Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

Specimen paper

Mathematics: analysis and approaches

Higher level

Paper 1

16 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

R Marks awarded for clear **Reasoning**.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme eg ***M1****,* ***A2****.*

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***M2***, ***A2***, *etc*., do **not** split the marks, unless there is a note.
- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

Examples

	Correct answer seen	Further working seen	Action
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 Implied marks

*Implied marks appear in **brackets e.g. (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

4 Follow through marks (only applied after an error is made)

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) or subpart(s). Usually, to award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if the only marks awarded in a subpart are for the answer (i.e. there is no working expected), then **FT** marks should be awarded if appropriate.*

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*e.g.* probability greater than 1, use of $r > 1$ for the sum of an infinite GP, $\sin\theta = 1.5$, non integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- Exceptions to this rule will be explicitly noted on the markscheme.
- If a candidate makes an error in one part, but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the question says hence. It is often possible to use a different approach in subsequent parts that does not depend on the answer to previous parts.

5 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (**MR**). Apply a **MR** penalty of 1 mark to that question*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates’ own work does **not** constitute a misread, it is an error.
- The ***MR*** penalty can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, ***accept*** *equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

8 Accuracy of Answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. There are two types of accuracy errors, and the final answer mark should not be awarded if these errors occur.

- **Rounding errors**: only applies to final answers not to intermediate steps.
- **Level of accuracy**: when this is not specified in the question the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures*.

9 Calculators

No calculator is allowed. The use of any calculator on paper 1 is malpractice, and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice. Examples: finding an angle, given a trig ratio of 0.4235.

Section A

1. attempt to substitute into $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$ ***(M1)***

Note: Accept use of Venn diagram or other valid method.

$0.6 = 0.5 + 0.4 - \mathrm{P}(A \cap B)$ ***(A1)***

$\mathrm{P}(A \cap B) = 0.3$ (seen anywhere) ***A1***

attempt to substitute into $\mathrm{P}(A \mid B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$ ***(M1)***

$= \dfrac{0.3}{0.4}$

$\mathrm{P}(A \mid B) = 0.75 \left(= \dfrac{3}{4} \right)$ ***A1***

Total [5 marks]

2. (a) attempting to expand the LHS ***(M1)***

$\mathrm{LHS} = (4n^2 - 4n + 1) + (4n^2 + 4n + 1)$ ***A1***

$= 8n^2 + 2 (= \mathrm{RHS})$ ***AG***

[2 marks]

(b) ***METHOD 1***

recognition that $2n-1$ and $2n+1$ represent two consecutive odd integers (for all odd integers n) ***R1***

$8n^2 + 2 = 2(4n^2 + 1)$ ***A1***

valid reason *eg* divisible by 2 (2 is a factor) ***R1***

so the sum of the squares of any two consecutive odd integers is even ***AG***

[3 marks]

METHOD 2

recognition, *eg* that n and $n+2$ represent two consecutive odd integers (for $n \in \mathbb{Z}$) ***R1***

$n^2 + (n+2)^2 = 2(n^2 + 2n + 2)$ ***A1***

valid reason *eg* divisible by 2 (2 is a factor) ***R1***

so the sum of the squares of any two consecutive odd integers is even ***AG***

[3 marks]

Total [5 marks]

3. attempt to integrate ***(M1)***

$u = 2x^2 + 1 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 4x$

$\int \frac{8x}{\sqrt{2x^2+1}}\,\mathrm{d}x = \int \frac{2}{\sqrt{u}}\,\mathrm{d}u$ ***(A1)***

