the free electrons j^i_- are

$$j^i_- = \begin{pmatrix} \rho'_- c \\ \rho'_- v' \\ 0 \\ 0 \end{pmatrix}_{\Sigma'}, \quad j^i_+ = \begin{pmatrix} \rho'_+ c \\ -\rho'_+ u \\ 0 \\ 0 \end{pmatrix}_{\Sigma'}.$$

We know that in the proper frame Σ of the conductor these four-vectors are

$$j^i_- = \begin{pmatrix} \rho_- c \\ \rho_- v \\ 0 \\ 0 \end{pmatrix}_{\Sigma}, \quad j^i_+ = \begin{pmatrix} \rho_+ c \\ 0 \\ 0 \\ 0 \end{pmatrix}_{\Sigma}.$$

The Lorentz transformation gives

$$\rho'_- c = \gamma_u \left(\rho_- c - \frac{u}{c}\rho_- v\right) = \gamma_u \rho_- c \left(1 - \frac{uv}{c^2}\right),$$
$$\rho'_+ c = \gamma_u \rho_+ c = \gamma_u \rho_- c$$

(because in Σ $\rho_+ = \rho_-$). The overall charge density in Σ' is

$$\rho' = \rho'_+ - \rho'_- = \gamma_u \rho_- \frac{uv}{c^2}.$$

This means that the conductor appears to be charged in Σ' although it is neutral in Σ! This fact is completely at odds with the Newtonian point of view and it is due to the fact that the charge density (not the charge!) is not Lorentz invariant in Special Relativity.

This charge of the conductor creates at the position of the moving charge the electric field

[13] From this result we can compute the force between two straight parallel conductors of infinite length carrying currents I_1 and I_2. Indeed the first conductor (the I_1 say) exerts to the elementary length dl_2 of the second conductor the force $\mathbf{F}_{21} = -\frac{dq_2 u \mu_0 I_1}{2\pi r}\widehat{\mathbf{e}}_r$. But the length dl_2 carries charge $dq_2 = I_2 dt$ so that $dq_2 u = I_2 dl_2$, therefore the force per unit length on the current I_2 due to the current I_1 is $\frac{\mathbf{F}_{21}}{dl_2} = -\frac{\mu_0 I_1 I_2}{2\pi r}\widehat{\mathbf{e}}_r$. This force is obviously attractive. If the currents have opposite direction of flow that force is repulsive, a phenomenon with important practical applications.

$$\mathbf{E}' = \frac{\rho'}{2\pi r'\varepsilon_0}\widehat{\mathbf{e}}_{r'} = \frac{\gamma_u \rho_- u v}{2\pi r'\varepsilon_0 c^2}\widehat{\mathbf{e}}_r.$$

But $r' = r$ because r is measured normal to the velocity u. Also $\varepsilon_0 c^2 = \frac{1}{\mu_0}$. Replacing we find

$$\mathbf{E}' = \gamma_u \frac{\mu_0(\rho_- v)u}{2\pi r}\widehat{\mathbf{e}}_r.$$

Now $\rho_- v = j_- = -I$ where I is the current through the conductor. Therefore

$$\mathbf{E}' = -\gamma_u \frac{\mu_0 u I}{2\pi r}\widehat{\mathbf{e}}_r = -\gamma_u \mathbf{E}.$$

The force which is exerted on the moving charge in Σ' is due to this electric field and equals

$$\mathbf{F}^+ = q\mathbf{E}' = -\gamma_u q\mathbf{E} = -\gamma_u \mathbf{F},$$

where $\mathbf{F}$ is the force in the charge in Σ, which is due to the magnetic field created by the current I.

This result is compatible with the transformation of the four-force. Indeed in Σ the product $\mathbf{F}\cdot\mathbf{u} = 0$, hence the four-force on the charge q has components

$$F^i = \begin{pmatrix} 0 \\ \gamma_u \mathbf{F}_\parallel = 0 \\ \gamma_u \mathbf{F}_\perp = -\gamma_u \mathbf{F} \end{pmatrix}_\Sigma .$$

In the LCF Σ' the four-force is

$$F^i = \begin{pmatrix} 0 \\ \mathbf{F}'_\parallel \\ \mathbf{F}'_\perp \end{pmatrix}_{\Sigma'} .$$

The Lorentz transformation relating Σ, Σ' gives

$$\begin{aligned} \mathbf{F}'_\parallel &= \gamma_u(0 - \beta\gamma_u \mathbf{F}_\parallel) = 0, \\ \mathbf{F}'_\perp &= -\gamma_u \mathbf{F}, \end{aligned}$$

which coincides with the previous result.

We note that in the proper frame Σ' of the moving charge *there does exist* magnetic field created by the current flowing through the conductor, but this field does not exert a force on the charge because in Σ' the charge is at rest.

In order to compute in Σ' the current which flows through the conductor we consider the currents of the positive ions and the free electrons and we have for the net total current I'

$$I' = I'_- + I'_+ = -\rho'_- v' + \rho'_+ v = -\gamma_u \rho_-(v-u) - \gamma_u \rho_- u = -\gamma_u \rho_- v = \gamma_u I,$$

where I is the current flowing through the conductor in Σ.
Question: Is this result compatible with the Lorentz transformation of the four-current?

13.5 The Electromagnetic Four-Potential

In the last section we wrote Maxwell equations for empty three-space in an arbitrary LCF Σ in terms of the potentials $\mathbf{A}, \phi$. In the present section we shall write these equations in terms of the four-vector formalism.

For convenience we collect the obtained results in the following table:

Maxwell equations	Electromagnetic potentials	
$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$	$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$	(13.13)
$\nabla \cdot \mathbf{B} = 0$	$\mathbf{B} = \nabla \times \mathbf{A}$	(13.11)
$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$	$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -\frac{\rho}{\varepsilon_0}$	(13.18)
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{j}$	(13.19)
Lorentz gauge	$\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t} = 0$	(13.17)

We note that the equations giving the potentials $\mathbf{A}, \phi$ in Σ are written ($\mu_0\varepsilon_0 = 1/c^2$) as

$$\text{Lorentz gauge}: \begin{pmatrix} -\frac{\partial}{c\partial t} \\ \nabla \end{pmatrix}_\Sigma \cdot \begin{pmatrix} \frac{\phi}{c} \\ \mathbf{A} \end{pmatrix}_\Sigma = 0, \tag{13.41}$$

$$\text{Maxwell equations}: \Box \begin{pmatrix} \frac{\phi}{c} \\ \mathbf{A} \end{pmatrix}_\Sigma = -\begin{pmatrix} \frac{\rho}{c\epsilon_0} \\ \mu_0 \mathbf{j} \end{pmatrix}_\Sigma = -\mu_0 j^i. \tag{13.42}$$

Equation (13.41) is a continuity equation with "current" $\begin{pmatrix} \frac{\phi}{c} \\ \mathbf{A} \end{pmatrix}_\Sigma$, which is compatible with the assumption $(-\phi/c, \mathbf{A})$ four-vector. Equation (13.42) implies (due to the quotient theorem) that $(-\phi/c, \mathbf{A})$ must be a four-vector, because the operator $\Box \equiv \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$ is Lorentz covariant and the rhs is the four-current. We conclude that there exists a new four-vector, which we denote Ω^i and call the *electromagnetic four-potential* which in the LCF Σ has components

$$\Omega^i = \begin{pmatrix} \frac{\phi}{c} \\ \mathbf{A} \end{pmatrix}_\Sigma \Leftrightarrow \Omega_i = \left(-\frac{\phi}{c}, \mathbf{A}\right)_\Sigma . \tag{13.43}$$

In terms of the electromagnetic four-potential Ω^i Maxwell equations are written in covariant form as follows:

$$\Omega^i{}_{,i} = 0, \tag{13.44}$$

$$\Box\Omega^i \equiv \Omega^{i,j}{}_{,j} = -\mu_0 j^i . \tag{13.45}$$

It is apparent that the knowledge of Ω^i is sufficient to determine the electromagnetic field. Indeed let us compute the fields **E**, **B** in Σ in terms of the components of Ω^i (in Σ!). If in (13.13) we consider

$$ct = x^0, \qquad x = x^1, \qquad y = x^2, \qquad z = x^3$$

and write for the partial derivative $\Omega_{i,j} = \frac{\partial \Omega_i}{\partial x^j}$ it follows easily that[14]

$$\frac{1}{c}E_x = \Omega_{0,1} - \Omega_{1,0},$$
$$\frac{1}{c}E_y = \Omega_{0,2} - \Omega_{2,0},$$
$$\frac{1}{c}E_z = \Omega_{0,3} - \Omega_{3,0}.$$

Similarly from (13.11) we have

$$B_x = \Omega_{3,2} - \Omega_{2,3},$$
$$B_y = \Omega_{1,3} - \Omega_{3,1},$$
$$B_z = \Omega_{2,1} - \Omega_{1,2}.$$

[14] In a compact formalism (13.13) is written as

$$E_\mu = -\phi_{,\mu} - cA_{\mu,0}.$$

Now (13.43) gives

$$\Omega_0 = -\frac{\phi}{c}, \quad \Omega_\mu = A_\mu$$

from which we compute

$$\Omega_{0,\mu} = -\frac{1}{c}\phi_{,\mu}, \quad \Omega_{\mu,0} = A_{\mu,0} \quad \Rightarrow \quad \Omega_{0,\mu} - \Omega_{\mu,0} = -\frac{1}{c}(\phi_{,\mu} + cA_{\mu,0}) = \frac{1}{c}E_\mu .$$

For the magnetic field we have from (13.11)

$$B^\mu = -\epsilon^{\mu\nu\rho} A_{[\nu\rho]} = -\epsilon^{\mu\nu\rho}\Omega_{[\nu\rho]}.$$

The above relations can be written collectively in the form of a matrix as follows[15]:

$$\begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix}_{\Sigma} = \Omega_{j,i} - \Omega_{i,j}\ . \tag{13.46}$$

The rhs of (13.46) is an antisymmetric tensor F_{ij} of order (0,2) which we name the *electromagnetic field tensor* and define as follows:

$$F_{ij} = \Omega_{j,i} - \Omega_{i,j}. \tag{13.47}$$

The components of F_{ij} in the LCF Σ are

$$F_{ij} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix}_{\Sigma}. \tag{13.48}$$

From the above we infer the following conclusions:

a. In an LCF Σ the fields **E**, **B** are parts (components) of a more general field which is the electromagnetic field tensor F_{ij}. This is similar to the Newtonian space and Newtonian time which are parts (components) of the relativistic spacetime.
b. The components of a tensor depend on the LCF where they are computed. Therefore it is possible that the electromagnetic field in an LCF is expressed in terms of an electric field, in another in terms of a magnetic field, and in another in terms of an electric and a magnetic field. For example the electromagnetic field of a charge in the proper frame of the charge is expressed by an electric field and in a frame where the charge is moving in terms of an electric and a magnetic field. This fact cannot be explained in terms of Newtonian Physics because there these vector fields are covariant, therefore if they vanish/do not vanish in one frame they vanish/do not vanish in all Newtonian frames.

In the following the electromagnetic field tensor due to the fields **E**, **B** shall be indicated as $F_{ij} \equiv (\mathbf{E}/c, \mathbf{B})$. Because the units of the components of a tensor must be the same, the magnetic field **B** must always be related to $\mathbf{E}/c$.

Exercise 58 Show that the potentials ϕ and **A** under a Lorentz transformation transform as follows:

[15] According to our convention the first index of a matrix counts rows and the second columns.

$$\begin{aligned}\mathbf{A}'_{\perp} &= \mathbf{A}_{\perp},\\ \mathbf{A}'_{||} &= \gamma\left[\mathbf{A}_{||} - \frac{\mathbf{u}}{c^2}\phi\right],\\ \phi' &= \gamma(\phi - \mathbf{u}\cdot\mathbf{A}_{||}).\end{aligned}$$

[Hint: Consider the electromagnetic four-potential Ω^i.]

13.6 The Electromagnetic Field Tensor F_{ij}

The electromagnetic field tensor is a powerful tool which enables us to use geometric methods and the Lorentz transformation to answer a number of fundamental questions concerning the electromagnetic field. In this section we shall deal with the following questions:

(a) Consider an electromagnetic field F_{ij} which in the LCF Σ is represented with the fields $(\mathbf{E}/c, \mathbf{B})$ and in the LCF Σ' with the fields $(\mathbf{E}'/c, \mathbf{B}')$. What is the transformation between the fields $(\mathbf{E}/c, \mathbf{B})$ and $(\mathbf{E}'/c, \mathbf{B}')$?
(b) How can one write the electromagnetic field equations (13.44), (13.45) in terms of the electromagnetic field tensor F_{ij}?
(c) What are the invariants of the electromagnetic field?

13.6.1 The Transformation of the Fields

The electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ are not parts of four-vectors, therefore one cannot use the Lorentz transformation relating Σ and Σ' to compute their transformation. Instead one must use the electromagnetic field tensor F_{ij} in which these fields are components. If $L^{i'}_i$ is the Lorentz transformation relating Σ and Σ' the components of F_{ij} are transformed as follows:

$$F_{i'j'} = L^i_{i'} L^j_{j'} F_{ij}. \tag{13.49}$$

Let us compute the transformation of the fields for the case of a boost along the common axis x, x' with velocity u. For a boost we have the matrices

$$L^i_{i'} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0\\ \beta\gamma & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}, \qquad L^{i'}_i = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0\\ -\beta\gamma & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}, \tag{13.50}$$

hence

$$-\frac{E_{x'}}{c} = F_{0'1'} = L^i_{0'} L^j_{1'} F_{ij} = L^0_{0'} L^1_{1'} F_{01} + L^1_{0'} L^0_{1'} F_{10} = -\gamma^2 \frac{E_x}{c} + \gamma^2 \beta^2 \frac{E_x}{c} = -\frac{E_x}{c},$$

$$B_{z'} = F_{1'2'} = L^i_{1'} L^j_{2'} F_{ij} = L^i_{1'} L^2_{2'} F_{i2} = L^0_{1'} F_{02} + L^1_{1'} F_{12} = \gamma \left(B_z - \beta \frac{E_y}{c} \right),$$

$$-B_{y'} = F_{1'3'} = L^i_{1'} L^j_{3'} F_{ij} = L^i_{1'} L^3_{3'} F_{i3} = L^0_{1'} F_{03} + L^1_{1'} F_{13} = -\gamma \left(B_y + \beta \frac{E_z}{c} \right).$$

Similarly we compute the remaining elements of the transformed matrix $F_{i'j'}$.

A practical method to compute the transformation of the components of the tensor F_{ij} in Σ' is the following.[16] The tensor F_{ij} in the LCFs Σ and Σ' is represented with the following matrices:

$$[F_{ij}] = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix}_{\Sigma},$$

$$[F_{i'j'}] = \begin{pmatrix} 0 & -E_{x'}/c & -E_{y'}/c & -E_{z'}/c \\ E_{x'}/c & 0 & B_{z'} & -B_{y'} \\ E_{y'}/c & -B_{z'} & 0 & B_{x'} \\ E_{z'}/c & B_{y'} & -B_{x'} & 0 \end{pmatrix}_{\Sigma'}.$$

Then the Lorentz transformation of the tensor F_{ij} is written as the following matrix equation:

$$[F_{i'j'}]_{\Sigma'} = [L^i_{i'}]^t [F_{ij}]_{\Sigma} [L^j_{j'}], \tag{13.51}$$

where the upper index t indicates transpose. The Lorentz metric for a boost is symmetric, therefore in the rhs we have to consider only the matrix $L^i_{i'}$. Introducing the matrices in the rhs we find

$$F_{i'j'} = \begin{pmatrix} 0 & -E_x/c & -\gamma(\frac{E_y}{c} - \beta B_z) & -\gamma(\frac{E_z}{c} + \beta B_y) \\ E_x/c & 0 & \gamma(B_z - \beta\frac{E_y}{c}) & -\gamma(B_y + \beta\frac{E_z}{c}) \\ \gamma(\frac{E_y}{c} - \beta B_z) & -\gamma(B_z - \beta\frac{E_y}{c}) & 0 & B_x \\ \gamma(\frac{E_z}{c} + \beta B_y) & \gamma(B_y + \beta\frac{E_z}{c}) & -B_x & 0 \end{pmatrix}. \tag{13.52}$$

[16] See also Exercise 6. This practical rule applies to tensors of type (0,2) and (2,0)! It is possible to be generalized to tensors with more indices; however, this is outside our interest.

If we replace $F_{i'j'}$ in terms of $\mathbf{E}', \mathbf{B}'$ we obtain the transformation equations of the electric and the magnetic fields:

$$\begin{aligned} E_{x'}/c &= E_x/c, & B_{x'} &= B_x, \\ E_{y'}/c &= \gamma(E_y/c - \beta B_z), & B_{y'} &= \gamma(B_y + \beta E_z/c), \\ E_{z'}/c &= \gamma(E_z/c + \beta B_y), & B_{z'} &= \gamma(B_z - \beta E_y/c). \end{aligned} \tag{13.53}$$

Another useful relation among the pairs $(\mathbf{E}/c, \mathbf{B})$, $(\mathbf{E}'/c, \mathbf{B}')$ relates the parallel and the normal projection of the fields along the direction of the relative velocity $\mathbf{u}$ of Σ and Σ'. Indeed by writing $\mathbf{u} = u\mathbf{i}$ we have from (13.53)

$$\begin{aligned} \mathbf{E}'_{||} &= E_{x'}\mathbf{i}' = E_x\mathbf{i} = \mathbf{E}_{||}, \\ \mathbf{E}'_{\perp} &= E_{y'}\mathbf{j}' + E_{z'}\mathbf{k}' = \gamma(E_y - uB_z)\mathbf{j} + \gamma(E_z + uB_y)\mathbf{k} \\ &= \gamma(E_y\mathbf{j} + E_z\mathbf{k}) + \gamma u(-B_z\mathbf{j} + B_y\mathbf{k}) = \gamma\left[\mathbf{E}_{\perp} + (\mathbf{u}\times\mathbf{B})_{\perp}\right]. \end{aligned}$$

Similarly we work with $\mathbf{B}'_{||}, \mathbf{B}'_{\perp}$. Finally we obtain the relations

$$\begin{aligned} \mathbf{E}'_{||} &= \mathbf{E}_{||}, \\ \mathbf{E}'_{\perp} &= \gamma\left[\mathbf{E}_{\perp} + \mathbf{u}\times\mathbf{B}\right], \\ \mathbf{B}'_{||} &= \mathbf{B}_{||}, \\ \mathbf{B}'_{\perp} &= \gamma\left[\mathbf{B}_{\perp} - \frac{1}{c^2}\mathbf{u}\times\mathbf{E}\right]. \end{aligned} \tag{13.54}$$

It follows that the projections of the fields $\mathbf{E}, \mathbf{B}$ along the direction of the velocity $\mathbf{u}$ do not change whereas the normal projections are changing by the factor $\mathbf{u}\times\mathbf{B}$, $-\mathbf{u}\times\mathbf{E}$, respectively. Relations (13.54) are more general than (13.53) because they apply to a general relative velocity $\mathbf{u}$ and not only a boost.

Exercise 59 Prove that the transformation equations (13.54) can be written equivalently as follows:

$$\mathbf{E}' = \gamma\mathbf{E} + \frac{1-\gamma}{u^2}(\mathbf{u}\cdot\mathbf{E})\mathbf{u} + \gamma(\mathbf{u}\times\mathbf{B}), \tag{13.55}$$

$$\mathbf{B}' = \gamma\mathbf{B} + \frac{1-\gamma}{u^2}(\mathbf{u}\cdot\mathbf{B})\mathbf{u} - \frac{1}{c^2}\gamma(\mathbf{u}\times\mathbf{E}). \tag{13.56}$$

13.6.2 Maxwell Equations in Terms of F_{ij}

From the definition $F_{ij,k} = -\Omega_{i,jk} + \Omega_{j,ik}$ we have by cyclic permutation of the indices

$$
\begin{aligned}
F_{ij,k} &= -\Omega_{i,jk} + \Omega_{j,ik}, \\
F_{jk,i} &= -\Omega_{j,ki} + \Omega_{k,ji}, \\
F_{ki,j} &= -\Omega_{k,ij} + \Omega_{i,kj}.
\end{aligned}
$$

Adding we find the equation

$$
F_{ij,k} + F_{jk,i} + F_{ki,j} = 0. \tag{13.57}
$$

This is one of the equations of the electromagnetic field which essentially is equivalent to the existence of the electromagnetic potential.[17]

The second equation relates F_{ij} with the four-current j^i and it corresponds to the main Maxwell equation $\Box\Omega^i = -\mu_0 j^i$. In order to find this new equation we write

$$
F_i{}^j = -\Omega_i{}^{,j} + \Omega^j{}_{,i}
$$

and differentiate wrt j

$$
F_i{}^{,j}{}_{,j} = -\Omega_i{}^{,j}{}_{,j} + \Omega^j{}_{,ij}.
$$

From the equation of continuity (13.44) the term $\Omega^j{}_{,ij} = \Omega^j{}_{,ji} = 0$ whereas from (13.45) the term $\Omega_i{}^{,j}{}_{,j} = \Box\Omega^i = -\mu_0 j^i$. It follows that the second Maxwell equation in terms of the electromagnetic field tensor F_{ij} is

$$
F^{ij}{}_{,j} = \mu_0 j^i. \tag{13.58}
$$

13.6.3 The Invariants of the Electromagnetic Field

The transformation relations (13.53), (13.54) enable us to compute the electromagnetic field in any LCF if we know it in one. This is a characteristic of Special Relativity, which makes possible the solution of a problem in a special LCF where the answer is either known or most convenient to be found and then transfer the solution to the required LCF. In electromagnetism this special LCF is selected (as a rule) by the invariants of the electromagnetic field.

The electromagnetic field is described completely by the electromagnetic field tensor, therefore the invariants of the field are given by the invariants of the tensor F_{ij}. The tensor F_{ij} has two, *and only two*, invariants given by the following expressions:

$$
X \equiv -\frac{1}{2} F_{ij} F^{ij} \qquad \text{and} \qquad Y \equiv -\frac{1}{8} \eta_{ijkl} F^{ij} F^{kl}, \tag{13.59}
$$

[17] It is the so-called Frobenius condition which need not worry us further.

where η_{ijkl} is the completely antisymmetric tensor with four indices.[18]

In an LCF Σ these invariants are computed in terms of the electric and the magnetic fields. Indeed considering the components of F_{ij} in Σ we have

$$\begin{aligned} X = -\frac{1}{2}F_{ij}F^{ij} &= -(F_{01}F^{01} + F_{02}F^{02} + F_{03}F^{03} + F_{12}F^{12} + F_{13}F^{13} + F_{23}F^{23}) \\ &= -\frac{1}{c^2}[E_x(-E_x) + E_y(-E_y) + E_z(-E_z)] - [B_x^2 + B_y^2 + B_z^2] \\ &= \frac{1}{c^2}\mathbf{E}^2 - \mathbf{B}^2, \end{aligned} \tag{13.60}$$

$$\begin{aligned} Y = -\frac{1}{8}\eta_{ijkl}F^{ij}F^{kl} &= 4\left[\eta_{0123}F^{01}F^{23} + \eta_{1230}F^{12}F^{30} + \eta_{2301}F^{23}F^{01}\right] \\ &= -4\left[\frac{E_x}{c}(-B_x) + \frac{E_y}{c}(-B_y) + \frac{E_z}{c}(B_z)\right] \\ &= \frac{1}{c}\mathbf{E}\cdot\mathbf{B}. \end{aligned} \tag{13.61}$$

In the next example we show that the quantities $\frac{1}{c^2}\mathbf{E}^2 - \mathbf{B}^2$ and $\frac{1}{c}\mathbf{E}\cdot\mathbf{B}$ are indeed invariant under Lorentz transformations.

Example 70 Consider two LCFs Σ and Σ' which are moving in the standard way and let $\mathbf{E}$, $\mathbf{B}$ and $\mathbf{E}'$, $\mathbf{B}'$ be the electric and the magnetic fields in Σ and Σ', respectively. Prove that

$$\mathbf{E}\cdot\mathbf{B} = \mathbf{E}'\cdot\mathbf{B}' \qquad \text{and} \qquad \mathbf{B}^2 - \frac{1}{c^2}\mathbf{E}^2 = \mathbf{B}'^2 - \frac{1}{c^2}\mathbf{E}'^2.$$

First Solution

From the transformation equations (13.53) we have

$$\begin{aligned} \frac{1}{c}\mathbf{E}'\cdot\mathbf{B}' &= \frac{1}{c}\left[E'_x B'_x + E'_y B'_y + E'_z B'_z\right] \\ &= \frac{1}{c}[E_x B_x + \gamma^2\left(E_y/c - \beta B_z\right)\left(B_y + \beta E_z\right) \\ &\quad + \gamma^2\left(E_z/c + \beta B_y\right)\left(B_z - \beta E_y/c\right)] \end{aligned}$$

$$= \frac{1}{c}\left[E_x B_x + \left(\gamma^2 - \gamma^2\beta^2\right)E_y B_y + \left(\gamma^2 - \gamma^2\beta^2\right)E_z B_z\right]$$

[18] η_{ijkl} $i, j, k, l = 0, 1, 2, 3$ equals zero if two of the indices have the same value and ± 1 if $(ijkl)$ is an even or an odd permutation of (0123). See Sect. 13.10.1.

$$
\begin{aligned}
&= \frac{1}{c}\left(E_x B_x + E_y B_y + E_z B_z\right) \\
&= \frac{1}{c}\mathbf{E}\cdot\mathbf{B}.
\end{aligned}
$$

Also

$$
\begin{aligned}
\frac{1}{c^2}(\mathbf{E}')^2 - (\mathbf{B}')^2 &= \frac{1}{c^2}(E_x'^2 + E_y'^2 + E_z'^2) - (B_x'^2 + B_y'^2 + B_z'^2) \\
&= \frac{1}{c^2}E_x^2 + \gamma^2\left(\frac{1}{c}E_y - \beta B_z\right)^2 + \gamma^2\left(\frac{1}{c}E_z + \beta B_y\right)^2 \\
&\quad - B_x^2 - \gamma^2\left(B_y + \beta\frac{1}{c}E_z\right)^2 - \gamma^2\left(B_z - \beta\frac{1}{c}E_y\right)^2 \\
&\quad + 2\gamma^2\beta\frac{1}{c}E_z B_y - 2\gamma^2\beta\frac{1}{c}E_z B_y \\
&= \frac{1}{c^2}E_x^2 + \left(\gamma^2 - \gamma^2\beta^2\right)\frac{1}{c^2}E_y^2 + \left(\gamma^2 - \gamma^2\beta^2\right)\frac{1}{c^2}E_z^2 \\
&\quad - B_x^2 - \left(\gamma^2 - \gamma^2\beta^2\right)B_y^2 - \left(\gamma^2 - \gamma^2\beta^2\right)B_z^2 \\
&= \frac{1}{c^2}(E_x^2 + E_y^2 + E_z^2) - (B_x^2 + B_y^2 + B_z^2) \\
&= \frac{1}{c^2}\mathbf{E}^2 - \mathbf{B}^2.
\end{aligned}
$$

Second Solution

In this solution we use the transformation equations (13.54) which define the general Lorentz transformation, therefore the proof is completely general. We have

$$
\begin{aligned}
\mathbf{E}'\cdot\mathbf{B}' &= \left[\mathbf{E}_\parallel + \gamma(\mathbf{E}_\perp + \mathbf{v}\times\mathbf{B})\right]\cdot\left[\mathbf{B}_\parallel + \gamma(\mathbf{B}_\perp - \frac{1}{c^2}\mathbf{v}\times\mathbf{E})\right] \\
&= \mathbf{E}_\parallel\cdot\mathbf{B}_\parallel + \gamma^2\mathbf{E}_\perp\cdot\mathbf{B}_\perp - \gamma^2\frac{1}{c^2}(\mathbf{v}\times\mathbf{B})\cdot(\mathbf{v}\times\mathbf{E}).
\end{aligned}
$$

From classical vector calculus we have the identity

$$
(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C}).
$$

Using the above identity we compute

$$
\frac{1}{c^2}(\mathbf{v}\times\mathbf{B})\cdot(\mathbf{v}\times\mathbf{E}) = \beta^2(\mathbf{E}\cdot\mathbf{B}) - \beta^2\mathbf{E}_\parallel\cdot\mathbf{B}_\parallel.
$$

Replacing we find

$$\begin{aligned}
\mathbf{E}' \cdot \mathbf{B}' &= (1 + \gamma^2\beta^2)\mathbf{E}_\parallel \cdot \mathbf{B}_\parallel + \gamma^2 \mathbf{E}_\perp \cdot \mathbf{B}_\perp - \gamma^2\beta^2(\mathbf{E} \cdot \mathbf{B}) \\
&= \gamma^2(\mathbf{E}_\parallel \cdot \mathbf{B}_\parallel + \mathbf{E}_\perp \cdot \mathbf{B}_\perp) - \gamma^2\beta^2(\mathbf{E} \cdot \mathbf{B}) = (\gamma^2 - \gamma^2\beta^2)(\mathbf{E} \cdot \mathbf{B}) \\
&= \mathbf{E} \cdot \mathbf{B}.
\end{aligned}$$

In the same manner one proves the second relation.

13.7 The Physical Significance of the Electromagnetic Invariants

The invariants of the electromagnetic field contain significant information concerning the character of the electromagnetic field and can be used to classify the electromagnetic fields in sets with similar dynamical properties.

As we have seen the electromagnetic field has two invariants, the quantities $X = \frac{1}{c^2}\mathbf{E}^2 - \mathbf{B}^2$ and $Y = \frac{1}{c}\mathbf{E} \cdot \mathbf{B}$. X measures the difference of the strengths of the fields and Y their angle. Therefore we consider the following four cases:

(i) $X = Y = 0$:
The electromagnetic fields with vanishing invariants are characterized by the properties

$$\frac{1}{c} \mid \mathbf{E} \mid = \mid \mathbf{B} \mid \qquad \text{and} \qquad \mathbf{E} \perp \mathbf{B},$$

that is, in all LCFs the electric and the magnetic fields have the same strength and they are normal to each other. These electromagnetic fields are called *null electromagnetic fields* and we assume that they describe electromagnetic waves.

(ii) $X \neq 0, \; Y = 0$:
The electromagnetic fields with invariants $X \neq 0, \; Y = 0$ are characterized by the fact that in an arbitrary LCF either the electric and the magnetic fields are perpendicular and have different strengths or one of them vanishes. Which of them vanishes we find from the sign of the invariant X. If $X > 0$ then only the magnetic field is possible to vanish and if $X < 0$ then only the electric field is possible to vanish.

(iii) $X = 0, \; Y \neq 0$:
The electric and the magnetic fields of these electromagnetic fields have equal strength in all LCFs ($\frac{1}{c}|\mathbf{E}| = |\mathbf{B}|$) and they are perpendicular in no LCFs. If $Y > 0$ then (in all LCFs!) their directions make an acute angle and if $Y < 0$ this angle is obtuse. In both cases there exists always a family of LCFs in which $\mathbf{E} \parallel \mathbf{B}$.

(iv) $X \neq 0, \quad Y \neq 0$:
This case is similar to (iii) with the difference that the strengths of the fields $\mathbf{E}, \; \mathbf{B}$ are different in all LCFs.

From the above analysis we conclude that we should consider three types of electromagnetic fields:

(a) Electromagnetic waves ($X = Y = 0$)
(b) Electromagnetic fields with electric or magnetic field only or both fields perpendicular and with different strength ($X \neq 0,\ Y = 0$)
(c) Electromagnetic fields with both electric and magnetic fields ($X = 0$ or $X \neq 0$ and $Y \neq 0$) for which there exists a family of LCFs in which $\mathbf{E} \parallel \mathbf{B}$

Because the invariant Y determines whether the electric and the magnetic fields are perpendicular or parallel, in our study we consider the cases $Y = 0$ and $Y \neq 0$.

13.7.1 The Case $Y = 0$

Consider an electromagnetic field whose invariant $Y = 0$. We prove that there exist families of LCF in which the electromagnetic field has only electric field (if $X > 0$) or only magnetic field (if $X < 0$).

Suppose that in some LCF Σ the electromagnetic field has $\mathbf{E} = 0$ (respectively, $\mathbf{B} = 0$). Then in all LCFs Σ' with velocities $\parallel \mathbf{B}$ (respectively, $\parallel \mathbf{E}$) it has only magnetic (respectively, electric) field.

Consider now an arbitrary LCF Σ, in which $\mathbf{E} \neq 0$ and $\mathbf{B} \neq 0$, and assume that $X < 0$. We consider the LCF Σ' with velocity

$$\mathbf{u} = \alpha(\mathbf{E} \times \mathbf{B}),$$

where α is a parameter which has to be determined. Because the velocity is perpendicular to the fields we have

$$\mathbf{E}_{\parallel} = \mathbf{B}_{\parallel} = 0.$$

From the transformation equations (13.54) we find for the field $\mathbf{E}'$ in Σ'

$$\begin{aligned}
\mathbf{E}'_{\parallel} &= \mathbf{B}'_{\parallel} = 0, \\
\mathbf{E}'_{\perp} &= \gamma \left\{\mathbf{E}_{\perp} + \mathbf{u} \times \mathbf{B}\right\} = \gamma \left\{\mathbf{E} + \alpha \left[(\mathbf{E} \times \mathbf{B}) \times \mathbf{B}\right]\right\} = \gamma(1 - \alpha B^2)\mathbf{E},
\end{aligned}$$

where we have taken into consideration $Y = \frac{1}{c}\mathbf{E} \cdot \mathbf{B} = 0$. Because we have assumed $X < 0$ we can define the LCF Σ' with the condition $\mathbf{E}' = 0$, that is, there exists only magnetic field. Then the last equation gives

$$\alpha = 1/B^2 \Rightarrow \mathbf{u} = \frac{1}{B^2}(\mathbf{E} \times \mathbf{B}).$$

The magnetic field $\mathbf{B}'$ in Σ' is given from the relation (13.54):

$$\mathbf{B}' = \gamma_u \left\{ \mathbf{B} - \frac{1}{c^2} \mathbf{u} \times \mathbf{E} \right\} = \gamma_u \left(1 - \frac{E^2}{c^2 B^2} \right) \mathbf{B}$$

and it is $\mathbf{B}' \parallel \mathbf{B}$ and also $\mathbf{B}' \perp \mathbf{u}$.

The LCF Σ' belongs to the LCF in which there exists only magnetic field and– as we have shown – in all LCFs Σ'' with velocities $\mathbf{u}' = \lambda \mathbf{B}'$ relative to Σ', where λ is a real parameter, the electromagnetic field has only magnetic part. In order to compute the velocity $\mathbf{v}$ say, of Σ'' wrt Σ, we apply the relativistic rule of composition of three-velocities. From Example 20 we have

$$\mathbf{v} = \frac{1}{\gamma_u Q} \left\{ \lambda \mathbf{B}' + \mathbf{u} \left[\frac{\mathbf{u} \cdot \lambda \mathbf{B}'}{u^2} (\gamma_u - 1) + \gamma_u \right] \right\},$$

where $Q = 1 - \frac{\mathbf{u} \cdot \lambda \mathbf{B}}{c^2} = 1$. Replacing we find

$$\mathbf{v} = \mathbf{u} + \lambda \left(1 - \frac{E^2}{c^2 B^2} \right) \mathbf{B} = \frac{1}{B^2} (\mathbf{E} \times \mathbf{B}) + \lambda \left(1 - \frac{E^2}{c^2 B^2} \right) \mathbf{B}.$$

In case the invariant $X > 0$ then working in a similar manner we find

$$\mathbf{B}' = \gamma \left\{ \mathbf{B} - \frac{1}{c^2} \mathbf{v} \times \mathbf{E} \right\} = \gamma \left(1 - \alpha \frac{E^2}{c^2} \right) \mathbf{B}$$

and requiring that $\mathbf{B}' = 0$ we find the velocity

$$\mathbf{u} = \frac{c^2}{E^2} (\mathbf{E} \times \mathbf{B}).$$

The electromagnetic field in Σ' has only electric field, which is given by the relation

$$\mathbf{E}' = \gamma \left(\mathbf{E} + \mathbf{v} \times \mathbf{B} \right) = \gamma \left(1 - \frac{c^2 B^2}{E^2} \right) \mathbf{E}.$$

In this case the LCF Σ' is not unique and all LCFs Σ'' with velocities $\mathbf{u} = \frac{c^2}{E^2} (\mathbf{E} \times \mathbf{B}) + \lambda \left(1 - \frac{B^2 c^2}{E^2} \right) \mathbf{E}$ have the same property.

13.7.2 The Case $Y \neq 0$

In this case the electric and the magnetic fields do not vanish and they are not perpendicular in any LCF. We shall show that in this case there exists a family of LCFs in which $\mathbf{E} \parallel \mathbf{B}$.

We consider a LCF Σ', which relative to the LCF Σ has velocity $\mathbf{u}$ normal to the plane defined by the fields $\mathbf{E}, \ \mathbf{B}$ in Σ. We write

$$\mathbf{u} = \alpha(\mathbf{E} \times \mathbf{B}),$$

where α is a parameter which must be determined. From the transformation relations (13.54) we obtain for the electric and the magnetic fields in Σ'

$$\mathbf{E}' = \gamma\left(\mathbf{E} + \mathbf{u} \times \mathbf{B}\right),$$
$$\mathbf{B}' = \gamma\left(\mathbf{B} - \frac{1}{c^2}\mathbf{u} \times \mathbf{E}\right).$$

We compute[19]

$$\mathbf{u} \times \mathbf{B} = \alpha(\mathbf{E} \times \mathbf{B}) \times \mathbf{B} = \alpha cY\mathbf{B} - \alpha B^2\mathbf{E},$$
$$\mathbf{u} \times \mathbf{E} = \alpha(\mathbf{E} \times \mathbf{B}) \times \mathbf{E} = -\alpha cY\mathbf{E} + \alpha E^2\mathbf{B}.$$

We replace this result and find

$$\mathbf{E}' = \gamma\left[(1 - \alpha B^2)\mathbf{E} + \alpha cY\mathbf{B}\right], \tag{13.62}$$
$$\mathbf{B}' = \gamma\left[\left(1 - \alpha\frac{E^2}{c^2}\right)\mathbf{B} + \alpha Y\frac{1}{c}\mathbf{E}\right]. \tag{13.63}$$

We note that the fields $\mathbf{E}'$ and $\mathbf{B}'$ are in the plane of $\mathbf{E}$, $\mathbf{B}$, therefore they are perpendicular to the velocity $\mathbf{u}$. The cross product

$$\mathbf{E}' \times \mathbf{B}' = \gamma^2\left[(1 - \alpha B^2)\left(1 - \alpha\frac{E^2}{c^2}\right) - \alpha^2 Y^2\right](\mathbf{E} \times \mathbf{B}).$$

The condition for the electric and the magnetic fields to be parallel in Σ' is

$$\mathbf{E}' \times \mathbf{B}' = 0,$$

which implies the condition

$$(1 - \alpha B^2)\left(1 - \alpha\frac{E^2}{c^2}\right) - \alpha^2 Y^2 = 0$$

or, after some easy algebra,

$$\left(\frac{1}{c^2}E^2B^2 - Y^2\right)\alpha^2 - \left(B^2 + \frac{E^2}{c^2}\right)\alpha + 1 = 0.$$

But

[19] We are using the identity $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{B} \cdot \mathbf{C})\mathbf{A}$.

$$Y = \frac{1}{c}\mathbf{B}\cdot\mathbf{E} = \frac{1}{c}BE\cos\varphi \Rightarrow \frac{1}{c^2}E^2B^2 - Y^2 = \frac{1}{c^2}E^2B^2\sin^2\varphi = \left|\frac{1}{c}\mathbf{E}\times\mathbf{B}\right|^2.$$

Also $\alpha = \frac{\beta}{|\frac{1}{c}\mathbf{E}\times\mathbf{B}|}$. Replacing we find the following equation for the parameter β:

$$\beta^2 - \frac{B^2 + \frac{E^2}{c^2}}{|\frac{1}{c}\mathbf{E}\times\mathbf{B}|}\beta + 1 = 0.$$

The allowable solution ($\beta < 1$) of this equation is

$$\beta = \frac{1}{2}\left[A - \sqrt{A^2 - 4}\right], \tag{13.64}$$

where $A = \frac{B^2+\frac{E^2}{c^2}}{|\frac{1}{c}\mathbf{E}\times\mathbf{B}|} \geq 2$. We conclude that the velocity of Σ' relative to Σ is

$$\mathbf{u} = \beta c \frac{\frac{1}{c}\mathbf{E}\times\mathbf{B}}{|\frac{1}{c}\mathbf{E}\times\mathbf{B}|},$$

where β is given in (13.64).

It remains to compute the proportionality factor relating the strengths of the two fields. We write

$$\mathbf{B}' = \lambda\frac{1}{c}\mathbf{E}' \Rightarrow \lambda = \frac{\mathbf{E}'\cdot c\mathbf{B}'}{\mathbf{E}'^2} = \frac{c^2Y}{\mathbf{E}'^2}.$$

For the length $\mathbf{E}'^2$ we have from (13.62)

$$\begin{aligned}\mathbf{E}'^2 &= \gamma^2\left[E^2 - 2\alpha B^2E^2 + 2\alpha c^2Y^2 + \alpha^2E^2B^4 - \alpha^2c^2Y^2B^2\right]\\ &= \gamma^2E^2\left[1 - 2\alpha B^2\sin^2\varphi + \alpha^2B^4\sin^2\varphi\right],\end{aligned}$$

hence

$$\lambda = \frac{cB\cos\phi}{\gamma^2E\left[1 - 2\alpha B^2\sin^2\varphi + \alpha^2B^4\sin^2\varphi\right]}.$$

Σ' is not the unique LCF in which the electric and the magnetic fields are parallel. Indeed it can be shown easily that the same holds in any other LCF in which the relative velocity of Σ' is $\mathbf{u}' = \kappa\hat{\mathbf{e}}'$, where κ is a real parameter and $\hat{\mathbf{e}}'$ is the unit vector in the common direction of the fields in Σ'. In order to compute the velocity, $\mathbf{u}$ say, of these LCFs relative to Σ we consider the relativistic rule of composition of three-velocities and have

$$\mathbf{u} = \frac{1}{\gamma_v Q}\left\{\mathbf{u}' + \mathbf{v}\left[\frac{\mathbf{v}\cdot\mathbf{u}'}{v^2}(\gamma_v - 1) + \gamma_v\right]\right\},$$

where $Q = 1 - \frac{\mathbf{u}\cdot\mathbf{v}'}{c^2}$. But $\mathbf{v}\cdot\mathbf{u}' = 0$ because the vector $\hat{\mathbf{e}}'$ is normal to the vectors $\mathbf{E}', \mathbf{B}'$, hence $Q = 1$, and

$$\mathbf{u} = \frac{1}{\gamma_v}\left(\kappa\frac{\mathbf{E}'}{E'} + \mathbf{v}\gamma_v\right) = \mathbf{v} + \frac{\kappa}{\gamma_v}\frac{\mathbf{E}'}{E'}.$$

13.8 Motion of a Charge in an Electromagnetic Field – The Lorentz Force

Consider a charge q and assume that in the proper frame of the charge Σ^+ there exists an electric and a magnetic field given by the vectors $\frac{1}{c}\mathbf{E}^+$, $\mathbf{B}^+$. We define the three-force on the charge in Σ^+ due to this electromagnetic field to be $\mathbf{f}^+ = q\mathbf{E}^+$. This definition coincides with the Newtonian force on a charge resting in a electric and a magnetic field.[20] Then in the proper frame of the charge the four-force $F^i = \begin{pmatrix} 0 \\ q\mathbf{E}^+ \end{pmatrix}_{\Sigma^+}$ is completely determined and can be computed in any other LCF using the appropriate Lorentz transformation.

However, instead of using the Lorentz transformation to compute the three-force in an arbitrary LCF, Σ say, it is more fundamental to write the four-force F^i covariantly in an arbitrary LCF and then compute the three-force on the charge in Σ by taking the components of the quantities involved in Σ.

In order to do that we note that the four-velocity u^i of the charge q in Σ^+ has components $[u^i] = \begin{pmatrix} c \\ \mathbf{0} \end{pmatrix}_{\Sigma^+}$ and similarly the electromagnetic field tensor in Σ^+

$$[F_{ij}] = \begin{pmatrix} 0 & -E_x^+/c & -E_y^+/c & -E_z^+/c \\ E_x^+/c & 0 & B_z^+ & -B_y^+ \\ E_y^+/c & -B_z^+ & 0 & B_x^+ \\ E_z^+/c & B_y^+ & -B_x^+ & 0 \end{pmatrix}_{\Sigma^+}.$$

Then using the definition of the four-force in Σ^+ we have (in Σ^+) the relation

$$F_i = qF_{ij}u^j. \tag{13.65}$$

[20] Recall that the definitions of physical quantities in Special Relativity are made in the proper frame and coincide with the corresponding Newtonian quantities (provided they exist). This is the general rule – strategy – for defining physical quantities in Special Relativity and it is justified by the facts that (a) the physical quantities are understood and manipulated in the Newtonian world and (b) Special Relativity must be understood physically in the Newtonian world. The difference introduced by Special Relativity is in the transformation of these quantities from LCF to LCF.

But $j^i = qu^i$ is the four-current due to the charge. Therefore the four-force acting on the charge q (in Σ^+!) is written covariantly as

$$F_i = F_{ij} J^j. \tag{13.66}$$

This relation has been proved in Σ^+ but because it is covariant it is valid in any other LCF, therefore it defines the four-force on the charge due to the electromagnetic field.[21]

In order to compute the force on a charge moving with velocity $\mathbf{u}$ in an electromagnetic field $\mathbf{E}/c, \mathbf{B}$ in an LCF Σ we work as follows. In Σ the electromagnetic field tensor is

$$[F_{ij}] = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix}_\Sigma$$

and let the four-velocity of the charge $u^i = \begin{pmatrix} \gamma_u c \\ \gamma_u \mathbf{u} \end{pmatrix}_\Sigma$. We know that the four-force F^i due to a three-force $\mathbf{f}$ in Σ is given by the expression $F^i = \begin{pmatrix} \gamma_u \frac{1}{c}\dot{\mathcal{E}} \\ \gamma_u \mathbf{f} \end{pmatrix}_\Sigma$, where $\dot{\mathcal{E}}$ is the rate of change of the energy of the charge in Σ under the action of the force $\mathbf{f}$ (in Σ!). Replacing this in the lhs of (13.65) we find

$$\begin{aligned} \begin{pmatrix} \gamma_u \frac{1}{c}\dot{\mathcal{E}} \\ \gamma_u \mathbf{f} \end{pmatrix}_\Sigma &= q \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix}_\Sigma \begin{pmatrix} \gamma_u c \\ \gamma_u \mathbf{u} \end{pmatrix}_\Sigma \\ &= q\gamma_u \begin{pmatrix} \frac{1}{c}\mathbf{E}\cdot\mathbf{u} \\ \mathbf{E} + \mathbf{u}\times\mathbf{B} \end{pmatrix}_\Sigma . \end{aligned}$$

From this result follow two important conclusions:

- The magnetic field does not change the energy of the charge (does not produce work). Work is produced only by the electric field $\mathbf{E}$ and it is given by the relation

$$\frac{d\mathcal{E}}{dt} = q\mathbf{E}\cdot\mathbf{u}. \tag{13.67}$$

- The three-force on the charge q in Σ due to the electromagnetic field $\mathbf{E}/c, \mathbf{B}$ (in Σ!) is given by the formula

[21] In case we have a charge density ρ instead of a single charge, then the four-vector F^i is the density of four-force and J^i is the density of the four-current of the charge $J^i = \begin{pmatrix} \rho c \\ \mathbf{j} \end{pmatrix}_\Sigma$.

$$\mathbf{f} = q\left(\mathbf{E} + \mathbf{u} \times \mathbf{B}\right). \tag{13.68}$$

This force is known as the *Lorentz force* to remind Lorentz who first gave this formula *prior to the invention of Special Relativity*. We note that this equation does not contain the factor γ_u and this fact sometimes obscures its relativistic origin. However, as we have shown this is a purely relativistic formula.

The equation of motion of a relativistic point, P say, under the action of a four-force F^i is given by the relativistic generalization of the Second Law of Newton:

$$F^i = \frac{dp^i}{d\tau}, \tag{13.69}$$

where p^i is the four-momentum of the mass point and τ is its proper time. In an LCF Σ $F^i = \begin{pmatrix} \gamma \frac{1}{c}\dot{\mathcal{E}} \\ \gamma \mathbf{f} \end{pmatrix}_\Sigma$, $p^i = \begin{pmatrix} \mathcal{E}/c \\ \mathbf{p} \end{pmatrix}_\Sigma$ and this equation gives

$$\begin{pmatrix} \gamma_u \frac{1}{c}\dot{\mathcal{E}} \\ \gamma_u \mathbf{f} \end{pmatrix}_\Sigma = \gamma \frac{d}{dt} \begin{pmatrix} \mathcal{E}/c \\ \mathbf{p} \end{pmatrix}_\Sigma,$$

where t is time in Σ and $\gamma = \frac{dt}{d\tau}$. The zeroth coordinate gives

$$\dot{\mathcal{E}} = \mathbf{f} \cdot \mathbf{v} \tag{13.70}$$

and the space coordinate gives the equation of motion

$$\mathbf{f} = \frac{d\mathbf{p}}{dt}. \tag{13.71}$$

The first relation expresses the conservation of kinetic energy, that is, *a motion in which the three-force is always normal to the velocity, the energy – and consequently the speed – is a constant of motion*. The second equation is the equation of motion and constitutes the generalization of Newton's Second Law in Special Relativity.

In accordance with the above the equation of motion of a charge q with velocity $\mathbf{v}$ in an LCF Σ in which the electromagnetic field is given by the vectors $(\frac{1}{c}\mathbf{E}, \mathbf{B})$ is

$$\frac{d\mathbf{p}}{dt} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right). \tag{13.72}$$

The equation of motion (13.72) can be written using the results of Exercise 43:

$$q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) = m\dot{\gamma}\mathbf{v} + m\gamma\mathbf{a} = m\gamma(\gamma^2\mathbf{a}_{||} + \mathbf{a}_\perp). \tag{13.73}$$

Another equivalent form of this equation is in terms of the proper time of the charge. In this case we use the relation $\gamma = \frac{dt}{d\tau}$ and find

$$\frac{d\mathbf{p}}{d\tau} = \gamma\mathbf{f} = q\left(\frac{dt}{d\tau}\mathbf{E} + \frac{d\mathbf{r}}{d\tau} \times \mathbf{B}\right). \tag{13.74}$$

13.9 Motion of a Charge in a Homogeneous Electromagnetic Field

In this section we study the motion of a charge q of mass m in an LCF Σ, in which there exists a homogeneous electromagnetic field with (constant) $\frac{1}{c}\mathbf{E},\ \mathbf{B}$. The equation of motion of the charge is (13.72) or (13.74), whose solution gives the orbit of the charge in Σ. In general the solution of the equations of motion is difficult and one looks for simplifications and shortcuts which could make the solution possible. This is indeed the case and will be shown below.

As we have shown in Sect. 13.7 for every electromagnetic field there are LCFs in which the electric and the magnetic vectors of the field have one of the following forms, depending on the values of the invariants of the field:

a. Only electric field ($X > 0,\ Y = 0$)
b. Only magnetic field ($X < 0,\ Y = 0$)
c. Electric and magnetic fields perpendicular and of equal strength ($X = 0,\ Y = 0$)
d. Electric and magnetic fields parallel and with different strength ($X \neq 0,\ Y \neq 0$)

This means that it is enough to solve the equations of motion of the charge in one of these cases in order to cover all possible homogeneous electromagnetic fields.

In practice the algorithm for studying the motion of a charge in an arbitrary homogeneous electromagnetic field consists of the following steps:

- Calculate the invariants.
- Determine the type of the electromagnetic field, determine the appropriate LCF, and calculate the electromagnetic field in that LCF.
- Write and solve the equations of motion in that LCF.
- Transfer the solution by means of the appropriate Lorentz transformation to the original LCF.

In the following we solve the equations of motion for every one of the special cases (a)–(d) above.

13.9.1 The Case of a Homogeneous Electric Field

In this case we use the equation of motion (13.72)

$$\frac{d\mathbf{p}}{dt} = q\mathbf{E} \tag{13.75}$$

and take the initial conditions $\mathbf{r}_0, \mathbf{p}_0$. Obviously the motion develops in the plane spanned by the vectors $\mathbf{E},\ \mathbf{p}_0$.

