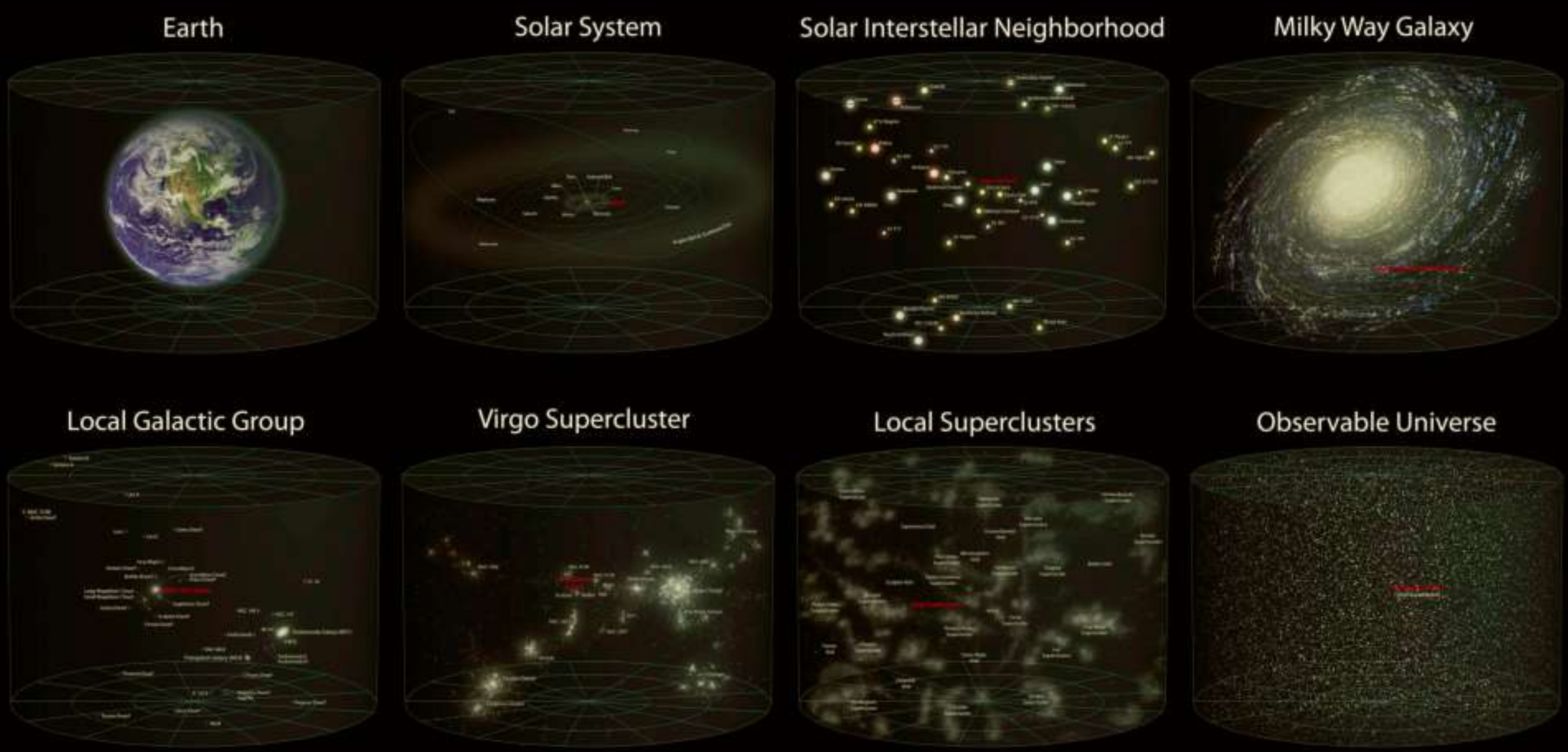
# Space Engineering

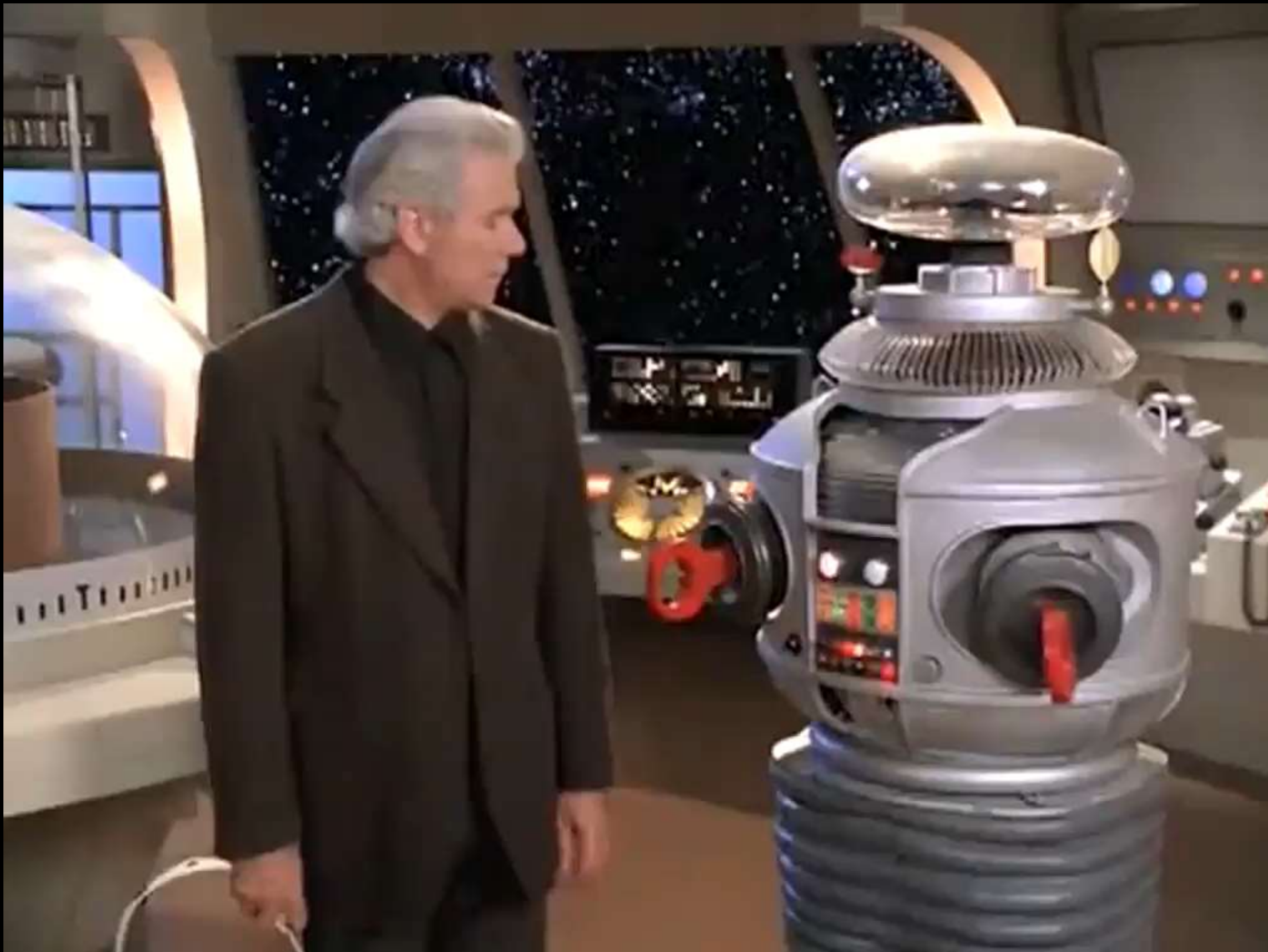
## Astrodynamics

THE UNIVERSITY OF SYDNEY

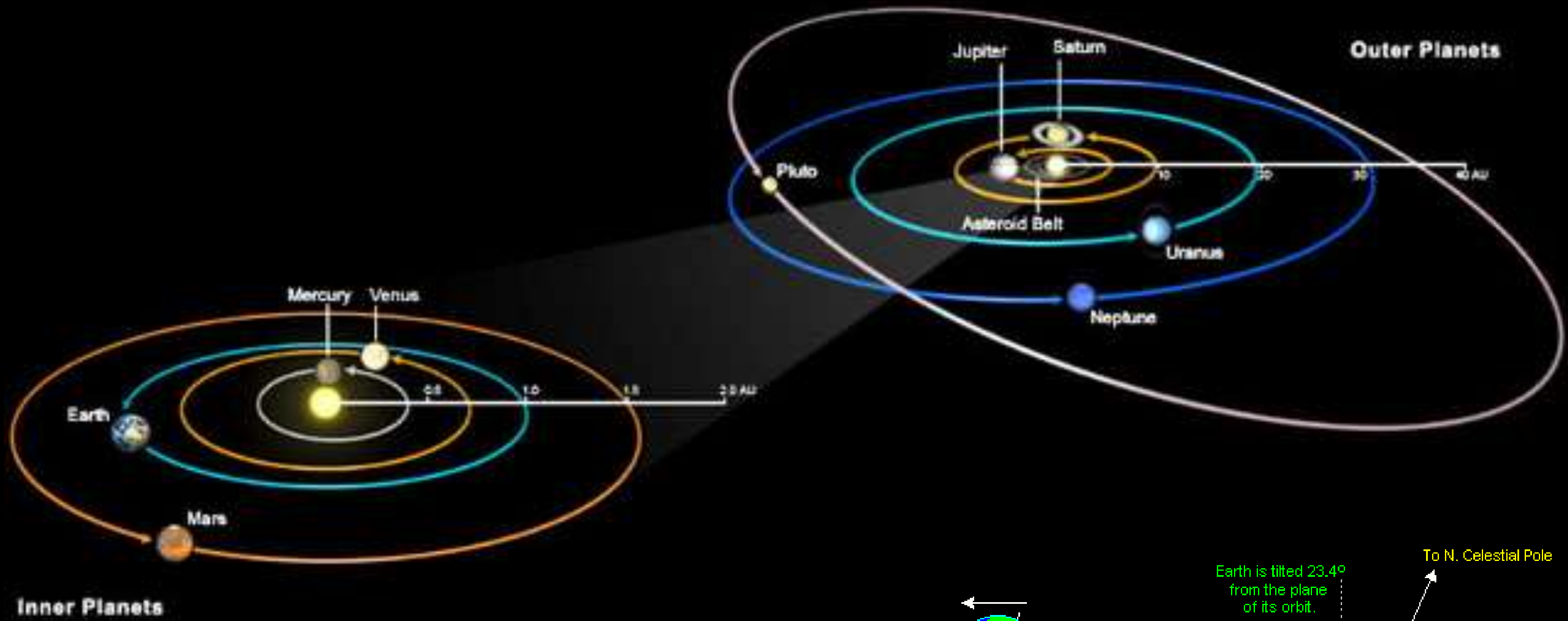
