

Research Article

Spherically Symmetric Fluid Cosmological Model with Anisotropic Stress Tensor in General Relativity

D. D. Pawar,¹ V. R. Patil,² and S. N. Bayaskar³

¹ *Department of Mathematics, Government Vidarbha Institute of Science and Humanities, Amravati 444404, India*

² *Department of Mathematics, Arts, Science and Commerce College, Chikhaldara 444807, India*

³ *Department of Mathematics, Adarsha Science, J. B. Art and Birla Commerce College, Dhamangaon Rly 444709, India*

Correspondence should be addressed to D. D. Pawar, dypawar@yahoo.com

Received 23 April 2012; Accepted 1 July 2012

Academic Editors: M. Rasetti and W.-H. Steeb

Copyright © 2012 D. D. Pawar et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

This paper deals with the cosmological models for the static spherically symmetric spacetime for perfect fluid with anisotropic stress energy tensor in general relativity by introducing the generating functions $g(r)$ and $w(r)$ and also discussing their physical and geometric properties.

1. Introduction

The study of anisotropic fluid sphere and gravitational collapse problem is important in relativistic astrophysics. Ruderman [1] investigated relativistic stellar model and shows that the stellar matter may be anisotropic at very high density ranges. Anisotropy in fluid pressure could be introduced by the existences of solid core by the presence of type 3A superfluid. Rago [2] presented the procedure to obtain the solution of the field equations by using two arbitrary functions in Einstein general relativity where two arbitrary functions are introduced: the anisotropic function that measures the degree of anisotropy and a generating function. These functions determine the relevant physical variables as well as metric coefficients. Kandalkar and Khadekar [3] have obtained analytical solutions for anisotropic matter distribution in the context of bimetric theory of gravitation. The dynamical equations governing the gravitational nonadiabatic collapse of a shear-free spherical distribution of anisotropic matter in the presence of charge have been studied by

Tikekar and Patel [4]. According to Ruderman [1] and Canuto [5], the pressure in the various gravitational collapse of spherical distribution consisting of super dense matter distribution, may not be isotropic throughout for such stars; the core region may be anisotropic. Gair [6] obtained the spherical universes with anisotropic pressure. Thomas and Ratanpal [7] studied various aspects of gravitational collapse by using analytical and numerical methods by considering the gravitational collapse for spherical distributions, consisting of superdense matter distribution. In the last few years there has been increasing interest in the interior solutions of Einstein field equations corresponding to fluid distributions with anisotropic pressures Letelier [8], Maharaj and Maartens [9], Bondi [10], Coley and Tupper [11], and Singh et al. [12]. The matter distribution is adequately described by perfect fluid due to the large-scale distribution of galaxies in our universe. Hence a relativistic treatment of the problem requires the consideration of material distribution other than the perfect fluid.

In this paper, we have obtained cosmological models for static spherically symmetric spacetime with anisotropic stress energy tensor by introducing two generating functions $g(r)$ and $w(r)$ and also discussed their physical properties.

2. Field Equations

Consider the static spherically symmetric space-time:

$$ds^2 = e^\gamma dt^2 - e^\lambda dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (2.1)$$

where λ and γ being the function of r alone.

The energy momentum tensor for perfect fluid with anisotropic stress energy with heat flux is given by

$$T_{ij} = (p + \rho)u_i u_j - p g_{ij} + \pi_{ij} + q_i u_j + q_j u_i, \quad (2.2)$$

where ρ , p , q_i , and u^i denote the matter density, fluid pressure, heat conduction vector orthogonal to u^i , and components of unit time-like flow vector field of matter, respectively. And the anisotropic stress energy tensor π_{ij} is given by

$$\pi_{ij} = \sqrt{3}S \left[c_i c_j - \frac{1}{3} (u_i u_j - g_{ij}) \right], \quad (2.3)$$

where $S = S(r)$ symbolizes the magnitude of the anisotropic stress tensor and the radial vector c^i is obtained as

$$c^i = (-e^{-\lambda/2}, 0, 0, 0). \quad (2.4)$$

By choosing the commoving system, $u_i u^i = 1$ yields

$$u^i = (0, 0, 0, e^{-\gamma/2}). \quad (2.5)$$

The energy momentum tensor (2.2) with (2.3) has the following nonvanishing components:

$$T_1^1 = -\left(p + \frac{2S}{\sqrt{3}}\right), \quad T_2^2 = T_3^3 = -\left(p - \frac{S}{\sqrt{3}}\right), \quad T_4^4 = \rho. \quad (2.6)$$