EITHER

$= 4\sqrt{u}\,(+C)$ ***A1***

OR

$= 4\sqrt{2x^2+1}\,(+C)$ ***A1***

THEN

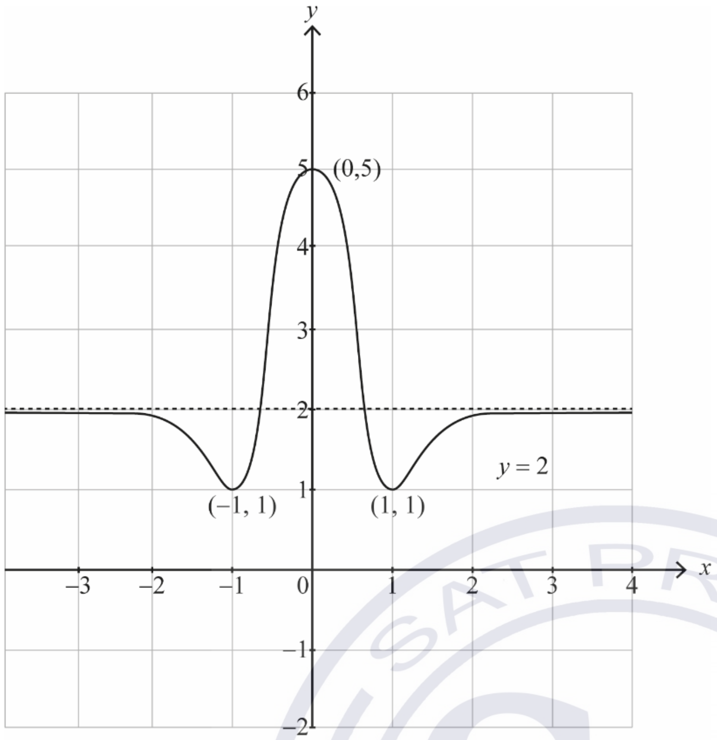
correct substitution into **their** integrated function (must have C) ***(M1)***

$5 = 4 + C \Rightarrow C = 1$

$f(x) = 4\sqrt{2x^2+1} + 1$ ***A1***

Total [5 marks]

4.



no y values below 1 ***A1***

horizontal asymptote at $y = 2$ with curve approaching from below as $x \to \pm\infty$ ***A1***

$(\pm 1, 1)$ local minima ***A1***

$(0, 5)$ local maximum ***A1***

smooth curve and smooth stationary points ***A1***

Total [5 marks]

5. (a) attempt to form composition ***M1***

correct substitution $g\left(\frac{x+3}{4}\right) = 8\left(\frac{x+3}{4}\right) + 5$ ***A1***

$(g \circ f)(x) = 2x + 11$ ***AG***

[2 marks]

(b) attempt to substitute 4 (seen anywhere) ***(M1)***

correct equation $a = 2 \times 4 + 11$ ***(A1)***

$a = 19$ ***A1***

[3 marks]

Total [5 marks]

6. (a) attempting to use the change of base rule ***M1***

$\log_9(\cos 2x+2)=\dfrac{\log_3(\cos 2x+2)}{\log_3 9}$ ***A1***

$=\dfrac{1}{2}\log_3(\cos 2x+2)$ ***A1***

$=\log_3\sqrt{\cos 2x+2}$ ***AG***

[3 marks]

(b) $\log_3(2\sin x)=\log_3\sqrt{\cos 2x+2}$

$2\sin x=\sqrt{\cos 2x+2}$ ***M1***

$4\sin^2 x=\cos 2x+2$ (or equivalent) ***A1***

use of $\cos 2x=1-2\sin^2 x$ ***(M1)***

$6\sin^2 x=3$

$\sin x=(\pm)\dfrac{1}{\sqrt{2}}$ ***A1***

$x=\dfrac{\pi}{4}$ ***A1***

Note: Award ***A0*** if solutions other than $x=\dfrac{\pi}{4}$ are included.

[5 marks]

Total [8 marks]

7. attempting integration by parts, *eg*

$u = \frac{\pi x}{36}, \mathrm{d}u = \frac{\pi}{36}\mathrm{d}x, \mathrm{d}v = \sin\left(\frac{\pi x}{6}\right)\mathrm{d}x, v = -\frac{6}{\pi}\cos\left(\frac{\pi x}{6}\right)$ ***(M1)***

$\mathrm{P}\left(0 \le X \le 3\right) = \frac{\pi}{36}\left(\left[-\frac{6x}{\pi}\cos\left(\frac{\pi x}{6}\right)\right]_0^3 + \frac{6}{\pi}\int_0^3 \cos\left(\frac{\pi x}{6}\right)\mathrm{d}x\right)$ (or equivalent) ***A1A1***

Note: Award ***A1*** for a correct uv and ***A1*** for a correct $\int v\,\mathrm{d}u$.