The first integration of the equation of motion gives

$$\mathbf{p} = q\mathbf{E}t + \mathbf{p}_0.$$

We decompose the three-momentum $\mathbf{p}_0$ parallel and perpendicular to the direction of the electric field $\mathbf{E}$ and have

$$\mathbf{p} = (qEt + p_{0\|})\frac{\mathbf{E}}{E} + \mathbf{p}_{0\perp}.$$

The total energy $\mathcal{E}$ of the charge is

$$\mathcal{E} = \sqrt{\mathbf{p}^2c^2 + m^2c^4} = \sqrt{(q\mathbf{E}t + \mathbf{p}_0)^2c^2 + m^2c^4} = \sqrt{A + \mathbf{p}_{0\perp}^2}\, c,$$

where $A = (qEt + p_{0\|})^2 + m^2c^2$. The three-velocity

$$\mathbf{v} = \frac{c^2\mathbf{p}}{\mathcal{E}} = \frac{q\mathbf{E}t + \mathbf{p}_0}{\sqrt{A + \mathbf{p}_{0\perp}^2}}\, c$$

and the position vector

$$\begin{aligned}
\mathbf{r}(t) &= c\int \frac{q\mathbf{E}t + \mathbf{p}_0}{\sqrt{A + \mathbf{p}_{0\perp}^2}}\, dt \\
&= c\int \frac{(qEt + p_{0\|})dt}{\sqrt{A + \mathbf{p}_{0\perp}^2}}\frac{\mathbf{E}}{E} + c\int \frac{p_{0\perp}dt}{\sqrt{A + \mathbf{p}_{0\perp}^2}}\,\hat{\mathbf{e}}_\perp,
\end{aligned}$$

where $\hat{\mathbf{e}}_\perp$ is the unit normal to $\mathbf{E}$ in the plane defined by the vectors $\mathbf{E}$, $\mathbf{p}_0$. The first integral gives

$$\frac{\mathbf{E}}{2qE^2}\int \frac{dA}{\sqrt{A + \mathbf{p}_{0\perp}^2}} = \frac{\mathbf{E}}{qE^2}\sqrt{A + \mathbf{p}_{0\perp}^2}$$

and the second

$$\int \frac{\mathbf{p}_{0\perp} dt}{\sqrt{A + \mathbf{p}_{0\perp}^2}} = \frac{\mathbf{p}_{0\perp}}{qE} \ln \left[qEt + p_{0\|} + \sqrt{(qEt + p_{0\|})^2 + \mathbf{p}_{0\perp}^2 + m^2c^2} \right].$$

Replacing we find

$$\begin{aligned} \mathbf{r}(t) = &\frac{\mathbf{E}c}{qE^2}\sqrt{(qEt + p_{0\|})^2 + m^2c^2 + \mathbf{p}_{0\perp}^2} \\ &+ \frac{p_{0\perp}c}{qE} \ln \left| qEt + p_{0\|} + \sqrt{(qEt + p_{0\|})^2 + p_{0\perp}^2 + m^2c^2} \right| \hat{\mathbf{e}}_\perp + \mathbf{C}. \end{aligned} \tag{13.76}$$

Taking into account the initial conditions we compute for the constant $\mathbf{C}$

$$\mathbf{C} = \mathbf{r}_0 - \frac{c}{qE}\left(\frac{\mathcal{E}_0 \mathbf{E}}{Ec} + p_{0\perp} ln \left| \frac{\mathcal{E}_0}{c} + p_{0\|} \right| \hat{\mathbf{e}}_\perp \right).$$

Second Solution

Without restricting generality we consider the x-axis along the direction of the electric field and the y-axis in the plane spanned by the vectors $\mathbf{E}, \mathbf{p}_0$. Then $\mathbf{E} = E\mathbf{i}$ and the initial conditions become

$$\begin{aligned} &x(0) = y(0) = z(0) = 0, \\ &\frac{dct}{d\tau} = \frac{\mathcal{E}_0}{mc}, \ \frac{dx}{d\tau} = \frac{p_{0x}}{m} \ , \ \frac{dy}{d\tau} = \frac{p_{0y}}{m}, \end{aligned}$$

where $\mathcal{E}_0 = \sqrt{p_0^2c^2 + m^2c^4}$ is the initial energy of the charge and τ its proper time. We consider the equations of motion in the form (13.69) and write

$$m\frac{dx^{i2}}{d\tau^2} = F^i = \begin{pmatrix} \gamma \frac{d\mathcal{E}}{cdt} \\ \gamma q \mathbf{E} \end{pmatrix}.$$

But

$$\gamma \frac{d\mathcal{E}}{dt} = \gamma \mathbf{f} \cdot \mathbf{v} = \gamma q E \frac{dx}{dt} = qE\frac{dx}{d\tau},$$

hence the equations of motion in the LCF we are working become

$$\begin{aligned}\frac{d^2ct}{d\tau^2} &= \frac{qE}{mc}\frac{dx}{d\tau},\\ \frac{d^2x}{d\tau^2} &= \frac{qE}{mc}\frac{dct}{d\tau},\\ \frac{d^2y}{d\tau^2} &= 0,\\ \frac{d^2z}{d\tau^2} &= 0.\end{aligned} \tag{13.77}$$

The solution of these equations, with the initial conditions we have considered, is

$$\begin{aligned}x(\tau) &= \frac{\mathcal{E}_0}{qE}\left(\cosh\frac{qE\tau}{mc} - 1\right) + \frac{cp_{0x}}{qE}\sinh\frac{qE\tau}{mc},\\ y(\tau) &= \frac{p_{0y}\tau}{m},\\ z(\tau) &= 0,\\ ct(\tau) &= \frac{\mathcal{E}_0}{qE}\sinh\frac{qE\tau}{mc} + \frac{cp_{0x}}{qE}\left(\cosh\frac{qE\tau}{mc} - 1\right).\end{aligned} \tag{13.78}$$

In order to calculate the motion of the particle in Σ we express the proper time τ in terms of the time t of Σ. This is done as follows. The x-component of the three-momentum is

$$p_x = mu_x = m\frac{dx}{d\tau} = \frac{\mathcal{E}_0}{c}\sinh\frac{qE\tau}{mc} + p_{0x}\cosh\frac{qE\tau}{mc}.$$

Similarly the energy (=zeroth coordinate of the four-momentum) is

$$\frac{\mathcal{E}}{c} = mu^0 = \frac{1}{c}\left(\mathcal{E}_0\cosh\frac{qE\tau}{mc} + cp_{0x}\sinh\frac{qE\tau}{mc}\right).$$

We add

$$\begin{aligned}\mathcal{E} + cp_x &= (\mathcal{E}_0 + cp_{0x})\exp\frac{qE\tau}{mc} \Rightarrow\\ \tau &= \frac{mc}{qE}\ln\left|\frac{\mathcal{E} + cp_x}{\mathcal{E}_0 + cp_{0x}}\right|.\end{aligned} \tag{13.79}$$

In detail this relation is written as follows:

$$\tau = \frac{mc}{qE}\ln\left|\frac{p_{0x} + qEt + \sqrt{(p_{0x} + qEt)^2 + m^2c^2 + p_{0y}^2}}{p_{0x} + \mathcal{E}_0/c}\right|. \tag{13.80}$$

Replacing τ in (13.78) the motion of the particle in Σ is found to be

$$x(t) = \frac{c}{qE}\left(\sqrt{(p_{0x} + qEt)^2 + m^2c^2 + p_{0y}^2} - \frac{\mathcal{E}_0}{c}\right), \qquad (13.81)$$

$$y(t) = \frac{cp_{0y}}{qE} \ln \left| \frac{p_{0x} + qEt + \sqrt{(p_{0x} + qEt)^2 + m^2c^2 + p_{0y}^2}}{p_{0x} + \mathcal{E}_0/c} \right|,$$

$$z(t) = 0.$$

It is of interest to examine the Newtonian and the relativistic limit of this result. In the Newtonian limit $p_0 \ll mc$ and $t \ll \frac{mc}{qE}$ from which follows easily that the motion of the charge in Σ is given by the classical solution

$$x(t) = \frac{p_{0x}t}{m} + \frac{qE}{2m}t^2,$$

$$y(t) = \frac{p_{0y}t}{m},$$

$$z(t) = 0.$$

In the relativistic limit for large times $t \gg \frac{mc}{qE}$ the speed of the charge approaches the value c even if originally small. In this limit the motion of the charge is

$$x(t) = ct - \frac{mc^2}{qE},$$

$$y(t) = \frac{cp_{0y}}{qE} \ln \frac{2|q|Et}{mc}, \qquad (13.82)$$

$$z(t) = 0.$$

In Fig. 13.2 it is shown the distance $x(t)$ for the initial conditions $p_{0x} = p_{0y} = 0$ and in addition the Newtonian ($t \ll \frac{mc}{qE}$) and the relativistic limit ($t \gg \frac{mc}{qE}$) of motion are indicated.

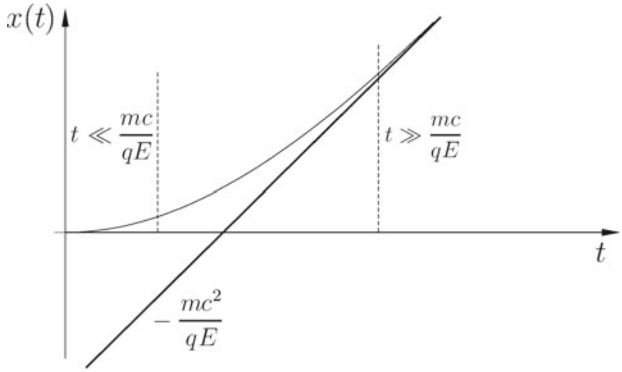

Fig. 13.2 Motion of a Charge in a Homogeneous Electric Field

In order to compute the equation of the trajectory of the charge in Σ we eliminate the proper time τ from (13.78) and find

$$x = \frac{\mathcal{E}_0}{qE}\left(\cosh\frac{qEy}{cp_{0y}} - 1\right) + \frac{cp_{0x}}{qE}\sinh\frac{qEy}{cp_{0y}}. \tag{13.83}$$

In the Newtonian limit $\mathcal{E}_0 = mc^2$, $p_0 \ll mc$ and $\frac{|q|Ey}{cp_{0y}} \ll 1$ because the momentum of the charge must be small compared to mc. Using these results one obtains the well-known Newtonian result

$$x = \frac{mqEy^2}{2p_{0y}^2} + \frac{p_{0x}}{p_{0y}}y.$$

13.9.2 The Case of a Homogeneous Magnetic Field

The equation of motion of a charge in a homogeneous magnetic field is

$$\frac{d\mathbf{p}}{dt} = q\mathbf{v} \times \mathbf{B}. \tag{13.84}$$

Multiplying with $\mathbf{p}$ we find

$$\frac{d\mathbf{p}}{dt}\mathbf{p} = 0 \Rightarrow \mathbf{p}^2 = \text{constant} \Rightarrow \mathcal{E} = \sqrt{\mathbf{p}_0^2 + m^2c^4} = \text{constant}.$$

The speed $v = \frac{|\mathbf{p}_0|c^2}{\mathcal{E}} = $ constant. Using the constant energy we rewrite the equation of motion as follows:

$$\frac{\mathcal{E}}{c^2}\dot{\mathbf{v}} = q\mathbf{v} \times \mathbf{B}$$

and we study the motion parallel and perpendicular to the magnetic field. Integrating (13.84) parallel to the magnetic field we find

$$\mathbf{r}_{\|} = \mathbf{v}_{\|}(0)t + \mathbf{r}_{\|}(0).$$

The speed[22]

[22] It is possible to integrate the equations of motion directly and not use the first integrals. This is done as follows. Without restricting the generality we consider the z-axis along the direction of the magnetic field ($\mathbf{B} = B\mathbf{k}$) in which case the equations of motion in the direction perpendicular to the magnetic field give

$$\mathbf{v}^2 = \mathbf{v}_\perp^2 + \mathbf{v}_\parallel^2 \Rightarrow |\mathbf{v}_\perp(t)| = \text{constant} = |\mathbf{v}_\perp(0)|\,.$$

This result is an additional integral of motion and implies that

$$\mathbf{v}_\perp(t) = \mathbf{v}_\perp(0)e^{\omega t},$$

where ω is a real parameter. In order to determine ω we differentiate $\mathbf{v}_\perp(t)$ and use the equation of motion. We find

$$\dot{\mathbf{v}}_\perp(t) = \mathbf{v}_\perp(0)\omega e^{\omega t} = \omega \mathbf{v}_\perp(t),$$

therefore

$$\frac{\mathcal{E}}{c^2}\omega v_\perp(t) = q v_\perp(t) B \Rightarrow \omega = \frac{qc^2 B}{\mathcal{E}} = \frac{qB}{m\gamma}.$$

The parameter ω is called the *cyclotronic frequency* and it is known from Newtonian Physics. Integrating once more we find the motion perpendicular to the magnetic field:

$$\mathbf{r}_\perp(t) = \frac{\mathbf{v}_\perp(0)}{\omega} e^{\omega t} + \mathbf{r}_\perp(0).$$

From the solution of the equations of motion we conclude that the motion of a charge in a constant magnetic field is a combination of two motions: a translational

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$$\left.\begin{aligned} m\gamma_0 \dot{\upsilon}_x &= qB\upsilon_y \\ m\gamma_0 \dot{\upsilon}_y &= -qB\upsilon_x \end{aligned}\right\} \Rightarrow \begin{aligned} \dot{\upsilon}_x &= \omega \upsilon_y \\ \dot{\upsilon}_y &= -\omega \upsilon_x \end{aligned}\;,$$

where $\omega = \frac{qB}{m\gamma_0}$. To solve this system of simultaneous equations we multiply the second with i and add. It follows:

$$\left(\dot{\upsilon}_x + i\dot{\upsilon}_y\right) = -i\omega\left(\upsilon_x + i\upsilon_y\right) \Rightarrow$$
$$w(t) = w(0)e^{-i\omega t}, \quad \text{where} \qquad w(t) = \upsilon_x + i\upsilon_y.$$

Equating the real and the imaginary parts we find

$$\begin{aligned} \upsilon_x(t) &= \upsilon_x(0)\cos\omega t + \upsilon_y(0)\sin\omega t, \\ \upsilon_y(t) &= \upsilon_x(0)\sin\omega t - \upsilon_y(0)\cos\omega t. \end{aligned}$$

Let us assume for simplicity the initial conditions $\upsilon_x(0) = \upsilon, \upsilon_y(0) = 0$. Then $\upsilon_x(t) = \upsilon\cos\omega t, \upsilon_y(t) = \upsilon\sin\omega t$. Integrating these last relations we find for the motion perpendicular to the direction of the magnetic field

$$\begin{aligned} x(t) &= x_0 + \frac{\upsilon}{\omega}\sin\omega t, \\ y(t) &= y_0 - \frac{\upsilon}{\omega}\cos\omega t. \end{aligned}$$

motion with constant speed parallel to the magnetic field and a uniform circular motion in the normal plane to the magnetic field with frequency ω.

In the Newtonian limit $\omega = \frac{qB}{m}$.

13.9.3 The Case of Two Homogeneous Fields of Equal Strength and Perpendicular Directions

This case concerns the motion of a charge in an electromagnetic field in which the electric and the magnetic fields are related as follows: $\frac{E}{c} = B$ and $\mathbf{E} \cdot \mathbf{B}$=0. Without restricting generality we consider $\mathbf{E} = E\,\mathbf{j}$ and $\mathbf{B} = \frac{E}{c}\,\mathbf{k}$. Then the equations of motion

$$\dot{\mathbf{p}} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

become

$$\begin{aligned} \dot{p}_x &= \frac{q}{c} v_y E, \\ \dot{p}_y &= qE\left(1 - \frac{v_x}{c}\right), \\ \dot{p}_z &= 0. \end{aligned}$$

The last equation gives

$$P_z = p_{0z} = \text{constant}.$$

In order to solve the remaining two equations we are looking for first integrals. We note that the energy $\mathcal{E}$ is written as

$$\mathcal{E}^2 - p_x^2 c^2 = p_y^2 c^2 + C_1^2, \tag{13.85}$$

where $C_1^2 = p_z^2 c^2 + m^2 c^4$. But

$$\dot{\mathcal{E}} = \mathbf{f} \cdot \mathbf{v} = qEv_y = \dot{p}_x c \Rightarrow \mathcal{E} - p_x c = a = \text{constant},$$

where in the first step we have used the equation of motion for p_y. Combining this result with (13.85) we find

$$\mathcal{E} + p_x c = \frac{p_y^2 c^2 + C_1^2}{a}. \tag{13.86}$$

From the last two relations we find by adding and subtracting

$$\mathcal{E} = \frac{a}{2} + \frac{p_y^2 c^2 + C_1^2}{2a}, \tag{13.87}$$

$$p_x = -\frac{a}{2c} + \frac{p_y^2 c^2 + C_1^2}{2ac}. \tag{13.88}$$

The second equation of motion gives for the quantity $\dot{p}_y$

$$\dot{p}_y = qE\left(1 - \frac{p_x c}{\mathcal{E}}\right) = \frac{qE}{\mathcal{E}}(\mathcal{E} - p_x c) = \frac{qEa}{\mathcal{E}} \Rightarrow \mathcal{E}\dot{p}_y = qEa.$$

Replacing $\mathcal{E}$ from (13.87) we have

$$\int \left(\frac{a^2 + C_1^2}{2a} + \frac{c^2}{2a}p_y^2\right) dp_y = qEat$$

and finally

$$\frac{c^2}{3a^2}p_y^3 + \left(1 + \frac{C_1^2}{a^2}\right) p_y = 2qEt + \text{constant.} \tag{13.89}$$

In order to compute the motion (that is the functions $x(t)$, $y(t)$, $z(t)$) we use (13.89) and replace time with the variable p_y. This is done as follows.

We consider initially the coordinate $x(t)$ and note that the velocity v_x can be expressed in two ways, either as

$$v_x = \frac{p_x c^2}{\mathcal{E}}$$

or as

$$v_x = \frac{dx}{dt} = \frac{dx}{dp_y}\frac{dp_y}{dt} = \frac{dx}{dp_y}\frac{qaE}{\mathcal{E}}.$$

Equating these two expressions we find the differential equation

$$\frac{dx}{dp_y} = \frac{p_x c^2}{qaE} = \frac{c^2}{qaE}\left(-\frac{a}{2c} + \frac{p_y^2 c^2 + C_1^2}{2ac}\right) = \frac{c}{2qE}\left(-1 + \frac{p_y^2 c^2 + C_1^2}{a^2}\right),$$

whose solution is

$$x(p_y) = \frac{c}{2qE}\left[\left(-1 + \frac{C_1^2}{a^2}\right) p_y + \frac{c^2}{3a^2}p_y^3\right] + x(0). \tag{13.90}$$

Similarly for the $y(t)$ coordinate we compute

$$v_y = \frac{p_y c^2}{\mathcal{E}},$$
$$v_y = \frac{dy}{dt} = \frac{dy}{dp_y}\frac{dp_y}{dt} = \frac{dy}{dp_y}\frac{qaE}{\mathcal{E}}$$

from which follows

$$\frac{dy}{dp_y} = \frac{c^2}{qaE}p_y \Rightarrow y(p_y) = \frac{c^2}{2qaE}p_y^2 + y(0). \tag{13.91}$$

Finally for the $z(t)$ coordinate we have

$$\frac{dz}{dp_y} = \frac{p_{z0}c^2}{qaE} \Rightarrow z(p_y) = \frac{p_{z0}c^2}{qaE}p_y + z(0). \tag{13.92}$$

13.9.4 The Case of Homogeneous and Parallel Fields $\mathbf{E} \parallel \mathbf{B}$

In this case the electric and the magnetic fields are parallel and affect independently the motion of the charge, the electric field along the common direction of the fields, and the magnetic field normal to this direction. Therefore the motion can be considered as a combination of two motions: one motion under the action of a homogeneous electric field and a second motion under the action of a homogeneous magnetic field. These two motions are related via the energy of the charge, which depends on its speed (contrary to the Newtonian case).

Without restricting generality we assume the z-axis along the common direction of the fields and write $\mathbf{E} = E\mathbf{k}$, $\mathbf{B} = B\mathbf{k}$. The equations of motion are

$$\dot{p}_x = qv_y B, \ \dot{p}_y = -qv_x B, \ \dot{p}_z = qE. \tag{13.93}$$

Because $v_x = \frac{p_x c^2}{\mathcal{E}}$ and similarly for v_y, the equations of motion become

$$\dot{p}_x = \frac{qBc^2}{\mathcal{E}}\,p_y, \ \dot{p}_y = -\frac{qBc^2}{\mathcal{E}}\,p_x, \ \dot{p}_z = qE. \tag{13.94}$$

The third equation concerns the motion under the action of a homogeneous electric field and has been studied in Sect. 13.9.1. The solution is (see (13.81))

$$z(t) = \frac{c}{qE}\left(\sqrt{(p_{0z} + qEt)^2 + m^2c^2 + p_{0\perp}^2} - \frac{\mathcal{E}_0}{c}\right), \tag{13.95}$$

where $\mathbf{p}_{0\perp} = p_{0x}\mathbf{i} + p_{0y}\mathbf{j}$ and $\mathcal{E}_0$ is the energy of the charge at the moment $t = 0$. We also have

$$p_z(t) = qEt.$$

Differentiating $\mathcal{E}^2 = \mathbf{p}^2c^2 + m^2c^4$ wrt time and using the equations of motion we find $\dot{\mathcal{E}}\mathcal{E} = \dot{p}_z p_z$ from which follows

$$\mathcal{E} = \sqrt{p_z^2c^2 + \mathcal{E}_0^2} = \sqrt{(qEct)^2 + \mathcal{E}_0^2}. \tag{13.96}$$

Concerning the solution of the remaining two equations we cannot apply the previous study dealing with the motion of a charge in a homogeneous magnetic field because now the energy of the charge is not constant. From the equations of motion we have

$$p_x\dot{p}_x + p_y\dot{p}_y = 0 \;\Rightarrow\; p_x^2 + p_y^2 = \text{constant} = \mathbf{p}_\perp^2 = p_\perp^2, \tag{13.97}$$

therefore we have the first integral

$$p_x + ip_y = p_\perp e^{-i\phi}, \tag{13.98}$$

where ϕ is a real parameter. Differentiating (13.98) wrt t and replacing $\dot{p}_x, \dot{p}_y$ from the equations of motion we find for the parameter ϕ the equation

$$\dot{\phi} = \frac{qBc^2}{\mathcal{E}} = \frac{qBc^2}{\sqrt{(qEct)^2 + \mathcal{E}_0^2}}. \tag{13.99}$$

Integrating wrt t we find eventually ($\phi(0) = 0$)

$$ct = \frac{\mathcal{E}_0}{qE}\sinh\frac{\phi E}{Bc}.$$

Using this equation we can change the variable t with ϕ facilitating the integration of the equations of motion. Indeed we have

$$\begin{aligned} p_x + ip_y &= \frac{\mathcal{E}}{c^2}(v_x + iv_y) = \frac{\mathcal{E}}{c^2}\frac{d(x+iy)}{dt} = \frac{\mathcal{E}}{c^2}\frac{d\phi}{dt}\frac{d(x+iy)}{d\phi} \\ &= \frac{\mathcal{E}}{c^2}\frac{qBc^2}{\mathcal{E}}\frac{d(x+iy)}{d\phi} = qB\frac{d(x+iy)}{d\phi}. \end{aligned}$$

Integrating wrt ϕ

$$x + iy = \frac{p_\perp}{qB}\int_0 e^{-i\phi}d\phi = i\frac{p_\perp}{qB}(e^{-i\phi} - 1) + x(0) + iy(0)$$

from which follows

$$x(\phi) = \frac{p_\perp}{qB}\sin\phi + x(0),\;\; y(\phi) = \frac{p_\perp}{qB}(\cos\phi - 1) + y(0).$$

Concerning the $z(t)$ coordinate we find in a similar fashion

$$qEt = \frac{\mathcal{E}}{c^2}\dot{z} = \frac{E}{c^2}\frac{d\phi}{dt}\frac{dz}{d\phi} \Rightarrow \frac{dz}{d\phi} = \frac{\mathcal{E}_0}{qBc}\sinh\frac{\phi E}{Bc}.$$

Integrating

$$z(\phi) = \frac{\mathcal{E}_0}{qE}\left(\cosh\frac{\phi E}{Bc} - 1\right) + z(0).$$

The orbit of the charge is a helix with constant radius $\frac{p_\perp}{qB}$ and a pace which increases monotonically. The angular speed of rotation equals $\frac{d\phi}{dt} = \frac{qBc^2}{\mathcal{E}}$ and diminishes with time whereas the speed of the translational motion along the common direction of the fields increases continually with limiting value c.

13.10 The Relativistic Electric and Magnetic Fields

In the previous sections we developed the theory of the electromagnetic field using the three-vector notation. This formulation, although it is more tangible to the newcomer in Special Relativity, lacks the consistency and the power of the four-vector formalism. Furthermore it cannot be used in General Relativity in which the four-formalism is a must. For these reasons in the present and the following sections we discuss the theory of the electromagnetic field using the four-formalism. In particular we consider the case of a homogeneous and isotropic material, the empty space being the extreme particular case. Although it is not necessary, we shall keep c in the formulae in order to make them applicable to numerical calculations. This means that the four-velocity of a comoving observer is $u^a = c\delta^a_0$ $(u_a = -c\delta^0_a)$. All frames are assumed to be Lorentz orthonormal frames (i.e., LCF) so that the metric $\eta_{ab} = (-1, 1, 1, 1)$.

13.10.1 The Levi-Civita Tensor Density

In the following we shall use the antisymmetric tensor F_{ab}, hence it is necessary that we shall discuss concisely the basic tool in the manipulation of such tensors, that is the *Levi-Civita tensor density*.

In Minkowski space the alternating tensor is defined as follows:

$$\eta^{abcd} = \eta^{[abcd]}, \quad \eta^{0123} = (-g)^{-1/2} = -\eta_{0123}, \tag{13.100}$$

where $g = \det(g_{ab})$. It has the following properties:

$$\begin{aligned}\eta^{cabd}\eta_{cmnr} &= -3!\delta_m^{[a}\delta_n^b\delta_r^{d]} \\ &= -\left(\delta_m^a\delta_n^b\delta_r^d - \delta_m^a\delta_n^d\delta_r^b + \delta_m^b\delta_n^d\delta_r^a - \delta_m^b\delta_n^a\delta_r^d + \delta_m^d\delta_n^a\delta_r^b - \delta_m^d\delta_n^b\delta_r^a\right),\end{aligned} \tag{13.101}$$

$$\eta^{abcd}\eta_{abst} = -4\delta_s^{[c}\delta_t^{d]} = -2(\delta_s^c\delta_t^d - \delta_s^d\delta_t^c), \tag{13.102}$$

$$\eta^{abcd}\eta_{abct} = -3!\delta_t^d, \tag{13.103}$$

$$\eta^{abcd}\eta_{abcd} = -4!. \tag{13.104}$$

In the Euclidean three-dimensional space the alternating tensor is defined as follows:

$$\eta^{\mu\nu\rho} = \eta^{[\mu\nu\rho]}, \quad \eta^{123} = h^{-1/2},$$

where $h = \det(h_{\mu\nu})$ and satisfies the properties

$$\eta^{\mu\nu\rho}\eta_{\mu\sigma\tau} = 2\delta_\sigma^{[\nu}\delta_\tau^{\rho]}, \quad \eta^{\mu\nu\rho}\eta_{\mu\nu\tau} = 2\delta_\tau^\rho. \tag{13.105}$$

Usually in Euclidean space $\eta_{\mu\nu\rho}$ is written as $\varepsilon_{\mu\nu\rho}$ because the determinant of the Euclidean metric equals 1, therefore the tensor density becomes a tensor. However, we should not worry about that and we shall keep the notation $\eta_{\mu\nu\rho}$ for uniformity. Examples

1. Consider the Euclidean vectors $\mathbf{u} = u^\mu\hat{\mathbf{e}}_\mu$, $\mathbf{v} = v^\mu\hat{\mathbf{e}}_\mu$ where $\{\hat{\mathbf{e}}\}$ is an orthonormal basis, e.g., the $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$. Then the cross product

$$\mathbf{u}\times\mathbf{v} = \eta^{\mu\nu\rho}u_\nu v_\rho\hat{\mathbf{e}}_\mu \qquad \text{or} \qquad (u\times v)^\mu = \eta^{\mu\nu\rho}u_\nu v_\rho.$$

2. Let us prove the well-known identity of vector calculus:

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}.$$

We have

$$\begin{aligned}[\mathbf{A}\times(\mathbf{B}\times\mathbf{C})]^\mu &= \eta^{\mu\nu\rho}A_\nu(\mathbf{B}\times\mathbf{C})_\rho = \eta^{\mu\nu\rho}A_\nu\eta_{\rho\sigma\tau}B^\sigma C^\tau \\ &= \eta^{\mu\nu\rho}\eta_{\rho\sigma\tau}A_\nu B^\sigma C^\tau \\ &= (\delta_\sigma^\mu\delta_\tau^\nu - \delta_\sigma^\nu\delta_\tau^\mu)A_\nu B^\sigma C^\tau = (A_\nu C^\nu)B^\mu - (A_\nu B^\nu)C^\mu \\ &= [(\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}]^\mu.\end{aligned}$$

3. For every antisymmetric tensor $A_{\mu\nu} = \frac{1}{2}\left(A_{\mu\nu} - A_{\nu\mu}\right)$ we define the vector

$$R^\mu = \frac{1}{2}\eta^{\mu\nu\rho}A_{\nu\rho}. \tag{13.106}$$

Conversely for every vector R^μ we define the antisymmetric tensor

$$A_{\mu\nu} = \eta_{\mu\nu\rho} R^{\rho}. \tag{13.107}$$

4. The curl of a vector field B^{μ} is the antisymmetric tensor $A_{\mu\nu} = B_{[\mu,\nu]}$.

In Minkowski space the above remain the same except that we have more terms and changes of sign. The relation which we shall use frequently is the definition of a vector from an antisymmetric tensor:

$$\omega^{a} = \frac{1}{2c^{2}} \eta^{abcd} \omega_{bc} u_{d}, \tag{13.108}$$

where u^{d} is the four-velocity. The four-vector ω^{a} is spacelike, that is $\omega^{a} u_{a} = 0$. The inverse relation is

$$\omega_{ab} = \eta_{abcd} \omega^{c} u^{d}. \tag{13.109}$$

These are the basics one should know in order to be able to follow the subsequent calculations. It is strongly advised that the reader read and understand these concepts properly before he/she continues.

13.10.2 The Case of Vacuum

As we have seen the electromagnetic field in vacuum for a given LCF Σ is described by the three-electric and the three-magnetic fields $\mathbf{E}$, $\mathbf{B}$ (in Σ). These fields are the components of the electromagnetic field tensor F_{ab} in Σ. The tensor F_{ab}, although determined in Σ, is independent of Σ and characterizes the electromagnetic field only. This means that the vector fields $\mathbf{E}$, $\mathbf{B}$ essentially require two tensors to be defined, i.e., the tensor F_{ab} and the (relativistic) observer with four-velocity u^{a} observing the electromagnetic field. Using this observation we introduce a (relativistic) electric and a (relativistic) magnetic induction field E^{a}, B^{a}, respectively, as follows:

$$E_{a} = F_{ab} u^{b}, \quad B_{a} = \frac{1}{2c} \eta_{abcd} F^{bc} u^{d}, \tag{13.110}$$

where η^{abcd} is the Levi-Civita density η_{abcd}. For reasons which we shall explain soon we name these new four-vectors the *four-electric field* and the *four-magnetic induction*. The following relations are obvious:

$$E_{a} u^{a} = B_{a} u^{a} = 0, \tag{13.111}$$

that is, the four-vectors E^{a}, H^{a} are spacelike.

Let us compute the components of E^{a}, B^{a} in the proper frame of the observer, Σ^{+} say, in which we assume that the electromagnetic field tensor is given by

$$[F_{ij}] = \begin{pmatrix} 0 & -E_x^+/c & -E_y^+/c & -E_z^+/c \\ E_x^+/c & 0 & B_z^+ & -B_y^+ \\ E_y^+/c & -B_z^+ & 0 & B_x^+ \\ E_z^+/c & B_y^+ & -B_x^+ & 0 \end{pmatrix}_{\Sigma^+}$$

while $u^a = \begin{pmatrix} c \\ \mathbf{0} \end{pmatrix}_{\Sigma^+}$. Replacing these coordinate expressions in (13.110) we compute easily[23]

$$E^a = \begin{pmatrix} 0 \\ \mathbf{E}^+ \end{pmatrix}_{\Sigma^+}, \qquad B^a = \begin{pmatrix} 0 \\ \mathbf{B}^+ \end{pmatrix}_{\Sigma^+}.$$

What is the difference among the fields E^a, H^a and $\mathbf{E}$, $\mathbf{B}$? The difference is that they coincide only in the proper frame Σ^+ of the observer while in the frame of any other LCF Σ the E^a, H^a are computed by the proper Lorentz transformation relating Σ, Σ^+ while the $\mathbf{E}$, $\mathbf{B}$ are computed from the transformed *components* of the tensor F_{ij} via the relations

$$\mathbf{E} = c \begin{pmatrix} F_{01} \\ F_{02} \\ F_{03} \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} -F_{23} \\ -F_{31} \\ -F_{12} \end{pmatrix}. \tag{13.112}$$

We remark that the fields E^a, H^a depend on the observer u^i while the tensor F_{ab} is the same for all observers and characterizes the electromagnetic field only.

We consider the LCF Σ in which there exists the electromagnetic field F_{ab} given by the following matrix:

$$[F_{ij}] = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix}_{\Sigma}. \tag{13.113}$$

In order to compute the components of the electric field observed by an observer u^a whose four-velocity in Σ is $u^a = \begin{pmatrix} \gamma c \\ \gamma \mathbf{v} \end{pmatrix}_{\Sigma}$ we multiply the matrices

$$[E_a] = [F_{ab}][u^b]$$

[23] It will help if we demonstrate the computation of the components of the four-vector for the magnetic field. For example for the B_x coordinate we have

$$B_x = \frac{1}{2c}\eta_{xjkl}F^{jk}u^l = \frac{1}{2c}c\,\eta_{xtjk}F^{jk} = -\frac{1}{2}\eta_{txyz}(F^{yz} - F^{zy}) = F^{yz} = B_x^+.$$

from which follows

$$E^a = \left(\gamma \frac{\mathbf{E} \cdot \mathbf{v}}{c}, \gamma (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \right)_\Sigma . \tag{13.114}$$

Exercise 60 Show that the components of the magnetic field B^a in Σ are as follows:

$$B^a = \left(\gamma \frac{\mathbf{B} \cdot \mathbf{v}}{c}, \gamma \left(\mathbf{B} - \frac{1}{c^2} \mathbf{v} \times \mathbf{E} \right) \right)_\Sigma . \tag{13.115}$$

Are these results compatible with the ones we found in (13.55) and (13.56)?

It is possible to express the tensor F_{ab} in terms of the vector fields E^a, B^a. The simplest way to do this is to consider the fields E^a, B^a in the proper frame of the observer u^i and show that

$$F_{ab} = -\frac{1}{c^2} (E_a u_b - E_b u_a) + \frac{1}{c} \eta_{abcd} B^c u^d . \tag{13.116}$$

This equation although proved in one frame holds in all frames because it is a tensor equation.

Exercise 61 Verify (13.116) by replacing F_{ij} in (13.110).

The four-force F^a on a four-current J^a which is moving under the action of the electromagnetic field F_{ab} is given by the expression (see Sect. 13.8)

$$F_a = F_{ab} j^b , \tag{13.117}$$

hence the equation of motion of a charge of mass m is

$$\frac{d}{d\tau} (m u^a) = F^a_b j^b , \tag{13.118}$$

where u^a is the four-velocity of the proper observer of the charge and τ is the proper time of that observer.

Exercise 62 An observer with four-velocity $u^a = \begin{pmatrix} \gamma c \\ \gamma \mathbf{v} \end{pmatrix}_\Sigma$ observes the electric field $E^a = \begin{pmatrix} \gamma \mathbf{E} \cdot \mathbf{v}/c \\ \gamma\,(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \end{pmatrix}_\Sigma$. Show that the equation of motion of a charge q in Σ is as follows:

$$\gamma \dot{\gamma} = \frac{q}{m} \gamma \frac{\mathbf{E} \cdot \mathbf{v}}{c^2} \Leftrightarrow \frac{d\gamma}{d\tau} = \frac{q}{m} \gamma \frac{\mathbf{E} \cdot \mathbf{v}}{c^2} , \tag{13.119}$$

$$\mathbf{v} \dot{\gamma} + \gamma \mathbf{a} = \frac{q}{m} \gamma\,(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \Leftrightarrow \frac{d(\gamma \mathbf{v})}{d\tau} = \frac{q}{m} \gamma\,(\mathbf{E} + \mathbf{v} \times \mathbf{B}) , \tag{13.120}$$

where τ is the proper time of the charge.

[Hint: Use that the four-acceleration of the observer in Σ is given by $a^a = \begin{pmatrix} \gamma\dot{\gamma} \\ \mathbf{v}\dot{\gamma} + \gamma\mathbf{a} \end{pmatrix}_\Sigma$ where $\dot{\gamma} = \frac{d\gamma}{dt}$, $\mathbf{a} = \frac{d\mathbf{v}}{dt}$, and t is time in Σ.]

In Sect. 13.6.2 we have seen that Maxwell equations in vacuum in terms of the electromagnetic field tensor F_{ab} are written as follows:

$$F_{\{ab,c\}} = 0, \tag{13.121}$$

$$F^{ab}{}_{,b} = \mu_0 J^a, \tag{13.122}$$

where $\{ab, c\}$ means cyclic sum in all enclosed indices. Furthermore the continuity equation for the charge is expressed by the equation

$$J^a{}_{,a} = 0. \tag{13.123}$$

The above equations are the basic equations of the electromagnetic field for the vacuum in the four-formalism.

13.10.3 The Electromagnetic Theory for a General Medium

As it has been said at the beginning of this chapter the electromagnetic field for an LCF Σ in a general medium is specified by the following four three-vectors:

1. Electric field **E**
2. Magnetic field **B**
3. Electric displacement field **D**
4. Magnetic displacement field **H**

As the vectors **E**, **B** define in Σ the antisymmetric tensor F_{ab} (13.113), in the same manner the fields **D**, **H** define in Σ the antisymmetric tensor K_{ab} as follows:

$$K_{ab} = \begin{pmatrix} 0 & -D_x & -D_y & -D_z \\ D_x & 0 & H_z/c & -H_y/c \\ D_y & -H_z/c & 0 & H_x/c \\ D_z & H_y/c & -H_x/c & 0 \end{pmatrix}_\Sigma . \tag{13.124}$$

Therefore in a general medium the electromagnetic field is expressed by two different (antisymmetric) tensors, the electromagnetic field tensor F_{ab} and the *induction field tensor* K_{ab}. This means that for a general material an observer with four-velocity u^a defines the following four spacelike four-vectors:

1. The electric field:

$$E^a = F^{ab} u_b. \tag{13.125}$$

2. The electric displacement field:

$$D^a = \frac{1}{c} K^{ab} u_b. \tag{13.126}$$

3. The magnetic induction field:

$$B_a = \frac{1}{2c} \eta_{abcd} F^{bc} u^d. \tag{13.127}$$

4. The magnetic field strength:

$$H_a = \frac{1}{2} \eta_{abcd} K^{bc} u^d. \tag{13.128}$$

These four-vector fields express the "interaction" of the observer with the electromagnetic field and they are in general different for the same electromagnetic field and different observers. In order to compute the components of these fields in an LCF Σ in which the four-velocity of the observes $u^a = \begin{pmatrix} \gamma c \\ \gamma \mathbf{v} \end{pmatrix}_\Sigma$ we use (13.124) and multiply with the matrix representation of u^a. For example the components of the electric induction D^a in Σ are computed from the multiplication of the matrices:

$$[D^a] = [K_{ab}][u^b].$$

Replacing we find

$$D^a = \left(\gamma \mathbf{D} \cdot \mathbf{v}, \gamma c \left(\mathbf{D} + \frac{1}{c^2} \mathbf{v} \times \mathbf{H} \right) \right)_\Sigma . \tag{13.129}$$

Exercise 63 Show that the components of the magnetic field H^a in Σ are as follows:

$$H^a = \left(\gamma \frac{\mathbf{H} \cdot \mathbf{v}}{c}, \gamma \left(\mathbf{H} - \mathbf{v} \times \mathbf{D} \right) \right)_\Sigma . \tag{13.130}$$

Relations (13.125) to (13.128) can be inverted to express F_{ab} and K_{ab} in terms of the vector fields E^a, B^a, D^a, H^a as follows:

$$F_{ab} = \frac{1}{c^2} (-E_a u_b + E_b u_a) + \frac{1}{c} \eta_{abcd} B^c u^d, \tag{13.131}$$

$$K_{ab} = \frac{1}{c} (-D_a u_b + D_b u_a) + \frac{1}{c^2} \eta_{abcd} H^c u^d. \tag{13.132}$$

Exercise 64 Prove (13.132) by replacing the expressions of K_{ab} in (13.128). See Exercise 61.

In the literature use is made of the dual form of the tensors F_{ab} and K_{ab} by applying the theory of bivectors. A bivector is any second-order antisymmetric tensor $X_{ab} = -X_{ba.}$ A bivector is called *simple* if it can be written in the form $X_{ab} = A_{[a}B_{b]}$ where A^a, B^a are vectors.

The *dual bivector* X^{*ab} of a bivector X^{ab} is defined as follows:

$$X^{*ab} = \frac{1}{2}\eta^{abcd}X_{cd} \Leftrightarrow X^{ab} = -\frac{1}{2}\eta^{abcd}X^*_{cd}. \tag{13.133}$$

It is easy to show that in the LCF Σ the components of the dual bivectors F^{*ab}, K^{*ab} are

$$[F^*_{ab}] = \begin{pmatrix} 0 & -B_x/c & -B_y/c & -B_x/c \\ B_x/c & 0 & E_z & -E_y \\ B_y/c & -E_z & 0 & E_x \\ B_x/c & E_y & -E_x & 0 \end{pmatrix}_\Sigma, \tag{13.134}$$

$$[K^*_{ab}] = \begin{pmatrix} 0 & -H_x/c & -H_y/c & -H_z/c \\ H_x/c & 0 & D_z & -D_y \\ H_y/c & -D_z & 0 & D_x \\ H_z/c & D_y & -D_x & 0 \end{pmatrix}_\Sigma. \tag{13.135}$$

Exercise 65 Show that in the comoving frame the magnetic induction and the magnetic field intensity are given by the relation

$$F^*_{ab}u^b = (0, B_x, B_y, B_z)_{\Sigma^+}, \qquad K^*_{ab}u^b = (0, H_x, H_y, H_z)_{\Sigma^+}.$$

Then prove that the dual bivectors F^*_{ab}, K^*_{ab} in terms of the vector fields of the electromagnetic field are written as follows:

$$F^*_{ab} = -\frac{1}{c}B_a u_b + \frac{1}{c}B_b u_a - \frac{1}{c^2}\eta_{abcd}E^c u^d, \tag{13.136}$$

$$K^*_{ab} = -\frac{1}{c^2}H_a u_b + \frac{1}{c^2}H_b u_a - \frac{1}{c}\eta_{abcd}D^c u^d. \tag{13.137}$$

Exercise 66 Prove the following formulae:

$$B^a = F^{*ab}u_b, \quad H^a = K^{*ab}u_b, \tag{13.138}$$

$$E_a = -\frac{1}{2}\eta_{abcd}F^{*bc}u^d, \quad D_a = \frac{1}{2c}\eta_{abcd}K^{*bc}u^d. \tag{13.139}$$

13.10.4 The Electric and Magnetic Moments

Consider an electromagnetic field which is described by the tensors $F_{ab},\ K_{ab}$. We define the magnetization tensor

$$M_{ab} = -\sqrt{\frac{\varepsilon_0}{\mu_0}}F_{ab} + K_{ab}. \tag{13.140}$$

If u^a is the four-velocity of an observer we define the four-vectors

$$P_a = \frac{1}{c}M_{ab}u^b, \tag{13.141}$$
$$M_a = \frac{1}{2}\eta_{abcd}M^{bc}u^d. \tag{13.142}$$

P^a is called the *polarization* four-vector and M_a the *magnetization four-vector*.

Exercise 67 a. Show that for a general medium

$$P_a = D_a - \varepsilon_0 E_a, \tag{13.143}$$
$$M_a = H_a - \frac{1}{\mu_0}B_a. \tag{13.144}$$

Deduce that in empty space $P^a = M^a = 0$, hence the polarization and the magnetization fields measure properties of matter.

b. Show that in a homogeneous and isotropic medium the polarization and the magnetization vectors are given by the following formulae:

$$P^a = (\varepsilon - \varepsilon_0)E^a, \tag{13.145}$$
$$M^a = \left(1 - \frac{\mu}{\mu_0}\right)H^a. \tag{13.146}$$

c. Making use of (13.114),(13.115),(13.129),(13.130) show that:

$$P^a = \gamma\left(\left(\mathbf{D}-\frac{\varepsilon_0}{c}\mathbf{E}\right)\cdot\mathbf{v},\, c\left(\mathbf{D}-\frac{\varepsilon_0}{c}\mathbf{E}\right) + \mathbf{v}\times\left(\frac{1}{c}\mathbf{H}-\varepsilon_0\mathbf{E}\right)\right)_\Sigma, \tag{13.147}$$
$$M^a = \frac{1}{c}\gamma\left(\frac{1}{c}\left(\mathbf{H}-\frac{1}{\mu_0}\mathbf{B}\right)\cdot\mathbf{v},\, \mathbf{H}-\frac{1}{\mu_0}\mathbf{B} - \mathbf{v}\times\left(\mathbf{D}-\frac{\varepsilon_0}{c}\mathbf{E}\right)\right)_\Sigma. \tag{13.148}$$

13.10.5 Maxwell Equations for a General Medium

For a general medium Maxwell equations take the form

$$K^{ab}_{\ \ ;b} = J^a, \tag{13.149}$$
$$F^{*ab}_{\ \ \ ;b} = 0, \tag{13.150}$$

where J^a is the four-current density and $F^{*ab} = \frac{1}{2}\eta^{abcd}F_{cd}$ is the dual bivector of the electromagnetic field tensor F^{ab}.

Exercise 68 Show that (13.149), (13.150) reduce to the three-dimensional Maxwell equations (13.1), (13.2), and (13.3) in the proper frame of the material. More specifically show that

$$F^{*ab}{}_{;b} = 0 \text{ reduces to the equations } \mathrm{div}\mathbf{B} = 0, \ \mathrm{curl}\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$K^{ab}{}_{;b} = J^a \text{ reduces to the equations } \mathrm{div}\mathbf{D} = \rho, \ \mathrm{curl}\mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}.$$

Exercise 69 Prove that the field equation (13.150) can be written as follows:

$$F^{*ab}{}_{;b} = 0 \Leftrightarrow F_{\{ab;c\}} = 0, \tag{13.151}$$

where $\{ab;c\}$ means cyclic sum in all indices enclosed.
[Hint: $F^{*ab}{}_{;b} = 0 \Rightarrow \eta^{abcd}F_{cd;b} = 0$. Multiply with η_{arst} and expand the product $\eta_{arst}\eta^{abcd}$ to get the result. The inverse is obvious.]

The four-current by means of the four-velocity u^a is decomposed as follows:

$$J^a = \frac{1}{c^2}\rho u^a + j^a, \tag{13.152}$$

where

$$\rho = -u^a J_a, \quad j^a = h^a_b J^b. \tag{13.153}$$

ρ is the charge density and j^a is the *conduction current*. The current ρu^a is called the *convection current*. More on the 1+3 decomposition of the conduction current is discussed in Sect. 13.11.

The antisymmetry of K^{ab} implies $K^{ab}{}_{,ab} = 0$ which leads to the continuity equation for the charge

$$J^a_{;a} = 0. \tag{13.154}$$

13.10.6 The $1 + 3$ Decomposition of Maxwell Equations

In order to write Maxwell equations in terms of the various vector fields we must decompose them wrt the four-velocity u^a of the observer who observes the electromagnetic field. These equations are the relativistic covariant generalization of the three-dimensional Newtonian equations and consist of two sets of equations, called the *constraint* and the *propagation equations* for the electromagnetic field.

Before we proceed with the computation of these equations we need to recall the 1+3 decomposition of the tensor $u_{i;j}$. This is given by the identity (not equation!)[24]

$$u_{i;j} = \omega_{ij} + \sigma_{ij} + \frac{1}{3}\theta h_{ij} - \frac{1}{c^2}u_{i;k}u^k u_j, \tag{13.155}$$

where the (symmetric) tensor $h_{ij} = \eta_{ij} + \frac{1}{c^2}u_i u_j$ is the standard tensor which projects perpendicular to u^i, that is $h_{ij}u^j = 0$, and the various tensors involved are defined as follows:

$$\omega_{ij} = h_i^r h_j^s u_{[r;s]},\ \omega^i = \frac{1}{2c}\eta^{ijkl}u_j\omega_{kl},\ \theta = u^a{}_{;a},\ \sigma_{ij} = h_i^r h_j^s u_{(r;s)} - \frac{1}{3}\theta h_{ij}. \tag{13.156}$$

Proposition 8 *The 1+3 decomposition of the equation $F^{*ab}{}_{;b} = 0$ gives the equations*

$$h_b^a B^b{}_{;a} = \frac{2}{c}\omega^a E_a, \tag{13.157}$$

$$h_b^a \dot{B}^b = u^a{}_{;b}B^b - \theta B^a + I^a(E) = \left(\sigma^a{}_b + \omega^a{}_b - \frac{2}{3}\theta h^a{}_b\right) B^b + I^a(E), \tag{13.158}$$

where the "current"

$$I^a(E) = \eta^{abcd}u_b(\dot{u}_c E_d - E_{c;d}) \tag{13.159}$$

and a dot over a symbol denotes covariant differentiation wrt u^a.

The 1+3 decomposition of the equation $K^{ab}{}_{;b} = J^a$ gives the equations

$$h_b^a D^b{}_{;a} = \frac{\rho}{c} + \frac{2}{c}\omega^a H_a, \tag{13.160}$$

$$h_b^a \dot{D}^b = u^a{}_{;b}D^b - \theta D^a + I^a(H) - J^a = \left(\sigma^a{}_b + \omega^a{}_b - \frac{2}{3}\theta h^a{}_b\right) D^b + I^a(H) - J^b, \tag{13.161}$$

where the "current"

$$I^a(H) = \frac{1}{c}\eta^{abcd}u_b\left(\frac{1}{c^2}\dot{u}_c H_d - H_{c;d}\right). \tag{13.162}$$

---

[24] We use ; to indicate the partial derivative and not, as we do for the rest of the book. The reason is that the results we derive hold also in General Relativity where we have the Riemannian covariant derivative, which is indicated with semicolon. These results are important, therefore we see no reason for not giving them in all their generality. If the reader finds it hard to follow he/she can replace semicolon with comma and all results go through without any change.

Proof

We make extensive use of the formulae (13.138) and (13.139).
Equation $F^{*ab}{}_{;b} = 0$
u^a-projection

$$u_a F^{*ab}{}_{;b} = 0.$$

We compute

$$u_a F^{*ab}{}_{;b} = \left(u_a F^{*ab}\right)_{;b} - u_{a;b} F^{*ab} = -cB^b{}_{;b} - F^{*ab}\omega_{ab} + \frac{1}{c^2} F^{*ab}\dot{u}_a u_b$$
$$= -cB^b{}_{;b} - F^{*ab}\omega_{ab} + c\frac{1}{c^2} B^a \dot{u}_a. \quad (13.163)$$

The *vorticity vector* ω^a is defined by the relation (see (13.108), (13.109))

$$\omega^a = \frac{1}{2c^2}\eta^{abcd} u_b \omega_{cd} \Leftrightarrow \omega_{ab} = \eta_{abcd}\omega^c u^d.$$

This gives

$$F^{*ab}\omega_{ab} = -\eta^{abrs}\eta_{abcd} E_r u_s \omega^c u^d = 2\left(\delta^r_c \delta^s_d - \delta^s_c \delta^r_d\right) E_r u_s \omega^c u^d = -2c^2 E_a \omega^a.$$

Also the term

$$B^b{}_{;b} - \frac{1}{c^2} B^a \dot{u}_a = B^b{}_{;b} + \frac{1}{c^2}\dot{B}^b u_b = B^{a,b}\left(\eta_{ab} + \frac{1}{c^2} u_a u_b\right) = h^a_b B^b_{;\,a}.$$

Replacing in (13.163) we find the constraint equation for B^a:

$$h^a_b B^b{}_{;a} = \frac{2}{c}\omega^a E_a. \quad (13.164)$$

Spatial Projection

$$h^a_b F^{*bc}{}_{;c} = 0.$$

We compute

$$2h^a_b F^{*bc}{}_{;c} = h^a{}_b \eta^{bcde} F_{cd;e}$$
$$= h^a{}_b \eta^{bcde}[-2(E_c u_d)_{;e} - \eta_{cdrs}(u^r B^s)_{;e}]$$
$$= -2h^a{}_b \eta^{bcde}(E_{c;e} u_d + E_c u_{d;e}) - h^a{}_b \eta^{bcde}\eta_{cdrs}(u^r{}_{;e} B^s + u^r B^s{}_{;e}).$$

The first two terms give

$$
\begin{aligned}
A^a &\equiv -2h^a{}_b\eta^{bcde}(E_{c;e}u_d + E_c u_{d;e}) = \\
&= -2h^a{}_b\eta^{bcde}E_{c;e}u_d - 2h^a{}_b\eta^{bcde}E_c\left(\omega_{de} - \frac{1}{c^2}\dot{u}_d u_e\right) \\
&= -2\eta^{acde}E_{c;e}u_d - 2h^a{}_b\eta^{bcde}E_c\omega_{de} + \frac{2}{c^2}h^a{}_b\eta^{bcde}E_c\dot{u}_d u_e. \\
&= 2\eta^{abcd}E_{c;d}u_b - \frac{2}{c^2}\eta^{abcd}u_b\dot{u}_c E_d = -2\eta^{abcd}u_b\left(\frac{1}{c^2}\dot{u}_c E_d - E_{c;d}\right).
\end{aligned}
$$

where we have used that

$$2h^a{}_b E_c(-2)(\delta^b_r\delta^c_s - \delta^c_r\delta^b_s)u^r\omega^s = 0.$$

The last term gives

$$
\begin{aligned}
C^a &\equiv -h^a{}_b\eta^{bcde}\eta_{cdrs}(u^r{}_{;e}B^s + u^r B^s{}_{;e}) = \\
&= 2h^a{}_b(\delta^b_r\delta^e_s - \delta^e_r\delta^b_s)(u^r{}_{;e}B^s + u^r B^s{}_{;e}) \\
&= 2h^a{}_b\left(u^b{}_{;e}B^e + u^b B^e{}_{;e} - u^r{}_{;r}B^b - u^e B^b{}_{;e}\right) \\
&= 2h^a{}_b\left(\sigma^b{}_e + \omega^b{}_e + \frac{1}{3}\theta h^b{}_e - \frac{1}{c^2}\dot{u}^b u_e\right)B^e - 2\theta B^a - 2h^a{}_b\dot{B}^b \\
&= 2\left(\sigma^a{}_e + \omega^a{}_e - \frac{2}{3}\theta^a{}_e\right)B^e - 2h^a{}_b\dot{B}^b.
\end{aligned}
$$

Combining the last two results we have the propagation equation for B^a:

$$h^a{}_b\dot{B}^b = (\sigma^a{}_c + \omega^a{}_c - \frac{2}{3}\theta h^a{}_c)B^c + I^a(E), \tag{13.165}$$

where the E-current is given by the relation

$$I^a(E) \equiv \eta^{abcd}u_b\left(\frac{1}{c^2}\dot{u}_c E_d - E_{c;d}\right). \tag{13.166}$$

Equation $K^{ab}_{;b} = J^a$
u^a-projection

$$u_a K^{ab}{}_{;b} = -\rho.$$

We compute

$$
\begin{aligned}
u_a K^{ab}{}_{;b} &= \left(u_a K^{ab}\right)_{;b} - u_{a;b} K^{ab} = \\
&= -cD^b{}_{;b} - \left(\omega_{ab} - \frac{1}{c^2}\dot{u}_a u_b\right) K^{ab} = -cD^b{}_{;b} - \omega_{ab}K^{ab} + \frac{1}{c}\dot{u}_a D^a.
\end{aligned}
$$

The term

$$K^{ab}\omega_{ab} = K^{ab}\eta_{abcd}\omega^c u^d = 2H^b\omega_b.$$

Therefore we have the constraint equation for D^a:

$$D^a{}_{;a} - \frac{1}{c^2}D^a\dot{u}_a = \frac{\rho}{c} + \frac{2}{c}\omega^a H_a. \tag{13.167}$$

Spatial projection

$$h^a{}_b K^{bc}{}_{;c} = h^a{}_b J^b = j^a.$$

The lhs of the equation gives

$$
\begin{aligned}
& h^a{}_b\left[\frac{1}{c}(-D^b u^c + D^c u^b)_{;c} - \frac{1}{c^2}\eta^{bcde}(u_d H_e)_{;c}\right] = \\
&= \frac{1}{c}h^a{}_b\left[-\dot{D}^b - D^b u^c{}_{;c} + D^c{}_{;c}u^b + D^c u^b{}_{;c} - \frac{1}{c}\eta^{bcde}(u_{d;c}H_e + u_d H_{e;c})\right] \\
&= -\frac{1}{c}h^a{}_b\dot{D}^b - D^a u^c{}_{;c} + D^c\left(\sigma^a{}_c + \frac{1}{3}h^a{}_c\theta + \omega^a{}_c - \frac{1}{c^2}\dot{u}^a u_c\right) - \frac{1}{c}h^a{}_b\eta^{bcde}\omega_{dc}H_e \\
&\quad + \frac{1}{c^3}\eta^{acde}\dot{u}_d u_c H_e - \frac{1}{c}\eta^{acde}u_d H_{e;c} \\
&= \frac{1}{c}\left[-h^a{}_b\dot{D}^b + D^c\left(\sigma^a{}_c + \omega^a{}_c - \frac{2}{3}\theta h^a{}_c\right) - \frac{1}{c}\eta^{acde}\omega_{dc}H_e - u^a 2\omega^b H_b\right. \\
&\quad \left. + \frac{1}{c^3}\eta^{acde}u_c\dot{u}_d H_e - \frac{1}{c}\eta^{abcd}u_b H_{c;d}\right].
\end{aligned}
$$

The term

$$
\begin{aligned}
-\eta^{acde}\omega_{dc}H_e = \eta^{abcd}\omega_{bc}H_d &= -c\eta^{abcd}\eta_{bcrs}u^r\omega^s H_d u^r\omega^s H_d \\
&= 2c\left(\delta^a_r\delta^d_s - \delta^d_r\delta^a_s\right) = 2cu^a\omega^b H_b.
\end{aligned}
$$

Replacing in the last relation we find the propagation equation for D^a :

$$h^a_b\dot{D}^b = u^a{}_{;b}D^b - \theta D^a + I^a(H) - j^a = \left(\sigma^a{}_c + \omega^a{}_c - \frac{2}{3}\theta h^a{}_c\right)D^c + I^a(H) - j^a, \tag{13.168}$$

where the H-current is defined as follows:

$$I^a(H) \equiv \frac{1}{c}\eta^{abcd}u_b\left(\frac{1}{c^2}\dot{u}_c H_d - H_{c;d}\right). \tag{13.169}$$

Equations (13.157) and (13.160) are called the *constraint equations* (because they contain divergences) and (13.158) and (13.161) the *propagation equations* (because they express the derivative of the fields along u^a) for the electric and the magnetic fields.

