4

Spectral Analysis of Linear Systems

In this chapter the central theme is the decomposition of the abstract linear equation $\mathbf{Tx} = \mathbf{y}$ into sets of simple linear equations which can be solved independently. Our initial purpose for exploring this decomposition is to obtain conceptual simplification of the system model. It is easier to think about the behavior of one scalar variable at a time than to think about the behavior of a vector variable. Furthermore, the solutions to the decomposed pieces of the original equation usually have physical meanings which provide insight into the behavior of the system. (See for example, P&C 4.7 or the discussion of the analysis of three-phase power systems by the method of symmetrical components.)

There are also computational reasons for examining the decomposition process. Generally speaking, decomposition provides an alternative to inversion as a technique for solving or analyzing the equations which describe a system. In particular, decomposition provides a practical technique for computing solutions to linear differential equations with arbitrary inputs (Section 5.5). In some instances decomposition provides both solutions and insight at no additional computational expense as compared to inversion. (Again, see the discussion of symmetrical components mentioned above.)

The ability to combine the solutions to small subproblems into a solution for the full system equation depends on the principle of linearity. Consequently, we restrict ourselves to linear models in this chapter in order to be able to fully develop the decomposition principle. We find that we can decompose most linear systems into sets of simple scalar multiplications. We refer to such "completely decomposable" systems as "diagonalizable" systems. A few systems are not diagonalizable or are so nearly nondiagonalizable that we cannot accurately compute fully decomposed solutions. We still split them into as small pieces as possible. Nondiagonalizable finite-dimensional systems are discussed in Sections 4.4 and 4.5. In Section 4.6 we explore the concept of functions of matrices for

both the diagonalizable and nondiagonalizable cases. We encountered several such matrix functions in Chapter 3; we find the need for others in later chapters. The discussion of diagonalization of infinite-dimensional systems and of functions of linear operators on infinite dimensional spaces is begun in Section 4.6, but is not completed until Section 5.5.

4.1 System Decomposition

In this section we explore the subdivision of the system equation $\mathbf{Tx} = \mathbf{y}$ into a set of "smaller" equations which can be solved independently. Our ability to subdivide a linear equation in this manner is based partly on the fact that the effect of a linear transformation $\mathbf{T}$ on a basis determines the effect of $\mathbf{T}$ on all vectors in the space. In finding the matrix of a transformation, for instance, we simplified the process of determining the matrix elements by examining the effect of the transformation on the basis vectors. Consequently, we begin our investigation of decomposition by subdividing the vector space on which the transformation $\mathbf{T}$ acts. We can think of the space as a sum of smaller subspaces.

Definition. Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be subspaces of the vector space $\mathcal{V}$. We call $\mathcal{V}$ the **direct sum** $\mathcal{W}_1 \oplus \mathcal{W}_2$ of $\mathcal{W}_1$ and $\mathcal{W}_2$ if*

(*a*) $\mathcal{V} = \mathcal{W}_1 + \mathcal{W}_2$ ($\mathcal{W}_1$ and $\mathcal{W}_2$ **span** $\mathcal{V}$) and
(*b*) $\mathcal{W}_1 \cap \mathcal{W}_2 = \boldsymbol{\theta}$ ($\mathcal{W}_1$ and $\mathcal{W}_2$ are **linearly independent**)

Example 1. Direct Sum in Arrow Space. The two-dimensional arrow space is the direct sum of two different lines which intersect at the origin (Figure 4.1). If the two lines are identical, they are not independent and do not span the arrow space.

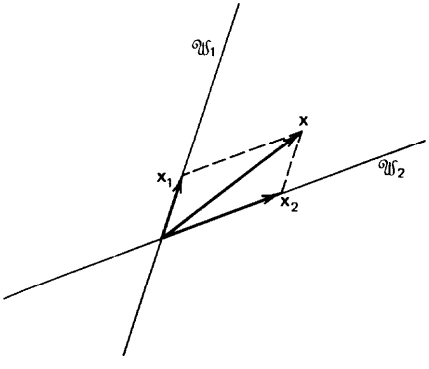

Figure 4.1. Direct sum in arrow space.

*See P&C 2.1 for definitions of the sum and intersection of subspaces.

This arrow space is also the sum of *three* lines which intersect at the origin. However, that sum is not direct; only two of the lines can be independent.

It is apparent from Figure 4.1 that for any finite-dimensional space every splitting of a basis into two parts determines a direct sum; that is, if $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is a basis for $\mathcal{V}$, $\text{span}\{\mathbf{x}_1, \ldots, \mathbf{x}_n\} = \text{span}\{\mathbf{x}_1, \ldots, \mathbf{x}_k\} \oplus \text{span}\{\mathbf{x}_{k+1}, \ldots, \mathbf{x}_n\}$. It is apparent that the two subspaces can also be subdivided. Although we have not yet defined a basis for an infinite-dimensional space, the concept of splitting a basis applies as well to direct sums in infinite-dimensional spaces (Sections 5.3-5.5).

Example 2. Direct Sum in a Function Space. Let $\mathcal{C}(-1,1)$ be the space of continuous functions defined on [-1,1]. Let $\mathcal{W}_1$ be the even functions in $\mathcal{C}(-1,1)$; $\mathbf{f}_e(-t) = \mathbf{f}_e(t)$. Let $\mathcal{W}_2$ be the odd functions in $\mathcal{C}(-1,1)$; $\mathbf{f}_o(-t) = -\mathbf{f}_o(t)$. Any function $\mathbf{f}$ in $\mathcal{C}(-1,1)$ decomposes into even and odd components:

$$\mathbf{f}(t) = \frac{\mathbf{f}(t) + \mathbf{f}(-t)}{2} + \frac{\mathbf{f}(t) - \mathbf{f}(-t)}{2}$$

Thus $\mathcal{W}_1$ and $\mathcal{W}_2$ span $\mathcal{C}(-1,1)$. The even and odd components of $\mathbf{f}$ are unique; for if $\mathbf{f}_e$ and $\mathbf{f}_o$ are even and odd functions, respectively, such that $\mathbf{f} = \mathbf{f}_e + \mathbf{f}_o$, then

$$\frac{\mathbf{f}(t) + \mathbf{f}(-t)}{2} = \frac{[\mathbf{f}_e(t) + \mathbf{f}_o(t)] + [\mathbf{f}_e(-t) + \mathbf{f}_o(-t)]}{2} = \mathbf{f}_e(t)$$

$$\frac{\mathbf{f}(t) - \mathbf{f}(-t)}{2} = \frac{[\mathbf{f}_e(t) + \mathbf{f}_o(t)] - [\mathbf{f}_e(-t) + \mathbf{f}_o(-t)]}{2} = \mathbf{f}_o(t)$$

Only the zero function is both even and odd; therefore, $\mathcal{W}_1 \cap \mathcal{W}_2 = \boldsymbol{\theta}$, and $\mathcal{C}(-1,1) = \mathcal{W}_1 \oplus \mathcal{W}_2$.

Example 2 demonstrates an important property of the direct sum. Using bases for $\mathcal{W}_1$ and $\mathcal{W}_2$, it is easily shown that $\mathcal{V} = \mathcal{W}_1 \oplus \mathcal{W}_2$ if and only if each $\mathbf{x}$ in $\mathcal{V}$ decomposes *uniquely* into a sum, $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$, with $\mathbf{x}_1$ in $\mathcal{W}_1$ and $\mathbf{x}_2$ in $\mathcal{W}_2$.

It is a small step to extend the direct sum concept to several subspaces. We merely redefine independence of subspaces: $\mathcal{W}_1, \ldots, \mathcal{W}_p$ are linearly independent if each subspace is disjoint from the sum of the rest,

$$\mathcal{W}_i \cap \left(\sum_{j \neq i} \mathcal{W}_j \right) = \boldsymbol{\theta} \tag{4.1}$$

With the modification (4.1) we say $\mathcal{V}$ is the direct sum of $\{\mathcal{W}_i\}$ if the subspaces $\{\mathcal{W}_i\}$ are linearly independent and span $\mathcal{V}$. We denote the direct sum by

$$\mathcal{V} = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \cdots \oplus \mathcal{W}_p \tag{4.2}$$

The previous comments concerning splitting of bases and unique decomposition of vectors also extend to the direct sum of several subspaces.

Exercise 1. Demonstrate in the two-dimensional arrow space that pairwise disjointness is not sufficient to guarantee independence of $\mathcal{W}_1, \ldots, \mathcal{W}_p$.

Example 3. Direct Sum of Three Subspaces. Let $f_1(t) = 1 + t$, $\mathbf{f}_2(t) = t + t^2$, and $\mathbf{f}_3(t) = 1 + t^2$ be a basis for $\mathcal{P}^3$. Define $\mathcal{W}_i = \text{span}\{\mathbf{f}_i\}$, $i = 1,2,3$. Then $\mathcal{P}^3 = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$. Let $\mathbf{f}(t) \triangleq \eta_1 + \eta_2 t + \eta_3 t^2$ be a specific vector in $\mathcal{P}^3$. By the process of determining coordinates of $\mathbf{f}$ relative to the basis $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ for $\mathcal{P}^3$, we decompose $\mathbf{f}$ uniquely into

$$\mathbf{f} = \left(\frac{\eta_1 + \eta_2 - \eta_3}{2}\right)\mathbf{f}_1 + \left(\frac{-\eta_1 + \eta_2 + \eta_3}{2}\right)\mathbf{f}_2 + \left(\frac{\eta_1 - \eta_2 + \eta_3}{2}\right)\mathbf{f}_3,$$

a sum of vectors from $\mathcal{W}_1$, $\mathcal{W}_2$, and $\mathcal{W}_3$, respectively.

Projection Operators

We can express the direct-sum decomposition of a space in terms of linear operators on the space. Suppose $\mathcal{V} = \mathcal{W}_1 \oplus \mathcal{W}_2$; any vector $\mathbf{x}$ in $\mathcal{V}$ can be written uniquely as $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$ with $\mathbf{x}_i$ in $\mathcal{W}_i$. We define the **projector** (or **projection operator) $\mathbf{P}_1$ on $\mathcal{W}_1$ along $\mathcal{W}_2$** by $\mathbf{P}_1\mathbf{x} \triangleq \mathbf{x}_1$ (see Figure 4.1). We call the vector $\mathbf{x}_1$ the **projection of $\mathbf{x}$ on $\mathcal{W}_1$ along $\mathcal{W}_2$**. Similarly $\mathbf{P}_2\mathbf{x} \triangleq \mathbf{x}_2$ defines the projector on $\mathcal{W}_2$ along $\mathcal{W}_1$.

Example 4. Projector on $\mathcal{P}^3$. Let $\mathbf{f}_1$, $\mathbf{f}_2$, and $\mathbf{f}_3$ be the functions defined in Example 3. Redefine $\mathcal{W}_1 \triangleq \text{span}\{\mathbf{f}_1\}$ and $\mathcal{W}_2 \triangleq \text{span}\{\mathbf{f}_2, \mathbf{f}_3\}$. Then $\mathcal{P}^3 = \mathcal{W}_1 \oplus \mathcal{W}_2$. In Example 3, the general vector $\mathbf{f}(t) = \eta_1 + \eta_2 t + \eta_3 t^2$ in $\mathcal{P}^3$ is decomposed into a linear combination of $\mathbf{f}_1$, $\mathbf{f}_2$, and $\mathbf{f}_3$. From that decomposition we see that the projections of $\mathbf{f}$ on $\mathcal{W}_1$ and $\mathcal{W}_2$, respectively, are

$$\mathbf{P}_1\mathbf{f} = \left(\frac{\eta_1 + \eta_2 - \eta_3}{2}\right)\mathbf{f}_1$$

$$\mathbf{P}_2\mathbf{f} = \left(\frac{-\eta_1 + \eta_2 + \eta_3}{2}\right)\mathbf{f}_2 + \left(\frac{\eta_1 - \eta_2 + \eta_3}{2}\right)\mathbf{f}_3$$

The bases for $\mathcal{W}_1$ and $\mathcal{W}_2$ combine to provide a basis which is particularly appropriate for matrix representation of the projectors. Using (2.48), the matrix of the projector $\mathbf{P}_1$ relative to the basis $\mathcal{F} \triangleq \{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ is

$$[\mathbf{P}_1]_{\mathcal{F}\mathcal{F}} = ([\mathbf{P}_1\mathbf{f}_1]_{\mathcal{F}} \vdots [\mathbf{P}_1\mathbf{f}_2]_{\mathcal{F}} \vdots [\mathbf{P}_1\mathbf{f}_3]_{\mathcal{F}})$$

$$= ([\mathbf{f}_1]_{\mathcal{F}} \vdots [\boldsymbol{\theta}]_{\mathcal{F}} \vdots [\boldsymbol{\theta}]_{\mathcal{F}})$$

$$= \left(\begin{array}{c:c:c} 1 & 0 & 0 \\ \hdashline 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 \end{array}\right)$$

Similarly, the matrix of $\mathbf{P}_2$ with respect to $\mathcal{F}$ is

$$[\mathbf{P}_2]_{\mathcal{F}\mathcal{F}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example 4 emphasizes the fact that a projector acts like the identity operator on its "own" subspace, the one *onto* which it projects, but like the zero operator on the subspace *along* which it projects. The following properties of projectors can be derived from the definition and verified by the matrices of Example 4. Assume $\mathcal{V} = \mathcal{W}_1 \oplus \mathcal{W}_2$. Let $\mathbf{P}_i$ be the projector on $\mathcal{W}_i$ along $\mathcal{W}_j$ $(j \neq i)$, and $\mathbf{x}_i = \mathbf{P}_i\mathbf{x}$. Then

$$\begin{aligned} &(a)\ \mathbf{P}_i \text{ is linear} \\ &(b)\ \ \mathbf{P}_i^2 = \mathbf{P}_i \text{ (i.e., } \mathbf{P}_i\mathbf{x}_i = \mathbf{x}_i) \\ &(c)\ \ \mathbf{P}_i\mathbf{P}_j = \mathbf{\Theta} \text{ (i.e., } \mathbf{P}_i\mathbf{x}_j = \boldsymbol{\theta} \quad \text{for } j \neq i) \\ &(d)\ \ \text{range } (\mathbf{P}_i) = \mathcal{W}_i \\ &(e)\ \ \sum_i \mathbf{P}_i = \mathbf{I} \left(\text{i.e., } \sum_i \mathbf{P}_i\mathbf{x} = \mathbf{x} \right) \end{aligned} \tag{4.3}$$

If $\mathcal{V} = \mathcal{W}_1 \oplus \cdots \oplus \mathcal{W}_k$, we can define the projector $\mathbf{P}_i$ on $\mathcal{W}_i$ along $\Sigma_{j \neq i} \mathcal{W}_j$, for $i = 1, \ldots, k$. The properties (4.3) apply to this set of projectors as well.

Reduced Operators

The projectors in Example 4 act like scalar multiplication on certain vectors in $\mathcal{V}$; $\mathbf{P}_i$ acts like multiplication by 1 on all vectors in the subspace $\mathcal{W}_i$, and like multiplication by zero on $\mathcal{W}_j$, $j \neq i$. Other operators also act in a simple manner on certain subspaces. Define the nonlinear operator $\mathbf{G}$: $\mathcal{R}^2 \to \mathcal{R}^2$ by

$$\mathbf{G}(\xi_1, \xi_2) \triangleq \left((\xi_1 - \xi_2)^2 + 2\xi_2, 2\xi_2 \right)$$

On the subspace $\mathcal{W}_1 \triangleq \text{span}\{(1,0)\}$, $\mathbf{G}$ acts like the simple "squaring" operation, $\mathbf{G}(a,0) = (a^2, 0)$. On the subspace $\mathcal{W}_2 \triangleq \text{span}\{(1,1)\}$, $\mathbf{G}$ acts like the "doubling" operation $\mathbf{G}(b,b) = (2b, 2b)$. In point of fact, as far as vectors in $\mathcal{W}_1$ and $\mathcal{W}_2$ are concerned we can replace $\mathbf{G}$ by the "simpler" operators $\mathbf{G}_1$: $\mathcal{W}_1 \to \mathcal{W}_1$ defined by $\mathbf{G}_1(\xi, 0) \triangleq (\xi^2, 0)$ and $\mathbf{G}_2$: $\mathcal{W}_2 \to \mathcal{W}_2$ defined by $\mathbf{G}_2(\xi, \xi) \triangleq 2(\xi, \xi)$. We are able to reduce $\mathbf{G}$ to these simpler operators because the action of $\mathbf{G}$ on $\mathcal{W}_1$ produces only vectors in $\mathcal{W}_1$ and the action of $\mathbf{G}$ on $\mathcal{W}_2$ produces only vectors in $\mathcal{W}_2$.

Definition. Let $\mathbf{G}$ be an operator (perhaps nonlinear) on $\mathcal{V}$. The subspace $\mathcal{W}$ (of $\mathcal{V}$) is **invariant under G** if for each $\mathbf{x}$ in $\mathcal{W}$, $\mathbf{Gx}$ is also in $\mathcal{W}$; that is, if $\mathbf{G}(\mathcal{W})$ is contained in $\mathcal{W}$.

Example 5. Invariance of the Nullpace and Range. Let $\mathbf{G}: \mathcal{V} \to \mathcal{V}$. Then range $(\mathbf{G})$ is invariant under $\mathbf{G}$, for $\mathbf{G}$ takes all vectors in $\mathcal{V}$, including those in **range(G)**, into **range(G)**. By definition, $\mathbf{G}$ takes **nullspace(G)** into $\boldsymbol{\theta}$. If $\mathbf{G}(\boldsymbol{\theta}) = \boldsymbol{\theta}$, then $\boldsymbol{\theta}$ is in **nullspace(G)**. In this case, **nullspace(G)** is also invariant under $\mathbf{G}$. These subspaces are pictured abstractly in Figure 2.6.

If $\mathbf{G}: \mathcal{V} \to \mathcal{V}$, and $\mathcal{W}$ is a subspace of $\mathcal{V}$ which is invariant under $\mathbf{G}$, then we can define a **reduced operator** $\mathbf{G}_{\mathcal{W}}: \mathcal{W} \to \mathcal{W}$ by $\mathbf{G}_{\mathcal{W}}\,\mathbf{x} \triangleq \mathbf{Gx}$ for all $\mathbf{x}$ in $\mathcal{W}$. The operators $\mathbf{G}_1$ and $\mathbf{G}_2$ discussed earlier are examples of reduced operators. The following illustration shows that the reduced operator $\mathbf{G}_{\mathcal{W}}$ is truly different from $\mathbf{G}$.

Example 6. Reduced Linear Operators. We define $\mathbf{T}: \mathcal{R}^2 \to \mathcal{R}^2$ by

$$\mathbf{T}(\xi_1, \xi_2) \triangleq (2\xi_1 + 3\xi_2, 4\xi_2) \tag{4.4}$$

The matrix of $\mathbf{T}$ relative to the standard basis $\mathcal{E}$ is

$$[\mathbf{T}]_{\mathcal{E}\,\mathcal{E}} = ([\mathbf{T}\varepsilon_1]_{\mathcal{E}} \vdots [\mathbf{T}\varepsilon_2]_{\mathcal{E}}) = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}$$

The subspaces $\mathcal{W}_1 \triangleq \text{span}\{(1,0)\}$ and $\mathcal{W}_2 \triangleq \text{span}\{(3,2)\}$ are invariant under $\mathbf{T}$. Therefore, we can define the reduced operators $\mathbf{T}_1: \mathcal{W}_1 \to \mathcal{W}_1$ by $\mathbf{T}_1(\xi, 0) \triangleq \mathbf{T}(\xi, 0) = 2(\xi, 0)$ and $\mathbf{T}_2: \mathcal{W}_2 \to \mathcal{W}_2$ by $\mathbf{T}_2(3\xi, 2\xi) \triangleq \mathbf{T}(3\xi, 2\xi) = 4(3\xi, 2\xi)$. Using $\mathcal{X} \triangleq \{(1,0)\}$ as a basis for $\mathcal{W}_1$ and $\mathcal{Y} \triangleq \{(3,2)\}$ as a basis for $\mathcal{W}_2$ we find

$$[\mathbf{T}_1]_{\mathcal{X}\,\mathcal{X}} = ([\mathbf{T}_1(1,0)]_{\mathcal{X}}) = (2)$$

$$[\mathbf{T}_2]_{\mathcal{Y}\,\mathcal{Y}} = ([\mathbf{T}_2(3,2)]_{\mathcal{Y}}) = (4)$$

The reduced operators $\mathbf{T}_1$ and $\mathbf{T}_2$ are scalar operators, represented by 1×1 matrices. They are very different from $\mathbf{T}$, which is represented by a 2×2 matrix. Clearly the domain and range of definition of a transformation are necessary parts of its definition.

Solution of Equations by Decomposition

The combination of three basic concepts—direct sum, invariance, and linearity—leads to the spectral decomposition, a decomposition of an

operator or an equation into a set of scalar multipliers or scalar single-variable equations. The decomposition provides considerable insight into the nature of linear models. It also provides a technique for solving equations which is an alternative to inverting the equations.

If $\mathbf{T}$ is a linear operator on $\mathcal{V}$, if $\mathcal{V} = \mathcal{W}_1 \oplus \cdots \oplus \mathcal{W}_p$, and if each $\mathcal{W}_i$ is invariant under $\mathbf{T}$, then the set $\{\mathcal{W}_i\}$ of subspaces **decomposes** $\mathbf{T}$ into a set of reduced linear operators $\mathbf{T}_i: \mathcal{W}_i \to \mathcal{W}_i$ defined by $\mathbf{T}_i\mathbf{x} \triangleq \mathbf{T}\mathbf{x}$ for all $\mathbf{x}$ in $\mathcal{W}_i$. Analysis of a system represented by $\mathbf{T}$ reduces to analysis of a set of *independent subsystems* represented by $\{\mathbf{T}_i\}$; that is, we can solve the equation $\mathbf{T}\mathbf{x} = \mathbf{y}$ by the following process.

The Spectral Decomposition Process (4.5)

1. Using the direct sum, decompose $\mathbf{y}$ into the unique combination

$$\mathbf{y} = \mathbf{y}_1 + \cdots + \mathbf{y}_p \quad \text{with } \mathbf{y}_i \text{ in } \mathcal{W}_i$$

2. Using the invariance of $\mathcal{W}_i$ under $\mathbf{T}$, solve the subsystems

$$\mathbf{T}\mathbf{x}_i = \mathbf{y}_i \qquad i = 1, 2, \ldots, p$$

(in effect solving the reduced equations $\mathbf{T}_i\mathbf{x}_i = \mathbf{y}_i$).

3. Using the linearity of $\mathbf{T}$, get the solution $\mathbf{x}$ by adding

$$\mathbf{x} = \mathbf{x}_1 + \cdots + \mathbf{x}_p$$

If the reduced operators $\mathbf{T}_i$ are simple scalar multipliers like those of Example 6, then solution of the subsystem equations is trivial; that is, if $\mathbf{T}\mathbf{x}_i = \lambda_i\mathbf{x}_i$ for each $\mathbf{x}_i$ in $\mathcal{W}_i$, then $\lambda_i\mathbf{x}_i = \mathbf{y}_i$ and parts (2) and (3) of (4.5) can be expressed as

$$\mathbf{x} = \left(\frac{1}{\lambda_1}\right)\mathbf{y}_1 + \cdots + \left(\frac{1}{\lambda_p}\right)\mathbf{y}_p \tag{4.6}$$

If we know the invariant subspaces $\mathcal{W}_i$ and the scalars λ_i, the primary effort required to carry out this procedure is that in decomposing $\mathbf{y}$.

Example 7. Solution of an Equation by Decomposition. Let $\mathbf{T}: \mathcal{R}^2 \to \mathcal{R}^2$ be as in (4.4):

$$\mathbf{T}(\xi_1, \xi_2) \triangleq (2\xi_1 + 3\xi_2, 4\xi_2)$$

From Example 6, we know the subspaces $\mathcal{W}_1 \triangleq \text{span}\{(1,0)\}$ and $\mathcal{W}_2 \triangleq \text{span}\{(3,2)\}$ are invariant under $\mathbf{T}$; furthermore, $\mathbf{T}$ acts like $\mathbf{T}_1\mathbf{x} \triangleq 2\mathbf{x}$ for $\mathbf{x}$ in $\mathcal{W}_1$, and like $\mathbf{T}_2\mathbf{x} \triangleq 4\mathbf{x}$ for $\mathbf{x}$ in $\mathcal{W}_2$. Also $\mathcal{V} = \mathcal{W}_1 \oplus \mathcal{W}_2$. Therefore, we can solve the

equation

$$\mathbf{Tx} = \mathbf{y} \triangleq (\eta_1, \eta_2)$$

by the process (4.5). We decompose $\mathbf{y}$ by solving $(\eta_1, \eta_2) = c_1(1,0) + c_2(3,2)$ to find

$$(\eta_1, \eta_2) = \left(\eta_1 - \frac{3\eta_2}{2}\right)(1,0) + \left(\frac{\eta_2}{2}\right)(3,2)$$

$$\triangleq \mathbf{y}_1 + \mathbf{y}_2$$

By (4.6)

$$(\xi_1, \xi_2) = \left(\frac{1}{2}\right)\left(\eta_1 - \frac{3\eta_2}{2}\right)(1,0) + \left(\frac{1}{4}\right)\left(\frac{\eta_2}{2}\right)(3,2)$$

$$= \left(\frac{\eta_1}{2} - \frac{3\eta_2}{8}, \frac{\eta_2}{4}\right)$$

The procedure (4.5) is essentially the one we use to determine the steady-state solution of a constant-coefficient differential equation by Fourier series. It is well known that a continuous function $\mathbf{f}$ can be expanded uniquely as a Fourier series of complex exponentials of the form $e^{i2\pi kt/b}$, where $i = \sqrt{-1}$ and b is the length of the interval over which $\mathbf{f}$ is defined. Each such exponential spans a subspace $\mathcal{W}_k$. The Fourier series expansion is possible because the space of continuous functions is in some sense the direct sum of $\{\mathcal{W}_k\}$. But each subspace $\mathcal{W}_k$ is invariant under any linear constant-coefficient differential operator; for instance, $(\mathbf{D}^2 + \mathbf{D})e^{\mu t} = (\mu^2 + \mu)e^{\mu t}$, a scalar multiple of $e^{\mu t}$. Thus the solution to certain differential equations can be found by an extension of (4.6). See P&C 5.35.

The Spectrum

The real goal of most systems analyses is insight into the system structure. Most linear models have a structure which permits decomposition into a set of scalar operations. It is not yet clear what effect the subdivision of a linear operator $\mathbf{T}$ has on the overall computation. In fact, since one result of the decomposition is valuable insight into the structure of the system represented by $\mathbf{T}$, perhaps we should expect an increase in total computation. Although this expectation is justified, we find that under certain circumstances the decomposition information is known a priori. Then decomposition can also lead to reduced computation (Section 5.2).

Definition. An **eigenvalue** (or **characteristic value)** of a linear operator $\mathbf{T}$ on a vector space $\mathcal{V}$ is a scalar λ such that $\mathbf{Tx} = \lambda\mathbf{x}$ for some nonzero

vector $\mathbf{x}$ in $\mathcal{V}$. Any nonzero $\mathbf{x}$ for which $\mathbf{T}\mathbf{x} = \lambda\mathbf{x}$ is called an **eigenvector** of $\mathbf{T}$ associated with the eigenvalue λ.

The eigenvector $\mathbf{x}$ spans a subspace of $\mathcal{V}$. Each member of this subspace (or **eigenspace)** is also an eigenvector for the same eigenvalue. In fact, because $\mathbf{T}$ is linear, any one-dimensional subspace which is invariant under $\mathbf{T}$ must be an eigenspace of $\mathbf{T}$. The identity operator $\mathbf{I}$ clearly has only one eigenvalue; the whole space $\mathcal{V}$ is the eigenspace for $\lambda = 1$. Similarly, for the zero operator $\mathbf{\Theta}$, $\mathcal{V}$ is the eigenspace for $\lambda = 0$. If $\mathcal{V} = \mathcal{W}_1 \oplus \mathcal{W}_2$, then for the projector $\mathbf{P}_i$ of (4.3), $\mathcal{W}_i$ is the eigenspace for $\lambda = 1$ and $\mathcal{W}_j$ is the eigenspace for $\lambda = 0$.

The eigenvectors of an operator which acts on a function space are often called **eigenfunctions.** We will refer to the eigenvalues and eigenvectors (or eigenfunctions) of $\mathbf{T}$ as the **eigendata for T**. The eigendata usually have some significant physical interpretation in terms of the system represented by $\mathbf{T}$.

Example 8. Eigendata for a Transformation in $\mathcal{R}^2$. The operator $\mathbf{T}$: $\mathcal{R}^2 \to \mathcal{R}^2$ of (4.4) is

$$\mathbf{T}(\xi_1, \xi_2) \triangleq (2\xi_1 + 3\xi_2, 4\xi_2)$$

It has two eigenvalues: $\lambda_1 = 2$ and $\lambda_2 = 4$. The corresponding eigenspaces are span{(1,0)} for λ_1 and span{(3,2)} for λ_2.

Example 9. Eigendata for Differential Operators. The exponential function $e^{\mu t}$ and its multiples form an eigenspace for any linear constant-coefficient differential operator *without boundary conditions.* For instance, since

$$\frac{d^n}{dt^n}e^{\mu t} + a_1\frac{d^{n-1}}{dt^{n-1}}e^{\mu t} + \cdots + a_n e^{\mu t} = (\mu^n + a_1\mu^{n-1} + \cdots + a_n)e^{\mu t}$$

for any complex scalar μ, the differential operator $\mathbf{D}^n + a_1\mathbf{D}^{n-1} + \cdots + a_n\mathbf{I}$ has the eigenfunction $e^{\mu t}$ corresponding to the eigenvalue $\lambda = \mu^n + a_1\mu^{n-1} + \cdots + a_n$. A differential operator without boundary conditions possesses a continuum of eigenvalues.

Example 10. An Operator Without Eigenvalues. A linear differential operator with homogeneous boundary conditions need not have any eigenvalues. For example, the only vector that satisfies

$$\frac{d\mathbf{f}(t)}{dt} = \lambda\mathbf{f}(t), \qquad \mathbf{f}(0) = 0$$

is the zero function, regardless of the value we try for the eigenvalue λ. Thus the operator $\mathbf{D}$ acting on the space of differentiable functions $\mathbf{f}$ which satisfy $\mathbf{f}(0) = 0$ has no eigenvalues. Furthermore, any nth order linear differential operator with n independent one-point homogeneous boundary conditions is without eigenvalues. [See the discussion following (3.28).]

The problem of finding eigenvalues for a linear operator $\mathbf{T}: \mathcal{V} \to \mathcal{V}$ is basically the problem of determining values of λ for which the equation

$$(\mathbf{T} - \lambda\mathbf{I})\mathbf{x} = \boldsymbol{\theta} \tag{4.7}$$

has nonzero solutions $\mathbf{x}$; that is, we seek the values of λ for which the operator $\mathbf{T} - \lambda\mathbf{I}$ is singular. Once we have a specific eigenvalue, say, λ_1, obtaining the corresponding eigenvectors involves the determination of $\text{nullspace}(\mathbf{T} - \lambda_1\mathbf{I})$ —the solution of (4.7) with $\lambda = \lambda_1$. The determination of eigendata and the use of eigendata in practical analysis are explored for finite-dimensional systems in Section 4.2 and for infinite-dimensional systems in Section 4.3.

4.2 Spectral Analysis in Finite-Dimensional Spaces

In this section we convert (4.7) to a matrix equation for the case where $\mathcal{V}$ is finite-dimensional. We also examine the spectral (eigendata) properties of matrix equations. Practical computation of eigendata for finite-dimensional problems, a more difficult task than appears on the surface, is discussed at the end of the section.

In Section 2.5 we found we could convert any equation involving a linear operator on a finite-dimensional space into an equivalent matrix equation. If $\mathbf{T}: \mathcal{V} \to \mathcal{V}$, we simply pick a basis $\mathcal{X}$ for $\mathcal{V}$. The basis converts the equation $\mathbf{Tx} = \mathbf{y}$ into the equation $[\mathbf{T}]_{\mathcal{X}\mathcal{X}}[\mathbf{x}]_{\mathcal{X}} = [\mathbf{y}]_{\mathcal{X}}$. We generally define $\mathbf{A} \triangleq [\mathbf{T}]_{\mathcal{X}\mathcal{X}}$, and use the simpler matrix notation $\mathbf{A}[\mathbf{x}]_{\mathcal{X}} = [\mathbf{y}]_{\mathcal{X}}$. The eigenvalues and eigenvectors for $\mathbf{T}$ are then specified by the matrix equivalent of (4.7):

$$(\mathbf{A} - \lambda\mathbf{I})[\mathbf{x}]_{\mathcal{X}} = [\boldsymbol{\theta}]_{\mathcal{X}} \tag{4.8}$$

The values of λ for which (4.8) has nonzero solutions constitute the eigenvalues of $\mathbf{T}$. We also refer to them as the **eigenvalues of the matrix A.**

From Section 1.5 we know that the square-matrix equation (4.8) has nonzero solutions if and only if

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \tag{4.9}$$

Equation (4.9) is known as the **characteristic equation of the matrix A** (or of the operator $\mathbf{T}$ which $\mathbf{A}$ represents). If $\mathbf{A}$ is an $n \times n$ matrix, then

$$c(\lambda) \triangleq \det(\lambda\mathbf{I} - \mathbf{A}) = (-1)^n \det(\mathbf{A} - \lambda\mathbf{I}) \tag{4.10}$$

is an nth order polynomial in A called the **characteristic polynomial of A** (or of **T**). An nth order polynomial has precisely n (possibly complex) roots. (This fact follows from the fundamental theorem of algebra.) The set $\{\lambda_1, \ldots, \lambda_n\}$ of roots of $c(\lambda)$ constitutes the complete set of eigenvalues of **A** (or **T**); the set is called the **spectrum of A** (or **T**). We often refer to an analysis which involves eigenvalues as a *spectral analysis.* Since $\lambda = \lambda_i$ makes $\mathbf{A} - \lambda\mathbf{I}$ singular, there must be at least one nonzero eigenvector for each different eigenvalue. A solution $[\mathbf{x}]_{\mathcal{X}}$ of (4.8) for $\lambda = \lambda_i$ is an eigenvector of **A** for λ_i. The corresponding vector $\mathbf{x}$ is an eigenvector of **T** for λ_i.