NASA

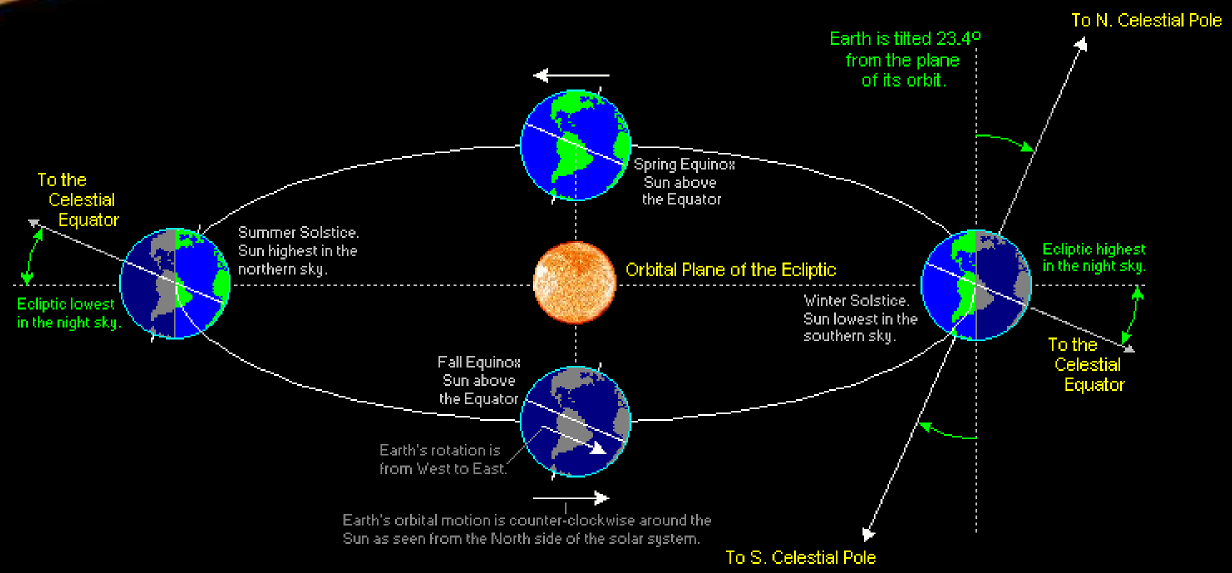
# Where are we?


Earth
Solar System
Solar Interstellar Neighborhood
Milky Way Galaxy
Local Galactic Group