13.11 The Four-Current of Conductivity and Ohm's Law

We consider a charge q moving with four-velocity u^a in a region of space where there exists an electromagnetic field F_{ab}. The charge "sees" an electric field E^a and a magnetic field B^a given by (13.110)

$$E^a = F^a{}_b u^b, \tag{13.170}$$

$$B^a = \frac{1}{2c}\eta^{abcd} u_b F_{cd} \tag{13.171}$$

while the tensor F_{ab} in terms of the fields E^a, B^a, and u^a is given by relation (13.117):

$$F_{ab} = \frac{1}{c^2}(u_a E_b - u_b E_a) - \frac{1}{c}\eta_{abcd} u^c B^d. \tag{13.172}$$

Finally we recall the equation of continuity for charge:

$$J^a{}_{;a} = 0. \tag{13.173}$$

In order to study the four-vector J^a we 1+3 decompose it along the four-velocity u^a of the charge and then we give a physical meaning to each irreducible part. From the standard analysis of 1+3 decomposition we have

$$J^a = -\frac{1}{c^2}(J^b u_b)u^a + h^a_b J^b, \tag{13.174}$$

where h_{ab} is the tensor projecting perpendicularly to u^a. It follows that in every LCF the four-current J^a determines and it is determined by one invariant (the $J^b u_b$) and one spacelike vector (the $h^a{}_b J^b$). Consequently in order to give a physical interpretation to the four-current J^a we must give an interpretation to the invariant $J^b u_b$ and the spacelike vector $h^a{}_b J^b$. As we have emphasized the physical interpretation of a relativistic quantity is made by its identification with a corresponding Newtonian physical quantity in the proper frame of the observer. We consider the following interpretation/identification:

- The invariant $-J^a u_a$ is the charge density of the charge in the proper frame of the charge.
- The spacelike four-vector $h^a_b J^b$ is the three-vector of electrical conductivity in the proper frame of the charge.

Let us compute the components of the four-current J^a in an LCF Σ in which the four-velocity $u^a = \begin{pmatrix} \gamma c \\ \gamma \mathbf{v} \end{pmatrix}_\Sigma$. We write

$$J^a = \begin{pmatrix} j^0 \\ \mathbf{j} \end{pmatrix}_\Sigma, \tag{13.175}$$

where the components j^0 and $\mathbf{j}$ have to be computed. From the definition of the tensor h_{ab} we have

$$\begin{aligned} h_{ab}J^b &= (\eta_{ab} + \frac{1}{c^2}u_a u_b)J^b = J_a + \frac{J^c u_c}{c^2}u_a \\ &= (-j^0, \mathbf{j})_\Sigma + \frac{1}{c^2}(-\gamma c j^0 + \gamma \mathbf{j}\cdot\mathbf{v})(-\gamma c, \gamma \mathbf{v})_\Sigma \\ &= \left((\gamma^2 - 1)j^0 - \gamma^2 \frac{\mathbf{j}\cdot\mathbf{v}}{c}, \mathbf{j} - \frac{\gamma^2}{c} j^0 \mathbf{v} + \frac{\gamma^2}{c^2}(\mathbf{j}\cdot\mathbf{v})\mathbf{v}\right)_\Sigma \\ &= \left(\beta^2\gamma^2 j^0 - \gamma^2 \frac{\mathbf{j}\cdot\mathbf{v}}{c}, \mathbf{j} - \frac{\gamma^2}{c} j^0 \mathbf{v} + \frac{\gamma^2}{c^2}(\mathbf{j}\cdot\mathbf{v})\mathbf{v}\right)_\Sigma . \end{aligned}$$

Here we have exhausted all our assumptions. Essentially all equations we have obtained up to now are mathematical identities because we have not used any physical law. In order to continue we consider Ohm's Law which is stated as follows:

$$h_{ab}J^b = k_{ab}F^b{}_c u^c. \tag{13.176}$$

k_{ab} is a tensor which we call the *electric conductivity tensor*. For a homogeneous and isotropic medium the tensor $k_{ab} = k\delta_{ab}$ where k is a constant called the *electric conductivity* of the medium. Subsequently for such a medium Ohm's Law reads

$$h^a{}_b J^b = kF^a{}_b u^b = kE^a. \tag{13.177}$$

Ohm's Law makes possible the calculation of the charge density and the conduction current in Σ. Indeed replacing in (13.177) the components in Σ of the various quantities involved we find

$$k\left(\gamma\frac{\mathbf{E}\cdot\mathbf{v}}{c}, \gamma(\mathbf{E} + \mathbf{v}\times\mathbf{B})\right)_\Sigma = \left(\beta^2\gamma^2 j^0 - \gamma^2\frac{\mathbf{j}\cdot\mathbf{v}}{c}, \mathbf{j} - \frac{\gamma^2}{c}j^0\mathbf{v} + \frac{\gamma^2}{c^2}(\mathbf{j}\cdot\mathbf{v})\mathbf{v}\right)_\Sigma . \tag{13.178}$$

The zeroth component gives

$$\frac{\mathbf{j}\cdot\mathbf{v}}{c} = \beta^2 j^0 - \frac{k}{\gamma}\frac{\mathbf{E}\cdot\mathbf{v}}{c} \tag{13.179}$$

and the spatial component

$$\mathbf{j} = \left(\frac{\gamma^2}{c} j^0 - \frac{\gamma^2}{c^2} (\mathbf{j} \cdot \mathbf{v}) \right) \mathbf{v} + k\gamma (\mathbf{E} + \mathbf{v} \times \mathbf{B}). \tag{13.180}$$

We note that the zeroth component follows from the spatial part if the later is scalar multiplied with $\mathbf{v}$, hence the zeroth component is included in the spatial part and we ignore it.

We continue with the spatial part of the four-vector. We note that the quantity in parenthesis gives

$$\begin{aligned} \frac{\gamma^2}{c} j^0 - \frac{\gamma^2}{c^2} (\mathbf{j} \cdot \mathbf{v}) = \frac{\gamma^2}{c} \left(j^0 - \frac{1}{c} (\mathbf{j} \cdot \mathbf{v}) \right) &= \frac{\gamma^2}{c} \left(j^0 - \beta^2 j^0 + \frac{k}{\gamma} \frac{\mathbf{E} \cdot \mathbf{v}}{c} \right) \\ &= \frac{1}{c} \left(j^0 + k\gamma \frac{\mathbf{E} \cdot \mathbf{v}}{c} \right). \end{aligned}$$

Replacing in (13.180) we find

$$\mathbf{j} = \frac{1}{c} j^0 \mathbf{v} + k\gamma \left(\mathbf{E} + \mathbf{v} \times \mathbf{B} + \frac{\mathbf{E} \cdot \mathbf{v}}{c^2} \mathbf{v} \right). \tag{13.181}$$

We conclude that in a charged homogeneous and isotropic medium the three-current $\mathbf{j}$ in an LCF Σ has two parts:

1. The *convection current*:

$$\mathbf{j}_{conv} = \frac{1}{c} j^0 \mathbf{v}, \tag{13.182}$$

which depends on the motion of the charged medium and it is due to the charge density in Σ.

2. The *conduction current*:

$$\mathbf{j}_{cond} = k\gamma \left(\mathbf{E} + \mathbf{v} \times \mathbf{B} + \frac{\mathbf{E} \cdot \mathbf{v}}{c^2} \mathbf{v} \right), \tag{13.183}$$

which is due to the fields $\mathbf{E}$, $\mathbf{B}$ in Σ and depends on the electric conductivity k of the charged medium.

The physical meaning of this result is that if a conductor is not charged (that is $J^a u_a = 0$) and moves in an LCF with velocity $\mathbf{v}$, it appears to be charged with charge $\frac{\mathbf{E} \cdot \mathbf{v}}{c^2}$. (More on this aspect can be found in Exercise 69.)

We compute next the components of the four-vector J^a in Σ in terms of the zeroth component j^0 in Σ.[25] From the previous considerations we have

[25] Note that the components are not the same with the coefficients of the 1+3 decomposition because the former are just components whereas the second tensors.

$$\begin{aligned}
J^a u_a &= -\gamma c j^0 + \gamma \mathbf{j} \cdot \mathbf{v} \\
&= \gamma c \left(-j^0 + \beta^2 j^0 - \frac{k}{\gamma} \frac{\mathbf{E} \cdot \mathbf{v}}{c} \right) \\
&= -\frac{c}{\gamma} j^0 - k \mathbf{E} \cdot \mathbf{v}.
\end{aligned}$$

Therefore the component j^0 of the four-vector J^a in Σ is

$$j^0 = \frac{\gamma}{c} \left(-J^a u_a \right) - k\gamma \frac{\mathbf{E} \cdot \mathbf{v}}{c}. \tag{13.184}$$

Concerning the spatial part of the four-current J^a in Σ we have from (13.181)

$$\begin{aligned}
\mathbf{j} &= \frac{1}{c} j^0 \mathbf{v} + k\gamma \left(\mathbf{E} + \mathbf{v} \times \mathbf{B} + \frac{\mathbf{E} \cdot \mathbf{v}}{c^2} \mathbf{v} \right) \\
&= \frac{1}{c} \left(j^0 + k\gamma \frac{\mathbf{E} \cdot \mathbf{v}}{c} \right) \mathbf{v} + k\gamma (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \\
&= \frac{\gamma}{c^2} \left(-J^a u_a \right) \mathbf{v} + k\gamma (\mathbf{E} + \mathbf{v} \times \mathbf{B}).
\end{aligned}$$

The conduction current can be written in an alternative form which helps us understand its physical meaning. We project (13.183) along and perpendicularly to the three-velocity vector $\mathbf{v}$ of the charged medium in Σ and find

$$\begin{aligned}
k\gamma \left(\mathbf{E} + \mathbf{v} \times \mathbf{B} - \frac{\mathbf{E} \cdot \mathbf{v}}{c^2} \mathbf{v} \right) \cdot \mathbf{v} &= k\gamma \left(\mathbf{E} \cdot \mathbf{v} - \frac{\mathbf{E} \cdot \mathbf{v}}{c^2} \mathbf{v}^2 \right) \\
&= \frac{k}{\gamma} \mathbf{E} \cdot \mathbf{v} = \frac{kv}{\gamma} \mathbf{E}_{\|} = \frac{kv}{\gamma} \mathbf{E}'_{\|}
\end{aligned}$$

and

$$k\gamma \left(\mathbf{E}_{\perp} + \mathbf{v} \times \mathbf{B} \right) = k\gamma \mathbf{E}'_{\perp},$$

that is, the two currents are expressed in terms of the corresponding parts of the electric field in the proper frame of the charged medium. We conclude that the conduction current in Σ is

$$\mathbf{j}_{cond} = \frac{kv}{\gamma} \mathbf{E}'_{\|} + k\gamma \mathbf{E}'_{\perp}. \tag{13.185}$$

From this relation it becomes clear that the conduction current is due to the electromagnetic field and not due to the charge density of the medium.

Example 71 A charged isotropic and homogeneous medium with electric conductivity k moves in an LCF O with velocity $\mathbf{u}$. In "Newtonian" electrodynamics Ohm's Law states that $\mathbf{j} = k\mathbf{E}$, where $\mathbf{j}$ is the conduction current in Σ and $\mathbf{E}$ the electric field in Σ. Assuming that the scalar quantity k is Lorentz invariant prove that the Lorentz covariant form of Ohm's Law in Special Relativity is $h_{ab}J^b = kE_a$ where

a. u^a is the four-velocity of the charge defined by the three-velocity $\mathbf{u}$ in Σ, that is $u^a = \begin{pmatrix} \gamma c \\ \gamma \mathbf{v} \end{pmatrix}_\Sigma$.
b. J^a is a four-vector with components $J^a = \begin{pmatrix} \rho/c^2 \\ \mathbf{j} \end{pmatrix}_\Sigma$ where ρ is the charge density in Σ.
c. $E^a = F^a_b u^b$ is the relativistic electric field which corresponds to the four-velocity u^a.

Solution

It is enough to prove that the covariant expression holds in one LCF and specifically in the proper frame Σ^+ of the charged medium. In this frame the four-velocity $u^a = \begin{pmatrix} c \\ \mathbf{0} \end{pmatrix}_{\Sigma^+}$ and the electric field $E^a = F^{ab}u_b = \begin{pmatrix} 0 \\ \mathbf{E}^+ \end{pmatrix}_{\Sigma^+}$. Suppose that in this frame $J^a = \begin{pmatrix} \rho^+/c^2 \\ \mathbf{j}^+ \end{pmatrix}_{\Sigma^+}$. We replace in the given expression the various quantities in terms of their components and find $\mathbf{j}^+ = k\mathbf{E}^+$ which coincides with the "Newtonian" $\mathbf{j} = k\mathbf{E}$ written in Σ^+.

13.11.1 The Continuity Equation $J^a{}_{;a} = 0$ for an Isotropic Material

The equation $J^a{}_{;a} = 0$ is a direct consequence of Maxwell equations, therefore it does not give information which is not already included in Maxwell equations. However, it can be used in order to express the electric conductivity through Maxwell equations.

We have

$$J^a{}_{;a} = \left(\frac{\rho_0}{c^2}u^a + kE^a\right)_{;a} = 0 \tag{13.186}$$

from which follows

$$\frac{1}{c^2}(\dot{\rho}_0 + \rho_0 u^a{}_{;a}) + k_{;a}E^a + kE^a{}_{;a} = 0, \tag{13.187}$$

where $\dot{\rho}_0 = \rho_{,a}u^a$ and have assumed that the material is isotropic but not homogeneous so that k need not be constant. We compute the divergence of the field E^a. We have

$$E^a{}_{;a} = (F^{ab}u_b)_{;a} = F^{ab}{}_{;a}u_b + F^{ab}u_{b;a}. \tag{13.188}$$

But Maxwell equations give

$$F^{ab}{}_{;a}u_b = -\mu J^b u_b = \mu\rho_0. \tag{13.189}$$

Also from the decomposition (13.155) of $u_{a;b}$ and the antisymmetry of F^{ab} we have

$$F^{ab}u_{b;a} = F^{ab}\left(\omega_{ba} - \frac{1}{c^2}\dot{u}_b u_a\right) = F^{ab}\omega_{ba} + \frac{1}{c^2}\dot{u}_b E^b. \tag{13.190}$$

Replacing in (13.188) we find

$$E^a{}_{;a} = \mu\rho_0 - F^{ab}\omega_{ab} + \frac{1}{c^2}\dot{u}_b E^b. \tag{13.191}$$

Finally the conservation equation (13.186) gives

$$k_{,a}E^a + k\left(\mu\rho_0 - F^{ab}\omega_{ab} + \frac{1}{c^2}\dot{u}_a E^a\right) + \frac{1}{c^2}(\dot{\rho}_0 + \rho_0\theta) = 0. \tag{13.192}$$

Equation (13.192) holds for a general velocity and a general electromagnetic field. It is a differential equation which determines the electric conductivity along the direction of the field E^a.

13.12 The Electromagnetic Field in a Homogeneous and Isotropic Medium

The fields E^a, B^a, D^a, H^a and (13.157), (13.160), (13.158), (13.161) describe the electromagnetic field in a general medium. Since it is impossible to solve these equations for an arbitrary medium we consider special cases by assuming relations amongst these fields. We call these relations *constitutive relations*. We used one such relation at the beginning of this chapter to define (in the rest frame of the material only!) the homogeneous and isotropic material by the requirements

$$\mathbf{D} = \varepsilon\mathbf{E}, \qquad \mathbf{B} = \mu\mathbf{H}, \tag{13.193}$$

where the coefficients $\varepsilon,\ \mu$ are constants, characteristic of the material, and satisfy the relation

$$\varepsilon\mu = \frac{1}{v^2}, \tag{13.194}$$

where v is the (phase) velocity of the electromagnetic field within the material. For vacuum these coefficients are the ε_0, μ_0, and satisfy the relation (13.9):

$$\varepsilon_0\mu_0 = \frac{1}{c^2}. \tag{13.195}$$

The *refraction index* n of the material is defined by the ratio

$$n = \frac{c}{v} > 1 \tag{13.196}$$

and it is given by the relation

$$n = \sqrt{\frac{\varepsilon\mu}{\varepsilon_0\mu_0}} = c\sqrt{\varepsilon\mu}. \tag{13.197}$$

The index of refraction is the physical quantity which differentiates a homogeneous and isotropic material from the vacuum (the extreme of such material).

This definition of the homogeneous and isotropic material is not suitable in Special Relativity because it is given in the proper frame only and in terms of three-vectors in that frame. The proper way to state a constitutive relation in relativity is in terms of the four-vectors E^a, B^a, D^a, H^a. Indeed in that case this relation is covariant, hence observer independent. Following this remark we define a (relativistic) *homogeneous and isotropic material* as one in which the electromagnetic field vectors E^a, B^a, D^a, H^a, which are assigned by some observers u^a, satisfy the following requirements:

$$D^a = \varepsilon E^a, \qquad B^a = \mu H^a, \qquad J^a = kH^a, \tag{13.198}$$

where the constants ε, μ are the dielectric constant and the magnetic permeability of the material, respectively, and k is the electric conductivity of the material.

Example 72 i. Show that condition (13.198) implies the following conditions on the tensors F_{ab}, K_{ab} of the electromagnetic field:

$$\frac{1}{c}K_{ab}u^b = \varepsilon F_{ab}u^b, \tag{13.199}$$

$$\mu K_{\{ab}u_{c\}} = \frac{1}{c}F_{\{ab}u_{c\}}. \tag{13.200}$$

ii. Define the tensor

$$e_{abcd} = \frac{1}{\mu c}\left(\eta_{ac} - \frac{\chi}{c^2}u_a u_c\right)\left(\eta_{bd} - \frac{\chi}{c^2}u_b u_d\right), \tag{13.201}$$

where $\chi = n^2 - 1$ is the *electric susceptibility* of the material and n is the refraction index of the material. Show that for a homogeneous and isotropic material the induction tensor K_{ab} is related to the electromagnetic field tensor F^{ab} as follows:

$$K_{ab} = e_{abcd}F^{cd}. \tag{13.202}$$

Proof

ii.

$$\begin{aligned}
&\frac{1}{\mu c}\left(\eta_{ac}-\frac{\chi}{c^2}u_a u_c\right)\left(\eta_{bd}-\frac{\chi}{c^2}u_b u_d\right)F^{cd}\\
&=\frac{1}{\mu c}\left(F_a{}^d+\frac{\chi}{c^2}u_a E^d\right)\left(\eta_{bd}-\frac{\chi}{c^2}u_b u_d\right)\\
&=\frac{1}{\mu c}\left(F_{ab}+\frac{\chi}{c^2}u_a E_b-\frac{\chi}{c^2}u_b E_a\right)\\
&=\frac{1}{\mu c}\left[\frac{1}{c^2}\left(-E_a u_b+E_b u_a\right)+\frac{1}{c}\eta_{abcd}B^c u^d+\frac{\chi}{c^2}u_a E_b-\frac{\chi}{c^2}u_b E_a\right]\\
&=\frac{1}{\mu c}\left[\frac{1}{c^2}(1+\chi)\left(-E_a u_b+E_b u_a\right)+\frac{\mu}{c}\eta_{abcd}H^c u^d\right]\\
&=\frac{1}{\mu c}\frac{n^2}{c^2}\left(-E_a u_b+E_b u_a\right)+\frac{1}{c^2}\eta_{abcd}H^c u^d\\
&=\frac{1}{\mu c}\frac{\varepsilon\mu c^2}{c^2}\left(-E_a u_b+E_b u_a\right)+\frac{1}{c^2}\eta_{abcd}H^c u^d\\
&=\frac{1}{c}\left(-D_a u_b+D_b u_a\right)+\frac{1}{c^2}\eta_{abcd}H^c u^d\\
&=K_{ab}.
\end{aligned}$$

In order to express (13.198) in terms of the three-vector fields we consider an LCF Σ in which the four-vectors E^a, B^a, D^a, H^a are given by (13.114),(13.115) and (13.129), (13.130), respectively, and find

$$\mathbf{D}+\frac{1}{c^2}\mathbf{v}\times\mathbf{H}=\frac{\varepsilon}{c}(\mathbf{E}+\mathbf{v}\times\mathbf{B}) \tag{13.203}$$

$$\frac{\mu}{c}(\mathbf{H}-\mathbf{v}\times\mathbf{D})=\mathbf{B}-\frac{1}{c^2}\mathbf{v}\times\mathbf{E}. \tag{13.204}$$

Replacing $\mathbf{B}$ into the first equation we find

$$\mathbf{D}+\frac{1}{c^2}\mathbf{v}\times\mathbf{H}=\frac{\varepsilon}{c}\mathbf{E}+\frac{\varepsilon\mu}{c}\mathbf{v}\times\mathbf{H}-\frac{\varepsilon\mu}{c}\mathbf{v}\times(\mathbf{v}\times\mathbf{D})+\frac{1}{c^3}\varepsilon\mathbf{v}\times(\mathbf{v}\times\mathbf{E}). \tag{13.205}$$

Using the identity of vector calculus

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=(\mathbf{A}\cdot\mathbf{C})\mathbf{B}-(\mathbf{A}\cdot\mathbf{B})\mathbf{C}$$

and relations (13.194), (13.195) we end up with the conditions

$$\mathbf{D}_{\|}=\frac{\varepsilon}{c}\mathbf{E}_{\|} \tag{13.206}$$

$$\left(1-\frac{n^2\beta^2}{c^2}\right)\mathbf{D}_{\perp}=\frac{\varepsilon}{c\gamma^2}\mathbf{E}_{\perp}+\left(\frac{n^2}{c}-1\right)\varepsilon_0\mu_0(\mathbf{v}\times\mathbf{H}). \tag{13.207}$$

We see that in Σ condition (13.193) does not hold for a homogeneous and isotropic material (except if $n = 1$, i.e., in empty space!).

Exercise 70 Show that for a homogeneous and isotropic material

$$\mathbf{B}_{\parallel} = \frac{\mu}{c}\mathbf{H}_{\parallel} \tag{13.208}$$

$$\left(1 - \frac{n^2\beta^2}{c^2}\right)\mathbf{B}_{\perp} = \frac{\mu}{c\gamma^2}\mathbf{H}_{\perp} + \frac{1}{c^2}\left(1 - \frac{n^2}{c}\right)\mathbf{v}\times\mathbf{E}. \tag{13.209}$$

For $\beta \ll 1$ the above expressions (13.206), (13.207) and (13.208), (13.209) reduce to

$$\mathbf{D} = \varepsilon\mathbf{E}_{\perp} + \frac{n^2-1}{c^2}\mathbf{v}\times\mathbf{H}, \tag{13.210}$$

$$\mathbf{B}=\mu\mathbf{H} + \frac{n^2-1}{c^2}\mathbf{v}\times\mathbf{E}. \tag{13.211}$$

These relations are used widely in the study of the electromagnetic field in a homogeneous and isotropic medium.

Exercise 71 Show that the polarization four-vector and the magnetization four-vector under a Lorentz transformation transform as follows:

$$\mathbf{P}_{\parallel} = \mathbf{P}^0_{\parallel}, \quad \mathbf{P}_{\perp} = \gamma\left(\mathbf{P}^0 - \frac{1}{c^2}\mathbf{v}\times\mathbf{M}^0\right), \tag{13.212}$$

$$\mathbf{M}_{\parallel} = \mathbf{M}^0_{\parallel}, \quad \mathbf{M}_{\perp} = \gamma\left(\mathbf{M}^0 - \mathbf{v}\times\mathbf{P}^0\right), \tag{13.213}$$

where $\mathbf{P}^0, \mathbf{M}^0$ are these vectors in the proper frame Σ^+ of the medium and $u^a = \begin{pmatrix}\gamma c\\ \gamma\mathbf{v}\end{pmatrix}_{\Sigma}$ is the four-velocity in Σ^+. Deduce that a medium which is polarized but not magnetized in one LCF is polarized and magnetized for another LCF.

For small velocities $\beta \to 0$ relations (13.212), (13.213) give

$$\mathbf{P} = \mathbf{P}^0, \quad \mathbf{M} = \mathbf{P}^0 \times \mathbf{v}, \tag{13.214}$$

which means that polarized material appears magnetized.

Let us consider a magnet resting in Σ^+ so that $\mathbf{P}^0 = 0$, $\mathbf{M}^0 \neq 0$. Then in another LCF Σ (13.212), (13.213) give

$$\mathbf{P} = -\gamma\frac{1}{c^2}\mathbf{v}\times\mathbf{M}^0, \quad \mathbf{M} = \mathbf{M}^0. \tag{13.215}$$

This means that a moving permanent magnet carries an electric moment giving rise to the phenomenon of *homopolar induction* utilized widely in electrical engineering.

There remain the propagation and the constraint equations for a homogeneous and isotropic material.

From the corresponding equations for a general medium, i.e., (13.157), (13.158), (13.160), (13.161), it follows:

$$h^a{}_b H^b{}_{;a} = \frac{1}{\mu c} 2\omega^a E_a, \tag{13.216}$$

$$h^a{}_b E^b{}_{;a} = \frac{\rho}{\varepsilon c} + \frac{1}{\varepsilon c} 2\omega^a H_a, \tag{13.217}$$

$$h^a{}_b \dot{H}^b = u^a{}_{;b} H^b - \theta H^a + \frac{1}{\mu} I^a(E) = \left(\sigma^a{}_b + \omega^a{}_b - \frac{2}{3}\theta h^a{}_b\right) H^b + \frac{1}{\mu} I^a(E), \tag{13.218}$$

$$\begin{aligned} h^a{}_b \dot{E}^b &= u^a{}_{;b} E^b - \theta E^a + \frac{1}{\varepsilon} I^a(H) - \frac{1}{\varepsilon} j^a \\ &= \left(\sigma^a{}_b + \omega^a{}_b - \frac{2}{3}\theta h^a{}_b\right) E^b + \frac{1}{\varepsilon} I^a(H) - \frac{1}{\varepsilon} j^a, \end{aligned} \tag{13.219}$$

where

$$I^a(E) = \eta^{abcd} u_b \left(\frac{1}{c^2} \dot{u}_c E_d - E_{c;d}\right), \tag{13.220}$$

$$I^a(H) = \eta^{abcd} u_b \left(\frac{1}{c^2} \dot{u}_c H_d - H_{c;d}\right) \tag{13.221}$$

are the electric field and the magnetic field currents. The constraint and the propagation equations correspond to the three-dimensional equations of the electromagnetic field as follows:

$h^{ab} E_{a;b} - \frac{1}{\varepsilon c} 2\omega^a H_a = \frac{\rho}{\varepsilon c}$	$\Leftrightarrow$	$\mathrm{div}\mathbf{E} = \rho/\varepsilon c$
$h^{ab} H_{a;b} - \frac{1}{\mu c} 2\omega^a E_a = 0$	$\Leftrightarrow$	$\mathrm{div}\mathbf{H} = 0$
$h^a_b \dot{E}^b = \left(\sigma^a{}_b + \omega^a{}_b - \frac{2}{3}\theta h^a{}_b\right) E^b + \frac{1}{\varepsilon}(I^a(H) - j^a)$	$\Leftrightarrow$	$\frac{\partial \mathbf{E}}{\partial t} = -\frac{1}{\varepsilon}\mathbf{j} + \frac{1}{\varepsilon}\nabla \times \mathbf{H}$
$h^a_b \dot{H}^b = \left(\sigma^a{}_b + \omega^a{}_b - \frac{2}{3}\theta h^a{}_b\right) H^b + \frac{1}{\mu} I^a(E)$	$\Leftrightarrow$	$\frac{\partial \mathbf{H}}{\partial t} = -\frac{1}{\mu}\nabla \times \mathbf{E}.$

13.13 Electric Conductivity and the Propagation Equation for E^a

The electric conductivity is defined by Ohm's Law which in its simplest form is stated as follows:

$$h^a_b J^b = j^a = k E^a. \tag{13.222}$$

We consider the propagation equation of D^a:

$$h^a_b \dot{D}^b = \left(\sigma^a{}_b + \omega^a{}_b - \frac{2}{3}\theta h^a{}_b\right) D^b + I^a(H) - j^a$$

and contract with D^a to get

$$\dot{D}^a D_a = \left(\sigma^a{}_b + \omega^a{}_b - \frac{2}{3}\theta h^a{}_b\right) D^b D_a + D_a I^a(H) - k\varepsilon E^2.$$

Solving this for k we find

$$k = \frac{1}{\varepsilon E^2}\left[-\frac{1}{2}\dot{D}^2 + \sigma_{ab} D^a D^b - \frac{2}{3}\theta D^2 + \varepsilon E_a I^a(H)\right].$$

The term

$$\begin{aligned} E_a I^a(H) &= \frac{1}{c}\eta^{abcd} u_b \left[\frac{1}{c^2}\dot{u}_c H_d - H_{c;d}\right] E_a \\ &= \frac{1}{c^3}\eta^{abcd} u_b \dot{u}_c H_d E_a - \frac{1}{c}\eta^{abcd} u_b H_{c;d} E_a \\ &= -\frac{1}{c^2}\dot{u}_a S^a - \frac{1}{c}\eta^{abcd} u_b H_{c;d} E_a, \end{aligned}$$

where $S^a = \frac{1}{c}\eta^{abcd} u_b E_c H_d$ is the *Poynting vector*. We conclude that according to Ohm's Law the electric conductivity is given by

$$k = \frac{1}{\varepsilon E^2}\left[-\frac{1}{2}\dot{D}^2 + \sigma_{ab} D^a D^b - \frac{2}{3}\theta D^2 - \frac{1}{c^2}\dot{u}_a S^a - \frac{1}{c}\eta^{abcd} u_b H_{c;d} E_a\right]. \tag{13.223}$$

Assuming a homogeneous and isotropic medium we write $D^a = \varepsilon E^a$ and (13.223) reduces to

$$k = -\left(\dot{\varepsilon} + \frac{2}{3}\theta\varepsilon\right) + \frac{1}{\varepsilon E^2}\left[-\frac{\varepsilon^2}{2}(E^2)^{\cdot} + \varepsilon^2 \sigma_{ab} E^a E^b - \frac{1}{c^2}\dot{u}_a S^a - \eta^{abcd} u_b H_{c;d} E_a\right]. \tag{13.224}$$

This relation can be written differently. Indeed

$$\begin{aligned} \eta^{abcd} u_b H_{c;d} E_a &= \left(\eta^{abcd} u_b H_c E_a\right)_{;d} - \eta^{abcd} u_{b;d} H_c E_a - \eta^{abcd} u_b H_c E_{a;d} \\ &= -cS^d{}_{;d} - \eta^{abcd}\left(\omega_{bd} - \frac{1}{c^2}\dot{u}_{[b} u_{d]}\right) H_c E_a - \eta^{abcd} u_b H_c E_{a;d} \\ &= -cS^d{}_{;d} - \eta^{abcd}\omega_{bd} E_a H_c - \frac{1}{c} S^b \dot{u}_b - \eta^{abcd} u_b H_c E_{a;d}. \end{aligned}$$

The term

$$\eta^{abcd}\omega_{bd} E_a H_c = \eta^{abcd}\eta_{bdrs} u^r \omega^s E_a H_c = 2\left(\delta^a_r \delta^c_s - \delta^c_r \delta^a_s\right) u^r \omega^s E_a H_c = 0.$$

Replacing in (13.224) we obtain the final result:

$$k = -\left(\dot{\varepsilon} + \frac{2}{3}\theta\varepsilon\right) - \frac{1}{\varepsilon E^2}\left[\frac{1}{2}\varepsilon^2 (E^2)^{\cdot} - \varepsilon^2 \sigma_{ab} E^a E^b + S^d{}_{;d} + \frac{1}{c}\eta^{abcd} u_b H_c E_{a;d}\right].$$

13.14 The Generalized Ohm's Law

Ohm's Law concerns the current due to the motion of a charge in a medium in which there exists an electromagnetic field. The standard form of this law in a homogeneous and isotropic medium is

$$j^\mu = kE^\mu, \tag{13.225}$$

where the scalar quantity k is the electric conductivity of the medium and $\mathbf{E}$ is the electric field. This expression of Ohm's Law is not general in Newtonian Physics because it takes account only of the conduction and the convection currents, whereas it is known that in a conducting medium there are more types of electric current. One such current is the *Hall current*.

In Newtonian Physics Ohm's Law which incorporates the conduction current and the Hall current is defined by the following relation:

$$\mathbf{j} = k\mathbf{E} + \lambda \mathbf{j} \times \mathbf{B}. \tag{13.226}$$

The new coefficient λ is called the *transverse conductivity*.

In this section we determine the relativistic form of Ohm's Law when the Hall current is taken into consideration.

The technique we follow in our calculations is a good working example of how one can transfer an expression of Newtonian Physics into a corresponding expression of Relativistic physics. This technique consists of two steps:

a. Write the Newtonian equation in tensor form.
b. Develop a correspondence between the Newtonian and the relativistic tensors and transfer this expression in covariant form in Special Relativity. There is a (small) possibility that the relativistic expression is not determined uniquely from the corresponding Newtonian, hence before one accepts the result it is advisable to examine its physical significance.

The tensor form of the Newtonian expression (13.226) is

$$j^\mu = kE^\mu + \lambda \varepsilon^{\mu\nu\rho} j_\nu B_\rho. \tag{13.227}$$

We note that the current j^μ is involved in two terms, therefore we have to solve this relation in terms of j^μ. We multiply with the Levi-Civita antisymmetric tensor and get

$$\varepsilon_{\mu\sigma\tau} j^{\mu} = \varepsilon_{\mu\sigma\tau} k E^{\mu} + \lambda \varepsilon_{\mu\sigma\tau} \varepsilon^{\mu\nu\rho} j_{\nu} B_{\rho}.$$

The term (see (13.105))

$$\varepsilon_{\mu\sigma\tau} \varepsilon^{\mu\nu\rho} = \delta^{\nu}_{\sigma} \delta^{\rho}_{\tau} - \delta^{\rho}_{\sigma} \delta^{\nu}_{\tau},$$

thus

$$\varepsilon_{\mu\sigma\tau} j^{\mu} = \varepsilon_{\mu\sigma\tau} k E^{\mu} + \lambda (j_{\sigma} B_{\tau} - j_{\tau} B_{\sigma}).$$

We multiply with the magnetic field B^{τ} and find the expression

$$\varepsilon_{\mu\sigma\tau} j^{\mu} B^{\tau} = \varepsilon_{\mu\sigma\tau} k E^{\mu} B^{\tau} - \lambda \left[(\mathbf{j} \cdot \mathbf{B}) B_{\sigma} - \mathbf{B}^2 j_{\sigma} \right]. \tag{13.228}$$

Replacing in the original equation (13.227) we find

$$\frac{1}{\lambda} (j_{\sigma} - k E_{\sigma}) = k \varepsilon_{\mu\sigma\tau} E^{\mu} B^{\tau} + \lambda \left[(\mathbf{j} \cdot \mathbf{B}) B_{\sigma} - \mathbf{B}^2 j_{\sigma} \right]. \tag{13.229}$$

We note that in this expression the current appears in the inner product $\mathbf{j} \cdot \mathbf{B}$. We multiply (13.229) with B_{σ} and find

$$\mathbf{j} \cdot \mathbf{B} = k (\mathbf{E} \cdot \mathbf{B}). \tag{13.230}$$

Replacing we get the required Newtonian tensor expression[26]:

$$(1 + \lambda^2 \mathbf{B}^2) j_{\mu} = k E_{\mu} - \lambda k \varepsilon_{\mu\sigma\tau} E^{\sigma} B^{\tau} + \lambda^2 k (\mathbf{E} \cdot \mathbf{B}) B_{\mu}. \tag{13.231}$$

Having found the tensor form of the law in Newtonian Physics we continue with its relativistic generalization. The first step to take is to find the corresponding four-vectors. For the current we have the four-current J^a. Concerning the coordinate system we consider a relativistic observer u^a, who interacts with (observes) the electromagnetic field. We 1+3 decompose J^a wrt u^a and get

$$J^a = \frac{1}{c^2} \rho u^a + h^a{}_b J^b \tag{13.232}$$

where ρ is the charge density as measured by the observer u^a. Subsequently

[26] A proof using the classical vector calculus is the following. We take the cross and the inner product of (13.226) with $\mathbf{B}$:

$$\mathbf{j} \times \mathbf{B} = k(\mathbf{E} \times \mathbf{B}) + \lambda (\mathbf{j} \cdot \mathbf{B}) \mathbf{B} - \lambda B^2 \mathbf{j}, \qquad \mathbf{j} \cdot \mathbf{B} = k(\mathbf{E} \cdot \mathbf{B}).$$

Subsequently we replace in (13.226) and get the required expression.

a. We consider the correspondence:

$$\varepsilon_{\mu\sigma\tau} \to \frac{1}{c}\eta_{abcd}u^d$$

b. We identify the spatial part $h^a{}_b J^b$ with the three-current j_μ which we calculated above.

Finally we obtain the required[27] relativistic form of the *generalized Ohm's Law* :

$$J^a = \frac{\rho}{c^2}u^a + \frac{1}{(1+\lambda^2 B^c B_c)}\left[kE^a + \frac{1}{c}\lambda k\eta^{abcd}E_b u_c B_d + \lambda^2 k(E^c B_c)B^a\right]. \tag{13.233}$$

Equation (13.233) gives the four-current in terms of the (relativistic) electric and magnetic fields and takes into account the conduction current as well as the Hall effect.

13.15 The Energy Momentum Tensor of the Electromagnetic Field

Consider an electromagnetic field described by the tensors F^{*ab}, K_{ab} which satisfy the field equations (13.149), (13.150):

$$K^{ab}{}_{,b} = J^a, \tag{13.234}$$

$$F^{*ab}{}_{,b} = 0, \tag{13.235}$$

where J^a is the four-current density vector and $F^{*ab} = \frac{1}{2}\eta^{abcd}F_{cd}$ is the dual bivector of F^{ab}. The four-force on the current J^a due to the electromagnetic field is

$$F^a = F^{ab}J_b = F^{ab}K_{b..,c}^{\ \ c} = \frac{1}{c}\left(F^{ab}K_b{}^c\right)_{,c} - F^{ab}{}_{,c}K_b{}^c.$$

We consider the tensor

$$T_{ab} = -\frac{1}{c}\left[F_{ac}K^c_{.b} + \frac{1}{4}g_{ab}\left(F_{cd}K^{cd}\right)\right] \tag{13.236}$$

and compute its divergence. We have

$$\begin{aligned} T_a{}^b{}_{,b} &= -\frac{1}{c}\left[\left(F_{ac}K^{cb}\right)_{,b} + \frac{1}{4}\left(F_{cd}K^{cd}\right)_{,a}\right] \\ &= -\frac{1}{c}F_a - \frac{1}{c}F^{ab}_{,c}K_b{}^c - \frac{1}{4c}\left(F_{cd}K^{cd}\right)_{,a}. \end{aligned} \tag{13.237}$$

[27] See *Bekenstein and Oron Phys. Rev. D.* ***18****, 1809* (1978).

The term

$$\frac{1}{c}F^{ab}{}_{,c}K_b{}^c = \frac{1}{2c}\left[F^a{}_{b,c} - F^a{}_{c,b}\right]K^{bc} = \frac{1}{2c}K^{bc}\left[F_{ab,c} + F_{ca,b}\right]$$
$$= -\frac{1}{2c}K^{bc}F_{bc,a} \tag{13.238}$$

where in the last step we have used Maxwell equation in the form (13.151).

We specialize our study to a homogeneous and isotropic material. In this case

$$D^a = \varepsilon E^a \Rightarrow \frac{1}{c}K_{ab}u^b = \varepsilon F_{ab}u^b \Leftrightarrow \frac{1}{c}K_{a0} = \varepsilon F_{a0}, \tag{13.239}$$

$$B^a = \mu H^a \Rightarrow \eta_{abcd}\left(\frac{1}{c}F^{cd} - \mu K^{cd}\right)u^b \Leftrightarrow F_{b\nu} = \mu c K_{b\nu}. \tag{13.240}$$

Replacing in (13.238) we find

$$\begin{aligned}
-\frac{1}{2c}K^{bc}F_{bc,a} &= -\frac{1}{2c}K^{b0}F_{b0,a} - \frac{1}{2c}K^{b\mu}\;F_{b\mu,a}\\
&= -\frac{1}{2c}K^{b0}\frac{1}{c\varepsilon}K_{b0,a} - \frac{1}{2c}K^{b\mu}\mu c K_{b\mu,a}\\
&= -\frac{1}{4c}\left[\frac{1}{c\varepsilon}\left(K^{b0}K_{b0}\right)_{,a} + \mu c\left(K^{b\nu}K_{b\nu}\right)_{,a}\right]\\
&= -\frac{1}{4c}\left[\left(K^{b0}F_{b0}\right)_{,a} + \left(K^{b\nu}F_{b\nu}\right)_{,a}\right]\\
&= -\frac{1}{4c}\left(K^{bc}F_{bc}\right)_{,a}.
\end{aligned}$$

Replacing in (13.237) we obtain the final result

$$T^{ab}{}_{,b} = \frac{1}{c}F_a \tag{13.241}$$

that is, for a homogeneous and isotropic material the divergence of the tensor T^{ab} is the four-force on the current J^a. Because the vacuum is a special homogeneous and isotropic material this relation holds also in vacuum. However, there is a difference between the vacuum and a homogeneous and isotropic material. Indeed in vacuum the tensor T^{ab} is symmetric whereas for a homogeneous and isotropic material this tensor is not symmetric.

Let us compute the expression of T^{ab} in terms of the vector fields of the electromagnetic field in a homogeneous and isotropic material. The invariant

$$F^{ab}K_{ab} = \left[\frac{1}{c^2}(-E^a u^b + E^b u^a) + \frac{1}{c}\eta^{abcd}B_c u_d\right]$$

$$
\begin{aligned}
&\times \left[\frac{1}{c}\left(-D_a u_b + D_b u_a\right) + \frac{1}{c^2}\eta_{abrs} H^r u^s\right] \\
&= \frac{2}{c^3}(E^a D_a)(u^b u_b) + \frac{1}{c^3}\eta^{abcd}\eta_{abrs} B_c u_d H^r u^s \\
&= -\frac{2}{c}(E^a D_a) - \frac{2}{c^3}\left(\delta^c_r \delta^d_s - \delta^d_r \delta^c_s\right) B_c u_d H^r u^s \\
&= -\frac{2}{c}(E^a D_a) - \frac{2}{c^3}(B_c H^c)(u_d u^d) \\
&= \frac{2}{c}\left[B_a H^a - E^a D_a\right].
\end{aligned} \tag{13.242}
$$

The term

$$
\begin{aligned}
F_{ac} K^c_{.b} &= \left[\frac{1}{c^2}\left(-E_a u_c + E_c u_a\right) + \frac{1}{c}\eta_{acde} B^d u^e\right] \\
&\qquad \left[\frac{1}{c}\left(-D^c u_b + D_b u^c\right) + \frac{1}{c^2}\eta^c{}_{brs} H^r u^s\right] \\
&= \frac{1}{c^3}\left[-E_a D_b(-c^2) - (E_c D^c) u_a u_b\right] \\
&\quad - \frac{1}{c^4}\eta_{bcrs} E^c H^r u^s u_a - \frac{1}{c^2}\eta_{acde} D^c B^d u^e u_b \\
&\quad + \frac{1}{c^3}\eta_{acde}\eta^{cfrs}\eta_{fb} H_r u_s B^d u^e .
\end{aligned}
$$

The term

$$
\begin{aligned}
\eta_{acde}\eta^{cfrs}\eta_{fb} H_r u_s B^d u^e &= -\eta_{cade}\eta^{cfrs}\eta_{fb} H_r u_s B^d u^e \\
&= \left[\delta^f_a \delta^r_d \delta^s_e - \delta^f_a \delta^s_d \delta^r_e + \delta^r_a \delta^s_d \delta^f_e - \delta^r_a \delta^f_d \delta^s_e + \delta^s_a \delta^f_d \delta^r_e - \delta^s_a \delta^r_d \delta^f_e\right] \eta_{fb} H_r u_s B^d u^e \\
&= \eta_{ab}(H_r B^r)(-c^2) - H_a B_b(-c^2) - (H_r B^r) u_a u_b \\
&= c^2\left[-h_{ab}(H_c B^c) + H_a B_b\right].
\end{aligned}
$$

We introduce the Poynting four-vector and the polarization four-vector by the formulae

$$
S_a = \frac{1}{c}\eta_{abcd} E^b H^c u^d, \tag{13.243}
$$

$$
P_a = \frac{1}{c}\eta_{abcd} D^b B^c u^d \tag{13.244}
$$

and have

$$
F_{ac} K^c_{.b} = \frac{1}{c}\left[E_a D_b + H_a B_b - \frac{1}{c^2}(E_c D^c) u_a u_b - h_{ab}(H_c B^c) - \frac{1}{c^3} S_b u_b - P_a u_b\right].
$$

Adding these results we find

$$
\begin{aligned}
T_{ab} &= -\frac{1}{c}\left[F_{ac}K^{c}{}_{b} + \frac{1}{4}\eta_{ab}\left(F_{cd}K^{cd}\right)\right] \\
&= -\frac{1}{c^2}\left[E_a D_b + H_a B_b - \frac{1}{c^2}(E_c D^c)u_a u_b - h_{ab}(H_c B^c) - \frac{1}{c^3}S_b u_a - P_a u_b\right] \\
&- \frac{1}{2c^2}\eta_{ab}\left[B_c H^c - E^c D_c\right].
\end{aligned}
$$

This can be written differently as follows:

$$
\begin{aligned}
T_{ab} = -\frac{1}{c^2}\Bigg[&(E_a D_b + H_a B_b) - \frac{1}{c^2}(E_c D^c)u_a u_b - \eta_{ab}(H_c B^c) - \frac{1}{c^2}(H_c B^c)u_a u_b \\
&+ \frac{1}{2}\eta_{ab}\left[B_c H^c - E^c D_c\right] - \frac{1}{c^3}S_a u_b - P_a u_b\Bigg] \\
= -\frac{1}{c^2}\Bigg[&(E_a D_b + H_a B_b) - \frac{1}{c^2}\left(H_c B^c + E_c D^c\right)u_a u_b \\
&- \frac{1}{2}\eta_{ab}\left(H_c B^c + E_c D^c\right) - \frac{1}{c^3}S_a u_b - P_a u_b\Bigg],
\end{aligned}
$$

hence

$$
T_{ab} = \frac{1}{c^2}\left[-(E_a D_b + H_a B_b) + \frac{1}{2}\left(h_{ab} + \frac{1}{c^2}u_a u_b\right)(H_c B^c + E_c D^c) + \frac{1}{c^3}S_b u_a + P_a u_b\right]. \tag{13.245}
$$

This tensor has the following irreducible parts wrt the four-velocity u^a (see (12.11):

$$
w = T_{ab}u^a u^b = \frac{1}{2}\left(H_c B^c + E_c D^c\right), \tag{13.246}
$$

$$
S_a = -c h_a^c T_{bc} u^b, \tag{13.247}
$$

$$
P_a = -h_a^b T_{bc} u^c, \tag{13.248}
$$

$$
\Pi_{ab} = h_a^c h_b^d T_{cd} = -\frac{1}{c^2}\left[E_a D_b + H_a B_b - \frac{1}{2}h_{ab}\left(H_c B^c + E_c D^c\right)\right]. \tag{13.249}
$$

It is obvious that the tensor T_{ab} is not symmetric. In order to find the physical significance of each of the irreducible parts we consider the proper frame of the observer, Σ^+ say, in which the four-velocity $u^a = c\delta_0^a$. Let us assume that in this frame the electromagnetic field is given by the three-vectors $\mathbf{E}$, $\mathbf{B}$, $\mathbf{D}$, $\mathbf{H}$ so that from (13.114),(13.115),(13.129),(13.130) we have

$$
E^a = (0, \mathbf{E})_{\Sigma^+},\ B^a = (0, \mathbf{B})_{\Sigma^+},\ D^a = (0, \mathbf{D})_{\Sigma^+},\ H^a = (0, \mathbf{H})_{\Sigma^+},
$$

whereas from (13.243), (13.244) we compute[28] for the spacelike four-vectors S^a, P^a:

$$S^a = (0, \frac{1}{c}\eta^a_{.bc0} E^b H^c u^0) = (0, \mathbf{E} \times \mathbf{H})_{\Sigma^+}, \tag{13.250}$$

$$P^a = (0, \frac{1}{c}\eta^a_{.bc0} D^b B^c u^0) = (0, \mathbf{D} \times \mathbf{B})_{\Sigma^+}. \tag{13.251}$$

Replacing the expressions of E^a, B^a, D^a, H^a in (13.246) and (13.249) we find

$$w = \frac{1}{2}(\mathbf{H} \cdot \mathbf{B} + \mathbf{E} \cdot \mathbf{D}), \tag{13.252}$$

$$\Pi_{ab} = -\frac{1}{c^2}\left[E_\mu D_\nu + H_\mu B_\nu - w\delta_{\mu\nu}\right]\delta^\mu_a \delta^\nu_b, \tag{13.253}$$

where in (13.253) we have assumed a Cartesian coordinate system. The quantity w expresses the energy density of the electromagnetic field in Σ^+, the vectors S^a, P^a measure the momentum transfer, and the tensor Π_{ab} is the stress tensor for the electromagnetic field when considered as a "fluid".

Exercise 72 Show that for a homogeneous and isotropic material of dielectric constant ε and magnetic permeability μ

i. The polarization vector:

$$P^a = \varepsilon\mu S^a. \tag{13.254}$$

ii. The energy density:

$$w = \frac{1}{2}(\mu H^2 + \varepsilon E^2). \tag{13.255}$$

iii. The stress tensor:

$$\Pi_{ab} = -\frac{1}{c^2}\left[\varepsilon E_\mu E_\nu + \mu H_\mu H_\nu - w\delta_{\mu\nu}\right]\delta^\mu_a \delta^\nu_b. \tag{13.256}$$

iv. The energy momentum tensor:

$$\begin{aligned} T_{ab} = \frac{1}{c^2}\Bigg[&-(\varepsilon E_a E_b + \mu H_a H_b) + \frac{1}{2}\left(h_{ab} + \frac{1}{c^2}u_a u_b\right)\left(\mu H^2 + \varepsilon E^2\right) \\ &+ \frac{1}{c^3} S_b u_a + \varepsilon\mu S_a u_b\Bigg]. \end{aligned} \tag{13.257}$$

[28] Recall that $u^0 = c$, $u_0 = -c$.

Exercise 73 The energy conservation law for charges and the electromagnetic field in a homogeneous and isotropic material ($c = 1$).

i. Consider the vector identity

$$\operatorname{div}(\mathbf{A} \times \mathbf{B}) = \mathbf{A} \cdot (\nabla \times \mathbf{B}) + \mathbf{B} \cdot (\nabla \times \mathbf{A}) \tag{13.258}$$

and using Maxwell equations for a general medium show that

$$\operatorname{div}(\mathbf{E} \times \mathbf{H}) = -\left(\frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{H} + \frac{\partial \mathbf{D}}{\partial t} \cdot \mathbf{E}\right) - \mathbf{j} \cdot \mathbf{E}. \tag{13.259}$$

ii. Show that for a homogeneous and isotropic medium of dielectric constant ε and magnetic permeability μ

$$\operatorname{div} \mathbf{S} = -\frac{\partial w}{\partial t} - \mathbf{j} \cdot \mathbf{E}, \tag{13.260}$$

where $w = \frac{1}{2}(\mu H^2 + \varepsilon E^2)$ is the energy density of the electromagnetic field and $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ is the Poynting vector.

iii. Consider the current to be a conduction current $\mathbf{j} = \rho \mathbf{u}$ where ρ is the charge density and $\mathbf{u}$ is the velocity of the charges. Then $\mathbf{j} \cdot \mathbf{E} = \rho \mathbf{E} \cdot \mathbf{u} = \mathbf{F} \cdot \mathbf{u}$ where $\mathbf{F}$ is the force density on the charge. The quantity $\mathbf{F} \cdot \mathbf{u} = \frac{\partial T}{\partial t}$ where T is the kinetic energy density of the charges, so that (13.260) reads

$$\operatorname{div} \mathbf{S} = -\frac{\partial (w + T)}{\partial t}. \tag{13.261}$$

iv. Consider a volume V in which the charge has kinetic energy T and the electromagnetic field energy density w. Then integrate (13.261) to obtain

$$\int_V \frac{\partial (w + T)}{\partial t} dV = -\int_V \operatorname{div} \mathbf{S} dV.$$

Apply Gauss Theorem to write this equation as

$$w + T = -\oint \mathbf{S} \cdot d\boldsymbol{\sigma}. \tag{13.262}$$

In this form this equation expresses the conservation of energy for charges and the electromagnetic field. As a consequence of (13.262) we may interpret the Poynting vector as the flux of energy per unit time through a unit area oriented normally to the Poynting vector. However, this interpretation is not fully justified by Maxwell equations because if we add to the vector $\mathbf{S}$ another vector $\mathbf{S}'$ satisfying the condition $\operatorname{div} \mathbf{S}' = 0$ the above result does not change.

Exercise 74 The momentum conservation law for charges and the electromagnetic field in a homogeneous and isotropic material.

i. Consider the following identity of vector calculus:

$$\mathbf{grad}(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} + \mathbf{A}\times\mathbf{curl}\,\mathbf{B} + \mathbf{B}\times\mathbf{curl}\,\mathbf{A} \tag{13.263}$$

and take $\mathbf{A} = \mathbf{B}$ to find

$$\frac{1}{2}\nabla\mathbf{A}^2 = (\mathbf{A}\cdot\nabla)\mathbf{A} + \mathbf{A}\times\mathbf{curl}\,\mathbf{A}. \tag{13.264}$$

Show next that the term

$$(\mathbf{A}\cdot\nabla)\mathbf{A} = \frac{\boldsymbol{\partial}}{\boldsymbol{\partial x^a}}(\boldsymbol{A_a A_b})\hat{\mathbf{e}}_{\boldsymbol{b}} - \mathbf{A}\cdot\mathrm{div}\,\mathbf{A}, \tag{13.265}$$

where $\mathbf{A} = A_a\hat{\mathbf{e}}_a$ in the basis $\{\hat{\mathbf{e}}_a\}$. Conclude that identity (13.264) is written as

$$\mathbf{A}\cdot\mathrm{div}\,\mathbf{A} - \mathbf{A}\times\mathrm{curl}\,\mathbf{A} = \frac{\partial}{\partial x^a}\left(A_a A_b - \frac{1}{2}\mathbf{A}^2\delta_{ab}\right)\hat{\mathbf{e}}_b. \tag{13.266}$$

ii. Consider Maxwell equations for a general medium and show that

$$\mathbf{B}\times(\nabla\times\mathbf{H}) + \mathbf{D}\times(\nabla\times\mathbf{E}) = \mathbf{B}\times\mathbf{j} + \frac{\partial}{\partial t}(\mathbf{B}\times\mathbf{D}).$$

From Maxwell equations we also have $\mathrm{div}\,\mathbf{D} = \rho$, $\mathrm{div}\,\mathbf{B} = 0$ where ρ is the charge density. Use these and the last relation to show that for a homogeneous and isotropic material of dielectric constant ε and magnetic permeability μ

$$\frac{\partial}{\partial x^a}\left(E_a D_b + H_a B_b + \frac{1}{2}(\mathbf{H}\cdot\mathbf{B} + \mathbf{E}\cdot\mathbf{D})\delta_{ab}\right)\hat{\mathbf{e}}_b = \rho\mathbf{E} + \mathbf{j}\times\mathbf{B} + \mathbf{P},$$

where $\mathbf{P}$ is the polarization vector. The term $\rho\mathbf{E} + \mathbf{j}\times\mathbf{B} = \mathbf{F}$ where $\mathbf{F}$ is the Lorentz force density.

iii. Show that the last relation can be written as

$$\frac{\partial\Pi_{ab}}{\partial x^b}\hat{\mathbf{e}}_b = \mathbf{F} + \frac{\partial\mathbf{P}}{\partial t}, \tag{13.267}$$

where the tensor

$$\Pi_{ab} = E_a D_b + H_a B_b - w\delta_{ab}. \tag{13.268}$$

iv. Consider a volume V and integrate (13.267) over the volume V to find

$$\int_V \frac{\partial \Pi_{ab}}{\partial x^b}\hat{\mathbf{e}}_b dV = \int_V \mathbf{F} dV + \frac{\partial}{\partial t}\int_V \mathbf{P} dV. \tag{13.269}$$

The term $\int_V \mathbf{F} dV = \frac{\partial \mathbf{p}}{\partial t}$ where $\mathbf{p}$ is the linear momentum density of the charges enclosed in the volume V.[29] Write (13.269) in the form

$$\oint_S \frac{\partial \Pi_{ab}}{\partial x^b}\hat{\mathbf{e}}_b \cdot d\boldsymbol{\sigma} = \frac{\partial}{\partial t}\int_V (\mathbf{p} + \mathbf{P}) dV \tag{13.270}$$

and conclude that (a) the quantity $\frac{\partial \Pi_{ab}}{\partial x^b}\hat{\mathbf{e}}_b \cdot d\boldsymbol{\sigma}$ represents the force acting on an infinitesimal surface area $d\sigma$ normal to the vector $\hat{\mathbf{e}}_b$ and (b) the quantity $\mathbf{P}$ is the field momentum density.

v. Show that for a homogeneous and isotropic material the polarization $\mathbf{P} = \varepsilon\mu \mathbf{S} = \frac{n^2}{c}\mathbf{S}$ where n is the index of refraction of the medium. Note that in this case the Poynting vector has also the interpretation of the field momentum density. Furthermore

$$\Pi_{ab} = \varepsilon E_a E_b + \mu H_a H_b - w\delta_{ab},$$

i.e., Π_{ab} is symmetric. Note that even in this case the energy momentum tensor T_{ab} is *not symmetric* (it is symmetric only for vacuum).