attempting to substitute limits ***M1***

$\frac{\pi}{36}\left[-\frac{6x}{\pi}\cos\left(\frac{\pi x}{6}\right)\right]_0^3 = 0$ ***(A1)***

so $\mathrm{P}\left(0 \le X \le 3\right) = \frac{1}{\pi}\left[\sin\left(\frac{\pi x}{6}\right)\right]_0^3$ (or equivalent) ***A1***

$= \frac{1}{\pi}$ ***A1***

Total [7 marks]

8. recognition that the angle between the normal and the line is $60°$ (seen anywhere) ***R1***

attempt to use the formula for the scalar product ***M1***

$\cos 60° = \frac{\left|\begin{pmatrix}2\\1\\2\end{pmatrix} \cdot \begin{pmatrix}1\\-2\\p\end{pmatrix}\right|}{\sqrt{9} \times \sqrt{1+4+p^2}}$ ***A1***

$\frac{1}{2} = \frac{|2p|}{3\sqrt{5+p^2}}$ ***A1***

$3\sqrt{5+p^2} = 4|p|$

attempt to square both sides ***M1***

$9\left(5+p^2\right) = 16p^2 \Rightarrow 7p^2 = 45$

$p = \pm 3\sqrt{\frac{5}{7}}$ (or equivalent) ***A1A1***

Total [7 marks]

9. (a) attempt to differentiate and set equal to zero ***M1***

$f'(x) = 2e^{2x} - 6e^{x} = 2e^{x}(e^{x} - 3) = 0$ ***A1***

minimum at $x = \ln 3$

$a = \ln 3$ ***A1***

[3 marks]

(b)

Note: Interchanging x and y can be done at any stage.

$y = \left(e^{x} - 3\right)^{2} - 4$ ***(M1)***

$e^{x} - 3 = \pm\sqrt{y+4}$ ***A1***

as $x \le \ln 3$, $x = \ln\left(3 - \sqrt{y+4}\right)$ ***R1***

so $f^{-1}(x) = \ln\left(3 - \sqrt{x+4}\right)$ ***A1***

domain of f^{-1} is $x \in \mathbb{R}, -4 \le x < 5$ ***A1***

[5 marks]

Total [8 marks]

Section B

10. (a) attempt to use quotient rule *(M1)*

correct substitution into quotient rule

$$f'(x) = \frac{5kx\left(\frac{1}{5x}\right) - k\ln 5x}{(kx)^2} \text{ (or equivalent)}$$ ***A1***

$$= \frac{k - k\ln 5x}{k^2x^2}, \left(k \in \mathbb{R}^+\right)$$ ***A1***

$$= \frac{1 - \ln 5x}{kx^2}$$ ***AG***

[3 marks]

(b) $f'(x) = 0$ ***M1***

$$\frac{1 - \ln 5x}{kx^2} = 0$$

$\ln 5x = 1$ ***(A1)***

$x = \frac{e}{5}$ ***A1***

[3 marks]

(c) $f''(x) = 0$ ***M1***

$$\frac{2\ln 5x - 3}{kx^3} = 0$$

$\ln 5x = \frac{3}{2}$ ***A1***

$5x = e^{\frac{3}{2}}$ ***A1***

so the point of inflexion occurs at $x = \frac{1}{5}e^{\frac{3}{2}}$ ***AG***

[3 marks]

continued…

Question 10 continued

(d) attempt to integrate ***(M1)***

$u = \ln 5x \Rightarrow \frac{du}{dx} = \frac{1}{x}$

$\int \frac{\ln 5x}{kx} dx = \frac{1}{k}\int u \, du$ ***(A1)***

EITHER

$= \frac{u^2}{2k}$ ***A1***

so $\frac{1}{k}\int_1^{\frac{3}{2}} u \, du = \left[\frac{u^2}{2k}\right]_1^{\frac{3}{2}}$ ***A1***

OR

$= \frac{(\ln 5x)^2}{2k}$ ***A1***

so $\int_{\frac{e}{5}}^{\frac{1}{5}e^{\frac{3}{2}}} \frac{\ln 5x}{kx} dx = \left[\frac{(\ln 5x)^2}{2k}\right]_{\frac{e}{5}}^{\frac{1}{5}e^{\frac{3}{2}}}$ ***A1***