***Example 1. Finding Eigendata from* [T].** Let $\mathbf{T}: \mathcal{R}^2 \to \mathcal{R}^2$ be defined as in (4.4) by

$$\mathbf{T}(\xi_1, \xi_2) \triangleq (2\xi_1 + 3\xi_2, 4\xi_2)$$

Using the standard basis $\mathcal{E}$ for $\mathcal{R}^2$ as in Example 6, (4.8) becomes

$$\left(\begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)[\mathbf{x}]_{\mathcal{E}} = [\boldsymbol{\theta}]_{\mathcal{E}}$$

or

$$\begin{pmatrix} 2-\lambda & 3 \\ 0 & 4-\lambda \end{pmatrix}[\mathbf{x}]_{\mathcal{E}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The characteristic equation is

$$\begin{vmatrix} 2-\lambda & 3 \\ 0 & 4-\lambda \end{vmatrix} = (2-\lambda)(4-\lambda) = 0$$

The eigenvalues of **A** (and **T**) are $\lambda_1 = 2$ and $\lambda_2 = 4$. We find the eigenvectors of **A** for λ_i by solving (4.8) with $\lambda = \lambda_i$:

$$(\mathbf{A} - 2\mathbf{I})\,[\mathbf{x}_1]_{\mathcal{E}} = \begin{pmatrix} 0 & 3 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad [\mathbf{x}_1]_{\mathcal{E}} = \begin{pmatrix} c_1 \\ 0 \end{pmatrix}$$

$$(\mathbf{A} - 4\mathbf{I})\,[\mathbf{x}_2]_{\mathcal{E}} = \begin{pmatrix} -2 & 3 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad [\mathbf{x}_2]_{\mathcal{E}} = \begin{pmatrix} 3d_1 \\ 2d_1 \end{pmatrix}$$

The scalars c_1 and d_1 are arbitrary; there is a one-dimensional eigenspace for each eigenvalue. The eigenvectors of **T** for λ_i are found from the relationship between a vector and its coordinates relative to the basis $\mathcal{E}$:

$$[\mathbf{x}]_{\mathcal{E}} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \Leftrightarrow \mathbf{x} = c_1(1,0) + c_2(0,\ 1)$$

Therefore, the eigenvectors of **T** corresponding to λ_1 and λ_2 are

$$\mathbf{x}_1 = c_1(1,0) + 0(0,1) = c_1(1,0)$$

$$\mathbf{x}_2 = 3d_1(1,0) + 2d_1(0,1) = d_1(3,2)$$

In our previous discussions of vector spaces we have been able to allow freedom in the type of scalars which we use. We have thought primarily in terms of real numbers. However, in the discussion of eigenvalues this freedom in choice of scalars can cause difficulty. A real polynomial need not have real roots. Thus an operator on a space with real scalars may not have real eigenvalues; on the other hand, a complex eigenvalue has no meaning for such a space. The usual engineering practice is to accept the complex scalars whenever they appear, and assign them an appropriate meaning if necessary. We follow this approach, and assume, whenever we speak of eigenvalues, that the characteristic equation has a full set of roots.

Exercise 1. Define the operator $\mathbf{T}$ on $\mathcal{R}^2$ by

$$\mathbf{T}(\xi_1, \xi_2) = (\xi_1 \cos\phi - \xi_2 \sin\phi, \xi_2 \cos\phi + \xi_1 \sin\phi) \tag{4.11}$$

This operator describes "rotation through the angle ϕ" in $\mathcal{R}^2$. Show that the eigendata for $\mathbf{T}$ are

$$\lambda_1 = \cos\phi + i \sin\phi = e^{i\phi}, \qquad \mathbf{x}_1 = (1, -i)$$

$$\lambda_2 = \cos\phi - i \sin\phi = e^{-i\phi}, \qquad \mathbf{x}_2 = (1, i)$$

where $i = \sqrt{-1}$. The vector $(1, \pm i)$ is not a real 2-tuple; it is not in $\mathcal{R}^2$.

We could have used any basis in Example 1. The eigenvalues and eigenvectors of $\mathbf{T}$ are properties of $\mathbf{T}$; they do not depend upon the basis. Suppose we use the invertible change of coordinate matrix $\mathbf{S}^{-1}$ to convert (4.8) from the $\mathcal{Z}$ coordinate system to a new coordinate system $\mathcal{X}$ as in (2.54):

$$[\mathbf{x}]_{\mathcal{X}} = \mathbf{S}^{-1}[\mathbf{x}]_{\mathcal{Z}}$$

The effect of the change of coordinates on the matrix of $\mathbf{T}$ is represented by the similarity transformation (2.62): $[\mathbf{T}]_{\mathcal{Z}\mathcal{Z}} = \mathbf{S}[\mathbf{T}]_{\mathcal{X}\mathcal{X}}\mathbf{S}^{-1}$. Recalling that $\mathbf{A} = [\mathbf{T}]_{\mathcal{Z}\mathcal{Z}}$, we find that (4.8) can be expressed as $([\mathbf{T}]_{\mathcal{Z}\mathcal{Z}} - \lambda\mathbf{I})[\mathbf{x}]_{\mathcal{Z}} = (\mathbf{S}[\mathbf{T}]_{\mathcal{X}\mathcal{X}}\mathbf{S}^{-1} - \lambda\mathbf{I})[\mathbf{x}]_{\mathcal{Z}} = \mathbf{S}([\mathbf{T}]_{\mathcal{X}\mathcal{X}} - \lambda\mathbf{I})\mathbf{S}^{-1}[\mathbf{x}]_{\mathcal{Z}} = [\boldsymbol{\theta}]_{\mathcal{Z}}$. Multiplying by the invertible matrix $\mathbf{S}^{-1}$, we find

$$([\mathbf{T}]_{\mathcal{X}\mathcal{X}} - \lambda\mathbf{I})[\mathbf{x}]_{\mathcal{X}} = [\boldsymbol{\theta}]_{\mathcal{X}} \tag{4.12}$$

Clearly, any λ which is an eigenvalue of $\mathbf{A}$ is also an eigenvalue of any other matrix $[\mathbf{T}]_{\mathcal{X}\mathcal{X}}$ which represents $\mathbf{T}$. The similarity transformation, $[\mathbf{T}]_{\mathcal{X}\mathcal{X}} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$, results in a change in the *coordinates* of the eigenvectors of $\mathbf{T}$ corresponding to λ, but it does not change either the eigenvectors of $\mathbf{T}$ or the characteristic polynomial of $\mathbf{T}$.

Example 2. Invariance of Eigenvalues under a Change of Coordinates. The transformation T: $\mathcal{R}^2 \to \mathcal{R}^2$ of Example 1 is

$$\mathbf{T}(\xi_1, \xi_2) \triangleq (2\xi_1 + 3\xi_2, 4\xi_2)$$

The eigenvectors (1,0) and (3,2) found for $\mathbf{T}$ in Example 1 form a basis for $\mathcal{R}^2$; denote this basis by $\mathcal{X}$. With respect to this basis,

$$[\mathbf{T}]_{\mathcal{X}\mathcal{X}} = \left([\mathbf{T}(1,0)]_{\mathcal{X}} \vdots [\mathbf{T}(3,2)]_{\mathcal{X}} \right)$$
$$= \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$$

Then

$$\det([\mathbf{T}]_{\mathcal{X}\mathcal{X}} - \lambda \mathbf{I}) = \begin{vmatrix} 2-\lambda & 0 \\ 0 & 4-\lambda \end{vmatrix}$$
$$= (2-\lambda)(4-\lambda)$$

The characteristic polynomial and the eigenvalues are those found in Example 1.

Diagonalization

It is apparent that the matrix of any linear operator $\mathbf{T}$ with respect to a basis of eigenvectors for $\mathbf{T}$ is of the form demonstrated in Example 2. If $\mathcal{X}$ is a basis of eigenvectors, $[\mathbf{T}]_{\mathcal{X}\mathcal{X}}$ has the eigenvalues of $\mathbf{T}$ on its diagonal; the rest of the matrix is zero. We call a linear operator $\mathbf{T}$: $\mathcal{V} \to \mathcal{V}$ **diagonalizable** if there is a basis $\mathcal{X}$ for $\mathcal{V}$ which is composed of eigenvectors of $\mathbf{T}$. We refer to the diagonal matrix $[\mathbf{T}]_{\mathcal{X}\mathcal{X}}$ as the **spectral matrix** of $\mathbf{T}$, and denote it by the symbol $\mathbf{\Lambda}$. If $\mathbf{A}$ is the matrix of $\mathbf{T}$ relative to some other basis, say $\mathcal{Z}$, for $\mathcal{V}$, we will also refer to $\mathbf{\Lambda}$ as the **diagonal form of A**.

A basis of eigenvectors converts the operator equation $\mathbf{Tx} = \mathbf{y}$ to the matrix equation

$$\mathbf{\Lambda}[\mathbf{x}]_{\mathcal{X}} = [\mathbf{y}]_{\mathcal{X}} \tag{4.13}$$

Equation (4.13) is actually a matrix version of the process (4.5) for solving an equation by decomposition. Finding an eigenvector basis $\mathcal{X}$ corresponds to finding a direct-sum decomposition of the space into subspaces $\mathcal{W}_i$ which are invariant under $\mathbf{T}$. Finding a coordinate matrix $[\mathbf{y}]_{\mathcal{X}}$ is equivalent to the decomposition of y in (4.5). Inverting the diagonal (or "uncoupled") matrix $\mathbf{\Lambda}$ amounts to solving the reduced equations, $\mathbf{T}_i\mathbf{x}_i = \lambda_i\mathbf{x}_i = \mathbf{y}_i$. When we find $\mathbf{x}$ from the coordinates $[\mathbf{x}]_{\mathcal{X}}$, we are merely

combining the subsystem solutions as in (4.6). The process of computing eigenvalues and eigenvectors of matrices has been automated using a digital computer. Furthermore, the process of diagonalizing a matrix equation is more mnemonic than the decomposition process (4.5); the visual manner in which the eigenvalues and eigenvectors interact is easy to remember. Equation (4.13) is a clear and simple model for the system it represents.

What types of linear operators are diagonalizable? That is, for what finite-dimensional systems is there a basis of eigenvectors for the space? Since the existence of an eigenvalue λ_i implies the existence of a corresponding eigenvector $\mathbf{x}_i$, we expect the eigenvectors of an operator $\mathbf{T}$ on an n-dimensional space $\mathcal{V}$ to form a basis if its n eigenvalues are distinct. We verify that the n eigenvectors are independent if the eigenvalues are distinct by the test (2.1 I). Let

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n = \boldsymbol{\theta}$$

where $\mathbf{x}_i$ is an eigenvector of $\mathbf{T}$ for the eigenvalue λ_i. Operating with $(\mathbf{T} - \lambda_1\mathbf{I})$ we obtain

$$c_1(\lambda_1 \cancelto{0}{- \lambda_1})\mathbf{x}_1 + c_2(\lambda_2 - \lambda_1)\mathbf{x}_2 + \cdots + c_n(\lambda_n - \lambda_1)\mathbf{x}_n = \boldsymbol{\theta}$$

Successively operating with $(\mathbf{T} - \lambda_2\mathbf{I}), \ldots, (\mathbf{T} - \lambda_{n-1}\mathbf{I})$ eliminates all terms but

$$c_n(\lambda_n - \lambda_1)(\lambda_n - \lambda_2)\cdots(\lambda_n - \lambda_{n-1})\mathbf{x}_n = \boldsymbol{\theta}$$

since $\lambda_i \neq \lambda_j$, $c_n = 0$. By backtracking, we can successively show that $c_{n-1} = \cdots = c_1 = 0$; the eigenvectors are independent and form a basis for the n-dimensional space.

In the above proof we applied the operator $(\mathbf{T} - \lambda_1\mathbf{I})(\mathbf{T} - \lambda_2\mathbf{I})\cdots(\mathbf{T} - \lambda_{n-1}\mathbf{I})$ to a general vector in the space $\mathfrak{M}^{n\times 1}$ (i.e., to a linear combination, $\mathbf{x} = \sum c_i\mathbf{x}_i$, of the eigenvectors in the basis). Suppose we operate once more, using the factor $(\mathbf{T} - \lambda_n\mathbf{I})$. Then, for any $\mathbf{x}$, we obtain

$$c_n(\lambda_n - \lambda_1)(\lambda_n - \lambda_2)\cdots(\lambda_n \cancelto{0}{- \lambda_n})\mathbf{x}_n = \boldsymbol{\theta}$$

That is,

$$(\mathbf{T} - \lambda_1\mathbf{I})(\mathbf{T} - \lambda_2\mathbf{I})\cdots(\mathbf{T} - \lambda_n\mathbf{I}) = \boldsymbol{\Theta} \tag{4.14}$$

Recall from (4.10) that if $\mathbf{A}$ is a matrix of $\mathbf{T}$, the characteristic polynomial

for $\mathbf{T}$ is $c(\lambda) = \det(\lambda\mathbf{I} - \mathbf{A}) = (\lambda - \lambda_1)\cdots(\lambda - \lambda_n)$. Thus (4.14) is an operator analogue of $c(\lambda)$ which we denote by $c(\mathbf{T})$. The characteristic polynomial in $\mathbf{T}$ annihilates all vectors in the space. This fact is commonly known as the **Cayley-Hamilton theorem.** It applies as well to matrices—a square matrix satisfies its own characteristic equation:

$$c(\mathbf{A}) = \mathbf{\Theta} \tag{4.15}$$

Although we have proved the theorem only for an operator which is diagonalizable, it holds for all square matrices [see (4.85)].

Example 3. A Nondiagonalizable Matrix. Suppose

$$[\mathbf{T}]_{\mathfrak{X}\mathfrak{X}} = \mathbf{A} = \begin{pmatrix} \lambda_1 & 2 \\ 0 & \lambda_1 \end{pmatrix}$$

Then

$$\begin{aligned} c(\lambda) &= \det(\lambda\mathbf{I} - \mathbf{A}) \\ &= (\lambda - \lambda_1)^2 \end{aligned}$$

The only eigenvalue for $\mathbf{A}$ is $\lambda = \lambda_1$. Using (4.8) we solve for the associated eigenvectors of $\mathbf{A}$:

$$(\mathbf{A} - \lambda_1\mathbf{I})[\mathbf{x}_1]_{\mathfrak{X}} \triangleq \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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$$[\mathbf{x}_1]_{\mathfrak{X}} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$$

There are not enough independent eigenvectors of $\mathbf{A}$ to form a basis for $\mathfrak{M}^{2\times 1}$. The characteristic polynomial in $\mathbf{A}$ is

$$\begin{aligned} c(\mathbf{A}) &= (\mathbf{A} - \lambda_1\mathbf{I})^2 \\ &= \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}^2 = \mathbf{\Theta} \end{aligned}$$

It is apparent that the Cayley-Hamilton theorem also applies to matrices which do not possess distinct eigenvalues.

Although repeated eigenvalues can signal difficulty, it is possible for the eigenvectors to form a basis even though the eigenvalues are not distinct. A notable example is the identity operator; any vector in the space is an

eigenvector for the eigenvalue $\lambda = 1$. In Section 4.4 we discuss further those operators that are not diagonalizable.

Most matrices have distinct eigenvalues, and are thus diagonalizable. For a diagonalizable matrix $\mathbf{A}$, the eigenvalues by themselves (or the equivalent spectral matrix A) give a rough idea of the manner in which the system operates. However, in order to be specific about the operation of the system, we need to know what $\mathbf{A}$ does to specific vectors $[\mathbf{x}]_{\mathcal{Z}}$ on which it operates. Thus we need the eigenvectors of $\mathbf{A}$. In the process of finding the eigenvectors, we relate $\mathbf{A}$ and $\mathbf{\Lambda}$. A change of basis is the key. Let $\mathbf{T}$ act on a finite-dimensional space $\mathcal{V}$. Assume $\mathbf{A} = [\mathbf{T}]_{\mathcal{Z}\mathcal{Z}}$. Let $\mathcal{X} \triangleq \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ be a basis for $\mathcal{V}$ composed of eigenvectors of $\mathbf{T}$. Let $\{[\mathbf{x}_1]_{\mathcal{Z}}, \ldots, [\mathbf{x}_n]_{\mathcal{Z}}\}$ be the corresponding basis for $\mathcal{M}^{n \times 1}$ composed of eigenvectors of $\mathbf{A}$. Define the change of basis matrix $\mathbf{S}$ by

$$\mathbf{S}[\mathbf{x}]_{\mathcal{X}} = [\mathbf{x}]_{\mathcal{Z}} \tag{4.16}$$

Then, by (2.55),

$$\mathbf{S} = \left([\mathbf{x}_1]_{\mathcal{Z}} \vdots \cdots \vdots [\mathbf{x}_n]_{\mathcal{Z}} \right) \tag{4.17}$$

Furthermore, by (2.62),

$$\begin{aligned} [\mathbf{T}]_{\mathcal{X}\mathcal{X}} &= \mathbf{S}^{-1}[\mathbf{T}]_{\mathcal{Z}\mathcal{Z}}\mathbf{S} \\ &= \mathbf{S}^{-1}\mathbf{A}\mathbf{S} \\ &= \mathbf{\Lambda} \end{aligned} \tag{4.18}$$

We call the matrix $\mathbf{S}$, the columns of which are eigenvectors of $\mathbf{A}$, a **modal** matrix for $\mathbf{A}$.* Of course, the definition of S in (4.16) is arbitrary; the roles of $\mathbf{S}$ and $\mathbf{S}^{-1}$ can be reversed. In order to help keep in mind which of the matrices S and $\mathbf{S}^{-1}$ is the modal matrix, we note that $\mathbf{A}$ in (4.18) multiplies the eigenvectors of $\mathbf{A}$ in the modal matrix.

An engineer often generates a system model directly in matrix form. The matrix form follows naturally from the use of standard models and standard physical units. When the underlying transformation is not explicitly stated, it becomes cumbersome to carry the coordinate notation $[\mathbf{x}]_{\mathcal{Z}}$ for the vectors on which the $n \times n$ matrix $\mathbf{A}$ operates. Under these circumstances, we will change the notation in (4.8) to

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \boldsymbol{\theta} \tag{4.19}$$

*In some contexts the eigenvectors are referred to as modes of the system.

where $\mathbf{x}$ is a vector in $\mathfrak{M}^{n\times 1}$. This new notation can cause confusion—we are using the same notation $\mathbf{x}$ for both a vector (on which $\mathbf{T}$ operates) and its coordinate matrix (which $\mathbf{A}$ multiplies.) We must keep in mind that $\mathbf{A}$ and $\mathbf{x}$ may be representatives of an underlying transformation $\mathbf{T}$ and a vector $\mathbf{x}$ on which it operates.

Example 4. Diagonalization of a Matrix. Let

$$\mathbf{A}=\begin{pmatrix} 4 & -2 & 1 \\ -2 & 1 & 2 \\ 1 & 2 & 4 \end{pmatrix}$$

Then $c(\lambda)=\det(\lambda\mathbf{I}-\mathbf{A})=(\lambda-5)^2(\lambda+1)=0$. The eigenvalues of $\mathbf{A}$ are $\lambda_1=5$, $\lambda_2=5$, $\lambda_3=-1$. The eigenvectors for $\lambda = 5$ satisfy

$$(\mathbf{A}-5\mathbf{I})\mathbf{x}=\begin{pmatrix} -1 & -2 & 1 \\ -2 & -4 & 2 \\ 1 & 2 & -1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

or $\xi_3 = \xi_1 + 2\xi_2$. The eigenspace of $\mathbf{A}$ for $\lambda = 5$ is two-dimensional; one basis for this space is

$$\mathbf{x}_1=\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad \mathbf{x}_2=\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

The eigenvectors for $\lambda = -1$ satisfy

$$(\mathbf{A}+\mathbf{I})\mathbf{x}=\begin{pmatrix} 5 & -2 & 1 \\ -2 & 2 & 2 \\ 1 & 2 & 5 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

or, by row reduction, $\xi_1 = -\xi_3$ and $\xi_2 = -2\xi_3$. The eigenspace of $\mathbf{A}$ for $\lambda = -1$ is one-dimensional. We choose

$$\mathbf{x}_3=\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

as a basis for this eigenspace. We use the eigenvectors $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ of the matrix $\mathbf{A}$ as the columns of a modal matrix $\mathbf{S}$ for $\mathbf{A}$. We find $\mathbf{S}^{-1}$ from $\mathbf{S}$ by row reduction:

$$\mathbf{S}=\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & -1 \end{pmatrix} \qquad \mathbf{S}^{-1}=\frac{1}{6}\begin{pmatrix} 5 & -2 & 1 \\ -2 & 2 & 2 \\ 1 & 2 & -1 \end{pmatrix}$$

The diagonal form of $\mathbf{A}$ is:

$$\mathbf{\Lambda} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

The eigenvalues appear on the diagonal of $\mathbf{\Lambda}$ in the same order as their corresponding eigenvectors appear in the modal matrix.

Eigendata and Inverse Operators

If $\mathbf{T}$ is an invertible operator and $\mathbf{x}$ is an eigenvector of $\mathbf{T}$ for the eigenvalue λ, it follows from the definition $(\mathbf{T}\mathbf{x} = \lambda\mathbf{x})$ that

$$\mathbf{T}^{-1}\mathbf{x} = \left(\frac{1}{\lambda}\right)\mathbf{x} \tag{4.20}$$

That is, $\mathbf{x}$ is also an eigenvector for $\mathbf{T}^{-1}$ corresponding to the eigenvalue $1/\lambda$. Furthermore, $\mathbf{T}$ is invertible if and only if $\lambda = 0$ is not an eigenvalue of $\mathbf{T}$. This fact is easily seen if $\mathbf{T}$ acts on a finite-dimensional space: suppose $\mathbf{A}$ is a matrix of $\mathbf{T}$ (relative to some basis). Then $\lambda = 0$ is an eigenvalue of $\mathbf{T}$ if and only if

$$\det(\mathbf{A} - 0\mathbf{I}) = 0 \tag{4.21}$$

But (4.21) is just the condition for noninvertibility of $\mathbf{A}$ (and $\mathbf{T}$). If $\mathbf{\Lambda}$ is a diagonal form of $\mathbf{A}$, the relationship between the eigenvalues and invertibility is even more transparent. If $\lambda = 0$ is an eigenvalue of $\mathbf{A}$, then $\mathbf{\Lambda}$ has a zero row, and $\mathbf{A}$ and $\mathbf{T}$ are not invertible.

Example 5. Eigendata for an Inverse Matrix. The inverse of the matrix $\mathbf{A}$ of Example 4 is

$$\mathbf{A}^{-1} = \frac{1}{5}\begin{pmatrix} 0 & -2 & 1 \\ -2 & -3 & 2 \\ 1 & 2 & 0 \end{pmatrix}$$

Using the spectral matrix $\mathbf{\Lambda}$ and the modal matrix $\mathbf{S}$ for $\mathbf{A}$ (from Example 4), we find the spectral matrix for $\mathbf{A}^{-1}$ by

$$\mathbf{\Lambda}_{\mathbf{A}^{-1}} = \mathbf{S}^{-1}\mathbf{A}^{-1}\mathbf{S} = (\mathbf{S}^{-1}\mathbf{A}\mathbf{S})^{-1} = \mathbf{\Lambda}_{\mathbf{A}}{}^{-1}$$

or

$$\mathbf{\Lambda}_{\mathbf{A}^{-1}} = \begin{pmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Thus $\mathbf{A}$ and $\mathbf{A}^{-1}$ have inverse eigenvalues, but the same eigenvectors (modal matrices).

Computation of Eigendata for Matrices

Computation of the eigenvalues and eigenvectors of a square matrix appears straightforward. We need only solve for the roots λ_i of the characteristic polynomial, $c(\lambda) = \det(\lambda\mathbf{I} - \mathbf{A})$, then solve the equation $(\mathbf{A} - \lambda_i\mathbf{I})\mathbf{x} = \boldsymbol{\theta}$ for the eigenvectors associated with λ_i. For the selected low-order matrices used in the examples and in the Problems and Comments, the eigendata can be computed exactly using this approach. As a practical matter, however, the process is difficult for an arbitrary diagonalizable matrix. For a matrix larger than, say, 3×3, we resort to the digital computer.

Determination of the characteristic polynomial of the matrix by computing the determinant of $\lambda\mathbf{I} - \mathbf{A}$ is an expensive process. Computation of a simple $n \times n$ determinant requires $n^3/3$ multiplications, without the complication of the unspecified variable λ.* A more efficient approach for finding $c(\lambda)$ is **Krylov's method,** which is based on the Cayley-Hamilton theorem (4.15).† The characteristic equation for the $n \times n$ matrix $\mathbf{A}$ can be written

$$c(\lambda) = \lambda^n + b_1\lambda^{n-1} + \cdots + b_n = 0 \tag{4.22}$$

where the coefficients $\{b_i\}$ are, as yet, unknown. By (4.15),

$$c(\mathbf{A}) = \mathbf{A}^n + b_1\mathbf{A}^{n-1} + \cdots + b_n\mathbf{A} = \boldsymbol{\Theta}$$

Then for an arbitrary vector $\mathbf{x}$ in $\mathfrak{M}^{n\times 1}$,

$$\mathbf{A}^n\mathbf{x} + b_1\mathbf{A}^{n-1}\mathbf{x} + \ldots + b_n\mathbf{x} = \boldsymbol{\theta} \tag{4.23}$$

For a specific $\mathbf{x}$, the vector equation (4.23) can be solved by row reduction to obtain the coefficients $\{b_i\}$. Note that the powers of $\mathbf{A}$ need not be formed. Rather, $\mathbf{x}$ is multiplied by $\mathbf{A}$ n times. The method requires approximately n^3 multiplications to compute (4.23), then $n^3/3$ multiplications to solve for the coefficients $\{b_i\}$ by Gaussian elimination.

Example 6. Computing $c(\lambda)$ by Krylov's Method Let $\mathbf{A}$ be the system matrix of Example 1, Section 3.4:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$$

*See Appendix 1 for a discussion of determinants and their evaluation.

†Ralston [4.13]. Refer also to P&C 1.3*c*.

The characteristic equation is second order:

$$c(\lambda)=\lambda^2+b_1\lambda+b_2=0$$

$$c(\mathbf{A})=\mathbf{A}^2+b_1\mathbf{A}+b_2\mathbf{I}=\boldsymbol{\Theta}$$

Let $\mathbf{x} = (1\ 1)^T$. Then

$$\mathbf{A}^2\mathbf{x}+b_1\mathbf{A}\mathbf{x}+b_2\mathbf{x}=\boldsymbol{\theta}$$

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$$\begin{pmatrix}-1\\1\end{pmatrix}+b_1\begin{pmatrix}1\\-1\end{pmatrix}+b_2\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

The solution to these equations is $b_1 = 1$, $b_2=0$. Therefore,

$$c(\lambda)=\lambda^2+\lambda$$

Suppose that in Example 6 we had let $\mathbf{x}=(1\ -\ 1)^{\mathrm{T}}$, the eigenvector of $\mathbf{A}$ for $\lambda=$ -1. Then (4.23) would have been

$$\begin{pmatrix}1\\-1\end{pmatrix}+b_1\begin{pmatrix}-1\\1\end{pmatrix}+b_2\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

an underdetermined set of equations. The difficulty arises because $\mathbf{A}+\mathbf{I}$, one of the two factors of $c(\mathbf{A})$, is sufficient to annihilate $\mathbf{x}$. If we use an eigenvector of $\mathbf{A}$ in (4.23), we can determine only those factors of $c(\mathbf{A})$ that annihilate the eigenvector. Thus is it possible to make a poor choice for $\mathbf{x}$ in (4.23); try another! If the eigenvalues are not distinct, similar difficulties arise. (Try Krylov's method for $\mathbf{A}=\mathbf{I}$.)

Once we have $c(\lambda)$, we still need a scheme for finding its roots. A suitable method for finding the real roots is the iterative technique known as Newton's method. This method is discussed in detail in Section 8.1. If we need only the eigenvalues of $\mathbf{A}$ [as in evaluating functions of matrices by (4.108)], and if these eigenvalues are real, Krylov's method together with Newton's method is a reasonable approach to obtaining them.

Denote the eigenvalue of $\mathbf{A}$ which is of largest magnitude by λ_{L}. If λ_{L} is real, the **power method** obtains directly from $\mathbf{A}$ both its largest eigenvalue λ_{L} and a corresponding eigenvector $\mathbf{x}_{\mathrm{L}}$. The method relies on the "dominance" of the eigenvalue λ_{L}. Suppose eigenvectors of an $n \times n$ matrix $\mathbf{A}$ form a basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ for $\mathfrak{M}^{n\times 1}$. Then any vector $\mathbf{x}$ in $\mathfrak{M}^{n\times 1}$ can be expressed as $\mathbf{x}=\sum_{i=1}^{n} c_i\mathbf{x}_i$. Repeated multiplication of $\mathbf{x}$ by $\mathbf{A}$ yields $\mathbf{A}^k\mathbf{x} =\sum_{i=1}^{n} c_i\mathbf{A}^k\mathbf{x}_i=\sum_{i=1}^{n} c_i\lambda_i^k\mathbf{x}_i$. If one of the eigenvalues λ_{L} is larger in magni-

tude that the rest, then for large enough k, $\mathbf{A}^k\mathbf{x} \approx c_L\lambda_L^k\mathbf{x}_L$, an eigenvector for λ_L. Furthermore, λ_L is approximately equal to the ratio of the elements of $\mathbf{A}^{k+1}\mathbf{x}$ to those of $\mathbf{A}^k\mathbf{x}$. We explore the use of the power method in P&C 4.17. The method can be extended, by a process known as deflation, to obtain all the eigendata for $\mathbf{A}$. However, computational errors accumulate; the method is practical only for a few dominant eigenvalues. See Wilkinson [4.19].

Practical computation of the full set of eigenvectors of an arbitrary matrix is more difficult than is computation of the eigenvalues. The eigenvalues $\{\lambda_i\}$, by whatever method they are obtained, will be inexact, if only because of computer roundoff. Therefore, $(\mathbf{A} - \lambda_i\mathbf{I})$ is not quite singular; we need to compute the "near nullspace" of $(\mathbf{A} - \lambda_i\mathbf{I})$ (i.e., the "near solution" to $(\mathbf{A} - \lambda_i\mathbf{I})\mathbf{x} = \boldsymbol{\theta}$). In Section 2.4 we describe the **inverse iteration method** for determining a vector in the "near nullspace" of a nearly singular matrix. We now justify that method. If a matrix $\mathbf{B}$ is nearly singular, its near nullspace is precisely the eigenspace for its smallest (least dominant) eigenvalue, λ_s. Then the near nullspace of $\mathbf{B}$ is also the eigenspace for the largest (dominant) eigenvalue $1/\lambda_s$ of $\mathbf{B}^{-1}$. If λ_s is real, we can determine an eigenvector $\mathbf{x}_s$ corresponding to λ_s by applying the power method to $\mathbf{B}^{-1}$. We pick an arbitrary vector $\mathbf{z}_0$, and repetitively determine $\mathbf{z}_{k+1} = \mathbf{B}^{-1}\mathbf{z}_k$; for large enough k, the vector $\mathbf{z}_k$ is a good approximation to $\mathbf{x}_s$; the ratio of the components of $\mathbf{z}_k$ to those of $\mathbf{z}_{k+}$, is essentially λ_s. Thus the inverse iteration method is just the power method applied to the inverse matrix. In practice, rather than explicitly computing $\mathbf{B}^{-1}$, we would repetitively solve $\mathbf{B}\mathbf{z}_{k+1} = \mathbf{z}_k$, a less expensive operation.