The energy momentum tensor for the electromagnetic field we have considered is due to H. Minkowski. Its derivation is based on the assumption $T^{ab}_{;b} = \frac{1}{c}F^a$ and not to symmetry (i.e., $T^{ab} = T^{ba}$). Because in general we "assume" the energy momentum tensor to be symmetric, soon after Minkowski, M. Abraham suggested another energy momentum tensor which was similar to Minkowski – in fact it is based on the Minkowski energy momentum tensor – and is supposed to hold within a medium only (in the empty space they coincide). Both tensors are correct and still today there is a discussion going on as to which energy momentum tensor should be considered as more appropriate. Abraham uses the equation $T^{ab}_{;b} = \frac{1}{c}F^a$ to define the four-force, which inevitably is different from the Lorentz force $\frac{1}{c}F_{ab}u^b$.

Closing we should remark that the four-dimensional formulation of electromagnetism is not an academic luxury but a practical necessity because this formalism leads safely to correct and consistent results which can always be translated into practical working equations (i.e., written in terms of three-dimensional quantities) for a given LCF.

[29] The partial derivative $\frac{\partial}{\partial t}$ indicates that the volume V is comoving, that is does not change in time.

13.16 The Electromagnetic Field of a Moving Charge

The determination of the electromagnetic field produced by a moving charge in an LCF Σ is an important problem with many applications. The solution of this problem with Newtonian methods is difficult and the results cannot be checked reliably. On the contrary the solution within the relativistic formalism gives a complete and substantiated answer and exhibits the power and the usefulness of this formalism.

We consider a charge q with world line $\mathcal{Q}$ whose equation is $c^i(\tau)$, where τ is the proper time of the charge. We wish to determine the electromagnetic field due to the charge at proper time τ at the spacetime point P with coordinates x^i. Suppose that at the moment τ the position vector of the point P relative to the charge is R^i. Then (see Fig. 13.3)

$$R^i = x^i - c^i(\tau). \tag{13.271}$$

We make the following two assumptions:

1. The electromagnetic field created by the charge propagates with speed c.
   This implies that the point P is on the light cone of the point $c^i(\tau)$ of the world line. Therefore the position vector R^i is a null four-vector:

$$R^i R_i = 0. \tag{13.272}$$

2. The electromagnetic field created by the charge in the proper frame of the charge consists only of an electric field, which is spherically symmetric with center at the charge.
   This means that in the proper frame of the charge, Σ' say, the potentials of the electromagnetic field are (SI system of units)

$$\phi' = \frac{1}{4\pi\epsilon_0}\frac{q}{r'}, \quad \mathbf{A}' = \mathbf{0}.$$

Then the four-potential Ω^i in Σ' is given by

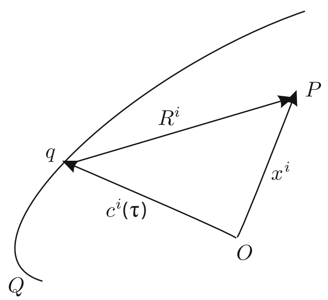

Fig. 13.3 The electromagnetic field of a moving charge

$$\Omega^i = \begin{pmatrix} \frac{1}{4\pi\epsilon_0}\frac{q}{cr'} \\ \mathbf{0} \end{pmatrix}_{\Sigma'}, \qquad \Omega_i = \left(-\frac{1}{4\pi\epsilon_0}\frac{q}{cr'}, \mathbf{0}\right)_{\Sigma'}.$$

Suppose that in Σ' the position vector R^i of the spacetime point P has components $R^i = \begin{pmatrix} ct' \\ \mathbf{r}' \end{pmatrix}_{\Sigma'}$. Because R^i is null we have

$$-c^2 t'^2 + \mathbf{r}'^2 = 0 \Rightarrow ct' = r', \quad \text{thus} \quad R^i = \begin{pmatrix} r' \\ \mathbf{r}' \end{pmatrix}_{\Sigma'},$$

where r' is the length of $\mathbf{r}'$. The four-velocity of the charge in Σ' is $u^i = \begin{pmatrix} c \\ \mathbf{0} \end{pmatrix}_{\Sigma'}$ and a simple calculation shows that Ω_i can be written covariantly as follows:

$$\Omega_i = \frac{1}{4\pi\epsilon_0}\frac{qu_i}{cD}, \tag{13.273}$$

where in order to save writing we have set $D = R^j u_j = -r'c$.

Having computed the four-potential we have practically solved the problem, because the tensor F_{ij} is given by

$$F_{ij} = -\Omega_{i,j} + \Omega_{j,i}.$$

In order to compute the derivative $\Omega_{i,j}$ at the point P we note that

$$\frac{\partial}{\partial x^i}_{|P} = \frac{\partial \tau}{\partial x^i}_{|c(\tau)}\frac{d}{d\tau} = \tau_i \frac{d}{d\tau}.$$

The quantity τ_i is computed from R^i as follows. From (13.271) we have $R_{i,j} = \eta_{ij} - u_i \tau_j$ so that

$$R^j R_j = 0 \Rightarrow R^j R_{j,i} = 0 \Rightarrow R^j(\eta_{ij} - u_j \tau_i) = 0 \Rightarrow \tau_j = \frac{R_j}{D}.$$

Accordingly the derivative of the four-velocity

$$u_{i,j} = \tau_j \frac{du_i}{d\tau} = \tau_j \dot{u}_i = \frac{1}{D} R_j \dot{u}_i$$

and that of the quantity D

$$D_{,j} = u_j + \frac{c^2}{D}\left(1 + \frac{1}{c^2} R^k \dot{u}_k\right) R_j.$$

Using these expressions we compute

$$\Omega_{i,j} = -\frac{1}{4\pi\epsilon_0}\frac{q}{cD^2}u_i u_j - \frac{1}{4\pi\epsilon_0}\frac{qc}{D^3}S_i R_j,$$

where

$$S_i = \left(1 + \frac{1}{c^2}R^k \dot{u}_k\right)u_i - \frac{D}{c^2}\dot{u}_i. \tag{13.274}$$

Finally

$$F_{ij} = -\Omega_{i,j} + \Omega_{j,i} = \frac{1}{4\pi\epsilon_0}\frac{qc}{D^3}\left(S_i R_j - S_j R_i\right). \tag{13.275}$$

The tensor F_{ij} contains all the information concerning the electromagnetic field due to the charge. Let us see why.

13.16.1 The Invariants

From (13.59) we have for the invariant X

$$X = -\frac{1}{2}F_{ij}F^{ij} = -\frac{1}{2}\frac{1}{(4\pi\epsilon_0)^2}\frac{q^2c^2}{D^6}\left[S_i R_j - S_j R_i\right]\left[S^i R^j - S^j R^i\right]$$

$$= -\frac{1}{(4\pi\epsilon_0)^2}\frac{q^2c^2}{D^6}\left[S^2R^2 - (S^i R_i)^2\right] = \frac{1}{(4\pi\epsilon_0)^2}\frac{q^2c^2}{D^6}\left(S^i R_i\right)^2$$

(because $R^2 = 0$). But

$$\left(S^i R_i\right)^2 = D^2\left(1 + \frac{1}{c^2}R^k \dot{u}_k\right)^2 - \frac{D^2}{c^4}\left(R^k \dot{u}_k\right)^2 = D^2,$$

hence

$$X = \frac{1}{(4\pi\epsilon_0)^2}\frac{q^2c^2}{D^4}. \tag{13.276}$$

Similarly from (13.59) we have for the invariant Y

$$4Y = -\frac{1}{2}\eta_{ijkl}F^{ij}F^{kl} = \frac{1}{(4\pi\eta_0)^2}\frac{2q^2c^2}{D^6}\eta_{ijkl}S^i R^j S^k R^l = 0.$$

From the values of the invariants we conclude that the electromagnetic field produced by a moving charge in an LCF Σ, say, either consists of an electric field **E** and a magnetic field **B** which are normal to each other and with different strength, or an electric field only (as it is the case in Σ').

13.16.2 *The Fields* $E^i,\ B^i$

The fields $E^i,\ B^i$ are computed from the tensor F_{ij} using the relations (13.110). We find

$$E_i = F_{ij}u^j = \frac{1}{4\pi\epsilon_0}\frac{qc}{D^3}\left[S_i D - (S_j u^j)R_i\right].$$

But

$$S_j u^j = -c^2\left(1 + \frac{1}{c^2}R^k \dot{u}_k\right),$$

therefore

$$E_i = \frac{1}{4\pi\epsilon_0}\frac{qc}{D^3}\left[\left(1 + \frac{1}{c^2}R^k \dot{u}_k\right)\left(Du_i + c^2 R_i\right) - \frac{D^2}{c^2}\dot{u}_i\right]. \tag{13.277}$$

For the magnetic field we have

$$B_i = \frac{1}{2c}\eta_{ijkl}F^{jk}u^l = \frac{1}{2c}\frac{1}{4\pi\epsilon_0}\frac{2qc}{D^3}\eta_{ijkl}S^j R^k u^l = \frac{1}{4\pi\epsilon_0}\frac{q}{D^2c^2}\eta_{ijkl}R^j \dot{u}^k u^l. \tag{13.278}$$

It is easy to prove that the fields $E^i,\ B^i$ are (Lorentz!) perpendicular:

$$E^i B_i = 0.$$

This result was expected because in the proper frame of the charge the fields $E^i,\ B^i$ coincide with the fields $\mathbf{E}^+,\ \mathbf{B}^+$, whose inner product vanishes, therefore in this system $E^i B_i = 0$. However, this relation is covariant hence it is valid in all LCFs.

13.16.3 *The Liénard–Wiechert Potentials and the Fields* E, B

Consider an arbitrary LCF Σ in which the charge has velocity **u** and acceleration **a**. We calculate in Σ the scalar and the vector potentials as well as the fields **E**, **B**.

In the LCF Σ we have the following components of the involved four-vectors:

$$R^i = \begin{pmatrix} r \\ \mathbf{r} \end{pmatrix}_\Sigma, u^i = \begin{pmatrix} \gamma c \\ \gamma \mathbf{u} \end{pmatrix}_\Sigma, \dot{u}^i = \begin{pmatrix} a_0 c \\ a_0 \mathbf{u} + \gamma^2 \mathbf{a} \end{pmatrix}_\Sigma,$$

where $\dot{u}^i$ is the four-acceleration of the charge and $a_0 = \gamma\dot{\gamma}$. We compute

$$D = R^i u_i = -\gamma(cr - \mathbf{r}\cdot\mathbf{u}),$$
$$R^k \dot{u}_k = -a_0(cr - \mathbf{r}\cdot\mathbf{u}) + \gamma^2(\mathbf{r}\cdot\mathbf{a}) = \frac{a_0 D}{\gamma} + \gamma^2(\mathbf{r}\cdot\mathbf{a}).$$

The four-potential is given from the covariant expression (13.273)

$$\Omega_i = \frac{1}{4\pi\epsilon_0}\frac{q u_i}{cD}.$$

The zeroth component and the spatial part of the four-potential are the scalar and the vector potentials, respectively. We call these potentials the *Liénard–Wiechert potentials*. In order to compute these potentials it is enough to calculate the components of $\Omega^i = \frac{1}{4\pi\epsilon_0}\frac{qu^i}{Dc}$. Using the above results we find easily that

$$\phi = -\frac{1}{4\pi\epsilon_0}\frac{q}{r\left(1 - \frac{u_r}{c}\right)}, \qquad \mathbf{A} = -\frac{1}{4\pi\epsilon_0}\frac{q\mathbf{u}}{c^2 r\left(1 - \frac{u_r}{c}\right)}, \tag{13.279}$$

where $u_r = \frac{\mathbf{u}\cdot\mathbf{r}}{r}$ is the component of the velocity in the direction $\mathbf{r}$.

From the Liénard–Wiechert potentials we cannot compute (directly) the fields $\mathbf{E}$, $\mathbf{B}$ from the relations

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{E} = -\nabla\phi + \frac{\partial \mathbf{A}}{\partial t}$$

and we must work with the relativistic formalism. From the components of the four-vectors in Σ we compute[30]

$$S_i = \left(-\frac{\gamma}{c}\left[c^2 + \gamma^2(\mathbf{r}\cdot\mathbf{a})\right], \frac{\gamma}{c^2}\left\{\left[c^2 + \gamma^2(\mathbf{r}\cdot\mathbf{a})\right]\mathbf{u} + \gamma^2(cr - \mathbf{r}\cdot\mathbf{u})\mathbf{a}\right\}\right).$$

For the electric field $\mathbf{E}$ we have (see footnote 14)

[30] S_i is defined in (13.274).

$$\frac{1}{c}\mathbf{E} = F_{0\mu} = -\frac{1}{4\pi\epsilon_0}\frac{qc}{\gamma^3(cr-\mathbf{r}\cdot\mathbf{u})^3}\left(S_0R_\mu - R_0S_\mu\right)$$
$$= -\frac{1}{4\pi\epsilon_0}\frac{qc}{\gamma^3(cr-\mathbf{r}\cdot\mathbf{u})^3}\left[-\frac{\gamma}{c}\left[c^2+\gamma^2(\mathbf{r}\cdot\mathbf{a})\right]\mathbf{r}\right.$$
$$\left.-(-r)\frac{\gamma}{c^2}\left\{\left[c^2+\gamma^2(\mathbf{r}\cdot\mathbf{a})\right]\mathbf{u}+\gamma^2(cr-\mathbf{r}\cdot\mathbf{u})\mathbf{a}\right\}\right].$$

After some simple algebra we find

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{c\gamma^2(cr-\mathbf{r}\cdot\mathbf{u})^3}\left\{\left[c^2+\gamma^2(\mathbf{r}\cdot\mathbf{a})\right](c\mathbf{r}-r\mathbf{u})-\gamma^2 r(cr-\mathbf{r}\cdot\mathbf{u})\mathbf{a}\right\}. \tag{13.280}$$

This relation can be written in a form which singles out completely the part of the field which is due to acceleration. Indeed using the identity $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B}-(\mathbf{A}\cdot\mathbf{B})\mathbf{C}$ we write the electric field as follows:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{(cr-\mathbf{r}\cdot\mathbf{u})^3}\left[\frac{c^2}{\gamma^2}(c\mathbf{r}-r\mathbf{u})+\mathbf{r}\times\left[(c\mathbf{r}-r\mathbf{u})\times\mathbf{a}\right]\right]. \tag{13.281}$$

For the magnetic field we have

$$\mathbf{B} = (-F_{23}, -F_{31}, -F_{12})$$
$$= \frac{qc/4\pi\epsilon_0}{\gamma^3(cr-\mathbf{r}\cdot\mathbf{u})^3}(S_2R_3-S_3R_2, S_3R_1-S_1R_3, S_1R_2-R_1S_2)$$
$$= \frac{qc/4\pi\epsilon_0}{\gamma^3(cr-\mathbf{r}\cdot\mathbf{u})^3}(\mathbf{S}\times\mathbf{R})$$
$$= \frac{qc/4\pi\epsilon_0}{\gamma^3(cr-\mathbf{r}\cdot\mathbf{u})^3}\left(\frac{\gamma}{c^2}\left\{\left[c^2+\gamma^2(\mathbf{r}\cdot\mathbf{a})\right]\mathbf{u}+\gamma^2(cr-\mathbf{r}\cdot\mathbf{u})\mathbf{a}\right\}\times\mathbf{r}\right)$$
$$= \frac{1}{cr}(\mathbf{r}\times\mathbf{E}),$$

that is

$$\mathbf{B} = \frac{1}{c}\frac{\mathbf{r}}{r}\times\mathbf{E}. \tag{13.282}$$

A special case with much interest is the motion of a charge with constant velocity ($\mathbf{a}=0$), e.g., the motion of free electrons within a conductor. In this case relations (13.281) and (13.282) give

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{qc^3}{\gamma^2(c^2t-\mathbf{r}\cdot\mathbf{u})^3}(\mathbf{r}-\mathbf{u}t), \tag{13.283}$$

$$\mathbf{B} = \frac{1}{4\pi\epsilon_0}\frac{qc}{\gamma^2(c^2t-\mathbf{r}\cdot\mathbf{u})^3}(\mathbf{u}\times\mathbf{r}), \tag{13.284}$$

where we have used that $r = ct$.

These expressions have an easier physical interpretation if they are written in terms of the angle θ of the vectors $\mathbf{u}$, $\mathbf{r}$ in Σ. Indeed we have

$$cr - \mathbf{r}\cdot\mathbf{u} = cr - ru\cos\theta = cr(1-\beta\cos\theta),$$

hence ($\epsilon_0\mu_0 = 1/c^2$)

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q\mathbf{r}'}{\gamma^2 r^3(1-\beta\cos\theta)^3}, \tag{13.285}$$

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{q(\mathbf{u}\times\mathbf{r}')}{\gamma^2 r^3(1-\beta\cos\theta)^3}, \tag{13.286}$$

where $\mathbf{r}' = \mathbf{r} - \mathbf{u}t$ is the vector in the three-space of Σ connecting the charge with the point in three-space where the electromagnetic field is created.

In order to draw geometric conclusions from (13.285) we have to express the rhs in terms of the vector $\mathbf{r}'$. If we call ϕ the angle between the vectors $\mathbf{r}'$, $\mathbf{u}$ in Σ we have $|\mathbf{r}'_{\|}| = r'\cos\phi$, $|\mathbf{r}'_{\perp}| = r'\sin\phi$ (see Fig. 13.4).

We note that $ut = ur/c = \beta r$ and calculate the quantity

$$\begin{aligned}
\gamma^2\mathbf{r}'^2_{\|} + \mathbf{r}'^2_{\perp} &= \gamma^2\left[\left(\frac{\mathbf{r}\cdot\mathbf{u}}{\mathbf{u}} - ut\right)^2 + \frac{1}{\gamma^2}\mathbf{r}^2_{\perp}\right]\\
&= \gamma^2\left[(r\cos\theta - ut)^2 + (1-\beta^2)r^2\sin^2\theta\right]\\
&= \gamma^2 r^2\left[1 + \beta^2(1-\sin^2\theta) - 2\beta\cos\theta\right]\\
&= \gamma^2 r^2(1-\beta\cos\theta)^2.
\end{aligned}$$

But in the proper frame of the charge

$$\begin{aligned}
\gamma^2\mathbf{r}'^2_{\|} + \mathbf{r}'^2_{\perp} &= \gamma^2 r'^2\cos^2\phi + r'^2\sin^2\phi\\
&= \gamma^2 r'^2[\cos^2\phi + (1-\beta^2)\sin^2\phi]\\
&= \gamma^2 r'^2(1-\beta^2\sin^2\phi)
\end{aligned}$$

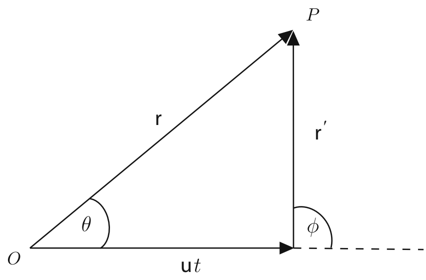

Fig. 13.4 Angles in the proper frame of the charge and in Σ

and finally

$$r(1-\beta\cos\theta) = r'\sqrt{1-\beta^2\sin^2\phi}.$$

The electric and the magnetic fields are written in terms of the elements of $\mathbf{r}'$:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q\mathbf{r}'}{\gamma^2 r'^3(1-\beta^2\sin^2\phi)^{3/2}}, \tag{13.287}$$

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{q\,\mathbf{u}\times\mathbf{r}'}{\gamma^2 r'^3(1-\beta^2\sin^2\phi)^{3/2}}. \tag{13.288}$$

From these relations we infer the following:

1. The electric field is not isotropic (consequently spherically symmetric) and (as we show below) its strength is largest normal to the direction of the velocity and takes the smallest value along the direction of the velocity. Consequently the lines of force of the electric field are denser in the plane perpendicular to the velocity.
2. The magnetic field which is produced from an electric current in Σ is normal to the plane defined by the current and the point at which we are looking for the field. Furthermore its strength is proportional to the value of the current ($\mathbf{j} = q\mathbf{u}$).
3. In the Newtonian limit $\gamma \approx 1$, $\beta \approx 0$, and also $ut \ll ct$ so that $r \approx r'$ and relations (13.285) and (13.286) reduce to

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q\mathbf{r}'}{r'^3}, \tag{13.289}$$

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{\mathbf{j}\times\mathbf{r}'}{r'^3}. \tag{13.290}$$

The first gives the electric field of the Newtonian approach and the second is the celebrated *Biot–Savart Law*.[31]

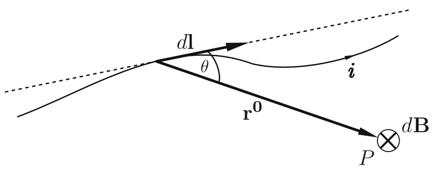

Fig. 13.5 Biot–Savart Law

[31] We recall that in the Newtonian approach the magnetic field of a current i satisfies two laws: the Ampére Law $\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0 i$ and the Biot–Savart Law $d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{i}{r^3}\,d\mathbf{l}\times\mathbf{r}$ where $d\mathbf{l}$ is an elementary length along the conductor and $\mathbf{r}$ is the point where one calculates the magnetic field (see Fig. 13.5). Ampére's Law is used in the cases where the magnetic field has high (geometric) symmetry whereas the Biot–Savart Law is used in more general cases in which the magnetic field is computed by integration along the conductor.

In order to study in depth the anisotropy of the electric field, which is a purely relativistic phenomenon, we write (13.287) as follows:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q\mathbf{r}'}{r'^3}\frac{1}{\gamma^2(1-\beta^2\sin^2\phi)^{3/2}}. \tag{13.291}$$

The first term in the rhs

$$\mathbf{E}(q) = \frac{1}{4\pi\epsilon_0}\frac{q\mathbf{r}'}{r'^3}$$

is the electric field which is created at the point P in Σ by the charge q. The second term is due to the effect of anisotropy and it is of the order β^2, hence absent in the Newtonian limit.

In order to estimate the effect of anisotropy we introduce the quantity $f(\beta,\phi) = \frac{1}{\gamma^2(1-\beta^2\sin^2\phi)^{\frac{3}{2}}}$ which we plot as a function of ϕ for various values of β. This plot is shown in Fig. 13.6 where the anisotropy of the quantity $f(\beta,\phi)$ in Σ and its dependence on the factor β (the speed of the charge in Σ) are apparent.

We note that when $\beta \to 0$ the curve tends to a straight line parallel to the x-axis, which means that in the Newtonian limit there is no dependence on the angle ϕ and the field becomes isotropic. For relativistic β the strength of the field in the equatorial plane tends to zero whereas it tends to infinity near the value $\phi = \pm\frac{\pi}{2}$ (δ-function).

In the following we discuss two examples of motion of a charge in an LCF and calculate the resulting electromagnetic field. The first example concerns uniform motion and the second uniform circular motion.

Example 73 A charge q is moving in an LCF Σ with constant velocity $\mathbf{u}$. If the charge is at the origin O of Σ at the moment $t = 0$ of Σ calculate the electromagnetic field at the point P of Σ.

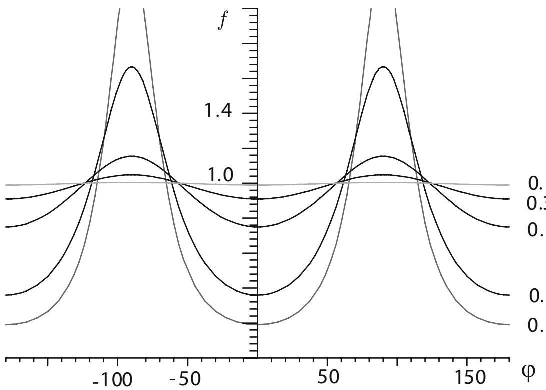

Fig. 13.6 The anisotropy of the electric field

Solution

Let $\mathbf{r}$ be the position vector of P in Σ. Because the electromagnetic field propagates with speed c, the field created by the charge when it was at the origin O of Σ will reach P at the moment $t = r/c$ of Σ. But then the charge will be at the position $\mathbf{u}t$ in Σ and the point P will have position vector wrt the charge $\mathbf{r}'_\Sigma = \mathbf{r} - \mathbf{u}t$.[32] Obviously

$$\mid \mathbf{r} \mid = r = ct.$$

In the proper frame Σ' of the charge the electromagnetic field has only electric field which is given from the relation

$$\mathbf{E}_{\Sigma'} = \frac{1}{4\pi\epsilon_0}\frac{q\mathbf{r}'_{\Sigma'}}{r'^3_{\Sigma'}}. \tag{13.292}$$

In order to calculate the electric field in Σ we consider the Lorentz transformation. However, we do not need to do that because we have already computed this field in (13.283), i.e.,

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{qc^3}{\gamma^2(cr - \mathbf{r}\cdot\mathbf{u})^3}(\mathbf{r} - \mathbf{u}t). \tag{13.293}$$

As an instructive exercise let us compute the electric field directly. We have in an obvious notation

$$\mathbf{E}_{\parallel\Sigma} = \mathbf{E}_{\parallel\Sigma'} = \frac{1}{4\pi\epsilon_0}\frac{q}{r'^3_{\Sigma'}}\mathbf{r}'_{\parallel\Sigma'} = \frac{1}{4\pi\epsilon_0}\frac{q}{r'^3_{\Sigma'}}\gamma\mathbf{r}'_{\parallel\Sigma},$$
$$\mathbf{E}_{\perp\Sigma} = \gamma\mathbf{E}_{\perp\Sigma'} = \gamma\frac{1}{4\pi\epsilon_0}\frac{q}{r'^3_{\Sigma'}}\mathbf{r}'_{\perp\Sigma'} = \frac{1}{4\pi\epsilon_0}\frac{q}{r'^3_{\Sigma'}}\gamma\mathbf{r}'_{\perp\Sigma}.$$

The electric field in Σ is

$$\mathbf{E}_\Sigma = \mathbf{E}_{\parallel\Sigma} + \mathbf{E}_{\perp\Sigma} = \frac{1}{4\pi\epsilon_0}\frac{q}{r'^3_{\Sigma'}}\gamma\mathbf{r}'_\Sigma.$$

The length $\mathbf{r}'^2_{\Sigma'}$ in Σ remains to be computed. We have

$$\mathbf{r}'^2_{\Sigma'} = \mathbf{r}'^2_{\parallel\Sigma'} + \mathbf{r}'^2_{\perp\Sigma'} = \gamma^2\mathbf{r}'^2_{\parallel\Sigma} + \mathbf{r}'^2_{\perp\Sigma} = \gamma^2\left[\left(\frac{\mathbf{r}\cdot\mathbf{u}}{u} - ut\right)^2 + \frac{1}{\gamma^2}\mathbf{r}'^2_{\perp\Sigma}\right] = \gamma^2\tilde{R}^2,$$

[32] For the observer on the charge the field at the point P at the moment t appears to come from the origin O of Σ. Because in the standard non-relativistic approach to electromagnetism time is understood in the Newtonian approach, the origin O is referred to as the *retarding point*. This terminology has no place in the relativistic approach where time is a mere coordinate and can take any value depending on the frame.

where $\tilde{R}^2 = \left(\frac{\mathbf{r}\cdot\mathbf{u}}{u} - ut\right)^2 + \frac{1}{\gamma^2}\mathbf{r}'^2_{\perp\Sigma}$. Finally

$$\mathbf{E}_\Sigma = \frac{1}{4\pi\epsilon_0}\frac{q}{\gamma^2 \tilde{R}^3}\mathbf{r}'_\Sigma. \tag{13.294}$$

This expression appears to differ from (13.283) but this is not true. Indeed we note that

$$\begin{aligned}\tilde{R}^2 &= (r_\parallel - ut)^2 + (1-\beta^2)r_\perp^2 = (r\cos\theta - ut)^2 + (1-\beta^2)r^2\sin^2\theta \\ &= r^2[\cos^2\theta - 2\beta\cos\theta + \beta^2 + \sin^2\theta - \beta^2\sin^2\theta] \\ &= r^2(1-\beta\cos\theta)^2,\end{aligned}$$

therefore

$$\tilde{R} = r(1-\beta\cos\theta) = \frac{1}{c}(rc - \mathbf{r}\cdot\mathbf{u}).$$

Replacing in (13.294) we recover (13.293).

We see once more that the anisotropy of the electric field increases as the angle ϕ tends to $\pm\pi/2$. For the limiting values $\phi = 0,\ \frac{\pi}{2}$ we have

$$\mathbf{E}_\parallel = \frac{1}{4\pi\epsilon_0}\frac{q}{\gamma^2 r'^3_\Sigma}\mathbf{r}'_\Sigma, \qquad \mathbf{E}_\perp = \frac{1}{4\pi\epsilon_0}\frac{q\gamma}{r'^3_\Sigma}\mathbf{r}'_\Sigma. \tag{13.295}$$

The magnetic field in Σ must satisfy the relation (why?)

$$\mathbf{B} = \frac{1}{c^2}\mathbf{u}\times\mathbf{E}$$

from which follows again relation (13.290).

Example 74 A charge q moves in the LCF Σ along the periphery of a circle of radius R with constant angular velocity ω. Calculate the electromagnetic field in Σ at the center P of the orbit.

Solution

We consider the origin of Σ to be at the center of the orbit, so that the coordinates of the point P we are interested in for the electromagnetic field are $x^i(t) = 0$ and $c^i(t) = \begin{pmatrix} ct \\ R\hat{\mathbf{e}}_r \end{pmatrix}_\Sigma$. The four-vector R^i is $R^i = \begin{pmatrix} r \\ \mathbf{r} \end{pmatrix}_\Sigma$ where $r = ct$, $\mathbf{r} = -R\hat{\mathbf{e}}_r$. But R^i is null, therefore $r = R$. The three-velocity of the charge in Σ is $\mathbf{u} = \omega R\hat{\mathbf{e}}_\theta$ and the three-acceleration $\mathbf{a} = -\omega^2 R\hat{\mathbf{e}}_r$. These give for the four-velocity $u^i = \begin{pmatrix} \gamma c \\ \gamma\omega R\hat{\mathbf{e}}_\theta \end{pmatrix}_\Sigma$ and for the invariant $D = R^i u_i = -R\gamma c$. We replace these in (13.273) and calculate the four-potential

$$\Omega^i = \frac{1}{4\pi\epsilon_0}\frac{qu^i}{Dc} = -\frac{q}{4\pi\epsilon_0 Rc^2}\begin{pmatrix} c \\ \omega R\hat{\mathbf{e}}_\theta \end{pmatrix}_\Sigma. \tag{13.296}$$

Having found the four-potential one calculates the antisymmetric tensor F_{ij} and subsequently the electric and the magnetic fields. However, one is possible to work directly, by replacing

$$\mathbf{r} = -\,R\hat{\mathbf{e}}_r,\ \mathbf{u} = \omega R\hat{\mathbf{e}}_\theta,\ \mathbf{a} = -\omega^2 R\hat{\mathbf{e}}_r$$

in the general relation (13.281). Indeed doing that we find for the electric field

$$\begin{aligned}\mathbf{E} &= -\frac{1}{4\pi\epsilon_0}\frac{q}{\gamma^2 c^3 R^3}\left[c^3 R\hat{\mathbf{e}}_r + R\omega(c^2+\gamma^2\beta^2\omega^2)\hat{\mathbf{e}}_\theta\right]\\ &= -\frac{1}{4\pi\epsilon_0}\frac{q}{\gamma^2 R^2}\left[\hat{\mathbf{e}}_r + \beta(1+\beta^2\gamma^2)\hat{\mathbf{e}}_\theta\right]\\ &= -\frac{1}{4\pi\epsilon_0}\frac{q}{\gamma^2 R^2}\left[\hat{\mathbf{e}}_r + \beta\gamma^2\hat{\mathbf{e}}_\theta\right].\end{aligned}$$

In order to find the Newtonian limit we set $\gamma = 1, \beta = 0$ and find the Coulomb field:

$$\mathbf{E} = -\frac{1}{4\pi\epsilon_0}\frac{q}{R^2}\hat{\mathbf{e}}_r.$$

Concerning the magnetic field we have from (13.282)

$$\begin{aligned}\mathbf{B} &= \frac{1}{c}\hat{\mathbf{e}}_r\times\mathbf{E} = -\,\frac{1}{4\pi\epsilon_0}\frac{q}{\gamma^2 R^2}\frac{1}{c}\hat{\mathbf{e}}_r\times\left[\hat{\mathbf{e}}_r + \beta\gamma^2\hat{\mathbf{e}}_\theta\right]\\ &= -\frac{1}{4\pi\epsilon_0 c}\frac{q}{\gamma^2 R^2}\beta\gamma^2\hat{\mathbf{e}}_k\\ &= -\frac{\mu_0 q\omega}{4\pi R}\hat{\mathbf{e}}_k,\end{aligned}$$

which coincides with the previous result.

This expression can be compared with the well-known result concerning the electromagnetic field of a circular conductor if we consider the charge as the current:

$$i = \frac{q}{T} = \frac{q\omega}{2\pi}.$$

Then

$$\mathbf{B} = -\frac{\mu_0}{2}\frac{i}{R}\hat{\mathbf{e}}_k,$$

which is the well-known result of non-relativistic electromagnetism. If we introduce the *magnetic dipole moment* of the loop

$$\boldsymbol{\mu} = \pi R^2 i\hat{\mathbf{e}}_k, \tag{13.297}$$

we have for the magnetic field the expression:

$$\mathbf{B} = -\frac{\mu_0}{2\pi R^3}\boldsymbol{\mu}. \tag{13.298}$$

Example 75 In an LCF Σ two charges q_1 and q_2 start moving uniformly along parallel directions with the same speed u. Calculate (in Σ!) the force between the charges when they are moving (a) along the same direction and (b) in opposite directions.
Solution

From Fig. 13.7(A) we have $\mathbf{r}'_{21}(t) = \mathbf{r}'_{21}(0) = \mathbf{l}_{21} =$ constant and $\phi = \frac{\pi}{2}$. Replacing in (13.291) we find for the electric field which is due to the charge q_1 at the position of the charge q_2

$$\mathbf{E}_{21} = \frac{1}{4\pi\epsilon_0}\frac{q_1}{\gamma^2 l_{12}^3\left(1-\beta^2\right)^{3/2}}\mathbf{l}_{21} = \frac{1}{4\pi\epsilon_0}\frac{q_1\gamma}{l_{21}^3}\mathbf{l}_{21} = A\mathbf{l}_{21},$$

where $A = \frac{1}{4\pi\epsilon_0}\frac{q_1\gamma}{l_{12}^3}$.

The magnetic field at the position of the charge q_2 is

$$\mathbf{B}_{21} = \frac{1}{c^2}\mathbf{u}\times\mathbf{E}_{21} = \frac{A}{c^2}\mathbf{u}\times\mathbf{l}_{21}.$$

The force on the charge q_2 is the Lorentz force:

$$\begin{aligned}\mathbf{F}_{21} &= q_2\left[\mathbf{E}_{21} + \frac{1}{c^2}\mathbf{u}\times\mathbf{B}_{21}\right] = q_2 A\left[\mathbf{l}_{21} + \frac{1}{c^2}\mathbf{u}\times(\mathbf{u}\times\mathbf{l}_{21})\right]\\ &= q_2 A(1-\beta^2)\mathbf{l}_{21}\\ &= \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{\gamma l_{21}^3}\mathbf{l}_{21}.\end{aligned}$$

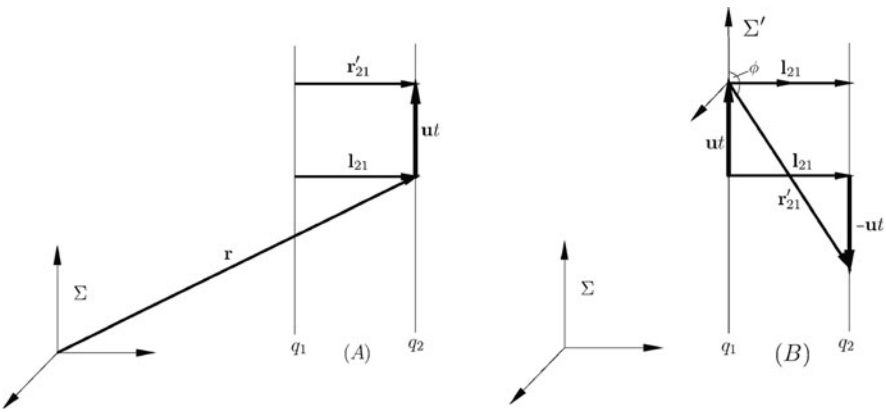

Fig. 13.7 The force between parallel currents

Second Solution

In the proper frame Σ_1 of charge q_1 the charge q_2 is fixed, therefore the applied force is

$$\mathbf{F}_{21,\Sigma_1} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{l_{21}^3}\mathbf{l}_{21}$$

(the l_{21} does not suffer Lorentz contraction because it is normal to the relative velocity $\mathbf{u}$). Obviously $\mathbf{F}_{21,\Sigma_1} \perp \mathbf{u}$ thus $\mathbf{F}_{21,\Sigma_1} \cdot \mathbf{u} = 0$ and the four-force on the charge q_2 is

$$F_{21}^i = \begin{pmatrix} 0 \\ \gamma \mathbf{F}_{21,\Sigma_1} \end{pmatrix}_{\Sigma_1}.$$

In order to calculate the force $\mathbf{F}_{21,\Sigma}$ in Σ we use the Lorentz transformation with speed $\mathbf{u}$. It is left as an exercise to the reader to show that one finds the result of the first solution.

(b) From Fig. 13.5(B) we have for the counter parallel motion

$$\mathbf{r}'_{21}(t) = \mathbf{l}_{21} - 2\mathbf{u}t,\ r'_{12} = \sqrt{l_{12}^2 + 4u^2t^2},\ \sin\phi = \frac{2ut}{\sqrt{l_{12}^2 + 4u^2t^2}}.$$

The electric field created by the charge q_1 at the position of charge q_2 is

$$\mathbf{E}_{21} = \frac{1}{4\pi\epsilon_0}\frac{q_1}{\gamma^2\left[l_{12}^2 + 4u^2t^2\right]^3\left(1 - \beta^2\frac{4u^2t^2}{l_{12}^2+4u^2t^2}\right)^{3/2}}(\mathbf{l}_{21} - 2\mathbf{u}t) = A(\mathbf{l}_{21} - 2\mathbf{u}t),$$

where $A = \frac{1}{4\pi\epsilon_0}\frac{q_1}{\gamma^2\left[l_{12}^2+4u^2t^2\right]^3\left(1-\beta^2\frac{4u^2t^2}{l_{12}^2+4u^2t^2}\right)^{3/2}}$. The magnetic field is given by

$$\mathbf{B}_{21} = \frac{1}{c^2}\mathbf{u} \times \mathbf{E}_{21} = \frac{A}{c^2}\mathbf{u} \times \mathbf{l}_{21}.$$

The force on the charge q_2 is (see (13.291))

$$\mathbf{F}_{21} = q_2[\mathbf{E}_{21} + (-\mathbf{u}) \times \mathbf{B}_{21}] = q_2 A[\mathbf{l}_{21} - \beta^2\mathbf{l}_{21}] = \frac{q_2 A}{\gamma^2}\mathbf{l}_{21}$$

or, replacing A,

$$\mathbf{F}_{21} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{\gamma^4\left[l_{12}^2 + 4u^2t^2\right]^3\left(1 - \beta^2\frac{4u^2t^2}{l_{12}^2+4u^2t^2}\right)^{3/2}}\mathbf{l}_{21}.$$

In this case too it is possible to calculate the force in the proper frame of the charge q_1 and then transfer the result in Σ using the appropriate Lorentz transformation. The details are left to the reader.

13.17 Special Relativity and Practical Applications

The Theory of Special Relativity is not a luxurious exercise of the mind which "helps" us to understand the world satisfying our metaphysical agonies. It is a theory for the engineer, a theory which leads us to construct new medical devices, new measuring instruments, and certainly new energy production plants and (unfortunately) new weapons. In order to just touch at this aspect of Special Relativity, in this section we discuss an application, which is used directly or indirectly in the design of counters of charged particles in the laboratory using the electromagnetic field they produce. The requirements we set for the performance of this machine are

- The reaction time (that is the time interval in which the instrument can distinguish between two particles) must be small in order to be possible to measure fast moving (i.e., relativistic) particles.
- The sensitivity of the instrument (that is the output which is produced for the maximum velocity and the minimum charge) must be adequate so that it will be possible to count various kinds of particles.
- The instrument must be capable to "see" small regions in space, because the radioactive sources used in the laboratory are of small size.

The above conditions are satisfied if we use the magnetic field created by the moving charged particles to produce electrical pulses. In practice this is achieved if we place a loop near the orbit of the particle. Indeed the passing of the particle creates a change in the flux of the magnetic field through the loop (zero–maximum–zero) which produces an electromotive force $E_\delta = -\frac{d\Phi}{dt}$ at the ends of the loop. We can measure this potential relatively easily.

Based on the above analysis we design the following construction. We consider a small plane loop of area dS which we place near the radioactive source (considered to be a point) and in such a way so that the source is in the plane of the loop. We consider an LCF Σ with origin the source, we assign the plane $x-z$ to be the plane of the loop and assume the velocity of the charged particles to be along the z-axis. We also place the center of the loop on the x-axis and at a distance x_0 from the source.

The change of the magnetic flux Φ is due to the normal component of the magnetic field to the plane of the loop, which according to our arrangement is the component B_y. In order to compute B_y we use the relation $\mathbf{B} = \frac{1}{c^2}\mathbf{u} \times \mathbf{E}$ and taking into account that $\mathbf{u} = (0, 0, u)$ we have

$$\mathbf{B} = \frac{u}{c^2}(E_x\mathbf{j} - E_y\mathbf{i}) \Rightarrow B_y = \frac{u}{c^2}E_x.$$

Hence

$$\Phi = B_y(x_0)dS = \frac{u}{c^2}E_x(x_0)dS,$$

where $E_x(x_0)$ is the x-component of the electric field at the position $(x_0, 0, 0)$. The electromotive force which is created in the loop due to the passing of the charge is

$$E_\delta = -\frac{d\Phi}{dt} = -\frac{u}{c^2}\frac{\partial E_x}{\partial t}dS.$$

But we have computed (see (13.295))

$$E_x(x_0) = \gamma\frac{q}{4\pi\epsilon_0}\frac{x_0}{\left(x_0^2 + u^2t^2\right)^{3/2}}$$

so that

$$\frac{\partial E_x(x_0)}{\partial t} = -\gamma\frac{3q}{4\pi\epsilon_0}\frac{u^2x_0t}{\left(x_0^2 + u^2t^2\right)^{5/2}}.$$

Therefore the electromotive force per unit of loop surface is

$$\Delta E_\delta = \gamma\frac{3q\beta^2u}{4\pi\epsilon_0}\frac{x_0t}{\left(x_0^2 + u^2t^2\right)^{5/2}}.$$

We compute that ΔE_δ has an extremum at the moments

$$t_0 = \pm\frac{|x_0|}{2\gamma u}.$$

Without restricting generality we consider $x_0 < 0$. Then we have that at the moment $t_{0,1} = -\frac{x_0}{2\gamma u}$ appears the maximum electromotive force and the moment $t_{0,2} = \frac{x_0}{2\gamma u}$ the minimum. (If $x_0 > 0$ the role of these time moments is interchanged.) These values are symmetric about the value $t = 0$. Therefore at the ends of the loop we have the voltage of Fig. 13.8 where we have assumed that the voltage pulse has the form of a Gaussian.

Having discussed the basic structure and operation of the instrument counter we continue with its precision. A Gaussian pulse is characterized by two parameters: the time interval ρ between the maximum and the minimum of the pulse and the time interval d between two successive pulses (Fig. 13.9).

Observation has shown that an instrument can distinguish two successive pulses if $d > \rho$ (see Fig. 13.9A), whereas in the case ($d \leq \rho$) it considers the pulses as one (see Fig. 13.9B). In the counter under consideration the maximum occurs at the moment $t_{0,1}$ and the minimum the moment $t_{0,2}$. If we consider that the particle is radiated at the moment $t = 0 = \frac{1}{2}(t_{0,2} + t_{0,1})$ then we have that $\rho = |t_{0,1}| = \frac{|x_0|}{2\gamma u}$.

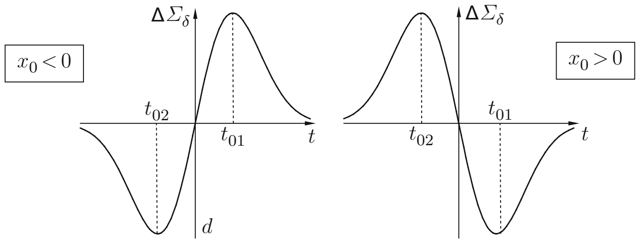

Fig. 13.8 The form of the voltage pulse

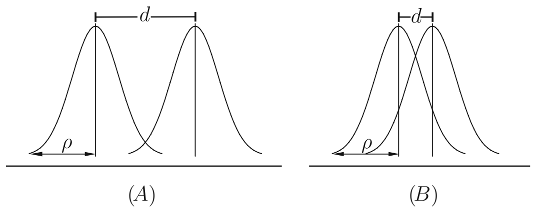

Fig. 13.9 Precision of an instrument

Using this result and the fact that the speed of the electromagnetic field covers the distance x_0 with finite speed c, we define the precision of the instrument as follows:

$$\Delta t = \frac{x_0}{2\gamma u}.$$

We note that the precision depends

a. On the speed of the radiated particles (as the speed increases the precision is reduced, which is logical and expected)
b. On the distance of the loop from the source (as the loop moves away from the source the precision increases, assuming that the signal of the source remains detectable and not influenced from other interferences)

The above analysis must be given to an industrial physicist who will explain it to the designing engineers and together will start the designing of the instrument. This activity involves the construction design, the development of the construction plans, the construction of the prototype, the evaluation of the prototype with reference to prototype sources or other similar reference instruments, its precision, etc. When this procedure has been completed the project is passed on to the team of designing the appearance of the instrument and after cost analysis, market research it is possible that the instrument will appear in the market.

Physics of the "thinking room" is not possible today. We all must be actively involved in the process of economy and the development of society. However difficult and disappointing such an action may be for a traditional physicist, it is a necessity which has to be faced.

13.18 The Systems of Units SI and Gauss in Electromagnetism

The main systems of units which are used in the (non-industrial) applications of electromagnetism are the SI and the Gauss system. The use of two different systems differentiates the constants in Maxwell equations causing confusion as to which form corresponds to which system of units and how an equation given in one system can be taken over to the other. In this book we have used the SI system only, therefore this confusion is not possible. However, it is possible that one will wish for some reason to write an equation in the Gauss system of units. In the present section we give simple rules to how this can be done.

There are two approaches. One, which is the most reliable, is to carry out the dimensional analysis of an equation and then apply the necessary "factors" as for example we do with the speed of light when we set $c = 1$ and then we add the required c at the end, so that the dimensions of all factors match properly. The second, and easier, method is to use the basic equations of electromagnetism and find the correspondence in the two systems between the fundamental quantities. Because this correspondence is independent of the equation used to derive it, it must be applicable to all equations, therefore one can use it to transfer any equation from one system to the other by transferring term by term. We note that the correspondence between the electric and the magnetic quantities is done by the fundamental relation $\epsilon_0\mu_0 = \frac{1}{c^2}$.

We shall work with the second method and start with the correspondence of the quantity ϵ_0. For this we consider the Coulomb Law, which in the two systems of units has the form

$$\text{SI System:} \qquad \mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{Q^2}{r^2}\hat{\mathbf{r}},$$

$$\text{Gauss System:} \qquad \mathbf{F} = \frac{Q^2}{r^2}\hat{\mathbf{r}}.$$

It follows that in order to write an expression involving ϵ_0 from SI to the Gauss system we must set

$$\epsilon_0 \leftrightarrow \frac{1}{4\pi}.$$

We find the transformation of the electric field from the equation $\mathbf{F} = Q\mathbf{E}$. This relation is identical in both systems of units, therefore the correspondence is

$$\mathbf{E} \leftrightarrow \mathbf{E}.$$

We find the correspondence of the magnetic field $\mathbf{H}$ and the electric induction $\mathbf{D}$ from Ampére's Law. We have

$$\text{SI system:} \qquad \nabla \times \mathbf{H} = \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t},$$
$$\text{Gauss system:} \qquad \nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{j} + \frac{\partial \mathbf{D}}{c\partial t},$$

therefore

$$\frac{4\pi}{c}\mathbf{H} \leftrightarrow \mathbf{H},$$
$$4\pi\mathbf{D} \leftrightarrow \mathbf{D}.$$

We compute the correspondence between the other basic physical quantities from the catastatic equations:

$$\text{SI system:} \qquad \mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}; \qquad \mathbf{B} = \mu_0\mathbf{H} + \mu_0\mathbf{M},$$
$$\text{Gauss system:} \qquad \mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}; \qquad \mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}.$$

The first gives

$$\mathbf{P} \leftrightarrow \mathbf{P}$$

and the second

$$c\mathbf{B} \leftrightarrow \mathbf{B},$$

where we have used the relation $\epsilon_0\mu_0 = \frac{1}{c^2}$.

We compute the correspondence for the vector potential $\mathbf{A}$ from its definition $\mathbf{A} = \nabla \times \mathbf{B}$, from which follows

$$c\mathbf{A} \leftrightarrow \mathbf{A}.$$

Similarly for the scalar potential we find ($\phi = -\nabla\mathbf{E}$)

$$\phi \leftrightarrow \phi.$$

For easy reference we collect these results in Table 13.1.

Example 76 In the SI system the magnetic moment of a current $\mathbf{j}$ at the point $\mathbf{r}$ is defined as follows: $\mathbf{m} = \frac{1}{2}\int \mathbf{r} \times \mathbf{j}$. Write the corresponding equation in the Gauss system of units given that the energy W of the current $\mathbf{j}$ in an external field $\mathbf{B}$ is defined by the relation $W = \mathbf{m} \cdot \mathbf{B}$.

Table 13.1 Table for the transformation of equations between the SI and the Gauss system of units

Quantity		Gauss		
	SI		SI	Gauss
ε_0	$4\pi\varepsilon_0$	1	$\mathbf{F} = \frac{1}{4\pi\varepsilon_0}\frac{Q^2}{r^2}\widehat{\mathbf{r}}$	$\mathbf{F} = \frac{Q^2}{r^2}\widehat{\mathbf{r}}$
$\mathbf{E}$	$\mathbf{E}$	$\mathbf{E}$	$\mathbf{F} = Q\mathbf{E}$	$\mathbf{F} = Q\mathbf{E}$
$\mathbf{H}$	$\frac{4\pi}{c}\mathbf{H}$	$\mathbf{H}$	$\nabla\times\mathbf{H} = \mathbf{j}+\frac{\partial\mathbf{D}}{\partial t}$	$\nabla\times\mathbf{H} = \frac{4\pi}{c}\mathbf{j}+\frac{1}{c}\frac{\partial\mathbf{D}}{\partial t}$
$\mathbf{D}$	$4\pi\mathbf{D}$	$\mathbf{D}$	$\nabla\times\mathbf{H} = \mathbf{j}+\frac{\partial\mathbf{D}}{\partial t}$	$\nabla\times\mathbf{H} = \frac{4\pi}{c}\mathbf{j}+\frac{1}{c}\frac{\partial\mathbf{D}}{\partial t}$
$\mathbf{P}$	$\mathbf{P}$	$\mathbf{P}$	$\mathbf{D} = \varepsilon_0\mathbf{E}+\mathbf{P}$	$\mathbf{D} = \mathbf{E}+4\pi\mathbf{P}$
$\mathbf{B}$	$c\mathbf{B}$	$\mathbf{B}$	$\mathbf{B} = \mu_0\mathbf{H}+\mu_0\mathbf{M}$	$\mathbf{B} = \mathbf{H}+4\pi\mathbf{M}$
$\mathbf{M}$	$\mathbf{M}/c$	$\mathbf{M}$	$\mathbf{B} = \mu_0\mathbf{H}+\mu_0\mathbf{M}$	$\mathbf{B} = \mathbf{H}+4\pi\mathbf{M}$
$\mathbf{A}$	$c\mathbf{A}$	$\mathbf{A}$	$\mathbf{A} = \nabla\times\mathbf{B}$	$\mathbf{A} = \nabla\times\mathbf{B}$
ϕ	ϕ	ϕ	$\phi = -\nabla\mathbf{E}$	$\phi = -\nabla\mathbf{E}$

Solution

Because the energy has the same units in both systems of units and because the correspondence of $\mathbf{B}$ is $c\mathbf{B} \leftrightarrow \mathbf{B}$ it follows that the correspondence for $\mathbf{m}$ is

$$\frac{1}{c}\mathbf{m} \leftrightarrow \mathbf{m}.$$

Hence in the Gauss system the magnetic moment is defined with the relation $\mathbf{m} = \frac{1}{2c}\int \mathbf{r}\times\mathbf{j}\,dV$.

Example 77 In the SI system of units the magnetic induction of a coil of length l, small radius r, and n turns per unit of length is given by the formula

$$L = \mu\mu_0 n^2 l\pi r^2.$$

Write this formula in the Gauss system of units given that the inductance L is transformed as the quantity $\mathbf{B}$.

Solution

In the SI system we have

$$L = \mu\frac{1}{\varepsilon_0 c^2}n^2 l\pi r^2 = \frac{4\pi\mu}{c^2}n^2 l\pi r^2.$$

The correspondence of L is $cL \leftrightarrow L$, therefore in the Gauss system this relation becomes

$$cL \to L = \frac{4\pi\mu}{c}n^2 l\pi r^2.$$

Chapter 14
Relativistic Angular Momentum

14.1 Introduction

In the previous chapters we considered in Special Relativity concepts which corresponded to Newtonian vector quantities. In this chapter we deal with the angular momentum tensor of Newtonian Physics, which is a physical quantity described by an antisymmetric second-order tensor. This necessitates the introduction of new mathematical concepts and tools, the main one being the bivector. We shall also make use of the basics of the antisymmetric tensor analysis discussed in Sect. 13.10.1. The reader should consult this section before attempting to read the present chapter.

14.2 Mathematical Preliminaries

A *bivector* is any second-order antisymmetric tensor $X_{ab} = -X_{ba.}$ A bivector is called *simple* if it can be written in the form $X_{ab} = A_{[a}B_{b]}$ where A^a, B^a are vectors. The *dual bivector* X^{*ab} of a bivector X^{ab} is defined as follows:

$$X^{*ab} = \frac{1}{2}\eta^{abcd}X_{cd} \Leftrightarrow X^{ab} = -\frac{1}{2}\eta^{abcd}X^{*}_{cd}. \tag{14.1}$$

Bivectors, being tensors, can be 1+3 decomposed wrt any timelike vector.

14.2.1 1 + 3 Decomposition of a Bivector X_{ab}

We consider a timelike vector field p^a ($p^a p_a = -p^2,\ p^2 > 0$) and the associated projection operator

$$h_{ab} = g_{ab} + \frac{1}{p^2}p_a p_b.$$

M. Tsamparlis, *Special Relativity*, DOI 10.1007/978-3-642-03837-2_14,

The 1+3 decomposition[1] of a general second-order tensor T_{ab} in Minkowski space has been computed in (12.11) as follows:

$$T_{ab}=\frac{1}{p^4}(T_{cd}p^c p^d)p_a p_b-\frac{1}{p^2}(h_a{}^c T_{cd}p^d)p_b-\frac{1}{p^2}(h_b{}^d T_{cd}p^c)p_a+h_a{}^c h_b{}^d T_{cd}$$

or in matrix form

$$\begin{pmatrix} \frac{1}{p^4}(T_{cd}p^c p^d) & -\frac{1}{p^2}(h_b{}^c T_{cd}p^d) \\ -\frac{1}{p^2}(h^{bd}T_{dc}p^c) & h_a{}^c h_b{}^d T_{cd} \end{pmatrix}. \tag{14.2}$$

This is a mathematical identity which expresses covariantly the tensor in terms of one scalar, two vectors, and one second-order tensor of order (2, 2) in the proper space of p^a. In the special case that the tensor $T_{ab}=X_{ab}$ where X_{ab} is a bivector, formula (14.2) simplifies. Indeed the term

$$X_{cd}p^c p^d=0. \tag{14.3}$$

Next we define the vector

$$E_a=\frac{1}{p^2}h_a{}^c X_{cd}p^d=\frac{1}{p^2}X_{ab}p^b. \tag{14.4}$$

This vector is called the *electric part* of the bivector X_{ab}. Then the terms in (14.2), which contain once the projection tensor, give

$$-E_a p_b+E_b p_a. \tag{14.5}$$

Concerning the last term which contains twice the projection tensor, we define another vector H^a as follows:

$$h_a{}^c h_b{}^d X_{cd}=-\eta_{abcd}p^c H^d \tag{14.6}$$

and have the final 1+3 decomposition of X_{ab} along p^a:

$$X_{ab}=-E_a p_b+E_b p_a-\eta_{abcd}p^c H^d. \tag{14.7}$$

We call the vector H^a the *magnetic part* of the bivector X_{ab}. We emphasize that (14.7) is a mathematical identity, therefore all information of X_{ab} is contained in the pair of vector fields $E^a,\ H^a$.