THEN

$= \frac{1}{2k}\left(\frac{9}{4} - 1\right)$

$= \frac{5}{8k}$ ***A1***

setting **their** expression for area equal to 3 ***M1***

$\frac{5}{8k} = 3$

$k = \frac{5}{24}$ ***A1***

[7 marks]

Total [16 marks]

11. (a) attempt to find modulus *(M1)*

$r = 2\sqrt{3}\left(=\sqrt{12}\right)$ ***A1***

attempt to find argument in the correct quadrant *(M1)*

$\theta = \pi + \arctan\left(-\frac{\sqrt{3}}{3}\right)$ ***A1***

$= \frac{5\pi}{6}$ ***A1***

$-3+\sqrt{3}\mathrm{i} = \sqrt{12}\mathrm{e}^{\frac{5\pi\mathrm{i}}{6}}\left(=2\sqrt{3}\mathrm{e}^{\frac{5\pi\mathrm{i}}{6}}\right)$

[5 marks]

(b) attempt to find a root using de Moivre's theorem ***M1***

$12^{\frac{1}{6}}\mathrm{e}^{\frac{5\pi\mathrm{i}}{18}}$ ***A1***

attempt to find further two roots by adding and subtracting $\frac{2\pi}{3}$ to the argument ***M1***

$12^{\frac{1}{6}}\mathrm{e}^{-\frac{7\pi\mathrm{i}}{18}}$ ***A1***

$12^{\frac{1}{6}}\mathrm{e}^{\frac{17\pi\mathrm{i}}{18}}$ ***A1***

Note: Ignore labels for u, v and w at this stage.

[5 marks]

continued…

Question 11 continued

(c) ***METHOD 1***

attempting to find the total area of (congruent) triangles UOV, VOW and UOW ***M1***

$$\text{Area} = 3\left(\frac{1}{2}\right)\left(12^{\frac{1}{6}}\right)\left(12^{\frac{1}{6}}\right)\sin\frac{2\pi}{3}$$ ***A1A1***

Note: Award ***A1*** for $\left(12^{\frac{1}{6}}\right)\left(12^{\frac{1}{6}}\right)$ and ***A1*** for $\sin\frac{2\pi}{3}$.

$$= \frac{3\sqrt{3}}{4}\left(12^{\frac{1}{3}}\right)$$ (or equivalent) ***A1***

[4 marks]

METHOD 2

$$\text{UV}^2 = \left(12^{\frac{1}{6}}\right)^2 + \left(12^{\frac{1}{6}}\right)^2 - 2\left(12^{\frac{1}{6}}\right)\left(12^{\frac{1}{6}}\right)\cos\frac{2\pi}{3}$$ (or equivalent) ***A1***

$$\text{UV} = \sqrt{3}\left(12^{\frac{1}{6}}\right)$$ (or equivalent) ***A1***

attempting to find the area of UVW using $\text{Area} = \frac{1}{2} \times \text{UV} \times \text{VW} \times \sin\alpha$ for example ***M1***

$$\text{Area} = \frac{1}{2}\left(\sqrt{3} \times 12^{\frac{1}{6}}\right)\left(\sqrt{3} \times 12^{\frac{1}{6}}\right)\sin\frac{\pi}{3}$$

$$= \frac{3\sqrt{3}}{4}\left(12^{\frac{1}{3}}\right)$$ (or equivalent) ***A1***

[4 marks]

(d) $u + v + w = 0$ ***R1***

$$12^{\frac{1}{6}}\left(\cos\left(-\frac{7\pi}{18}\right) + \mathrm{i}\sin\left(-\frac{7\pi}{18}\right) + \cos\frac{5\pi}{18} + \mathrm{i}\sin\frac{5\pi}{18} + \cos\frac{17\pi}{18} + \mathrm{i}\sin\frac{17\pi}{18}\right) = 0$$ ***A1***

consideration of real parts ***M1***

$$12^{\frac{1}{6}}\left(\cos\left(-\frac{7\pi}{18}\right) + \cos\frac{5\pi}{18} + \cos\frac{17\pi}{18}\right) = 0$$

$$\cos\left(-\frac{7\pi}{18}\right) = \cos\frac{7\pi}{18}$$ explicitly stated ***A1***

$$\cos\frac{5\pi}{18} + \cos\frac{7\pi}{18} + \cos\frac{17\pi}{18} = 0$$ ***AG***