The inverse iteration method can be used to obtain the eigenvectors of a matrix $\mathbf{A}$ which correspond to a previously computed real eigenvalue 4. Just repetitively solve $(\mathbf{A} - \lambda_i\mathbf{I})\mathbf{z}_{k+1} = \mathbf{z}_k$ for some initial vector $\mathbf{z}_0$; after several iterations, $\mathbf{z}_k$ will approximate an eigenvector $\mathbf{x}_i$ corresponding to 4. The ratio of the elements of $\mathbf{z}_{k+1}$ to those of $\mathbf{z}_k$ will approximate $1/\lambda_s$ where λ_s is the smallest eigenvalue of the matrix $\mathbf{B} = \mathbf{A} - \lambda_i\mathbf{I}$. The eigenvalue λ_s is a measure of the nonsingularity of $\mathbf{B}$ and, therefore, the inaccuracy in λ_i; a better approximation to the eigenvalue of $\mathbf{A}$ is $\lambda_i + \lambda_s$. A highly accurate value of λ_i implies a low value of λ_s and, consequently, rapid convergence. Of course, small λ_s also implies an ill-conditioned matrix $(\mathbf{A} - \lambda_i\mathbf{I})$; yet, as discussed in Section 2.4, the resulting uncertainty in the solution will be a vector in nullspace $(\mathbf{A} - \lambda_i\mathbf{I})$. The inverse iteration method works well as long as the eigenvalue 4 is "isolated." *Any* method will have trouble distinguishing between eigenvectors corresponding to nearly equal eigenvalues. *

*Wilkinson [4.19].

Example 7. Computing Eigenvectors by Inverse Iteration. Let $\mathbf{A}$ be the following matrix

$$\mathbf{A}=\begin{pmatrix}1 & 0\\ 1 & -1\end{pmatrix}$$

The exact eigendata of $\mathbf{A}$ are

$$\lambda_1=1,\quad \mathbf{x}_1=\begin{pmatrix}2\\ 1\end{pmatrix},\qquad \lambda_2=-1,\quad \mathbf{x}_2=\begin{pmatrix}0\\ 1\end{pmatrix}$$

Suppose we have computed the eigenvalue $\hat{\lambda}_1 = 1 + \epsilon$, perhaps by means of Krylov's method and Newton's method. The equation $(\mathbf{A}-\hat{\lambda}_1\mathbf{I})\mathbf{x}=\boldsymbol{\theta}$ has no nonzero solution. We use inverse iteration with the matrix $(\mathbf{A}-\hat{\lambda}_1\mathbf{I})$ to approximate the true eigenvector $\mathbf{x}_1$. Denote $\mathbf{z}_k=(\eta_1\eta_2)^{\mathrm{T}}$ and $\mathbf{z}_{k+1}=(\xi_1\ \xi_2)^{\mathrm{T}}$. Then

$$(\mathbf{A}-\lambda_1\mathbf{I})\mathbf{z}_{k+1}=\begin{pmatrix}-\epsilon & 0\\ 1 & -2-\epsilon\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}=\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}=\mathbf{z}_k$$

has the exact solution

$$\mathbf{z}_{k+1}=\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}=-\frac{1}{\epsilon}\begin{pmatrix}1 & 0\\ \frac{1}{2+\epsilon} & \frac{\epsilon}{2+\epsilon}\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}=(\mathbf{A}-\hat{\lambda}_1\mathbf{I})^{-1}\mathbf{z}_k$$

Let $\mathbf{z}_0=(1\ 1)^{\mathrm{T}}$. Then

$$\mathbf{z}_1=-\frac{1}{\epsilon}\begin{pmatrix}1\\ \frac{\epsilon+1}{\epsilon+2}\end{pmatrix},\qquad \mathbf{z}_2=\left(-\frac{1}{\epsilon}\right)^2\begin{pmatrix}1\\ \frac{\epsilon^2+2\epsilon+2}{(\epsilon+2)^2}\end{pmatrix}$$

This sequence rapidly approaches a true eigenvector for λ_1 even if the approximate eigenvalue $\hat{\lambda}_1$ contains significant error. If ϵ = 0.1, for instance, $\mathbf{z}_1$ = -10 (1 .52)$^{\mathrm{T}}$ and $\mathbf{z}_2$ = 100 (1 .501)$^{\mathrm{T}}$. The smallest eigenvalue of $(\mathbf{A}-\hat{\lambda}_1\mathbf{I})$ is clearly $\hat{\lambda}_s=-\epsilon$, which approaches zero as the error in $\hat{\lambda}_1$ approaches zero. It is apparent that for small ϵ, the elements of $\mathbf{z}_k$ would soon become very large. Practical computer implementations of the inverse iteration method avoid large numbers by normalizing $\mathbf{z}_k$ at each iteration.

If $\mathbf{A}$ is symmetric, the eigenvalues of $\mathbf{A}$ are real (P&C 5.28) and there is a basis of eigenvectors for the space.* The most efficient and accurate algorithms for determination of the full set of eigendata for a symmetric matrix avoid computation of the characteristic polynomial altogether. Rather, they perform a series of similarity transformations on $\mathbf{A}$, reducing the matrix to its diagonal form $\boldsymbol{\Lambda}$; the eigenvalues appear on the diagonal. Since $\boldsymbol{\Lambda} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$, where $\mathbf{S}$ is a matrix of eigenvectors, the sequence of

*See Section 5.4.

similarity transformations determines the eigenvectors of **A.** See P&C 4.11 for an example of such a method.

Because methods that produce the full set of eigendata for a matrix must, in effect, determine both **S** and $\mathbf{S}^{-1}$, we should expect the accuracy of the results to be related to the invertibility of the modal matrix **S.** In point of fact, it can be shown that if **S** is ill-conditioned, the eigenvalues are difficult to compute accurately; some of the eigenvalues are sensitive functions of the elements of **A.** As a general rule, symmetric matrices have easily determined eigenvalues, whereas unsymmetric matrices do not. For a full discussion of computer techniques for computing eigendata, see Wilkinson [4.19] and Forsythe [4.6].

Application of Spectral Decomposition-Symmetrical Components

Since a sinusoid of specified frequency is completely determined by two real numbers, its amplitude and phase, we can represent it by a single complex number; for example, the function $2\sin(\omega t + \phi)$ is equivalent to the complex number $2e^{i\phi}$, where $i = \sqrt{-1}$. Therefore, complex numbers adequately represent the steady-state 60-Hz sinusoidal voltages and currents in an electric power system (assuming physical units of volts and amperes, respectively).

Figure 4.2 is a simplified description of a three-phase electric power system. The complex amplitudes of the generated voltages, load voltages, and load currents are denoted by E_i, V_i, and I_i, respectively. These voltages and currents are related by the following matrix equations:

$$\mathbf{E} - \mathbf{V} = \mathbf{ZI} \tag{4.24}$$

$$\mathbf{V} = \mathbf{WI} \tag{4.25}$$

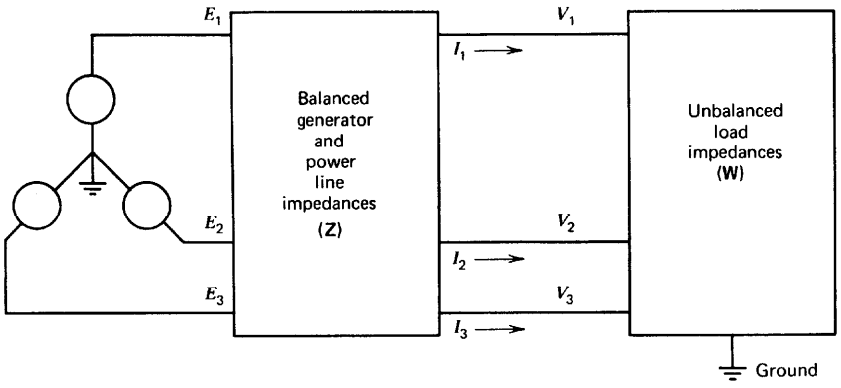


Figure 4.2. A three-phase electric power system.

where $\mathbf{E} = (E_1\ E_2\ E_3)^T$, $\mathbf{V} = (V_1\ V_2\ V_3)^T$, $\mathbf{I} = (I_1\ I_2\ I_3)^T$, and $\mathbf{Z}$ and $\mathbf{W}$ are 3×3 impedance matrices. In a typical power system, the generating system is balanced; that is, $\mathbf{Z}$ has the form

$$\mathbf{Z} = \begin{pmatrix} z_1 & z_2 & z_2 \\ z_2 & z_1 & z_2 \\ z_2 & z_2 & z_1 \end{pmatrix} \tag{4.26}$$

A useful approach to analyzing a three-phase power system is to change coordinates in (4.24)-(4.25) in order to diagonalize (4.24). The method is known to power system engineers as the **method of symmetrical components.**

Exercise 2. Show (or verify) that the eigenvalues λ_i and corresponding eigenvectors $\mathbf{x}_i$ of $\mathbf{Z}$ are

$$\lambda_0 = z_1 + 2z_2 \quad \lambda_+ = z_1 - z_2 \quad \lambda_- = z_1 - z_2 \tag{4.27}$$

$$\mathbf{x}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \mathbf{x}_+ = \begin{pmatrix} 1 \\ a \\ a^2 \end{pmatrix} \quad \mathbf{x}_- = \begin{pmatrix} 1 \\ a^2 \\ a \end{pmatrix} \tag{4.28}$$

where $a = e^{i2\pi/3}$, a $120°$ counterclockwise rotation in the complex plane. (Note that $a^2 + a + 1 = 0$.) Let $\mathbf{S} = (\mathbf{x}_0 \vdots \mathbf{x}_+ \vdots \mathbf{x}_-)$. Show (or verify) that

$$\mathbf{S}^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{pmatrix} \tag{4.29}$$

Each of the eigenvectors (4.28) represents the complex amplitudes of a symmetrical three-phase sinusoidal quantity (voltage or current). The subscripts indicate the relative placement of the elements of each vector in the complex plane. The generated voltage vector $\mathbf{E}$ typically has the form of $\mathbf{x}_+$. The eigenvalues (4.27) can be interpreted as impedances associated with the symmetrical (eigenvector) components of the voltage and current vectors.

The engineer usually needs to analyze the generation and distribution system under various loads. If the load impedance matrix $\mathbf{W}$ is an arbitrary matrix, it need not simplify during diagonalization. However, system loads are usually of a more specialized nature. For example, if the load is balanced (a goal of system planners), $\mathbf{W}$ is of the same form as $\mathbf{Z}$, both (4.24) and (4.25) diagonalize simultaneously, only positive sequence quantities appear in the equations, and the matrix equations reduce to two scalar

equations. Certain unbalanced loads (such as a line-to-line fault) also lead to specialized forms of $\mathbf{W}$ for which symmetrical component analysis is useful. A more complete discussion of symmetrical component analysis can be found in Rothe [4.15].

4.3 Spectral Analysis in Function Spaces

Spectral analysis is at least as helpful for understanding differential systems as it is for matrix equations. Furthermore, for many distributed systems (those described by partial differential equations) it provides the only reasonable approach to the determination of solutions. This section is devoted primarily to a discussion of spectral analysis of differential systems. We found in Example 9 of Section 4.1 that for a differential operator without boundary conditions, every scalar is an eigenvalue. The differential operators of real interest, however, are the ones we use in modeling systems. These ordinarily possess an appropriate number of boundary conditions. Suppose

$$\begin{aligned} \mathbf{Lf} &\triangleq g_0(t)\frac{d^n\mathbf{f}(t)}{dt^n} + \cdots + g_n(t)\mathbf{f}(t) = \mathbf{u}(t) \\ \boldsymbol{\beta}_i(\mathbf{f}) &= \alpha_i \qquad i = 1,\ldots,n \end{aligned} \tag{4.30}$$

It is convenient to decompose this differential system into two pieces:

$$\mathbf{Lf} = \mathbf{u} \quad \text{with } \boldsymbol{\beta}_i(\mathbf{f}) = 0, \qquad i = 1,\ldots,n \tag{4.31}$$

and

$$\mathbf{Lf} = \boldsymbol{\theta} \quad \text{with } \boldsymbol{\beta}_i(\mathbf{f}) = \alpha_i, \qquad i = 1,\ldots,n \tag{4.32}$$

Equation (4.32) is essentially finite dimensional in nature-by substituting for $\mathbf{f}$ the complementary function $\mathbf{f}_c = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$ of (3.19), we convert (4.32) to the matrix equation

$$\begin{pmatrix} \boldsymbol{\beta}_1(\mathbf{v}_1) & \cdots & \boldsymbol{\beta}_1(\mathbf{v}_n) \\ \vdots & & \vdots \\ \boldsymbol{\beta}_n(\mathbf{v}_1) & \cdots & \boldsymbol{\beta}_n(\mathbf{v}_n) \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \tag{4.33}$$

We examined the eigendata for matrix operators in Section 4.2. We focus now on the infinite-dimensional problem (4.31).

We seek the eigenvalues and eigenfunctions for the system $\mathbf{T}$ defined by $\mathbf{L}$ together with the homogeneous boundary conditions of (4.31). That is,

we only allow $\mathbf{L}$ to operate on functions which satisfy these boundary conditions. The equation which defines the eigendata is (4.7); thus

$$\begin{aligned}(\mathbf{L}-\lambda\mathbf{I})\mathbf{f}&=\boldsymbol{\theta}\\ \boldsymbol{\beta}_i(\mathbf{f})&=0 \qquad i=1,\ldots,n\end{aligned} \tag{4.34}$$

We introduce, by means of an example, a procedure for obtaining from (4.34) the eigenvalues and eigenfunctions associated with (4.31). The armature-controlled motor of (3.40)-(3.41) is modeled by $\mathbf{L}\boldsymbol{\phi} \triangleq \mathbf{D}^2\boldsymbol{\phi}+\mathbf{D}\boldsymbol{\phi}$, with $\boldsymbol{\beta}_1(\boldsymbol{\phi}) \triangleq \boldsymbol{\phi}(0)$ and $\boldsymbol{\beta}_2(\boldsymbol{\phi}) \triangleq \boldsymbol{\phi}(b)$. For this specific $\mathbf{L}$ and $\{\boldsymbol{\beta}_i\}$, (4.34) becomes

$$\begin{aligned}&\frac{d^2\phi(t)}{dt^2}+\frac{d\phi(t)}{dt}-\lambda\phi(t)=0\\ &\phi(0)=\phi(b)=0\end{aligned} \tag{4.35}$$

We first obtain a fundamental set of solutions for $(\mathbf{L}-\lambda\mathbf{I})$. The characteristic equation for $(\mathbf{L}-\lambda\mathbf{I})$, found by inserting $\phi(t) = e^{\mu t}$, is

$$\mu^2+\mu-\lambda=0$$

with roots

$$\mu=\frac{-1\pm\sqrt{1+4\lambda}}{2}$$

If $\lambda = -\frac{1}{4}$, then the fundamental solutions are

$$\mathbf{v}_1(t) = e^{-t/2} \qquad \mathbf{v}_2(t) = te^{-t/2}$$

Any nonzero solutions to (4.35) for $\lambda = -\frac{1}{4}$ must be of the form $\mathbf{f} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$ and must satisfy the boundary conditions:

$$\begin{pmatrix}\boldsymbol{\beta}_1(\mathbf{f})\\ \boldsymbol{\beta}_2(\mathbf{f})\end{pmatrix}=\begin{pmatrix}\boldsymbol{\beta}_1(\mathbf{v}_1) & \boldsymbol{\beta}_1(\mathbf{v}_2)\\ \boldsymbol{\beta}_2(\mathbf{v}_1) & \boldsymbol{\beta}_2(\mathbf{v}_2)\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix}=\begin{pmatrix}1 & 0\\ e^{-b/2} & be^{-b/2}\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

The boundary condition matrix is invertible; $c_1 = c_2 = 0$. There are no nonzero solutions for $\lambda = -\frac{1}{4}$, and $\lambda = -\frac{1}{4}$ is not an eigenvalue.

If $\lambda \neq -\frac{1}{4}$, a pair of fundamental solutions is

$$\mathbf{g}_1(t)=e^{-t/2}\exp\left(\frac{(1+4\lambda)^{1/2}t}{2}\right), \qquad \mathbf{g}_2(t)=e^{-t/2}\exp\left(\frac{-(1+4\lambda)^{1/2}t}{2}\right)$$

A different but equivalent pair is

$$\mathbf{h}_1(t) = e^{-t/2}\cos\left(\frac{-i(1+4\lambda)^{1/2}t}{2}\right), \qquad \mathbf{h}_2(t) = e^{-t/2}\sin\left(\frac{-i(1+4\lambda)^{1/2}t}{2}\right)$$

We let $\mathbf{g} = c_1\mathbf{g}_1 + c_2\mathbf{g}_2$, and again invoke the boundary conditions:

$$\begin{pmatrix} \boldsymbol{\beta}_1(\mathbf{g}) \\ \boldsymbol{\beta}_2(\mathbf{g}) \end{pmatrix} = \begin{pmatrix} \boldsymbol{\beta}_1(\mathbf{g}_1) & \boldsymbol{\beta}_1(\mathbf{g}_2) \\ \boldsymbol{\beta}_2(\mathbf{g}_1) & \boldsymbol{\beta}_2(\mathbf{g}_2) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ e^{-b/2}\exp\left(\frac{(1+4\lambda)^{1/2}b}{2}\right) & e^{-b/2}\exp\left(\frac{-(1+4\lambda)^{1/2}b}{2}\right) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

There is a nonzero solution $\mathbf{g}$ (or nonzero coefficients $\{c_i\}$) if and only if the boundary condition matrix is singular; thus, denoting the boundary condition matrix by $\mathbf{B}(\lambda)$,

$$\det(\mathbf{B}(\lambda)) = e^{-b/2}\exp\left(\frac{-(1+4\lambda)^{1/2}b}{2}\right) - e^{-b/2}\exp\left(\frac{(1+4\lambda)^{1/2}b}{2}\right) = 0$$

or

$$\exp\left[(1+4\lambda)^{1/2}b\right] = 1 \tag{4.36}$$

By analogy with the finite-dimensional case, we are inclined to refer to $\det(\mathbf{B}(\lambda)) = 0$ as the characteristic equation for the operator $\mathbf{T}$ ($\mathbf{L}$ with the homogeneous boundary conditions). However, the term characteristic equation is commonly used in reference to the equation (in the variable μ) used earlier to determine the fundamental solutions for $\mathbf{L}$. Therefore, we call $\det(\mathbf{B}(\lambda)) = 0$ the **eigenvalue equation** for $\mathbf{T}$. We may also refer to it as the eigenvalue equation for $\mathbf{L}$ if it is clear which homogeneous boundary conditions are intended. The eigenvalue equation (4.36) is a transcendental equation in λ. To find the roots, recall from the theory of complex variables that*

$$\ln(e^{\alpha+i\gamma}) = \alpha + i\gamma + i2\pi k, \qquad k = 0, \pm 1, \pm 2, \ldots$$

for real scalars α and γ. Thus (4.36) becomes

$$(1+4\lambda)^{1/2}b + i2\pi k = 0 \qquad k = 0, \pm 1, \pm 2, \ldots$$

*See Chapter 14 of Wylie [4.18].

and the eigenvalues (for which nonzero solutions exist) are

$$\lambda_k = -\tfrac{1}{4} - \left(\frac{k\pi}{b}\right)^2 \qquad k = 1, 2, 3, \ldots \tag{4.37}$$

Note that $k = 0$ has been deleted; it corresponds to $\lambda = -\frac{1}{4}$, for which case $\mathbf{g}_1$ and $\mathbf{g}_2$ are not a fundamental set of solutions. Since k is squared, the positive and negative values of k yield identical values of λ; thus, the positive values are sufficient.

We obtain the eigenfunctions ϕ_k corresponding to the eigenvalue λ_k by solving (4.35) with $\lambda = \lambda_k$. The solutions involve the roots μ_k of the characteristic equation:

$$\mu_k = \frac{-1 \pm (1 + 4\lambda_k)^{1/2}}{2} = -\tfrac{1}{2} \pm i\frac{k\pi}{b}$$

Since these roots are complex, we use the sinusoidal form $\{\mathbf{h}_i\}$ for the fundamental solutions:

$$\phi_k(t) = c_1 e^{-t/2}\cos\left(\frac{k\pi t}{b}\right) + c_2 e^{-t/2}\sin\left(\frac{k\pi t}{b}\right)$$

The boundary conditions yield

$$\begin{pmatrix} \beta_1(\phi_k) \\ \beta_2(\phi_k) \end{pmatrix} = \mathbf{B}(\lambda_k)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} e^0\cos(0) & e^0\sin(0) \\ e^{-b/2}\cos(k\pi) & e^{-b/2}\sin(k\pi) \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

It follows that $c_1 = 0$ and c_2 is arbitrary. Letting $c_2 = 1$, we obtain the eigenfunction

$$\phi_k(t) = e^{-t/2}\sin\left(\frac{k\pi t}{b}\right) \tag{4.38}$$

corresponding to the eigenvalue λ_k.

The eigenfunctions for the two-point boundary value operator of (4.35) are analogous to the modes of oscillation of a string which is tied at both ends. The modes are harmonics of the fundamental or lowest-order mode, $e^{-t/2}\sin(\pi t/b)$; that is, the frequencies of oscillation are integral multiples of the lowest-order frequency. The number μ_k is the complex "natural frequency" of the kth mode. The eigenvalue λ_k can be thought of as a "characteristic number" for the kth mode. It is not clear whether or not $\mathbf{T}$ is a diagonalizable operator. The eigenvalues are distinct; the set of eigenfunctions are suggestive of the terms of a Fourier series; however, we

wait until Chapter 5 to determine that there are sufficient eigenfunctions $\{\boldsymbol{\phi}_k, k = 1, 2, \ldots\}$ to form a basis for the space of functions $\mathbf{f}$ on which $\mathbf{T}$ (or L) operates. (See Example 3, Section 5.3.)

Finding Eigendata for Differential Operators

For general differential equations of the form (4.30) we find eigendata by following the procedure used for the specific operator of (4.35). We first seek values of λ (or **eigenvalues)** for which (4.34) has nonzero solutions **(eigenfunctions).** Then we determine the corresponding eigenfunctions. We occasionally refer to the eigendata for the differential equation when we really mean the eigendata for the differential operator which determines the equation. Let the functions $\mathbf{v}_1(\lambda), \ldots, \mathbf{v}_n(\lambda)$ be a fundamental set of solutions for $(\mathbf{L} - \lambda\mathbf{I})$; note that the functions depend on λ. The solutions to (4.34) consist in linear combinations

$$\mathbf{f}_c = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$$

which satisfy the boundary conditions. The coefficients are determined by the boundary condition matrix, whose λ dependency we denote explicitly by $\mathbf{B}(\lambda)$:

$$\mathbf{B}(\lambda)\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} \boldsymbol{\beta}_1(\mathbf{v}_1) & \cdots & \boldsymbol{\beta}_1(\mathbf{v}_n) \\ \vdots & & \vdots \\ \boldsymbol{\beta}_n(\mathbf{v}_1) & \cdots & \boldsymbol{\beta}_n(\mathbf{v}_n) \end{pmatrix}\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \tag{4.39}$$

There are nonzero solutions to (4.34) [or nonzero coefficients $\{c_i\}$ in (4.39)] only for λ such that

$$\det(\mathbf{B}(\lambda)) = 0 \tag{4.40}$$

As discussed beneath (4.36), we call (4.40) the **eigenvalue equation for** $\mathbf{T}$ (or for $\mathbf{L}$ with its boundary conditions). Its roots constitute the spectrum of $\mathbf{T}$ (or of $\mathbf{L}$ with its boundary conditions).

Determining the complementary function for $\mathbf{T} - \lambda\mathbf{I}$ is not necessarily a simple task. But it is the fundamental problem of differential equation analysis-standard techniques apply. The eigenvalue equation (4.40) is generally transcendental. Its solution, perhaps difficult, is a matter of algebra. Once we have determined a specific eigenvalue λ_k we return to (4.39) to determine those combinations of the fundamental solutions which are eigenfunctions for λ_k. The eigenfunctions are

$$\mathbf{f}_k = c_1\mathbf{v}_1(\lambda_k) + \cdots + c_n\mathbf{v}_n(\lambda_k) \tag{4.41}$$

where the scalars $c_1, \ldots, c_n$ satisfy

$$\mathbf{B}(\lambda_k)\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \boldsymbol{\theta}$$

As noted in the discussion following (3.28), the boundary condition matrix for a one-point boundary value problem is always invertible. Thus if the boundary conditions for $\mathbf{L}$ are all initial conditions, (4.40) has no roots, and the system $\mathbf{T}$ has no eigenvalues.

Exercise 1. Seek the eigenvalues for the operator $\mathbf{L}$ of (4.35) with the initial conditions $\phi(0) = \phi'(0) = 0$.

Example 1. Eigendata for a Heat-How Problem. Equation (3.1) is a steady-state description of a system wherein the heat generated within an insulated bar of length b diffuses toward heat sinks at the surfaces $t = 0$ and $t = b$. We now modify the second boundary condition. At $t = b$ we withdraw heat from the system by convection. The equation and modified boundary conditions for the temperature distribution $\mathbf{f}$ are as follows:

$$\begin{aligned} (\mathbf{L}f)(t) &\triangleq -\frac{d^2\mathbf{f}(t)}{dt^2} = \mathbf{u}(t) \\ \beta_1(\mathbf{f}) \triangleq \mathbf{f}(0) = \alpha_1, &\qquad \beta_2(\mathbf{f}) \triangleq \mathbf{f}'(b) + \mathbf{f}(b) = \alpha_2 \end{aligned} \tag{4.42}$$

The characteristic equation for $(\mathbf{L} - \lambda\mathbf{I})$ is

$$-\mu^2 - \lambda = 0$$

with roots $\mu = \pm i\sqrt{\lambda}$. We pick as a fundamental set of solutions (for $\lambda \neq 0$):

$$\mathbf{v}_1(t) = \cos\sqrt{\lambda}\, t, \qquad \mathbf{v}_2(t) = \sin\sqrt{\lambda}\, t$$

The eigenvalue equation is

$$\begin{aligned} \det(\mathbf{B}(\lambda)) &= \begin{vmatrix} 1 & 0 \\ -\sqrt{\lambda}\sin(\sqrt{\lambda}\, b) + \cos(\sqrt{\lambda}\, b) & \sqrt{\lambda}\cos(\sqrt{\lambda}\, b) + \sin(\sqrt{\lambda}\, b) \end{vmatrix} \\ &= \sqrt{\lambda}\cos(\sqrt{\lambda}\, b) + \sin(\sqrt{\lambda}\, b) = 0 \end{aligned}$$

or

$$\tan\sqrt{\lambda}\, b = -\sqrt{\lambda} \tag{4.43}$$

Making the substitution $r \triangleq \sqrt{\lambda}\, b$, (4.43) becomes

$$\tan r = -\frac{r}{b} \tag{4.44}$$

Figure 4.3 shows the two halves of the eigenvalue equation plotted versus r for $b = 2$. If $\{r_k, k = 0, \pm 1, \pm 2, \ldots\}$ are the roots of (4.44), then the eigenvalues for (4.42) are

$$\lambda_k = \frac{r_k^2}{b^2} \qquad k = 1,2,3,\ldots \tag{4.45}$$

The root r_0 has been eliminated. It corresponds to $\lambda = 0$, for which the sinusoids are not a fundamental set of solutions. That $\lambda = 0$ is not an eigenvalue is easily seen by repeating the above, using a fundamental set of solutions for $(\mathbf{L} - 0\mathbf{I})$. Since

$$(r_{-k})^2 = (-r_k)^2 = r_k^2$$

the negative values of k are unnecessary. We find the eigenfunctions $\mathbf{f}_k$ for λ_k by (4.41):

$$\mathbf{B}(\lambda_k)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{r_k}{b}\sin r_k + \cos r_k & \frac{r_k}{b}\cos r_k + \sin r_k \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or $c_1 = 0$ and c_2 is arbitrary. Therefore, letting $c_2 = 1$, we obtain only one independent eigenvector,

$$\mathbf{f}_k(t) = \sin\left(\frac{r_k}{b}\, t\right) \tag{4.46}$$

for each eigenvalue $\lambda_k = r_k^2/b^2$, $k = 1,2,3,\ldots$.

In this example, the modes are not harmonic; the frequencies r_k^2/b^2 are not integral multiples of the lowest frequency. Although the operator of (4.42) is diagonalizable (the eigenvectors (4.46) form a basis for the domain of $\mathbf{L}$), we are not presently prepared to show it.

Eigendata for Integral Operators

We found in (4.20) that if an operator $\mathbf{T}$ is invertible and $\mathbf{Tx} = \lambda\mathbf{x}$, then $\mathbf{T}^{-1}\mathbf{x} = (1/\lambda)\mathbf{x}$. That is, the eigenvectors of $\mathbf{T}$ and $\mathbf{T}^{-1}$ are identical and correspond to reciprocal eigenvalues. From (4.40) we know that a differential system $\mathbf{T}$ has the eigenvalue $\lambda = 0$ if and only if $\det(\mathbf{B}(\lambda)) = \det(\mathbf{B}(0)) = 0$. But this is just the opposite of the condition (3.28) for invertibility of $\mathbf{T}$. Thus a differential system $\mathbf{T}$ is invertible if and only if $\lambda = 0$ is not an eigenvalue for $\mathbf{T}$. If we think in terms of a diagonalized $(\infty \times \infty)$ matrix representation of $\mathbf{T}$, it is clear that a zero eigenvalue is equivalent to singularity of the operator. Thus if $\lambda = 0$ is an eigenvalue of $\mathbf{T}$, then the

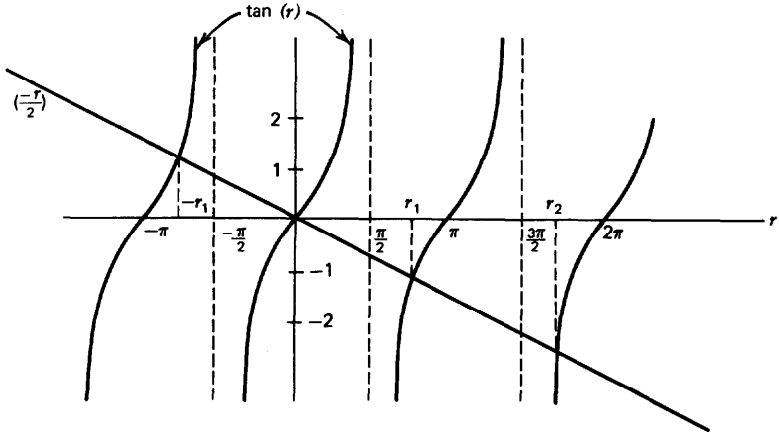


Figure 4.3. Roots of the eigenvalue equation (4.44) for $b = 2$.

Green's function for $\mathbf{T}$ does not exist. Invertible differential and integral equations come in pairs, one the inverse of the other. Because the properties of integration are theoretically and computationally less troublesome than those of differentation, we use the integral form to derive useful information about the eigenfunctions of operators and the solutions of equations (Sections 5.4 and 5.5). We also use the integral form for approximate numerical solution of equations. Yet because integral equations are difficult to solve, we often return to the differential form and standard differential equation techniques to determine the eigenfunctions of specific operators or the solutions of specific equations. In the following example, we obtain the eigendata for an integral operator from its differential inverse.

Example 2. Eigendata for an Integral Operator. The eigendata for the system $\mathbf{T}$ represented by the differential operator $\mathbf{L} = \mathbf{D}^2 + \mathbf{D}$ with $\phi(0) = 0$ and $\phi(b) = 0$ are given in (4.37) and (4.38). They are

$$\lambda_k = -\frac{1}{4} - \left(\frac{k\pi}{b}\right)^2, \quad \phi_k(t) = e^{-t/2}\sin\left(\frac{k\pi t}{b}\right), \qquad k = 1, 2, \ldots$$

Note that $\lambda = 0$ is not an eigenvalue. The Green's function for this operator is (3.42). Using this Green's function, we write the inverse of the differential system as

$$\begin{aligned}\phi(t) &= \frac{1 - e^b e^{-t}}{e^b - 1}\int_0^t (e^s - 1)\mathbf{u}(s)\,ds + \frac{1 - e^{-t}}{e^b - 1}\int_t^b (e^s - e^b)\mathbf{u}(s)\,ds \\ &= (\mathbf{T}^{-1}\mathbf{u})(t)\end{aligned} \tag{4.47}$$

We expect the eigenfunctions of $\mathbf{T}^{-1}$ to be the same as those of $\mathbf{T}$. Operating on ϕ_k with $\mathbf{T}^{-1}$, a complicated integration, we find

$$\begin{aligned}(\mathbf{T}^{-1}\phi_k)(t) &= \frac{1-e^b e^{-t}}{e^b-1}\int_0^t (e^s-1)e^{-s/2}\sin\left(\frac{k\pi s}{b}\right)ds \\ &\quad + \frac{1-e^{-t}}{e^b-1}\int_t^b (e^s-e^b)e^{-s/2}\sin\left(\frac{k\pi s}{b}\right)ds \\ &= \frac{1}{-1/4-(k\pi/b)^2}e^{-t/2}\sin\left(\frac{k\pi t}{b}\right) \\ &= \left(\frac{1}{\lambda_k}\right)\phi_k(t) \quad k=1,2,3,\ldots \end{aligned} \tag{4.48}$$

The eigenvalues of the integral operator $\mathbf{T}^{-1}$ are clearly $\{1/\lambda_k\}$.