In order to compute the vector H_a in terms of the bivector X_{ab} we contract (14.6) with η_{abcd} and get

[1] This result is general and holds for the $1+(n-1)$ decomposition of a second-order tensor.

$$\begin{aligned}
\eta_{abcd} p^b X^{cd} &= \eta_{abcd} p^b \left(-E^c p^d + E^d p^c - \eta^{cdrs} p_r H_s\right) \\
&= -\eta_{abcd} \eta^{cdrs} p^b p_r H_s \\
&= 2\left(\delta^r_a \delta^s_b - \delta^s_a \delta^r_b\right) p^b p_r H_s \\
&= -2(p^c p_c) H_a = 2p^2 H_a
\end{aligned}$$

from which follows

$$H_a = \frac{1}{2p^2} \eta_{abcd} p^b X^{cd}, \ \ p^2 > 0. \tag{14.8}$$

We note that both vectors E^a, H^a are spacelike:

$$E^a p_a = H^a p_a = 0. \tag{14.9}$$

Let Σ_p be the proper frame of p^a and suppose that in this frame $E^a = (E^1, E^2, E^3)^t$ and $H^a = (H^1, H^2, H^3)^t$. Then in Σ_p the components of X_{ab} are

$$[X_{ab}] = \left(\begin{matrix} 0 & -p^2 E^1 & -p^2 E^2 & -p^2 E^3 \\ p^2 E^1 & 0 & pH^3 & -pH^2 \\ p^2 E^2 & -pH^3 & 0 & pH^1 \\ p^2 E^3 & pH^2 & -pH^1 & 0 \end{matrix}\right)_{\Sigma_p}. \tag{14.10}$$

Finally we compute the invariants of the bivector X_{ab} in terms of the vectors E^a, H^a. We have

$$\begin{aligned}
2X \equiv -X_{ab} X^{ab} &= -(-E_a p_b + E_b p_a - \eta_{abcd} p^c H^d) \\
&\quad (-E^a p^b + E^b p^a - \eta^{abcd} p_c H_d) \\
&= -2(E^a E_a)(-p^2) - \eta_{abcd} \eta^{abmn} p^c p_m H^d H_n \\
&= 2p^2 E^2 + 2(\delta^m_c \delta^n_d - \delta^m_d \delta^n_c) p^c p_m H^d H_n \\
&= 2p^2 E^2 - 2p^2 H^2 = 2p^2 (E^2 - H^2)
\end{aligned} \tag{14.11}$$

and

$$\begin{aligned}
-8Y &= \eta_{abcd} X^{ab} X^{cd} \\
&= \eta_{abcd} (-E^a p^b + E^b p^a - \eta^{abrs} p_r H_s)(-E^c p^d + E^d p^c - \eta^{cdmn} p_m H_n) \\
&= 2\eta_{abcd} \eta^{cdmn} E^a p^b p_m H_n + 2\eta_{abcd} \eta^{abrs} p_r H_s E^c p^d \\
&= -4(\delta^m_a \delta^n_b - \delta^n_a \delta^m_b) E^a p^b p_m H_n - 4(\delta^r_c \delta^s_d - \delta^s_c \delta^r_d p_r H_s E^c p^d) && (14.12) \\
&= -8p^2 (E^a H_a). && (14.13)
\end{aligned}$$

Therefore

$$X = -\frac{1}{2}X_{ab}X^{ab} = p^2(E^2 - H^2), \tag{14.14}$$

$$Y = -\frac{1}{8}\eta_{abcd}X^{ab}X^{cd} = p^2(E^a H_a). \tag{14.15}$$

Exercise 75 Consider the bivector F_{ab} of the electromagnetic field tensor and let $p^a = u^a$, where u^a is the four-velocity of the observer. Show that the electric part and the magnetic part of this bivector defined in (14.4) and (14.8) as well as the invariants defined in (14.11) and (14.13), respectively, coincide with the corresponding quantities considered in Sect. 13.1.

14.3 The Derivative of X_{ab} Along the Vector p^a

Let s be an affine parameter along the trajectory of a particle and let $u^a = \frac{dx^a}{ds} = \frac{p^a}{p}$ be the four-velocity of the particle. We find from (14.7)

$$\frac{dX_{ab}}{ds} \equiv \dot{X}_{ab} = -\dot{E}_a p_b - E_a \dot{p}_b + \dot{E}_b p_a + E_b \dot{p}_a - \eta_{abcd}\dot{p}^c H^d - \eta_{abcd} p^c \dot{H}^d,$$

where a dot over a symbol indicates derivation wrt s. The tensor $\dot{X}_{ab}$ is also a bivector, therefore it can be decomposed as above in terms of a new pair of vectors (e^a, h^a). We compute the electric and the magnetic parts assuming $p^a \dot{p}_a = 0$, i.e., the proper mass is constant. We find using (14.4), (14.8) that

$$e^a = \frac{1}{p^2}\dot{X}^{ab} p_b = h^{ab}\dot{E}_b - \frac{1}{p^2}\eta^{abcd} p_b \dot{p}_c H_d. \tag{14.16}$$

Also

$$\begin{aligned}
h^a &= \frac{1}{2p^2}\eta^{abcd} p_b \dot{X}_{cd} \\
&= \frac{1}{2p^2}\eta^{abcd} p_b \left(-\dot{E}_c p_d - E_c \dot{p}_d + \dot{E}_d p_c + E_d \dot{p}_c - \eta_{cdrs}\dot{p}^r H^s - \eta_{cdrs} p^r \dot{H}^s\right) \\
&= \frac{1}{2p^2}\eta^{abcd} p_b \left(2E_d \dot{p}_c - \eta_{cdrs}\dot{p}^r H^s - \eta_{cdrs} p^r \dot{H}^s\right) \\
&= \frac{1}{2p^2}\eta^{abcd} p_b 2E_d \dot{p}_c - \frac{1}{2p^2}\eta^{abcd} p_b \eta_{cdrs}\dot{p}^r H^s - \frac{1}{2p^2}\eta^{abcd} p_b \eta_{cdrs} p^r \dot{H}^s.
\end{aligned}$$

The terms

$$\eta^{abcd} p_b \eta_{cdrs} \dot{p}^r H^s = -2(\delta^a_r \delta^b_s - \delta^b_r \delta^a_s) p_b \dot{p}^r H^s = 0,$$
$$\eta^{abcd} p_b \eta_{cdrs} p^r \dot{H}^s = -2(\delta^a_r \delta^b_s - \delta^b_r \delta^a_s) p_b p^r \dot{H}^s = -2p^2 h^a_b \dot{H}^b,$$

therefore finally

$$h^a = h^a_b \dot{H}^b + \frac{1}{p^2} \eta^{abcd} p_b \dot{p}_c E_d. \tag{14.17}$$

We consider now the orthonormal frame[2] $\{u^a, N(\rho)^a\}$ where $u^a = p^a/|p|$ and $N(\rho)^a N(\rho)_b = \delta^a_b$. In this frame we set

$$E^a = \sum_{\rho=1}^{3} p_\rho N(\rho)^a, \quad H^a = \sum_{\rho=1}^{3} (-1)^\rho p_{3+\rho} N(\rho)^a, \tag{14.18}$$

where p_ρ, $p_{3+\rho}$ are components. Then the derivative of E^a, H^a along u^a is

$$\dot{E}^a = \sum_{\rho=1}^{3} \left(\dot{p}_\rho N(\rho)^a + p_\rho \dot{N}(\rho)^a\right), \ \dot{H}^a = \sum_{\rho=1}^{3} (-1)^\rho \left(\dot{p}_{3+\rho} N(\rho)^a + p_{3+\rho} \dot{N}(\rho)^a\right), \tag{14.19}$$

that is, they are expressed in terms of the derivatives $\dot{N}(\rho)^a$. Let us assume that the vectors $N(\rho)^a$ are propagated along the particle trajectory according to the "law"

$$\dot{N}(\rho)^a = S^0_\rho u^a + \sum_{\mu=1}^{3} S^\mu_\rho N(\mu)^a. \tag{14.20}$$

Then we obtain

$$\dot{E}^a = \sum_{\rho=1}^{3} \left(\dot{p}_\rho N(\rho)^a + p_\rho \left(S^0_\rho u^a + \sum_{\mu=1}^{3} S^\mu_\rho N(\mu)^a \right) \right)$$
$$= \sum_{\rho=1}^{3} \left[p_\rho S^0_\rho u^a + \left(\dot{p}_\rho \delta^\mu_\rho + p_\rho \sum_{\mu=1}^{3} S^\mu_\rho \right) N(\mu)^a \right], \tag{14.21}$$

$$\dot{H}^a = \sum_{\rho=1}^{3} (-1)^\rho \left[p_{3+\rho} S^0_\rho u^a + \left(\dot{p}_{3+\rho} \delta^\mu_\rho + p_{3+\rho} \sum_{\mu=1}^{3} S^\mu_\rho \right) N(\mu)^a \right]. \tag{14.22}$$

[2] The calculations are general and hold for n dimensions.

These relations give us the propagation equation of the vectors E^a, H^a associated with the bivector X_{ab} relative to the unit four-vector u^a. The transport law of the frame $\{u^a, N^a\}$ is general and can be established in practice by means of a "parallel" transport law of the frame $\{u^a N^\mu\}$, that is by a derivation.[3] If we replace the expressions (14.21) and (14.22) into (14.16) and (14.17) we find

$$e^a = \left[\sum_{\rho=1}^{3} \dot{p}_\rho \delta^\mu_\rho + p_\rho \sum_{\mu=1}^{3} S^\mu_\rho - \kappa_1 \eta^{01\mu\nu} p_{3+\nu}\right] N(\mu)^a, \tag{14.23}$$

$$h^a = \left[\sum_{\rho=1}^{3} (-1)^\rho \left(\dot{p}_{3+\rho} \delta^\mu_\rho + p_{3+\rho} \sum_{\mu=1}^{3} S^\mu_\rho\right) + \kappa_1 \eta^{01\mu\nu} p_\nu\right] N(\mu)^a, \tag{14.24}$$

where κ_1 is the first (principal) normal of the trajectory defined by the equation

$$\left|\frac{du^a}{ds}\right| = \kappa_1. \tag{14.25}$$

Kinematically κ_1 is the length of the four-acceleration and geometrically the inverse of the radius of curvature of the orbit at the point where it is computed.

14.4 The Angular Momentum in Special Relativity

Having presented the basics of the theory of bivectors we are in a position to proceed with the generalization of the concept of Newtonian angular momentum in Special Relativity. The relativistic form of this concept is necessary because angular momentum is a fundamental quantity of Newtonian Physics and, as will be seen, leads to the important concept of spin, which is a purely relativistic physical quantity with no Newtonian analogue.

14.4.1 The Angular Momentum in Newtonian Theory

The Newtonian angular momentum of a particle with linear momentum p^μ with reference to a point with position vector a^μ is the (0, 2) tensor $l_{\mu\nu}$ defined as follows:

$$l_{\mu\nu} = (x_\mu - a_\mu)p_\nu - p_\mu(x_\nu - a_\nu). \tag{14.26}$$

We note that $l_{\mu\nu}$ is the same for all points with position vector $a_\mu + kx_\mu$ $(k \epsilon R)$. Definition (14.26) can be written as follows:

---

[3] E.g., a Fermi-propagated frame.

$$l_\mu(a) = \eta_{\mu\nu\rho}(r^\nu - a^\nu)p^\rho, \tag{14.27}$$

where l_ρ is a 1-form or pseudovector with components the three components $l_{12,}l_{13}, l_{23}$ of the antisymmetric tensor $l_{\mu\nu}$ of angular momentum.

In the following we consider the angular momentum wrt the origin (i.e., we take the point with $a_\mu = 0$ and $k = 0$). Then relations (14.26) and (14.27) read

$$l_{\mu\nu} = \eta_{\mu\nu\rho}l^\rho, \tag{14.28}$$
$$l_\mu = \eta_{\mu\nu\rho}r^\nu p^\rho. \tag{14.29}$$

In three-vector notation we write $l^\rho = \mathbf{l}$ and it is easy to show that $\mathbf{l}$ can be written as the cross product:

$$\mathbf{l} = \mathbf{r} \times \mathbf{p}, \tag{14.30}$$

where $\mathbf{r}$, $\mathbf{p}$ are the position vector and the linear momentum of the particle.

Newton's Second Law gives

$$\frac{dl_{\mu\nu}}{dt} = x_\mu f_\nu - f_\mu x_\nu, \tag{14.31}$$

where $f^\mu = \frac{dp^\mu}{dt}$ is the three-force on the particle. The three-vector form of this formula is

$$\frac{d\mathbf{l}}{dt} = \mathbf{r} \times \mathbf{f}. \tag{14.32}$$

The bivector

$$M_{\mu\nu} = x_\mu f_\nu - f_\mu x_\nu \tag{14.33}$$

is called the *net moment* or the *net torque* of the force acting on the particle. It can be represented by a 1-form M^ρ according to the formula

$$M_{\mu\nu} = \eta_{\mu\nu\rho}M^\rho. \tag{14.34}$$

Then the equation of motion of the angular momentum reads

$$\frac{dl_{\mu\nu}}{dt} = M_{\mu\nu} \tag{14.35}$$

and in terms of the corresponding 1-forms

$$\eta_{\mu\nu\rho}\left(M^\rho - \frac{dl^\rho}{dt}\right) = 0.$$

Contracting with $\eta^{\mu\nu\sigma}$ we find

$$M^{\rho} = \frac{dl^{\rho}}{dt}. \tag{14.36}$$

The three-vector notation of the above is as follows. If $\mathbf{F}$ is the three-force on the particle Newton's Second Law gives for the moment of force or the pseudovector (=1-form) of torque about the point with position vector $\mathbf{a}$

$$\mathbf{N}(\mathbf{a}) = (\mathbf{r} - \mathbf{a}) \times \mathbf{f} = \mathbf{r}\times\frac{d\mathbf{p}}{dt} - (\mathbf{a} \times \mathbf{f}) = \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) - \mathbf{L}(\mathbf{a}) = \frac{d\mathbf{L}}{dt} - \mathbf{L}(\mathbf{a}),$$

where $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is the angular momentum of the particle wrt the origin, which we call the *net angular momentum*. When we take the point to be the origin, then $\mathbf{L}(\mathbf{a}) = \mathbf{0}$ and the net angular momentum $\mathbf{L}$ is related to the *net torque* $\mathbf{N}$ as follows:

$$\mathbf{N} = \frac{d\mathbf{L}}{dt}. \tag{14.37}$$

From this relation it follows that if the net torque vanishes, the net angular momentum remains constant. This result is known as the *conservation of angular momentum*.

14.4.2 The Angular Momentum of a Particle in Special Relativity

Before we proceed, we remark that in the generalization of the angular momentum in Special Relativity we have a new situation. Indeed up to now all the physical quantities we have considered and generalized were vectors (e.g., the velocity, the acceleration, the momentum) whereas now we generalize a bivector or equivalently a pseudovector. This means that we do not anymore have the rather easy physical intuition of the vector quantities and we must rely more on the mathematical manipulations and "similarities," rather than on "plausible" physical grounds. Because of this we must be prepared to meet "strange" situations in the sense that we may end up with relativistic physical quantities with no Newtonian analogue. However, this is not news. Indeed, we recall that the four-velocity in the proper frame of the particle has one component only, the quantity c, a purely relativistic quantity with no Newtonian analogue. A more drastic situation is the case with the four-acceleration which in the proper frame was defined solely by the proper acceleration $\mathbf{a}^{+}$, again without Newtonian analogue. In both these cases, we *postulated* the physical nature of the new relativistic quantities. Therefore, it is reasonable to expect that in the proper frame of the particle, the angular momentum is possible to be reduced to an antisymmetric $(0, 2)$ tensor with no Newtonian analogue, whose physical significance will have to be postulated.

Let us consider a particle with position four-vector x^a and four-momentum p^a and let a spacetime point with position four-vector A^a. We define the *Relativistic Angular Momentum* of the particle wrt the point A^a to be the bivector

$$L_{ab}(A) = (x_a - A_a)p_b - p_a(x_b - A_b) = \eta_{abcd}(x^c - A^c)p^d. \tag{14.38}$$

The bivector

$$L_{ab} = x_a p_b - p_a x_b = \eta_{abcd} x^c p^d \tag{14.39}$$

is called the *net angular momentum* of the particle. We have

$$L_{ab}(A) = L_{ab} - (A_a p_b - p_a A_b), \tag{14.40}$$

that is $L_{ab}(A)$ equals L_{ab} minus the constant term $A_a p_b - p_a A_b$. This formal definition takes us to a situation analogue to that of Newtonian Physics, where the angular momentum depends on the point at which it is defined. Note that if A^a is replaced with $A^a + kp^a$ then $L_{ab}(A)$ does not change. This means that if the reference point "moves" along the world line of the instantaneous inertial observer of the particle, the angular momentum $L_{ab}(A)$ does not change. In the following we consider the angular momentum wrt the origin (i.e., we take $A^a = 0$) and we discuss the relativistic net angular momentum.

We compute the electric and magnetic four-vectors associated with the bivector L_{ab} in its 1+3 decomposition wrt the four-momentum p^a. For the electric part we have from (14.4)

$$E_a = \frac{1}{p^2} h_a{}^c L_{cd} p^d = \frac{1}{p^2} h_a{}^c (x_c p_d - p_c x_d) p^d = \frac{1}{p^2}(-p^2) h_a{}^c x_c = -h_a{}^c x_c, \tag{14.41}$$

that is, E^a is the spatial part of the position vector x^a. Concerning the magnetic part we find using (14.8)

$$H_a = -\frac{1}{2p^2} \eta_{abcd} p^b L^{cd} = -\frac{1}{2p^2} \eta_{abcd} p^b (x^c p^d - p^c x^d) = 0. \tag{14.42}$$

We conclude that the angular momentum tensor with reference to the four-momentum p^a has "electric" part only and this equals $-h_a{}^c x_x$. It is remarkable that the mass does not enter into the vector fields defined by the four-momentum.

Concerning the invariants of L_{ab} we compute using the general formulae (14.14), (14.15)

$$X = p^2 E^2, \quad Y = 0. \tag{14.43}$$

Next we consider the propagation of the net angular momentum along the world line of the particle, that is the derivative $\frac{dL_{ab}}{d\tau} = \dot{L}_{ab}$. This is also a bivector whose electric and magnetic parts are given by (14.16), (14.17), respectively. Assuming that the proper mass of the particle is constant (i.e., the particle does not radiate) we compute

$$\begin{aligned}
e^a(L) &= h^a_b \dot{E}^b = -h^{ab}\frac{d}{d\tau}(h^c_b x_c) = -h^{ab}(\dot{h}^c_b x_c + h^c_b u_c) \\
&= -h^{ab}\left[\frac{d}{d\tau}\left(\delta^c_b + \frac{1}{c^2}u^c u_b\right)x_c\right] = -\frac{1}{c^2}h^{ab}(\dot{u}^c u_b + u^c \dot{u}_b)x_c \\
&= -\left(\frac{1}{c^2}\right)\dot{u}^a(u^c x_c) = -\frac{1}{mc^2}(u^c x_c)F^a, \qquad (14.44) \\
h^a(L) &= \frac{1}{p^2}\eta^{abcd}p_b \dot{p}_c E_d = -\frac{1}{m^2c^2}\eta^{abcd}p_b F_c h^e_d x_e = -\frac{1}{m^2c^2}\eta^{abcd}p_b F_c x_d \\
&= -\frac{1}{mc^2}\eta^{abcd}u_b F_c x_d, \qquad (14.45)
\end{aligned}$$

where u^a is the four-velocity of the particle and we have applied Newton's generalized Second Law to write $F^a = \frac{dp^a}{d\tau} = \dot{p}^a$. We note that both e^a, h^a are spacelike four-vectors:

$$e^a = h^a_b e^b, \qquad h^a = h^a_b h^b. \qquad (14.46)$$

We define the (net relativistic) *torque tensor* of the four-force F^a by the formula

$$M^{ab} = x^a F^b - F^a x^b.$$

This is also a bivector, hence it has an "electric" and a "magnetic" part. We compute

$$\begin{aligned}
E^a(M) &= \frac{1}{p^2}h^a_c M^{cd}p_d = -\frac{1}{m^2c^2}(x^b p_b)F^a = -\frac{1}{mc^2}(x^b u_b)F^a, \\
H^a(M) &= \frac{1}{2p^2}\eta^{abcd}p_b(x_c F_d - F_c x_d) = \frac{1}{2mc^2}\eta^{abcd}u_b(x_c F_d - x_d F_c) \\
&= \frac{1}{mc^2}\eta^{abcd}u_b x_c F_d.
\end{aligned}$$

We note that $E^a(M) = e^a(L)$ and $H^a(M) = h^a(L)$, therefore[4]

[4] This equation is computed directly as follows:

$$\begin{aligned}
\frac{dL^{ab}}{d\tau} &= \frac{dx^a}{d\tau}p^b + x^a\frac{dp^b}{d\tau} - \frac{dx^b}{d\tau}p^a - x^b\frac{dp^a}{d\tau} \\
&= u^a p^b + x^a F^b - u^b p^a - x^b F^a \\
&= x^a F^b - F^a x^b \\
&= M^{ab}. \qquad (14.47)
\end{aligned}$$

$$\frac{dL_{ab}}{d\tau} = M_{ab}. \tag{14.48}$$

This is the equation of propagation (i.e., equation of motion) of the (relativistic) angular momentum L_{ab} of the particle.

We compute the invariant $u^c x_c$ in the proper frame of the particle. In that frame $x^a = \binom{c\tau}{\mathbf{0}}_{\Sigma^+}$, $u^a = \binom{c}{\mathbf{0}}_{\Sigma^+}$, hence $u^c x_c = -c^2\tau$. Therefore

$$e^a(L) = \frac{\tau}{m} F^a. \tag{14.49}$$

The components of the angular momentum in the frame of an LCF Σ in which $x^a = \binom{ct}{r^\mu}_\Sigma$, $p^a = \binom{\gamma mc}{\gamma m v^\mu}_\Sigma$ are computed as follows[5]:

$$\begin{aligned}
[L^{ab}]_\Sigma &= \begin{pmatrix} ct \\ r^\mu \end{pmatrix}_\Sigma \otimes \begin{pmatrix} m\gamma c \\ m\gamma v^\nu \end{pmatrix}_\Sigma - \begin{pmatrix} m\gamma c \\ m\gamma v^\mu \end{pmatrix}_\Sigma \otimes \begin{pmatrix} ct \\ r^\nu \end{pmatrix}_\Sigma \\
&= \begin{pmatrix} ct & r_\mu \end{pmatrix}_\Sigma \otimes \begin{pmatrix} m\gamma c \\ m\gamma v^\nu \end{pmatrix}_\Sigma - \begin{pmatrix} m\gamma c & m\gamma v_\mu \end{pmatrix}_\Sigma \otimes \begin{pmatrix} ct \\ r^\nu \end{pmatrix}_\Sigma \\
&= \left(ct \binom{m\gamma c}{m\gamma v^\nu} \; r_\mu \binom{m\gamma c}{m\gamma v^\nu} \right)_\Sigma - \left(m\gamma c \begin{pmatrix} ct \\ r^\nu \end{pmatrix} \; m\gamma v_\mu \begin{pmatrix} ct \\ r^\nu \end{pmatrix} \right)_\Sigma \\
&= \begin{pmatrix} c^2 t m\gamma c & m\gamma c r_\mu \\ m\gamma\, ct v^\nu & m\gamma r_\mu v^\nu \end{pmatrix} - \begin{pmatrix} c^2 t m\gamma c & m\gamma ct v_\mu \\ m\gamma c r^\nu & m\gamma v_\mu r^\nu \end{pmatrix} \\
&= \begin{pmatrix} 0 & -m\gamma c(t v_\mu - r_\mu) \\ m\gamma c(t v^\nu - r^\nu) & m\gamma(r^\mu v_\nu - v^\mu r_\nu) \end{pmatrix},
\end{aligned}$$

i.e., the components of L^{ab} in the frame Σ are

$$L^{00} = 0, \tag{14.50}$$

$$L^{0\mu} = -m\gamma c(t v^\mu - r^\mu), \tag{14.51}$$

$$L^{\mu 0} = m\gamma c(t v^\mu - r^\mu) = -L^{0\mu}, \tag{14.52}$$

$$L^{\mu\nu} = m\gamma(r^\mu v^\nu - v^\mu r^\nu) = m\gamma \ell^{\mu\nu}, \tag{14.53}$$

where $\ell^{\mu\nu} = m(r^\mu v^\nu - v^\mu r^\nu)$ is identified with the net Newtonian angular momentum.

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[5] Note the method we use to compute the components by the tensor product of the corresponding matrices.

We note that for a particle at rest $\ell^{\mu\nu} = 0$, but $L^{0\nu} = -mcr^{\nu} \neq 0$, i.e., the four-angular momentum does not vanish. However, in the *proper frame* of the particle – and only there because only there $r^{\mu} = 0$ – both $L_{ab} = 0$ and $M^{ab} = 0$. We conclude that it is possible to define the proper frame, i.e., the requirement $L^{0\mu} = 0$, by the condition

$$L_{ab} p^{b} = 0. \tag{14.54}$$

This observation is important and will be used subsequently.

14.5 The Intrinsic Angular Momentum – The Spin Vector

The angular momentum we considered in Sect. 14.4.2 is due to the motion of the particle, that is, it is of a kinematic nature. For this reason we call it the *orbital angular momentum*, mainly because originally it was used in the early model of the atom to study the motion of the rotating electrons around the nucleus. Its main characteristic is that it vanishes at the proper frame of the particle. Physical observations have shown that this type of angular momentum is not sufficient to cover the physical phenomena and one has to consider an additional type of angular momentum, however, of a dynamic nature. The sum of the orbital and the dynamic angular momentum makes the total angular momentum of the particle, which is the quantity one should consider in the study of motion of a particle in a magnetic field.

It is to be noted that the definition of the intrinsic angular momentum (and consequently the spin) given below does not apply to the photon and the neutrino, because both do not have proper frame. However, it is possible to define spin for these particles by a limiting process ($m \to 0$). It turns out that for these particles the spin is either parallel or antiparallel to the three-velocity in all frames.[6]

14.5.1 The Magnetic Dipole

To obtain a feeling of the physics of the "dynamic" angular momentum, we discuss briefly some well-known experiments of classical electromagnetism. It is well known that when an electric current i moves in a magnetic field $\mathbf{B}$, it suffers a force $\mathbf{F} = id\mathbf{l} \times \mathbf{B}$ where $d\mathbf{l}$ is a differential element of length along the conductor (more general along the path of the current). Consider a rectangular loop $ABCD$ of wire of length $AB = CD = a$ and width $BC = DA = b$ which is placed in a uniform magnetic field $\mathbf{B}$, so that the plane of the loop is always normal to the direction of the magnetic field (see Fig. 14.1(a)). The current is provided into the loop by a pair of wires which are twisted tightly together so that there will be no net magnetic force on the twisted pair, because the currents in the two wires are in opposite directions.

[6] See E. P. Wigner Rev. Mod. Phys. **29**, 255, (1957).

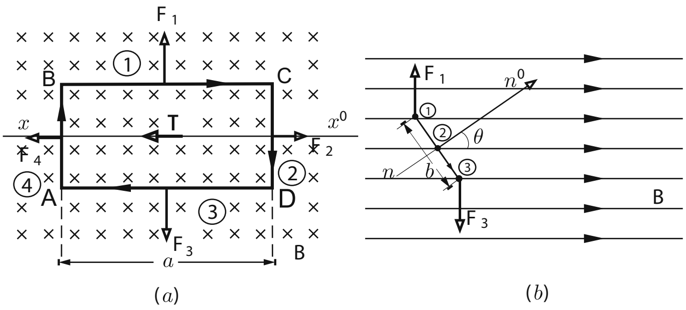

Fig. 14.1 Rectangular coil carrying current in a uniform magnetic field

Thus the lead wires may be ignored. The loop is suspended from a long inextensible string at its center of mass, so that it is free to turn, at least through a small angle.

The net force on the loop is the resultant of the forces on the four sides of the loop. Let us determine the force on side CD (see Fig. 14.1(b)). On the side CD the vector $d\mathbf{l}$ points in the direction of the current and has magnitude b. The angle between CD and B is $90 - \theta$, hence the magnitude of the force on this side is

$$F_{CD} = ibB\cos\theta$$

and its direction is out of the plane of Fig. 14.1(b). Working in a similar way, we show that the force on the opposite side AB has the same magnitude $F_{AB} = F_{CD}$ and points in the opposite direction to $\mathbf{F}_{CD}$. Thus $\mathbf{F}_{CD} + \mathbf{F}_{AB} = \mathbf{0}$ and these two forces taken together have no effect on the motion of the loop. The remaining forces $\mathbf{F}_{BC}$, $\mathbf{F}_{DA}$ have equal magnitude iaB and opposite directions, but they have different line of action. As a result the total force $\mathbf{F}_{BC} + \mathbf{F}_{DA} = \mathbf{0}$ but they produce a net torque which tends to rotate the loop about the axis xx' as shown in Fig. 14.1(a). This torque can be represented with a vector pointing along the xx' axis from right to left in Fig. 14.1(a). The magnitude of this torque, τ' say, equals twice the torque caused by $\mathbf{F}_{BC}$, that is

$$\tau' = 2(iaB)\left(\frac{b}{2}\right)\sin\theta = iabB\sin\theta = iSB\sin\theta, \tag{14.55}$$

where $S = ab$ is the area of the loop. It can be shown that this result holds for all plane loops of area S, whether they are rectangular or not. If we have N loops together, as in a coil, then the total torque on the coil is $\tau_{coil} = N\tau' = iNSB\sin\theta$. The quantity

$$\boldsymbol{\mu} = iNS\hat{\mathbf{e}}, \tag{14.56}$$

where $\hat{\mathbf{e}}$ is the unit vector along the direction xx', is called the *magnetic dipole moment* of the coil and the coil itself is called a magnetic dipole. In general by a *magnetic dipole* we understand any structure which interacts with the magnetic field and this interaction is characterized by the magnetic dipole moment of the magnetic dipole and the magnetic field **B**, producing the torque

$$\boldsymbol{\tau} = \boldsymbol{\mu}\times\mathbf{B}. \tag{14.57}$$

When a magnetic dipole is placed in a magnetic field its orientation changes so that a work (positive or negative) must be done by an external agent to restore the orientation of the magnetic dipole. Thus the magnetic dipole has potential energy U associated with its orientation in an external magnetic field. This energy may be taken to vanish for any arbitrary initial position of the magnetic dipole. If we assume that the potential energy vanishes when $\boldsymbol{\mu}$ and $\mathbf{B}$ are at right angles (that is, when $\theta = \pi/2$) in (14.55), then it can be shown that

$$U = -\boldsymbol{\mu}\cdot\mathbf{B}. \tag{14.58}$$

The conclusion from the above considerations is that a magnetic dipole acquires a torque in its center of mass which is of no kinematical character but it is due to its interaction with the magnetic field. This "dynamic" torque gives rise to an angular momentum, which we call the *intrinsic angular momentum* or *spin angular momentum* of the magnetic dipole. This angular momentum must be added to the kinematic angular momentum of the magnetic dipole (e.g., the particle) to make up the total angular momentum which modulates the motion of the magnetic dipole in a magnetic field.

Example 78 Consider the Bohr model of the atom of hydrogen in which the electron orbits in a circular path of radius r around the nucleus. This may be considered as a tiny current loop so that the atom itself is a magnetic dipole. The magnetic dipole moment of this atom is called *orbital magnetic dipole moment* and denoted by μ_l. Derive a relation between μ_l and the orbital angular momentum L_l of the electron. Compute μ_l if $r = 5.1 \times 10^{-11}$ m and the ratio $\frac{e}{m} = 1.76 \times 10^{11}$ Cb/kg.
Solution

The force on the electron due to the charge of the nucleus is the Coulomb force $F = k\frac{e^2}{r^2}$ where e is the charge of the electron (and the nucleus) and k is a constant depending on the system of units. This force is a centrifugal force, therefore $F = \frac{mv^2}{r}$ where m is the mass of the electron and v its speed. Equating the two expressions of the force we find

$$v = \sqrt{\frac{ke^2}{mr}}.$$

The angular velocity of rotation is

$$\omega = \frac{v}{r} = \sqrt{\frac{ke^2}{mr^3}}.$$

The current produced by the rotation of the electron is the rate at which it passes through any given point of the orbit, hence

$$i = e\nu = e\left(\frac{\omega}{2\pi}\right) = \frac{e^2}{2\pi}\sqrt{\frac{k}{mr^3}}.$$

The orbital dipole moment μ_I is given by (14.56) if we put $N = 1$ and $A = \pi r^2$, that is

$$\mu_I = \frac{e^2}{2\pi}\sqrt{\frac{k}{mr^3}}\pi r^2 = \frac{e^2}{2}\sqrt{\frac{kr}{m}}. \tag{14.59}$$

The orbital angular momentum L_I of the electron is

$$L_I = mvr = m\omega r^2 = m\sqrt{\frac{ke^2}{mr^3}}r^2 = \frac{2m}{e}\mu_I,$$

which shows that the orbital angular momentum of the electron is proportional to the magnetic dipole moment.

Introducing the data in (14.59) and taking $k = \frac{1}{4\pi\varepsilon_0}$ (MKS system) we compute $\mu_I = 9.1 \times 10^{-24}$A m^2.

It is an experimental fact that the elementary particles are magnetic dipoles, that is, they have an intrinsic angular momentum. Originally this was confirmed for the electron and then it was established for all other particles. This means that an electron in its proper frame creates an electric field due to its charge and a magnetic field due to its magnetic dipole moment. The field lines of these two fields are shown in Fig. 14.2 where the intrinsic angular momentum $\mathbf{L}_I$ is also shown. The fact that the elementary particles are magnetic dipoles, and not simply charged or neutral units of mass, indicates that they consist of "smaller" more "elementary" particles in the same way the atom is a magnetic dipole due to the rotation of the rotating electron. This is true even for the particles with zero intrinsic angular momentum,

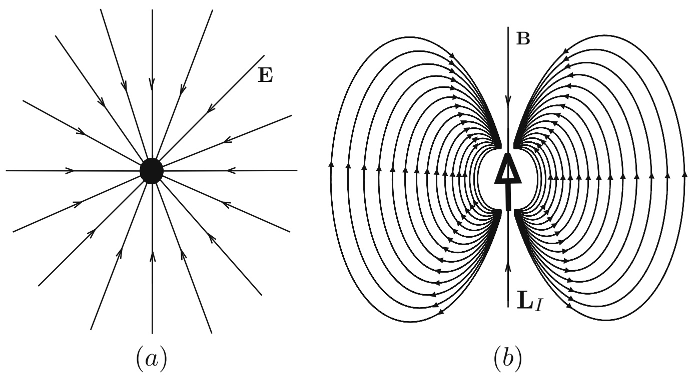

Fig. 14.2 Electric and magnetic field lines of electron

in the sense that the parts it consists of cancel the effects of each other. The following exercise could be an extreme mechanistic physical explanation of the magnetic dipole moment of the electron.

Exercise 76 Assume that the electron is a small sphere of radius R, its charge and mass being distributed uniformly throughout its volume. It has been measured that such an electron has an intrinsic angular momentum $L_I = 0.53 \times 10^{-34}$J s and a magnetic dipole moment $\mu_I = 9.1 \times 10^{-24}$A m^2. Show that the ratio $e/m = 2\mu_I/L_I$. To justify this result divide the spherical electron into infinitesimal current loops and find an expression for the magnetic dipole moment by integration.

14.5.2 The Relativistic Spin

In the last section we have shown that besides the kinematic angular momentum bivector and the torque tensor which correspond directly to the relevant concepts of Newtonian theory, and both vanish in the particle's proper frame, there is another angular momentum of non-kinematic nature which must be taken into account. Although the appropriate place to discuss this topic is quantum electrodynamics,[7] in the following we shall attempt a classic treatment which we find is of some physical value.

The question we have to answer is

> How could one incorporate the two types of angular momentum, kinematic and dynamic, in one, the total angular momentum?

The answer to this question is necessary because experiment has shown that the elementary particles are magnetic dipoles, therefore their motion in magnetic fields (which is a routine in experimental physics) will be modulated by the total angular momentum and not by the orbital angular momentum alone.

Looking for the answer we note that the orbital angular momentum bivector L_{ab} has only electric part. Therefore if we add a bivector S_{ab} to L_{ab}, which has only magnetic part, then we have the total angular momentum while we preserve the kinematic and the dynamic characters apart. The requirement that the electric part $E^a(S)$ of S_{ab} vanishes is (use (14.4) to see this)

$$S_{ab}p^b = 0 \Leftrightarrow S_a u^a = 0, \tag{14.60}$$

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[7] The reader may wonder why we bother to discuss the concept of spin within the limits of the non-quantum theory when we know that only a quantal description can be correct. The answer lies in the quantal theorem which states that the classical equation of motion of a dynamical variable is the quantal equation of motion of the mean value of that variable averaged over an ensemble of identical systems. Therefore the conclusions we shall draw, with the classical treatment, will apply to averages over many identical particles prepared in the same way, like the electrons or muons in a beam or the valence electrons in a gas of atoms in a glow tube.

where we have introduced the *spin vector*:

$$S^a = \frac{1}{2c^2}\eta^{abcd}u_b S_{cd} \tag{14.61}$$

or equivalently

$$S_{ab} = \eta_{abcd}S^c u^d. \tag{14.62}$$

Concerning the magnetic part $H^a(S)$ of S_{ab} from (14.8) we find

$$H^a(S) = \frac{1}{2p^2}\eta^{abcd}p_b S_{cd} = \frac{1}{m}S^a. \tag{14.63}$$

The length of the spin vector S^a is computed as follows:

$$\begin{aligned} S^2 = S^a S_a &= \frac{1}{4c^4}\eta^{abcd}u_b S_{cd}\eta_{arst}u^r S^{st} \\ &= -\frac{1}{4c^4}\left(\delta^b_r\delta^c_s\delta^d_t - \delta^b_r\delta^d_s\delta^c_t + \delta^c_r\delta^d_s\delta^b_t - \delta^c_r\delta^b_s\delta^d_t + \delta^d_r\delta^b_s\delta^c_t - \delta^d_r\delta^c_s\delta^b_t\right)u_b u^r S_{cd}S^{st} \\ &= -\frac{1}{4c^4}2(-c^2)S_{ab}S^{ab} = \frac{1}{2c^2}S_{ab}S^{ab}. \end{aligned} \tag{14.64}$$

The invariant S^2 is called the *spin of the particle*. It is independent of the mass and it is this quantity which is quantized in multiples of $\hbar/2$.

There still remains the propagation of the intrinsic angular momentum along the particle's world line, that is, the quantity $\frac{dS_{ab}}{d\tau} = \dot{S}_{ab}$. This is a bivector whose irreducible parts are the vectors $e^a(S)$ and $h^a(S)$ computed in the general relations (14.16) and (14.17). Substituting in these relations $E^a(S) = 0$, $H^a(S) = -\frac{1}{2mc^2}S^a$ we find

$$\begin{aligned} e^a(S) &= \frac{1}{p^2}\dot{S}_{ab}p^b = -\frac{1}{2p^2}\eta^{abcd}p_b\dot{p}_c H_d, \\ h^a(S) &= h^a_b\dot{H}^b = \frac{1}{m}h^a_b\dot{S}^b. \end{aligned} \tag{14.65}$$

From (14.7) we have then

$$\dot{S}_{ab} = -e_a(S)p_b + e_b(S)p_a - \eta_{abcd}p^c h^a(S) = -e_a(S)p_b + e_b(S)p_a - \eta_{abcd}u^c\dot{S}^d. \tag{14.66}$$

This is as far as one can go with the mathematics. Physics will give an expression for the quantity $\dot{S}_{ab}$ and there will result an equation of motion for the spin vector. Newton's Second Law cannot be used because S_{ab} is a non-Newtonian physical quantity, therefore Newtonian Physics cannot (and in fact need not) say anything about it.

In order for physics to make a statement it goes over to experiment and observation. Because S_{ab} is an angular momentum $\dot{S}_{ab}$ must be a torque. We have seen that with each magnetic dipole there is associated a magnetic dipole moment $\boldsymbol{\mu}_I$ and if a magnetic dipole is placed in a magnetic field $\mathbf{B}$ it suffers a torque $\boldsymbol{\tau} = \boldsymbol{\mu}_I \times \mathbf{B}$ (see (14.57)). Furthermore, experiment has shown that the elementary particles (in the standard sense of the term) behave as magnetic dipoles. These and the expected application of the theory of relativity to elementary particles lead us to relate the magnetic dipole moment with the magnetic part of the tensor $\dot{S}_{ab}$, and subsequently via (14.66), with the spin vector S^a.

Now, let $\tau_{ab}(S)$ be the torque tensor corresponding to the intrinsic angular momentum S_{ab}. Then we assume the equation of motion

$$\dot{S}_{ab} = \tau_{ab}(S). \tag{14.67}$$

The torque tensor has an "electric" part (which vanishes) and a "magnetic" part $H^a(\tau)$ which equals the magnetic part of $\dot{S}_{ab}$, that is we have

$$H^a(\tau) = h^a(S) = \frac{1}{m} h^a_b \dot{S}^b.$$

In analogy with the Newtonian result (14.57), we *define* $H^a(\tau)$ by the formula

$$H^a(\tau) = \frac{1}{m} \eta^{abcd} u_b \mu_{Ic} B_d, \tag{14.68}$$

where μ_{Ic} is the magnetic dipole moment of the particle. Then the equation of motion of the spin vector is

$$h^a_b \dot{S}^b = \eta^{abcd} u_b \mu_{Ic} B_d \tag{14.69}$$

or using (14.79) (see below)

$$h^a_b \dot{S}^b = g \frac{|q|}{2mc} \eta^{abcd} u_b S_c B_d. \tag{14.70}$$

In the proper frame of the particle the equation of motion (14.70) is written as

$$\frac{d\mathbf{S}^*}{d\tau} \overset{\&}{=} g \frac{|q|}{2mc} \mathbf{S}^* \times \mathbf{B}^*, \tag{14.71}$$

where the $*$ besides a symbol indicates that the quantity is computed in the proper frame of the particle and the & sign above the equality indicates that the equation holds in the proper frame of the particle only.

The equation of motion (14.70) specifies the spatial part $h^a_b \dot{S}^b$ of $\dot{S}^a$. To find the equation of motion of $\dot{S}^a$ we write[8]

$$\dot{S}^a = -(\dot{S}^b u_b)u^a + h^a_b \dot{S}^b \tag{14.72}$$

under the condition $S^a u_a = 0$. This equation can be written as

$$\dot{S}^a = (S^b \dot{u}_b)u^a + h^a_b \dot{S}^b, \tag{14.73}$$

where $\dot{u}^a$ is the four-acceleration of the particle. Using Newton's generalized second law we write $\dot{u}^a = F^a/m$ where F^a is the (inertial) four-force acting on the particle and m is its mass. These give

$$\dot{S}^a = \frac{1}{m}(S^b F_b)u^a + g\frac{|q|}{2mc}\eta^{abcd}u_b S_c B_d,$$

which is the general formula of the propagation of spin.

We derive a formula for $\dot{S}^a$ when the particle moves in a homogenous electromagnetic field[9] F_{ab}. In this case

$$F_a = \frac{q}{c^2}F_{ab}u^b, \tag{14.74}$$

hence

$$\dot{S}^a = \frac{q}{mc^2}(S^b F_{bc}u^c)u^a + g\frac{|q|}{2mc}\eta^{abcd}u_b S_c B_d. \tag{14.75}$$

We know that the magnetic field

$$B^a = \frac{1}{2c}\eta^{abcd}u_b F_{cd}. \tag{14.76}$$

Replacing in the second term of (14.75) we find

$$\begin{aligned}
&g\frac{|q|}{4mc^2}\eta^{abcd}u_b S_c \eta_{drst}u^r F^{st} \\
&= -g\frac{|q|}{4mc^2}(-1)2\left(\delta^a_r\delta^b_s\delta^c_t + \delta^b_r\delta^c_s\delta^a_t + \delta^c_r\delta^a_s\delta^b_t\right)u_b S_c u^r F^{st}
\end{aligned}$$

[8] This is the 1+3 decomposition of $\dot{S}^a$ wrt u^a. Note that $S^a u_a = 0$ does not imply $\dot{S}^a u_a = 0$.

[9] See (a) V. Bargmann, L. Michel and V. L. Telegdi "Recession of the polarization of particles moving in a homogeneous electromagnetic field" *Phys. Rev. Lett.* **2**, 435–436, (1959). (b) V. Henry and J. Silver "Spin and orbital motions of a particle in a homogeneous magnetic field" *Phys. Rev.* **180**, 1262–1263, (1969).

$$= g\frac{|q|}{4mc^2}2\left(u^a F^{bc} + u^b F^{ca} + u^c F^{ab}\right) u_b S_c$$

$$= g\frac{|q|/m}{2c^2}(F^{bc}u_b S_c u^a - c^2 F^{ca} S_c)$$

$$= g\frac{|q|/m}{2}F^{ac}S_c + g\frac{|q|/m}{2c^2}(F^{bc}u_b S_c)u^a.$$

Therefore

$$\dot{S}^a = \frac{q}{mc^2}(S^b F_{bc}u^c)u^a + g\frac{|q|/m}{2}F^{ac}S_c + g\frac{|q|/m}{2c^2}(F^{bc}u_b S_c)u^a \Rightarrow \tag{14.77}$$

$$\dot{S}^a = \frac{|q|}{m}\frac{g}{2}F^{ac}S_c + \frac{|q|}{mc^2}\left(\frac{g}{2} - \frac{q}{|q|}\right)(S^b F_{cb}u^c)u^a. \tag{14.78}$$

The experiments which confirm the proportionality of the magnetic dipole moment and the change of the intrinsic angular momentum are called "gyromagnetic" experiments. Relation (14.78) has been confirmed by such experiments on many different systems. The constant of proportionality is one of the parameters characterizing the particular system. It is normally specified by giving the *gyromagnetic ratio* or *g-factor*, defined by the relation

$$\mu_I^a = g\frac{|q|}{2mc}S^a, \tag{14.79}$$

where $|q|/m$ is the (measure of) charge to mass ratio of the particle. The first successful experiments to show this have been performed by Einstein and de Hass as early as 1915 and later on (1935) by Barnett.[10] It has been found that for the electron $g_{e^-} = -2$ and for the positron $g_{e^+} = 2$. Pion has $g_\pi = 0$. The magnetic dipole moment of the hydrogen atom results from both the electron orbital motion and the electron spin. These two interact and the value of the g-factor for the atom is between -1 (pure electron orbit) and -2 (pure electron spin). For electrons, positrons, and muons experiment has given the following values:

$$\begin{aligned}
g_{e^-} &= -2(1 + 1.1596 \times 10^{-3}),\\
g_{e^+} &= +2(1 + 1.17 \times 10^{-3}),\\
g_{\mu^-} &= -2(1 + 1.166 \times 10^{-3}),\\
g_{\mu^+} &= +2(1 + 1.16 \times 10^{-3}),
\end{aligned}$$

[10] A. Einstein and W. J. de Haas Verhandl. *Deut. Phys. Ges.* **17**, 152, (1915); S.J. Barnett *Rev. Mod. Phys.* **7**, 129, (1935)

therefore for all these cases we find the simple equation of motion for the spin vector:

$$\dot{S}^a \simeq \frac{|q|}{m}\frac{g}{2}F^{ac}S_c.$$

The difference

$$a = \frac{g}{2} - \frac{q}{|q|}$$

is called the *magnetic moment anomaly*. Note that the value of a is the same for electron and positron and this result holds in general for a particle and its antiparticle. This is a result of the ratio $\frac{q}{|q|}$ and the opposite signs of g for each particle.

From the general relation (14.78) we note that $\dot{S}^a S_a = 0 \Rightarrow S^2$ =constant. This result and $S^a u_a = 0$ show that the spin vector is a spacelike vector in the rest space of u^a which rotates about the origin of the proper frame of the particle. The rotation depends on the external magnetic field.

14.5.3 Motion of a Particle with Spin in a Homogeneous Electromagnetic Field

Consider a particle of mass m, charge q, and spin vector S^a moving in a homogeneous electromagnetic field F_{ab}. If u^a is the four-velocity of the particle the four-force on the particle is $\frac{q}{c^2}F_{ab}u^b$ and the magnetic field is $B_a = \frac{1}{2c}\eta_{abcd}u^b F^{cd}$. Newton's Second Law is the equation of motion of the four-velocity:

$$m\dot{u}^a = \frac{q}{c^2}F_{ab}u^b \tag{14.80}$$

and (14.78) is the equation of motion of the spin vector. Because the electromagnetic field is homogeneous, the four-acceleration is constant and equal to $\dot{u}^a = \frac{q}{mc^2}F_{ab}u^b$. To find the motion of the spin vector we consider (14.78) in the proper frame of the particle where this equation is reduced to (14.71), which we write in the form

$$\dot{S}^{*\mu} = a\eta^{\mu\nu\rho}S^*_\nu B^*_\rho, \tag{14.81}$$

where $a = g\frac{|q|}{2mc}$ and an $*$ indicates a three-vector in the proper frame of the particle.

From the equation of motion (14.81) we have

$$S^2 = \text{constant and } \dot{S}^{*\mu}B^*_\mu = 0 \Rightarrow (S^{*\mu}B^*_\mu)^{\cdot} = 0,$$

therefore the angle ζ between the three-vectors $S^{*\mu}$, $B^{*\mu}$ is constant in the proper frame of the particle. This implies that during the motion of the spin (not of the particle!) the vector $S^{*\mu}$ traces the surface of a right circular cone with axis along the magnetic field B^*_μ with opening angle ζ. The solution of (14.81) is written as follows:

$$S^{*\mu}(\tau) = S^* \sin\zeta(\cos\omega^*\tau e^\mu_{(1)} + \sin\omega^*\tau e^\mu_{(2)}) + \cos\zeta\, S^* e^\mu_{(3)}, \tag{14.82}$$

where $e_{(1)\mu}, e_{(2)\mu}, e_{(3)\rho}$ is an orthonormal basis and $B^*_\mu = B^* e_{(3)\mu}$. To determine the angular speed ω^* we compute the derivative $\dot{S}^{*\mu}$ and then use the equation of motion (14.81). We find

$$\omega^*\left(-\sin\omega^*\tau e^\mu_{(1)} + \cos\omega^*\tau e^\mu_{(2)}\right) = a\eta^{\mu\nu\rho}(\cos\omega^*\tau e^\mu_{(1)} + \sin\omega^*\tau e^\mu_{(2)})B^* e^\mu_{(3)}.$$

Taking $\mu = 1$ we find easily that $\omega^* = aB^*$, hence

$$\omega^{*\mu} = -aB^* e^\mu_{(3)}. \tag{14.83}$$

Therefore the solution of the equation of motion in the proper frame is

$$S^{*\mu} = S^* \sin\zeta(e_{(1)\mu}\cos aB^*\tau + e_{(2)\mu}\sin aB^*\tau) + \cos\zeta\, S^* e_{(3)\mu}. \tag{14.84}$$

This represents a regular precession in which the spin vector traces out a right circular cone with the direction of the magnetic field as axis and constant angular velocity $\boldsymbol{\omega}^* = \frac{g}{2}\frac{|q|\boldsymbol{B}^*}{mc}$. The quantity $\boldsymbol{\omega}^*_c = -\frac{|q|\mathbf{B}^*}{mc}$ is the cyclotronic frequency[11] in the proper frame of the particle (see Fig. 14.3).

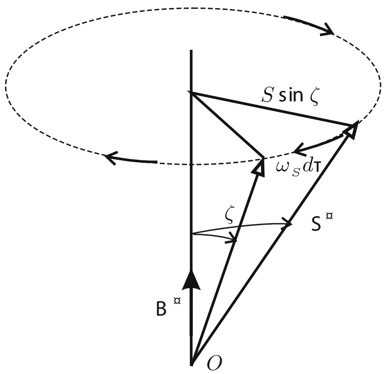

Fig. 14.3 Spin precession in a uniform magnetic field

[11] This is the angular speed of a particle of charge q which is introduced at right angles to a uniform magnetic field of magnetic induction $\mathbf{B}^*$.

14.5.4 Transformation of Motion in Σ

In order to find the motion of spin in another coordinate frame, the Σ say, we have to apply the appropriate Lorentz transformation to the various quantities involved. However, this is not enough. Indeed in the proper frame the spin precesses around the magnetic field with angular velocity $\boldsymbol{\omega}^* = -\frac{g}{2}\boldsymbol{\omega}_c^*$ whereas the particle accelerates (i.e., $a^+ \neq 0$) as it moves. This means that the spatial directions in the proper frame of the particle suffer the Thomas rotation which (in Σ!) is given by the angular velocity $\boldsymbol{\omega}_T = -\frac{\gamma_u - 1}{u^2}\mathbf{u}\times\mathbf{a}$, where $\mathbf{u}$, $\mathbf{a}$ are the velocity and the acceleration of the particle in Σ (see (14.70)). Therefore in Σ the spin vector executes two independent rotational motions with angular velocity $\boldsymbol{\omega}^*$ and $\boldsymbol{\omega}_T$, the net angular velocity being the composition of the two angular velocities. Let us compute the motion of the spin in Σ.

Choose the coordinates in Σ so that the z-axis is along the direction of the homogeneous magnetic field $\mathbf{B}$ and assume that the initial velocity of the particle is normal to the magnetic field so that the motion takes place in the plane $x,\ y$ with basis vectors $\hat{\mathbf{e}}_{1,}\hat{\mathbf{e}}_2$. The electromagnetic field induced in the proper frame (=local rest frame!) of the particle at each position along its trajectory is given by the transformation formulae

$$\mathbf{E}^* = \mathbf{E}_{||} + \gamma_u\left[\mathbf{E}_\perp + \mathbf{u}\times\mathbf{B}\right] = \gamma_u u B\hat{\mathbf{e}}_2, \tag{14.85}$$

$$\mathbf{B}^* = \mathbf{B}_{||} + \gamma_u\left[\mathbf{B}_\perp - \frac{1}{c^2}\mathbf{u}\times\mathbf{E}\right] = -\gamma_u B\hat{\mathbf{e}}_3, \tag{14.86}$$

where $\mathbf{u} = u\hat{\mathbf{e}}_1$ is the velocity of the particle in Σ. We see that in the proper frame the direction of the magnetic field is along the z-axis whereas the electric field is uniform and along the direction of the radius. The force due to the electric field is

$$\mathbf{F}^* = q\mathbf{E}^* = q\gamma_u u B\hat{\mathbf{e}}_2 \tag{14.87}$$

and it is a centripetal force (otherwise the particle would not rotate[12]) with acceleration

$$a^* = \omega_c^* u. \tag{14.88}$$

From the above we find

$$\boldsymbol{\omega}_c^* = \gamma_u\frac{|q|B}{mc} = \gamma_u\omega_0, \tag{14.89}$$

where $\omega_0 = \frac{|q|B}{mc}$ is the Newtonian cyclotronic angular speed in Σ. Therefore in the proper frame we have that the spin precesses with angular velocity

[12] We can always make that force centripetal by changing the direction of the speed $\mathbf{u}$.

$$\boldsymbol{\omega}^* = -\frac{g}{2}\gamma_u \omega_0 \hat{\mathbf{e}}_3. \tag{14.90}$$

From the transformation of the angular velocity[13] we have that in Σ

$$\boldsymbol{\omega} = -\frac{g}{2}\omega_0 \hat{\mathbf{e}}_3. \tag{14.91}$$

Concerning the Thomas precession we have shown in (6.72) that for a circular motion

$$\boldsymbol{\omega}_T = -(\gamma_u - 1)\omega_c \hat{\mathbf{e}}_3, \tag{14.92}$$

where ω_c is the angular (i.e., the cyclotronic) speed of rotation given by

$$\omega_c = \frac{|q|B}{\gamma_u mc} = \frac{1}{\gamma_u}\omega_0. \tag{14.93}$$

Eventually we have that in Σ the spin precesses around the direction of $\mathbf{B}$ with total angular velocity

$$\boldsymbol{\omega}_S = \boldsymbol{\omega} + \boldsymbol{\omega}_T = -\left(\frac{g}{2} + 1 - \frac{1}{\gamma_u}\right)\omega_0 \hat{\mathbf{e}}_3. \tag{14.94}$$

One important (and well-known) conclusion is that the difference

$$\boldsymbol{\omega}_D \equiv \boldsymbol{\omega}_S - \boldsymbol{\omega}_c = -\left(\frac{g}{2} + 1\right)\omega_0 \hat{\mathbf{e}}_3, \tag{14.95}$$

which is possible to be measured directly, is independent of the speed of the particle. This fact facilitates the measurement of the factor g because one can use beams of the same particles but different initial speeds.