Virgo Supercluster
Local Superclusters
Observable Universe

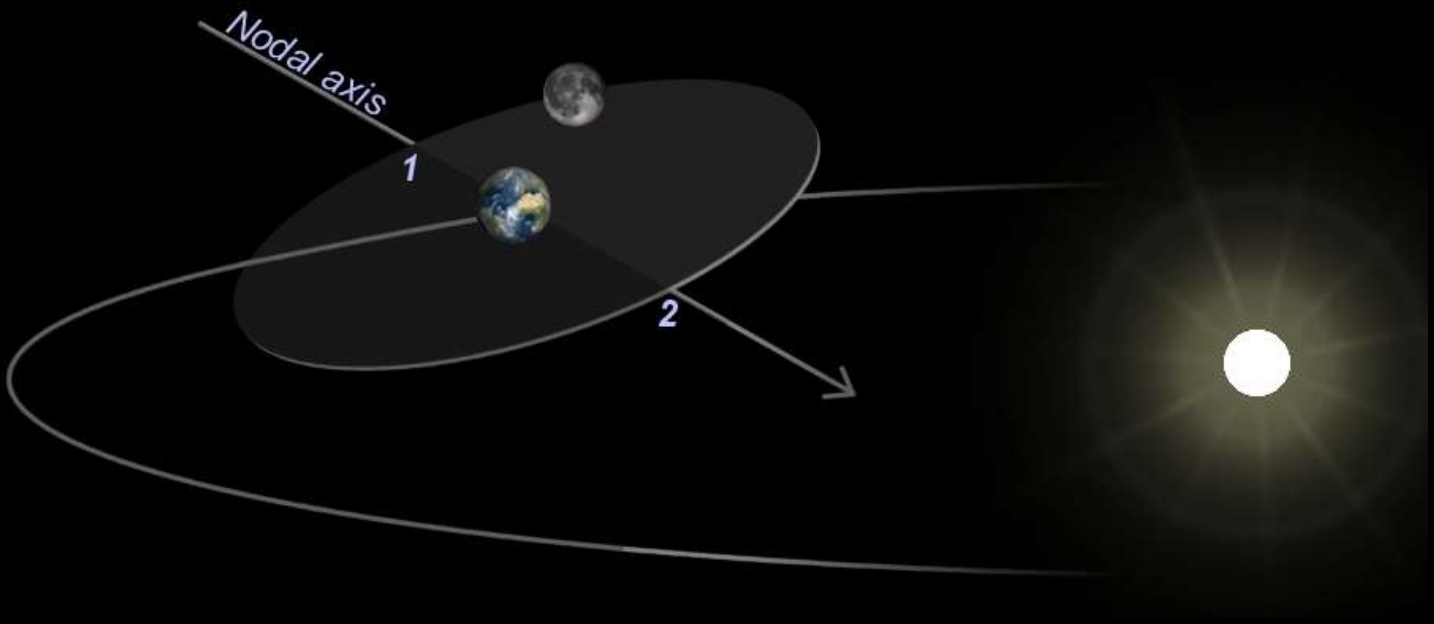


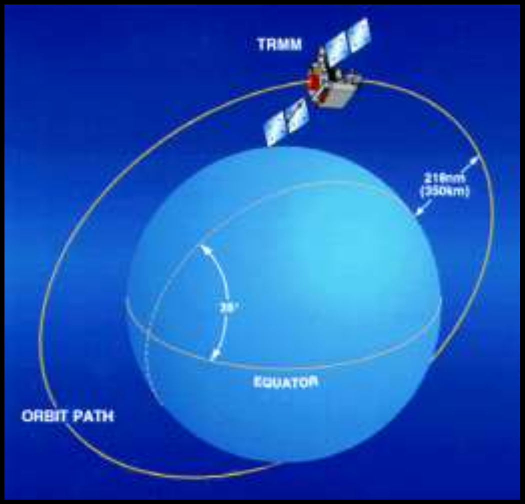
# Where are we going?