Eigenvalue Problems in State Space

We introduced the state space model for dynamic systems in Section 3.4. We reproduce it here:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{4.49}$$

where $\mathbf{A}$ is an $n \times n$ matrix multiplying the $n \times 1$ state vector $\mathbf{x}(t)$, and $\mathbf{B}$ is an $n \times m$ matrix multiplying the $m \times 1$ input vector $\mathbf{u}(t)$. We know the differential system of (4.49) has no eigenvalues—it is an initial-value problem.* However, there is a meaningful and interesting eigenvalue problem associated with (4.49). It has to do with the system matrix $\mathbf{A}$. We introduce the relationship between the eigendata for the system matrix and the solutions of (4.49) by examining the system matrix for the *n*th-order constant-coefficient differential equation, the companion matrix of (3.36). The eigenvalues of $\mathbf{A}$ are the roots of the equation $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$.

Exercise 2. Show that if $\mathbf{A}$ is the companion matrix for the *n*th-order constant-coefficient differential equation

$$\mathbf{D}^n\mathbf{f} + a_1\mathbf{D}^{n-1}\mathbf{f} + \cdots + a_n\mathbf{f} = \mathbf{u} \tag{4.50}$$

then the characteristic equation for $\mathbf{A}$ is

$$\det(\lambda\mathbf{I} - \mathbf{A}) = (\lambda^n + a_1\lambda^{n-1} + \cdots + a_n) = 0 \tag{4.51}$$

*If the initial condition vector is $\mathbf{x}(0) = \boldsymbol{\theta}$, then $\dot{\mathbf{x}}(t) - \mathbf{A}\mathbf{x}(t) - \lambda\mathbf{x}(t) = 0$ has *only* the zero solution, $\mathbf{x}(t) = \boldsymbol{\theta}$.

From (4.51), we see that if $\mathbf{A}$ is the system matrix corresponding to an nth-order constant-coefficient differential equation, the characteristic equation for $\mathbf{A}$ is the same as the characteristic equation (3.37) for the underlying nth-order differential equation. The eigenvalues of the system matrix are the exponents for a fundamental set of solutions to the differential equation. They are sometimes referred to as poles of the system. This relationship between the eigenvalues of the system matrix and the fundamental set of solutions to the underlying set of differential equations holds for any system matrix $\mathbf{A}$, not just for those in companion matrix form. [See the discussion below (4.94); refer also to P&C 4.16] Thus in the state-space equation (4.49) the concepts of matrix transformations and differential operators merge in an interesting way. The origin of the term "characteristic equation for the differential equation" is apparent. Fortunately, the state-space formulation is not convenient for boundary value problems. Thus eigenvalues of a system matrix and eigenvalues of a differential equation usually do not appear in the same problem.

Suppose we use the eigenvectors of the system matrix $\mathbf{A}$ as a new basis for the state space, assuming, of course, that $\mathbf{A}$ is diagonalizable. We change coordinates as in (4.16)-(4.18). (We can think of the state vector $\mathbf{x}(t)$ in $\mathcal{M}^{n\times 1}$ as representing itself relative to the standard basis for $\mathcal{M}^{n\times 1}$.) If $\{\mathbf{x}_1,\ldots,\mathbf{x}_n\}$ is a basis of eigenvectors for $\mathbf{A}$ corresponding to the eigenvalues $\{\lambda_1,\ldots,\lambda_n\}$, we transform $\mathbf{x}(t)$ into the new coordinates $\mathbf{y}(t)$ by the transformation

$$\mathbf{y}(t)=\mathbf{S}^{-1}\mathbf{x}(t) \tag{4.52}$$

where $\mathbf{S}$ is the modal matrix for $\mathbf{A}$:

$$\mathbf{S}=(\mathbf{x}_1\vdots\cdots\vdots\mathbf{x}_n) \tag{4.53}$$

Then, by (4.18), (4.49) becomes

$$\begin{aligned}\mathbf{S}\dot{\mathbf{y}}(t)&=\mathbf{A}\mathbf{S}\mathbf{y}(t)+\mathbf{B}\mathbf{u}(t), \qquad \mathbf{S}\mathbf{y}(0)=\mathbf{x}_0\\ \dot{\mathbf{y}}(t)&=\mathbf{S}^{-1}\mathbf{A}\mathbf{S}\mathbf{y}(t)+\mathbf{S}^{-1}\mathbf{B}\mathbf{u}(t)\\ &=\mathbf{\Lambda}\mathbf{y}(t)+\mathbf{S}^{-1}\mathbf{B}\mathbf{u}(t), \qquad \mathbf{y}(0)=\mathbf{S}^{-1}\mathbf{x}_0\end{aligned} \tag{4.54}$$

Equation (4.54) is a set of n uncoupled first-order differential equations which can be solved independently. The eigenvectors (or modes) of $\mathbf{A}$ in a sense express natural relationships among the state variables [the elements of $\mathbf{x}(t)$] at each instant t. By using these eigenvectors as a basis, we eliminate the interactions-the new state variables [the elements of $\mathbf{y}(t)$] do not affect each other.

Example 3. Diagonalizing a State Equation The state equation for the armature controlled dc motor of (3.40) was obtained in Example 1 of Section 3.4. It is

$$\dot{\mathbf{x}}(t)=\begin{pmatrix}0 & 1\\ 0 & -1\end{pmatrix}\mathbf{x}(t)+\begin{pmatrix}0\\ 1\end{pmatrix}\mathbf{u}(t), \qquad \mathbf{x}(0)=\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} \tag{4.55}$$

The eigendata for the system matrix are

$$\lambda_1=0, \quad \lambda_2=-1 \qquad \mathbf{x}_1=\begin{pmatrix}1\\ 0\end{pmatrix} \quad \mathbf{x}_2=\begin{pmatrix}1\\ -1\end{pmatrix} \tag{4.56}$$

The modal matrix is its own inverse

$$\mathbf{S}^{-1}=\begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}=\mathbf{S} \tag{4.57}$$

The decoupled state equation is

$$\dot{\mathbf{y}}(t)=\begin{pmatrix}0 & 0\\ 0 & -1\end{pmatrix}\mathbf{y}(t)+\begin{pmatrix}1\\ -1\end{pmatrix}\mathbf{u}(t), \quad \mathbf{y}(0)=\begin{pmatrix}\alpha_1+\alpha_2\\ -\alpha_2\end{pmatrix} \tag{4.58}$$

Denote the new state variables [elements of $\mathbf{y}(t)$] by $\mathbf{g}_1(t)$ and $\mathbf{g}_2(t)$. We can solve independently for $\mathbf{g}_1$ and $\mathbf{g}_2$. On the other hand, we can use (3.79) with $\mathbf{x}$, $\mathbf{A}$, and $\mathbf{B}$ replaced by $\mathbf{y}$, $\mathbf{\Lambda}$, and $\mathbf{S}^{-1}\mathbf{B}$, respectively. By either approach the result is

$$\mathbf{y}(t) \triangleq \begin{pmatrix}\mathbf{g}_1(t)\\ \mathbf{g}_2(t)\end{pmatrix}=\int_0^t\begin{pmatrix}1\\ -e^{-(t-s)}\end{pmatrix}\mathbf{u}(s)\,ds+\begin{pmatrix}1 & 0\\ 0 & e^{-t}\end{pmatrix}\begin{pmatrix}\alpha_1+\alpha_2\\ -\alpha_2\end{pmatrix} \tag{4.59}$$

Then

$$\begin{aligned}\mathbf{x}(t)&=\mathbf{S}^{-1}\mathbf{y}(t)\\ &=\int_0^t\begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}\begin{pmatrix}1\\ -e^{-(t-s)}\end{pmatrix}\mathbf{u}(s)\,ds+\begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & e^{-t}\end{pmatrix}\begin{pmatrix}\alpha_1+\alpha_2\\ -\alpha_2\end{pmatrix}\\ &=\int_0^t\begin{pmatrix}1-e^{-(t-s)}\\ e^{-(t-s)}\end{pmatrix}\mathbf{u}(s)\,ds+\begin{pmatrix}\alpha_1+\alpha_2-\alpha_2e^{-t}\\ \alpha_2e^{-t}\end{pmatrix}\end{aligned} \tag{4.60}$$

Compare this result with (3.80).

Note that the modal matrix in Example 3 is the Vandermond matrix for the system. Whenever the system matrix is in companion matrix form and the poles of the system are distinct, the Vandermond matrix is a modal matrix; then the eigenvectors of $\mathbf{A}$ need not be calculated, but follow directly from the eigenvalues. See P&C 4.16.

Eigenvalue Problems and Partial Differential Equations

As we found in Example 10 of Section 4.1, not all differential operators have eigenvalues. This statement applies to both ordinary and partial differential operators. However, the most common analytical method for solving partial differential equations, separation of variables, generally introduces an eigenvalue problem even if the partial differential operator itself does not have eigenvalues. In point of fact, an analytical solution to a partial differential equation and its associated boundary conditions is usually obtainable only by summing eigenfunctions of a related differential operator. See Wylie [4.18]. On the other hand, some partial differential operators do have eigenvalues. One example is the Laplacian operator ∇^2, defined by

$$\nabla^2 \mathbf{f}(s,t) \triangleq \frac{\partial^2 \mathbf{f}(s,t)}{\partial s^2} + \frac{\partial^2 \mathbf{f}(s,t)}{\partial t^2} \tag{4.61}$$

together with the "many-point" boundary conditions

$$\mathbf{f}(s,t) = 0 \quad \text{on} \quad \Gamma \tag{4.62}$$

where Γ is a closed curve in the (s, t) plane,

Exercise 3. Let Γ be the boundary of the rectangle with sides at $s = 0$, $s = a$, $t = 0$, and $t = b$. Show (by separation of variables) or verify that the eigenvalues and eigenfunctions for ∇^2 together with the boundary conditions (4.62) are:

$$\lambda_{mk} = -\left(\frac{m\pi}{a}\right)^2 - \left(\frac{k\pi}{b}\right)^2$$

$$\mathbf{f}_{mk}(s,t) = \sin\left(\frac{m\pi s}{a}\right)\sin\left(\frac{k\pi t}{b}\right) \tag{4.63}$$

$$m = 1,2,\ldots \qquad k = 1,2,\ldots$$

Notice that $\lambda = 0$ is not an eigenvalue of (4.61)-(4.62). Therefore the operator must be invertible, and we can expect to find a unique solution to Poisson's equation, $\nabla^2 \mathbf{f} = \mathbf{u}$, together with the boundary conditions of Example 3.

4.4 Nondiagonalizable Operators and Jordan Form

Most useful linear transformations are diagonalizable. However, there occasionally arises in practical analysis a system which is best modeled by a nondiagonalizable transformation. Probably the most familiar example is

a dynamic system with a pair of nearly equal poles. We use such an example to introduce the concept of nondiagonalizability.

Suppose we wish to solve the undriven differential equation $(\mathbf{D} + 1)(\mathbf{D} + 1 + \epsilon)\mathbf{f} = 0$ with the boundary conditions $\mathbf{f}(0) = \alpha_1$ and $\mathbf{f}'(0) = \alpha_2$, where ϵ is a small constant. The solution is of the form

$$\mathbf{f}(t) = c_1 e^{-t} + c_2 e^{-(1+\epsilon)t} \tag{4.64}$$

Applying the boundary conditions, we find

$$\begin{pmatrix} 1 & 1 \\ -1 & -(1+\epsilon) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

Since ϵ is small, this equation is ill-conditioned; it is difficult to compute accurately the multipliers c_1 and c_2 (see Section 1.5). The difficulty occurs because the poles of the system (or roots of the characteristic equation) are nearly equal; the functions e^{-t} and $e^{-(1+\epsilon)t}$ are nearly indistinguishable (see Figure 4.4). We resolve this computational difficulty by replacing e^{-t} and $e^{-(1+\epsilon)t}$ by a more easily distinguishable pair of functions; (4.64) becomes

$$\begin{aligned} \mathbf{f}(t) &= e^{-t}(c_1 + c_2 e^{-\epsilon t}) \\ &= e^{-t}\left[c_1 + c_2\left(1 - \epsilon t + \frac{(\epsilon t)^2}{2!} - \cdots\right)\right] \\ &\approx e^{-t}[(c_1 + c_2) - c_2 \epsilon t] \\ &= d_1 e^{-t} + d_2 t e^{-t} \end{aligned} \tag{4.65}$$

where $d_1 = c_1 + c_2$ and $d_2 = -\epsilon c_2$. Since ϵ is small, the functions e^{-t} and te^{-t} span essentially the same space as e^{-t} and $e^{-(1+\epsilon)t}$; yet this new pair of functions is clearly distinguishable (Figure 4.4b). The "new" function

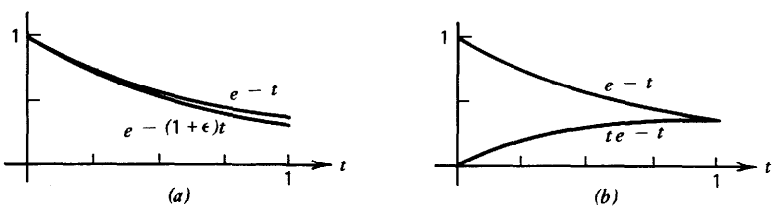


Figure 4.4. Alternative pairs of solutions to $(\mathbf{D} + 1)(\mathbf{D} + 1 + \epsilon)\mathbf{f} = \boldsymbol{\theta}$.

te^{-t} is essentially the difference between the two nearly equal exponentials. The boundary conditions now require

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} d_1 \\ d_2 \end{pmatrix}=\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

or $\mathbf{f}(t)=\alpha_1 e^{-t}+(\alpha_1+\alpha_2)te^{-t}$. We have eliminated the computational difficulty by equating the nearly equal poles of the system. When the roots of the characteristic equation are equal, (4.65) is the exact complementary function for the differential operator.

It is enlightening to view the differential system in state-space form. By writing the differential equation in the form $(\mathbf{D}^2 + (2 + \epsilon)\mathbf{D} + (1 + \epsilon)\mathbf{I})\mathbf{f} = \boldsymbol{\theta}$, we recognize from (3.63) that the state equation is

$$\mathbf{x}(t)=\begin{pmatrix} 0 & 1 \\ -(1+\epsilon) & -(2+\epsilon) \end{pmatrix}\mathbf{x}(t), \qquad \mathbf{x}(0)=\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

The nearly equal poles of the system appear now as nearly equal eigenvalues of the system matrix, $\lambda_1 = -1, \lambda_2 = -(1 + \epsilon)$. We know from P&C 4.16 that the modal matrix is the Vandermond matrix;

$$\mathbf{S}=\begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}=\begin{pmatrix} 1 & 1 \\ -1 & -(1+\epsilon) \end{pmatrix}$$

Since this matrix is ill-conditioned, we would have computational difficulty in finding $\mathbf{S}^{-1}$ in order to carry out a diagonalization of the system matrix $\mathbf{A}$. However, if we equate the eigenvalues (as we did above), the system matrix becomes

$$\mathbf{A}=\begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$$

which is not diagonalizable. Moreover, the earlier computational difficulty arose because we tried to diagonalize a "nearly nondiagonalizable" matrix.

The above example has demonstrated the need for dealing with nondiagonalizable transformations. In this section we explore nondiagonalizable finite-dimensional operators in detail. We discover that they can be represented by simple, nearly diagonal matrices which have the eigenvalues on the diagonal. Thus the conceptual clarity associated with the decoupling of system equations extends, to a great extent, to general linear operators.

To avoid heavy use of the cumbersome coordinate matrix notation, we focus throughout this section on matrices. However, we should keep in mind that an $n \times n$ matrix $\mathbf{A}$ which arises in a system model usually

represents an underlying linear operator $\mathbf{T}$. The eigenvectors of $\mathbf{A}$ are the coordinates of the eigenvectors of $\mathbf{T}$. Thus when we use a similarity transformation, $\mathbf{S}^{-1}\mathbf{AS}$, to convert $\mathbf{A}$ to a new form, we are merely changing the coordinate system for the space on which $\mathbf{T}$ operates.

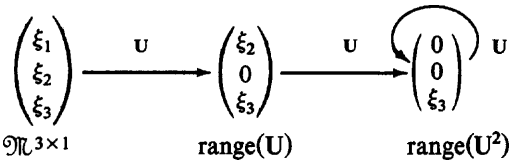
Generalized Nullspace and Range

Unlike a scalar, a linear operator $\mathbf{U}$ is generally neither invertible nor zero. It lies in a "gray region" in between; $\mathbf{U}$ takes some vectors to zero (acting like the zero operator); others it does not take to zero (thereby acting invertible). Perhaps even more significant is the fact that the nullspace and range of $\mathbf{U}$ may overlap. The second and higher operations by $\mathbf{U}$ may annihilate additional vectors. In some ways, the subspace annihilated by higher powers of $\mathbf{U}$ is more characteristic of the operator than is nullspace $(\mathbf{U})$.

Example 1. Overlapping Nullspace and Range. Define the operator $\mathbf{U}$ on $\mathfrak{R}^{3\times 1}$ by $\mathbf{Ux} \triangleq \mathbf{Bx}$, where

$$\mathbf{B} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then $\mathbf{U}$ has the following effect on a general vector in $\mathfrak{R}^{3\times 1}$:



The vectors annihilated by various powers of $\mathbf{U}$ are described by

$$\text{nullspace}(\mathbf{U}) = \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\right\}, \qquad \text{nullspace}(\mathbf{U}^2) = \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$$

The nullspace and range of $\mathbf{U}^k$ for $k > 2$ are the same as the nullspace and range of $\mathbf{U}^2$.

Definition. The **generalized nullspace** $\mathfrak{N}_g(\mathbf{U})$ of a linear operator $\mathbf{U}$ acting on an n-dimensional space $\mathcal{V}$ is the largest subspace of $\mathcal{V}$ annihilated by powers of $\mathbf{U}$. Since $\mathcal{V}$ is finite dimensional, the annihilation must terminate. Let q be that power of $\mathbf{U}$ required for maximum annihilation.

We call q the **index of annihilation** for $\mathbf{U}$. Then $\mathcal{N}_g(\mathbf{U}) = \text{nullspace}(\mathbf{U}^q)$. The generalized range $\mathcal{R}_g(\mathbf{U})$ of the operator $\mathbf{U}$ is defined by $\mathcal{R}_g(\mathbf{U}) = \text{range}(\mathbf{U}^q)$. Since multiplication by a square matrix is a linear operator, we speak also of the generalized nullspace and generalized range of square matrices.

In Example 1, the index of annihilation is $q = 2$. The generalized range and generalized nullspace are

$$\mathcal{R}_g(\mathbf{U}) = \text{span}\left\{\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}, \qquad \mathcal{N}_g(\mathbf{U}) = \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$$

Notice that $\mathcal{V}$ is the direct sum of the generalized range and the generalized nullspace of $\mathbf{U}$. It is proved in Theorem 1 of Appendix 3 that any linear operator on an n-dimensional space splits the space in this manner. It is further shown in that theorem that both $\mathcal{N}_g(\mathbf{U})$ and $\mathcal{R}_g(\mathbf{U})$ are invariant under $\mathbf{U}$, and that $\mathbf{U}$ acts like a reduced invertible operator on the generalized range of $\mathbf{U}$. These facts are verified by Example 1. An operator (or a square matrix) some power of which is zero is said to be **nilpotent;** $\mathbf{U}$ acts like a reduced nilpotent operator on the generalized nullspace of $\mathbf{U}$.

Exercise 1. Let $\mathbf{U}$ be the operator of Example 1. Define $\mathbf{U}_1$: $\mathcal{R}_g(\mathbf{U}) \to \mathcal{R}_g(\mathbf{U})$ by $\mathbf{U}_1\mathbf{x} \triangleq \mathbf{U}\mathbf{x}$ for all $\mathbf{x}$ in $\mathcal{R}_g(\mathbf{U})$; define $\mathbf{U}_2$: $\mathcal{N}_g(\mathbf{U}) \to \mathcal{N}_g(\mathbf{U})$ by $\mathbf{U}_2\mathbf{x} \triangleq \mathbf{U}\mathbf{x}$ for all $\mathbf{x}$ in $\mathcal{N}_g(\mathbf{U})$. Pick as bases for $\mathcal{R}_g(\mathbf{U})$, $\mathcal{N}_g(\mathbf{U})$, and $\mathcal{N}^{3\times 1}$ the standard bases

$$\mathcal{X}_1 = \left\{\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}, \qquad \mathcal{X}_2 = \left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\}, \qquad \text{and} \qquad \mathcal{X} = \{\mathcal{X}_1, \mathcal{X}_2\}$$

respectively. Show that

$$[\mathbf{U}]_{\mathcal{X}\mathcal{X}} = \begin{pmatrix}[\mathbf{U}_1]_{\mathcal{X}_1\mathcal{X}_1} & \mathbf{O}\\ \mathbf{O} & [\mathbf{U}_2]_{\mathcal{X}_2\mathcal{X}_2}\end{pmatrix} = \left(\begin{array}{c:c:c}1 & 0 & 0\\ \hdashline 0 & 0 & 1\\ \hdashline 0 & 0 & 0\end{array}\right)$$

What are the characteristics of $\mathbf{U}_1$ and $\mathbf{U}_2$? Why is the matrix of $\mathbf{U}$ in "block-diagonal" form? (See P&C 4.3.)

Generalized Eigendata

The characteristic polynomial of an $n \times n$ matrix $\mathbf{A}$ can be expressed in the form

$$c(\lambda) = \det(\lambda\mathbf{I} - \mathbf{A}) = (\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_p)^{m_p} \tag{4.66}$$

where p is the number of distinct eigenvalues, and $m_1 + \cdots + m_p = n$. We call m_i the **algebraic multiplicity** of λ_i. The eigenspace for λ_i is nullspace $(\mathbf{A} - \lambda_i\mathbf{I})$. The dimension of this eigenspace, the nullity of $(\mathbf{A} - \lambda_i\mathbf{I})$, we denote by k_i. We call k_i the **geometric multiplicity** of λ_i; it is the number of independent eigenvectors of $\mathbf{A}$ for λ_i. If the geometric multiplicity equals the algebraic multiplicity for each eigenvalue, it is reasonable to believe that there is a basis for $\mathcal{M}^{n \times 1}$ composed of eigenvectors for $\mathbf{A}$, and that $\mathbf{A}$ is diagonalizable.

If λ_i is deficient in eigenvectors $(k_i < m_i)$, we say $\mathbf{A}$ is **defective** at λ_i. If $\mathbf{A}$ has any defective eigenvalues, we must pick noneigenvectors to complete the basis. We seek $(m_i - k_i)$ additional independent vectors from the subspace associated with λ_i—from the generalized nullspace of $(\mathbf{A} - \lambda_i\mathbf{I})$. Define

$$\begin{aligned} \mathcal{W}_i &\triangleq \text{generalized nullspace of } (\mathbf{A} - \lambda_i\mathbf{I}) \\ &= \text{nullspace}(\mathbf{A} - \lambda_i\mathbf{I})^{q_i} \end{aligned} \tag{4.67}$$

where q_i is the **index of annihilation** for $(\mathbf{A} - \lambda_i\mathbf{I})$. It is shown in Theorem 2 of Appendix 3 that

$$\dim(\mathcal{W}_i) = m_i \tag{4.68}$$

We will think of all vectors in the generalized nullspace of $(\mathbf{A} - \lambda_i\mathbf{I})$ as generalized eigenvectors of $\mathbf{A}$ for λ_i. Specifically, we call $\mathbf{x}_r$ a **generalized eigenvector of rank r** for λ_i if

$$\begin{aligned} (\mathbf{A} - \lambda_i\mathbf{I})^r \mathbf{x}_r &= \boldsymbol{\theta} \\ (\mathbf{A} - \lambda_i\mathbf{I})^{r-1} \mathbf{x}_r &\neq \boldsymbol{\theta} \end{aligned} \tag{4.69}$$

If $\mathbf{x}_r$ is a generalized eigenvector of rank r for λ_i, then $(\mathbf{A} - \lambda_i\mathbf{I})\mathbf{x}_r$ is a

generalized eigenvector of rank r − 1; for (4.69) can be rewritten

$$(\mathbf{A}-\lambda_i\mathbf{I})^{r-1}(\mathbf{A}-\lambda_i\mathbf{I})\mathbf{x}_r=\boldsymbol{\theta}$$

$$(\mathbf{A}-\lambda_i\mathbf{I})^{r-2}(\mathbf{A}-\lambda_i\mathbf{I})\mathbf{x}_r\neq\boldsymbol{\theta}$$

Thus each vector in $\mathcal{W}_i$ is a member of some chain of generalized eigenvectors generated by repeated multiplication with $(\mathbf{A}-\lambda_i\mathbf{I})$; the last member of each chain is a true eigenvector (of rank 1). We think of $\mathcal{W}_i$ as the **generalized eigenspace** for λ_i; $\mathcal{W}_i$ contains precisely the m_i independent vectors associated with λ_i that we intuitively expect in a basis for $\mathcal{R}^{n\times 1}$.

In Theorem 3 of Appendix 3 we show that

$$\mathcal{R}^{n\times 1}=\mathcal{W}_1\oplus\cdots\oplus\mathcal{W}_p \tag{4.70}$$

Therefore, any bases which we pick for $\{\mathcal{W}_i\}$ combine to form a basis for $\mathcal{R}^{n\times 1}$. Any basis for $\mathcal{W}_i$ consists in m_i generalized eigenvectors. Furthermore, k_i of these m_i generalized eigenvectors can be true eigenvectors for λ_i.

Jordan Canonical Form

If $\mathbf{A}$ is diagonalizable, we can diagonalize it by the similarity transformation $\mathbf{S}^{-1}\mathbf{AS}$, where the columns of $\mathbf{S}$ are a basis for $\mathcal{R}^{n\times 1}$ composed of eigenvectors of $\mathbf{A}$. Suppose $\mathbf{A}$ is not diagonalizable. What form can we expect for the matrix $\mathbf{S}^{-1}\mathbf{AS}$ if the columns of $\mathbf{S}$ are a basis of generalized eigenvectors of $\mathbf{A}$? It depends on the way we pick the bases for the subspaces $\{\mathcal{W}_i\}$. We demonstrate, by example, a way to pick the bases which results in as simple a form for the matrix $\mathbf{S}^{-1}\mathbf{AS}$ as we can possibly get in the presence of multiple eigenvalues. In order that the form be as nearly diagonal as possible, we include, of course, the true eigenvectors for λ_i in the basis for $\mathcal{W}_i$.

Let

$$\mathbf{A}=\left(\begin{array}{ccc:ccc}2&3&0&&&\\0&2&4&&\mathbf{0}&\\0&0&2&&&\\\hdashline&&&2&-1&\\&\mathbf{0}&&0&2&\\&&&&&3\end{array}\right) \tag{4.71}$$

Then $c(\lambda)=(\lambda-2)^5(\lambda-3)$, or $p=2$, $\lambda_1=2$, $m_1=5$, $\lambda_2=3$, and $m_2=1$. Also,

$$(\mathbf{A}-2\mathbf{I})=\begin{pmatrix} 0 & 3 & 0 & & & \\ 0 & 0 & 4 & & \mathbf{0} & \\ 0 & 0 & 0 & & & \\ & & & 0 & -1 & \\ & \mathbf{0} & & 0 & 0 & \\ & & & & & 1 \end{pmatrix}$$

$$(\mathbf{A}-2\mathbf{I})^2=\begin{pmatrix} 0 & 0 & 12 & & & \\ 0 & 0 & 0 & & \mathbf{0} & \\ 0 & 0 & 0 & & & \\ & & & 0 & 0 & \\ & \mathbf{0} & & 0 & 0 & \\ & & & & & 1 \end{pmatrix}$$

$$(\mathbf{A}-2\mathbf{I})^3=\begin{pmatrix} 0 & 0 & 0 & & & \\ 0 & 0 & 0 & & \mathbf{0} & \\ 0 & 0 & 0 & & & \\ & & & 0 & 0 & \\ & \mathbf{0} & & 0 & 0 & \\ & & & & & 1 \end{pmatrix},$$

$$(\mathbf{A}-3\mathbf{I})=\begin{pmatrix} -1 & 3 & 0 & & & \\ 0 & -1 & 4 & & \mathbf{0} & \\ 0 & 0 & -1 & & & \\ & & & -1 & -1 & \\ & \mathbf{0} & & 0 & -1 & \\ & & & & & 0 \end{pmatrix}$$

It is apparent that

$$\begin{aligned} &\text{nullity}(\mathbf{A}-2\mathbf{I})=2=k_1 \\ &\text{nullity}(\mathbf{A}-2\mathbf{I})^2=4 \\ &\text{nullity}(\mathbf{A}-2\mathbf{I})^3=5 \\ &\text{nullity}(\mathbf{A}-3\mathbf{I})=1=k_2 \end{aligned} \tag{4.72}$$

The indices of annihilation for $(\mathbf{A}-\lambda_1\mathbf{I})$ and $(\mathbf{A}-\lambda_2\mathbf{I})$, respectively, are $q_1=3$ and $q_2=1$. The five-dimensional subspace $\mathfrak{M}_1$, the generalized

eigenspace for λ_1, consists in vectors of the form $(\xi_1\ \xi_2\ \xi_3\ \xi_4\ \xi_5\ 0)^T$; vectors in $\mathcal{W}_2$, the generalized eigenspace for λ_2, are of the form $(0\ 0\ 0\ 0\ 0\ \xi_6)^T$. [Note that (4.68) and (4.70) are verified in this example.]

Any eigenvector for $\lambda = 3$ will form a basis $\mathcal{C}_2$ for $\mathcal{W}_2$. Clearly, a basis $\mathcal{C}_1$ for $\mathcal{W}_1$ must contain five vectors. Since there are only two independent true eigenvectors (of rank 1), three of the vectors in the basis must be generalized eigenvectors of rank greater than 1.

Assume we pick a basis which reflects the nullity structure of (4.72); that is, we pick two generalized eigenvectors of rank 1 for $\lambda = 2$, two of rank 2 for $\lambda = 2$, one of rank 3 for $\lambda = 2$, and one of rank 1 for $\lambda = 3$. Also assume we pick the basis vectors in chains; that is, if $\mathbf{x}$ is a vector of rank 3 for $\lambda = 2$, and $\mathbf{x}$ is in the basis, $(\mathbf{A} - 2\mathbf{I})\mathbf{x}$ and $(\mathbf{A} - 2\mathbf{I})^2\mathbf{x}$ will also be in the basis. We express both the nullity structure and chain structure by the following subscript notation:

$$\begin{array}{lccc} & \text{rank 1} & \text{rank 2} & \text{rank 3} & \\ & \downarrow & \downarrow & \downarrow & \\ \mathcal{C}_1 = & \left\{ \begin{array}{l} \mathbf{x}_1 \\ \mathbf{x}_2 \end{array} \right. & \begin{array}{l} \mathbf{x}_{12} \\ \mathbf{x}_{22} \end{array} & \mathbf{x}_{13} & \begin{array}{l} \leftarrow \text{chain 1} \\ \leftarrow \text{chain 2} \end{array} \\ \mathcal{C}_2 = & \{\mathbf{x}_3 & & & \leftarrow \text{chain 3} \end{array} \tag{4.73}$$

This nullity and chain structure is expressed mathematically by the following equations:

$$\begin{aligned} (\mathbf{A}-2\mathbf{I})\mathbf{x}_{13} &= \mathbf{x}_{12} \\ (\mathbf{A}-2\mathbf{I})\mathbf{x}_{12} &= \mathbf{x}_1 \\ (\mathbf{A}-2\mathbf{I})\mathbf{x}_1 &= \boldsymbol{\theta} \\ (\mathbf{A}-2\mathbf{I})\mathbf{x}_{22} &= \mathbf{x}_2 \\ (\mathbf{A}-2\mathbf{I})\mathbf{x}_2 &= \boldsymbol{\theta} \\ (\mathbf{A}-3\mathbf{I})\mathbf{x}_3 &= \boldsymbol{\theta} \end{aligned} \tag{4.74}$$

We propose the union of the sets $\mathcal{C}_i$ as a basis, denoted $\mathcal{C}$, for $\mathcal{M}^{6\times 1}$. It can be shown that a set of vectors of this form can be constructed and is a basis for $\mathcal{M}^{6\times 1}$ (see Friedman [4.7]). Using the basis $\mathcal{C}$, we form the change of coordinates matrix as in (4.17):

$$\mathbf{S} = (\mathbf{x}_1 \vdots \mathbf{x}_{12} \vdots \mathbf{x}_{13} \vdots \mathbf{x}_2 \vdots \mathbf{x}_{22} \vdots \mathbf{x}_3) \tag{4.75}$$

As in (4.18), this change of coordinates transforms $\mathbf{A}$ into the matrix $\boldsymbol{\Lambda} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$. Recasting this similarity relation into the form $\mathbf{A}\mathbf{S} = \mathbf{S}\boldsymbol{\Lambda}$, we

recognize that

$$\begin{aligned}\mathbf{AS} &= \mathbf{A}(\mathbf{x}_1 \vdots \mathbf{x}_{12} \vdots \mathbf{x}_{13} \vdots \mathbf{x}_2 \vdots \mathbf{x}_{22} \vdots \mathbf{x}_3) \\ &= (2\mathbf{x}_1 \vdots 2\mathbf{x}_{12}+\mathbf{x}_1 \vdots 2\mathbf{x}_{13}+\mathbf{x}_{12} \vdots 2\mathbf{x}_2 \vdots 2\mathbf{x}_{22}+\mathbf{x}_2 \vdots 3\mathbf{x}_3) \\ &= (\mathbf{x}_1 \vdots \mathbf{x}_{12} \vdots \mathbf{x}_{13} \vdots \mathbf{x}_2 \vdots \mathbf{x}_{22} \vdots \mathbf{x}_3) \begin{pmatrix} 2 & 1 & 0 & & & \\ 0 & 2 & 1 & & \mathbf{0} & \\ 0 & 0 & 2 & & & \\ & & & 2 & 1 & \\ & \mathbf{0} & & 0 & 2 & \\ & & & & & 3 \end{pmatrix} \\ &= \mathbf{S}\Lambda \end{aligned} \tag{4.76}$$

The form of Λ is as simple and as nearly diagonal a representation of $\mathbf{A}$ as we can expect to obtain. The eigenvalues are on the diagonal. The off-diagonal l's specify in a simple manner the "rank structure" or "chain structure" inherent in $\mathbf{A}$.