In case the velocity of the particles is not normal to the magnetic field, we decompose it into two components, one parallel to the field and one normal to it. The above considerations hold for the normal component. Concerning the parallel component this stays constant (the magnetic field has no effect parallel to its direction) hence the motion parallel to the magnetic field is with uniform velocity. In conclusion the motion of the particle (not the spin!) is the combination of two motions: one motion with uniform velocity (drifting) parallel to the direction of $\mathbf{B}$ and a planar circular motion with uniform angular velocity $\boldsymbol{\omega}_S$ with axis along the direction of $\mathbf{B}$. The combination of the two motions results in a helical motion with axis along

[13] Note that the angular velocity equals $2\pi/T$, therefore its transformation is like T^{-1}, that is

$$\omega^* = \gamma_u \omega,$$

where ω is the angular speed in Σ and ω^* is the angular speed in Σ^*.

the direction of **B**. This means that the frame Σ we considered is drifting along the direction of **B** with constant speed $u_{||}$, therefore in order to find the motion in the original frame, the Σ' say, we have to apply the appropriate boost along the z-axis. This means that the circular frequencies we have found above must be multiplied with the factor $\frac{1}{\gamma_{||}}$. For example in Σ' the spin precesses with angular velocity

$$\boldsymbol{\omega}'_S = -\left(\frac{g}{2} + 1 - \frac{1}{\gamma_u}\right)\frac{1}{\gamma_{||}}\omega_0 \hat{\mathbf{e}}_3. \tag{14.96}$$

Chapter 15
The Covariant Lorentz Transformation

15.1 Introduction

It can hardly be emphasized that one of the most important elements of Special Relativity is the Lorentz transformation. This is the reason why we have spent so much space and effort to derive and study the Lorentz transformation in the early chapters of the book. One could naturally ask "After all these different derivations of the Lorentz transformation why are we not yet finished with it?" The reason is the following. Special Relativity is a geometric theory of physics, which can be written and studied covariantly in terms of Lorentz tensors (four-vectors, etc.) without the need to consider a coordinate system until the very end, when one has to compute explicitly the components of the physical quantities of a problem for some observer.

All derivations of the Lorentz transformations so far used either coordinates or three-vectors, that is, they were not covariant. This does not mean that these transformations are not the general ones or that they are insufficient to deal with all relativistic problems. The point is that, although the covariant Lorentz transformation is not necessary for the development and the application of the theory, in cases where one has to deal with general problems of a qualitative character or with involved problems, the covariant formalism significantly simplifies the calculations because one is able to apply geometric techniques to get answers which would not be feasible to get with the standard form of the transformation. Furthermore, if one describes a problem in covariant formalism then it is possible to use one of the well-known algebraic computing programs to perform the calculations. This is of practical importance because it makes easy (or even possible) complex calculations, which would be unbearable to be carried out by hand and, most important, without mistakes. Finally, we emphasize the aesthetic side of the matter and claim that since the Theory of Special Relativity is a geometric theory of motion, its description in any but the covariant formalism hides much of the power and the elegance of the theory.

In this chapter we derive the covariant form of Lorentz transformation and we refer to some of its applications. As it might be expected this chapter is of a more advanced level, therefore it is advisable to be studied after the reader has some experience with Special Relativity.

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Before we get into the details, let us take a quick promenade into the relevant literature. Based on the vector form of the Lorentz transformation, Fahnline[1] gave the covariant expression of the (proper) Lorentz transformation and the corresponding covariant transformation of a four-vector. He used this form of the transformation to compute the composition of successive Lorentz transformations in a brute yet direct way. This problem is not easy. Indeed, it is well known[2] that in order to compute the covariant form of the Euclidean rotation of this decomposition one has to invoke the homomorphism between the restricted Lorentz group SO(3,1) and SL(2,C). Fahnline showed that this is not necessary, provided that the covariant form of the transformation is used.

Finally, a different and very interesting approach to the covariant Lorentz transformation has been given by Krause[3] who expressed the Lorentz transformation in terms of two unit timelike vectors u^a, v^a corresponding to the four-velocities of inertial observers related by the transformation. He considered only the proper Lorentz transformation, whose covariant form he derived using the orthonormal tetrads associated with the observers u^a, v^a. He followed a similar technique developed by Basanski[4] who used null tetrads to express the Lorentz transformation in terms of spacetime rotations. The approach of Krause is more fundamental than that of Fahnline.

Finally, in a more recent work, Jantzen et al.[5] have introduced a covariant form of Lorentz transformation based on relative velocity and the fact that under a Lorentz transformation between two observers the timelike plane defined by the four-velocities of the observers and the normal two-plane to that plane are preserved. They obtained the proper Lorentz transformation (they call it "relative observer boost") and they discuss some of its properties, however again in a form which obscures the simple geometric significance of the Lorentz transformation. In a similar approach Urbantke[6] studied the Lorentz transformation and considered some applications of the proper Lorentz transformation in terms of spacetime reflections.

Before we present our approach, we emphasize once more that the Lorentz transformation (in vector or in covariant form) is of a pure mathematical origin and has nothing to do with Special Relativity or any other theory of physics. Its connection with physics is introduced by the Principle of Special Relativity, which demands that all physical quantities in Special Relativity must be covariant under the Lorentz

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[1] D. Fahnline *Am. J. Phys.* **50**, 50, 818–821, (1982).

[2] See F. R. Halpern (1968) "Special Relativity and Quantum Mechanics" (Prentice Hall, Englewood Cliffs, NJ and Goldstein Herbert "Classical Mechanics" Second Edition (1980) Addison-Wesley, chapter 7

[3] J. Krause "Lorentz transformations as space-time reflections" *J. Math. Physics* **18**, 879–893, (1977).

[4] S. L. Basanski "Decomposition of the Lorentz Transformation Matrix into Skew – Symmetric Tensors" *J. Math. Phys.* **6**,1201–1202, (1965).

[5] R. Jantzen, P. Capini, D. Bini *Ann Phys.* **215**, (1992) 1 and gr-qc/0106043.

[6] H. K. Urbantke *Found. Phys. Lett.* **16**, 111, (2003).

transformation (that is, they must be Lorentz tensors). This point is justified by the fact that we are able to derive/study the Lorentz transformation based on geometric assumptions only without making reference to any physical concepts or systems.

15.2 The Covariant Lorentz Transformation

In Chap. 1 we derived the Lorentz transformation $L : M^4 \rightarrow M^4$ as an endomorphism of Minkowski space satisfying the following two requirements:

(1) L is a linear transformation.
(2) L is an isometry of M^4 which in addition preservers the canonical form $\eta = diag(-1, 1, 1, 1)$ of the Lorentz metric, that is, it satisfies the equation $L^t \eta L = \eta$.

The derivation of the covariant Lorentz transformation will be based on geometric assumptions which are hidden in the above requirements. Indeed the linearity of the transformation means that L preserves the linear geometric elements (straight lines, two-planes and three-planes or hyperplanes) in M^4. The first part of the second assumption means that it also preserves the character of these linear elements, that is, a timelike straight line goes to a timelike straight line, a spacelike two-plane to a spacelike two-plane, etc. Since a timelike straight line is characterized by its unit tangent vector, let u^a, v^a be two unit timelike vectors related by the Lorentz transformation.

Then the preservation of the canonical form η of the Lorentz metric implies that L preserves the Euclidean planes normal to the four-vectors u^a, v^a. Equivalently, the Lorentz transformation relates the LCF associated with the four-vectors u^a, v^a. The above observations are sufficient in order to define the covariant form of the Lorentz transformation.

In the following we shall need the projection tensors h_{ab} and p_{ab} we defined in (12.2) and (12.21).

15.2.1 Definition of the Lorentz Transformation

Definition 14 Let u^a, v^a be two unit, non-collinear, timelike vectors, corresponding to the four-velocities of two (inertial) observers, which define a timelike two-plane. The (planar) Lorentz transformation defined by u^a, v^a is the map[7] $L : M^4 \rightarrow M^4$ specified by the following requirements:

(1) L is a linear transformation and preserves the timelike two-plane spanned by the vectors u^a, v^a and the spacelike two-plane normal to the u^a, v^a two-plane.
(2) L is an isometry of M^4.

[7] To be precise this is a transformation in the tangent space of M^4 but due to the flatness of M^4 it can be reduced unambiguously to a transformation in M^4.

(3) L is defined on an equal footing in terms of u^a, v^a (this is the so-called *reciprocity principle*[8]). This is formulated by the requirement that the inverse Lorentz transformation is the same with the direct but with u^a, v^a interchanged.

Let us write the verbal requirements of the definition in terms of equations.

Requirement 1 implies for the timelike two-plane spanned by the vectors u^a, v^a the equations

$$Lv^a = Au^a + Bv^a, \qquad Lu^a = Cu^a + Dv^a, \tag{15.1}$$

where A, B, C, D are quantities independent of the spacetime coordinates such that $AB - CD \neq 0$ (because u^a, v^a are not collinear) and $AD \neq 0$ (otherwise the transformation is singular).

For the spacelike two-plane normal to that plane the same requirement implies the equation

$$Lp_{ab}(u, v) = p_{ab}(u, v), \tag{15.2}$$

where p_{ab} is the projection tensor which projects normal to the timelike plane of u^a, v^a. Requirement 2 means that the inner products of vectors are preserved. For the two vectors u^a, v^a this requirement implies the equations

$$(Lu^a, Lu^a) = (Lv^a, Lv^a) = -1, \qquad (Lu^a, Lv^a) = (u^a, v^a). \tag{15.3}$$

Requirement 3 means that the inverse transformation applied to u^a has the same result as the direct transformation applied to the vector v^a and vice versa. This implies the equations

$$L^{-1}u^a = Av^a + Bu^a, \qquad L^{-1}v^a = Cv^a + Du^a. \tag{15.4}$$

We conclude that

(a) In order to specify the (planar) Lorentz transformation defined by the vectors u^a, v^a it is enough to compute the unknown coefficients A, B, C, D describing the transformation.
(b) The transformation is defined by (15.1), (15.2), (15.3), and (15.4).

15.2.2 Computation of the Covariant Lorentz Transformation

We introduce the quantity (we consider $c = 1$ in what follows)

$$\gamma = -u^a v_a \tag{15.5}$$

[8] The interested reader can find more information on this principle in V. Bruzzi and V. Gorini "Reciprocity principle and the Lorentz transformation" *J. Math. Phys.* **10**, 1518–1524, (1989).

(this is the gamma-factor relating the two observers) and use (15.3) to get

$$(Lu^a, Lv^a) = (u^a, v^a) = -\gamma \Rightarrow (AC + BD) + (BC + AD - 1)\gamma = 0.$$

From the lengths of the four-vectors we have

$$v^a v_a = -1 \Rightarrow A^2 + 2\gamma AB + B^2 = 1$$

and

$$u^a u_a = -1 \Rightarrow C^2 + 2\gamma CD + D^2 = 1.$$

The first equation of relation (15.4) gives

$$u^a = LL^{-1}u^a = L(Av^a + Bu^a) = (A^2 + BC)u^a + (AB + BD)v^a,$$

hence

$$\begin{aligned} A^2 + BC &= 1, \\ AB + BD &= 0. \end{aligned}$$

Similarly the second part of (15.4) gives

$$\begin{aligned} D^2 + BC &= 1, \\ AC + CD &= 0. \end{aligned}$$

Finally, after some simple manipulations, we have the following system of seven non-linear algebraic equations in the four unknowns $A,\ B,\ C,\ D$:

$$B(2\gamma A + B - C) = 0, \tag{15.6}$$
$$C(2\gamma D + C - B) = 0, \tag{15.7}$$
$$B(A + D) = 0, \tag{15.8}$$
$$C(A + D) = 0, \tag{15.9}$$
$$A^2 - D^2 = 0, \tag{15.10}$$
$$A^2 + BC = 1, \tag{15.11}$$
$$(AC + BD) + (BC + AD - 1)\gamma = 0. \tag{15.12}$$

We expect that the system has many solutions, therefore there are more than one Lorentz transformations satisfying the conditions of Definition 14. A simple analysis shows that the solutions of the system can be classified by the vanishing or not of the quantities $A + D$, $A - D$. It is an easy exercise to show that the system admits the following families of solutions ($AD \neq 0$):

(1) $A - D = 0,\ A + D \neq 0 \Longleftrightarrow\ A = D \neq 0.$

A	B	C	D
± 1	0	0	± 1

(2) $A + D = 0,\ A - D \neq 0 \Longleftrightarrow\ A = -D \neq 0.$

A	B	C	D
± 1	0	$\pm 2\gamma$	∓ 1
± 1	$\mp 2\gamma$	0	∓ 1

We end up with a total of $2 + 4 = 6$ solutions. Replacing the values of the coefficients A, B, C, D we find the action of the Lorentz transformation (not the transformation per se!) on the four-vectors u^a, v^a. The results are collected in Table 15.1.

We observe that cases V, VI are identical with cases IV, III, respectively, if we interchange $u^a \leftrightarrow v^a$. This is to be expected because cases V, VI express the inverse transformation from u^a to v^a and we have assumed that u^a, v^a are completely equivalent. Therefore, without loss of generality we may restrict our considerations to the first four cases $I - IV$ only. By doing so, we introduce an asymmetry, which, however, can be ignored if we assume that the *direct* Lorentz transformation is from $v^a \to u^a$. This implies then that the coefficient $B = 0$, that is, we define the generic Lorentz transformation by the relations

$$Lv^a = \pm u^a, \qquad Lu^a = Cu^a + Dv^a. \tag{15.13}$$

The coefficient ± 1 in the rhs of the first equation (15.13) is due to the fact that the Lorentz transformation preserves the magnitude and the direction of the four-vectors but not necessarily the sense of direction.

Table 15.1 The action of the generic Lorentz transformation

	A	B	C	D	Lorentz transformation and the inverse
I.	1	0	0	1	$Lu^a = v^a, Lv^a = u^a$ $L^{-1}u^a = v^a, L^{-1}v^a = u^a$
II.	-1	0	0	-1	$Lu^a = -v^a, Lv^a = -u^a$ $L^{-1}u^a = -v^a, L^{-1}v^a = -u^a$
III.	1	0	2γ	-1	$Lu^a = 2\gamma u^a - v^a, Lv^a = u^a$ $L^{-1}u^a = v^a, L^{-1}v^a = 2\gamma v^a - u^a$
IV.	-1	0	-2γ	1	$Lu^a = -2\gamma u^a + v^a, Lv^a = -u^a$ $L^{-1}u^a = -v^a, L^{-1}v^a = -2\gamma v^a + u^a$
V.	1	-2γ	0	-1	$Lu^a = -v^a, Lv^a = -2\gamma v^a + u^a$ $L^{-1}u^a = -2\gamma u^a + v^a, L^{-1}v^a = -u^a$
VI.	-1	2γ	0	1	$Lu^a = v^a, Lv^a = 2\gamma v^a - u^a$ $L^{-1}u^a = 2\gamma u^a - v^a, L^{-1}v^a = u^a$

We compute now the Lorentz transformation itself. From (15.13) we expect that the generic expression of the Lorentz transformation in terms of the four-vectors $u^a,\ v^a$ will be of the form

$$L^a_{\ b} = I^a_{\ b} + A_1 u^a v_b + B_1 u^a u_b + C_1 v^a v_b + D_1 v^a u_b, \tag{15.14}$$

where $I^a_{\ b}$ is the identity transformation and the coefficients $A_1,\ B_1,\ C_1,\ D_1$ are computed in terms of $A, B,\ C,\ D$ by means of the action of the transformation on $u^a,\ v^a$. For example, the action of L (on the left) on u^a gives

$$\begin{aligned} Lu^a &= u^a + A_1 u^a(-\gamma) + B_1 u^a(-1) + C_1 v^a(-\gamma) + D_1 v^a(-1) \\ &= (1 - A_1\gamma - B_1)u^a - (C_1\gamma + D_1)v^a. \end{aligned}$$

Comparing this with the second of (15.13) we find the equations

$$1 - A_1\gamma - B_1 = C, \tag{15.15}$$

$$C_1\gamma + D_1 = -D. \tag{15.16}$$

Similarly, the action of L on v^a gives the equations

$$1 - D_1\gamma - C_1 = B, \tag{15.17}$$

$$A_1 + B_1\gamma = -A. \tag{15.18}$$

The solution of the system of (15.15), (15.16), (15.17), and (15.18) is

$$\begin{aligned} A_1 &= \frac{1}{1-\gamma^2}\,(\gamma C - \gamma - A)\,, \\ B_1 &= \frac{1}{1-\gamma^2}\,(\gamma A + 1 - C)\,, \\ C_1 &= \frac{1}{1-\gamma^2}\,(\gamma D + 1 - B)\,, \\ D_1 &= \frac{1}{1-\gamma^2}\,(\gamma B - \gamma - D)\,. \end{aligned} \tag{15.19}$$

Using the values of the coefficients $A,\ B,\ C,\ D$ for each solution we compute the value of the coefficients $A_1,\ B_1,\ C_1,\ D_1$ and consequently the Lorentz transformation for each case. The result of the calculations are collected in Table 15.2.

We conclude that the covariant expression of the *generic* Lorentz transformation (that is, the one which covers all possible cases!) and its inverse in terms of the

Table 15.2 The generic Lorentz transformation

	A	B	C	D	A_1	B_1	C_1	D_1
I.	1	0	0	1	$-\frac{1}{1-\gamma}$	$\frac{1}{1-\gamma}$	$\frac{1}{1-\gamma}$	$-\frac{1}{1-\gamma}$
II.	-1	0	0	-1	$\frac{1}{1+\gamma}$	$\frac{1}{1+\gamma}$	$\frac{1}{1+\gamma}$	$\frac{1}{1+\gamma}$
III.	1	0	2γ	-1	$-\frac{1+2\gamma}{1+\gamma}$	$\frac{1}{1+\gamma}$	$\frac{1}{1+\gamma}$	$\frac{1}{1+\gamma}$
IV.	-1	0	-2γ	1	$\frac{1-2\gamma}{1-\gamma}$	$\frac{1}{1-\gamma}$	$\frac{1}{1-\gamma}$	$\frac{1}{1-\gamma}$
V.	Same as IV if $u \longleftrightarrow v$							
VI.	Same as III if $u \longleftrightarrow v$							

initial coefficients $A,\ B,\ C,\ D$ are

$$\begin{aligned} L^a_{\ b} &= \delta^a_b + \frac{1}{1-\gamma^2}(\gamma C - \gamma - A)\, u^a v_b + \frac{1}{1-\gamma^2}(\gamma A + 1 - C)\, u^a u_b \\ &\quad + \frac{1}{1-\gamma^2}(\gamma D + 1 - B)\, v^a v_b + \frac{1}{1-\gamma^2}(\gamma B - \gamma - D)\, v^a u_b, \quad (15.20) \\ (L^{-1})^a_{\ b} &= \delta^a_b + \frac{1}{1-\gamma^2}(\gamma B - \gamma - D)\, u^a v_b + \frac{1}{1-\gamma^2}(\gamma D + 1 - B)\, u^a u_b \\ &\quad + \frac{1}{1-\gamma^2}(\gamma A + 1 - C)\, v^a v_b + \frac{1}{1-\gamma^2}(\gamma C - \gamma - A)\, v^a u_b. \quad (15.21) \end{aligned}$$

We note that the transformation is defined solely in terms of the four-vectors u^a, v^a as requested. The expressions are important because

(1) They are fully covariant (independent of any coordinate frame)
(2) They depend only on the four-vectors u^a, v^a, that is, the four-velocities of the observers defining the transformation.

Exercise 77 (a) Prove that the generic Lorentz transformation does satisfy the isometry condition $(L^{-1})^c_{\ a}\eta_{cd}L^d_{\ b} = \eta_{ab}$ where η_{ab} is the Lorentz metric.
(b) Prove that the inverse generic Lorentz transformation is found from the direct by interchanging $u^a \longleftrightarrow v^a$.

Replacing in these expressions the values of the coefficients $A,\ B,\ C,\ D$ of Table 15.1 for each case (or using Table 15.2) we find the covariant expression for the corresponding Lorentz transformation. The results are given[9] in Table 15.3.

Exercise 78 (a) Prove that in cases I, II the Lorentz transformation L satisfies the property $L = L^{-1}$, i.e., $L^2 = I$. Such operators are called *spacetime reflections*.

In the last column of Table 15.3 we give the characterization of each type of Lorentz transformation with the one we computed in Chap. 1. The justification of this will be given below when we derive the action of each type of Lorentz transformation in terms of coordinates.

[9] The expression of the proper Lorentz transformation given in Table 15.2 coincides (after some rearrangements) with that of Krause.

Table 15.3 The covariant Lorentz transformations

	Lorentz transformation	Type
I.	$L^a_b = \delta^a_b + \frac{1}{1-\gamma}(u^a - v^a)(u_b - v_b)$ $L^{-1a}_{\ \ b} = \delta^a_b + \frac{1}{1-\gamma}(v^a - u^a)(v_b - u_b)$	Space inversion
II.	$L^a_b = \delta^a_b + \frac{1}{1+\gamma}(u^a + v^a)(u_b + v_b)$ $L^{-1a}_{\ \ b} = \delta^a_b + \frac{1}{1+\gamma}(v^a + u^a)(v_b + u_b)$	Time inversion
III.	$L^a_b = \delta^a_b + \frac{1}{1+\gamma}(u^a + v^a)(u_b + v_b) - 2u^a v_b$ $L^{-1a}_{\ \ b} = \delta^a_b + \frac{1}{1+\gamma}(v^a + u^a)(v_b + u_b) - 2v^a u_b$	Proper
IV.	$L^a_b = \delta^a_b + \frac{1}{1-\gamma}(u^a - v^a)(u_b - v_b) + 2u^a v_b$ $L^{-1a}_{\ \ b} = \delta^a_b + \frac{1}{1-\gamma}(v^a - u^a)(v_b - u_b) + 2v^a u_b$	Spacetime inversion

15.2.3 The Action of the Covariant Lorentz Transformation on the Coordinates

The expression of the generic Lorentz transformation we found treats the transformation on an equal footing wrt the four-vectors u^a, v^a. However, this is not the standard practice where one considers the Lorentz transformation from one LCF to another. To reconcile the two approaches we consider the action of the Lorentz transformation on the components of a four-vector. Indeed, using the expression (15.20) it is a straightforward matter to write the transformation equation and the inverse of any four-vector x^a in *generic* and *covariant* form (recall that $B = 0$):

$$\begin{aligned}
\overline{x}^a &= x^a + \frac{1}{1-\gamma^2}\left\{[(\gamma C - \gamma - A)\,v_b + (\gamma A + 1 - C)\,u_b]\,x^b\right\}u^a \\
&\quad + \frac{1}{1-\gamma^2}\left\{[(\gamma D + 1)\,v_b - (\gamma + D)\,u_b]\,x^b\right\}v^a, \qquad (15.22)\\
x^a &= \overline{x}^a + \frac{1}{1-\gamma^2}\left\{[(\gamma C - \gamma - A)\,u_b + (\gamma A + 1 - C)\,v_b]\,\overline{x}^b\right\}v^a \\
&\quad + \frac{1}{1-\gamma^2}\left\{[(\gamma D + 1)\,u_b - (\gamma + D)\,v_b]\,\overline{x}^b\right\}u^a. \qquad (15.23)
\end{aligned}$$

From this, we calculate in Table 15.4 the covariant expression of the action of each type of Lorentz transformation and its inverse on an arbitrary four-vector x^a.

In order to find the standard vector expression of the Lorentz transformation, we have to consider the proper frame of one of the defining four-vectors. Take u^a as the reference vector and denote its proper frame by Σ_u. Then we have

Table 15.4 The action of the covariant Lorentz transformation on a four-vector

	Transformation of four-vector	Type
$I.$	$L^a_b x^b = x^a + \frac{1}{1-\gamma}\left[(u_b - v_b)x^b\right](u^a - v^a)$ $(L^{-1})^a_b x^b = x^a + \frac{1}{1-\gamma}\left[(v_b - u_b)x^b\right](v^a - u^a)$	SI
$II.$	$L^a_b x^b = x^a + \frac{1}{1+\gamma}\left[(u_b + v_b)x^b\right](u^a + v^a)$ $(L^{-1})^a_b x^b = x^a + \frac{1}{1+\gamma}\left[(v_b + u_b)x^b\right](v^a + u^a)$	TI
$III.$	$L^a_b x^b = x^a + \frac{1}{1+\gamma}\left[(u_b + v_b)x^b\right](u^a + v^a) - 2\left(v_b x^b\right)u^a$ $(L^{-1})^a_b x^b = x^a + \frac{1}{1+\gamma}\left[(v_b + u_b)x^b\right](v^a + u^a) - 2\left(u_b x^b\right)v^a$	PLT
$IV.$	$L^a_b x^b = x^a + \frac{1}{1-\gamma}\left[(u_b - v_b)x^b\right](u^a - v^a) + 2\left(v_b x^b\right)u^a$ $(L^{-1})^a_b x^b = x^a + \frac{1}{1-\gamma}\left[(v_b - u_b)x^b\right](v^a - u^a) + 2\left(u_b x^b\right)v^a$	STI

Notation: SI = space inversion, TI = time inversion, PLT = proper Lorentz transformation, STI = spacetime inversion.

$$u^a = \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix}_{\Sigma_u}, \quad v^a = \begin{pmatrix} \gamma \\ \gamma\mathbf{v} \end{pmatrix}_{\Sigma_u}. \tag{15.24}$$

Let us consider x^a to be the position four-vector[10] and let us assume that $x^a = \begin{pmatrix} l \\ \mathbf{r} \end{pmatrix}_{\Sigma_u}$. Then from (15.22) we have

$$\begin{aligned}
\begin{pmatrix} l' \\ \mathbf{r}' \end{pmatrix}_{\Sigma_v} &= \begin{pmatrix} l \\ \mathbf{r} \end{pmatrix}_{\Sigma_u} \\
&+ \frac{1}{1-\gamma^2}\left\{(\gamma C - \gamma - A)\begin{pmatrix} \gamma \\ \gamma\mathbf{v} \end{pmatrix}_{\Sigma_u} \cdot \begin{pmatrix} l \\ \mathbf{r} \end{pmatrix}_{\Sigma_u}\right\}\begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix}_{\Sigma_u} \\
&+ \frac{1}{1-\gamma^2}\left\{(\gamma A + 1 - C)\begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix}_{\Sigma_u} \cdot \begin{pmatrix} l \\ \mathbf{r} \end{pmatrix}_{\Sigma_u}\right\}\begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix}_{\Sigma_u} \\
&+ \frac{1}{1-\gamma^2}\left\{(\gamma D + 1)\begin{pmatrix} \gamma \\ \gamma\mathbf{v} \end{pmatrix}_{\Sigma_u} \cdot \begin{pmatrix} l \\ \mathbf{r} \end{pmatrix}_{\Sigma_u}\right\}\begin{pmatrix} \gamma \\ \gamma\mathbf{v} \end{pmatrix}_{\Sigma_u} \\
&- \frac{1}{1-\gamma^2}\left\{(\gamma + D)\begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix}_{\Sigma_u} \cdot \begin{pmatrix} l \\ \mathbf{r} \end{pmatrix}_{\Sigma_u}\right\}\begin{pmatrix} \gamma \\ \gamma\mathbf{v} \end{pmatrix}_{\Sigma_u},
\end{aligned}$$

which leads to the following generic transformation equations

[10] This is not necessary but it will help the reader to associate the new approach with the standard formalism.

$$l' = \frac{1}{1-\gamma^2}\left(\gamma^2 D + \gamma C - A\right)\gamma(\mathbf{v}\cdot\mathbf{r}) + (\gamma D + C)l, \tag{15.25}$$

$$\mathbf{r}' = \mathbf{r} + \frac{\gamma^2(\gamma D + 1)(\mathbf{v}\cdot\mathbf{r})}{1-\gamma^2}\mathbf{v} + \gamma D l \mathbf{v}. \tag{15.26}$$

Exercise 79 Prove that the vector expression of the Lorentz transformation we derived in (1.75), (1.76), (1.77), and (1.78) of Chap. 1 is recovered from the generic relations (15.25) and (15.26) for the various values of the coefficients $A,\ B,\ C,\ D$.

To specialize further and obtain the boosts, we demand

$$\mathbf{v} = \begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix}_{\Sigma_u}, \quad \mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}_{\Sigma_u}.$$

Then the (generic) transformation equations become

$$\begin{aligned} l' &= \frac{1}{1-\gamma^2}\left(\gamma^2 D + \gamma C - A\right)\gamma x v + (\gamma D + C)l, \\ x' &= -\gamma D x + \gamma v D l, \\ y' &= y, \quad z' = z. \end{aligned} \tag{15.27}$$

Exercise 80 Compute the boost along the common x-axis for each type of Lorentz transformation. Show that the results coincide with those of Sect. 1.7 as well as with the results of Exercise 81 below.

Another useful representation of the covariant Lorentz transformation is in the form of a matrix in a specific coordinate system. For example we consider in the proper frame Σ_u of u^a the decomposition (15.24) of the four-vectors u^a, v^a and compute easily ($v^\mu = \mathbf{v}$)

$$L_u(v)^i{}_j = \begin{bmatrix} C + \gamma D & \dfrac{1}{1-\gamma^2}\left(\gamma C + \gamma^2 D - A\right)\gamma v_\mu \\ \\ \gamma D v^\mu & \delta^\mu_\nu + \dfrac{1}{1-\gamma^2}(\gamma D + 1)\gamma^2 v^\mu v_\nu \end{bmatrix}. \tag{15.28}$$

In writing (15.28) we followed the convention that the upper indices count columns and the lower indices rows, whereas the Greek indices take the values 1, 2, 3. For a boost this matrix becomes

$$L_u(v)^i{}_j = \begin{bmatrix} C+\gamma D & \frac{\gamma v}{1-\gamma^2}\left(\gamma C+\gamma^2 D - A\right) & 0 \\ \gamma D v & -\gamma D & 0 \\ 0 & 0 & \delta_K^L \end{bmatrix}, \tag{15.29}$$

where the indices K, L take the values 1, 2.

Exercise 81 Prove that the determinant of the generic boost (15.29) equals

$$\det\ L_u(v) = -AD. \tag{15.30}$$

Conclude that in the cases I, II (spatial and temporal inversion) the det $L_u(v) = -1$ whereas for the cases III, IV (proper transformation and spacetime reflection) det $L_u(v) = +1$. This shows which types of Lorentz transformation do not constitute (by themselves only!) a group (because they do not contain the identity).[11]

Exercise 82 Show that the matrix representation of each of the four types of Lorentz transformation is the following:
Case I (space inversion) ($C = 0$, $D = 1$, $A = 1$).
General:

$$L(u)^i_j = \begin{bmatrix} \gamma & -\gamma v_\mu \\ \gamma v^\mu & \delta^\mu_\nu + \frac{\gamma^2}{1-\gamma} v^\mu v_\nu \end{bmatrix}. \tag{15.31}$$

Boost:

$$L(u)^i_j = \begin{bmatrix} \gamma & -\gamma v & 0 & 0 \\ \gamma v & -\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{15.32}$$

Case II (time inversion) ($C = 0$, $A = 1$, $D = 1$).
General:

$$L(u)^i_j = \begin{bmatrix} -\gamma & \gamma v_\mu \\ -\gamma v^\mu & \delta^\mu_\nu + \frac{\gamma^2}{1+\gamma} v^\mu v_\nu \end{bmatrix}. \tag{15.33}$$

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[11] The set of all types of Lorentz transformations constitutes a group. This group has four components. Of those, only the subset of the proper Lorentz transformations form a group, which is a subgroup of the Lorentz group.

Boost:

$$L(u)^i_j = \begin{bmatrix} -\gamma & \gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{15.34}$$

Case III (proper Lorentz transformation) ($C = 2\gamma$, $A = 1$, $D = -1$).
General:

$$L_u(v)^i_j = \begin{bmatrix} \gamma & -\gamma v_\mu \\ & \\ -\gamma v^\mu & \delta^\mu_\nu + \dfrac{\gamma^2}{1+\gamma} v^\mu v_\nu \end{bmatrix}. \tag{15.35}$$

Boost:

$$L_u(v)^i_j = \begin{bmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{15.36}$$

Case IV (spacetime inversion) ($C = -2\gamma$, $A = -1$, $D = 1$).
General:

$$L(u)^i_j = \begin{bmatrix} -\gamma & \gamma v_\mu \\ & \\ \gamma v^\mu & \delta^\mu_\nu + \dfrac{\gamma^2}{1-\gamma} v^\mu v_\nu \end{bmatrix}. \tag{15.37}$$

Boost:

$$L(u)^i_j = \begin{bmatrix} -\gamma & \gamma v & 0 & 0 \\ \gamma v & -\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{15.38}$$

We note that all four types of Lorentz transformation can be written in covariant form. This means that it is not necessary to study the relativistic problems with the proper Lorentz transformation only, and one can use equally well all other types of Lorentz transformation. However, this is not done in practice and rarely in the existing literature.[12]

[12] See for example the book R. Sachs "*The Physics of the Time Reversal*" The University of Chicago Press, Chicago, (1987).

In the following sections we shall consider various simple but important applications of the covariant Lorentz transformation, keeping always in mind the level of the present book.

15.2.4 The Invariant Length of a Four-Vector

Consider a four-vector w^a and the Lorentz transformation $L(u, v)$ defined by the unit timelike (non-collinear) four-vectors u^a, v^a. Consider the decomposition

$$w^a = a_1 u^a + a_2 v^a + w^a_\perp,$$

where $w^a_\perp = p(u, v)^a_b w^b$ is the normal projection of w^a on the two-plane of u^a, v^a. We introduce the invariants $A_u = w^a u_a$, $A_v = w^a v_a$ (and $\gamma = -u^a v_a$) and compute easily

$$A_u = -a_1 - a_2\gamma,\ \ A_v = -a_1\gamma - a_2.$$

Reversing these relations we compute a_1, a_2 in terms of A_u, A_v and find that

$$w^a = \frac{1}{1-\gamma^2}\left\{(\gamma A_v - A_u)u^a + (\gamma A_u - A_v)\, v^a\right\} + w^a_\perp. \tag{15.39}$$

It follows that the length $w^2 = -w^a w_a$ of w^a is given by the relation

$$w^2 = \frac{1}{1-\gamma^2}\left(-A_u^2 - A_v^2 + 2\gamma_u A_v A_u\right) + w^a_\perp w_{\perp a}. \tag{15.40}$$

This result is important because w^2 is Lorentz invariant. This means that to any four-vector we have associated a Lorentz invariant quantity written in covariant form. In later sections we shall discuss the use of this invariant.

For later reference we note that the action of the generic Lorentz transformation on the (arbitrary) four-vector w^a reads

$$L^a_{\ b} w^b = \frac{1}{1-\gamma^2}\{(\gamma A - C)A_u + (\gamma C - A)\, A_v\}\, u^a + \frac{1}{1-\gamma^2} D\, (\gamma A_v - A_u)\, v^a + w^a_\perp. \tag{15.41}$$

15.3 The Four Types of the Lorentz Transformation Viewed as Spacetime Reflections

We have already pointed out that the first two types of the Lorentz transformation satisfy the property $L^2 = I$, that is their action twice produces no change. Such operations are called spacetime reflections. In this section, we take the subject further and show that all four types of the Lorentz transformation can be described in terms of spacetime reflections.

Let n^a be a unit vector so that $\epsilon(n) = \pm 1$. Then the tensor

$$N_{\epsilon(n)}{}^a_b(n) = \delta^a_b - 2\epsilon(n) n^a n_b \tag{15.42}$$

reflects n^a, that is, $N_{\epsilon(n)}{}^a_b(n) n^b = -n^a$.

From the two vectors u^a, v^a we can define two new vectors. The spacelike unit vector $w^a = \frac{1}{\sqrt{2(\gamma-1)}}(u^a - v^a)$ and the timelike unit vector $s^a = \frac{1}{\sqrt{2(\gamma+1)}}(u^a + v^a)$. Then it is easy to see that the spacetime reflection along w^a, i.e., $N_{+}{}^a_b(w)$, is the space inversion Lorentz transformation and that the spacetime reflection along s^a, i.e., $N_{-}{}^a_b(s)$, is the time inversion Lorentz transformation. This explains the spacetime reflection property $L^2 = I$ of these two types of transformation.

The proper Lorentz transformation cannot be described by means of a single spacetime reflection operator. To see this, let us denote this transformation as $L_3{}^a_b$ and assume that it can be written in the form

$$L_3{}^a_b = \delta^a_b + k m^a m_b \tag{15.43}$$

for some unit vector m^a and some factor k. Writing $m^a = \alpha u^a + \beta v^a +_{\perp} m^a$ and demanding that the resulting form of the transformation coincides with the one given in Table 15.3 we arrive at a contradiction. However, we can represent $L_3{}^a_b$ as the product of two spacetime reflections. Indeed, it is easy to show that L_3 can be written as

$$L_3{}^a_b = (\delta^a_c + 2u^a u_c)\left(\delta^c_b + \frac{1}{1+\gamma}(u^c + v^c)(u_b + v_b)\right), \tag{15.44}$$

that is[13]

$$L_3{}^a_b = N_{-}{}^a_b(u) N_{-}{}^a_b(s). \tag{15.45}$$

It is interesting to note that the product $N_{-}{}^a_b(v) N_{-}{}^a_b(s)$ produces the type V Lorentz transformation we have neglected and corresponds to the inverse $L_3^{-1}{}^a{}_b$.

Working similarly with the spacetime inversion $L_4{}^a_b$ we show that

$$L_4{}^a_b = (\delta^a_c + 2u^a u_c)\left(\delta^c_b + \frac{1}{1-\gamma}(u^c - v^c)(u_b - v_b)\right), \tag{15.46}$$

which implies

$$L_4{}^a_b = N_{-}{}^a_b(u) N_{+}{}^a_b(w). \tag{15.47}$$

[13] See J. Krause "Lorentz transformations as space-time reflections" *J. Math. Phys.* 18, 879–893, (1977).

Therefore, we have described all four types of the Lorentz transformation in terms of spacetime reflections along the reference vector u^a and the characteristic vectors w^a, s^a.

One application of this result (see Krause *ibd*) is to compute the transformation matrix $S(L)$ corresponding to the Lorentz transformation L in the Dirac four-spinor transformation law. We consider the Dirac 4×4 γ matrices which are defined by the condition

$$\gamma^a \gamma^b + \gamma^b \gamma^a = 2\eta^{ab}. \tag{15.48}$$

We introduce as usual the matrix $\gamma^5 = \gamma^0 \gamma^1 \gamma^2 \gamma^3$ which satisfies the well-known properties

$$\gamma^5 \gamma^a + \gamma^a \gamma^5 = 0, \qquad (\gamma^5)^2 = -I. \tag{15.49}$$

The invariance of Dirac's equation under a Lorentz transformation L^a_b implies the condition

$$\gamma^a = L^a_b S(L) \gamma^b S^{-1}(L), \tag{15.50}$$

where $S(L)$ is a non-singular matrix associated with the transformation L^a_b. Taking L^a_b to be a spacetime reflection $N_{\epsilon(n)}{}^a_b(n) = \delta^a_b - 2\epsilon(n) n^a n_b$, we compute the commutator:

$$[\gamma^a, S(L)] = -2\epsilon(n) n^a S(L) \not{n}, \tag{15.51}$$

where $\not{n} = n_a \gamma^a$. Multiplying (15.48) with n^b we get the anticommutator

$$\{\gamma^a \not{n}, \not{n} \gamma^a\} = 2n^a. \tag{15.52}$$

To make (15.52) a commutator we multiply with γ^5 and use (15.49) to get

$$[\gamma^a, \gamma^5 \not{n}] = -2\epsilon(n) \gamma^5 n^a, \tag{15.53}$$

which by virtue of the identity $\not{n} \not{n} = \epsilon(n)$ is written as

$$[\gamma^a, \gamma^5 \not{n}] = -2n^a (\gamma^5 \not{n}) \not{n}. \tag{15.54}$$

Comparison of (15.51) and (15.54) gives

$$S(L) = \gamma^5 \not{n} \tag{15.55}$$

up to a constant. Let us denote the space (time) reflection Lorentz transformation along the vector w^a, (resp. s^a) by L_1, (resp. L_2). Then we have

$$S(L_1) = \gamma^5 \not{w} = \frac{1}{\sqrt{2(\gamma-1)}}\gamma^5(\not{u} - \not{v}), \tag{15.56}$$

$$S(L_2) = \gamma^5 \not{s} = \frac{1}{\sqrt{2(\gamma+1)}}\gamma^5(\not{u} + \not{v}). \tag{15.57}$$

Concerning the proper Lorentz transformation we have from (15.45)

$$\begin{aligned} S(L_3) &= S(N_-(u))S(N_-(s)) = \gamma^5 \not{u}\frac{1}{\sqrt{2(\gamma+1)}}\gamma^5(\not{u} + \not{v}) \\ &= \frac{1}{\sqrt{2(\gamma+1)}}(-1 + \not{u}\not{v}). \end{aligned} \tag{15.58}$$

Similarly for the spacetime reflection we use (15.47) to find

$$\begin{aligned} S(L_4) &= S(N_-(u))S(N_-(w)) = \gamma^5 \not{u}\frac{1}{\sqrt{2(\gamma-1)}}\gamma^5(\not{u} - \not{v}) \\ &= -\frac{1}{\sqrt{2(\gamma-1)}}(1 + \not{u}\not{v}). \end{aligned} \tag{15.59}$$

These results are manifestly covariant.

15.4 Relativistic Composition Rule of Four-Vectors

In this section, we employ the covariant Lorentz transformation to discuss the relativistic composition rule of four-vectors. Usually we refer to the relativistic composition rule for three-velocities and three-acceleration but as we shall show, the composition rule is general and applies to all four-vectors and of course to all types of Lorentz transformation.

The reason why we pay so much attention to the composition rule of three-velocities is historic and is due to the fact that this rule was used to prove that the velocity of light was incompatible with the Newtonian composition rule of three-vectors and therefore a new theory of physics had to be introduced (of course none is so unwise to say that Newtonian Physics has to be abandoned!). Furthermore, it was shown that the behavior of the velocity of light was compatible with the composition rule proposed by Special Relativity, a fact that contributed to the acceptance and the further development of that theory.

Before we discuss the relativistic composition rule for four-vectors we examine the corresponding rule of Newtonian Physics.[14] Let us start with the Galileo transformation for the position vector[15]:

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[14] See also Sect. 6.3.

[15] Many times this transformation is written as

$$\mathbf{r}_{P'} = \Gamma_{\mathbf{v}}(O, O')\mathbf{r}_P. \tag{15.61}$$

As we know there are two ways to look at a transformation: the passive and the active view. According to the first view we consider one vector (more generally tensor) and two coordinate systems and the transformation transfers the components of the vector (respectively, tensor) from one system to the other leaving the vector (respectively, tensor) the same. In the second point of view, we consider one coordinate system and two vectors (respectively, tensors) and the transformation connects one vector (respectively, tensor) with another in the *same* coordinate system (see Fig. 15.1).

For example, the passive view of the Galileo transformation is

$$x' = x - v_x t, \quad y' = y - v_y t, \quad z' = z - v_z t \tag{15.62}$$

and the active view is equation (15.61). The active view contains more of the mathematical information of the transformation whereas the passive view is necessary in the computations. Concerning the Lorentz transformation in a similar manner, the passive view of a boost along the common *x-axis* is

$$l' = \gamma(l - vt), \quad x' = \gamma(x - vl), \quad y' = y, \quad z' = z \tag{15.63}$$

and the active view for the general Lorentz transformation is (see (15.22))

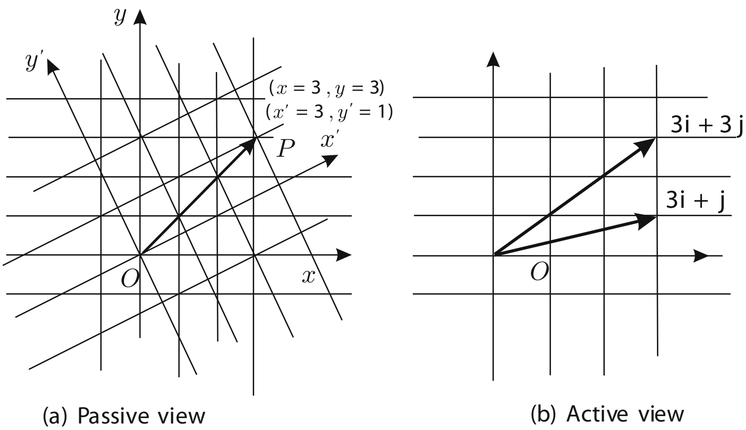

Fig. 15.1 Passive and active interpretation of a transformation

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$$\mathbf{r}' = A\mathbf{r} - \mathbf{v}t, \tag{15.60}$$

where A is a Euclidean (orthogonal) rotation matrix and $\mathbf{v}$ is the velocity of Σ' wrt Σ. This relation is not more general than (15.61), and the matrix A is not needed because relation (15.60) is a vector equation. The matrix A is needed only when (15.61) is written in a coordinate system in which case it describes the relative rotation of the three-axes.

$$x'^i = x^i + \frac{1}{1-\gamma^2}\left\{\left[(\gamma C - \gamma - A)\, v_j + (\gamma A + 1 - C)\, u_j\right] x^j\right\} u^i$$
$$+ \frac{1}{1-\gamma^2}\left\{\left[(\gamma D + 1)\, v_j - (\gamma + D)\, u_j\right] x^j\right\} v^i. \qquad (15.64)$$

After this detour into the two views of a transformation, let us return to the Euclidean three-space where we describe Newtonian motion and let us consider a moving point mass. Let O, O' be the origins of the coordinate systems of two Newtonian observers Π and Π' and let $\mathbf{v}_\Pi$ and $\mathbf{v}_{\Pi'}$ be the velocities of a point mass wrt Π and Π', respectively. The Newtonian law of composition of velocities requires

$$\mathbf{v}_{\Pi'} = \mathbf{v}_\Pi - \mathbf{v}_{O,O'}, \qquad (15.65)$$

where $\mathbf{v}_{O,O'}$ is the relative velocity of the observer Π wrt Π'. We can regard equation (15.65) as a transformation in a linear three-dimensional space whose vectors are the velocities (this is the tangent space of the Newtonian three-dimensional space). In this space the transformation (15.65) is the Galileo transformation! That is we have

$$\mathbf{v}_{\Pi'} = \Gamma_{\mathbf{u}}(O, O')\mathbf{v}_\Pi, \qquad (15.66)$$

where $\mathbf{u}$ is the relative velocity of Π' wrt Π.

A similar result holds for the acceleration, that is Newton's composition rule for acceleration is

$$\mathbf{a}_{\Pi'} = \Gamma_{\mathbf{a}}(O, O')\mathbf{a}_\Pi \qquad (15.67)$$

and similarly for any other vector. This is expected as according to the Galileo Principle of Covariance, the Galileo transformation concerns all Newtonian vectors (and tensors) and not only the position vector.

From the above analysis we conclude that

> The Newtonian composition rule of Euclidean vectors (respectively, tensors) is equivalent to the active view of the Galileo transformation in the corresponding linear space of the relevant vector (velocity space, momentum space, acceleration space, etc.) (respectively, tensor).

Based on this conclusion we define the Law of Composition of four-vectors in Special Relativity as follows:

Definition 15 Consider the LCFs Σ and Σ' with four-velocities u^a and v^a, respectively ($u^a u_a = v^a v_a = -1$, $u^a \neq \pm v^a$) and let $L^i_j(u, v)$ be the Lorentz transformation defined by the four-vectors u^a, v^a. Let $w^i_{P'}$ be a four-vector in the tangent space of Σ' at the point P', with position vector $x^{i'}_{P'}$, on the straight line (=the cosmic line of Σ') defined by the four-vector v^a (see Fig. 15.2). We define the point P on the straight line (=the cosmic line of Σ) defined by the four-vector u^a, by the position vector

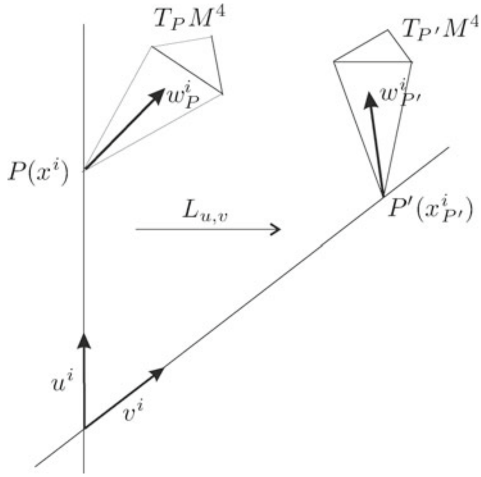

Fig. 15.2 Composition of Lorentz transformations

$$x^i_P = (L^{-1}_{u,v})^i_j x^j_{P'}. \tag{15.68}$$

The tangent vector w^i_P in the tangent space $T_P M$ which is defined by the relation

$$w^i_P = (L^{-1}_{u,v})^i_j w^j_{P'} \tag{15.69}$$

is called the composite vector under the Lorentz transformation $L^i_j(u, v)$ and we postulate that it defines the Composition Rule of four-vectors in spacetime.

In the following when the four-vectors u^i, v^i are understood we shall omit them, that is, instead of writing $L^{-1}_{u,v}$ we shall simply write L^{-1}.

15.4.1 Computation of the Composite Four-Vector

Using relation (15.23), which gives the inverse Lorentz transformation, we compute the generic form of the composite four-vector w^i_P:

$$\begin{aligned} w^i_P &= w^i_{P'} + \frac{1}{1-\gamma^2}\left\{\left[(\gamma C - \gamma - A)\, u_j + (\gamma A + 1 - C)\, v_j\right] w^j_{P'}\right\} v^i \\ &+ \frac{1}{1-\gamma^2}\left\{\left[(\gamma D + 1)\, u_j - (\gamma + D)\, v_j\right] w^j_{P'}\right\} u^i \\ &= w^i_{P'} + \frac{1}{1-\gamma^2}\left\{(\gamma C - \gamma - A)\, A'_u + (\gamma A + 1 - C)\, A'_v\right\} v^i \\ &+ \frac{1}{1-\gamma^2}\left\{(\gamma D + 1)\, A'_u - (\gamma + D)\, A'_v\right\} u^i, \end{aligned} \tag{15.70}$$

where we have introduced the quantities

$$A'_u = w^i_{P'} u_i, \quad A'_v = w^i_{P'} v_i. \tag{15.71}$$

Special attention must be paid to the computation of the quantity $A'_v = w^i_{P'} v_i$. Indeed, this quantity involves the four-vectors $w^i_{P'}, v_i$ which are defined at different points of M^4, therefore their contraction is meaningless. However, M^4 is a flat space, therefore we can transport parallel the four-vector $w^i_{P'}$ from the point P' to the point P along any path we wish, the parallel transport being independent of the path taken.[16] This parallel transport is not possible in a curved space in which transportation is path dependent. This is the reason why we do not use the composition of four-vectors in General Relativity. This does not mean of course that people have not tried to do so[17] however, as expected, without any success.

We continue with the computation of the zeroth component ($A_u = w^i_P u_i$) and the spatial part ($h^i_j(u) w^j_P$) of the composite four-vector in the proper frame of u^i. A simple computation gives for the first

$$A_u = \frac{1}{1-\gamma^2}\left[(\gamma A - \gamma^2 C - D\gamma)A'_u + (D - \gamma^2 A + \gamma C)A'_v\right]. \tag{15.72}$$

Concerning the spatial part we have

$$h^i_j(u) w^j_P = h^i_j(u) w^i_{P'} + \frac{1}{1-\gamma^2}\left[(\gamma C - \gamma - A)\, A'_u + (\gamma A + 1 - C)\, A'_v\right] h^i_j(u) v^j. \tag{15.73}$$

Replacing $h^i_j(u) = \delta^i_j + u^i u_j$ we find

$$\begin{aligned} h^i_j(u) w^j_P &= w^i_{P'} + A'_u u^i \\ &\quad + \frac{1}{1-\gamma^2}\left\{(\gamma C - \gamma - A)\, A'_u + (\gamma A + 1 - C)\, A'_v\right\}(v^i - \gamma u^i) \\ &= w^i_{P'} + \frac{1}{1-\gamma^2}\left\{\left(1 + \gamma A - \gamma^2 C\right) A'_u - \gamma\,(\gamma A + 1 - C)\, A'_v\right\} u^i \\ &\quad + \frac{1}{1-\gamma^2}\left\{(\gamma C - \gamma - A)\, A'_u + (\gamma A + 1 - C)\, A'_v\right\} v^i. \end{aligned} \tag{15.74}$$

The length of the spatial part is $h_{ij}(u) w^i_P w^j_P$ and it is computed to be

$$\begin{aligned} h_{ij}(u) w^i_P w^j_P &= w_{Pi} w^i_P + (w^i_P u_i)^2 \\ &= w_{P'i} w^i_{P'} + \frac{1}{(1-\gamma^2)^2}\left[(\gamma^2 A - \gamma C - D)A'_v + (\gamma D + \gamma^2 C - \gamma A)A'_u\right]^2, \end{aligned} \tag{15.75}$$

[16] The parallel transport is defined by the requirement that the transported vector at the point P has the same components with the original vector at the point P' in the *same* (global) coordinate system of M^4.

[17] For example see F. Felice "On the velocity composition law in General Relativity" *Lettere al Nuovo Cimento* **25**, 531–532 (1979).

where we have replaced the quantity $w^i_P u_i$ from (15.72) and used the fact that the Lorentz transformation is an isometry, therefore $w_{Pi} w^i_P = w_{P'i} w^i_{P'}$. We note that the rhs contains only $w^i_{P'}$ and not w^i_P.

Relations (15.72) and (15.75) are general and hold for an arbitrary four-vector (null, timelike, or spacelike) and *all* types of Lorentz transformation. Therefore, they contain all possible rules of composition of all four-vectors in Special Relativity.

15.4.2 The Relativistic Composition Rule for Three-Velocities

In order to convince the reader that the general relations derived in the last section contain all known results as special cases, we derive in the following the well-known relativistic rules for the composition of three-velocities and three-accelerations for the proper Lorentz transformation.

From Table 15.1 we have that the proper Lorentz transformation is defined by the values $A = 1$, $C = 2\gamma$, $D = -1$. Therefore, in the case of proper Lorentz transformation, relations (15.72), (15.73), and (15.75) become

$$A_u = 2\gamma A'_u - A'_v, \tag{15.76}$$

$$h^i_j(u) w^j_P = h^i_j(u) w^i_{P'} + \frac{1}{1+\gamma} \left\{ -(2\gamma + 1) A'_u + A'_v \right\} h^i_j(u) v^j, \tag{15.77}$$

and

$$h_{ij}(u) w^i_P w^j_P = w_{P'i} w^i_{P'} + \left[2\gamma A'_u - A'_v \right]^2 . \tag{15.78}$$

Suppose that the components of the four-vectors w^i_P, $w^i_{P'}$ in the proper frame Σ_u are

$$w^i_P = \begin{pmatrix} \gamma_w \\ \gamma_w \mathbf{w} \end{pmatrix}_{\Sigma_u}, \qquad w^i_{P'} = \begin{pmatrix} \gamma_{w'} \\ \gamma_{w'} \mathbf{w}' \end{pmatrix}_{\Sigma_u} \tag{15.79}$$

and those of the four-vectors u^i, v^i which define the transformation

$$u^i = \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix}_{\Sigma_u}, \quad v^i = \begin{pmatrix} \gamma_v \\ \gamma_v \mathbf{v} \end{pmatrix}_{\Sigma_u}.$$

We compute

$$\begin{aligned} A_u &= w^i u_i = -\gamma_w, \\ A'_v &= w^i_{P'} v_i = -\gamma_v \gamma_{w'} + \gamma_v \gamma_{w'} \mathbf{v} \cdot \mathbf{w}', \\ A'_u &= w^i_{P'} u_i = -\gamma_{w'}, \end{aligned}$$

where for emphasis we have replaced γ with γ_v.