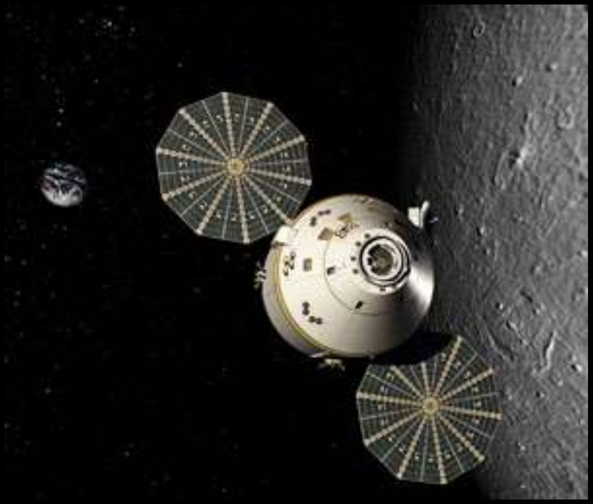


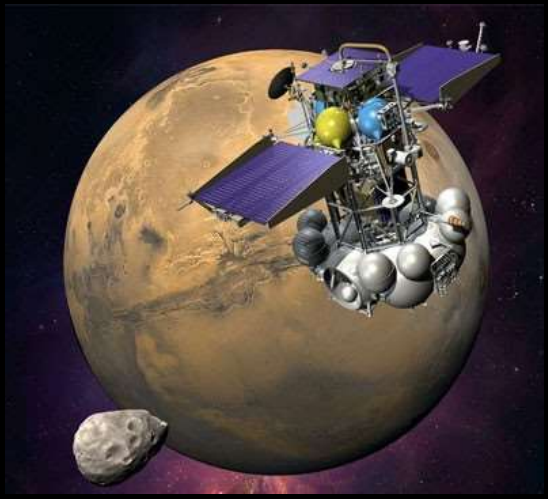



# How do we get there?

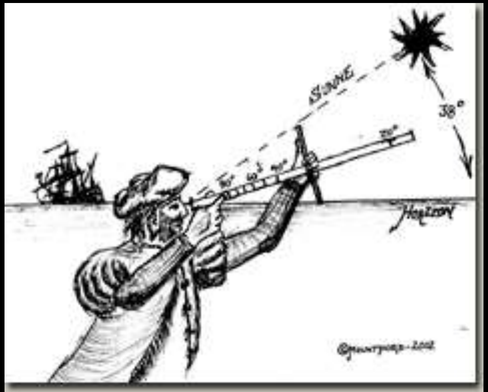




# Early Navigation

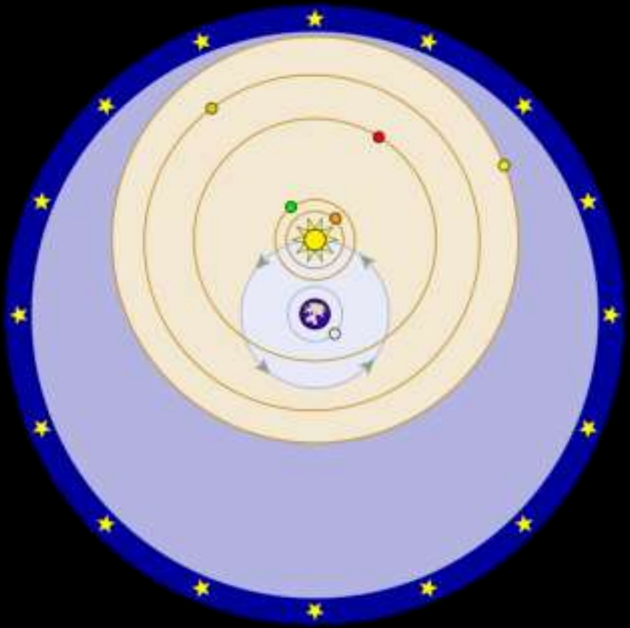



# Nicolaus Copernicus 1473-1543

## De revolutionibus orbium coelestium (1543)

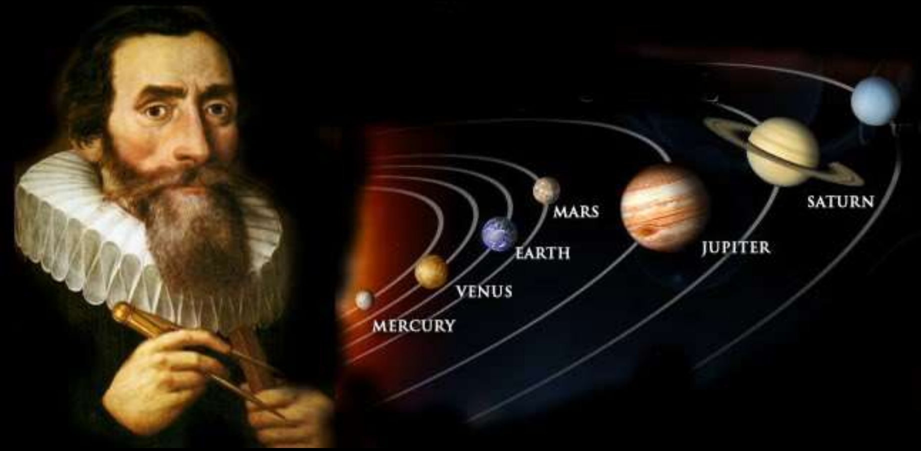
- The center of the earth is not the center of the universe, but only of gravity and of the lunar sphere.
- All the spheres revolve about the sun as their midpoint, and therefore the sun is the center of the universe.
- Whatever motion appears in the firmament arises not from any motion of the firmament, but from the earth's motion. The earth together with its circumjacent elements performs a complete rotation on its fixed poles in a daily motion, while the firmament and highest heaven abide unchanged.
- What appear to us as motions of the sun arise not from its motion but from the motion of the earth and our sphere, with which we revolve about the sun like any other planet.
- The apparent retrograde and direct motion of the planets arises not from their motion but from the earth's. The motion of the earth alone, therefore, suffices to explain so many apparent inequalities in the heavens.


# Tycho Brahe
# 1546-1601



**Tycho's geo-heliocentric astronomy**

- **Danish Nobleman, Astronomer, Astrologer, Alchemist**
- **Built two observatories – Hven, Prague**
- **Accurate and Comprehensive Astronomical Observations**
- **Published - De nova stella (1573)**
- **Combined geometric benefits of Copernican system with philosophical benefits of the Ptolemaic system**
- **Assisted by Johannes Kepler**




# Johannes Kepler
# 1571-1630

## Kepler's Laws

1. The orbits of planets are ellipses with the sun at one focus.
2. A line drawn from a planet to the sun sweeps out equal areas in equal intervals of time.