The pressure along radial direction,

$$p_r = \left(p + \frac{2S}{\sqrt{3}}\right), \quad (2.7)$$

is different from the pressure along the tangential direction,

$$p_\perp = \left(p - \frac{S}{\sqrt{3}}\right). \quad (2.8)$$

Using (2.7) and (2.8), the magnitude of anisotropic stress tensor is

$$S = \left(\frac{p_r - p_\perp}{\sqrt{3}}\right). \quad (2.9)$$

The Einstein field equations for space-time (2.1) with (2.6) yield

$$e^{-\lambda} \left(\frac{\gamma'}{r} + \frac{1}{r^2} \right) - \frac{1}{r^2} = 8\pi p_r, \quad (2.10)$$

$$e^{-\lambda} \left(\frac{\gamma''}{2} - \frac{\lambda' \gamma'}{4} + \frac{\gamma'^2}{4} - \frac{(\lambda' - \gamma')}{2r} \right) = 8\pi p_\perp, \quad (2.11)$$

$$e^{-\lambda} \left(-\frac{\lambda'}{r} + \frac{1}{r^2} \right) - \frac{1}{r^2} = -8\pi \rho, \quad (2.12)$$

where the prime over the letters indicates the derivative with respect to r .

The consequence of conservation of energy momentum tensor $T_{ij}^j = 0$ leads to

$$\frac{dp_r}{dr} = -(p_r + \rho) \frac{\gamma'}{2} + \frac{2}{r} (p_\perp - p_r). \quad (2.13)$$

From (2.12),

$$e^{-\lambda} = 1 - \frac{2m(r)}{r}, \quad (2.14)$$

where $m(r) = \int 4\pi \rho r^2 dr$ = mass function.

Again, from (2.13) we obtain

$$\gamma' = \frac{4(p_{\perp} - p_r)}{r(p_r + \rho)} - \frac{2p_r'}{(p_r + \rho)}. \quad (2.15)$$

Using (2.14), (2.15), and then (2.10) yields

$$(8\pi p_r r^2 + 1) = \left[1 - \frac{2m(r)}{r}\right] \left[\frac{4(p_{\perp} - p_r)}{(p_r + \rho)} - \frac{2rp_r'}{(p_r + \rho)} + 1\right]. \quad (2.16)$$

Now, we define generating function $g(r)$ and also introduce the anisotropic function $w(r)$, respectively, as

$$g(r) = \frac{1 - 2m(r)/r}{(8\pi p_r r^2 + 1)}, \quad (2.17)$$

$$w(r) = \frac{4(p_r - p_{\perp})}{(p_r + \rho)} g(r). \quad (2.18)$$

From (2.17) and (2.18), we can obtain ρ , p_r , p_{\perp} , S , and the metric potentials e^{λ} , e^{γ} are as follows.

Using (2.17) and (2.18) in (2.16),

$$(1 - g + w)8\pi(p_r + \rho) = -16\pi r p_r' g. \quad (2.19)$$

Differentiating (2.14) yields

$$e^{-\lambda} \lambda' = \frac{2m'(r)}{r} - \frac{2m(r)}{r^2}. \quad (2.20)$$

Adding $8\pi p_r$ on both sides of (2.12) and using (2.20) and (2.17), we obtain

$$8\pi(p_r + \rho) = \frac{2m'(r)}{r^2} - \frac{2m(r)}{r^3} + \frac{8\pi p_r r^2 + 1}{r^2} (1 - g). \quad (2.21)$$

Differentiating (2.17) yields

$$\frac{2m'(r)}{r} = -\left(8\pi p_r r^2 + 1\right) g' - 16\pi p_r r g - 8\pi p_r' r^2 g + \frac{2m(r)}{r^2}. \quad (2.22)$$

On simplifying (2.21) and (2.22), we get

$$8\pi p_r' + \frac{(1 - 3g - r g') (1 - g + w)}{r g (1 + g - w)} 8\pi p_r + \frac{(1 - g - r g') (1 - g + w)}{r^3 g (1 + g - w)} = 0. \quad (2.23)$$