Replacing in (15.76) we find for the zeroth coordinate the well-known transformation rule for $\gamma's$:

$$\gamma_w = \gamma_v \gamma_{w'}(1 + \mathbf{v} \cdot \mathbf{w}'). \tag{15.80}$$

Concerning the spatial coordinate we find from (15.77)

$$\gamma_w \mathbf{w} = \gamma_{w'} \mathbf{w}' + \frac{1}{1+\gamma_v}\left\{-(2\gamma_v + 1)(-\gamma_{w'}) - \gamma_v \gamma_{w'} + \gamma_v \gamma_{w'} \mathbf{v} \cdot \mathbf{w}'\right\} \gamma_v \mathbf{v}.$$

Making use of (15.80)[18] follows the well-known formula (c.f. with (6.33))

$$\mathbf{w} = \frac{1}{\gamma_v(1 + \mathbf{v} \cdot \mathbf{w}')}\left\{\mathbf{w}' + \gamma_v\left(1 + \frac{\gamma_v}{1+\gamma_v}\mathbf{v} \cdot \mathbf{w}'\right)\mathbf{v}\right\}. \tag{15.81}$$

Exercise 83 Consider the boost along the x-axis of Σ_u with velocity v and assume that the decomposition of the three-velocity $\mathbf{w}$ in Σ_u is $\mathbf{w} = \begin{pmatrix} w_x \\ w_y \\ w_z \end{pmatrix}_{\Sigma_u}$. Then prove that the relativistic composition rule for the three-velocities is given by the well-known relations

$$w_x = \frac{w'_x + v}{1 + vw'_x}, \quad w_y = \frac{w'_y}{\gamma_v(1 + vw'_x)}, \quad w_z = \frac{w'_z}{\gamma_v(1 + vw'_x)}. \tag{15.82}$$

Furthermore show that

$$w'_x = \frac{w_x - v}{1 - vw'_x}, \quad w'_y = \frac{w_y}{\gamma_v(1 - vw'_x)}, \quad w'_z = \frac{w_z}{\gamma_v(1 - vw_x)} \tag{15.83}$$

and verify the correspondence $\mathbf{v} \longleftrightarrow -\mathbf{v},\ \mathbf{w}' \longleftrightarrow \mathbf{w}$.

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[18] An equivalent expression is (see Ar Ben-Menahem, *Am. J. Phys* **53**, 62–66, (1985).

$$\mathbf{w} = \frac{1}{(1 + \mathbf{v} \cdot \mathbf{w}')}\left\{\frac{\mathbf{w}'}{\gamma_v} + \left(1 + \frac{\gamma_v - 1}{\gamma_v}\frac{\mathbf{v} \cdot \mathbf{w}'}{v^2}\right)\right\}\mathbf{v}.$$

The proof is simple. We have

$$1 + \frac{\gamma_v}{1+\gamma_v}(\mathbf{v}\mathbf{w}') = 1 + \frac{\gamma_v^2 v^2}{\gamma_v(1+\gamma_v)}\frac{(\mathbf{v}\mathbf{w}')}{v^2} = 1 + \frac{\gamma_v^2 - 1}{\gamma_v(1+\gamma_v)}(\mathbf{v}\mathbf{w}')v^2 = 1 + \frac{\gamma_v - 1}{\gamma_v}(\mathbf{v}\mathbf{w}')v^2.$$

15.4.3 Riemannian Geometry and Special Relativity

It is generally believed that Riemannian geometry has no place in Special Relativity since Minkowski space M^4 is flat (=has zero curvature). This is true but it is not the whole story. Indeed, as we have seen in Special Relativity, besides the spacetime, other linear spaces with physical significance are involved such as the three-velocity space, the three-momentum space. Using the length of the spatial part of the four-vectors one can define in any of these spaces a *Lorentz covariant, positive definite*, and symmetric (that is Riemannian) metric whose curvature does not (in general) vanish.

In the current section we study the case of the three-velocity space and obtain results which have been around for a long time. One can use the same approach to study the three-space of other four-vectors and obtain new results. From (15.78) we have for the length of the spatial part of the composite four-velocity[19]

$$\begin{aligned}\gamma_w^2 \mathbf{w}^2 &= -1 + \left(2\gamma_v(-\gamma_{w'}) + \gamma_v\gamma_{w'} - \gamma_v\gamma_{w'}\mathbf{v}\cdot\mathbf{w}'\right)^2 \\ &= -1 + \left[\gamma_v\gamma_{w'}(1+\mathbf{v}\cdot\mathbf{w}')\right]^2 \\ &= -1 + \gamma_w^2,\end{aligned}$$

where for emphasis we write γ_v in place of γ and we have used (15.80) for the transformation of the time component. This implies the relation

$$\mathbf{w}^2 = 1 - \frac{1}{\gamma_w^2}. \tag{15.84}$$

Exercise 84 (a) Prove that $\mathbf{w}^2$ can be written as follows:

$$\mathbf{w}^2 = 1 - \frac{1}{\gamma_v^2\gamma_{w'}^2 Q_{w'}^2}, \tag{15.85}$$

where $Q_{w'} = 1 + \mathbf{v}\cdot\mathbf{w}'$.

(b) Using (a) justify the following calculation:

$$\begin{aligned}\mathbf{w}^2 &= \frac{1}{Q_{w'}^2}\left[(1+\mathbf{v}\cdot\mathbf{w}')^2 - (1-\mathbf{v}^2)(1-\mathbf{w}'^2)\right] \\ &= \frac{1}{Q_{w'}^2}\left[(\mathbf{v}^2 + \mathbf{w}'^2 + 2\mathbf{v}\mathbf{w}') + (\mathbf{v}\cdot\mathbf{w}')^2 - \mathbf{v}^2\mathbf{w}'^2\right] \\ &= \frac{1}{Q_{w'}^2}\left[(\mathbf{v}+\mathbf{w}')^2 - (\mathbf{v}\times\mathbf{w}')^2\right].\end{aligned} \tag{15.86}$$

[Hint: Use the identity $|\mathbf{a}\times\mathbf{b}|^2 = \mathbf{a}^2\mathbf{b}^2 - (\mathbf{a}\cdot\mathbf{b})^2$.]

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[19] Note that w^i is a four-velocity, hence $w^i w_i = -1$.

(c) Using the correspondence $\mathbf{w}' \longleftrightarrow \mathbf{w},\ \mathbf{v} \longleftrightarrow -\mathbf{v}$ show that

$$(\mathbf{w}')^2 = 1 - \frac{1}{\gamma_{w'}^2}, \tag{15.87}$$

hence

$$(\mathbf{w}')^2 = 1 - \frac{1}{\gamma_v^2 \gamma_w^2 Q_w^2}, \tag{15.88}$$

where $Q_w = 1 - \mathbf{v} \cdot \mathbf{w}$. Finally show that

$$\begin{aligned}(\mathbf{w}')^2 &= \frac{1}{Q_w^2}\left[(1 - \mathbf{v}\cdot\mathbf{w})^2 - (1 - \mathbf{v}^2)(1 - \mathbf{w}^2)\right] \\ &= \frac{1}{Q_w^2}\left[(\mathbf{v}^2 + \mathbf{w}^2 - 2\mathbf{v}\mathbf{w}) + (\mathbf{v}\cdot\mathbf{w})^2 - \mathbf{v}^2\mathbf{w}^2\right] \\ &= \frac{1}{Q_w^2}\left[(\mathbf{v} - \mathbf{w})^2 - (\mathbf{v}\times\mathbf{w})^2\right]. \end{aligned} \tag{15.89}$$

The quantity $(\mathbf{w}')^2$ is positive definite and most important it is Lorentz invariant (because the quantity $\gamma_{w'} = -w'^a u_a$ is Lorentz invariant). Therefore, it can be used as a Lorentz invariant, positive definite distance in the space of three-velocities. This distance leads to a Lorentz covariant Riemannian metric which is not flat. Let us find this metric.

In the space of three-velocities we consider an “infinitesimal” change $\mathbf{w} = \mathbf{v} + d\mathbf{v}$ and have[20]

$$\begin{aligned}(\mathbf{v} - \mathbf{w})^2 &= (d\mathbf{v})^2, \\ (\mathbf{v}\times\mathbf{w})^2 &= |\mathbf{v}\times(\mathbf{v}+d\mathbf{v})|^2 = |\mathbf{v}\times d\mathbf{v}|^2 = \mathbf{v}^2(d\mathbf{v})^2 - (\mathbf{v}\cdot d\mathbf{v})^2, \\ Q_w &= 1 - \mathbf{v}\cdot\mathbf{w} = 1 - \mathbf{v}\cdot(\mathbf{v} + d\mathbf{v}) = 1 - \mathbf{v}^2 - \mathbf{v}\cdot d\mathbf{v}.\end{aligned}$$

Replacing the result in (15.89) we find

$$\begin{aligned}\mathbf{w}'^2 &= \frac{1}{\left(1 - \mathbf{v}^2 - \mathbf{v}\cdot d\mathbf{v}\right)^2}\left[(d\mathbf{v})^2 - \mathbf{v}^2(d\mathbf{v})^2 + (\mathbf{v}\cdot d\mathbf{v})^2\right] \\ &= \frac{1}{\left(1 - \mathbf{v}^2 - \mathbf{v}\cdot d\mathbf{v}\right)^2}\left[\left(1 - \mathbf{v}^2\right)(d\mathbf{v})^2 + (\mathbf{v}\cdot d\mathbf{v})^2\right]. \end{aligned} \tag{15.90}$$

[20] Note that $\mathbf{v}\cdot d\mathbf{v} = v\,dv$.

The term

$$\frac{1}{\left(1-\mathbf{v}^2-\mathbf{v}\cdot d\mathbf{v}\right)^2}=\frac{1}{\left(1-\mathbf{v}^2\right)^2}\frac{1}{\left[1-\frac{(\mathbf{v}\cdot d\mathbf{v})^2}{\left(1-\mathbf{v}^2\right)^2}\right]^2}$$
$$=\frac{1}{\left(1-\mathbf{v}^2\right)^2}\left[1+\frac{2(\mathbf{v}\cdot d\mathbf{v})}{1-\mathbf{v}^2}+\frac{3(\mathbf{v}\cdot d\mathbf{v})^2}{(1-\mathbf{v}^2)^2}+\cdots\right].$$

Replacing in (15.90) we find

$$\frac{\left(1-\mathbf{v}^2\right)(d\mathbf{v})^2+(\mathbf{v}\cdot d\mathbf{v})^2}{\left(1-\mathbf{v}^2\right)^2}+O((d\mathbf{v})^3).$$

We introduce in the space of three-velocities spherical coordinates (v,θ,ϕ) with the standard relation $\mathbf{v}=v(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta)$. In these coordinates

$$(d\mathbf{v})^2=(dv)^2+v^2d\theta^2+v^2\sin^2\theta d\phi^2,$$

hence

$$-\mathbf{v}^2(d\mathbf{v})^2+(\mathbf{v}\cdot d\mathbf{v})^2=-v^4(d\theta^2+\sin^2\theta d\phi^2).$$

Replacing the results in (15.90) we find

$$\begin{aligned}\mathbf{w}'^2&=\frac{(d\mathbf{v})^2}{\left(1-\mathbf{v}^2\right)^2}-\frac{1}{\left(1-\mathbf{v}^2\right)^2}v^4(d\theta^2+\sin^2\theta d\phi^2)+O((d\mathbf{v})^3)\\&=\frac{(dv)^2+v^2d\theta^2+v^2\sin^2\theta d\phi^2}{\left(1-v^2\right)^2}-\frac{v^4}{(1-v^2)^2}(d\theta^2+\sin^2\theta d\phi^2)+O((d\mathbf{v})^3)\\&=\frac{(dv)^2}{\left(1-v^2\right)^2}+\frac{v^2}{1-v^2}(d\theta^2+\sin^2\theta d\phi^2)+O((d\mathbf{v})^3).\end{aligned}\tag{15.91}$$

The quantity $\mathbf{w}'^2$ is the required distance in the space of three-velocities if we neglect third-order terms in dv. Then, the rhs defines a Riemannian metric ds^2 in that space as follows:

$$ds^2=\frac{(dv)^2}{\left(1-v^2\right)^2}+\frac{v^2}{1-v^2}(d\theta^2+\sin^2\theta d\phi^2)\tag{15.92}$$

or, in more standard notation,

$$g_{ij}=diag\left(\frac{1}{\left(1-v^2\right)^2},\frac{v^2}{1-v^2},\frac{v^2}{1-v^2}\sin^2\theta\right).\tag{15.93}$$

The contravariant metric is

$$g^{ij} = diag\left(\left(1-v^2\right)^2, \frac{1-v^2}{v^2}, \frac{1-v^2}{v^2\sin^2\theta}\right). \tag{15.94}$$

It can be shown that the space of three-velocities endowed with this metric becomes a Riemannian space of constant negative curvature. If we introduce the rapidity χ with the relation $\gamma = \cosh\chi$ we have

$$1-v^2 = \frac{1}{\cosh^2\chi}, \quad \frac{v^2}{1-v^2} = \sinh^2\chi, \quad dv = \frac{d\chi}{\cosh^2\chi}$$

from which follows

$$ds^2 = d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta d\phi^2). \tag{15.95}$$

For small χ the $\sinh\chi \sim \chi \sim v$ and the space of three-velocities is flat, as expected at the Newtonian level of small velocities.[21]

Exercise 85 Consider the metric (15.93) and define the Lagrangian

$$L = \frac{\dot{v}^2}{\left(1-v^2\right)^2} + \frac{v^2}{1-v^2}(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2), \tag{15.96}$$

where dot means derivation wrt an affine parameter along a geodesic in three-velocity space.

(a) Show that Lagrange equations are

$$\ddot{\theta} + \frac{2}{v(1-v^2)}\dot{v}\dot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 = 0, \tag{15.97}$$

$$\ddot{\phi} + 2\cot\theta\dot{\phi}\dot{\theta} + \frac{2}{v(1-v^2)}\dot{v}\dot{\phi} = 0, \tag{15.98}$$

$$\ddot{v} + \frac{2v}{1-v^2}\dot{v}^2 - v(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) = 0. \tag{15.99}$$

(b) From Lagrange equations conclude that the non-vanishing connection coefficients[22] are

[21] More information on the geometry of the three-velocity space can be found in V. Fock "The theory of space, time and gravitation" 2nd Revised Edition, Pergamon Press, (1976).

[22] One can also compute the connection coefficients directly from the metric by means of the formulae

$$\Gamma^2_{12} = \frac{1}{v(1-v^2)}, \qquad \Gamma^2_{33} = -\sin\theta\cos\theta,$$
$$\Gamma^3_{13} = \frac{1}{v(1-v^2)}, \qquad \Gamma^3_{23} = \cot\theta,$$
$$\Gamma^1_{11} = \frac{2v}{1-v^2}, \qquad \Gamma^1_{33} = -v\sin^2\theta, \qquad \Gamma^1_{22} = -v.$$

(c) The Ricci tensor R_{ij} is defined by the relation

$$R_{ij} = \Gamma^k_{ij,k} - \Gamma^k_{ik,j} - \Gamma^k_{lj}\Gamma^l_{ik} + \Gamma^k_{lk}\Gamma^l_{ij}. \tag{15.100}$$

Show that

$$R_{ij} = -2g_{ij}. \tag{15.101}$$

(d) The scalar curvature R is defined by the relation

$$R = g^{ij}R_{ij}. \tag{15.102}$$

Prove that $R = -6$.

(e) The curvature tensor R_{ijkl} is defined by the relation

$$R_{ijkl} = g_{ik}R_{jl} - g_{jk}R_{il} - g_{il}R_{jk} + g_{jl}R_{ik} - \frac{1}{2}(g_{ik}g_{jl} - g_{il}g_{jk})R. \tag{15.103}$$

Show that

$$R_{ijlk} = -(g_{ik}g_{jl} - g_{il}g_{jk}). \tag{15.104}$$

Conclude that the space of three-velocities endowed with the metric $\mathbf{w}'$ is a space of constant negative curvature ($R = -6$).

15.4.4 *The Relativistic Rule for the Composition of Three-Accelerations*

We recall that the components of the four-acceleration a^i of a relativistic particle in an LCF Σ are ($c = 1$)

$$\Gamma^i_{ij} = \frac{\partial}{\partial x^l}\log\sqrt{|g_{ii}|}, \quad \Gamma^i_{ii} = \frac{\partial}{\partial x^i}\log\sqrt{|g_{ii}|}, \quad \Gamma^i_{jj} = -\frac{1}{2g_{ii}}\frac{\partial g_{jj}}{\partial x^i},$$
$$\Gamma^i_{jk} = 0 \quad \text{when all indices are different.}$$

$$a^i_P = \begin{pmatrix} a_0 \\ a_0\mathbf{u} + \gamma_u^2 w \end{pmatrix}_{\Sigma_u},$$

where $\mathbf{u}$ is the three-velocity of the particle in Σ, $\mathbf{a} = \frac{d\mathbf{u}}{dt}$ is the three-acceleration of the particle in Σ, and $a_0 = \gamma_u^4(\mathbf{u}\cdot\mathbf{a})$.

Let P, P' be the points in Minkowski space related by the Lorentz transformation and let a^i_P and $a^i_{P'}$ the corresponding four-acceleration vectors resulting from the transformation. We consider an LCF Σ_u (the same for both accelerations!) in which we shall relate the components of the accelerations. In Σ_u we have the components

$$a^i_P = \begin{pmatrix} a_0 \\ a_0\mathbf{w} + \gamma_w^2\mathbf{a} \end{pmatrix}_{\Sigma_u}, \quad a^i_{P'} = \begin{pmatrix} a'_0 \\ a'_0\mathbf{w}' + \gamma_{w'}^2\mathbf{a}' \end{pmatrix}_{\Sigma_u},$$

where $a_0 = \gamma_w^4(\mathbf{w}\cdot\mathbf{a})$, $a'_0 = \gamma_{w'}^4(\mathbf{w}'\cdot\mathbf{a}')$, $\mathbf{a} = \frac{d\mathbf{w}}{dt}$, $\mathbf{a}' = \frac{d\mathbf{w}'}{dt'}$ (all quantities referring to Σ_u). We compute (see (15.71))

$$\begin{aligned} A_{u'} &= -a'_0, \\ A_u &= -a_0, \\ A_{v'} &= -a'_0\gamma_v + \gamma_v\mathbf{v}\cdot(a'_0\mathbf{w}' + \gamma_{w'}^2\mathbf{a}'). \end{aligned}$$

To find the transformation of the zeroth coordinate we replace in (15.76)

$$a_0 = \gamma_v(1+\mathbf{v}\cdot\mathbf{w}')\left[a'_0 + \frac{\gamma_{w'}^2}{1+\mathbf{v}\cdot\mathbf{w}'}(\mathbf{v}\cdot\mathbf{a}')\right]. \tag{15.105}$$

Concerning the spatial part, from (15.77) we have

$$\begin{aligned} a_0\mathbf{w} + \gamma_w^2\mathbf{a} &= a'_0\mathbf{w}' + \gamma_{w'}^2\mathbf{a}' \\ &\quad + \frac{1}{1+\gamma_v}\left\{-(2\gamma_v+1)(-a'_0) + (-a'_0)\gamma_v + \gamma_v\mathbf{v}\cdot(a'_0\mathbf{w}' + \gamma_{w'}^2\mathbf{a}')\right\}\gamma_v\mathbf{v}. \end{aligned}$$

Replacing $\mathbf{w}$ from (15.81) and remembering that $Q_{w'} = (1+\mathbf{v}\cdot\mathbf{w}')$ we find

$$\begin{aligned} \gamma_w^2\mathbf{a} &= a'_0\mathbf{w}' - \frac{a_0}{\gamma_v Q_{w'}}\left[\mathbf{w}' + \gamma_v\left(1 + \frac{\gamma_v}{1+\gamma_v}\mathbf{v}\cdot\mathbf{w}'\right)\mathbf{v}\right] + \gamma_{w'}^2\mathbf{a}' \\ &\quad + \frac{\gamma_v}{1+\gamma_v}\left[(1+\gamma_v)a'_0 + \gamma_v a'_0\mathbf{v}\cdot\mathbf{w}' + \gamma_v\gamma_{w'}^2\mathbf{v}\cdot\mathbf{a}'\right]\mathbf{v} \\ &= \left(a'_0 - \frac{a_0}{\gamma_v Q_{w'}}\right)\mathbf{w}' + \gamma_{w'}^2\mathbf{a}' \\ &\quad + \frac{\gamma_v}{1+\gamma_v}\left\{a'_0 + \gamma_v Q_{w'}a'_0 + \gamma_v\gamma_{w'}^2\mathbf{v}\cdot\mathbf{a}' - \frac{a_0}{\gamma_v Q_{w'}}(1+\gamma_v Q_{w'})\right\}\mathbf{v}. \end{aligned} \tag{15.106}$$

The term multiplying $\mathbf{w}'$ reads if we use (15.105):

$$a_0' - \frac{a_0}{\gamma_v Q_{w'}} = -\frac{\gamma_{w'}^2}{Q_{w'}}(\mathbf{v} \cdot \mathbf{a}'). \tag{15.107}$$

Similarly the term multiplying $\mathbf{v}$ gives if we replace a_0 from (15.107):

$$\frac{\gamma_v}{1+\gamma_v}\left\{a_0' + \gamma_v Q_{w'} a_0' + \gamma_v \gamma_{w'}^2 \mathbf{v} \cdot \mathbf{a}' - \left(a_0' + \frac{\gamma_{w'}^2}{Q_{w'}}(\mathbf{v} \cdot \mathbf{a}')\right)(1 + \gamma_v Q_{w'})\right\}$$
$$= -\frac{\gamma_v}{1+\gamma_v}\frac{\gamma_{w'}^2}{Q_{w'}}(\mathbf{v} \cdot \mathbf{a}').$$

Replacing these in (15.106) we obtain after some tedious yet standard calculations the result

$$\mathbf{a} = \frac{1}{\gamma_v^2 Q_{w'}^3}\left[Q_{w'}\mathbf{a}' - (\mathbf{v} \cdot \mathbf{a}')\mathbf{w}' - \frac{\gamma_v}{1+\gamma_v}(\mathbf{v} \cdot \mathbf{a}')\mathbf{v}\right]. \tag{15.108}$$

It is easy to check that (15.108) coincides with the transformation rule for three-acceleration (7.20) derived in Sect. 7.

15.5 The Composition of Lorentz Transformations

Another application of the composite four-vector is the computation of products of successive Lorentz transformations. We consider three linearly independent unit four-vectors (that is, the velocities of three relativistic observers in relative motion) $u^i,\ v^i,\ w'^i$ and the Lorentz transformations they define:

$$L_v = L(u, v), \quad L_{w'} = L(u, w'), \quad L_w = L(u, w).$$

From the defining equations (15.13) of the Lorentz transformation we have

$$L_v v^i = A u^i, \tag{15.109}$$
$$L_v u^i = C_v u^i + D_v v^i, \tag{15.110}$$
$$L_v^{-1} u^i = A_v v^i, \tag{15.111}$$
$$L_v^{-1} v^i = C_v v^i + D_v u^i, \tag{15.112}$$

where $A_v,\ C_v,\ D_v$ are constants given in Table 15.1. We recall that $A_v = \pm 1$. Similar relations hold for the rest two Lorentz transformations $L_{w'},\ L_w$. Of course in each case we have to change the index in the coefficients and write $A_{w'},\ C_{w'},\ D_{w'}$ for $L_{w'}$ and $A_w,\ C_w,\ D_w$ for L_w. Let w^i be the composite four-vector of w'^i under the transformation L_v. Then

$$w^i = L_v^{-1}\, w'^i, \quad w'^i = L_v\, w^i. \tag{15.113}$$

Let L_w be the Lorentz transformation defined by the four-vectors u^i, w^i. We consider the product of transformations $L_{w'} L_v L_w^{-1}$ and study its effect on the four-vector u^i. We have

$$\begin{aligned} L_{w'} L_v L_w^{-1} u^i &= L_{w'} L_v A_w u^i && \text{(Use (15.111))} \\ &= A_w L_{w'} L_v L_v^{-1} w'^i && \text{(Use (15.113))} \\ &= A_w L_{w'} w'^i && \text{(Use(15.109))} \\ &= A_w A_{w'} u^i. \end{aligned}$$

We conclude that the action of the composite transformation leaves the length and the direction of the four-vector u^i invariant but not the sense of direction (this changes when $A_w A_{w'} = -1$). This means that the effect of the composite transformation is a spatial rotation in the spatial plane normal to the four-vector u^a (that is, the rest space of the observer u^a). The set of all these transformations constitutes a group known as the *little group* or *isotropic group* of u^i. The dimension of this group equals 3 and it is this group which makes possible the covariant 1+3 decomposition of a four-vector in temporal and spatial parts. We write

$$L_{w'} L_v L_w^{-1} = R(u). \tag{15.114}$$

The computation of the composite transformation $R(u)$ is difficult and involved, especially if one follows the standard vector form of the Lorentz transformation. However, using the covariant Lorentz transformation one computes relatively easily $R(u)$ and certainly in covariant form! Furthermore the method of computation can be extended formally to the remaining three types of Lorentz transformation.

In the following, we compute $R(u)$ for proper Lorentz transformations (defined by $A_w = A_{w'} = 1$). First, we write $R(u)$ in the form of a block matrix as follows:

$$R(u) = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}, \tag{15.115}$$

where A is a Euclidean 3×3 matrix, that is $A^t A = I_3$, generating a rotation in the spatial plane normal to the four-vector u^i.

From Table 15.3 we have for the proper Lorentz transformation and its inverse the expressions

$$L^i_j = \delta^i_j + \frac{1}{1+\gamma}(u^i + v^i)(u_j + v_j) - 2u^i v_j, \tag{15.116}$$

$$L^{-1}{}^i_j = \delta^i_j + \frac{1}{1+\gamma}(v^i + u^i)(v_j + u_j) - 2v^i u_j. \tag{15.117}$$

From (15.113) and (15.117) it follows:

$$w^i = w'^i + \frac{1}{1+\gamma_v}\left[(1+\gamma_v)\gamma_{w'} - \psi\right] v^i - \frac{\gamma_{w'} + \psi}{1+\gamma_v} u^i, \tag{15.118}$$

where we have set $\psi = -w'^i v_i$.

We express γ_w in terms of $\gamma_{w'}$.

From the transformation equation (see 15.76)

$$A_u = 2\gamma A'_u - A'_v \tag{15.119}$$

of the zeroth component of the composite four-vector we have in the current notation

$$\begin{aligned} \gamma_w &= 2\ \gamma_v \gamma_{w'} - \psi \\ w^i &= w^{i'} + \frac{\gamma_w + \gamma_{w'}}{1+\gamma_v} v^i - \frac{\gamma_{w'} + \psi}{1+\gamma_v} u^i \end{aligned} \tag{15.120}$$

Now we are ready to work with the transformation $R(u)$. We have

$$\begin{aligned} R^i_j(u) &= L_{w'} L_v L_w^{-1} \\ &= L_{w'} L_v \left[\delta^i_j + \frac{1}{1+\gamma_w}\left(u^i + w^i\right)(u_j + w_j) - 2w^i u_j\right] \\ &= (L_{w'} L_v)^i_j + \frac{1}{1+\gamma_w}\left[L_{w'} L_v (u+w)\right]^i (u_j + w_j) - 2\left[L_{w'} L_v w\right]^i u_j. \end{aligned} \tag{15.121}$$

The term

$$-2L_{w'} L_v w^i = -2L_{w'} w'^i = -2u^i. \tag{15.122}$$

The term

$$\begin{aligned} L_{w'} L_v (u^i + w^i) &= L_{w'^i} L_v u^i + L_{w'^i} L_v w^i = L_{w'}(-v^i + 2\gamma_v u^i) + u^i \\ &= -L_{w'} v^i + 2\gamma_v L_{w'} u^i + u^i \\ &= -L_{w'} v^i + 2\gamma_v(-w'^i + 2\gamma_{w'^i} u^i) + u^i \\ &= -L_{w'} v^i - 2\gamma_v w'^i + (1 + 4\gamma_v \gamma_{w'^i}) u^i. \end{aligned} \tag{15.123}$$

Using (15.116) we compute for the four-vector w'^i

$$L_{w'} v^i = v^i + \frac{1}{1+\gamma_{w'}}(\psi + 2\psi\gamma_{w'} - \gamma_v) u^i - \frac{1}{1+\gamma_{w'}}(\gamma_v + \psi) w'^i. \tag{15.124}$$

Replacing we find that the term

$$
\begin{aligned}
L_{w'}L_v(u^i + w^i) = -v^i &+ \frac{1}{1+\gamma_{w'}}(4\gamma\gamma_{w'} + 4\gamma_v\gamma_{w'}^2 - \psi - 2\psi\gamma_{w'} + \gamma_v)u^i + u^i \\
&+ \frac{1}{1+\gamma_{w'}}(\psi - \gamma_v - 2\gamma_v\gamma_{w'})w'^i. \qquad (15.125)
\end{aligned}
$$

For the other four-vectors we find using (15.116)

$$
\begin{aligned}
(L_{w'}L_v)^i_j = \delta^i_j &+ \frac{1}{(1+\gamma_{w'})(1+\gamma_v)}\left[1+\gamma_{w'} + 2\gamma_v\gamma_{w'} + \gamma_v - \psi\right] w'^i v_j \\
&+ \frac{1}{1+\gamma_{w'}} w'^i w'_j + \frac{1}{1+\gamma_v} v^i v_j \\
&- \frac{-\gamma_{w'} - \psi}{(1+\gamma_v)(1+\gamma_{w'})} w'^i u_j + \frac{1}{1+\gamma_v} v^i u_j + \frac{-1-2\gamma_{w'}}{1+\gamma_{w'}} u^i w'_j \\
&+ \frac{\psi - \gamma_v + 2(\psi - 1 - 2\gamma_v)\gamma_{w'} - 2(1+2\gamma_v)\gamma_{w'}^2}{(1+\gamma_v)(1+\gamma_{w'})} u^i v_j \\
&+ \frac{1 + \psi + 2(1+\psi)\gamma_{w'} + 2\gamma_{w'}^2}{(1+\gamma_v)(1+\gamma_{w'})} u^i u_j.
\end{aligned}
$$

Collecting the above intermediate results and after quite a load of calculations we obtain the final result[23]

$$
\begin{aligned}
R(u)^i_j = \delta^i_j &+ \frac{1}{P}[(1-\gamma_{w'}^2)v^i v_j - (1+\gamma_v)(1+\gamma_{w'})v^i w'_j \\
&+ (1-\gamma_v^2)w'^i w'_j + (\gamma_{w'} + 1 + \gamma_v + 3\gamma_v\gamma_{w'} - 2\psi)w'^i v_j], \qquad (15.126)
\end{aligned}
$$

where we have set

$$
P = (1+\gamma_v)(1+\gamma_{w'})(1+\gamma_w) = (1+\gamma_v)(1+\gamma_{w'})(1+2\gamma_v\gamma_{w'} - \psi). \qquad (15.127)
$$

In order to compute the Euclidean angle θ introduced by $R(u)$ we use the standard formula[24]

$$
\mathrm{Tr}\, R(u) = 1 + 2\cos\theta, \qquad (15.128)
$$

where Tr $R(u)$ is the trace of $R(u)$. Using (15.126) we find

[23] In the calculations we can omit the terms containing u^i because they vanish except the term $u^i u_j$ which gives -1.

[24] See, e.g., page 163 in H. Goldstein, C. Poole, J. Safko *"Classical Mechanics" Third Edition* Addison-Wesley Publishing Company, (2002).

$$\cos\theta = \frac{1}{2}(R^{\mu}_{\mu} - 1) = \frac{1}{2P}[(1-\gamma_{w'}^2)v^{\mu}v_{\mu} - (1+\gamma_v)(1+\gamma_{w'})v^{\mu}w'_{\mu}$$
$$+ (1-\gamma_v^2)w'^{\mu}w'_{\mu} + (\gamma_{w'} + 1 + \gamma_v + 3\gamma_v\gamma_{w'} - 2\psi)w'^{\mu}v_{\mu}] + 1.$$

For the terms involved in this relation we compute

$$v^{\mu}v_{\mu} = -1 + \gamma_v^2,$$
$$w'^{\mu}w'_{\mu} = -1 + \gamma_{w'}^2,$$

$$w'^{\mu}v_{\mu} = \sqrt{-1+\gamma_{w'}^2}\sqrt{-1+\gamma_v^2}\cos\phi,$$
$$\psi = -w'^{i}v_i = w'^{\mu}v_{\mu} - v^0 w'^0 = \gamma_v\gamma_{w'} - \sqrt{-1+\gamma_{w'}^2}\sqrt{-1+\gamma_v^2}\cos\phi.$$

Replacing we find the cosine of the angle θ:

$$\cos\theta = 1 - \frac{(\gamma_v - 1)(\gamma_{w'} - 1)\sin^2\phi}{1 + \gamma_v\gamma_{w'} + \sqrt{-1+\gamma_{w'}^2}\sqrt{-1+\gamma_v^2}\cos\phi}. \tag{15.129}$$

To compare this result with existing results in the literature we introduce the quantity $\tau = \sqrt{\dfrac{(1+\gamma_w)(1+\gamma_{w'})}{(\gamma_w - 1)(\gamma_{w'} - 1)}}$. Then it can be shown that (15.129) reads

$$\cos\theta = 1 - \frac{2\sin^2\phi}{1 + \tau^2 + 2\tau\cos\phi}, \tag{15.130}$$

which coincides with the existing result in the literature.[25]

[25] See equation (10) of the paper "Wigner's rotation revisited" Ar Ben-Menahem, *Am. J. Phys.* **53**, 62–66, (1985).

Chapter 16
Geometric Description of Relativistic Interactions

16.1 Collisions and Geometry

There is a fundamental difference concerning the concept of particle in Newtonian and in relativistic physics. In Newtonian Physics a particle is a "thing" which has been created once and since then exists as an absolute unit for ever. Concerning the physical quantities associated with a particle they are divided into two classes: the ones which are inherent in the structure of the particle such as mass, charge and characterize the identity of the Newtonian particle and those which depend on the motion of the particle in a reference system such as velocity, linear momentum. Newtonian particles are assumed to interact by collisions creating larger systems. This interaction of particles happens in a way that the overall inherent quantities of the particles are conserved (i.e., mass, charge) while some of the motion dependent physical quantities such as energy, mass, and linear momentum are also conserved. Finally the systems consisting of many particles have more macroscopic physical quantities such as temperature, pressure.

In Special Relativity the scenario is drastically different. A relativistic particle is not a "thing" but a set of physical quantities which share the same frame as proper or as characteristic frame depending if they are timelike or spacelike, respectively. In this sense a "particle" can appear as a n or as three particles $p, e^-, \bar{\nu}_\beta$ according to the reaction $n \rightarrow p + e^- + \bar{\nu}_\beta$. In this sense, for example, an electron is a set of two scalars (mass, charge), one vector (spin), and other physical fields making up a catalogue which is not necessarily complete. That is, it is possible that an experiment will indicate that the electron has associated a new physical property which we did not know and which we have to consider. This new quantity will not change the concept of the electron; however, it might lead us to consider more types of electron (as we do with the positron) but this is it. In the relativistic approach we have discovered only partially the entities of creation and we should be open to new discoveries.

As in Newtonian theory in Special (and General) Relativity the relativistic particles, besides their inherent physical quantities, also have physical quantities which depend on their motion in a LCF. These quantities are the four-velocity, four-momentum, etc. Furthermore they are allowed to interact creating larger systems

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which, as in Newtonian Physics, introduce new physical quantities such as (relativistic) temperature. During their interaction we assume that some of the inherent and the non-inherent physical quantities of the particles are conserved. For example, the total charge and the total four-momentum are conserved.

It becomes clear that in Special Relativity one is possible to look upon a relativistic interaction as a transformation of a set of Lorentz tensors (scalars, four-vectors, tensors) to another set of Lorentz tensors (scalars, four-vectors, etc.) which may differ both in number and in type. This view is useful and important because it makes possible to geometrize particle interactions. One might ask

Why should we want to geometrize relativistic interactions?

The answer is simple

Because if we manage to do so, then we gain twofold, that is,

(a) We shall be able to “explain” various relativistic results and show that what is new is not their mathematical expression but the physical explanation we have given to mathematical expressions. In this sense we let “physics justify geometry”.
(b) It would be possible to produce new results which will be consistent mathematically and possibly lead to new physical phenomena we have not thought of yet. In a sense we let “geometry propose physics”.

In this chapter we discuss relativistic reactions in the above sense. However, we shall restrict our considerations to systems of four-momentum only, mainly because at the level we work this is the physical quantity we are most interested. However, the approach is otherwise general so we shall speak of interacting four-vectors which need not be four-momenta!

16.2 Geometric Description of Collisions in Newtonian Physics

From the previous discussion it is clear that behind the physics of relativistic collisions there exists a geometry, which would be interesting and useful to be recognized and studied. The same holds for Newtonian collisions, although this is not widely known. In this section we discuss briefly the Newtonian case for a simple model and in the next section we consider the relativistic counterpart.

We consider two similar smooth (solid) spheres with masses m_1, m_2 which are moving along the same direction along the x-axis with velocities v_1, v_2, respectively. We assume that at some moment the spheres collide centrally and after their collision they move again along the x-axis with corresponding velocities v_1', v_2'. Newtonian conservation laws of linear momentum and energy give

$$m_1(v_1 - v_1') = -m_2(v_2 - v_2') \qquad \text{(conservation of momentum)}, \tag{16.1}$$

$$m_1(v_1^2 - v_1'^2) = -m_2(v_2^2 - v_2'^2) \qquad \text{(conservation of energy)}. \tag{16.2}$$

Combining these two equations we find

$$v_1 - v_2 = v_2' - v_1', \tag{16.3}$$

that is, the relative velocity of the spheres is preserved during the elastic collision. Solving (16.1) and (16.2) with respect to v_1', v_2' we find

$$v_1' = \frac{1-k}{1+k}\,v_1 + \frac{2k}{1+k}\,v_2, \tag{16.4}$$

$$v_2' = \frac{2}{1+k}\,v_1 + \frac{k-1}{1+k}\,v_2, \tag{16.5}$$

where $k = m_2/m_1$. These equations can be written in the following form:

$$\begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} \dfrac{1-k}{1+k} & \dfrac{2k}{1+k} \\[2ex] \dfrac{2}{1+k} & \dfrac{-1+k}{1+k} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{16.6}$$

or

$$\begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = M \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \tag{16.7}$$

where M is the matrix:

$$M = \begin{pmatrix} \dfrac{1-k}{1+k} & \dfrac{2k}{1+k} \\[2ex] \dfrac{2}{1+k} & \dfrac{-1+k}{1+k} \end{pmatrix}. \tag{16.8}$$

The matrix M contains all the information concerning the collision and it is the geometric expression of the laws of conservation of linear momentum and energy. In order to understand the geometric significance of the matrix M (equivalently, of the laws of conservation of linear momentum and energy), we consider M as a transformation matrix in a linear space in which we interpret geometrically the collision. We assume that in the specific example this space is an Euclidean two-dimensional space, whose vectors are of the form$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$. In this space the state of the system of masses is described by the position vector $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, which after the collision is transformed to the vector $\begin{pmatrix} v_1' \\ v_2' \end{pmatrix}$. Therefore, in that space, the collision is described by the linear transformation

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \xrightarrow{M} \begin{pmatrix} v_1' \\ v_2' \end{pmatrix}.$$

We note that geometrically the collision is not characterized by the velocities, but with the matrix M which is defined in terms of the ratio $k = \frac{m_2}{m_1}$ of the masses. This observation is very important and has many consequences among which we note the following:

(1) If $\begin{pmatrix} \bar{v}_1 \\ \bar{v}_2 \end{pmatrix}$ is another initial point (i.e., different initial velocities), then for the same masses we can write the result of a central collision without any further calculations, as the image point under the linear transformation M, that is,

$$\begin{pmatrix} \bar{v}_1' \\ \bar{v}_2' \end{pmatrix} = M \begin{pmatrix} \bar{v}_1 \\ \bar{v}_2 \end{pmatrix}$$

(2) It is possible to study qualitatively the collision without any reference to velocities, by studying the geometric properties of the transformation matrix M.

The second point is more important, because it geometrizes the physical process of collision and allows us to employ the powerful mathematical methods of differential geometry in the study of the problem. Of course the problem we considered is simple but the method can be generalized to more complex problems and make possible the study of physical problems, which otherwise would be very difficult to do. More generally, the geometrization of mechanics is a long-standing study which recently has been extended to the study of dynamical systems, which involve practically most branches of modern science ranging from physics to economics and medicine (chaos, Hamiltonian dynamics, etc.).

Exercise 86 Show that the transformation matrix M satisfies the following geometric properties.

(1) M is symmetric, if and only if $m_1 = m_2$.
(2) $traceM = 0$
(3) $\det M = -1$
(4) $M^2 = I$ or $M^{-1} = M$ (M is a projective operator)
(5) $diag(m_1, m_2) = M^t diag(m_1, m_2) M$.

16.3 Geometric Description of Relativistic Reactions

The geometric description of Newtonian collisions cannot be transferred to Special Relativity as such, for the following two reasons:

(1) The photons have zero mass, therefore their description is not possible in a collision matrix containing masses.

(2) The number and the identity of the reacting particles in the relativistic inelastic collisions are not preserved. This means that the collision matrix is not square hence there is no inverse. This implies that the relativistic collision cannot be seen as a transformation matrix of masses in a properly dimensioned linear space. However, this might be possible for certain elastic collisions.

There are ways we can circumvent the first point and we can deal with the second, but the required methods are new and outside the simple Newtonian approach described in the last section.

However, this does not worry us because, as we explained, relativistic reactions should be understood as transformations between sets of timelike (i.e., for us in this restricted view we have by the level of the book) four-vectors under the constraint of conservation of their sum (=conservation of four-momentum). Therefore the geometrization of relativistic reactions will be realized by the establishment of geometric relations among the momenta of the reacting particles and possibly some momenta of the produced particles. This approach will be realized in two stages: the first stage involves the expression of the relativistic quantities in terms of the invariants built from the four-vectors (mainly their length which corresponds to the masses of the particles). The second stage concerns the description of the reaction with a matrix of three-momenta. In the following, we shall deal briefly with the first stage, because the detailed development of both stages is involved and beyond the level of this book.

16.4 The General Geometric Results

Before we discuss the details of relativistic systems we consider some general geometric results which shall be used. The main result is the following theorem (see also Proposition 4).

Theorem 1 *The sum of a set of future-directed timelike and/or null four-vectors is a future-directed timelike four-vector except if and only if all four-vectors are null and parallel in which case the sum is a null four-vector parallel to the other null vectors.*

Proof

Let $A^i_{(1)}, \ldots, A^i_{(n)}$ be a finite set of future-directed particle four-vectors. Then we have

$$A^0_{(I)} > 0, \quad A^a_{(I)}A_{(I)a} \leq 0 \qquad I = 1, .., n.$$

The sum

$$\left[\sum_{I=1}^{n} \left(A^a_{(I)}\right)\right]^2 = \sum_{I=1}^{n} \left(A^a_{(I)}\right)^2 + 2 \sum_{1\leq I<J\leq n} A^a_{(I)}A_{(J)a}, \tag{16.9}$$

where $\left(A^a_{(I)}\right)^2 = A^a_{(I)}A_{(I)a}$.

We consider the terms in the rhs. For the first term we have

$$\sum_{I=1}^{n}\left(A^a_{(I)}\right)^2 \leq 0,$$

where the equality holds if and only if all $A^a_{(I)}$ are null.

Concerning the second term let us assume that there exist a $A^a_{(I)}$ which is timelike. In the proper frame $\Sigma(A_{(I)})$ of $A^a_{(I)}$, the components of $A^a_{(I)}$ are

$$A^a_{(I)} = \begin{pmatrix} A^{0+}_{(I)} \\ \mathbf{0} \end{pmatrix}_{\Sigma(A_{(I)})}.$$

Let $A^a_{(I)}A_{(J)a}$ be an arbitrary element in the second term and assume that in $\Sigma(A_{(I)})$ the four-vector $A^a_{(J)}$ has components.

$$A^a_{(J)} = \begin{pmatrix} A^{0}_{(J)} \\ \mathbf{A}_{(J)} \end{pmatrix}_{\Sigma(A_{(I)})}. \tag{16.10}$$

Then in $\Sigma(A^a_{(I)})$ we have

$$\left[A^a_{(I)}A_{(J)a}\right]_{\Sigma\left(A_{(I)}\right)} = -A^{0+}_{(I)}A^0_{(J)} < 0 \tag{16.11}$$

because the four-vectors are future directed. But $A^a_{(I)}A^a_{(J)}$ is invariant, therefore

$$A^a_{(I)}A_{(J)a} < 0$$

in all LCF.

Working similarly with the rest of the terms which contain $A^a_{(I)}$ we show that

$$\sum_{J=1}^{n} A^a_{(I)}A_{(J)a} < 0.$$

But $A^a_{(I)}$ is an arbitrary timelike four-vector, hence

$$\sum_{1\leq I,J\leq n} A^a_{(I)}A_{(J)a} < 0. \tag{16.12}$$

We conclude that if all four-vectors $A^a_{(I)}, \quad I = 1,\ldots,n$ are not null, then

$$\left[\sum_{I=1}^{n}\left(A^a_{(I)}\right)\right]^2 < 0, \tag{16.13}$$

that is, the sum is a timelike four-vector.

We assume now that all four-vectors $A^a_{(I)}$, $I = 1, \ldots, n$ are null. Then $\sum_{I=1}^{n} \left(A^a_{(I)}\right)^2 = 0$ and furthermore

$$\sum_{1 \le I < J \le n} A^a_{(I)} A_{(J)a} = \left\{ \begin{array}{l} = 0 \text{ if and only if } A^a_{(1)} \parallel \ldots \parallel A^a_{(n)} \\ < 0 \text{ otherwise} \end{array} \right\}.$$

The first conclusion is profound. In order to prove the second we note that in an arbitrary LCF Σ the arbitrary (future directed) null four-vector $A^a_{(I)}$ has components

$$A^a_{(I)} = E_{(I)} \begin{pmatrix} 1 \\ \hat{\mathbf{e}}_{(I)} \end{pmatrix} \qquad I = 1, .., n,$$

where $E_{(I)} > 0$. Therefore, in Σ we have

$$A^a_{(I)} A_{(J)a} = E_{(I)} E_{(J)} \left(-1 + \hat{\mathbf{e}}_{(I)} \cdot \hat{\mathbf{e}}_{(J)}\right).$$

But $\hat{\mathbf{e}}_{(I)} \cdot \hat{\hat{\mathbf{e}}}_{(J)} \le 1$ and "=" holds if and only if $\hat{\mathbf{e}}_{(I)} = \hat{\mathbf{e}}_{(J)}$, $I, J = 1, \ldots, n$. This implies $A^a_{(I)} \parallel A^a_{(J)}$ $I, J = 1, \ldots, n$ which proves the second assertion and completes[1] the proof of the theorem.

16.4.1 The 1+3 Decomposition of a Particle Four-Vector wrt a Timelike Four-Vector

In Sect. 12.2.2 we discussed the 1+3 decomposition of a vector wrt a unit timelike vector. In this section we slightly generalize this discussion and consider the 1+3 decomposition of a particle four-vector A^i wrt a timelike four-vector B^i which is not necessarily unit. We start again from the identity

$$A^i = \delta^i_j A^j = \left(\delta^i_j + \frac{1}{B^2} B^i B_j\right) A^j - \frac{1}{B^2} B^i \left(B_j A^j\right) \tag{16.14}$$

and write:

$$A^i = A^i_{\parallel} + A^i_{\perp}, \tag{16.15}$$

[1] Why we do not have to consider the case that one subset of four-vectors is null and the subset of the remaining four-vectors is timelike? Is this case covered?

where:

$$A^i_{\parallel} = -\frac{1}{B^2}\left(B_j A^j\right) B^i, \tag{16.16}$$

$$A^i_{\perp} = \left(\delta^i_j + \frac{1}{B^2} B^i B_j\right) A^j. \tag{16.17}$$

The four-vector $A^i_{\parallel}$ is the parallel component and the four-vector $A^i_{\perp}$ the normal projection of A^i along B^i. The tensor

$$h^i_j(B) = \delta^i_j + \frac{1}{B^2} B^i B_j \tag{16.18}$$

is the *projection tensor* associated with the four-vector B^i. We have:

$$A^i_{\perp} = h^i_j(B) A^j. \tag{16.19}$$

As we have seen the two-tensor $h^i_j(B)$ is very important in Relativity and it is used extensively in all calculations. In the rest frame Σ_B of B^i the components of $h^i_j(B)$ are

$$h^i_j(B) = diag(0, 1, 1, 1)_{\Sigma_B} \tag{16.20}$$

and can be thought as the Euclidean metric of the rest space of B^i. It satisfies the properties:

$$h_{ij}(B) = h_{ji}(B),\ h^i_j(B) h^j_k(B) = h^i_k(B),\ h^i_i(B) = 3. \tag{16.21}$$

Summarizing we have

$$A^a = \frac{-1}{B^2}(A_b B^b) B^a + h^a_b(B) A^b \tag{16.22}$$

and in matrix form[2]:

$$A^a = \begin{pmatrix} -\frac{A^b B_b}{B} \\ \\ h(B)_{ab} A^b \end{pmatrix}_{\Sigma_B}.$$

Having given the basic facts about the decomposition of four-vectors we are ready to discuss the geometry of systems of particle four-vectors. It will help if we keep in mind that the results we shall obtain hold for the four-momentum, but

[2] The unit in the direction of B^a is B^a/B hence the B in the denominator.

they apply to any set of interacting four-vectors. In the rest of the chapter we shall consider two special four-vector systems. The system $A + B \to C$, which is the generic reaction, and the system $A + B \to C + D$, which is more general that the first. The results we shall obtain will be generic in the sense that they apply to *all reactions* of the type we consider. We shall illustrate the results by specific examples, which demonstrate how one applies the general ideas in practice. Finally, it should be remarked that the results we derive are covariant, therefore, it is possible to develop proper software (for algebraic computing), which will give the answer to *any problem* for sufficiently given data!

16.5 The System of Two to One Particle Four-Vectors

The simpler system of particle four-vectors is the system consisting of two future-directed particle four-vectors A^a, B^a and corresponds to the generic reaction $A + B \to C$. From these two four-vectors we define another two four-vectors the $A^a + B^a$, $A^a - B^a$. The four-vector $A^a + B^a$ is a future-directed particle four-vector, the Center System (CS) four-vector (see Sect. 1.12). In case A^a, B^a are four-momenta the four-vector $A^a + B^a$ is the center of momenta four-vector and the particle it defines is the center of momentum particle.

We shall express the inner products between the four-vectors A^a, B^a in terms of their lengths. The results will be used to compute the zeroth component of A^a in the proper frame of one of the rest four-vectors. The zeroth component, equivalently the inner product $A^a B_a$, is computed from the identity

$$(A^a + B^a)^2 = (A^a)^2 + (B^a)^2 + 2A^a B_a. \tag{16.23}$$

If we replace the lengths

$$(A^a + B^a)^2 = -M^2,\ (A^a)^2 = -A^2,\ (B^a)^2 = -B^2 \qquad (M, A, B > 0) \tag{16.24}$$

we find

$$A^a B_a = \frac{1}{2}\left(-M^2 + A^2 + B^2\right). \tag{16.25}$$

Concerning the spatial part of A^a in Σ_B we have

$$h_{ab}(B)A^b = A_a + \frac{(A^b B_b)}{B^2} B_a = A_a + \frac{1}{2B^2}\left(-M^2 + A^2 + B^2\right) B_a. \tag{16.26}$$

The spatial part can be decomposed further in the direction $\widehat{\mathbf{A}}_{(B)}$ and the length of $\mathbf{A}_{(B)}$ in Σ_B:

$$\mathbf{A}_{(B)} = \sqrt{h_{ab}(B)A^a A^b}\,\widehat{\mathbf{A}}_{(B)}. \tag{16.27}$$

Finally for the four-vector A^a we have the decomposition

$$A^a = \begin{pmatrix} \frac{M^2-A^2-B^2}{2B} \\ \\ \sqrt{h_{ab}(B)A^a A^b}\ \widehat{\mathbf{A}}_{(B)} \end{pmatrix}_{\Sigma_B} . \tag{16.28}$$

We see that the four-vector A^a is determined from the invariants A, B, M and the spatial direction $\widehat{\mathbf{A}}_{(B)}$.

Exercise 87 For every non-null four-vector A^i the symmetric tensor $h_{ij}(A) = \eta_{ij} - \frac{1}{A^k A_k} A_i A_j$ projects normal to the vector A^i, that is, $h_{ij}(A)A^j = 0$.

(a) Let p^i the four-momentum of a particle and p_1^i the four-momentum of another particle. Show the identity

$$p_1^i = \frac{p_1^j p_j}{p^k p_k} p^i + h^i_j(p) p_1^j .$$

This identity defines the $1+3$ decomposition of the four-vector p_1^i wrt the four-vector p^i. The part $p^i_{1\|} \equiv \frac{p_1^j p_j}{p^k p_k} p^i$ is the parallel part and the part $p^i_{1\perp} \equiv h^i_j(p) p_1^j$ is the normal part. Show that the inner product $p_1^j p_j = \frac{1}{2}\{(p^i + p_1^i)^2 - p^i p_i - p_1^i p_{1i}\}$, that is, it is expressed in terms of the length of the four-vectors. In order to give the above a physical interpretation we consider the masses $p^i p_i = -m^2 c^2$, $p_1^i p_{1i} = -m_1^2 c^2$ and then the inner product $p_1^j p_j = -m E_1^\Sigma$, where E_1^Σ is the energy of the particle of four-momentum p_1^i in the proper frame Σ of the particle with four-momentum p^i. Define $p_2^i = p^i - p_1^i$ and show that

$$E_1^\Sigma = \frac{m^1 + m_2^2 - m_1^2}{2m} c^2,$$

where $p_2^i p_{2i} = -m_2^2 c^2$. Also show that the length of the normal part $p^i_{1\perp} \cdot p^i_{1\perp}$ is given by the relation

$$p^i_{1\perp} \cdot p^i_{1\perp} \equiv \mathbf{p}_1{}^2 c^2 = E_1^{*2} - m_1^2 c^4 = \frac{[m^2 - (m_1 + m_2)^2][m^2 - (m_1 - m_2)^2]}{2M} c^2 .$$

Finally collect the results of the 1+3 decomposition of the four-vector p_1^i in Σ as follows:

$$p_1^i = \begin{pmatrix} E_1^*/c \\ \mathbf{p}_1 \end{pmatrix}_\Sigma = \begin{pmatrix} \dfrac{m^2 + m_2^2 - m_1^2}{2m} c \\ \\ \dfrac{[m^2 - (m_1 + m_2)^2][m^2 - (m_1 - m_2)^2]}{2M} c\hat{\mathbf{e}}^\Sigma \end{pmatrix}_\Sigma ,$$

where $\hat{\mathbf{e}}^\Sigma$ is the unit of the spatial part of p_1^i in Σ.

16.5.1 The Triangle Function of a System of Two Particle Four-Vectors

The length $h_{ab}(B)A^a A^b$ of $\mathbf{A}_{(B)}$ is an invariant therefore it is a characteristic quantity of the system of the four-vectors A^a, B^a. In order to determine the exact dependence of this quantity on the four-vectors A^a, B^a we compute it. From (16.26) we have

$$\begin{aligned} h_{ab}(B)A^a A^b &= \left[A_a + \frac{1}{2B^2}\left[-M^2 + A^2 + B^2\right] B_a \right] A^a \\ &= \frac{1}{4B^2}\lambda^2(M^2, A^2, B^2), \end{aligned} \tag{16.29}$$

where

$$\begin{aligned} \lambda(M^2, A^2, B^2) &= \sqrt{\left[-M^2 + A^2 + B^2\right]^2 - 4A^2 B^2} \\ &= \sqrt{M^4 + A^4 + B^4 - 2M^2A^2 - 2M^2B^2 - 2A^2B^2}. \end{aligned} \tag{16.30}$$

We conclude the following:

(1) The function $\lambda(M^2, A^2, B^2)$ is an invariant depending only on the lengths of the vectors $(A + B)^a$, A^a, B^a.
(2) The function $\lambda(M^2, A^2, B^2)$ is symmetric in all its arguments.