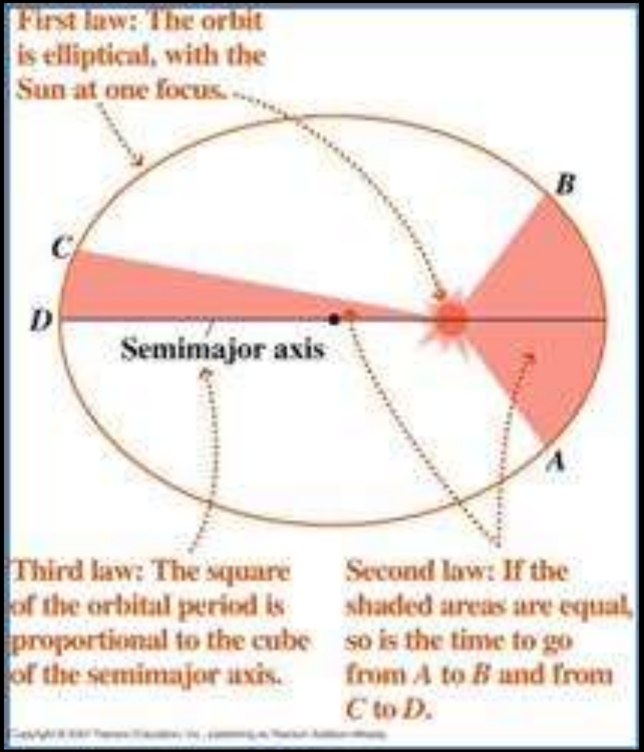
   A planet must move rapidly when it is close to the sun and more slowly when it is far from the sun.

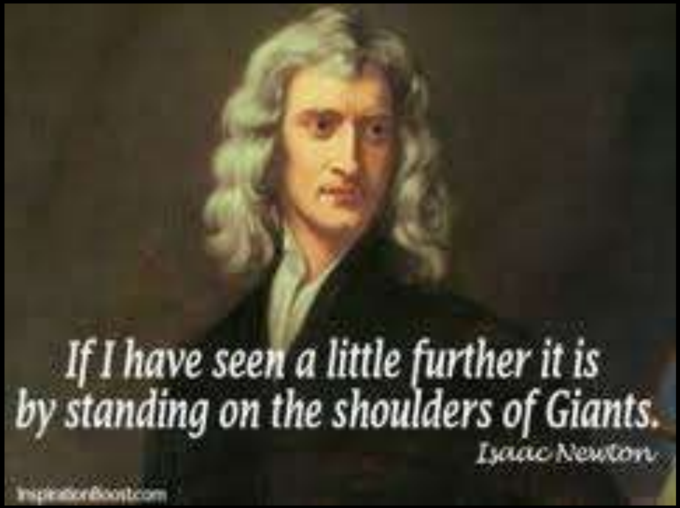
3. The square of a planet's orbital period is proportional to the cube of its average distance from the sun:

$$P_{yr}^2 = a_{AU}^3$$

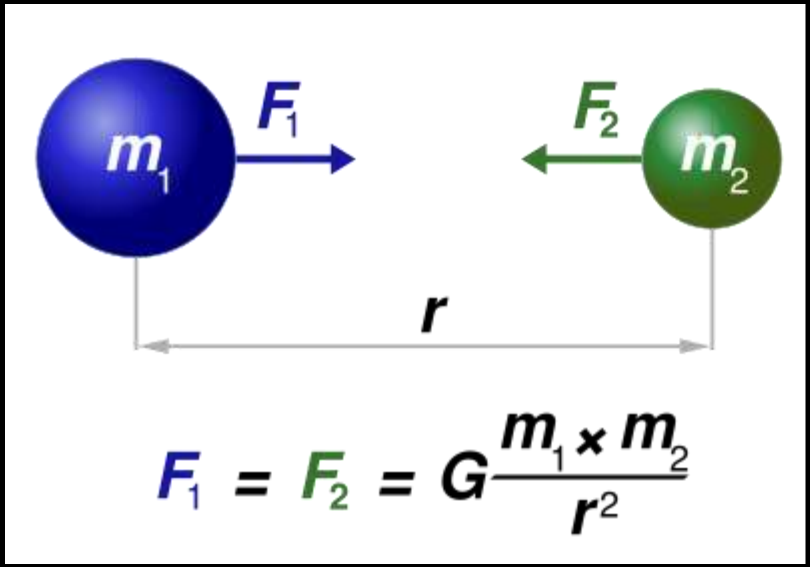
(when P is in years and a in AU)

Planets in large orbits take much longer to orbit the sun than do planets in small orbits.