Equation (2.23) is linear differential equation in p_r . We obtain its solution as

$$8\pi p_r = e^{-\int B(r)dr} \left[\alpha_0 + \int C(r) \left[e^{\int B(r)dr} \right] dr \right], \quad (2.24)$$

where α_0 is constant of integration and $B(r)$, $C(r)$ are,

$$B(r) = \frac{(1 - 3g - rg')(1 - g + w)}{rg(1 + g - w)}, \quad C(r) = \frac{(1 - g - rg')(1 - g + w)}{r^3g(1 + g - w)}. \quad (2.25)$$

Equation (2.14) yields

$$\frac{m'(r)}{r} = 4\pi\rho r. \quad (2.26)$$

Putting this value in (2.22) and using (2.17), we obtain

$$8\pi\rho = (1 - g)\frac{1}{r^2} - 8\pi(3p_r + rp'_r)g - \left(8\pi p_r + \frac{1}{r^2}\right)rg', \quad (2.27)$$

which is the expression for effective density ρ .

Equation (2.18) yields,

$$p_{\perp} = p_r - \frac{w}{4g}(p_r + \rho). \quad (2.28)$$

From (2.14) and (2.17), we have

$$e^{-\lambda} = \left(8\pi p_r r^2 + 1\right)g. \quad (2.29)$$

From (2.10) and (2.29),

$$e^{\gamma} = \frac{A^2}{r} e^{\int 1/rg dr}, \quad (2.30)$$

where A is constant of integration.

By using (2.28) and (2.29) the space-time (2.1) becomes

$$ds^2 = \frac{A^2}{r} e^{\int (1/rg) dr} dt^2 - \left[1 - \frac{2m(r)}{r}\right]^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (2.31)$$

The cosmological model (2.31) is physically meaningful with (2.9), (2.24), (2.25), and (2.27).

Here we consider the following three cases.

Case 1. We define the generating function from (2.17) and (2.18) as

$$g(r) = 1 - \alpha r^2, \quad (2.32)$$

$$w(r) = -\alpha r^2, \quad (2.33)$$

Where α is a constant such that $g - w = 1 \neq 0$, and this choice should lead to a physically reasonable model since the function $g(r) \sim 1$ as $r \rightarrow 0$ that implies the Minkowskian space via (2.29), then the (2.25) yields,

$$B(r) = 0, \quad C(r) = 0. \quad (2.34)$$

Equation (2.24) and hence (2.7) yield

$$p_r = \frac{\alpha_0}{8\pi} \implies p = \frac{\alpha_0}{8\pi} - \frac{2S}{\sqrt{3}}. \quad (2.35)$$

If the constant $\alpha_0 = 0$, then $p_r = 0$.

Hence from (2.25),

$$\rho = \frac{3\alpha}{8\pi}. \quad (2.36)$$

Also from (2.27) and (2.35) we obtain

$$S = \frac{1}{8\pi\sqrt{3}} \left[\alpha_0 - \frac{3\alpha^2 r^2}{4(1 - \alpha r^2)} \right]. \quad (2.37)$$

Using (2.32) and (2.33), (2.29) and (2.30) give

$$\begin{aligned} e^{-\lambda} &= (1 - \alpha r^2) (\alpha_0 r^2 + 1), \\ e^{\gamma} &= \left[\frac{A^2}{c_1} + \frac{A^2}{r} e^{\hbar} \right], \end{aligned} \quad (2.38)$$

where $\hbar = \sqrt{\alpha/2} \tan^{-1} \sqrt{\alpha r}$.

Using (2.38), the cosmological model for the space-time (2.1) is,

$$ds^2 = \left[\frac{A^2}{c_1} + \frac{A^2}{r} e^{\hbar} \right] dt^2 - \left[(1 - \alpha r^2) (\alpha_0 r^2 + 1) \right]^{-1} dr^2 - r^2 (d\theta^2 + \sin^2\theta d\phi^2). \quad (2.39)$$

Case 2. We choose the generating function as

$$g(r) = \beta, \quad w(r) = \text{constant}. \quad (2.40)$$

From (2.25),

$$B(r) = \frac{D}{r}, \quad C(r) = \frac{(1 - \beta + w)(1 - \beta)}{r^3 \beta (1 + \beta - w)}, \quad (2.41)$$

where $D = ((1 - 3\beta)(1 - \beta + w)) / (\beta(1 + \beta - w))$.