It is apparent that whenever the columns of $\mathbf{S}$ form a basis for $\mathfrak{M}^{n\times 1}$ composed of generalized eigenvectors of $\mathbf{A}$, and these basis vectors consist in chains of vectors which express the nullity structure of $\mathbf{A}$ as in (4.73)-(4.74), then $\mathbf{S}^{-1}\mathbf{AS}$ will be of the simple form demonstrated in (4.76). It will consist in a series of blocks on the diagonal; each block will be of the form

$$\begin{pmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & & \lambda_i & 1 \\ 0 & \cdots & & 0 & \lambda_i \end{pmatrix}$$

By analogy with (4.16)-(4.18) in our discussion of diagonalization, we call $\mathbf{S}$ the **modal matrix for A**. We also call the near-diagonal matrix Λ the **spectral matrix for A** (or for the underlying transformation **T**). The spectral matrix is also referred to as the **Jordan canonical form of A**. Each square block consisting in a repeated eigenvalue on the diagonal and an unbroken string of l's above the diagonal is called a **Jordan block.** There is one Jordan block in A for each chain of generalized eigenvectors in the basis. The dimension of each block equals the length of the corresponding chain. Thus we can tell from the nullity structure (4.71) alone, the form of the basis (4.73) and the precise form of Λ (4.76). Observe that the Jordan form is not unique. We can choose arbitrarily the order of the Jordan blocks by choosing the order in which we place the generalized eigenvectors in the basis.

Example 2. Nullities Determine the Jordan Form Suppose $\mathbf{A}$ is a 9×9 matrix for which

$$c(\lambda) = \det(\lambda\mathbf{I} - \mathbf{A}) = (\lambda - \lambda_1)^6(\lambda - \lambda_2)^2(\lambda - \lambda_3)$$

$$\text{nullity}(\mathbf{A} - \lambda_1\mathbf{I}) = 3$$

$$\text{nullity}(\mathbf{A} - \lambda_1\mathbf{I})^2 = 5$$

$$\text{nullity}(\mathbf{A} - \lambda_1\mathbf{I})^3 = 6$$

$$\text{nullity}(\mathbf{A} - \lambda_2\mathbf{I}) = 1$$

$$\text{nullity}(\mathbf{A} - \lambda_2\mathbf{I})^2 = 2$$

$$\text{nullity}(\mathbf{A} - \lambda_3\mathbf{I}) = 1$$

From (4.68), the factored characteristic polynomial, and the nullities stated above, we know that

$$m_1 = \dim(\mathcal{W}_1) = 6, \qquad k_1 = 3$$

$$m_2 = \dim(\mathcal{W}_2) = 2, \qquad k_2 = 1$$

$$m_3 = \dim(\mathcal{W}_3) = 1, \qquad k_3 = 1$$

It follows that $q_1 = 3$, $q_2 = 2$, and $q_3 = 1$; higher powers than $(\mathbf{A} - \lambda_i\mathbf{I})^{q_i}$ do not have higher nullities. The form of the basis of generalized eigenvectors of $\mathbf{A}$ which will convert $\mathbf{A}$ to its Jordan form is

$$\mathcal{Q} = \begin{cases} \mathcal{Q}_1 = & \begin{cases} \mathbf{x}_1 & \mathbf{x}_{12} & \mathbf{x}_{13} \\ \mathbf{x}_2 & \mathbf{x}_{22} & \\ \mathbf{x}_3 & & \end{cases} \\ \mathcal{Q}_2 = & \{\mathbf{x}_4 \quad \mathbf{x}_{42} \\ \mathcal{Q}_3 = & \{\mathbf{x}_5 \end{cases}$$

The Jordan form of $\mathbf{A}$ is

$$\mathbf{\Lambda} = \begin{pmatrix} \lambda_1 & 1 & 0 & & & & & & \\ 0 & \lambda_1 & 1 & & & & & & \\ 0 & 0 & \lambda_1 & & & & & & \\ & & & \lambda_1 & 1 & & & & \\ & & & 0 & \lambda_1 & & & & \\ & & & & & \lambda_1 & & & \\ & & & & & & \lambda_2 & 1 & \\ & & & & & & 0 & \lambda_2 & \\ & & & & & & & & \lambda_3 \end{pmatrix}$$

Bases of Generalized Eigenvectors

We now generate a specific basis for $\mathfrak{M}^{6\times 1}$ which is composed of generalized eigenvectors of the matrix $\mathbf{A}$ of (4.71). That is, we find a basis of the form (4.73) by satisfying (4.74). We use (4.69) to find the highest rank vector in each chain. We first seek the vector $\mathbf{x}_{13}$ of (4.73). All five of the basis vectors in $\mathcal{C}_1$ satisfy $(\mathbf{A}-2\mathbf{I})^3\mathbf{x} = \boldsymbol{\theta}$. But only $\mathbf{x}_{13}$ satisfies, in addition, $(\mathbf{A}-2\mathbf{I})^2\mathbf{x} \neq \boldsymbol{\theta}$. Therefore, we let $\mathbf{x}_{13} = (c_1\ c_2\ c_3\ c_4\ c_5\ 0)^T$, the general solution to $(\mathbf{A}-2\mathbf{I})^3\mathbf{x} = \boldsymbol{\theta}$. Then

$$(\mathbf{A}-2\mathbf{I})^2\mathbf{x}_{13} = \begin{pmatrix} 12c_3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \neq \boldsymbol{\theta} \tag{4.77}$$

or $c_3 \neq 0$. Thus any vector in $\mathfrak{M}^{6\times 1}$ which has a zero sixth element and a nonzero third element is a generalized eigenvector of rank 3 for $\lambda = 2$. We have a lot of freedom in picking $\mathbf{x}_{13}$. Arbitrarily, we let $c_3 = 1$, and $c_1 = c_2 = c_4 = c_5 = 0$. Then

$$\mathbf{x}_{13} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{x}_{12} = (\mathbf{A}-2\mathbf{I})\mathbf{x}_{13} = \begin{pmatrix} 0 \\ 4 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{x}_1 = (\mathbf{A}-2\mathbf{I})\mathbf{x}_{12} = \begin{pmatrix} 12 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{4.78}$$

Notice that in (4.77) we looked at the eigenvector, $\mathbf{x}_1 = (\mathbf{A}-2\mathbf{I})^2\mathbf{x}_{13}$, at the end of the chain in order to determine the vector $\mathbf{x}_{13}$ at the head of the chain.

To find the remaining vectors of $\mathcal{C}_1$, we look for the vector $\mathbf{x}_{22}$ at the head of the second chain. By (4.69), all vectors $(d_1\ d_2\ d_3\ d_4\ d_5\ d_6)^T$ of rank 2 or less satisfy

$$(\mathbf{A}-2\mathbf{I})^2 \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{pmatrix} = \begin{pmatrix} 12d_3 \\ 0 \\ 0 \\ 0 \\ 0 \\ d_6 \end{pmatrix} = \boldsymbol{\theta}$$

or $d_6 = d_3 = 0$. The vectors which are precisely of rank 2 also satisfy

$$(\mathbf{A}-2\mathbf{I})\begin{pmatrix} d_1 \\ d_2 \\ 0 \\ d_4 \\ d_5 \\ 0 \end{pmatrix} = \begin{pmatrix} 3d_2 \\ 0 \\ 0 \\ -d_5 \\ 0 \\ 0 \end{pmatrix} \neq \boldsymbol{\theta} \tag{4.79}$$

Again we are looking at the eigenvector at the end of the chain as we pick the constants. We must pick d_2 and d_5, not both zero, such that $\mathbf{x}_2$ is independent of the eigenvector $\mathbf{x}_1$ selected above (i.e., $d_2 = 1$, $d_5 = 0$ will not do). Arbitrarily, we let $d_5 = 1$, $d_1 = d_2 = d_4 = 0$; d_3 is already zero. Thus

$$\mathbf{x}_{22} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{x}_2 = (\mathbf{A}-2\mathbf{I})\mathbf{x}_{22} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} \tag{4.80}$$

The five vectors of (4.78) and (4.80) satisfy (4.73), and they are a basis for $\mathfrak{W}_1$. The equation $(\mathbf{A} - 3\mathbf{I})\mathbf{x} = 0$ determines the form of eigenvectors for $\lambda = 3$: $\mathbf{x} = (0\ 0\ 0\ 0\ 0\ b_6)^T$. We arbitrarily let $b_6 = 1$ to get

$$\mathbf{x}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

a basis for $\mathfrak{W}_2$. By (4.76), this basis of generalized eigenvectors generates the modal matrix $\mathbf{S}$:

$$\mathbf{S} = \begin{pmatrix} 12 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{S}^{-1} = \begin{pmatrix} \frac{1}{12} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The spectral matrix is

$$\mathbf{\Lambda} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \begin{pmatrix} 2 & 1 & 0 & & & \\ 0 & 2 & 1 & & \mathbf{0} & \\ 0 & 0 & 2 & & & \\ & & & 2 & 1 & \\ & \mathbf{0} & & 0 & 2 & \\ & & & & & 3 \end{pmatrix} \tag{4.81}$$

as we concluded earlier in (4.76).

Clearly, the chains of generalized eigenvectors which make up a basis are not unique. In fact, many different chains end in the same true eigenvector. It can be shown that any set of chains which possesses the structure of (4.73)-(4.74) will constitute a basis for $\mathfrak{M}^{6\times 1}$ if the eigenvectors at the ends of the chains are independent. Because of this fact, we might be led to find the true eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$ first, and then find the rest of the basis by "backing up" each chain. This approach need not work. The vectors $\mathbf{x}_1 = (1\ 0\ 0\ 1\ 0\ 0)^T$ and $\mathbf{z}_2 = (1\ 0\ 0\ -1\ 0\ 0)^T$ are independent eigenvectors of $\mathbf{A}$. However, they are both of the form (4.79) of eigenvectors at the end of chains of length 2. Neither is of the form (4.77) of an eigenvector at the end of a chain of length 3. Although these two eigenvectors can be used as part of a basis for $\mathfrak{M}^{6\times 1}$, the basis cannot be of the form (4.73).

Exercise 2. Attempt to determine a basis for $\mathfrak{M}^{6\times 1}$ which is of the form (4.73) and yet includes the eigenvectors $\mathbf{x}_1 = (1\ 0\ 0\ 1\ 0\ 0)^T$ and $\mathbf{x}_2 = (1\ 0\ 0\ -1\ 0\ 0)^T$.

Procedure for Construction of the Basis

We summarize the procedure for generating a basis of generalized eigenvectors. Suppose the $n \times n$ matrix $\mathbf{A}$ has the characteristic polynomial (4.66). Associated with the eigenvalue λ_i is an m_i-dimensional subspace $\mathfrak{W}_i$ (Theorem 1, Appendix 3). This subspace contains k_i independent eigenvectors for λ_i. Assume the basis vectors are ordered by decreasing chain length, with each chain ordered by increasing rank. We denote this basis for $\mathfrak{W}_i$ by

$$\mathcal{Q}_i = \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_{12} & \cdots & & \mathbf{x}_{1q_i} \\ \mathbf{x}_2 & \mathbf{x}_{22} & \cdots & \mathbf{x}_{2l_2} & \\ \vdots & & & & \\ \mathbf{x}_{k_i} & \mathbf{x}_{k_i 2} & \cdots & \mathbf{x}_{k_i l_{k_i}} & \end{pmatrix} \begin{matrix} \leftarrow \text{longest chain} \\ \\ \\ \leftarrow \text{shortest chain} \end{matrix} \tag{4.82}$$

$$\uparrow \text{ rank 1} \qquad \uparrow \text{ rank } q_i$$

where l_j is the length of the jth chain for λ_i and q_i is the index of annihilation for $(\mathbf{A}-\lambda_i\mathbf{I})$; thus q_i is the length of the longest chain. The nullities of various powers of $(\mathbf{A}-\lambda_i\mathbf{I})$ determine the structure of (4.82) just as (4.73) is determined by (4.72). The procedure for construction of the basis $\mathcal{Q}_i$ is as follows:

1. Determine the form of vectors of rank q_i or less by solving $(\mathbf{A}-\lambda_i\mathbf{I})^{q_i}\mathbf{x}=\boldsymbol{\theta}$.
2. Observe the true eigenvectors $(\mathbf{A}-\lambda_i\mathbf{I})^{q_i-1}\mathbf{x}$; choose from the vectors found in (1) a total of $(\text{nullity}(\mathbf{A}-\lambda_i\mathbf{I})^{q_i}-\text{nullity}(\mathbf{A}-\lambda_i\mathbf{I})^{q_i-1})$ vectors which lead to independent eigenvectors. These vectors are of rank q_i, and are the highest rank generalized eigenvectors in their respective chains.
3. Multiply each vector chosen in (2) by $(\mathbf{A}-\lambda_i\mathbf{I})$, thereby obtaining a set of generalized eigenvectors of $\text{rank}(q_i-1)$, which is part of the set of basis vectors of rank (q_i-1).
4. Complete the set of basis vectors of $\text{rank}(q_i-1)$ by adding enough vectors of rank (q_i-1) to obtain a total of $(\text{nullity}(\mathbf{A}-\lambda_i\mathbf{I})^{q_i-1}-\text{nullity}(\mathbf{A}-\lambda_i\mathbf{I})^{q_i-2})$ vectors which lead to independent eigenvectors. This step requires work equivalent to steps I and 2 with q_i replaced by (q_i-1). The vectors which are added are highest rank vectors in new chains.
5. Repeat steps 3 and 4 for lower ranks until a set of k_i eigenvectors is obtained.

Because $\mathcal{M}^{n\times 1}=\mathcal{W}_1\oplus\cdots\oplus\mathcal{W}_p$, we can obtain a basis $\mathcal{Q}$ for $\mathcal{M}^{n\times 1}$ consisting of generalized eigenvectors of $\mathbf{A}$ by merely combining the bases for the subspaces $\mathcal{W}_i$:

$$\mathcal{Q}=\{\mathcal{Q}_1,\ldots,\mathcal{Q}_p\}$$

Proceeding as in the example of (4.71), we can use the basis $\mathcal{Q}$ to convert $\mathbf{A}$ to its nearly diagonal Jordan canonical form $\boldsymbol{\Lambda}$.

Example 3. A Basis of Generalized Eigenvectors. Let

$$\mathbf{A}=\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0\\ -1 & 4 & 0 & 0 & 0 & 0\\ -1 & 1 & 2 & 1 & 0 & 0\\ -1 & 1 & -1 & 4 & 0 & 0\\ -1 & 1 & -1 & 1 & 3 & 0\\ -1 & 1 & -1 & 1 & 1 & 2 \end{pmatrix}$$

The process of finding and factoring the characteristic polynomial is complicated. We merely state it in factored form:

$$c(\lambda)=\det(\lambda\mathbf{I}-\mathbf{A})=(\lambda-3)^5(\lambda-2)$$

Therefore, $\lambda_1 = 3$, $m_1 = 5$, $\lambda_2 = 2$, and $m_2 = 1$. Furthermore,

$$(\mathbf{A}-3\mathbf{I}) = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1-1 & 1 & 0 & 0 & \\ -1 & 1-1 & 1 & 0 & 0 & \\ -1 & 1-1 & 1 & 0 & 0 & \\ -1 & 1-1 & 1 & 1-1 & & \end{pmatrix}, \quad (\mathbf{A}-3\mathbf{I})^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0-1 & 1 & \end{pmatrix}$$

Clearly, $\text{nullity}(\mathbf{A}-3\mathbf{I}) = 3$ and $\text{nullity}(\mathbf{A}-3\mathbf{I})^2 = 5 = m_1$. It is also apparent that $\text{nullity}(\mathbf{A}-3\mathbf{I})^3 = 5$. Thus $k_1 = 3$, $q_1 = 2$, and $\dim(\mathcal{W}_1) = 5$. Moreover,

$$(\mathbf{A}-2\mathbf{I}) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 1-1 & 2 & 0 & 0 & \\ -1 & 1-1 & 1 & 1 & 0 & \\ -1 & 1-1 & 1 & 1 & 0 & \end{pmatrix}$$

and $\text{nullity}(\mathbf{A}-2\mathbf{I}) = 1$. As a result, $k_2 = 1$, $q_2 = 1$, and $\dim(\mathcal{W}_2) = 1$. [Note that $\dim(\mathcal{W}_1) + \dim(\mathcal{W}_2) = \dim(\mathcal{R}^{6\times 1})$.] From the nullity information above, we know that the Jordan form of $\mathbf{A}$ is

$$\Lambda = \begin{pmatrix} 3 & 1 & & & & \\ 0 & 3 & & & & \\ & & 3 & 1 & & \\ & & 0 & 3 & & \\ & & & & 3 & \\ & & & & & 2 \end{pmatrix}$$

We find a basis $\mathcal{Q}$ for $\mathcal{R}^{6\times 1}$ consisting in chains of generalized eigenvectors with the following structure:

$$\mathcal{Q} = \begin{cases} \mathcal{Q}_1 = \begin{cases} \mathbf{x}_1 & \mathbf{x}_{12} \\ \mathbf{x}_2 & \mathbf{x}_{22} \\ \mathbf{x}_3 & \end{cases} \\ \mathcal{Q}_2 = \{\mathbf{x}_4 \end{cases}$$

We first seek $\mathbf{x}_{12}$ and $\mathbf{x}_{22}$, the vectors at the heads of the two longest chains. All generalized eigenvectors for $\lambda = 3$ satisfy $(\mathbf{A}-3\mathbf{I})^2\mathbf{x} = 8$. The solutions to this equation are of the form $\mathbf{x} = (c_1\ c_2\ c_3\ c_4\ c_5\ c_5)^T$. The vectors of rank 2 also satisfy

$$(\mathbf{A}-3\mathbf{I}) \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_5 \end{pmatrix} = (c_2-c_1) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + (c_4-c_3) \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

We are looking at the true eigenvector at the end of the most general chain of

length 2. We must select two different sets of constants in order to specify both $\mathbf{x}_{12}$ and $\mathbf{x}_{22}$. Furthermore, we must specify these constants in such a way that the eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$ (which are derived from $\mathbf{x}_{12}$ and $\mathbf{x}_{22}$, respectively) are independent. It is clear by inspection of the above equation that precisely two independent eigenvectors are available. By choosing $c_2 = 1$ and $c_1 = c_3 = c_4 = c_5 = 0$, we make

$$\mathbf{x}_{12} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

By selecting $c_4 = 1$ and $c_1 = c_2 = c_3 = c_5 = 0$ we get

$$\mathbf{x}_{22} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Of course, many other choices of $\mathbf{x}_{12}$ and $\mathbf{x}_{22}$ would yield the same $\mathbf{x}_1$ and $\mathbf{x}_2$. Furthermore, other choices of $\mathbf{x}_1$ and $\mathbf{x}_2$ would also have been appropriate. We now seek $\mathbf{x}_3$, a third true eigenvector for $\lambda = 3$ which is independent of $\mathbf{x}_1$ and $\mathbf{x}_2$. The eigenvectors for $\lambda = 3$ satisfy $(\mathbf{A} - 3\mathbf{I}) = \boldsymbol{\theta}$. From the matrix $\mathbf{A} - 3\mathbf{I}$ we recognize that $c_1 = c_2$ and $c_3 = c_4$, as well as $c_5 = c_6$ for all eigenvectors for $\lambda = 3$. Letting $c_1 = c_2 = c_3 = c_4 = 0$ and $c_5 = c_6 = 1$, we obtain

$$\mathbf{x}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

an eigenvector independent of the other two. It is a simple matter to determine $\mathbf{x}_4$, an eigenvector for $\lambda = 2$; we choose

$$\mathbf{x}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Exercise 3. Continuing Example 3, let

$$\mathbf{S} = \left(\mathbf{x}_1 \vdots \mathbf{x}_{12} \vdots \mathbf{x}_2 \vdots \mathbf{x}_{22} \vdots \mathbf{x}_3 \vdots \mathbf{x}_4 \right)$$

Show that $\boldsymbol{\Lambda} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$.

Generalized Eigenvectors in Function Spaces

Our discussion of generalized eigenvectors has been directed primarily toward matrices and, through matrices of transformations, toward any linear operator on an n-dimensional vector space. However, the concepts apply also to transformations on infinite-dimensional spaces. We have already noted that for the operator $\mathbf{D}$ acting on the space $\mathcal{C}^1(0, 1)$, any scalar λ is an eigenvalue, and that $e^{\lambda t}$ is a corresponding eigenfunction. Furthermore, there is no other eigenfunction for λ which is independent from $e^{\lambda t}$—the geometric multiplicity of λ is one.

We have not to this point explored the generalized nullspace for λ. In point of fact, powers of $(\mathbf{D}-\lambda\mathbf{I})$ do annihilate additional functions. Specifically, $(\mathbf{D}-\lambda\mathbf{I})^r$ annihilates the r-dimensional subspace of functions of the form $c_1e^{\lambda t} + c_2te^{\lambda t} + c_3t^2e^{\lambda t} + \cdots + c_rt^{r-1}e^{\lambda t}$. The annihilation does not terminate as r increases; the index of annihilation is infinite. It is apparent that the following functions constitute an infinite chain of generalized eigenfunctions of $\mathbf{D}$ for the eigenvalue λ:

$$e^{\lambda t}, te^{\lambda t}, \frac{1}{2!}t^2e^{\lambda t}, \frac{1}{3!}t^3e^{\lambda t}, \ldots \tag{4.83}$$

Generally, differential operators are accompanied by boundary conditions. The eigenvalues of a differential operator $\mathbf{L}$ (with its boundary conditions) are the roots of the eigenvalue equation (4.40), $\det(\mathbf{B}(\lambda))=0$. As in (4.41), the eigenfunctions corresponding to the eigenvalue λ_i are linear combinations of a set of fundamental solutions for $\mathbf{L}$, where the multipliers in the linear combination satisfy

$$\mathbf{B}(\lambda_i)\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

The **algebraic multiplicity** of the eigenvalue λ_i is the multiplicity of λ_i as a root of the eigenvalue equation. The nullity of $\mathbf{B}(\lambda_i)$ equals the number of independent eigenfunctions of $\mathbf{L}$ for the single eigenvalue λ_i; we call this number the **geometric multiplicity** of λ_i. It can be shown that $k_i \leqslant m_i$, just as we found for matrices (see Ince [4.10]). In the above example, where no boundary conditions were applied to the operator $\mathbf{D}$, these definitions do not apply. However, it seems appropriate in that case to assume that $m_i = \infty$ and $k_i = 1$ for each scalar λ_i, since there is an infinite string of generalized eigenfunctions associated with each λ_i. See P&C 4.12*d* for a differential operator (with boundary conditions) which possesses multiple eigenvalues.

The Minimal Polynomial

We showed in (4.15) that if an $n \times n$ matrix $\mathbf{A}$ has distinct roots, its characteristic polynomial in $\mathbf{A}$ is $\mathbf{\Theta}$; that is, $c(\mathbf{A}) = (\mathbf{A} - \lambda_i\mathbf{I}) \cdots (\mathbf{A} - \lambda_n\mathbf{I}) = \mathbf{\Theta}$. We are now in a position to extend this result to all square matrices. The fact that $\mathcal{M}^{n\times 1} = \mathcal{W}_1 \oplus \cdots \oplus \mathcal{W}_p$ is proved in Theorem 3 of Appendix 3. By definition (4.67), $(\mathbf{A} - \lambda_i\mathbf{I})^{q_i}$ annihilates $\mathcal{W}_i$. Furthermore, $\mathcal{W}_j$ is invariant under $(\mathbf{A} - \lambda_i\mathbf{I})^{q_i}$ if $j \neq i$. Therefore, the matrix

$$(\mathbf{A}-\lambda_1\mathbf{I})^{q_1} \cdots (\mathbf{A}-\lambda_p\mathbf{I})^{q_p}$$

annihilates the whole space $\mathcal{M}^{n\times 1}$. We call

$$m(\lambda) \triangleq (\lambda-\lambda_1)^{q_1} \cdots (\lambda-\lambda_p)^{q_p}$$

the **minimal polynomial** for $\mathbf{A}$. The minimal polynomial in $\mathbf{A}$ satisfies

$$m(\mathbf{A}) \triangleq (\mathbf{A}-\lambda_1\mathbf{I})^{q_1} \cdots (\mathbf{A}-\lambda_p\mathbf{I})^{q_p} = \mathbf{\Theta} \tag{4.84}$$

If $r \triangleq q_1 + \cdots + q_p$, then $m(\mathbf{A}) = \mathbf{A}^r + a_1\mathbf{A}^{r-1} + \cdots + a_r\mathbf{I}$, an rth-order polynomial in $\mathbf{A}$. In fact, $m(\mathbf{A})$ is the lowest-order polynomial in $\mathbf{A}$ which annihilates the whole space. It is apparent that polynomials in $\mathbf{A}$ which include higher powers of $(\mathbf{A}-\lambda_i\mathbf{I})$ also annihilate the space. For instance, recalling that $m_i \geqslant q_i$, the characteristic polynomial in $\mathbf{A}$ satisfies

$$c(\mathbf{A}) = (\mathbf{A}-\lambda_1\mathbf{I})^{m_1} \cdots (\mathbf{A}-\lambda_p\mathbf{I})^{m_p} = \mathbf{\Theta} \tag{4.85}$$

for any square matrix $\mathbf{A}$. Equation (4.85) is the Cayley-Hamilton theorem. Equations (4.84) and (4.85) find considerable use in computing. See, for example, Krylov's method (4.23) for finding the characteristic equation; see also the computation of functions of matrices via (4.108).

Example 4. A Minimal PolynomiaL Let

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then $p = 1$, $\lambda_1 = 1$, and $c(\lambda) = (\lambda - 1)^3$. Since

$$(\mathbf{A}-\mathbf{I}) = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and $(\mathbf{A}-\mathbf{I})^2 = \mathbf{\Theta}$, $q_i = 2$, and $m(\lambda) = (\lambda - 1)^2$. It is apparent that $c(\mathbf{A}) = m(\mathbf{A}) = \mathbf{\Theta}$.

4.5 Applications of Generalized Eigendata

The concept of the Jordan form of a matrix is useful partly because it is mnemonic—it helps us remember and categorize the fundamental properties of the matrix (or the linear transformation which the matrix represents). The diagonal form of a diagonalizable matrix is merely a special case of the Jordan form. Whether an operator is diagonalizable or not, a complete eigenvalue analysis-obtaining eigenvalues and eigenvectors—is a computationally expensive process. Thus computational efficiency alone does not ordinarily justify the use of spectral decomposition (decomposition by means of eigenvectors) as a technique for solving an operator equation. However, our reason for analyzing an operator is usually to gain insight into the input-output relation which it describes. Spectral analysis of a model does develop intuitive insight concerning this input-output relation. In some instances a basis of eigenvectors is known a priori, and it need not be computed (e.g., the symmetrical components of (4.28), the Vandermond matrix of P&C 4.16, and the complex exponential functions of Fourier series expansions). In these instances, we gain the insight of spectral decomposition with little more effort than that involved in solution of the operator equation.

Nearly Equal Eigenvalues

True multiple eigenvalues rarely appear in physical systems. But nearly equal eigenvalues are often accompanied by near singularity of the linear operator and, therefore, by computational difficulty. This difficulty can sometimes be avoided by equating the nearly equal eigenvalues and computing generalized eigenvectors in the manner described earlier.

Example 1. Nearly Equal Eigenvalues. In the introduction to Section 4.4 we described a dynamic system with nearly equal poles: $(\mathbf{D} + 1)(\mathbf{D} + 1 + \epsilon)\mathbf{f} = \boldsymbol{\theta}$ with $\mathbf{f}(0) = \alpha_1$ and $\mathbf{f}'(0) = \alpha_2$. As we found in our earlier discussion, the near equality of the poles causes computational difficulty which we remove by equating the poles. But equating the nearly equal poles is equivalent to replacing the nearly dependent set of solutions $\{e^{-t}, e^{-(1+\epsilon)t}\}$ by the easily distinguishable pair of functions $\{e^{-t}, te^{-t}\}$. Since the poles are made identical ($\epsilon = 0$), the state-space representation of the system becomes $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, where

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \tag{4.86}$$

This system matrix is not diagonalizable. The pair of vectors $\mathbf{x}_1 = (1\ -1)^T$ and $\mathbf{x}_{12} = (\frac{1}{2}\ \ \frac{1}{2})^T$ is a two-vector chain of generalized eigenvectors of $\mathbf{A}$ for the single

eigenvalue $\lambda = -1$. This pair of vectors is a basis for the state space. Therefore, the matrix

$$\mathbf{S} = \begin{pmatrix} 1 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{pmatrix} \tag{4.87}$$

is a modal matrix for the system. Note that $\mathbf{S}$ is well conditioned. There will be no computational difficulty in inverting $\mathbf{S}$. The nondiagonal spectral matrix for the system is

$$\mathbf{\Lambda} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \tag{4.88}$$

Example 1 demonstrates the practical value of the concepts of generalized eigenvectors and Jordan form. Even though these concepts are important, the full generality of the Jordan form is seldom, if ever, needed. We are unlikely to encounter, in practice, a generalized eigenspace more complex than that characterized by the single two-vector chain of generalized eigenvectors of Example 1. In Example 1, the system matrix $\mathbf{A}$ is nondiagonalizable only for $\epsilon = 0$. We focused on this nondiagonalizable case because it characterizes the situation for small ϵ better than does the true barely diagonalizable case.* It seems that diagonalizability is the rule in models which represent nature, except at the boundary between certain regions or at the limit of certain approximations. In Example 1, diagonalizability broke down completely only at the boundary between the two regions defined by $\epsilon > 0$ and $\epsilon < 0$. Yet from a practical point of view the boundary is a fuzzy, "small ϵ" transition region.

Pease [4.12, p. 81] presents a spectral analysis of the transmission of electrical signals through a 2-port system. His analysis illustrates the way that nondiagonalizability characterizes the boundary between different regions. The 2×2 system matrix which describes the transmission of signals through the 2-port network is diagonalizable for all sinusoidal signals except signals at the upper or lower cutoff frequencies. At these two frequencies the spectral analysis breaks down because of nondiagonalizability of the matrix of 2-port parameters. However, the analysis can be salvaged by using generalized eigenvectors. Even for frequencies *near* the cutoff frequencies, the spectral analysis is aided by the use of generalized eigenvectors because of the *near* nondiagonalizability of the system matrix.

* Forsythe [4.6] explores other problems in which accuracy is improved by treating near singularity as true singularity.

Application of Jordan Form—Feedback Control

The most common model for a linear time-invariant dynamic system is the state equation (3.67):

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \qquad \mathbf{x}(0) \text{ given} \tag{4.89}$$

where $\mathbf{x}(t)$ is the state (or condition) of the system at time t, and $\mathbf{u}(t)$ is the control (or input) at time t; $\mathbf{A}$ and $\mathbf{B}$ are arbitrary $n \times n$ and $n \times m$ matrices, respectively. In (3.79) we inverted the state equation, obtaining

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-s)}\mathbf{B}\mathbf{u}(s)\,ds \tag{4.90}$$

where the state transition matrix (or matrix exponential) $e^{\mathbf{A}t}$ is defined as the sum of an infinite series of matrices (3.72).

Equations (4.89) and (4.90) are generalizations of the simple first-order linear constant-coefficient differential equation

$$\dot{\mathbf{f}}(t) = a\mathbf{f}(t) + b\mathbf{u}(t), \qquad \mathbf{f}(0) \text{ given} \tag{4.91}$$

which has the solution

$$\mathbf{f}(t) = e^{at}\mathbf{f}(0) + \int_0^t e^{a(t-s)}b\mathbf{u}(s)\,ds$$

Another approach to the solution of (4.91) is through frequency domain analysis.* Taking the Laplace transform of (4.91), we obtain

$$s\mathbf{F}(s) - \mathbf{f}(0) = a\mathbf{F}(s) + b\mathbf{U}(s)$$

or

$$\mathbf{F}(s) = \left(\frac{1}{s-a}\right)\mathbf{f}(0) + \left(\frac{b}{s-a}\right)\mathbf{U}(s) \tag{4.92}$$

where the symbols $\mathbf{F}$ and $\mathbf{U}$ are the Laplace transforms of $\mathbf{f}$ and $\mathbf{u}$, respectively. The function $(s-a)^{-1}$ is known as the transfer function of the system (4.91). The pole of the transfer function $(s = a)$ characterizes the time response of the system. In fact, the transfer function is the Laplace transform of the impulse response of the system, e^{at}.