[4 marks]

Total [18 marks]

12. (a) attempting to use the chain rule to find the first derivative ***M1***

$f'(x)=(\cos x)\mathrm{e}^{\sin x}$ ***A1***

attempting to use the product rule to find the second derivative ***M1***

$f''(x)=\mathrm{e}^{\sin x}\left(\cos^2 x-\sin x\right)$ (or equivalent) ***A1***

attempting to find $f(0)$, $f'(0)$ and $f''(0)$ ***M1***

$f(0)=1;\ f'(0)=(\cos 0)\mathrm{e}^{\sin 0}=1;\ f''(0)=\mathrm{e}^{\sin 0}\left(\cos^2 0-\sin 0\right)=1$ ***A1***

substitution into the Maclaurin formula $f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\ldots$ ***M1***

so the Maclaurin series for $f(x)$ up to and including the x^2 term is $1+x+\frac{x^2}{2}$ ***A1***

[8 marks]

(b) ***METHOD 1***

attempting to differentiate $f''(x)$ ***M1***

$f'''(x)=(\cos x)\mathrm{e}^{\sin x}\left(\cos^2 x-\sin x\right)-(\cos x)\mathrm{e}^{\sin x}(2\sin x+1)$ (or equivalent) ***A2***

substituting $x=0$ into **their** $f'''(x)$ ***M1***

$f'''(0)=1(1-0)-1(0+1)=0$

so the coefficient of x^3 in the Maclaurin series for $f(x)$ is zero ***AG***

METHOD 2

substituting $\sin x$ into the Maclaurin series for e^x ***(M1)***

$\mathrm{e}^{\sin x}=1+\sin x+\frac{\sin^2 x}{2!}+\frac{\sin^3 x}{3!}+\ldots$

substituting Maclaurin series for $\sin x$ ***M1***

$$\mathrm{e}^{\sin x}=1+\left(x-\frac{x^3}{3!}+\ldots\right)+\frac{\left(x-\frac{x^3}{3!}+\ldots\right)^2}{2!}+\frac{\left(x-\frac{x^3}{3!}+\ldots\right)^3}{3!}+\ldots$$ ***A1***

coefficient of x^3 is $-\frac{1}{3!}+\frac{1}{3!}=0$ ***A1***

so the coefficient of x^3 in the Maclaurin series for $f(x)$ is zero ***AG***

[4 marks]

continued…

Question 12 continued

(c) substituting $3x$ into the Maclaurin series for e^x ***M1***

$$e^{3x} = 1 + 3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \ldots$$ ***A1***

substituting $\left(e^{3x} - 1\right)$ into the Maclaurin series for $\arctan x$ ***M1***

$$\arctan\left(e^{3x} - 1\right) = \left(e^{3x} - 1\right) - \frac{\left(e^{3x} - 1\right)^3}{3} + \frac{\left(e^{3x} - 1\right)^5}{5} - \ldots$$

$$= \left(3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \ldots\right) - \frac{\left(3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \ldots\right)^3}{3} + \ldots$$ ***A1***

selecting correct terms from above ***M1***

$$= \left(3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!}\right) - \frac{(3x)^3}{3}$$

$$= 3x + \frac{9x^2}{2} - \frac{9x^3}{2}$$ ***A1***

[6 marks]

(d) ***METHOD 1***

substitution of **their** series ***M1***

$$\lim_{x \to 0} \frac{x + \frac{x^2}{2} + \ldots}{3x + \frac{9x^2}{2} + \ldots}$$ ***A1***

$$= \lim_{x \to 0} \frac{1 + \frac{x}{2} + \ldots}{3 + \frac{9x}{2} + \ldots}$$

$$= \frac{1}{3}$$ ***A1***

METHOD 2

use of l'Hôpital's rule ***M1***

$$\lim_{x \to 0} \frac{(\cos x)e^{\sin x}}{\frac{3e^{3x}}{1 + \left(e^{3x} - 1\right)^2}}$$ (or equivalent) ***A1***

$$= \frac{1}{3}$$ ***A1***

[3 marks]

Total [21 marks]