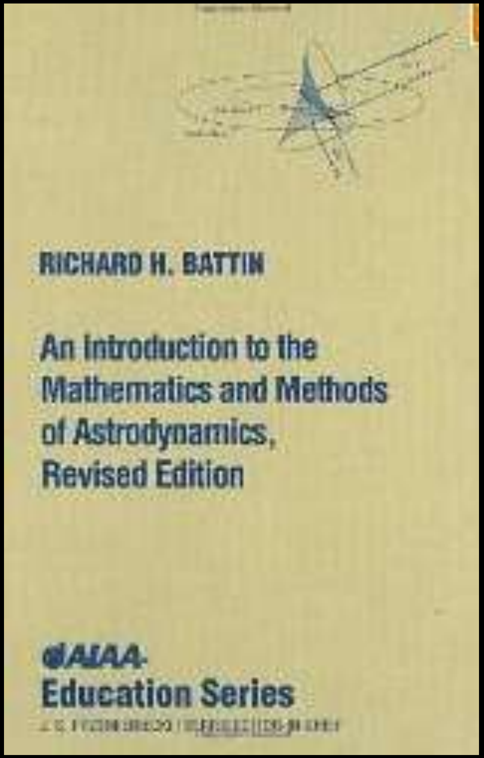



# Issac Newton 1642-1727

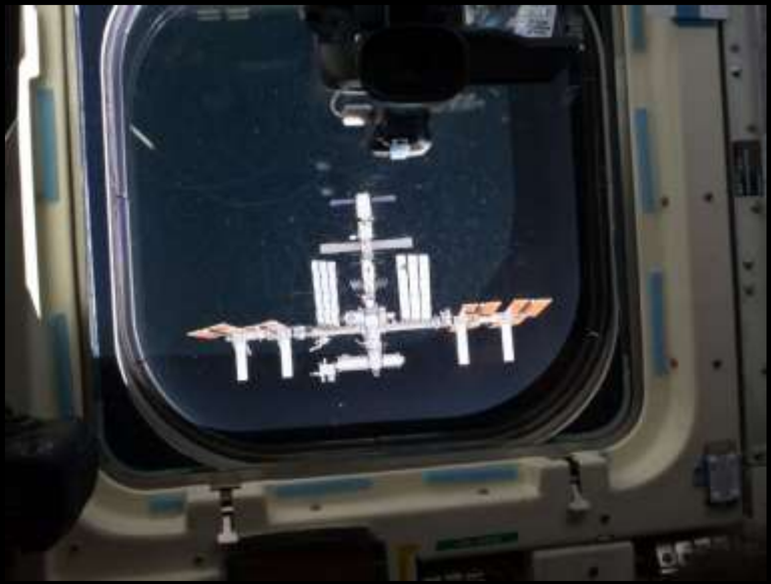



- Principia Mathematica (1687)
- Newton derived Kepler's laws of planetary motion from his mathematical description of gravity, removing the last doubts about the validity of the heliocentric model of the cosmos.