The relationships among the variables in a linear equation can be represented pictorially by means of a signal flow graph. A signal flow

*For an introduction to frequency domain analysis, see Appendix 2. Refer also to Schwartz and Friedland [4.16] or DeRusso, Roy, and Close [4.3].

graph for (4.91) is shown in Figure 4.5. The variables in the system are associated with nodes in the graph. The arrows indicate the flow of information (or the relationships among the variables). The encircled symbols contained in each arrow are multipliers. Thus the variable $\mathbf{f}(t)$ is multiplied by a as it flows to the node labeled $\dot{\mathbf{f}}(t)$. The symbol $1/s$ represents an integration operation on the variable $\dot{\mathbf{f}}$ (multiplication of the Laplace transform of $\dot{\mathbf{f}}$ by $1/s$ yields the Laplace transform of $\mathbf{f}$). Nodes are treated as summing points for all incoming signals. Thus the node labeled $\dot{\mathbf{f}}(t)$ is a graphic representation of the differential equation (4.91). The primary information about the system, the position of the pole, is contained in the feedback path. The signal flow graph focuses attention on the feedback nature of the system represented by the differential equation.

We can also obtain a transformed equation and a signal flow graph corresponding to the vector state equation (4.89). Suppose the state variables [or elements of $\mathbf{x}(t)$] are denoted by $\mathbf{f}_i(t)$, $i = 1, \ldots, n$. Then we define the Laplace transform of the vector $\mathbf{x}$ of (4.89) by

$$\mathbf{X} \triangleq \mathfrak{L}(\mathbf{x}) \triangleq \begin{pmatrix} \mathfrak{L}(\mathbf{f}_1) \\ \vdots \\ \mathfrak{L}(\mathbf{f}_n) \end{pmatrix} \tag{4.93}$$

Exercise 1. Show that $\mathfrak{L}(\mathbf{Ax}) = \mathbf{A}\,\mathfrak{L}(\mathbf{x})$ for any $n \times n$ matrix $\mathbf{A}$.

Using definition (4.93) and Exercise 1, we take the Laplace transform of (4.89):

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{AX}(s) + \mathbf{BU}(s)$$

Solving for $\mathbf{X}(s)$, we obtain the following generalization of (4.92):

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{BU}(s) \tag{4.94}$$

The matrix $(s\mathbf{I} - \mathbf{A})^{-1}$ is called the **matrix transfer function** for the system represented by (4.89). The poles of the transfer function are those values of

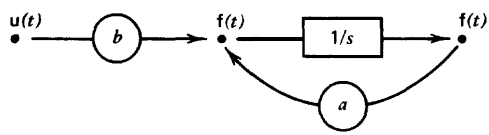


Figure 4.5. Signal flow graph for (4.91).

s for which $(s\mathbf{I}-\mathbf{A})$ is singular. Therefore, the poles of the system are the eigenvalues of the system matrix $\mathbf{A}$, a fact which we discovered for a restricted class of system matrices in (4.51). Because of the formal similarity between the results for the first-order system equation and for the n-dimensional state equation, we suspect that

$$\mathcal{L}(e^{\mathbf{A}t}) = (s\mathbf{I}-\mathbf{A})^{-1} \tag{4.95}$$

Equation (4.95) is easily verified by comparing (4.90) and (4.94). We can think of the state transition matrix $e^{\mathbf{A}t}$ as a **matrix impulse response** [see (3.77)-(3.78)]. The vector signal flow graph is formally the same as that for the scalar equation (Figure 4.6). However, individual nodes now represent vector variables. Again, the feedback nature of the system is emphasized by the flow graph model. The feedback path in Figure 4.6 contains all the information peculiar to the particular system, although the poles of the system are stated only implicitly as the eigenvalues of $\mathbf{A}$. The graph would be more specific if we were to use a separate node for each element of each vector variable; however, the diagram would be much more complicated. We draw such a detailed flow graph for a special case in Figure 4.8.

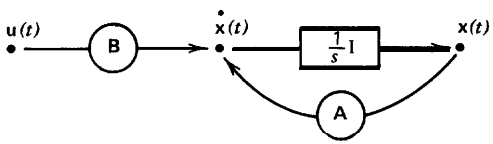


Figure 4.6. Vector signal flow graph for (4.89).

In order to obtain as much insight concerning the feedback nature of the state equation as we did for the scalar case, we change to a coordinate system which emphasizes the poles of the system. Let $\mathbf{x} = \mathbf{S}\mathbf{z}$, where $\mathbf{S}$ is an invertible $n \times n$ matrix. Then $\mathbf{z}(t)$ describes the state of the system relative to a new set of coordinates, and (4.89) becomes

$$\dot{\mathbf{z}}(t) = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}\mathbf{z}(t) + \mathbf{S}^{-1}\mathbf{B}\mathbf{u}(t), \qquad \mathbf{z}(0) = \mathbf{S}^{-1}\mathbf{x}(0) \text{ given} \tag{4.96}$$

We choose $\mathbf{S}$ so that $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{\Lambda}$, the spectral matrix (or Jordan form) of $\mathbf{A}$. Thus $\mathbf{S}$ consists in a basis for the state space composed of generalized eigenvectors of $\mathbf{A}$ as in (4.76). The new signal flow graph is Figure 4.7.

In order to see that this new signal flow graph is particularly informative, we must examine the interconnections between the individual elements of $\mathbf{z}(t)$. We do so for a particular example.

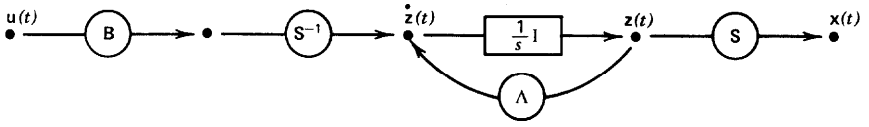


Figure 4.7. Signal flow graph for (4.96).

Example 2. A Specific Feedback System. Let the system and input matrices be

$$\mathbf{A}=\begin{pmatrix} 1 & 1 & 0 \\ -1 & 3 & 0 \\ -1 & 1 & 2 \end{pmatrix} \qquad \mathbf{B}=\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

It is easily verified that the Jordan form of $\mathbf{A}$ is

$$\mathbf{\Lambda}=\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

and that this nearly decoupled spectral matrix can be obtained using

$$\mathbf{S}=\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \mathbf{S}^{-1}=\begin{pmatrix} 0 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

There is only one element $\mathbf{u}(t)$ in the input vector ($\mathbf{B}$ is 3×1). Letting $\mathbf{f}_i$ and $\mathbf{v}_i$ represent the elements of $\mathbf{x}$ and $\mathbf{z}$, respectively, the flow graph corresponding to Figure 4.7 can be given in detail (Figure 4.8). We will refer to the new variables $\mathbf{v}_i(t)$ as the **canonical state variables** [as contrasted with the state variables $\mathbf{f}_i(t)$].

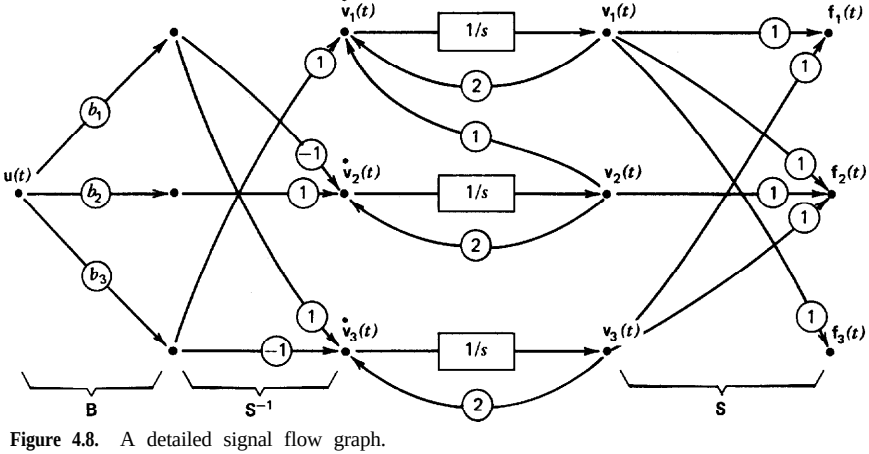


Figure 4.8. A detailed signal flow graph.

In the flow graph of Figure 4.8 the vector system is viewed as a set of nearly uncoupled scalar systems. The poles of the system (the eigenvalues of $\mathbf{A}$) appear in the main feedback paths in the graph. The only other feedback paths are those corresponding to the off-diagonal 1's in $\mathbf{\Lambda}$. It is these off-diagonal 1's that give rise to nonexponential terms (te^{2t}) in the response of the system. Specifically, if the input function $\mathbf{u}$ is zero,

$$\mathbf{v}_3(t) = \mathbf{v}_3(0)e^{2t}$$

$$\mathbf{v}_2(t) = \mathbf{v}_2(0)e^{2t}$$

$$\mathbf{v}_1(t) = \mathbf{v}_1(0)e^{2t} + \mathbf{v}_2(0)te^{2t}$$

The extra term in $\mathbf{v}_1$ arises because the scalar system which determines $\mathbf{v}_1$ is driven by $\mathbf{v}_2$.

It is evident that the Jordan form of a system matrix is a convenient catalog of the information available concerning the system. The modal matrix $\mathbf{S}$ describes the interconnections between the canonical variables and the state variables. Suppose the above system is undriven $[\mathbf{u}(t) = 0]$ and the initial values of the canonical variables are $\mathbf{v}_1(0) = \mathbf{v}_2(0) = 0$ and $\mathbf{v}_3(0) = 1$. Then $\mathbf{v}_1(t) = \mathbf{v}_2(t) = 0$ and $\mathbf{v}_3(t) = e^{2t}$. The corresponding output vector $\mathbf{x}(t)$ is

$$\mathbf{x}(t) = \begin{pmatrix} \mathbf{f}_1(t) \\ \mathbf{f}_2(t) \\ \mathbf{f}_3(t) \end{pmatrix} = \begin{pmatrix} e^{2t} \\ e^{2t} \\ 0 \end{pmatrix} = e^{2t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

At each instant, the output vector is proportional to the third column of $\mathbf{S}$, one of the eigenvectors of $\mathbf{A}$. Under these circumstances, we say only one "mode of response" of the system has been excited. There is one mode of response corresponding to each canonical variable; corresponding to the variable $\mathbf{v}_i(t)$ is the mode where $\mathbf{x}(t)$ is proportional to the i column of $\mathbf{S}$.

We call the system represented by (4.89) **controllable** if there is some input $\mathbf{u}(t)$ that will drive the system $[\mathbf{z}(t)$ or $\mathbf{x}(t)]$ from one arbitrary state to another arbitrary state in a finite amount of time. It should be apparent from Example 2 that in order to be able to control all the canonical state variables in the system, the input variables must be coupled to the inputs of each chain in the flow graph, namely, $\dot{\mathbf{v}}_2(t)$ and $\dot{\mathbf{v}}_3(t)$ in Figure 4.8. If in the above example $\mathbf{B} = (0\ 1\ 0)^T$, $\mathbf{u}(t)$ is not coupled to (and has no influence on) $\mathbf{v}_3(t)$. On the other hand, if $\mathbf{B} = (1\ 0\ 0)^T$, the input is coupled to all the canonical state variables; the system appears to be controllable. However, the variables $\mathbf{v}_2(t)$ and $\mathbf{v}_3(t)$ respond identically to $\mathbf{u}$—they are associated with identical poles. As a result, $\mathbf{v}_2(t)$ and $\mathbf{v}_3(t)$ cannot be

controlled independently. In point of fact, we cannot consider the single input system of Example 2 fully controllable regardless of which input matrix $\mathbf{B}$ we use. A system can be fully controlled only if we can influence identical subsystems independently. In Example 2, the use of a *pair* of inputs with the input matrix

$$\mathbf{B} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

yields a controllable system.

In physical systems we may not be able to measure the state variables directly. Perhaps we can only measure variables $\{\mathbf{g}_i(t)\}$ which are related to the state variables by

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

where $\mathbf{y}(t) = (\mathbf{g}_1(t) \cdots \mathbf{g}_p(t)^{\mathrm{T}}$ and $\mathbf{C}$ is $p \times n$. The matrix $\mathbf{C}$ would appear in the flow graph of Figure 4.8 as a set of connections between the state variables $\{\mathbf{f}_i(t)\}$ and the output (or measurable) variables $\{\mathbf{g}_i(t)\}$. Clearly, we cannot fully determine the state of the system from the measurements unless the output variables are coupled to the output of each chain; namely, $\mathbf{v}_1(t)$ and $\mathbf{v}_3(t)$. Furthermore, in this specific example, measurement of a single output variable $\mathbf{g}_i(t)$ is not sufficient to distinguish between the variables $\mathbf{v}_2(t)$ and $\mathbf{v}_3(t)$, because their behavior is identical. In general, we call a system **observable** if by observing the output $\mathbf{y}(t)$ of the undriven system for a finite interval of time, we get enough information to determine the initial state $\mathbf{x}(0)$. See Brown [4.2] or Zadeh and Desoer [4.20] for convenient tests for controllability and observability.

4.6 Functions of Matrices and Linear Operators

In previous examples we have encountered several functions of square matrices; namely, $\mathbf{A}^k$, $e^{\mathbf{A}t}$, and $(s\mathbf{I} - \mathbf{A})^{-1}$. In later sections we encounter additional matrix functions. The actual computation of such functions of matrices is a problem of practical importance, especially in the analysis of dynamic systems. In this section we develop a definition for functions of matrices which applies in essentially all situations where we might expect such functions to be meaningful. The definition applies to diagonalizable and nondiagonalizable matrices, and also to the linear operators that these matrices represent. (Functions of diagonalizable linear operators on infinite-dimensional spaces are considered in Section 5.5.) Much of this section

is devoted to the development of techniques for analyzing and evaluating functions of matrices.

Two of the matrix functions mentioned above, $\mathbf{A}^k$ and $(s\mathbf{I} - \mathbf{A})^{-1}$, are defined in terms of ordinary matrix operations-addition, scalar multiplication, and inversion. The third matrix function, $e^{\mathbf{A}t}$, represents the sum of an infinite polynomial series in $\mathbf{A}$, as defined in (3.72). This latter function suggests an approach to the definition of general functions of the square matrix $\mathbf{A}$. Polynomial functions of matrices are clearly defined; they can be evaluated by matrix multiplications and additions. Suppose the non-polynomial function f can be expanded in the power series*

$$f(\lambda) = \sum_{k=0}^{\infty} a_k \lambda^k$$

One reasonable way to define $f(\mathbf{A})$ is by using the same power series in $\mathbf{A}$,

$$f(\mathbf{A}) \triangleq \sum_{k=0}^{\infty} a_k \mathbf{A}^k \tag{4.97}$$

Each term of the series can be evaluated using ordinary matrix operations. Of course, the definition (4.97) is useful only if the series converges and we can evaluate the sum of the series. We explore the question of convergence of (4.97) shortly. The essential properties of $\mathbf{A}$ are displayed in its spectral matrix $\mathbf{\Lambda}$ and its modal matrix $\mathbf{S}$. Substituting $\mathbf{A} = \mathbf{S}\mathbf{\Lambda}\mathbf{S}^{-1}$ into (4.97) we find

$$\begin{aligned} f(\mathbf{A}) &= f(\mathbf{S}\mathbf{\Lambda}\mathbf{S}^{-1}) \\ &= \sum_k a_k (\mathbf{S}\mathbf{\Lambda}\mathbf{S}^{-1})^k \\ &= \sum_k a_k \mathbf{S}\mathbf{\Lambda}^k\mathbf{S}^{-1} \\ &= \mathbf{S}\left(\sum_k a_k \mathbf{\Lambda}^k\right)\mathbf{S}^{-1} \\ &= \mathbf{S} f(\mathbf{\Lambda})\mathbf{S}^{-1} \end{aligned} \tag{4.98}$$

*The power series used in (4.97) is a Taylor series expansion about the origin. The matrix function could have been defined in terms of a Taylor series or Laurent series expansion about some other point in the complex plane. See Wylie [4.18] a discussion of such power series expansions.

(We are able to take the similarity transformation outside the infinite sum because matrix multiplication is a continuous operator; see Section 5.4.) Thus if $f(\mathbf{A})$ as given in (4.97) is well-defined, then evaluation of $f(\mathbf{A})$ reduces to evaluation of $f(\mathbf{\Lambda})$. We again apply the power series definition to determine $f(\mathbf{\Lambda})$. If $\mathbf{A}$ is diagonalizable, then $\mathbf{\Lambda}$ is diagonal, and

$$
\begin{aligned}
f(\mathbf{\Lambda}) &= \sum_k a_k \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}^k \\
&= \sum_k a_k \begin{pmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{pmatrix} \\
&= \begin{pmatrix} \sum_k a_k \lambda_1^k & & \\ & \ddots & \\ & & \sum_k a_k \lambda_n^k \end{pmatrix} \\
&= \begin{pmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{pmatrix}
\end{aligned} \tag{4.99}
$$

On the other hand, if $\mathbf{A}$ is not diagonalizable, $f(\mathbf{\Lambda})$ differs from (4.99) only as a result of the off-diagonal l's in $\mathbf{\Lambda}$. By the same logic, we can express $f(\mathbf{\Lambda})$ as

$$
f(\mathbf{\Lambda}) = \sum_k a_k \begin{pmatrix} \mathbf{J}_1 & & \\ & \ddots & \\ & & \mathbf{J}_r \end{pmatrix}^k = \begin{pmatrix} f(\mathbf{J}_1) & & \\ & \ddots & \\ & & f(\mathbf{J}_r) \end{pmatrix} \tag{4.100}
$$

where $\mathbf{J}_i$ is the ith Jordan block in $\mathbf{\Lambda}$. Thus calculation of $f(\mathbf{A})$ reduces to the determination of $f(\mathbf{J}_i)$.

We explore $f(\mathbf{J}_i)$ by means of an example. For a 4×4 Jordan block we have

$$\mathbf{J} = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix} \qquad \mathbf{J}^2 = \begin{pmatrix} \lambda^2 & 2\lambda & 1 & 0 \\ 0 & \lambda^2 & 2\lambda & 1 \\ 0 & 0 & \lambda^2 & 2\lambda \\ 0 & 0 & 0 & \lambda^2 \end{pmatrix}$$

$$\mathbf{J}^3 = \begin{pmatrix} \lambda^3 & 3\lambda^2 & 3\lambda & 1 \\ 0 & \lambda^3 & 3\lambda^2 & 3\lambda \\ 0 & 0 & \lambda^3 & 3\lambda^2 \\ 0 & 0 & 0 & \lambda^3 \end{pmatrix}, \qquad \mathbf{J}^4 = \begin{pmatrix} \lambda^4 & 4\lambda^3 & 6\lambda^2 & 4\lambda \\ 0 & \lambda^4 & 4\lambda^3 & 6\lambda^2 \\ 0 & 0 & \lambda^4 & 4\lambda^3 \\ 0 & 0 & 0 & \lambda^4 \end{pmatrix}$$

Observe that in each matrix the element which appears on the jth "superdiagonal" is (l/j!) times the jth derivative (with respect to λ) of the element on the main diagonal. Thus, continuing the example,

$$f(\mathbf{J}) = \sum_k a_k \mathbf{J}^k$$

$$= \begin{pmatrix} \sum_k a_k \lambda^k & \sum_k \frac{a_k}{1!} \frac{d\lambda^k}{d\lambda} & \sum_k \frac{a_k}{2!} \frac{d^2\lambda^k}{d\lambda^2} & \sum_k \frac{a_k}{3!} \frac{d^3\lambda^k}{d\lambda^3} \\ 0 & \sum_k a_k \lambda^k & \sum_k \frac{a_k}{1!} \frac{d\lambda^k}{d\lambda} & \sum_k \frac{a_k}{2!} \frac{d^2\lambda^k}{d\lambda^2} \\ 0 & 0 & \sum_k a_k \lambda^k & \sum_k \frac{a_k}{1!} \frac{d\lambda^k}{d\lambda} \\ 0 & 0 & 0 & \sum_k a_k \lambda^k \end{pmatrix}$$

Relying on the term-by-term differentiability of power series (Kaplan [4.11, p. 353]), we take all derivatives outside the summations to obtain

$$f(\mathbf{J}) = \begin{pmatrix} f(\lambda) & \frac{f'(\lambda)}{1!} & \frac{f''(\lambda)}{2!} & \frac{f^{(3)}(\lambda)}{3!} \\ 0 & f(\lambda) & \frac{f'(\lambda)}{1!} & \frac{f''(\lambda)}{2!} \\ 0 & 0 & f(\lambda) & \frac{f'(\lambda)}{1!} \\ 0 & 0 & 0 & f(\lambda) \end{pmatrix} \tag{4.101}$$

Corresponding to each Jordan block $\mathbf{J}_i$ of $\mathbf{\Lambda}$ (with eigenvalue λ_i), $f(\mathbf{\Lambda})$ contains a block which has $f(\lambda_i)$ on the main diagonal. The upper elements in the block are filled with appropriately scaled derivatives off (evaluated at λ_i). The elements on the jth super-diagonal are

$$\frac{1}{j!}\frac{d^j f(\lambda_i)}{d\lambda^j}$$

Surprisingly, $f(\mathbf{\Lambda})$ is not in Jordan form.

Example 1. Matrix Inversion as a Matrix Function. Suppose $f(\lambda) = 1/\lambda$. If $\mathbf{A}$ is an invertible $n \times n$ matrix, we use (4.98) and (4.99) to find

$$\begin{aligned}\mathbf{A}^{-1} &= \mathbf{S}\mathbf{\Lambda}^{-1}\mathbf{S}^{-1} \\ &= \mathbf{S}\begin{pmatrix} 1/\lambda_1 & & \\ & \ddots & \\ & & 1/\lambda_n \end{pmatrix}\mathbf{S}^{-1}\end{aligned}$$

Suppose

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then $\mathbf{S} = \mathbf{S}^{-1} = \mathbf{I}$, and

$$\mathbf{A}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \infty \end{pmatrix}$$

It is clear that $\mathbf{A}^{-1}$ does not exist if zero is an eigenvalue of $\mathbf{A}$. The function $1/\lambda$ is not defined at $\lambda = 0$, and (4.99) cannot be evaluated.

Example 2. A Function of a Nondiagonalizable Matrix. As in Example 1, if $\mathbf{A} = \mathbf{S}\mathbf{\Lambda}\mathbf{S}^{-1}, \mathbf{A}^{-1} = \mathbf{S}\mathbf{\Lambda}^{-1}S^{-1}$. Suppose

$$\mathbf{\Lambda} = \begin{pmatrix} \lambda_1 & 1 & 0 & \vdots & \\ 0 & \lambda_1 & 1 & \vdots & \\ 0 & 0 & \lambda_1 & \vdots & \\ \cdots & \cdots & \cdots & \lambda_1 & \cdots \\ & & & \vdots & \lambda_2 \end{pmatrix}$$

Letting $f(\lambda) = 1/\lambda$ we find that $f'(\lambda) = -1/\lambda^2$ and $f''(\lambda)/2! = 1/\lambda^3$. Thus, using

(4.101) for each Jordan block,

$$\Lambda^{-1} = \begin{pmatrix} 1/\lambda_1 & -1/\lambda_1^2 & 1/\lambda_1^3 & \vdots & & \\ 0 & 1/\lambda_1 & -1/\lambda_1^2 & \vdots & & \\ 0 & 0 & 1/\lambda_1 & \vdots & & \\ \cdots & \cdots & \cdots & 1/\lambda_1 & \cdots & \\ & & & \vdots & & 1/\lambda_2 \end{pmatrix}$$

An Alternative Definition

Although we have used (4.97) to define $f(\mathbf{A})$, we have used (4.98) and (4.100) to perform the actual evaluation of $f(\mathbf{A})$. [Note that (4.99) is a special case of (4.100).] It can be shown that our original definition of $f(\mathbf{A})$, (4.97), converges if and only if f is analytic in a circle of the complex plane which contains all the eigenvalues of $\mathbf{A}$.* Yet (4.98) and (4.100), which we derived from (4.97), provide a correct evaluation of $f(\mathbf{A})$ in cases which do not satisfy this criterion. For example,

$$\text{if } \mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \quad \text{then} \quad \mathbf{A}^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

The function $f(\lambda) = \lambda^{-1}$ is not analytic at $\lambda = 0$. No circle encloses the points 2 and -2 while excluding the point 0, yet (4.98) and (4.99) provide the correct inverse. It is apparent that (4.98)-(4.101) provide a *more general definition* of $f(\mathbf{A})$ than does (4.97).

The definition (4.98)-(4.101) applies to all functions f and matrices $\mathbf{A}$ for which $f(\mathbf{J}_i)$ can be evaluated for each Jordan block $\mathbf{J}_i$. If $\mathbf{A}$ is diagonalizable, this evaluation requires only that f be *defined on the spectrum;* that is, that f be defined at all the eigenvalues of $\mathbf{A}$. If $\mathbf{A}$ is not diagonalizable, the evaluation of $f(\mathbf{A})$ requires the existence of derivatives of f at some of the eigenvalues of $\mathbf{A}$. Thus the definition of $f(\mathbf{A})$ given in (4.98)-(4.101) certainly applies to all f and $\mathbf{A}$ for which f is not only defined on the spectrum of $\mathbf{A}$ but also analytic at those eigenvalues of $\mathbf{A}$ for which $\mathbf{A}$ is defective (i.e., for which the corresponding Jordan blocks $\mathbf{J}_i$ are larger than 1×1). In every case where the definition (4.97) applies, the evaluation of $f(\mathbf{A})$ which results is identical to the evaluation provided by (4.98)-(4.101).

As illustrated in (4.101), the actual evaluation of $f(\mathbf{A})$ leads to evaluation of

$$f(\lambda_i), f'(\lambda_i), \ldots, f^{(q_i-1)}(\lambda_i), \qquad i = 1, \ldots, p \tag{4.102}$$

*Rinehart [4.14]. A function $f(\lambda)$ is said to be analytic at λ_1 if it is differentiable (as a function of a complex variable λ) in a neighborhood of λ_1 (see Wylie [4.18]).

We refer to this set of evaluations as **evaluation on the spectrum** of $\mathbf{A}$. It is apparent that any two functions that have the same evaluation on the spectrum lead to the same function of $\mathbf{A}$.

Exercise 1. Compare $f(\Lambda)$ and $g(\Lambda)$ for $f(\lambda) \triangleq 4\lambda - 8$, $g(\lambda) \triangleq \lambda^2 - 4$, and

$$\Lambda = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

Equations (4.98)-(4.101) provide a suitable definition of $f(\mathbf{A})$ for most choices of f and $\mathbf{A}$. Rinehart [4.14] shows that with this definition of $f(\mathbf{A})$ and with single-valued functions g and h for which $g(\mathbf{A})$ and $h(\mathbf{A})$ exist,

1. If $f(\lambda) = c$ then $f(\mathbf{A}) = c\mathbf{I}$
2. If $f(\lambda) = \lambda$ then $f(\mathbf{A}) = \mathbf{A}$
3. If $f(\lambda) = g(\lambda) + h(\lambda)$ then $f(\mathbf{A}) = g(\mathbf{A}) + h(\mathbf{A})$
4. If $f(\lambda) = g(\lambda) \cdot h(\lambda)$ then $f(\mathbf{A}) = g(\mathbf{A}) \cdot h(\mathbf{A})$
5. If $f(\lambda) = g(h(\lambda))$ then $f(\mathbf{A}) = g(h(\mathbf{A}))$

If g or h is not single valued, then the matrix $f(\mathbf{A})$ depends upon which branches of g and h are used in the evaluation on the spectrum of $\mathbf{A}$. From these properties it follows that scalar functional identities extend to matrices. For example, $\sin^2(\mathbf{A}) + \cos^2(\mathbf{A}) = \mathbf{I}$ and $e^{\ln \mathbf{A}} = \mathbf{A}$.

The Fundamental Formula for Matrices

Let $\mathbf{A}$ be a 3×3 diagonalizable matrix with only two distinct eigenvalues; that is, $c(\lambda) = (\lambda - \lambda_1)^2(\lambda - \lambda_2)$, and the eigenspace for λ_1 is two-dimensional. Suppose also that the function f is defined at λ_1 and λ_2. Then we can express $f(\Lambda)$ in the manner of Example 1:

$$f(\Lambda) = \begin{pmatrix} f(\lambda_1) & 0 & 0 \\ 0 & f(\lambda_1) & 0 \\ 0 & 0 & f(\lambda_2) \end{pmatrix}$$

In order to express $f(\Lambda)$ in a manner that clearly separates the essential properties of Λ from those off, we introduce the following notation. Let $\mathbf{E}_{i0}^{\Lambda}$ be a matrix which has a one wherever $f(\Lambda)$ has $f(\lambda_i)$, and zeros elsewhere. (The second subscript, "0," is used only to provide consistency with the nondiagonalizable case introduced later.) Specifically,

$$\mathbf{E}_{10}^{\Lambda} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{E}_{20}^{\Lambda} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then we can express $f(\mathbf{\Lambda})$ by

$$f(\mathbf{\Lambda}) = f(\lambda_1)\mathbf{E}_{10}^{\Lambda} + f(\lambda_2)\mathbf{E}_{20}^{\Lambda}$$

Since $f(\mathbf{A}) = \mathbf{S}f(\mathbf{\Lambda})\,\mathbf{S}^{-1}$ to obtain $f(\mathbf{A})$ we simply perform the similarity transformations $\mathbf{E}_{i0}^{\Lambda} = \mathbf{S}\mathbf{E}_{i0}^{\Lambda}\mathbf{S}^{-1}$ to obtain

$$f(\mathbf{A}) = f(\lambda_1)\mathbf{E}_{10}^{\Lambda} + f(\lambda_2)\mathbf{E}_{20}^{\Lambda}$$

It is evident that we can express any well-defined function of the specific matrix $\mathbf{A}$ by means of this formula. Once we have the matrices $\mathbf{E}_{i0}^{\Lambda}$, evaluation of $f(\mathbf{A})$ requires only evaluation of f on the spectrum of $\mathbf{A}$. By a derivation similar to that above, we can show that for any $n \times n$ diagonalizable matrix $\mathbf{A}$ and any f defined on the spectrum of $\mathbf{A}$, $f(\mathbf{A})$ can be expressed as

$$f(\mathbf{A}) = \sum_{i=1}^{p} f(\lambda_i)\mathbf{E}_{i0}^{\Lambda} \tag{4.103}$$

where p is the number of distinct eigenvalues of $\mathbf{A}$. We call (4.103) the **fundamental formula for $f(\mathbf{A})$.** The matrices $\mathbf{E}_{i0}^{\Lambda}$ are called the **constituent matrices** (or **components)** of $\mathbf{A}$. (We drop the superscript Λ when confusion seems unlikely.) Notice that (4.103) separates the contributions of f and $\mathbf{A}$. In fact, (4.103) is a satisfactory definition of $f(\mathbf{A})$, equivalent to (4.98-(4.99).

The definition of the fundamental formula (4.103) can be extended to nondiagonalizable matrices as well. Suppose f is analytic at λ_1 and defined at λ_2. Then we can write $f(\mathbf{\Lambda})$ for the matrix $\mathbf{\Lambda}$ of Example 2 as

$$f(\mathbf{\Lambda}) = \begin{pmatrix} f(\lambda_1) & f'(\lambda_1) & \dfrac{f''(\lambda_1)}{2} & \vdots & & \\ 0 & f(\lambda_1) & f'(\lambda_1) & \vdots & & \\ 0 & 0 & f(\lambda_1) & \vdots & & \\ \cdots & \cdots & \cdots & f(\lambda_1) & \cdots & \\ & & & \vdots & f(\lambda_2) & \end{pmatrix}$$

In order to separate the essential properties of $\mathbf{\Lambda}$ from those off, we define $\mathbf{E}_{ik}^{\Lambda}$ to be a matrix which has a one wherever $f(\mathbf{\Lambda})$ has $(1/k!)\,f^{(k)}(\lambda_i)$, $k = 0$,

1, and 2, and zeros elsewhere. Thus

$$\mathbf{E}^{\Lambda}_{10} = \left(\begin{array}{ccc:c:c} 1 & 0 & 0 & & \\ 0 & 1 & 0 & & \\ 0 & 0 & 1 & & \\ \hdashline & & & 1 & \\ \hdashline & & & & 0 \end{array}\right) \qquad \mathbf{E}^{\Lambda}_{11} = \left(\begin{array}{ccc:c:c} 0 & 1 & 0 & & \\ 0 & 0 & 1 & & \\ 0 & 0 & 0 & & \\ \hdashline & & & 0 & \\ \hdashline & & & & 0 \end{array}\right)$$

$$\mathbf{E}^{\Lambda}_{12} = \left(\begin{array}{ccc:c:c} 0 & 0 & 1 & & \\ 0 & 0 & 0 & & \\ 0 & 0 & 0 & & \\ \hdashline & & & 0 & \\ \hdashline & & & & 0 \end{array}\right) \qquad \mathbf{E}^{\Lambda}_{20} = \left(\begin{array}{ccc:c:c} 0 & 0 & 0 & & \\ 0 & 0 & 0 & & \\ 0 & 0 & 0 & & \\ \hdashline & & & 0 & \\ \hdashline & & & & 1 \end{array}\right)$$

Then we can express $f(\mathbf{\Lambda})$ by

$$f(\mathbf{\Lambda}) = f(\lambda_1)\mathbf{E}^{\Lambda}_{10} + f'(\lambda_1)\mathbf{E}^{\Lambda}_{11} + \frac{f''(\lambda_1)}{2!}\mathbf{E}^{\Lambda}_{12} + f(\lambda_2)\mathbf{E}^{\Lambda}_{20}$$

As in the diagonalizable case, we perform the similarity transformations $\mathbf{E}^{\mathbf{A}}_{ij} = \mathbf{S}\mathbf{E}^{\Lambda}_{ij}\mathbf{S}^{-1}$ to obtain

$$f(\mathbf{A}) = f(\lambda_1)\mathbf{E}^{\mathbf{A}}_{10} + f'(\lambda_1)\mathbf{E}^{\mathbf{A}}_{11} + \frac{f''(\lambda_1)}{2!}\mathbf{E}^{\mathbf{A}}_{12} + f(\lambda_2)\mathbf{E}^{\mathbf{A}}_{20}$$

We can compute any well-defined function of the matrix $\mathbf{A}$ of Example 2 by means of this formula. By a derivation similar to that above, we can show that for any $n \times n$ matrix $\mathbf{A}$ and any f which is defined on the spectrum of $\mathbf{A}$ and analytic at eigenvalues where $\mathbf{A}$ is defective, $f(\mathbf{A})$ can be expressed as

$$f(\mathbf{A}) = \sum_{i=1}^{p} \left[f(\lambda_i)\mathbf{E}^{\mathbf{A}}_{i0} + \frac{f'(\lambda_i)}{1!}\mathbf{E}^{\mathbf{A}}_{i1} + \cdots + \frac{f^{(q_i-1)}(\lambda_i)}{(q_i-1)!}\mathbf{E}^{\mathbf{A}}_{i(q_i-1)} \right] \tag{4.104}$$

where p is the number of distinct eigenvalues of $\mathbf{A}$, and q_i is the index of annihilation for λ_i [see (4.66) and (4.67)]. Equation (4.104) is the general form of the **fundamental formula** for $f(\mathbf{A})$. Again, we refer to the matrices $\mathbf{E}^{\mathbf{A}}_{ij}$ as constituent matrices (or components) of $\mathbf{A}$.*

*The constituent matrices are sometimes defined as $\mathbf{E}^{\mathbf{A}}_{ij}/j!$.