From (2.40) we obtain

$$p_r = \frac{1}{8\pi} \left[\frac{\alpha_0}{r^D} + \frac{V}{r^2} \right]. \quad (2.42)$$

From (2.7) we get

$$p = \frac{1}{8\pi} \left[\frac{\alpha_0}{r^D} + \frac{V}{r^2} \right] - \frac{2S}{\sqrt{3}}, \quad (2.43)$$

where $V = ((1 - \beta + w)(1 - \beta)) / (\beta(1 + \beta - w)(D - 2))$.

From (2.25),

$$8\pi\rho = (1 - \beta - \beta V) \frac{1}{r^2} + \alpha_0 \beta (D - 3) r^{-D}, \quad (2.44)$$

$$S = \frac{1}{32\sqrt{3}} \left(\frac{w}{\pi\beta} \right) \left\{ \frac{(V + 1)(1 - \beta)}{r^2} + \alpha_0 \frac{(1 + \beta D - 3\beta)}{r^D} \right\}. \quad (2.45)$$

The metric potentials in (2.29) and (2.30) become

$$e^{-\lambda} = (\beta\alpha_0 r^{-D+2} + \beta V + \beta), \quad (2.46)$$

$$e^{\gamma} = A^2 r^{(1/\beta-1)}. \quad (2.47)$$

The space-time (2.1) can be written as

$$ds^2 = [A^2 r^{(1/\beta-1)}] dt^2 - [(\beta\alpha_0 r^{-D+2} + \beta V + \beta)]^{-1} dr^2 - r^2 (d\theta^2 + \sin^2\theta d\phi^2). \quad (2.48)$$

Case 3. When $w(r) = 0$, then (2.18) gives

$$p_r = p_{\perp} = p \implies S = 0. \quad (2.49)$$

On (2.10) and (2.11), we have

$$e^{-\lambda} \left(\frac{\gamma''}{2} - \frac{\lambda' \gamma'}{4} + \frac{\gamma'^2}{4} - \frac{\lambda' \gamma'}{2r} - \frac{1}{r^2} \right) + \frac{1}{r^2} = 0. \quad (2.50)$$

With (2.13), (2.49), and (2.50), we obtain

$$\begin{aligned} \frac{dp}{dr} &= -(p + \rho) \frac{\gamma'}{2}, \\ \Rightarrow \frac{dp}{(p + \rho)} &= -\frac{d\gamma'}{2}. \end{aligned} \quad (2.51)$$

On integrating, we get

$$8\pi(p + \rho) = c_2 e^{-\gamma/2}, \quad (2.52)$$

where $c_2 = 8\pi c_1$, c_1 is the constant of integration.

Subtracting (2.12) from (2.10), we get

$$8\pi(p + \rho) = e^{-\lambda} \left(\frac{\gamma' + \lambda'}{r} \right). \quad (2.53)$$

Equations From (2.52) and (2.53) yield

$$e^{-\lambda} \left(\frac{\gamma' + \lambda'}{r} \right) = c_2 e^{-\gamma/2}. \quad (2.54)$$

On differentiating and simplifying (2.12), we get

$$e^{-\lambda} = 1 - \frac{r^2}{R^2}, \quad (2.55)$$

where $1/R^2 = 8\pi\rho/3$.

Using (2.54) and (2.55) we have

$$e^{\gamma/2} = A' - B \sqrt{\left(1 - \frac{r^2}{R^2}\right)}, \quad (2.56)$$

where $A' = c_3 = (c_2/2)R^2$ and $B = c_4R$, c_3 and c_4 is the constant of integration.