We have met the function λ before, when we were studying the collision $A + B \to C$. Here we simply recover it in a more general set up and, furthermore, we give its covariant geometric meaning. The properties of the function λ are given in Exercise 34.

Example 79 Show that the quantity

$$E = \frac{1}{4}\sqrt{-\lambda(x, y, z)} \tag{16.31}$$

equals the area of an Euclidean triangle of sides $\sqrt{x}$, $\sqrt{y}$, $\sqrt{z}$. Prove that the triangle inequality of Euclidean geometry assures that in a Euclidean space $\lambda(x, y, z) < 0$.

Solution

The semi-perimeter τ of a Euclidean triangle of sides $\alpha = \sqrt{x}, \beta = \sqrt{y}, \gamma = \sqrt{z}$ equals $\tau = (\alpha + \beta + \gamma)/2$ and the area is given by the Heron's formula $E = \sqrt{\tau(\tau - \alpha)(\tau - \beta)(\tau - \gamma)}$. We find

$$\begin{aligned} E^2 &= \frac{1}{16}(\alpha+\beta+\gamma)(-\alpha+\beta+\gamma)(\alpha-\beta+\gamma)(\alpha+\beta-\gamma) \\ &= \frac{1}{16}\left(-a^2+(\beta+\gamma)^2\right)\left(a^2-(\beta-\gamma)^2\right) \\ &= -\frac{1}{16}\left(x-(\sqrt{y}+\sqrt{z})^2\right)\left(x-(\sqrt{y}-\sqrt{z})^2\right) \\ &= -\frac{1}{16}\lambda(x,y,z). \end{aligned}$$

Because in Euclidean geometry the area $E \geq 0$ it follows that $\lambda(x, y, z) \leq 0$. This creates a contradiction because in Minkowski space $\lambda(x, y, z)$ equals the measure $|\mathbf{A}^*|$ which is positive, hence in Minkowski space $\lambda(x, y, z) \geq 0$! However, there is no problem because in Euclidean geometry the triangle inequality implies that $\lambda(x, y, z) \leq 0$ and in Minkowski space the same inequality assures that $\lambda(x, y, z) \geq 0$. To prove the latter, we consider the four-vectors A^a, B^a, $(A+B)^a$ and from (10.21) we have for $x = (A+B)^2$, $y = A^2$, $z = B^2$

$$\left[(A+B)^2 - \left(\sqrt{A^2}+\sqrt{B^2}\right)\right]\left[(A+B)^2 - \left(\sqrt{A^2}-\sqrt{B^2}\right)\right] \geq 0$$

$$(A+B)^2 \geq \left(\sqrt{A^2}+\sqrt{B^2}\right)^2, \quad (A+B)^2 \geq \left(\sqrt{A^2}-\sqrt{B^2}\right)^2$$

or

$$(A+B)^2 \leq \left(\sqrt{A^2}+\sqrt{B^2}\right)^2, \quad (A+B)^2 \leq \left(\sqrt{A^2}-\sqrt{B^2}\right)^2.$$

But $\sqrt{A^2}+\sqrt{B^2} \geq \sqrt{A^2}-\sqrt{B^2}$ because $A, B > 0$. Hence

$$(A+B) \geq \sqrt{A^2}+\sqrt{B^2} \tag{16.32}$$

for all pairs of particle four-vectors A^a, B^a. Therefore the condition

$$\lambda(M^2, A^2, B^2) \geq 0$$

is satisfied and it is equivalent to the triangle inequality in Minkowski space.

In words relation (16.32) means that the particle four-vector $(A+B)^a$ is something "more" than the aggregate of the particle four-vectors A^a, B^a. This "something" is the structure which couples the two four-vectors into the system – particle four-vector $(A+B)^a$. To see what this implies we examine its effect in one well-known issue of relativistic physics, the mass loss. We assume the four-vectors A^a, B^a to be four-momenta. Then $(A+B)^a$ represents the center momentum particle whose mass is M. Then inequality (16.32) implies that M is larger than the sum of the masses m_A, m_B of the individual particles A^a, B^a, the difference counting for the potential (or internal) energy of the particle $(A+B)^a$.

An interesting special case is $\lambda(x, y, z) = 0$. Then $M = A + B$ and $|\mathbf{A}_M| = |\mathbf{B}_M| = 0$, which implies that the proper frame of the four-vectors A^a, B^a coincides with the proper frame of M^a. We call the condition $\lambda(x, y, z) = 0$ the threshold of the interaction of the four-vectors A^a, B^a. Note that the above do not apply only to four-momentum but to *interacting* triplets of four-vectors A^a, B^a, $(A + B)^a$.

16.5.2 Extreme Values of the Four-Vectors $(A \pm B)^2$

The lengths $(A + B), (A - B)$ of a system of two particle four-vectors A^a, B^a of length A, B find application in many cases and especially in relativistic reactions (collisions) ($(A + B)$ is the mass of the center of momentum particle and $(A - B)$ is the amount of transfer of four-momenta). It is of interest to determine the extreme values of the quantities $(A + B), (A - B)$ when the direction of A^a, B^a changes while their length remains constant.

From (16.23) we have $-(A \pm B)^2 = -A^2 - B^2 \mp 2A^a B_a$ therefore the extremum of $-(A \pm B)^2$ occurs when the term $A^a B_a$ is an extremum. The term $A^a B_a$ is invariant therefore it is possible to be computed in any LCF. We choose the proper frame Σ_A of A^a and write

$$A^a = \begin{pmatrix} A \\ \mathbf{0} \end{pmatrix}_{\Sigma_A}, \quad B^a = \begin{pmatrix} B^0_{(A)} \\ \mathbf{B}_{(A)} \end{pmatrix}_{\Sigma_A}$$

from which follows

$$A^a B_a = -AB^0_{(A)}.$$

But $B^0_{(A)} = \sqrt{B^2 + \mathbf{B}^2_{(A)}}$ hence

$$A^a B_a = -A\sqrt{B^2 + \mathbf{B}^2_{(A)}}. \tag{16.33}$$

We note that in the rhs the only quantity which changes is $\mathbf{B}^2_{(A)}$, therefore the extremum (maximum) of $A^a B_a$ occurs if

$$\frac{\partial(A^a B_a)}{\partial(\mathbf{B}^2_{(A)})} = 0.$$

This condition gives

$$-A\frac{\mathbf{B}_{(A)}}{\sqrt{B^2 + \mathbf{B}^2_{(A)}}} = 0 \Rightarrow \mathbf{B}_{(A)} = 0$$

hence

$$(A^a B_a)_{\max} = -AB.$$

It follows

$$(A+B)^2_{\max} = A^2 + B^2 + 2AB = (A+B)^2,$$
$$(A-B)^2_{\min} = A^2 + B^2 - 2AB = (A-B)^2.$$

Condition $\mathbf{B}_{(A)} = 0$ means that the proper frames of the four-vectors A^a, B^a coincide or $A^a = \alpha B^a$, where α is an invariant. To compute a we multiply this equation with A^a and find

$$-A^2 = \alpha(A^a B_a) = -\alpha AB \Rightarrow \alpha = \frac{A}{B}.$$

Therefore, the condition for the extremum is[3] $\frac{A^a}{A} = \frac{B^a}{B}$ or $A^a \parallel B^a$.

16.5.3 The System A^a, B^a, $(A+B)^a$ of Particle Four-Vectors in CS

Let A^a, B^a be two particle four-vectors, which are not null and parallel and let Σ^* be the CS of the system of A^a, B^a. We shall denote the components of the four-vectors in CS with an asterisk, e.g., for the vector A^a we write

$$A^a = \begin{pmatrix} A^{0*} \\ \mathbf{A}^* \end{pmatrix}_{\Sigma^*}. \tag{16.34}$$

In order to compute A^{0*} we note that in Σ^* $(A+B)^a = \begin{pmatrix} M \\ 0 \end{pmatrix}_{\Sigma^*}$, hence

$$A^a(A+B)_a = -A^{0*}M. \tag{16.35}$$

But from (16.25) we have, if we replace B with M (why we can do this?)

$$A^a(A+B)_a = \frac{1}{2}\left(-M^2 - A^2 + B^2\right)$$

[3] It is possible to compute the extremals of the quantity (16.33) without any calculations if we note that the quantity under the square root is non-negative. Therefore the maximum value occurs for the minimum value of the denominator which is $\mathbf{B}^2_{(A)} = 0$ and the minimum when the denominator is maximum, that is, when $\mathbf{B}^2_{(A)}$ is infinite.

from which follows

$$-A^{0*}M = \frac{1}{2}\left(-M^2 - A^2 + B^2\right) \Rightarrow A^{0*} = \frac{1}{2M}\left(M^2 + A^2 - B^2\right). \tag{16.36}$$

Concerning the length of the spatial part $\mathbf{A}^*$ from (16.29) we have

$$(\mathbf{A}^*)^2 = h_{ab}(A+B)A^a A^b = \frac{1}{4M^2}\lambda^2(M^2, A^2, B^2). \tag{16.37}$$

It is instructive to compute the components of the four-vector A^a in the CS directly by making use of the decomposition (16.28). To do this we write $(A+B)^a = A^a + B^a$ as $-B^a = A^a - (A+B)^a$, which shows that $-B^a$ is the CS of the four-vectors A^a and $-(A+B)^a$. Therefore relation (16.28) applies if we make the correspondence

$$\begin{aligned} M &\longleftrightarrow B, \\ \Sigma_B &\longleftrightarrow \Sigma^*, \\ \widehat{\mathbf{A}}_{(B)} &\longleftrightarrow -\mathbf{A}^*, \\ A^0_{(B)} &\longleftrightarrow -A^{0*} \end{aligned}$$

It follows

$$A^a = \begin{pmatrix} \dfrac{M^2 + A^2 - B^2}{2M} \\ \frac{1}{2M}\sqrt{\lambda(M^2, A^2, B^2)}\widehat{\mathbf{A}}^* \end{pmatrix}_{\Sigma^*}. \tag{16.38}$$

Concerning the decomposition of B^a we have from (16.38), if we interchange $A \longleftrightarrow B$ and note that $\widehat{\mathbf{A}}^* + \widehat{\mathbf{B}}^* = 0$

$$B^a = \begin{pmatrix} \dfrac{M^2 - A^2 + B^2}{2M} \\ -\frac{1}{2M}\sqrt{\lambda(M^2, A^2, B^2)}\widehat{\mathbf{A}}^* \end{pmatrix}_{\Sigma^*}. \tag{16.39}$$

In order to check our results we compute the angle θ^*_{AB} between the vectors $\widehat{\mathbf{A}}^*, \widehat{\mathbf{B}}^*$ in Σ^*. Obviously we expect to find $\theta^*_{AB} = \pi$ since $\widehat{\mathbf{A}}^* = -\widehat{\mathbf{B}}^*$. We compute

$$\mathbf{A}^* \cdot \mathbf{B}^* = h_{ab}(A+B)A^a B^a = -\frac{1}{4M^2}\lambda^2(M^2, A^2, B^2) = |\mathbf{A}^*||\mathbf{B}^*|\cos\theta^*_{AB}.$$

But $|\mathbf{A}^*| = |\mathbf{B}^*| = \frac{1}{2M}\sqrt{\lambda(M^2, A^2, B^2)}$ therefore $\cos\theta^*_{AB} = -1 \Rightarrow \theta^*_{AB} = \pi$.

16.5.4 The System A^a, B^a, $(A+B)^a$ in the Lab

The LCF in which we study the "motion" of systems of particle four-vectors are

(a) The CS
(b) The proper frame of one of the particle four-vectors, which we call the *target system*
(c) The *Laboratory System* (lab) which usually coincides with the proper frame of one of the particle four-vectors

In Sect. 16.5.3 we studied the system of two particle four-vectors A^a, B^a in the CS. In this section we study the same system in the lab which we assume that it coincides with the proper frame of particle B^a.

We denote the components of a four-vector in the lab with an L and write

$$A^a = \begin{pmatrix} A^{0L} \\ \mathbf{A}^L \end{pmatrix}_{\Sigma^L}, \quad B^a = \begin{pmatrix} B \\ 0 \end{pmatrix}_{\Sigma^L}, \ (A+B)^a = \begin{pmatrix} (A+B)^{0L} \\ (\mathbf{A}+\mathbf{B})^L \end{pmatrix}_{\Sigma^L}. \qquad (16.40)$$

In order to compute the components A^{0L}, $(\mathbf{A}^L)^2$ we apply relations (16.36) and (16.37) provided we change $A+B$ with B. From (16.36) we find

$$A^{0L} = -\frac{1}{2B}\left(-B^2 - A^2 + M^2\right) = \frac{1}{2B}\left(B^2 + A^2 - M^2\right) \qquad (16.41)$$

and from (16.37)

$$(\mathbf{A}^L)^2 = h_{ab}(B)A^a A^b = \frac{1}{4B^2}\lambda^2(M^2, A^2, B^2), \qquad (16.42)$$

where we have used the fact the triangle function is symmetric in all its arguments.

Let us assume that Σ^* is moving wrt $\Sigma^L = \Sigma_B$ with velocity $\boldsymbol{\beta}^*$. Then we have

$$B^a = \begin{pmatrix} B \\ \mathbf{0} \end{pmatrix}_{\Sigma^L} = \begin{pmatrix} \dfrac{M^2 - A^2 + B^2}{2M} \\ \frac{1}{2M}\sqrt{\lambda(M^2, A^2, B^2)}\widehat{\mathbf{B}}^* \end{pmatrix}_{\Sigma^*}.$$

The two expressions are related with the Lorentz transformation which relates Σ^*, Σ^L. Therefore (see (1.75)).

$$\frac{M^2 - A^2 + B^2}{2M} = \gamma B \Rightarrow \gamma = \frac{M^2 - A^2 + B^2}{2BM} \qquad (16.43)$$

$$\frac{1}{2M}\sqrt{\lambda(M^2, A^2, B^2)}\widehat{\mathbf{B}}^* = \gamma\boldsymbol{\beta} \Rightarrow \boldsymbol{\beta} = \frac{\sqrt{\lambda(M^2, A^2, B^2)}}{2M\gamma B}\widehat{\mathbf{B}}^*. \qquad (16.44)$$

Let us consider the application of the above general results in special two particle systems.

Example 80 Consider an electron and a positron with four-momenta $p^a_{e^-}$, $p^a_{e^+}$, respectively. Assume $c = 1$.

(1) Determine the energy of each particle in the CS.
(2) Show that in the CS the spatial momenta are antiparallel.
(3) Assume that the positron rests in the lab and compute the velocity of the CS in the lab.

Solution

Let $m_{e^-} = m_{e^+} = m$ the masses of the particles involved. The four-momentum of the center of momentum particle is $(p_{e^-} + p_{e^+})^a = p^a_{e^-} + p^a_{e^+}$ and let its mass be M. Identifying A^a, B^a with $p^a_{e^-}$, $p^a_{e^+}$, respectively, we find from (16.38)

$$p^a_{e^-} = \begin{pmatrix} \dfrac{M^2 - m^2 + m^2}{2M} \\ \frac{1}{2M}\sqrt{\lambda(M^2, m^2, m^2)}\widehat{\mathbf{p}}^*_{e^-} \end{pmatrix}_{\Sigma^*} = \begin{pmatrix} \dfrac{M^2}{2} \\ \frac{1}{2}\sqrt{M^2 - 4m^2}\widehat{\mathbf{p}}^*_{e^-} \end{pmatrix}_{\Sigma^*} \tag{16.45}$$

$$p^a_{e^+} = \begin{pmatrix} \dfrac{M^2 + m^2 - m^2}{2M} \\ \frac{1}{2M}\sqrt{\lambda(M^2, m^2, m^2)}\widehat{\mathbf{p}}^*_{e^+} \end{pmatrix}_{\Sigma^*} = \begin{pmatrix} \dfrac{M^2}{2} \\ -\frac{1}{2}\sqrt{M^2 - 4m^2}\widehat{\mathbf{p}}^*_{e^-} \end{pmatrix}_{\Sigma^*}. \tag{16.46}$$

The energy is the zeroth component of the four-momentum, hence

$$E^*_{e^-} = E^*_{e^+} = \frac{M^2}{2}.$$

The three-momenta are the spatial part of the four-momentum. It follows

$$|\mathbf{p}^*_{e^-}| = |\mathbf{p}^*_{e^+}| = \frac{1}{2}\sqrt{M^2 - 4m^2}.$$

Concerning the angle $\theta^*_{e^-e^+}$ between the two three-momenta in the CS we find $\theta^*_{e^-e^+} = \pi$.

From (16.43) we find that the $\gamma-$factor of the CS in the lab (=proper frame of p_{e^+}) is

$$\gamma = \frac{M^2 - m^2 + m^2}{2mM} = \frac{M}{2m}.$$

Finally from (16.44) we find for the $\boldsymbol{\beta}^*$-factor ($c = 1$)

$$\boldsymbol{\beta}^* = \frac{\sqrt{\lambda(M^2, m^2, m^2)}}{2M\gamma m}\widehat{\mathbf{p}}^*_{e^-} = \frac{\sqrt{M^2 - 4m^2}}{M}\widehat{\mathbf{p}}^*_{e^-}.$$

The three-vector $\widehat{\mathbf{p}}^*_{e^-}$ is a space direction in the lab which shall be defined from the initial conditions (it is the direction of the bullet particle in the lab).

In the next exercise we compute the same results using direct calculation, so that the reader will gain experience with this type of problems.

Exercise 88 Assume $p^a_{e^-} = A^a, \quad p^a_{e^+} = B^a,\ A^2 = B^2 = m^2$ and that the length of the momentum of the center of momenta particle is M. Verify the following calculations:

(a) Energies:

$$E^*_{e^-} = \frac{M^2 - m^2 + m^2}{2Mc^2}c^4 = \frac{M}{2}c^2 = E^*_{e^+}.$$

(b) three-momenta:

$$h_{ab}(p_{e^-} + p_{e^+})p^a_{e^-}p^b_{e^-} = -m^2c^2 + \frac{1}{4M^2}\left[M^2 - m^2 + m^2\right]c^2 = \left(\frac{M^2}{4} - m^2\right)c^2$$

$$\mathbf{p}^{*2}_{e^-} = \mathbf{p}^{*2}_{e^+} = \frac{1}{4}(M^2 - 4m^2)c^2.$$

(c) Angle:

$$h_{ab}(p_{e^-} + p_{e^+})p^a_{e^-}p^b_{e^+} = -\frac{1}{4M^2}\left[M^4 - 4m^2M^2\right]c^2 = -\frac{1}{4}\left[M^2 - 4m^2\right]c^2$$
$$= -|\mathbf{p}^*_{e^-}||\mathbf{p}^*_{e^+}| = |\mathbf{p}^*_{e^-}||\mathbf{p}^*_{e^+}|\cos\pi.$$

Example 81 In the LCF Σ the null four-vectors A^a, B^a have components $A_a = A\,(1, 1, 0, 0)_\Sigma$, $B_a = B\,(1, 0, 1, 0)_\Sigma$.

(1) If A^a, B^a are four-momenta of photons determine the energy and the direction of motion (the speed is known!) of the photons in the LCF Σ.
(2) A particle Γ of mass $\sqrt{3}$ moves in the plane x, y of Σ with factor $\beta = \frac{1}{2}$ in a direction which makes an angle 45^0 with the x-axis. Determine the energy and the four-momentum of particle Γ in Σ.
(3) Compute the angle between the direction of motion of the photons A^a, B^a in the proper frame of the particle Γ.

Solution

(1) The energy of the photon A^a in Σ is

$$^A_\Sigma E/c = A \Rightarrow ^A_\Sigma E = Ac$$

and the three-momentum is

$$^A_\Sigma \mathbf{p} = A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

The direction of motion of the photon A^a in Σ is

$$^A_\Sigma \widehat{\mathbf{p}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Prove that the photon B^a moves in Σ in a direction normal to the direction of A^a.

(2) The γ-factor of the particle Γ in Σ is

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{2}{\sqrt{3}}, \quad \beta = \frac{1}{2}.$$

Therefore the energy and the three-momentum of Γ in Σ are

$$\begin{aligned} ^\Gamma_\Sigma E &= m\gamma c^2 = 2c^2, \\ ^\Gamma_\Sigma \mathbf{p} &= m\gamma\beta c \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \frac{c}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}. \end{aligned}$$

The four-momentum of Γ in Σ is

$$p^a_\Gamma = \begin{pmatrix} ^\Gamma_\Sigma E/c \\ \\ ^\Gamma_\Sigma \mathbf{p} \end{pmatrix}_\Sigma = \begin{pmatrix} 2c \\ \frac{c}{\sqrt{2}} \\ \frac{c}{\sqrt{2}} \\ 0 \end{pmatrix}_\Sigma .$$

We check the results by showing that $p^a_\Gamma p_{\Gamma a} = -m_\Gamma c^2$. Indeed

$$p^a_\Gamma p_{\Gamma a} = -4c^2 + \frac{c^2}{2} + \frac{c^2}{2} = -3c^2.$$

(3) To find the angle of the space directions of motion of the photons A^a, B^a in the proper frame of Γ we consider the inner product

$$\begin{aligned}
h_{ab}(\Gamma)A^a B^b &= \left(\eta_{ab} + \frac{1}{3c^2} p_{\Gamma a} p_{\Gamma b}\right) A^a B^b \\
&= A^a B_a + \frac{1}{3c^2}\left(p_{\Gamma a} A^a\right)\left(p_{\Gamma b} B^b\right) \\
&= 0 + \frac{1}{3c^2} c^2 \left(-2 + \frac{1}{\sqrt{2}}\right)^2 \\
&= \frac{1}{3}\left(-2 + \frac{1}{\sqrt{2}}\right)^2 .
\end{aligned}$$

But

$$h_{ab}(\Gamma)A^a B^b = |{}^A_\Sigma\mathbf{p}||{}^B_\Sigma\mathbf{p}| \cos\theta_{AB,\Sigma}.$$

The $|{}^A_\Sigma\mathbf{p}| = |{}^B_\Sigma\mathbf{p}| = 1$ therefore $\theta_{AB,\Sigma} = \cos^{-1}\{\frac{1}{3}(-2 + \frac{1}{\sqrt{2}})^2\}$.

16.6 The Relativistic System $A^a + B^a \to C^a + D^a$

Let M be the length of the common Center System four-vector of the pairs (A^a, B^a), (C^a, D^a). From (16.25) we have

$$-M^2 = -A^2 - B^2 + 2(AB) = -C^2 - D^2 + 2(CD), \tag{16.47}$$

where $(AB) = A^a B_a, (CD) = C^a D_a$. We assume that the lengths A, B, C, D of the particle four-vectors are given (e.g., they are the masses of the corresponding particles) and also that we are given enough data to compute one of the inner products (AB) or (CD). We shall show that with these data we can compute the remaining quantities involved.

In order to do that we consider the decomposition of the four-vectors

(a) In the lab, which we assume that it coincides with the proper frame of the (non-null!) particle four-vector B^a and
(b) In the CS.

Before we continue our discussion we recall that if we are given the four-vectors of one pair, e.g., the pair (A^a, B^a) in a LCF Σ then the $\boldsymbol{\beta}^*$-factor and the γ^*-factor of the CS in Σ are given by the relations

$$\boldsymbol{\beta}^*_\Sigma = \frac{\mathbf{A}_\Sigma + \mathbf{B}_\Sigma}{A^0_\Sigma + B^0_\Sigma}, \qquad \gamma^*_\Sigma = \frac{A^0_\Sigma + B^0_\Sigma}{M}. \tag{16.48}$$

In case Σ is the proper frame of B^a (=lab) then these relations read

$$\boldsymbol{\beta}^*_L = \frac{\mathbf{A}^L}{A^{L0} + B}, \quad \gamma^*_L = \frac{A^{0L} + B}{M}. \tag{16.49}$$

In the calculations it will be useful to write relations (16.48) and (16.49) in covariant form, that is, in terms of tensor quantities. If Σ is determined by the unit timelike four-vector s^a ($s^a s_a = -1$) which is the unit in the direction of the center four-vector $A^a + B^2$, then (16.48) is written as

$$\boldsymbol{\beta}^*_\Sigma = \frac{h_{ab}(s)(A^b + B^b)}{-s_b(A^b + B^b)}, \quad \gamma^*_\Sigma = -\frac{s_b(A^b + B^b)}{M}. \tag{16.50}$$

In the lab these relations give

$$\boldsymbol{\beta}^*_L = \frac{h_{ab}(B)(A^b + B^b)}{-\frac{1}{B}B_b(A^b + B^b)} = \frac{\frac{1}{2B}\sqrt{\lambda(M^2, A^2, B^2)}}{-\frac{1}{B}\left(A_b B^b - B^2\right)}\widehat{\mathbf{A}} = \frac{1}{2}\frac{\sqrt{\lambda(M^2, A^2, B^2)}}{M^2 - A^2 + B^2}\widehat{\mathbf{A}}, \tag{16.51}$$

$$\gamma^*_L = -\frac{B_b(A^b + B^b)}{BM} = \frac{-B_b A^b + B^2}{BM} = \frac{M^2 - A^2 + B^2}{2BM}, \tag{16.52}$$

where $\widehat{\mathbf{A}}$ is the unit of the spatial direction of A^a in the proper frame of B^a.

From relations (16.43) and (16.44) we have the following decompositions of the four-vectors in the lab (Σ^L) and in the CS (Σ^*).[4]

$$A^a = \begin{pmatrix} \dfrac{M^2 + A^2 - B^2}{2M} \\ \frac{1}{2M}\sqrt{\lambda(M^2, A^2, B^2)}\widehat{\mathbf{A}}^* \end{pmatrix}_{\Sigma^*} = \begin{pmatrix} \dfrac{M^2 - A^2 - B^2}{2B} \\ \frac{1}{2B}\sqrt{\lambda(M^2, A^2, B^2)}\widehat{\mathbf{A}}_B \end{pmatrix}_{\Sigma^L}, \tag{16.53}$$

$$B^a = \begin{pmatrix} \dfrac{M^2 - A^2 + B^2}{2M} \\ -\frac{1}{2M}\sqrt{\lambda(M^2, A^2, B^2)}\widehat{\mathbf{A}}^* \end{pmatrix}_{\Sigma^*} = \begin{pmatrix} B \\ \mathbf{0} \end{pmatrix}_{\Sigma^L}, \tag{16.54}$$

$$C^a = \begin{pmatrix} \dfrac{M^2 + C^2 - D^2}{2M} \\ \frac{1}{2M}\sqrt{\lambda(M^2, C^2, D^2)}\widehat{\mathbf{C}}^* \end{pmatrix}_{\Sigma^*} = \begin{pmatrix} -\dfrac{(CB)}{B} \\ h^a_b(B)C^b \end{pmatrix}_{\Sigma^L}, \tag{16.55}$$

[4] The "=" does not mean that we can equate the corresponding components of the four-vectors because the decompositions/components refer to different coordinate frames. It simply indicates that they refer to the decomposition of the *same* four-vector in different LCF. The components of each vector are related to the other via the Lorentz transformation which relates the corresponding LCF frames.

$$D^a = \begin{pmatrix} \dfrac{M^2 - C^2 + D^2}{2M} \\ -\frac{1}{2M}\sqrt{\lambda(M^2, C^2, D^2)}\widehat{\mathbf{C}}^* \end{pmatrix}_{\Sigma^*} = \begin{pmatrix} -\dfrac{(DB)}{B} \\ h^a_b(B)D^b \end{pmatrix}_{\Sigma^L} . \tag{16.56}$$

In order to calculate the zeroth component (i.e., (CB)) of the four-vector C^a in the lab, we note that the inner product is invariant, therefore we can compute it in any frame we wish. We choose the CS Σ^* where we know the components of the four-vectors. We have

$$\begin{aligned}(CB) = -\frac{1}{4M^2}[&\left(M^2 - A^2 + B^2\right)\left(M^2 + C^2 - D^2\right) \\ &+ \sqrt{\lambda(M^2, A^2, B^2)}\sqrt{\lambda(M^2, C^2, D^2)}(\widehat{\mathbf{A}}^* \cdot \widehat{\mathbf{C}}^*)].\end{aligned} \tag{16.57}$$

Similarly for the four-vector D^a we have

$$\begin{aligned}(DB) = -\frac{1}{4M^2}[&\left(M^2 - A^2 + B^2\right)\left(M^2 - C^2 + D^2\right) \\ &- \sqrt{\lambda(M^2, A^2, B^2)}\sqrt{\lambda(M^2, C^2, D^2)}(\widehat{\mathbf{A}}^* \cdot \widehat{\mathbf{C}}^*)].\end{aligned} \tag{16.58}$$

We conclude that in order to determine the zeroth components of the " daughter" four-vectors C^a, D^a in lab we need to know the angle between $\widehat{\mathbf{A}}^*$, $\widehat{\mathbf{C}}^*$ in the CS Σ^*.

In the following we compute the various angles which enter in the geometry of the interaction.

(a) Computation of the angle θ^L_{CD} between the spatial parts of the four-vectors C^a, D^a in the lab frame.
We have

$$\begin{aligned}h_{ab}(B)C^aD^b &= \left(\eta_{ab} + \frac{1}{B^2}B_aB_b\right)C^aD^b \\ &= (CD) + \frac{1}{B^2}(CB)(DB) = \frac{1}{2}(-M^2 + C^2 + D^2) \\ &\quad + \frac{1}{B^2}(CB)(DB) \\ &= |\mathbf{C}^L||\mathbf{D}^L|\cos\theta^L_{CD} \Rightarrow \\ \cos\theta^L_{CD} &= \frac{1}{|\mathbf{C}^L||\mathbf{D}^L|}\left[\frac{1}{2}(-M^2 + C^2 + D^2) + \frac{1}{B^2}(CB)(DB)\right],\end{aligned} \tag{16.59}$$

where

$$|\mathbf{C}^L| = \sqrt{h_{ab}(B)C^aC^b} = \sqrt{\left[-C^2 + \frac{1}{B^2}(CB)^2\right]}, \tag{16.60}$$

$$|\mathbf{D}^L| = \sqrt{h_{ab}(B)D^aD^b} = \sqrt{\left[-D^2 + \frac{1}{B^2}(DB)^2\right]}. \tag{16.61}$$

(b) Computation of the angle θ^L_{AC} between the spatial part of the four-vectors C^a and A^a in the lab frame.
The spatial part of the four-vectors A^a, C^a in the lab is given by the relations $h_{ab}(B)A^b$, $h_{cd}(B)C^d$, respectively. Therefore the angle $\cos\theta^L_{AC}$ of the spatial parts in the lab is

$$|\mathbf{C}^L||\mathbf{A}^L|\cos\theta^L_{AC} = h_{ab}(B)A^a h^b_d(B)C^d = h_{ab}(B)A^aC^b. \tag{16.62}$$

The term

$$\begin{aligned} h_{ab}(B)C^aA^b &= (CA) + \frac{1}{B^2}(AB)(CB) \\ &= (CA) + \frac{-M^2 + A^2 + B^2}{2B^2}(CB). \end{aligned}$$

In order to compute the inner product (CA) we use the conservation equation $A^a + B^a = C^a + D^a$ which we multiply with C^a and get

$$\begin{aligned} (AC) + (CB) = -C^2 + (CD) &\Rightarrow \\ (AC) &= -(CB) - C^2 + \frac{1}{2}(-M^2 + C^2 + D^2) \\ &= -(CB) - \frac{1}{2}(M^2 + C^2 - D^2). \end{aligned} \tag{16.63}$$

Replacing we find

$$h_{ab}(B)C^aA^b = -\frac{1}{2}\left[(M^2 + C^2 - D^2) + \frac{1}{B^2}\left(M^2 - A^2 + B^2\right)(CB)\right]. \tag{16.64}$$

The term $|\mathbf{C}^L|$ has been calculated in (16.60) and the term $|\mathbf{A}^L|$ in (16.42). Introducing the above results in (16.62) we find

$$\cos\theta^L_{CA} = -\frac{B^2}{\sqrt{\lambda(M^2, A^2, B^2)}\sqrt{-B^2C^2 + (CB)^2}} \left[(M^2 + C^2 - D^2) + \frac{1}{B^2}\left(M^2 - A^2 + B^2\right)(CB)\right]. \tag{16.65}$$

(c) Computation of the angle θ^*_{AC} of the spatial part of the four-vector C^a with the spatial part of the four-vector A^a in the CS.

We note that normal to the direction of $\boldsymbol{\beta}^*_L$ the $\mathbf{C}_L$ does not change. The condition for this is

$$\mathbf{C}_L \times \boldsymbol{\beta}^*_L = \mathbf{C}^* \times \boldsymbol{\beta}^*_L \tag{16.66}$$

which gives

$$|\mathbf{C}_L| \sin\theta^L_{AC} = |\mathbf{C}^*| \sin\theta^*_{AC} \Rightarrow$$
$$\sin\theta^L_{AC} = \frac{|\mathbf{C}^*|}{|\mathbf{C}_L|}\sin\theta^*_{AC} = \frac{\sqrt{\lambda(M^2, C^2, D^2)}}{2M\sqrt{-C^2 + \frac{1}{B^2}(CB)^2}}\sin\theta^*_{AC}, \tag{16.67}$$

where we have made use of (16.60).

With the calculation of the angle $\sin\theta^*_{AC}$ we have completed the various quantities concerning the interaction $A + B \longrightarrow C + D$.

Obviously it is necessary that we organize the above results in order to make clear their internal coherence and, most important, to make them usable in practice.

(A) Data We take as data the quantities $A, B, C, D, A^{0L}, \widehat{\mathbf{A}}$, which practically means the masses of the particles A, B, C, D the energy E^L_A of A in the lab (=proper frame of B) and the direction of motion of particle A in the lab.
(B) Computed quantities

(1) The mass M of the center of momentum particle:

$$M^2 = A^2 + B^2 + 2BE^L_A. \tag{16.68}$$

(2) The λ functions of the mother and the daughter particles:

$$\lambda^2(M^2, A^2, B^2) = M^4 + A^4 + B^4 - 2M^2A^2 - 2M^2B^2 - 2A^2B^2, \tag{16.69}$$
$$\lambda^2(M^2, C^2, D^2) = M^4 + C^4 + D^4 - 2M^2C^2 - 2M^2D^2 - 2C^2D^2. \tag{16.70}$$

(3) The factors $\boldsymbol{\beta}^*_L,\ \gamma^*_L$ of the CS in the lab system:

$$\boldsymbol{\beta}^*_L = \frac{1}{2}\frac{\sqrt{\lambda(M^2, A^2, B^2)}}{M^2 - A^2 + B^2}\widehat{\mathbf{A}}, \tag{16.71}$$

$$\gamma^*_L = \frac{M^2 - A^2 + B^2}{2BM}, \tag{16.72}$$

$$\gamma^*_L\boldsymbol{\beta}^*_L = \frac{\sqrt{\lambda(M^2, A^2, B^2)}}{4MB}\widehat{\mathbf{A}}. \tag{16.73}$$

The knowledge of these quantities fixes the Lorentz transformation between $\Sigma^*\ \Sigma^L$:

$$\mathbf{r}^L = \mathbf{r}^* + \left[\frac{\gamma^*_L - 1}{\boldsymbol{\beta}^{*2}_L}(\boldsymbol{\beta}^*_L \cdot \mathbf{r}^*) - \gamma^*_L l^*\right]\boldsymbol{\beta}^*_L, \tag{16.74}$$

$$l^L = \gamma^*_L(l^* - \boldsymbol{\beta}^*_L \cdot \mathbf{r}^*). \tag{16.75}$$

(4) The energies of the mother and the daughter particles in the CS Σ^*:

$$A^{0*} = E^*_A = \frac{M^2 + A^2 - B^2}{2M}, \tag{16.76}$$

$$B^{0*} = E^*_B = \frac{M^2 - A^2 + B^2}{2M}, \tag{16.77}$$

$$C^{0*} = E^*_C = \frac{M^2 + C^2 - D^2}{2M}, \tag{16.78}$$

$$D^{0*} = E^*_D = \frac{M^2 - C^2 + D^2}{2M}. \tag{16.79}$$

(5) The lengths of the three-momenta of the mother and the daughter particles in the CS Σ^*:

$$|\mathbf{A}^*| = |\mathbf{B}^*| = \frac{1}{2M}\sqrt{\lambda(M^2, A^2, B^2)} \tag{16.80}$$

$$|\mathbf{C}^*| = |\mathbf{D}^*| = \frac{1}{2M}\sqrt{\lambda(M^2, C^2, D^2)}. \tag{16.81}$$

It is not possible to compute any additional quantities because none of the four-vectors C^a, D^a is completely known either in Σ^* or in Σ^L. It is required an additional datum and as such we consider the angle between the spatial directions $\widehat{\mathbf{A}}^*,\ \widehat{\mathbf{C}}^*$ in the CS, that is, we assume we know the (Euclidean) inner product $\widehat{\mathbf{A}}^* \cdot \widehat{\mathbf{C}}^*$. With this new datum we compute the following quantities:

(6) The invariants (CB), (DB), (CA), (DA):

$$
\begin{aligned}
(CB) = &-\frac{1}{4M^2}\left[\left(M^2 - A^2 + B^2\right)\left(M^2 + C^2 - D^2\right)\right] \\
&- \frac{1}{4M^2}\left[\sqrt{\lambda(M^2, A^2, B^2)}\sqrt{\lambda(M^2, C^2, D^2)}(\widehat{\mathbf{A}}^* \cdot \widehat{\mathbf{C}}^*)\right],
\end{aligned}
\tag{16.82}
$$

$$
(DB) = -(CB) - \frac{1}{2}\left(M^2 + B^2 - A^2\right), \tag{16.83}
$$

$$
(CA) = -(CB) - \frac{1}{2}(M^2 + C^2 - D^2), \tag{16.84}
$$

$$
(DA) = -(CB) + \frac{1}{2}\left(M^2 + B^2 - A^2\right) + \frac{1}{2}(M^2 - C^2 + D^2). \tag{16.85}
$$

(7) The energies of the daughter particles in the lab Σ^L:

$$
E_C^L = -\frac{1}{B}(CB) \qquad [(CB) < 0 \text{ because } E_C^L > 0], \tag{16.86}
$$

$$
E_D^L = -\frac{1}{B}(DB) \qquad [(DB) < 0 \text{ because } E_D^L > 0]. \tag{16.87}
$$

(8) The length of the three-momentum of the daughter particles in the lab Σ^L:

$$
|\mathbf{C}^L| = \sqrt{-C^2 + \frac{1}{B^2}(CB)^2}, \tag{16.88}
$$

$$
|\mathbf{D}^L| = \sqrt{-D^2 + \frac{1}{B^2}(DB)^2}. \tag{16.89}
$$

(9) The angle θ_{CD}^L between the direction of motion of the daughter particles in the lab Σ^L:

$$
\cos\theta_{CD}^L = \frac{1}{|\mathbf{C}^L||\mathbf{D}^L|}\left[\frac{1}{2}(-M^2 + C^2 + D^2) + \frac{1}{B^2}(CB)(DB)\right]. \tag{16.90}
$$

(10) The angle θ_{AC}^L of the three-momentum $\mathbf{C}^L$ of the daughter particle C and the direction of motion of the mother particle A in the LAB Σ^L :

$$
\begin{aligned}
\cos\theta_{AC}^L = &-\frac{B}{|\mathbf{C}^L|\sqrt{\lambda(M^2, A^2, B^2)}} \\
&\left[(M^2 + C^2 - D^2) + \frac{1}{B^2}\left(M^2 - A^2 + B^2\right)(CB)\right],
\end{aligned}
\tag{16.91}
$$

$$
\sin\theta_{AC}^L = \frac{|\mathbf{C}^*|}{|\mathbf{C}^L|}\sin\theta_{AC}^* = \frac{\sqrt{\lambda(M^2, C^2, D^2)}}{2M|\mathbf{C}^L|}\sin\theta_{AC}^*. \tag{16.92}
$$

The above general relations can be used directly in an algebraic computing program to develop software that would solve automatically collision problems provided the correct data have been introduced. A hint on how this can be done we give in the examples below.

Example 82 An electron with four-momentum A^a interacts with a positron of four-momentum B^a producing two photons with four-momenta C^a and D^a, respectively. Given that (a) the positron rests in the laboratory; and (b) the electron moves in the laboratory along the direction specified by the unit vector $\hat{\mathbf{e}}_A$, with speed factor γ_A compute

(1) The mass M of the center of momentum particle
(2) The factor $\boldsymbol{\beta}^*_L$ of the CS in the lab system lab
(3) The energy of the particles and the measure of their three-momenta in the CS
(4) The energy of the photons and the measure of their three-momenta in the lab if the angle of direction of the photon with the direction of motion of the electron in the CS is $\theta^*_{AC} = \frac{\pi}{2}$
(5) The angle θ^L_{AC} of the direction of motion of the photon with that of the electron in the lab
(6) The angle between the direction of motion of the photons in the lab

Solution

From the data of the problem we have $A = B = m,\ C = D = 0$. Furthermore it is given that the energy of the photon A in the lab is $(c = 1)$

$$E^L = A^L = m\gamma_A$$

and its direction $\widehat{\mathbf{A}} = \mathbf{e}_A$. Finally it is given that in the CS the direction or propagation of the emitted photon is perpendicular to the direction of motion of the electron, therefore $\widehat{\mathbf{A}}^* \cdot \widehat{\mathbf{C}}^* = \cos\frac{\pi}{2} = 0$. Using these data we compute directly from the previous formulae the required quantities.

(1) From (16.68) we find M

$$M = m\sqrt{2(1 + \gamma_A)}.$$

(2) From (16.69) and (16.70) we find the λ function:

$$\lambda^2(M^2, A^2, B^2) = M^4 + 2m^4 - 4M^2m^2 - 2m^4 = M^2(M^2 - 4m^2),$$
$$\lambda^2(M^2, C^2, D^2) = M^4.$$

(3) From (16.71), (16.72), and (16.73) the velocity factors of the CS in the lab:

$$\begin{aligned}
\boldsymbol{\beta}_L^* &= \frac{1}{2}\frac{M\sqrt{M^2-4m^2}}{M^2} = \frac{1}{2}\sqrt{1-4\left(\frac{m}{M}\right)^2} \\
&= \frac{1}{2}\sqrt{1-4\frac{1}{2(1+\gamma_A)}} = \frac{\gamma_A-1}{\gamma_A+1}\mathbf{e}_A, \\
\gamma_L^* &= \frac{M^2-m^2+m^2}{2mM} = \frac{1}{2}\frac{M}{m} = \sqrt{\frac{1+\gamma_A}{2}}.
\end{aligned}$$

(4) From (16.76) to (16.79) the energy of the particles in the CS:

$$\begin{aligned}
E_A^* = E_B^* &= \frac{M^2+m^2-m^2}{2M} = \frac{M}{2} = m\sqrt{\frac{(1+\gamma_A}{2}}, \\
E_C^* = E_D^* &= \frac{M^2}{2M} = \frac{M}{2} = m\sqrt{\frac{(1+\gamma_A}{2}}.
\end{aligned}$$

(5) From (16.80) we have for the length of the three-momentum of the electron and the positron in the CS:

$$|\mathbf{A}^*| = |\mathbf{B}^*| = \frac{1}{2M}\lambda(M^2, A^2, B^2) = \frac{m}{2}\sqrt{\left(\frac{M}{m}\right)^2-4} = m\sqrt{\frac{\gamma_A-1}{2}}$$

and from relation (16.81)

$$E_C^* = E_D^* = \frac{M}{2} = m\sqrt{\frac{1+\gamma_A}{2}}.$$

Since C^a, D^a are null vectors the energy equals the measure of the three-momenta, that is, $|C^*| = |D^*| = m\sqrt{\frac{1+\gamma_A}{2}}$.

(6) In order to compute the energy and the measure of the three-momenta of the daughter particles in the lab we have to compute the inner product of the four-vectors involved. Equations (16.82) and (16.83) give for the inner products (CB), (DB)

$$(CB) = (DB) = -\frac{1}{4M^2}\left[M^2M^2\right] = -\frac{M^2}{4}$$

and (16.84) and (16.91) give for the inner products (AC), (AD)

$$\begin{aligned}
(CA) &= \frac{M^2}{4} - \frac{M^2}{2} = -\frac{M^2}{4}, \\
(DA) &= -\frac{M^2}{4} + \frac{M^2}{2} + \frac{M^2}{2} = \frac{3M^2}{4}.
\end{aligned}$$

From relations (16.86) and (16.87) we compute

$$E_C^L = E_D^L = -\frac{1}{m}\frac{-M^2}{4} = \frac{1+\gamma_A}{2}m.$$

Since C^a, D^a are null $|\mathbf{p}_C^L| = |\mathbf{p}_D^L| = m\frac{1+\gamma_A}{2}$.

(7) Using (16.88) and (16.89) we compute the magnitude of the three-momentum of the daughter particles in the lab:

$$|\mathbf{C}^L| = |\mathbf{D}^L| = \frac{M^2}{4m}.$$

(8) Relation (16.90) gives the angle between the daughter particles in the lab:

$$\cos\theta_{CD}^L = \frac{16m^2}{M^4}\left[-\frac{M^2}{2} + \frac{1}{m^2}\frac{M^4}{16}\right] = 1 - \frac{8}{\left(\frac{M}{m}\right)^2} = \frac{\gamma_A - 3}{1+\gamma_A}.$$

(9) From (16.91) we compute the angle θ_{AC}^L:

$$\cos\theta_{AC}^L = -\frac{m^2}{M\sqrt{M^2 - 4m^2}\frac{M^2}{4}}\left[M^2 + \frac{1}{m^2}M^2 - \frac{M^2}{4}\right] = \frac{\sqrt{\left(\frac{M}{m}\right)^2 - 4}}{\frac{M}{m}}$$
$$= \sqrt{\frac{\gamma_A - 1}{\gamma_A + 1}}.$$

(10) Finally from (16.92) we compute the angle θ_{AC}^L:

$$\sin\theta_{AC}^L = \frac{M^2}{2M\frac{M^2}{4m}}\sin\frac{\pi}{2} = \sqrt{\frac{2}{1+\gamma_A}}.$$

Example 83 (Compton scattering) A photon is scattered by an electron which rests in the laboratory. If the energy of the scattered photon in the laboratory is E_A and the scattering angle (in the laboratory) is θ^L, calculate the energy of the scattered photon in the laboratory.[5]

Solution

The reaction is

$$\gamma + e^- \longrightarrow \gamma + e^-.$$

Considering the reaction $A + B \longrightarrow C + D$ we identify the following data for the present problem:

[5] For the standard treatment see Example 51

$$A = C = 0,\ B = D = m,\ E_A^L = E_A,\ \theta_{AC}^L = \theta^L.$$

From relation (16.68) we compute

$$M^2 = m^2 + 2mE_A. \tag{16.93}$$

For the triangle function $\lambda(M^2, 0, m^2)$ we have from (16.30)

$$\lambda(M^2, 0, m^2) = (-M^2 + m^2)^2.$$

Replacing the data in (16.91) we end up with one equation with unknown inner product (CB):

$$cos\theta^L = -\frac{m^2\left[M^2 - m^2 + \frac{1}{m^2}(M^2 + m^2)(CB)\right]}{(M^2 - m^2)(CB)}.$$

We solve in terms of (CB) and find

$$(CB) = -\frac{(M^2 - m^2)m^2}{M^2 + m^2 - (M^2 - m^2)\cos\theta^L}.$$

Replacing M^2 from (16.93) we find finally

$$(CB) = -\frac{m^2 E_A}{m + E_A(1 - \cos\theta^L)}. \tag{16.94}$$

Having computed (CB) we replace in the general relations and calculate the rest of the elements of the reaction. For example, the energy of C in the laboratory is from (16.86)

$$E_C^L = -\frac{(CB)}{m} = \frac{E_A m}{m + (1 - \cos\theta^L)E_A}. \tag{16.95}$$

If the data of the problem are different then we work in a similar manner (see Example 84).

Exercise 89 In Example 83 consider as given the energies E_A, E_C in the lab system of the falling and the scattered photon and prove that the angle of scattering θ_{AC}^L is given by the expression

$$\sin^2\frac{\theta_{AC}^L}{2} = \frac{1}{2}m\left(\frac{1}{E_C} - \frac{1}{E_A}\right). \tag{16.96}$$

[Hint: In the expression $E_C = \frac{-(CB)}{m}$ replace (CB) from (16.94) and solve for $\cos\theta_{AC}^L$].

Example 84 Study the reaction $A + B \longrightarrow C + D$ considering as data the lengths A, B, C, D of the particle four-vectors and the scattering angles θ^L_{AC}, θ^L_{AD}. Apply your results to the case of final state of two photons.

Solution

Normal to the direction of motion of the particle A we have

$$|\mathbf{C}^L|\sin\theta^L_{AC} = |\mathbf{D}^L|\sin\theta^L_{AD} \Rightarrow |\mathbf{D}^L| = \frac{\sin\theta^L_{AC}}{\sin\theta^L_{AD}}|\mathbf{C}^L|. \tag{16.97}$$

The inner products give

$$\begin{aligned}
\mathbf{A}^L \cdot \mathbf{C}^L &= |\mathbf{A}^L||\mathbf{C}^L|\cos\theta^L_{AC},\\
\mathbf{A}^L \cdot \mathbf{D}^L &= |\mathbf{A}^L||\mathbf{D}^L|\cos\theta^L_{AD},\\
\mathbf{A}^L \cdot (\mathbf{C}^L + \mathbf{D}^L) &= |\mathbf{A}|^L\left[|\mathbf{C}^L|\cos\theta^L_{AC} + |\mathbf{D}^L|\cos\theta^L_{AD}\right].
\end{aligned}$$

From the conservation of four-momentum we have $\mathbf{A}^L = \mathbf{C}^L + \mathbf{D}^L$. Replacing we find

$$|\mathbf{A}|^L = \frac{\sin(\theta^L_{AC} + \theta^L_{AD})}{\sin\theta^L_{AD}}|\mathbf{C}^L|. \tag{16.98}$$

We conclude that we can express the measure of the spatial parts of the four-momenta A^a, D^a in terms of the spatial part of the four-momentum C^a.

Conservation of four-momentum gives for the zeroth component

$$E^L_A + B = E^L_C + E^L_D. \tag{16.99}$$

On the other hand for the particle A we have

$$\begin{aligned}
-A^2 = -(E^L_A)^2 + \left(|\mathbf{A}|^L\right)^2 \Rightarrow (E^L_A)^2 &= A^2 + \left(|\mathbf{A}|^L\right)^2\\
&= A^2 + \frac{\sin^2(\theta^L_{AC} + \theta^L_{AD})}{\sin^2\theta^L_{AD}}|\mathbf{C}^L|^2
\end{aligned}$$

and for the particles $C,\ D$

$$\begin{aligned}
(E^L_C)^2 &= C^2 + \left(|\mathbf{C}|^L\right)^2,\\
(E^L_D)^2 &= D^2 + \left(|\mathbf{D}|^L\right)^2 = D^2 + \frac{\sin^2\theta^L_{AC}}{\sin^2\theta^L_{AD}}|\mathbf{C}^L|^2.
\end{aligned}$$

Replacing in (16.99) we find the equation

$$\sqrt{A^2 + \frac{\sin^2(\theta^L_{AC} + \theta^L_{AD})}{\sin^2 \theta^L_{AD}} |\mathbf{C}^L|^2} + B = \sqrt{C^2 + \left(|\mathbf{C}|^L\right)^2} + \sqrt{D^2 + \frac{\sin^2 \theta^L_{AC}}{\sin^2 \theta^L_{AD}} |\mathbf{C}^L|^2} \tag{16.100}$$

which we solve and determine $|\mathbf{C}^L|$. From (16.97) we determine $|\mathbf{D}^L|$. Then from (16.88)

$$|\mathbf{C}^L|^2 = -C^2 + \frac{(CB)^2}{B^2}$$

we determine the inner product (CB) and from (16.89) the inner product (DB). Then (16.86) and (16.87) give the energies E^L_C, E^L_D of the daughter particles in the laboratory. Finally from (16.90) we compute M^2 and consequently any other quantity we wish. Obviously the calculations are involved and this indicates the usefulness of the covariant study of the relativistic collisions which makes possible the solution of a problem with the use of algebraic computing programmes.

Application

Final state of two photons means that the daughter particles are photons. This implies $C = D = 0$ and from the above analysis we find

$$|\mathbf{C}|^L = E^L_C,$$
$$E^L_D = |\mathbf{D}|^L = \frac{\sin \theta^L_{AC}}{\sin \theta^L_{AD}} E^L_C,$$
$$E^L_A = \sqrt{A^2 + \frac{\sin^2(\theta^L_{AC} + \theta^L_{AD})}{\sin^2 \theta^L_{AD}} \left(E^L_C\right)^2}.$$

Replacing in (16.100) we find an equation with sole unknown the energy E^L_C:

$$\left[\frac{\sin^2(\theta^L_{AC} + \theta^L_{AD})}{\sin^2 \theta^L_{AD}} - \left(1 + \frac{\sin \theta^L_{AC}}{\sin \theta^L_{AD}}\right)^2\right] \left(E^L_C\right)^2 + 2B\left(1 + \frac{\sin \theta^L_{AC}}{\sin \theta^L_{AD}}\right) E^L_C + A^2 - B^2 = 0.$$

Solving we determine E^L_C and from this all the quantities of the reaction.

The above solution is crude. We present a second solution in accordance to the previous considerations. We replace the data in (16.91) and find an equation with unknowns the quantities (CB) and M^2:

$$\cos \theta^L_C = -\frac{B^2\left[M^2 + \frac{1}{B^2}(M^2 - A^2 + B^2)(CB)\right]}{\sqrt{\lambda(M^2, A^2, B^2)(CB)}}. \tag{16.101}$$

We solve in terms of (CB):

$$(CB) = -\frac{M^2 B^2}{M^2 - A^2 + B^2 + \sqrt{\lambda(M^2, A^2, B^2)}\cos\theta_C^L}. \tag{16.102}$$

If we replace C with D in the above relation we find without any further calculations

$$(DB) = -\frac{M^2 B^2}{M^2 - A^2 + B^2 + \sqrt{\lambda(M^2, A^2, B^2)}\cos\theta_D^L}. \tag{16.103}$$

From the conservation equation we have if we contract with B^a

$$(AB) - B^2 = (CB) + (DB) \Longrightarrow \frac{-M^2 + A^2 - B^2}{2} = (CB) + (DB). \tag{16.104}$$

Replacing in this last equation the inner products (CD), (DB) from (16.102) and (16.103) we end up with one equation which contains only M^2.

16.6.1 The Reaction $B \longrightarrow C + D$

The reaction $B \longrightarrow C + D$ is a special case of $A + B \longrightarrow C + D$ if A "disappears." This means two conditions:

(a) $A^a = 0$ and
(b) The energies $E_A^C = E_A^D$ of the daughter particles in the "proper" frame of A vanish.

We conclude that in order to study reactions of the form $B \longrightarrow C + D$ we have to use simply the general formulae we have derived and consider as data $A = 0$, $E_A^L = E_A^C = E_A^D = 0$.

Condition: $E_A^L = -\frac{1}{B}(AB) = 0 \Rightarrow M = B$.

Condition: $E_A^C = -\frac{1}{B}(AC) = 0 \Rightarrow (AC) = 0$. Then from (16.82) follows that

$$(CB) = -\frac{1}{4B^2}(B^2 + B^2)(B^2 - C^2 + D^2) = -\frac{1}{2}(B^2 - C^2 + D^2).$$

Working similarly we see that condition E_A^D implies

$$(DB) = -\frac{1}{2}(B^2 + C^2 - D^2).$$

Concerning the energies of the daughter particles in the laboratory from relations (16.86) and (16.87) we have

$$E_C^L = \frac{1}{2B}(B^2 + C^2 - D^2),$$
$$E_C^L = \frac{1}{2B}(B^2 - C^2 + D^2).$$

From relations (16.88) and (16.89) we have that the magnitude of the three-momentum of the daughter particles is the same and equal to

$$|\mathbf{C}^L| = |\mathbf{D}^L| = \frac{1}{2B}\sqrt{\lambda(B^2, C^2, D^2)}.$$

We conclude that the reactions $1 + 2 \rightarrow 3$ and $1 \rightarrow 2 + 3$ are completely characterized by five parameters. One set of such parameters is the three masses of the particles and the angles of motion of one particle in the proper frame of some other. Obviously, there are other sets of five parameters.

To check the above we replace the above results in (16.90) and find $\cos\theta_{CD}^L = -1$, hence $\theta_{CD}^L = \pi$ as expected.

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