The fundamental formula can be used to generate a **spectral decomposition of A.** If we let $f(\lambda) = \lambda$ in (4.104), we obtain

$$\mathbf{A} = \sum_{i=1}^{p} (\lambda_i \mathbf{E}_{i0}^{\wedge} + \mathbf{E}_{i1}^{\wedge}) \tag{4.105}$$

If $\mathbf{A}$ is diagonalizable, $q_i = 1$ for each i, and (4.105) becomes

$$\mathbf{A} = \sum_{i=1}^{p} \lambda_i \mathbf{E}_{10}^{\wedge}$$

It is apparent that in the diagonalizable case $\mathbf{E}_{i0}^{\wedge}$ describes the projection onto the eigenspace associated with λ_i. That is, if $\mathbf{x} = \mathbf{x}_1 + \cdots + \mathbf{x}_p$, where $\mathbf{x}_i$ is the component of $\mathbf{x}$ in the eigenspace for λ_i, then $\mathbf{x}_i = \mathbf{E}_{i0}^{\wedge}\mathbf{x}$ and $\mathbf{A}$ acts like λ_i on $\mathbf{x}_i$. In the nondiagonalizable case, $\mathbf{E}_{i0}^{\wedge}$ describes the projection onto the *generahzed* eigenspace for λ_i. Furthermore, $\mathbf{E}_{ik}^{\wedge}$ acts like the nilpotent operator $(\mathbf{A} - \lambda_i \mathbf{I})^k$ on the generalized eigenspace for λ_i; that is, $\mathbf{E}_{ik}^{\wedge} = (\mathbf{A} - \lambda_i \mathbf{I})^k \mathbf{E}_{i0}^{\wedge}$.

Exercise 2. Verify that the matrices $\mathbf{E}_{10}^{\wedge}$ and $\mathbf{E}_{20}^{\wedge}$ of Example 2 satisfy the properties (4.3) for projectors. Show also that $\mathbf{E}_{ik}^{\wedge} = (\mathbf{A} - \lambda_i \mathbf{I})^k \mathbf{E}_{10}^{\wedge}$.

Functions of Linear Operators

The fundamental formula also serves to define functions of the underlying operator represented by $\mathbf{A}$. If $\mathbf{T}$ operates on an n-dimensional vector space $\mathcal{V}$, if $\mathbf{P}_{i0}$ is the operator which projects onto $\mathcal{W}_i$ (the generalized eigenspace for λ_i) along $\Sigma_{j \neq i} \mathcal{W}_j$, and if $\mathbf{P}_{ik} \triangleq (\mathbf{T} - \lambda_i \mathbf{I})^k \mathbf{P}_{i0}$, then the **fundamental formula for $f(\mathbf{T})$** is

$$f(\mathbf{T}) \triangleq \sum_{i=1}^{p} \left[f(\lambda_i)\mathbf{P}_{i0} + \frac{1}{1!} f'(\lambda_i)\mathbf{P}_{i1} + \cdots + \frac{1}{(q_i - 1)!} f^{(q_i - 1)}(\lambda_i)\mathbf{P}_{i(q_i - 1)} \right] \tag{4.106}$$

If $\mathcal{X}$ is a basis for $\mathcal{V}$ and we define $\mathbf{A} \triangleq [\mathbf{T}]_{\mathcal{X}\mathcal{X}}$, then $\mathbf{E}_{ij}^{\wedge} = [\mathbf{P}_{ij}]_{\mathcal{X}\mathcal{X}}$. As a result, (4.104) and (4.106) require that $[f(\mathbf{T})]_{\mathcal{X}\mathcal{X}} = f([\mathbf{T}]_{\mathcal{X}\mathcal{X}})$. For diagonalizable $\mathbf{T}$ ($\mathbf{T}$ for which there exists a basis for $\mathcal{V}$ composed of eigenvectors for $\mathbf{T}$), (4.106) simplifies to $f(\mathbf{T}) = \Sigma_{i=1}^{p} f(\lambda_i)\mathbf{P}_{i0}$. This simple result is extended to certain infinite-dimensional operators in (5.90).

Example 3. A Function of a Linear Operator. Consider $\mathbf{D}$: $\mathcal{P}^3 \to \mathcal{P}^3$. We first find the eigendata for $\mathbf{D}$ (as an operator on $\mathcal{P}^3$). The set $\mathcal{R} \triangleq \{\mathbf{f}_i(t) = t^{i-1}, i = 1, 2, 3\}$ is a basis for $\mathcal{P}^3$. In Example 2 of Section 2.5 we found that

$$\mathbf{A} = [\mathbf{D}]_{\mathcal{R}\mathcal{R}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

This matrix has only one eigenvalue, $\lambda_1 = 0$; a basis of generalized eigenvectors for $[\mathbf{D}]_{\mathcal{R}\mathcal{R}}$ is

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{x}_{12} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{x}_{13} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix}$$

Thus

$$\mathbf{\Lambda} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{S} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}, \qquad \text{and} \quad \mathbf{S}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

The generalized eigenfunctions of $\mathbf{D}$ corresponding to $\mathbf{x}_1$, $\mathbf{x}_{12}$, and $\mathbf{x}_{13}$ are

$$\mathbf{g}_1(t) = 1, \qquad \mathbf{g}_{12}(t) = t, \qquad \mathbf{g}_{13}(t) = \frac{t^2}{2}$$

Because the chain of generalized eigenvectors is of length 3, $q_1 = 3$. Therefore, in order to evaluate $\mathbf{f}(\mathbf{D})$, we must determine three operators: $\mathbf{P}_{10}$, $\mathbf{P}_{11}$, and $\mathbf{P}_{12}$. Since the generalized eigenspace of $\mathbf{D}$ for $\lambda_i = 0$ is the whole space $\mathcal{P}^3$, the projector $\mathbf{P}_{10}$ onto the generalized eigenspace for λ_1 is $\mathbf{P}_{10} = \mathbf{I}$. We find the other two operators by

$$\mathbf{P}_{11} = (\mathbf{D} - \lambda_1 \mathbf{I})\mathbf{P}_{10} = \mathbf{D}\mathbf{I} = \mathbf{D}$$

$$\mathbf{P}_{12} = (\mathbf{D} - \lambda_1 \mathbf{I})^2 \mathbf{P}_{10} = \mathbf{D}^2 \mathbf{I} = \mathbf{D}^2$$

By (4.106), if f is analytic at $\lambda = 0$,

$$f(\mathbf{D}) = f(0)\mathbf{I} + f'(0)\mathbf{D} + \frac{f''(0)}{2}\mathbf{D}^2$$

Let $f(\lambda) = \lambda$. Then $f(\mathbf{D})$ reduces to

$$\mathbf{D} = (0)\mathbf{I} + (1)\mathbf{D} + (0)\mathbf{D}^2$$

which verifies the formula for $f(\mathbf{D})$. Let $f(\lambda) = e^{\lambda}$. Then

$$e^{\mathbf{D}} = e^0\mathbf{I} + e^0\mathbf{D} + \tfrac{1}{2}e^0\mathbf{D}^2$$

$$= \mathbf{I} + \mathbf{D} + \tfrac{1}{2}\mathbf{D}^2$$

Returning to $\mathbf{A} = [\mathbf{D}]_{\mathfrak{R}\mathfrak{R}}$, we generate those functions of $\mathbf{A}$ which correspond to the functions $f(\mathbf{D})$, $\mathbf{D}$, and $e^{\mathbf{D}}$ above. By inspection of $\mathbf{\Lambda}$ we find that

$$\mathbf{E}_{10}^{\Lambda} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \mathbf{E}_{11}^{\Lambda} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{E}_{12}^{\Lambda} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Using the similarity transformation $\mathbf{E}_{ij}^{\mathbf{A}} = \mathbf{S}\mathbf{E}_{ij}^{\Lambda}\mathbf{S}^{-1}$, we obtain

$$\mathbf{E}_{10}^{\mathbf{A}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I}, \qquad \mathbf{E}_{11}^{\mathbf{A}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{A}, \qquad \mathbf{E}_{12}^{\mathbf{A}} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{A}^2$$

These constituents of $\mathbf{A}$ are $[\mathbf{P}_{10}]_{\mathfrak{R}\mathfrak{R}}$, $[\mathbf{P}_{11}]_{\mathfrak{R}\mathfrak{R}}$, and $[\mathbf{P}_{12}]_{\mathfrak{R}\mathfrak{R}}$, respectively. By (4.104),

$$f(\mathbf{A}) = f(0)\mathbf{I} + f'(0)\mathbf{A} + \frac{f''(0)}{2}\mathbf{A}^2$$

If $f(\lambda) = \lambda$, we find

$$\mathbf{A} = (0)\mathbf{I} + (1)\mathbf{A} + (0)\mathbf{A}^2$$

Let $f(\lambda) = e^{\lambda}$. Then

$$\begin{aligned} e^{\mathbf{A}} &= e^0\mathbf{I} + e^0\mathbf{A} + \tfrac{1}{2}e^0\mathbf{A}^2 \\ &= \mathbf{I} + \mathbf{A} + \tfrac{1}{2}\mathbf{A}^2 \\ &= \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

We easily verify that $e^{\mathbf{A}} = [e^{\mathbf{A}}]_{\mathfrak{R}\mathfrak{R}}$. These results are consistent with the definition (3.72) of $e^{\mathbf{A}t}$, because $\mathbf{A}^k = \mathbf{\Theta}$ for $k > 2$.

Computation of Functions of Matrices

We have already derived a method for computing $f(\mathbf{A})$ which relies on a complete eigenvalue analysis of $\mathbf{A}$. We summarize the method.

Computation of $f(\mathbf{A})$ *by eigenvalue analysis of* $\mathbf{A}$ (4.107)

1. Determine the Jordan form $\mathbf{\Lambda}$, the modal matrix $\mathbf{S}$, and $\mathbf{S}^{-1}$ such that $\mathbf{A} = \mathbf{S}\mathbf{\Lambda}\mathbf{S}^{-1}$.
2. Determine $\mathbf{E}_{ij}^{\Lambda}$ by inspection of $\mathbf{\Lambda}$.
3. Determine $\mathbf{E}_{ij}^{\mathbf{A}}$ by the similarity transformation $\mathbf{E}_{ij}^{\mathbf{A}} = \mathbf{S}\mathbf{E}_{ij}^{\Lambda}\mathbf{S}^{-1}$.
4. Evaluate f on the spectrum of $\mathbf{A}$.
5. Determine $f(\mathbf{A})$ from the fundamental formula, (4.103) or (4.104).

Example 4. Computing e^{At} Using Complete Eigenvalue Analysis **Let $f(\lambda) = e^{\lambda t}$.** Let **A** be the matrix of Example 2, Section 4.5:

$$A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 3 & 0 \\ -1 & 1 & 2 \end{pmatrix}$$

Then $f(\mathbf{A}) = e^{\mathbf{A}t}$ is the state transition matrix for that example. We found in that example that

$$(1) \quad \Lambda = \left(\begin{array}{cc:c} 2 & 1 & 0 \\ 0 & 2 & 0 \\ \hdashline 0 & 0 & 2 \end{array}\right), \quad \mathbf{S} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{S}^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

Following the other steps outlined above,

$$(2) \quad \mathbf{E}_{10}^{\Lambda} = \left(\begin{array}{cc:c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hdashline 0 & 0 & 1 \end{array}\right), \quad \mathbf{E}_{11}^{\Lambda} = \left(\begin{array}{cc:c} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 \end{array}\right)$$

$$(3) \quad \mathbf{E}_{10}^{\wedge} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{E}_{11}^{\wedge} = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix}$$

$$(4) \quad f(2) = e^{2t}, \quad f'(2) = te^{2t}$$

$$(5) \quad e^{\mathbf{A}t} = e^{2t}\mathbf{E}_{10}^{\wedge} + te^{2t}\mathbf{E}_{11}^{\wedge}$$

$$= \begin{pmatrix} e^{2t} - te^{2t} & te^{2t} & 0 \\ -te^{2t} & e^{2t} + te^{2t} & 0 \\ -te^{2t} & te^{2t} & e^{2t} \end{pmatrix}$$

Determination of $f(\mathbf{A})$ using complete eigenvalue analysis is lengthy and computationally expensive. The eigenvalue analysis serves only to determine constituents of **A.** [Of course, it provides considerable insight into the structure of the matrix **A** in addition to producing f **(A)]**. We can eliminate most of this computation by employing the fundamental formula in evaluating the constituents. If we substitute several different functions into (4.103)-(4.104), we obtain several equations involving the constituents as unknowns. By a judicious choice of functions, we can obtain equations that allow us to determine each constituent independently. If the minimal polynomial $m(\lambda)$ is evaluated on the spectrum, the evaluations are all zero. If one factor is cancelled from $m(\lambda)$ and the resulting polynomial evaluated on the spectrum, precisely one evaluation is nonzero; if we evaluate this same polynomial in **A,** precisely one constituent will remain in the fundamental formula. By successively cancelling factors from $m(\lambda)$, and evaluat-

ing the resulting polynomials in $\mathbf{A}$, we obtain the constituents in an efficient manner.*

Computation of $f(\mathbf{A})$ by evaluating factors of $m(\lambda)$ (4.108)

1. Find and factor $m(\lambda)$, the minimal polynomial for $\mathbf{A}$.
2. Cancel one factor from $m(\lambda)$. Denote the resulting polynomial $g_1(\lambda)$. Evaluating $g_1(\mathbf{A})$ will determine precisely one constituent matrix.
3. Cancel an additional factor from $m(\lambda)$. Let $g_i(\lambda)$ denote the polynomial which results from cancelling i factors from $m(\lambda)$. Evaluation of $g_i(\mathbf{A})$ determines precisely one constituent matrix in terms of previously determined constituents. This step is repeated until all the constituents $\mathbf{E}_{ij}^{\wedge}$ are known.
4. Evaluate f on the spectrum of $\mathbf{A}$.
5. Compute $f(\mathbf{A})$ from the fundamental formula, (4.103) or (4.104).

Example 5. Computing $e^{\mathbf{A}t}$ by Evaluating Factors of the Minimal Polynomial. Let $f(\lambda) = e^{\lambda t}$. Assume $\mathbf{A}$ is the matrix given in Example 4. We compute the state transition matrix $e^{\mathbf{A}t}$ by the steps outlined above:

1. The characteristic polynomial for $\mathbf{A}$ is $c(\lambda) = \det(\lambda\mathbf{I} - \mathbf{A}) = (\lambda - 2)^3$. The only eigenvalue is $\lambda_1 = 2$. By investigating the nullities of $(\mathbf{A} - 2\mathbf{I})$ and $(\mathbf{A} - 2\mathbf{I})^2$, we find that $q_1 = 2$ and $m(\lambda) = (\lambda - 2)^2$. Thus

$$f(\mathbf{A}) = f(2)\mathbf{E}_{10}^{\wedge} + f'(2)\mathbf{E}_{11}^{\wedge}$$

2. $g_1(\lambda) = (\lambda - 2)$ and $g_1'(\lambda) = 1$. Therefore,

$$g_1(A) \triangleq (\mathbf{A} - 2\mathbf{I}) = g_1(2)\mathbf{E}_{10}^{\wedge} + g_1'(2)\mathbf{E}_{11}^{\wedge}$$
$$= (0)\mathbf{E}_{10}^{\wedge} + (1)\mathbf{E}_{11}^{\wedge}$$

and $\mathbf{E}_{11}^{\wedge} = \mathbf{A} - 2\mathbf{I}$.

3. $g_2(\lambda) = 1$ and $g_2'(\lambda) = 0$. Then,

$$g_2(\mathbf{A}) \triangleq \mathbf{I} = g_2(2)\mathbf{E}_{10}^{\wedge} + g_2'(2)\mathbf{E}_{11}^{\wedge}$$
$$= (1)\mathbf{E}_{10}^{\wedge} + (0)\mathbf{E}_{11}^{\wedge}$$

and $\mathbf{E}_{10}^{\wedge} = \mathbf{I}$.

4. $f(2) = e^{2t}$, and $f'(2) = te^{2t}$.

5. $e^{\mathbf{A}t} = e^{2t}\mathbf{I} + te^{2t}(\mathbf{A} - 2\mathbf{I})$

$$= \begin{pmatrix} e^{2t} - te^{2t} & te^{2t} & 0 \\ -te^{2t} & e^{2t} + te^{2t} & 0 \\ -te^{2t} & te^{2t} & e^{2t} \end{pmatrix}$$

*From Zadeh and Desoer [4.20].

Evaluating factors of $m(\lambda)$ is probably the most efficient known method for computing $f(\mathbf{A})$. A suitable sequence of functions can also be obtained by successively cancelling factors from the characteristic polynomial $c(\lambda)$, thereby avoiding determination of the nullities of powers of $(\mathbf{A} - \lambda_i \mathbf{I})$. If $c(\lambda)$ had been used in Example 5, we would have found that $\mathbf{E}_{12}^{\mathbf{A}} = \mathbf{\Theta}$.

From our computation of $f(\mathbf{A})$ by evaluating factors of the minimal polynomial, we recognize that each of the constituents $\mathbf{E}_{ij}^{\mathbf{A}}$ equals a polynomial in $\mathbf{A}$; the order of the polynomial is, in each case, less than that of the minimal polynomial. Therefore, by the fundamental formula, $f(\mathbf{A})$ is also equal to a polynomial in $\mathbf{A}$. Since powers of $\mathbf{A}$, and thus polynomials in $\mathbf{A}$, commute with each other, functions of $\mathbf{A}$ commute with each other also. See P&C 4.29 for properties of commuting matrices. Additional techniques for computing $f(\mathbf{A})$ are given in P&C 4.25-4.27.

Application of Functions of Matrices—Modes of Oscillation

Figure 4.9 is an idealized one-dimensional representation of a piece of spring-mounted equipment. The variables $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{u}$ represent the positions, relative to their respective references, of the two identical masses (labeled m) and the frame which holds the equipment. The three springs have identical spring constants k. We treat the position (or vibration) of the frame as an independent variable; we seek the motions, $\mathbf{v}_1(t)$ and $\mathbf{v}_2(t)$, of the spring-mounted objects. The dynamic equations which describe these motions are

$$\begin{aligned} m\ddot{\mathbf{v}}_1(t) &= -2k\mathbf{v}_1(t) + k\mathbf{v}_2(t) + k\mathbf{u}(t) \\ m\ddot{\mathbf{v}}_2(t) &= k\mathbf{v}_1(t) - 2k\mathbf{v}_2(t) + k\mathbf{u}(t) \end{aligned} \tag{4.109}$$

We could convert (4.109) to a four-dimensional first-order state equation. However, emboldened by the formal analogy which we found between the solution to the state equation and its scalar counterpart, we develop a second-order vector equation which is equivalent to (4.109) and which keeps explicit the second-order nature of the individual equations.

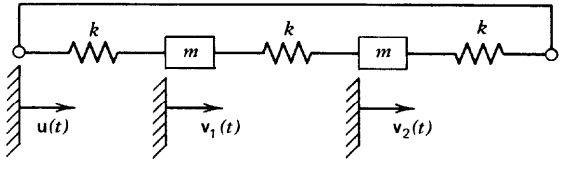


Figure 4.9. A model for spring-mounted equipment.

Let $\mathbf{x} = (v_1\ v_2)^T$. Then (4.109) becomes

$$\ddot{\mathbf{x}}(t)+\begin{pmatrix} 2k/m & -k/m \\ -k/m & 2k/m \end{pmatrix}\mathbf{x}(t)=\begin{pmatrix} k/m \\ k/m \end{pmatrix}\mathbf{u}(t) \tag{4.110}$$

The 2×2 matrix in (4.110) is known as the **stiffness matrix** for the system. Equation (4.110) is a special case of the general vector equation

$$\ddot{\mathbf{x}}(t)+\mathbf{A}\mathbf{x}(t)=\mathbf{B}\mathbf{u}(t) \tag{4.111}$$

where $\mathbf{x}(t)$ is $n \times 1$, $\mathbf{u}(t)$ is $m \times 1$, $\mathbf{B}$ is $n \times m$, and $\mathbf{A}$ is an $n \times n$ diagonalizable matrix with positive eigenvalues.* Equation (4.111) is a convenient way to express many conservative systems; for example, a frictionless mechanical system which contains n masses coupled by springs; or a lossless electrical network containing interconnected inductors and capacitors. We solve (4.110) and (4.111) by analogy with the scalar case.

The scalar counterpart of (4.111) is

$$\ddot{\mathbf{f}}(t)+\omega^2\mathbf{f}(t)=\mathbf{u}(t) \tag{4.112}$$

We found in P&C 3.6 that the inverse of (4.112), in terms of the initial conditions $\mathbf{f}(0)$ and $\dot{\mathbf{f}}(0)$, is

$$\mathbf{f}(t)=\mathbf{f}(0)\cos\omega t+\frac{\dot{\mathbf{f}}(0)}{\omega}\sin\omega t+\int_0^t\frac{\sin\omega(t-s)}{\omega}\mathbf{u}(s)\,ds \tag{4.113}$$

The solution consists in an undamped oscillation of frequency ω plus a term affected by the input vibration $\mathbf{u}$.

Comparing (4.111) and (4.112), we recognize that $\mathbf{x}$ is the vector analog of $\mathbf{f}$, and $\mathbf{A}$ plays the same role as ω^2. Therefore, we expect the solution to (4.111) to be

$$\begin{aligned}\mathbf{x}(t) = \cos(\sqrt{\mathbf{A}}\,t)\mathbf{x}(0) + (\sqrt{\mathbf{A}})^{-1}\sin(\sqrt{\mathbf{A}}\,t)\dot{\mathbf{x}}(0)\\ + \int_0^t(\sqrt{\mathbf{A}})^{-1}\sin\left[\sqrt{\mathbf{A}}\,(t-s)\right]\mathbf{B}\mathbf{u}(s)\,ds\end{aligned} \tag{4.114}$$

By $\sqrt{\mathbf{A}}$ we mean any matrix whose square equals $\mathbf{A}$. As with the scalar square root, $\sqrt{\mathbf{A}}$ is not unique. The fundamental formula (4.103) indicates that $\sqrt{\mathbf{A}}$ depends on the square roots of the eigenvalues of $\mathbf{A}$. We use in

*The matrix $\mathbf{A}$ is symmetric and positive definite. Such a matrix necessarily has positive real eigenvalues. See P&C 5.9 and 5.28.

(4.114) the principal square root of **A**—the one involving positive square roots of the eigenvalues (P&C 4.28). Recall from the discussion following Example 5 that functions of **A** commute with each other; the order of multiplication of $(\sqrt{\mathbf{A}})^{-1}$ and $\sin(\sqrt{\mathbf{A}}\,t)$ is arbitrary.

Equation (4.114) can be derived by finding a matrix Green's function and matrix boundary kernel for (4.111) (P&C 4.32). Or it can be verified by showing that it is a solution to the differential equation (4.111).

Exercise 3. Verify (4.114) by substituting $\mathbf{x}(t)$ into (4.111). Hint:

$$\frac{d}{dt} f(\mathbf{A}t) = \mathbf{A}\dot{f}(\mathbf{A}t) \qquad \text{(P\&C 4.30)}$$

$$\frac{d}{dt}\int_a^t g(t,s)\,ds = \int_a^t \frac{\partial}{\partial t} g(t,s)\,ds + g(t,t)$$

We now evaluate the solution (4.114) for the specific case (4.110) using the techniques derived for determining functions of matrices.

Exercise 4. Show that the eigendata for the 2×2 stiffness matrix **A** of (4.110) are

$$\lambda_1 = \frac{k}{m}, \qquad \lambda_2 = \frac{3k}{m}, \qquad \mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Exercise 5. Show that for **A** of (4.110),

$$f(\mathbf{A}) = f\left(\frac{k}{m}\right)\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} + f\left(\frac{3k}{m}\right)\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

It follows from Exercise 5 that

$$\cos\sqrt{\mathbf{A}}\,t = \begin{pmatrix} \dfrac{\cos\sqrt{k/m}\,t + \cos\sqrt{3k/m}\,t}{2} & \dfrac{\cos\sqrt{k/m}\,t - \cos\sqrt{3k/m}\,t}{2} \\ \dfrac{\cos\sqrt{k/m}\,t - \cos\sqrt{3k/m}\,t}{2} & \dfrac{\cos\sqrt{k/m}\,t + \cos\sqrt{3k/m}\,t}{2} \end{pmatrix}$$

$$(\sqrt{\mathbf{A}})^{-1}\sin\sqrt{\mathbf{A}}\,t = \begin{pmatrix} \dfrac{\sin\sqrt{k/m}\,t}{2\sqrt{k/m}} + \dfrac{\sin\sqrt{3k/m}\,t}{2\sqrt{3k/m}} & \dfrac{\sin\sqrt{k/m}\,t}{2\sqrt{k/m}} - \dfrac{\sin\sqrt{3k/m}\,t}{2\sqrt{3k/m}} \\ \dfrac{\sin\sqrt{k/m}\,t}{2\sqrt{k/m}} - \dfrac{\sin\sqrt{3k/m}\,t}{2\sqrt{3k/m}} & \dfrac{\sin\sqrt{k/m}\,t}{2\sqrt{k/m}} + \dfrac{\sin\sqrt{3k/m}\,t}{2\sqrt{3k/m}} \end{pmatrix}$$

$$(\sqrt{\mathbf{A}})^{-1}\sin[\sqrt{\mathbf{A}}\,(t-s)]\mathbf{B} = \sqrt{k/m}\,\sin\sqrt{k/m}\,(t-s)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

These three matrices can be substituted into (4.114) to obtain $\mathbf{x}(t)$ explicitly as a complicated function of the input data $\mathbf{u}(t)$, $\mathbf{x}(0)$, and $\dot{\mathbf{x}}(0)$.

Even though the general form of $\mathbf{x}(t)$ is complicated, we can provide a simple physical interpretation of the eigendata of the stiffness matrix of (4.110). Let $\mathbf{x}(0)=\mathbf{x}_1$, $\dot{\mathbf{x}}(0) = \boldsymbol{\theta}$, and $\mathbf{u}(t)=0$. Then recalling that $\mathbf{A}$ and $f(\mathbf{A})$ have the same eigenvectors,

$$\mathbf{x}(t) \triangleq \begin{pmatrix} \mathbf{v}_1(t) \\ \mathbf{v}_2(t) \end{pmatrix} = \cos\sqrt{\mathbf{A}}\, t \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \cos\sqrt{k/m}\, t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The first eigenvector initial condition excites a sinusoidal oscillation of frequency $\sqrt{k/m} = \sqrt{\lambda_1}$. In this first mode of oscillation, both masses move together-the center spring is not stressed. The system acts like a single mass with a spring-mass ratio of $2k/2m = k/m=\lambda_1$. A second mode of oscillation can be excited by the conditions $\mathbf{x}(0)=\mathbf{x}_2$, $\dot{\mathbf{x}}(0) = \boldsymbol{\theta}$, $\mathbf{u}(t)=0$;

$$\mathbf{x}(t) \triangleq \begin{pmatrix} \mathbf{v}_1(t) \\ \mathbf{v}_2(t) \end{pmatrix} = \cos\sqrt{\mathbf{A}}\, t \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \cos\sqrt{3k/m}\, t \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The second eigenvector initial condition excites a sinusoidal oscillation of frequency $\sqrt{3k/m} = \sqrt{\lambda_2}$. In this mode of oscillation, the masses move in opposite directions-the midpoint of the center spring does not move. The system acts like a pair of mirror images, each with a spring-mass ratio of $(k + 2k)m = 3\ k/m =\lambda_2$. Thus the eigenvectors and eigenvalues of $\mathbf{A}$ are natural modes of oscillation and squares of natural frequencies of oscillation, respectively.

The initial conditions $\dot{\mathbf{x}}(0) =\mathbf{x}_1$ or $\dot{\mathbf{x}}(0) =\mathbf{x}_2$ also excite the above two natural modes of oscillation. We note that for this particular example $\mathbf{Bu}(t)$ is of the form of $\mathbf{x}_1$. The motion excited by the input vibration $\mathbf{u}(t)$ can only be proportional to $\mathbf{x}_1$. Whether or not the motion is a sinusoidal oscillation is determined by the form of $\mathbf{u}(t)$.

4.7 Problems and Comments

4.1 Let $\mathcal{W}_1=\text{span}\{(1,0,1)\}$ and $\mathcal{W}_2=\text{span}\{(1,0,0),\ (0,1,0)\}$ in $\mathcal{R}^3$.

(a) Show that an arbitrary vector $\mathbf{x}$ in $\mathcal{R}^3$ can be decomposed into a unique pair of components $\mathbf{x}_1$ and $\mathbf{x}_2$ from $\mathcal{W}_1$ and $\mathcal{W}_2$, respectively.

(b) Let $\mathbf{P}_1$ be the projector onto $\mathcal{W}_1$ along $\mathcal{W}_2$, and $\mathbf{P}_2$ the

projector onto $\mathcal{W}_2$ along $\mathcal{W}_1$. Let $\mathcal{E}$ be the standard basis for $\mathcal{R}^3$. Find $[\mathbf{P}_1]_{\mathcal{X}\mathcal{X}}$ and $[\mathbf{P}_2]_{\mathcal{X}\mathcal{X}}$.

4.2 Let the linear operator $\mathbf{T}$ defined by $\mathbf{Tx} \triangleq \mathbf{Ax}$ operate on the space $\mathcal{M}^{n\times 1}$. Let the subspaces $\mathcal{W}_1$ and $\mathcal{W}_2$ of $\mathcal{M}^{n\times 1}$ be composed of vectors of the form

$$\begin{pmatrix} \xi_1 \\ \vdots \\ \xi_m \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \eta_{m+1} \\ \vdots \\ \eta_n \end{pmatrix}$$

respectively. Determine the form of $\mathbf{A}$ if

(*a*) $\mathcal{W}_1$ is invariant under $\mathbf{T}$.

(*b*) $\mathcal{W}_2$ is invariant under $\mathbf{T}$.

(*c*) Both $\mathcal{W}_1$ and $\mathcal{W}_2$ are invariant under $\mathbf{T}$.

Hint: investigate an example where $m = 1$, $n = 3$.

4.3 The *Cartesian product* is useful for *building up* complicated vector spaces from simple ones. The *direct sum*, on the other hand, is useful for *subdividing* complicated vector spaces into smaller subspaces.

(*a*) Define $\mathbf{T}_a : \mathcal{R}^2 \to \mathcal{R}^2$ by $\mathbf{T}_a(\xi_1, \xi_2) \triangleq (\xi_1 - \xi_2, \xi_1)$.
Let $\mathcal{X}_a = \{(1,0), (0,1)\}$. Find $[\mathbf{T}_a]_{\mathcal{X}_a\mathcal{X}_a}$.
Define $\mathbf{T}_b: \mathcal{R}^1 \to \mathcal{R}^1$ by $\mathbf{T}_b(\xi_3) \triangleq (-\xi_3)$.
Let $\mathcal{X}_b = \{(1)\}$. Find $[\mathbf{T}_b]_{\mathcal{X}_b\mathcal{X}_b}$.