Thus the space-time (2.1) becomes

$$ds^2 = \left(1 - \frac{r^2}{R^2}\right)^{-1} dt^2 - \left[A' - B \sqrt{\left(1 - \frac{r^2}{R^2}\right)} \right]^2 dr^2 - r^2 (d\theta^2 + \sin^2\theta d\phi^2). \quad (2.57)$$

Equation (2.57) perfectly matches with Schwarzschild interior solution with

$$8\pi p = \frac{\{3B\sqrt{(1-r^2/R^2)} - A'\}}{R^2\{A' - B\sqrt{(1-r^2/R^2)}\}},$$

$$8\pi\rho = \frac{\{c_2R^2 + A' - 3B\sqrt{(1-r^2/R^2)} - A'\}}{R^2\{A' - B\sqrt{(1-r^2/R^2)}\}}. \quad (2.58)$$

3. Discussion

The cosmological model (2.31) is physically meaningful with radial pressure (p_r), tangential pressure (p_\perp), and energy density (ρ) being given by (2.24), (2.28), and (2.27) respectively. The model has initial singularity at $r = 0$.

Here we discuss the following three cases.

In Case 1, we consider generating function and anisotropic function as defined in (2.32) and (2.33) such that $g - w = 1 \neq 0$ and the radial pressure (p_r) and energy density (ρ) become constant.

In Case 2, as $r \rightarrow 0$ and the pressure (p), energy density (ρ), and stress tensor S all are infinite, the model starts with big bang. As $r \rightarrow \infty$, $p = \rho = 0$, the model (2.48) represents a vacuum model.

While in Case 3, we consider $w(r) = 0 \Rightarrow S = 0$ this gives; $p_r = p_\perp$ which implies that the cosmological model (2.57) is isotropic with pressure (p) density (ρ) given by (2.58), and our result perfectly matches with Schwarzschild interior solution.

4. Conclusion

We have investigated the spherically symmetric cosmological model for perfect fluid with anisotropic stress tensor in general relativity. Here we discuss the three different cases in which the last case for $w(r) = 0$ matches with the Schwarzschild interior solution.

Acknowledgement

The authors are grateful to the referee for his valuable comments and suggestions.

References

- [1] R. Ruderman, "Pulsars: Structure and dynamics," *Annual Review of Astronomy and Astrophysics*, vol. 10, p. 427, 1972.
- [2] H. Rago, "Anisotropic spheres in general relativity," *Astrophysics and Space Science*, vol. 183, no. 2, pp. 333–338, 1991.
- [3] S. P. Kandalkar and G. S. Khadekar, "Anisotropic fluid distribution in bimetric theory of relativity," *Astrophysics and Space Science*, vol. 293, no. 4, pp. 415–422, 2004.
- [4] R. Tikekar and L. K. Patel, "Non-adiabatic gravitational collapse of charged radiating fluid spheres," *Pramana*, vol. 39, no. 1, pp. 17–25, 1992.

- [5] V. Canuto, "Equation of state at ultrahigh densities," *Annual Review of Astronomy and Astrophysics*, vol. 12, pp. 167–214, 1974.
- [6] J. R. Gair, "Spherical universes with anisotropic pressure," *Classical and Quantum Gravity*, vol. 18, no. 22, pp. 4897–4919, 2001.
- [7] V. O. Thomas and B. S. Ratanpal, "Non-adiabatic gravitational collapse with anisotropic core," *International Journal of Modern Physics D*, vol. 16, no. 9, pp. 1479–1495, 2007.
- [8] P. S. Letelier, "Anisotropic fluids with two-perfect-fluid components," *Physical Review D*, vol. 22, no. 4, pp. 807–813, 1980.
- [9] S. D. Maharaj and R. Maartens, "Anisotropic spheres with uniform energy density in general relativity," *General Relativity and Gravitation*, vol. 21, no. 9, pp. 899–905, 1989.
- [10] H. Bondi, "Addendum-Anisotropic Spheres in General Relativity," *Monthly Notices of the Royal Astronomical Society*, vol. 262, p. 1088, 1993.
- [11] A. Coley and B. Tupper, "Spherically symmetric anisotropic fluid ICKV spacetimes," *Classical and Quantum Gravity*, vol. 11, no. 10, pp. 2553–2574, 1994.
- [12] T. Singh, P. Singh, and A. Helmi, "New solutions for charged anisotropic fluid spheres in general relativity," *Nuovo Cimento B*, vol. 110, no. 4, pp. 387–393, 1995.



Hindawi

Submit your manuscripts at
<http://www.hindawi.com>