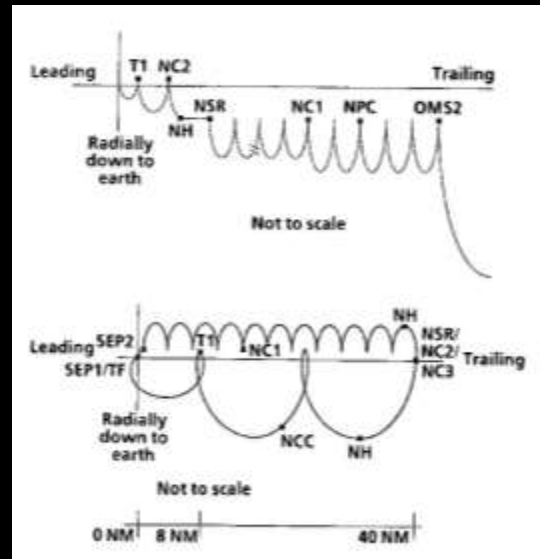
# Richard Battin
# 1925-2014

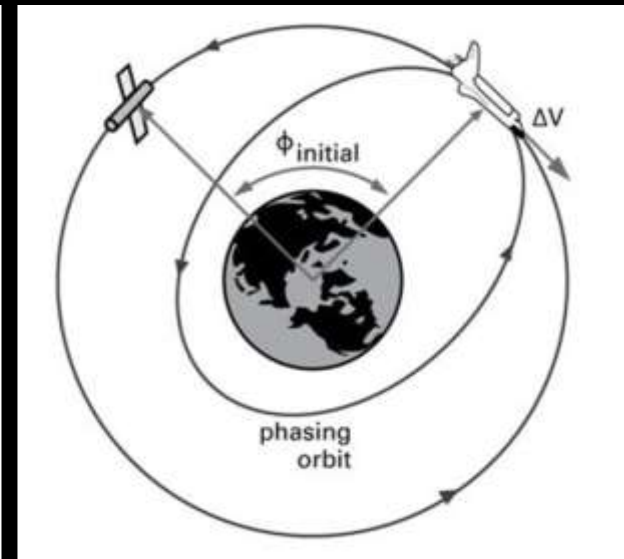


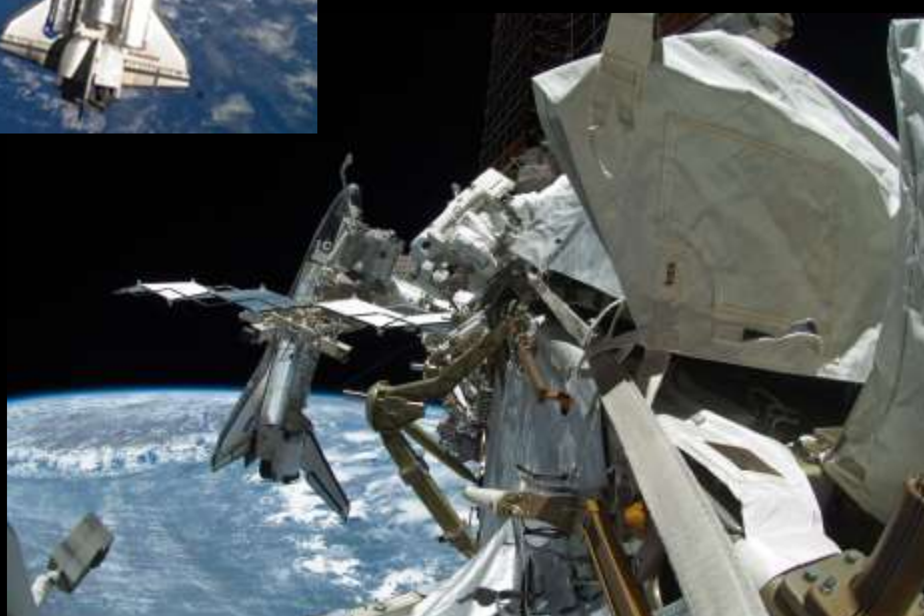

# Astronauts & Orbital Mechanics

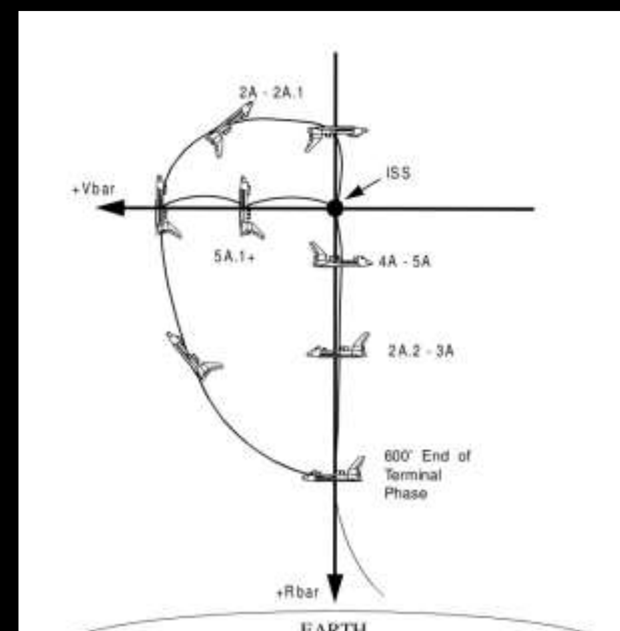


# Rendezvous


Leading
T1 NC2
Trailing
NSR
NC1 NPC
OMS2
NH
Radially down to earth
Not to scale
Leading SEP2
SEP1/TF
T1
NC1
NH
NSR/ NC2/ NC3
Trailing
Radially down to earth
NCC
NH
Not to scale
0 NM
8 NM
40 NM


ΔV
φ initial
phasing orbit




2A - 2A.1
ISS
+Vbar
5A.1+
4A - 5A
2A.2 - 3A
600' End of Terminal Phase
+Rbar
EARTH

# EVA Rescue

# EVA Rescue (2)




# Elements of Astrodynamics

## Launch into desired orbit

- Launch Window, Inclination
- LEO/GEO/Departure

## Orbital Maneuvers

- Feasible Trajectories/Orbit Types
- Minimize Propulsion Required
- Orbit/Plane Changes

## Interplanetary Transfers

- Hyperbolic Orbits
- Changing Reference Frames
- Orbital Insertion

## Rendezvous/Proximity Operations

- Relative Motion

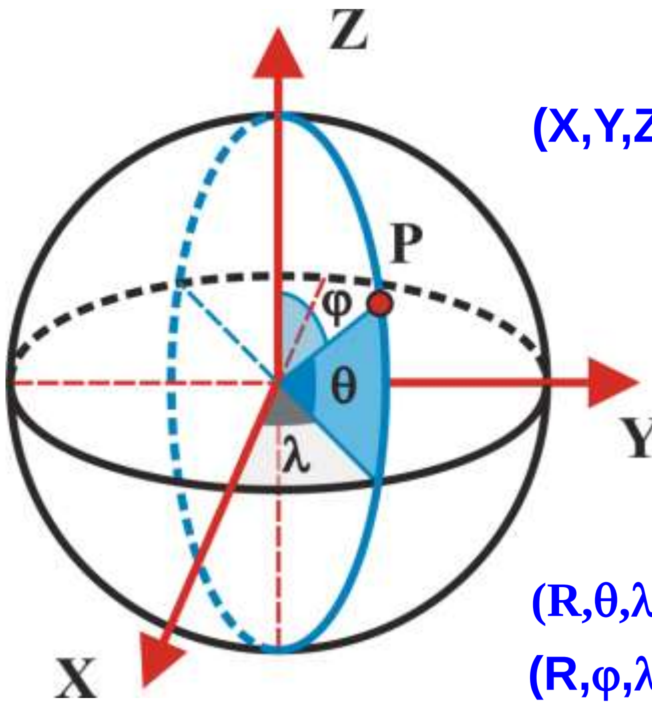
## Observations/Targeting/Entry/Landing

- Ground Coverage (ground track/swath)
- Deorbit Burn

# Mission to Mars (Spirit & Opportunity)

# Astrodynamics
## Reference Frames

# Coordinate Systems




(X,Y,Z)

(R,θ,λ)

(R,φ,λ)

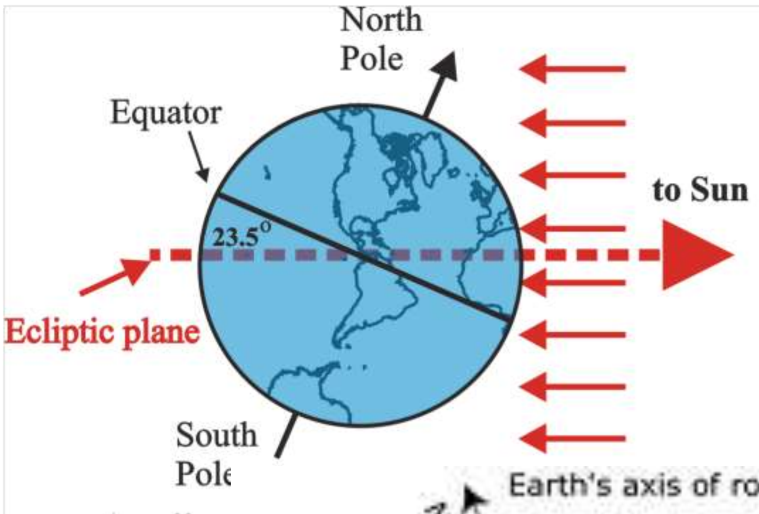
Origin?

- Center of Earth
- Sun or a Star
- Center of a planetary body
- Others….