(*b*) If we do not distinguish between $((\xi_1, \xi_2), (\xi_3))$ and (ξ_1, ξ_2, ξ_3), then $\mathcal{R}^3 = \mathcal{R}^2 \times \mathcal{R}^1$. Define $\mathbf{T}: \mathcal{R}^3 \to \mathcal{R}^3$ by $\mathbf{T}((\xi_1, \xi_2), (\xi_3)) \triangleq (\mathbf{T}_a(\xi_1, \xi_2), \mathbf{T}_b(\xi_3))$. Let $\mathcal{X} = \{((1, 0), (0)), ((0, 1), (0)), ((0,0), (1))\}$. Find $[\mathbf{T}]_{\mathcal{X}\mathcal{X}}$. What is the relationship between $[\mathbf{T}]_{\mathcal{X}\mathcal{X}}$, $[\mathbf{T}_a]_{\mathcal{X}_a\mathcal{X}_a}$, $[\mathbf{T}_b]_{\mathcal{X}_b\mathcal{X}_b}$?

(*c*) Let $\mathcal{W}_1 = \mathcal{R}^2 \times \{(0)\}$ and $\mathcal{W}_2 = \{(0,0)\} \times \mathcal{R}^1$. Then $\mathcal{R}^3 = \mathcal{W}_1 \oplus \mathcal{W}_2$. Appropriate bases for $\mathcal{W}_1$ and $\mathcal{W}_2$ are $\mathcal{X}_1 = \{((1,0), (0)), ((0,1), (0))\}$ and $\mathcal{X}_2 = \{((0,0), (1))\}$. Define $\mathbf{T}_1: \mathcal{W}_1 \to \mathcal{W}_1$ by $\mathbf{T}_1(\xi_1, \xi_2, 0) \triangleq (\xi_1 - \xi_2, \xi_1, 0)$. Define $\mathbf{T}_2: \mathcal{W}_2 \to \mathcal{W}_2$ by $\mathbf{T}_2(0, 0, \xi_3) \triangleq (0, 0, -\xi_3)$. Find $[\mathbf{T}_1]_{\mathcal{X}_1\mathcal{X}_1}$ and $[\mathbf{T}_2]_{\mathcal{X}_2\mathcal{X}_2}$. What is the relationship between $[\mathbf{T}]_{\mathcal{X}\mathcal{X}}$, $[\mathbf{T}_1]_{\mathcal{X}_1\mathcal{X}_1}$, and $[\mathbf{T}_2]_{\mathcal{X}_2\mathcal{X}_2}$?

(*d*) In general, if $\mathcal{V} = \mathcal{W}_1 \oplus \cdots \oplus \mathcal{W}_p$, with each subspace $\mathcal{W}_i$ invariant under $\mathbf{T}$, then $\{ \mathcal{W}_i \}$ decomposes $\mathbf{T}$ into $\{\mathbf{T}_i: \mathcal{W}_i \to \mathcal{W}_i\}$. Let $\mathcal{X}_i$ be a basis for $\mathcal{W}_i$. Then $\mathcal{X} = \{ \mathcal{X}_1, \ldots, \mathcal{X}_p \}$ is a basis for $\mathcal{V}$. If $\mathcal{V}$ is finite-dimensional, then

$$[\mathbf{T}]_{\mathcal{X}\mathcal{X}} = \begin{pmatrix} [\mathbf{T}_1]_{\mathcal{X}_1\mathcal{X}_1} & & \\ & \ddots & \\ & & [\mathbf{T}_p]_{\mathcal{X}_p\mathcal{X}_p} \end{pmatrix}$$

with zeros everywhere except in the blocks on the diagonal. Show that the transformation $\mathbf{T}: \mathcal{R}^3 \to \mathcal{R}^3$ defined by $\mathbf{T}(\xi_1, \xi_2, \xi_3) \triangleq (\xi_1 + \xi_2, 2\xi_1 + \xi_2 - \xi_3, \xi_1 + \xi_3)$ is decomposed by $\mathcal{W}_1$ and $\mathcal{W}_2$, where $\mathcal{W}_1$ consists in vectors of the form $(\xi_1, \xi_2, \xi_1 + \xi_2)$ and $\mathcal{W}_2$ consists in vectors of the form (ξ_1, ξ_1, ξ_1). Note that there is no Cartesian product which corresponds to this invariant direct-sum decomposition in the same manner as (*b*) corresponds to (*c*).

4.4 Find the eigenvalues and eigenvectors of the following matrices:

(*a*) $\begin{pmatrix} -3 & 0 & 0 \\ -5 & 2 & 0 \\ -5 & 1 & 1 \end{pmatrix}$ (*b*) $\begin{pmatrix} -2 & 0 & 0 \\ -3 & 1 & 3 \\ 0 & 0 & -2 \end{pmatrix}$

4.5 Let $\mathbf{A}$ be an $n \times n$ matrix. Denote the characteristic polynomial for $\mathbf{A}$ by $c(\lambda) = \lambda^n + b_1\lambda^{n-1} + \cdots + b_n$. The trace of a matrix is defined as the sum of its diagonal elements, an easily computed quantity. An iterative method based on the trace function has been proposed for computing the coefficients $\{ b_i \}$ in the characteristic polynomial [4.3, p. 296]. The iteration is:

$$b_1 = -\operatorname{Trace}(\mathbf{A})$$

$$b_2 = -\tfrac{1}{2}\left[b_1 \operatorname{Trace}(\mathbf{A}) + \operatorname{Trace}(\mathbf{A}^2) \right]$$

$$b_3 = -\frac{1}{3}\left[b_2 \operatorname{Trace}(\mathbf{A}) + b_1 \operatorname{Trace}(\mathbf{A}^2) + \operatorname{Trace}(\mathbf{A}^3) \right]$$

$$\vdots$$

$$b_n = -\frac{1}{n}\left[b_{n-1} \operatorname{Trace}(\mathbf{A}) + \cdots + b_1 \operatorname{Trace}(\mathbf{A}^{n-1}) + \operatorname{Trace}(\mathbf{A}^n) \right]$$

(*a*) How many multiplications are required to compute the characteristic polynomial by means of this trace iteration? Compare the iteration with Krylov's method.

(*b*) Compute the characteristic polynomial by Krylov's method and by the trace iteration for the matrix

$$\mathbf{A}=\begin{pmatrix}1 & 2 & 3\\ 2 & 1 & 0\\ 1 & 1 & 1\end{pmatrix}$$

*4.6 Let $\mathbf{A}$ be an $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Then

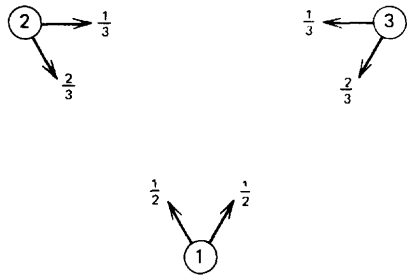
(*a*) $\text{Det}(\mathbf{A})=\lambda_1 \cdot \lambda_2 \cdot \cdots \cdot \lambda_n$

(*b*) $\text{Trace}(\mathbf{A}) \triangleq a_{11}+a_{22}+\cdots+a_{nn}=\lambda_1+\lambda_2+\cdots+\lambda_n$

(*c*) If $\mathbf{A}$ is triangular (i.e, if all elements to one side of the main diagonal are zero), then the diagonal elements of $\mathbf{A}$ are $\mathbf{A}_{ii}=\lambda_i$.

4.7 Three men are playing ball. Every two seconds the one who has the ball tosses it to one of the others, with the probabilities shown in the diagram. Let $p_n(i)$ be the probability that the ball is held by the ith player (or is in the ith state) after the nth toss. Let p_{ij} be the probability with which player j throws the ball to player i. The theory of conditional probability requires that

$$p_n(i)=\sum_{j=1}^{3} p_{ij}p_{n-1}(j) \quad \text{for} \quad i=1,2,3$$



Let $\mathbf{x}_n \triangleq (p_n(1)\ p_n(2)\ p_n(3))^T$. We call $\mathbf{x}_n$ a state probability vector. Let Ω denote the set of all possible 3×1 state probability vectors. The elements of each vector in Ω are non-negative and sum to one. Note that Ω is a *subset* of $\mathfrak{R}^{n\times 1}$, rather than a subspace. The game is an example of a Markov process. The future state probability vectors depend only on the present state, and not on the past history.

(*a*) A matrix whose columns are members of Ω is called a *transition probability matrix*. Find the transition probability matrix $\mathbf{A}$ such that $\mathbf{x}_n = \mathbf{A}\mathbf{x}_{n-1}$. Note that $\mathbf{x}_n = \mathbf{A}^n\mathbf{x}_0$; we refer to $\mathbf{A}^n$ as the n-step transition probability matrix.

(*b*) Determine the eigenvalues and eigenvectors of $\mathbf{A}$. What do they tell us about the game? (Hint: $\lambda = 1$ is an eigenvalue.)

(*c*) Find the spectral matrix $\mathbf{\Lambda}$ and the modal matrix $\mathbf{S}$ such that $\mathbf{A} = \mathbf{S}\mathbf{\Lambda}\mathbf{S}^{-1}$. Show that every transition probability matrix has $\lambda = 1$ as an eigenvalue.

(*d*) In the game described previously, the state probability vector $\mathbf{x}_n$ becomes independent of the initial state as n becomes large. Find the form of the limiting state probability vector. (Hint: find $\lim_{n\to\infty}\mathbf{A}^n$ using the substitution $\mathbf{A} = \mathbf{S}\mathbf{\Lambda}\mathbf{S}^{-1}$.) We note that the eigenvalues of every transition probability matrix satisfy $\lambda_i \leqslant 1$ [4.4, p. 4291.

(*e*) A transition probability matrix wherein the elements of each row also sum to one is called a *stochastic matrix*. What is the limiting state probability vector, $\lim_{n\to\infty}\mathbf{x}_n$, if the transition probabilities in the above game are modified to yield a stochastic matrix?

4.8 Let

$$\mathbf{A} = \begin{pmatrix} 3 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Find a matrix $\mathbf{S}$ for which $\mathbf{S}^{-1}\mathbf{A}\mathbf{S}$ is a diagonal matrix.

4.9 Find a nondiagonal matrix $\mathbf{A}$ which has as its diagonal form the matrix

$$\mathbf{\Lambda} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

What are eigenvectors of $\mathbf{A}$?

4.10 We wish to compute the eigendata of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$$

Assume that numerical computations have produced the following

approximations to the eigenvalues: $\lambda_1 \approx 0.99$ and $\lambda_2 \approx -1.01$. Use the inverse iteration method to compute more accurate eigenvalues and corresponding eigenvectors. Start the iterations with the initial vector $\mathbf{z}_0 = (1 \quad -1)^T$.

4.11 The **Jacobi method** for determining the eigenvalues and eigenvectors of a symmetric matrix $\mathbf{A}$ consists in performing a sequence of similarity transformations which reduce the off-diagonal elements of $\mathbf{A}$ to zero. In order to avoid a sequence of matrix inversions, we perform the similarity transformations with orthogonal matrices (matrices for which $\mathbf{S}^{-1} = \mathbf{S}^T$). Thus we let $\mathbf{A}_1 = \mathbf{S}_1^T\mathbf{A}\mathbf{S}_1$ and $\mathbf{A}_k = \mathbf{S}_k^T\mathbf{A}_{k-1}\mathbf{S}_k$ for $k = 2, 3, \ldots$. The eigenvalues of a matrix are not changed by similarity transformations. Consequently, the resulting diagonal matrix must be the spectral matrix (with the eigenvalues of $\mathbf{A}$ on its diagonal); that is,

$$\lim_{k\to\infty} \mathbf{A}_k = \lim_{k\to\infty} (\mathbf{S}_1\mathbf{S}_2\cdots\mathbf{S}_k)^T\mathbf{A}(\mathbf{S}_1\mathbf{S}_2\cdots\mathbf{S}_k) = \mathbf{\Lambda}$$

Furthermore, the matrix $\mathbf{S} = \lim_{k\to\infty}(\mathbf{S}_1\mathbf{S}_2\cdots\mathbf{S}_k)$ must be a modal matrix for $\mathbf{A}$ (with the eigenvectors of $\mathbf{A}$ as its columns). Let $a_{ij} = (\mathbf{A}_{k-1})_{ij}$. It is shown in [4.13] that a_{ij} and a_{ji} can be driven to zero simultaneously by a similarity transformation which uses the orthogonal matrix $\mathbf{S}_k$ which differs from the identity matrix only in the following elements:

$$(\mathbf{S}_k)_{ii} = (\mathbf{S}_k)_{jj} = \sqrt{(\gamma + |\beta|)/2\gamma} = \cos\phi$$

$$(\mathbf{S}_k)_{ij} = -(\mathbf{S}_k)_{ji} = \alpha \operatorname{sign}(\beta)/(2\gamma\cos\phi) = \sin\phi$$

where $\alpha = -a_{ij}$, $\beta = (a_{ii} - a_{ji})/2$, and $\gamma = (\alpha^2 + \beta^2)^{1/2}$. (Multiplication by the matrix $\mathbf{S}_k$ can be interpreted as a rotation of the axes of the i and j coordinates through an angle ϕ.) In the Jacobi method we pick an $\mathbf{S}_k$ of the above form which drives the largest pair of off-diagonal elements of $\mathbf{A}_{k-1}$ to zero. Although later transformations will usually make these elements nonzero again, the sum of the squares of the off-diagonal elements is reduced at each iteration.

(*a*) Use the Jacobi method to compute (to slide rule accuracy) the eigenvalues and eigenvectors of the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$$

(*b*) Calculate the eigenvalues of $\mathbf{A}$ by solving the characteristic polynomial. Determine the corresponding eigenvectors. Compare the results with (*a*).

4.12 Let $\mathbf{L}$ be the differential operator defined by $\mathbf{Lf} \triangleq \mathbf{f}''$. Assume $\mathbf{L}$ acts on the subspace of functions in $\mathcal{C}^2(0,\pi)$ which satisfy the boundary conditions $\boldsymbol{\beta}_1(\mathbf{f}) = \boldsymbol{\beta}_2(\mathbf{f}) = 0$. Find all the eigenvalues and corresponding eigenfunctions of $\mathbf{L}$ for each of the following definitions of the boundary conditions:

(*a*) $\boldsymbol{\beta}_1(\mathbf{f}) = \mathbf{f}(0), \quad \boldsymbol{\beta}_2(\mathbf{f}) = \mathbf{f}(\pi)$

(*b*) $\boldsymbol{\beta}_1(\mathbf{f}) = \mathbf{f}(0) + \mathbf{f}(\pi), \quad \boldsymbol{\beta}_2(\mathbf{f}) = \mathbf{f}'(0) - \mathbf{f}'(\pi)$

(*c*) $\boldsymbol{\beta}_1(\mathbf{f}) = \mathbf{f}(0) + 2\mathbf{f}(\pi), \quad \boldsymbol{\beta}_2(\mathbf{f}) = \mathbf{f}'(0) - 2\mathbf{f}'(\pi)$

(*d*) $\boldsymbol{\beta}_1(\mathbf{f}) = \mathbf{f}(0) - \mathbf{f}(\pi), \quad \boldsymbol{\beta}_2(\mathbf{f}) = \mathbf{f}'(0) - \mathbf{f}'(\pi)$

4.13 Find the eigenvalues and eigenfunctions associated with the differential system $\mathbf{f}'' - c\mathbf{f} = \mathbf{u}$, $\mathbf{f}(0) = \mathbf{f}'(1) = 0$. Hint: $\ln(-1) = i(\pi + 2k\pi)$, $k = 0, \pm 1, \pm 2, \ldots$. For what values of the constant c is the system invertible?

4.14 Let $\mathcal{V}$ be a space of functions $\mathbf{f}$ whose values $\mathbf{f}(n)$ are defined only for integer values of *n*. Define the forward difference operator $\boldsymbol{\Delta}$ on $\mathcal{V}$ by

$$(\boldsymbol{\Delta}\mathbf{f})(n) \triangleq \mathbf{f}(n+1) - \mathbf{f}(n)$$

(This operator can be used to approximate the differential operator $\mathbf{D}$.) Find the eigenvalues and eigenfunctions of $\boldsymbol{\Delta}$.

4.15 Define $\boldsymbol{\nabla}^2\mathbf{f}(s,t) \triangleq (\partial^2\mathbf{f}/\partial s^2) + (\partial^2\mathbf{f}/\partial t^2)$ in the rectangular region $0 \leqslant s \leqslant a$ and $0 \leqslant t \leqslant b$. Let $\mathbf{f}$ satisfy the boundary conditions

$$\frac{\partial \mathbf{f}}{\partial s}(0,t) = \frac{\partial \mathbf{f}}{\partial s}(a,t) = \frac{\partial \mathbf{f}}{\partial t}(s,0) = \frac{\partial \mathbf{f}}{\partial t}(s,b) = 0$$

Show that the partial differential operator $\boldsymbol{\nabla}^2$ and the given boundary conditions have the eigendata

$$\lambda_{km} = -\left(\frac{m\pi}{a}\right)^2 - \left(\frac{k\pi}{b}\right)^2$$

$$\mathbf{f}_{km}(s,t) = \cos\left(\frac{m\pi s}{a}\right)\cos\left(\frac{k\pi t}{b}\right)$$

for $k, m = 0, 1, 2, \ldots$.

*4.16 Let $\mathbf{A}$ be the companion matrix for an *n*th order constant-coefficient differential operator. Denote the eigenvalues of $\mathbf{A}$ by $\lambda_1, \ldots, \lambda_n$.

(*a*) Show that the vector $\mathbf{z}_i = (1\ \lambda_i\ \lambda_i^2\ \cdots\ \lambda_i^{n-1})^T$ is an eigenvector of $\mathbf{A}$ for the eigenvalue λ_i. Show further that there is only one independent eigenvector for each distinct eigenvalue.

(*b*) Show that the Vandermond matrix

$$\begin{pmatrix} 1 & \cdots & 1 \\ \lambda_1 & \cdots & \lambda_n \\ \vdots & \vdots & \vdots \\ \lambda_1^{n-1} & \cdots & \lambda_n^{n-1} \end{pmatrix}$$

is a modal matrix for $\mathbf{A}$ if and only if the eigenvalues of $\mathbf{A}$ are all distinct.

*4.17 *The power method:* the inverse of the differential operator $\mathbf{L} = \mathbf{D}^2$ with the boundary conditions $\mathbf{f}(0) = \mathbf{f}(1) = 0$ is the integral operator $\mathbf{T}$ defined by

$$(\mathbf{Tu})(t) = \int_0^t (t-1)s\mathbf{u}(s)\,ds + \int_t^1 t(s-1)\mathbf{u}(s)\,ds$$

The functions $\mathbf{f}_n(t) = \sin n\pi t$, $n = 1, 2, \ldots$, are eigenfunctions for both the differential and integral operators. We can find the dominant eigenvalue and the corresponding eigenfunction of $\mathbf{T}$ by the power method. We just compute the sequence of functions $\mathbf{u}_k = \mathbf{T}^k\mathbf{u}_0$, for some initial function $\mathbf{u}_0$, until $\mathbf{u}_k$ is a sufficiently good approximation to the dominant eigenfunction.

(*a*) Let $\mathbf{u}_0(t) = 1$, and compute $\mathbf{u}_1$ and $\mathbf{u}_2$.

(*b*) Compare $\mathbf{u}_1$ and $\mathbf{u}_2$ with the true dominant eigenfunction. Use the iterates $\{\mathbf{u}_k\}$ to determine an approximation to the dominant eigenvalue.

4.18 (*a*) Determine an ordered basis of generalized eigenvectors for the matrix

$$\mathbf{A} = \begin{pmatrix} 5 & -1 & 1 & 1 & 0 & 0 \\ 1 & 3 & -1 & -1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 1 & 1 \\ 0 & 0 & 0 & 4 & -1 & -1 \\ 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{pmatrix}$$

Hint: $\det(\mathbf{A} - \lambda\mathbf{I}) = (4-\lambda)^5(2-\lambda)$.

(*b*) Determine the Jordan canonical form of $\mathbf{A}$ [relative to the basis found in (*a*)].

(*c*) Determine the "change of coordinates" matrix $\mathbf{S}$ which would be used in a similarity transformation on $\mathbf{A}$ in order to obtain the Jordan form found in (*b*). (Obtain only the obvious matrix, not its inverse.)

4.19 Find a matrix $\mathbf{S}$ such that $\mathbf{S}^{-1}\mathbf{B}\mathbf{S}$ is in Jordan form, for

$$\mathbf{B} = \begin{pmatrix} 3 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 1 & 3 & 1 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

Hint: $c(\lambda) = (2-\lambda)^3(3-\lambda)$.

4.20 The minimal polynomial $m(\lambda)$ and the characteristic polynomial $c(\lambda)$ are useful for reducing effort in matrix computations. Assume $f(\mathbf{A})$ is a polynomial in the $n \times n$ matrix $\mathbf{A}$, and $f(\mathbf{A})$ includes powers of $\mathbf{A}$ higher than n. We divide $f(\lambda)$ by $m(\lambda)$ to determine a quotient $g(\lambda)$ and a remainder $r(\lambda)$; that is, $f(\lambda) = g(\lambda)\, m(\lambda) + r(\lambda)$. If we replace λ by $\mathbf{A}$, and use the fact that $m(\mathbf{A}) = \mathbf{\Theta}$, we observe that $f(\mathbf{A}) = r(\mathbf{A})$. The remainder $r(\mathbf{A})$ is of lower degree (in $\mathbf{A}$) than $m(\mathbf{A})$, regardless of the degree of $f(\mathbf{A})$. Consequently, $r(\mathbf{A})$ is easier to compute than is $f(\mathbf{A})$. The same procedure can be carried out using the more easily determined characteristic polynomial rather than the minimal polynomial. Use this "remainder" method to compute the matrix $\mathbf{A}^5$ for

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

4.21 Assume f is analytic at the eigenvalues of the matrix $\mathbf{A}$. Find the component matrices of $\mathbf{A}$ and express $f(\mathbf{A})$ as a linear combination of these components for:

$$(a) \quad \mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix} \qquad (b) \quad \mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

4.22 The gamma function $\Gamma(p)$ is defined for all positive values of the scalar p. If p is a positive integer, $\Gamma(p) = (p-1)!$ Find $\Gamma(\mathbf{A})$, where

$$\mathbf{A} = \begin{pmatrix} 3 & 0 & -2 \\ 0 & 3 & -1 \\ 0 & 0 & 2 \end{pmatrix}$$

4.23 Let

$$f(\lambda) \triangleq 0, \qquad \lambda \leqslant c$$

$$\triangleq (\lambda - c)^2, \qquad \lambda \geqslant c$$

and

$$\mathbf{\Lambda} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(*a*) Find $f(\mathbf{\Lambda})$.

(*b*) Consider various values of c. Is the resulting matrix what you would expect?

4.24 If $\mathbf{A}$ is invertible, the inverse can be computed by evaluating $f(\mathbf{A})$ for $f(\lambda) \triangleq 1/\lambda$. By modifying f, we can compute a "pseudoinverse" for a matrix which has zero eigenvalues. We merely change the definition of f to

$$\hat{f}(\lambda) \triangleq \frac{1}{\lambda}, \qquad \lambda \neq 0$$

$$\triangleq 0, \quad \lambda = 0$$

(See P&C 6.22 for an interpretation of this "pseudoinverse.")

(*a*) Find the inverse of the matrix $\mathbf{A}$ of P&C 4.21 *a* by evaluating $f(\mathbf{A})$.

(*b*) Find the "pseudoinverse" of the following matrix by evaluating $\hat{f}(\mathbf{B})$:

$$\mathbf{B} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

4.25 The constituent matrices of a square matrix A can be determined by partial fraction expansion of the *resolvant matrix*, $(s\mathbf{I} - \mathbf{A})^{-1}$ (the resolvant matrix is the Laplace transform of $e^{\mathbf{A}t}$). Let

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 3 & 0 \\ -1 & 1 & 2 \end{pmatrix}$$

(*a*) Determine the resolvant matrix $(s\mathbf{I} - \mathbf{A})^{-1}$ by inverting $(s\mathbf{I} - \mathbf{A})$.

(*b*) Perform a partial fraction expansion of $(s\mathbf{I} - \mathbf{A})^{-1}$; that is, perform a partial fraction expansion of each term of $(s\mathbf{I} - \mathbf{A})^{-1}$, and arrange the expansion into a sum of terms with multipliers which are constant 3×3 matrices.

(*c*) Let $f(\lambda) \triangleq 1/(s - \lambda)$; then $f(\mathbf{A}) = (s\mathbf{I} - \mathbf{A})^{-1}$. Express the fundamental formula for $f(\mathbf{A})$ in terms of $\{\mathbf{E}_{ij}^{\wedge}\}$, the constituent matrices for $\mathbf{A}$. (The form of the fundamental formula is determined by the minimal polynomial for $\mathbf{A}$.) Determine the constituent matrices by comparing the fundamental formula for $f(\mathbf{A})$ with the partial fraction expansion obtained in (*b*).

(*d*) Use the fundamental formula and the constituent matrices to evaluate $\mathbf{A}^5$.

4.26 Let f be a scalar-valued function of a scalar variable. Assume f is defined on the spectrum of the $n \times n$ matrix $\mathbf{A}$. Then $f(\mathbf{A})$ can be expressed as a polynomial in $\mathbf{A}$ of lower degree than the minimal polynomial for $\mathbf{A}$. That is, if r is the degree of the minimal polynomial, then $f(\mathbf{A}) = a_0\mathbf{I} + a_1\mathbf{A} + \cdots + a_{r-1}\mathbf{A}^{r-1}$. The coefficients $\{a_i\}$ can be determined by evaluating the corresponding scalar equation, $f(\lambda) = a_0 + a_1\lambda + \cdots + a_{r-1}\lambda^{r-1}$, on the spectrum of $\mathbf{A}$; the resulting equations are always solvable.

(*a*) Find the minimal polynomial for the matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ -1 & 3 & 0 \\ -1 & 1 & 2 \end{pmatrix}$$

(*b*) For the matrix $\mathbf{A}$ introduced in (*a*), evaluate the matrix function $f(\mathbf{A}) \triangleq \mathbf{A}^5$ by the technique described above.

4.27 Let the $n \times n$ matrix $\mathbf{A}$ be diagonalizable. Then, the fundamental formula is $f(\mathbf{A}) = \sum_{i=1}^{p} f(\lambda_i)\mathbf{E}_{i0}^{\wedge}$, where p is the number of distinct eigenvalues. The constituent matrix $\mathbf{E}_{i0}^{\wedge}$ is the projector on the eigenspace for λ_i along the sum of the other eigenspaces. It can be expressed as

$$\mathbf{E}_{i0}^{\wedge} = \prod_{j \neq i} \left(\frac{\mathbf{A} - \lambda_j \mathbf{I}}{\lambda_i - \lambda_j} \right)$$

($\mathbf{E}_{i0}^{\wedge}$ acts like $\mathbf{I}$ on the eigenspace for λ_i and like $\mathbf{0}$ on the eigenspace

for λ_j.) The scalar equivalent of the fundamental formula,

$$f(\lambda)=\sum_{1=1}^{p} f(\lambda_i)\prod_{j\neq i}\left(\frac{\lambda-\lambda_j}{\lambda_i-\lambda_j}\right)$$

is known as the *Lagrange interpolation formula* for the data points $\lambda_1\ldots,\lambda_p$.

(*a*) Let

$$\mathbf{A}=\begin{pmatrix}1 & 1 & 2\\ 0 & 2 & -1\\ 0 & 0 & 3\end{pmatrix}$$

Find the constituent matrices $\mathbf{E}_{i0}^{\mathbf{A}}$ by evaluating the polynomial expressions given above.

(*b*) Use the fundamental formula to evaluate the matrix exponential, $e^{\mathbf{A}t}$, for the matrix $\mathbf{A}$ given in (*a*).

4.28 Use the fundamental formula to find four square roots of the matrix

$$\mathbf{A}=\begin{pmatrix}20 & -8\\ 48 & -20\end{pmatrix}$$

*4.29 (*a*) *Commuting matrices:* if $\mathbf{A}$ and $\mathbf{B}$ commute (i.e., $\mathbf{AB}=\mathbf{BA}$), then

$$(\mathbf{A}+\mathbf{B})^n=\sum_{k=0}^{n}\binom{n}{k}\mathbf{A}^{n-k}\mathbf{B}^k,\qquad n=0,1,2,\ldots$$

where

$$\binom{n}{k}=\frac{n!}{k!(n-k)!}$$

That is, the binomial theorem is satisfied.

(*b*) The algebra of matrices is essentially the same as the algebra of scalars if the matrices commute with each other. Therefore, a functional relation which holds for scalars also holds for commuting matrices if the required matrix functions are defined. For example, $e^{\mathbf{A}+\mathbf{B}}=e^{\mathbf{A}}e^{\mathbf{B}}$, $\cos(\mathbf{A}+\mathbf{B})=\cos\mathbf{A}\cos\mathbf{B}-\sin\mathbf{A}\sin\mathbf{B}$; the binominal theorem is satisfied; etc.

(*c*) If $\mathbf{A}$ and $\mathbf{B}$ are diagonalizable, then they are commutable if and only if they are diagonalizable by the same similarity

transformation (i.e., if and only if they have the same eigenvectors).

4.30 Use the fundamental formula to show that $(d/dt)f(\mathbf{A}t) = \mathbf{A}\dot{f}(\mathbf{A}t)$ for any square matrix $\mathbf{A}$ and any function f which is analytic on the spectrum of $\mathbf{A}$.

4.31 Let $\mathbf{f}'' + 6\mathbf{f}' + 5\mathbf{f} = \mathbf{u}$, $\mathbf{f}(0) = \mathbf{f}'(0) = 0$.

(*a*) Express the differential system in state-space form.

(*b*) Diagonalize the state equation found in (*a*).

(*c*) Draw a signal flow diagram which relates the original state variables, the canonical state variables, and the input.

(*d*) Find the state transition matrix and invert the state equation.

4.32 Let $\ddot{\mathbf{x}} + \mathbf{A}\mathbf{x} = \mathbf{B}\mathbf{u}$, where $\mathbf{x}(t)$ is $n \times 1$, $\mathbf{u}(t)$ is $m \times 1$, $\mathbf{B}$ is $n \times m$, and $\mathbf{A}$ is $n \times n$ with positive eigenvalues. Assume $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$ are known.

(*a*) Use the power series method of Frobenius to show that the complementary function for this vector differential equation is

$$\mathbf{F}_c(t) = \cos(\sqrt{\mathbf{A}}\,t)\mathbf{C}_0 + (\sqrt{\mathbf{A}})^{-1}\sin(\sqrt{\mathbf{A}}\,t)\mathbf{C}_1$$

where $\mathbf{C}_0$ and $\mathbf{C}_1$ are arbitrary $n \times n$ matrices.

(*b*) The inverse of the differential equation is of the form

$$\mathbf{x}(t) = \int_0^\infty \mathbf{K}(t,s)\mathbf{B}\mathbf{u}(s)\,ds + \mathbf{R}_1(t)\mathbf{x}(0) + \mathbf{R}_2(t)\dot{\mathbf{x}}(0)$$

Show that the Green's function $\mathbf{K}(t,s)$ and boundary kernel $\mathbf{R}_j(t)$ satisfy:

$$\frac{d^2}{dt^2}\mathbf{K}(t,s) + \mathbf{A}\mathbf{K}(t,s) = \delta(t-s)\mathbf{I}$$

$$\mathbf{K}(0,s) = \frac{d}{dt}\mathbf{K}(0,s) = \mathbf{\Theta}$$

$$\frac{d^2}{dt^2}\mathbf{R}_j(t) + \mathbf{A}\mathbf{R}_j(t) = \mathbf{\Theta}, \qquad j = 1,2$$

$$\mathbf{R}_1(0) = \mathbf{I}, \qquad \dot{\mathbf{R}}_1(0) = \mathbf{\Theta}$$

$$\mathbf{R}_2(0) = \mathbf{\Theta}, \qquad \dot{\mathbf{R}}_2(0) = \mathbf{I}$$

(*c*) Show that

$$\mathbf{K}(t,s) = \boldsymbol{\Theta}, \qquad t \leqslant s$$
$$= (\sqrt{\mathbf{A}})^{-1} \sin\left(\sqrt{\mathbf{A}}\,(t-s)\right), \qquad t \geqslant s$$
$$\mathbf{R}_1(t) = \cos\sqrt{\mathbf{A}}\,t$$
$$\mathbf{R}_2(t) = (\sqrt{\mathbf{A}})^{-1} \sin(\sqrt{\mathbf{A}}\,t)$$

4.33 In optimal control problems we often need to solve a pair of simultaneous state equations. Suppose the equations are $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{B}^{\mathrm{T}}\boldsymbol{\lambda}$ and $\dot{\boldsymbol{\lambda}} = -\mathbf{A}^{\mathrm{T}}\boldsymbol{\lambda}$, where

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(*a*) Write the pair of equations as a single state equation $\dot{\mathbf{y}} = \mathbf{Q}\mathbf{y}$, where $\mathbf{y} \triangleq \begin{pmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{pmatrix}$.

(*b*) Find the eigenvalues and-constituent matrices of $\mathbf{Q}$.

(*c*) Find the solution $\mathbf{y}$ to the state equation as a function of $\mathbf{y}(0)$.

4.8 References

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