Reference Axes

- Axis of rotation or revolution
- Earth spin axis
  - Equatorial Plane
- Plane of the Earth's orbit around the Sun
  - Ecliptic Plane
- Need to pick two axes and then 3rd one is determined

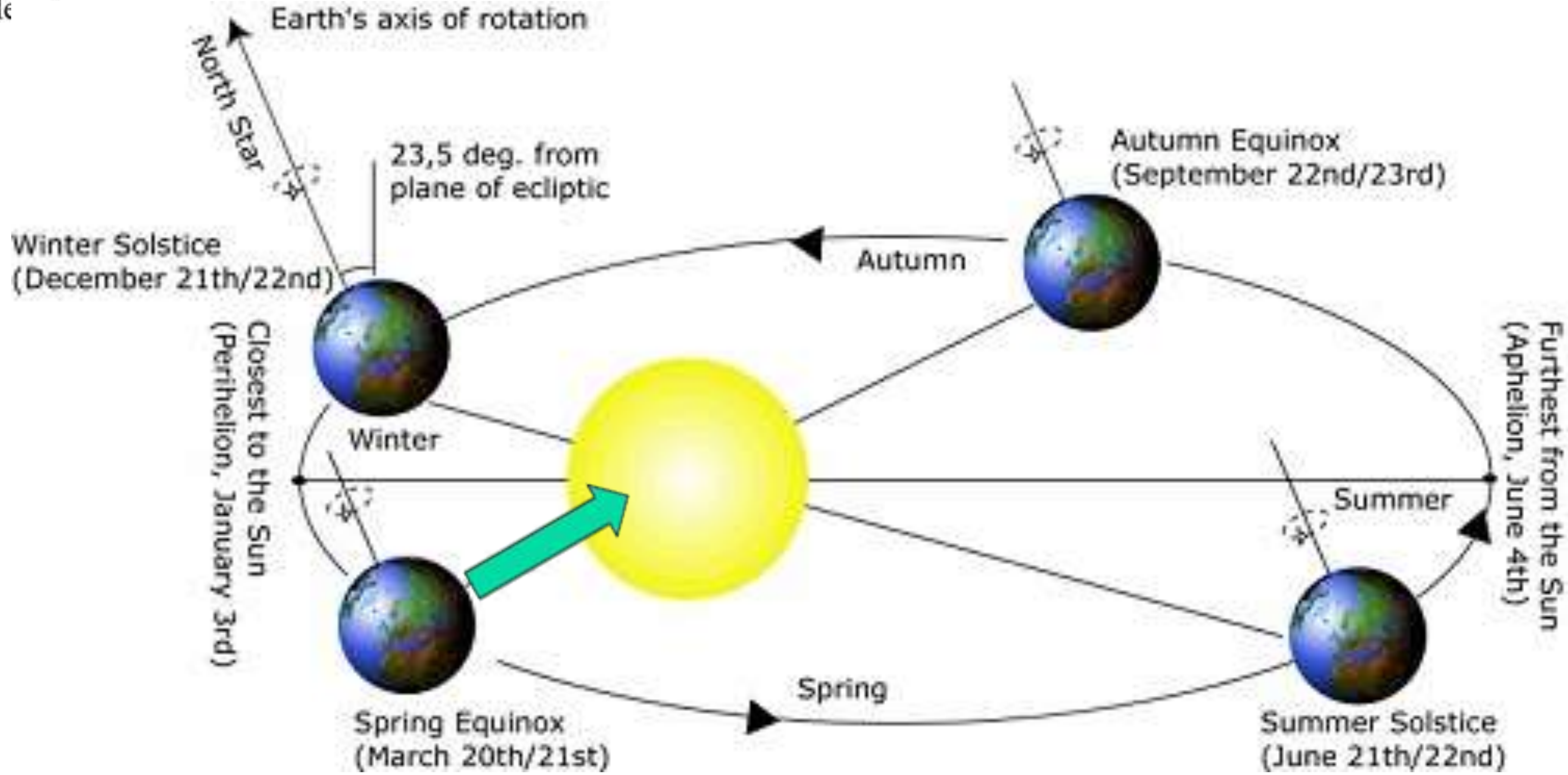
# Ecliptic and Equatorial Planes



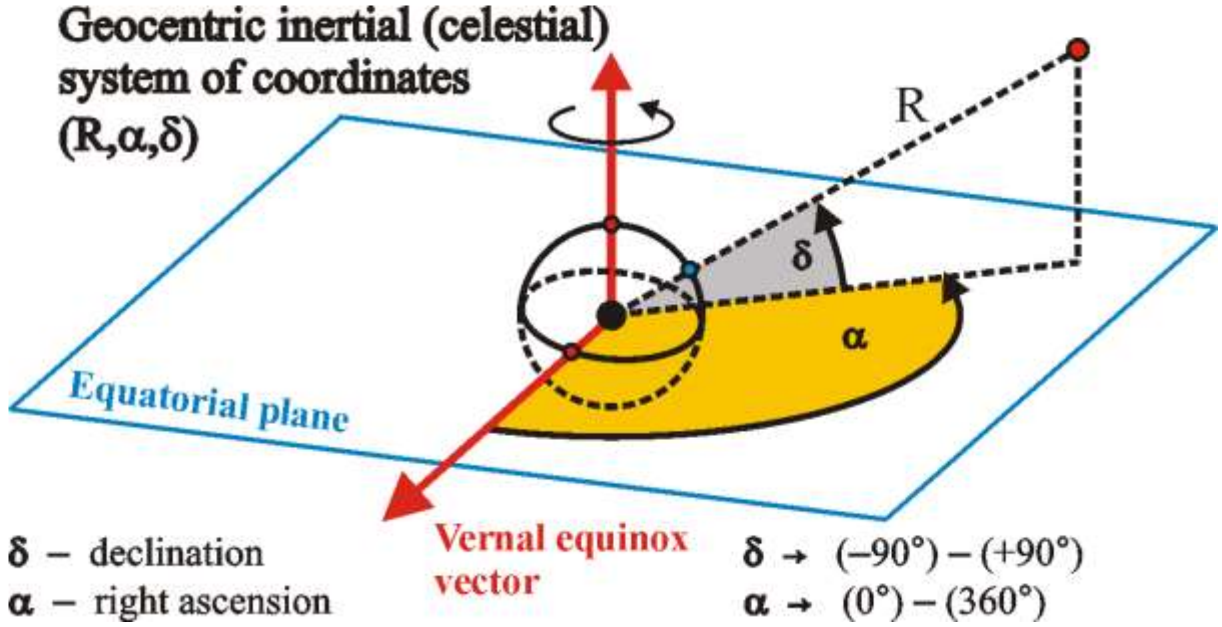

Obliquity of the Ecliptic = 23.44 °

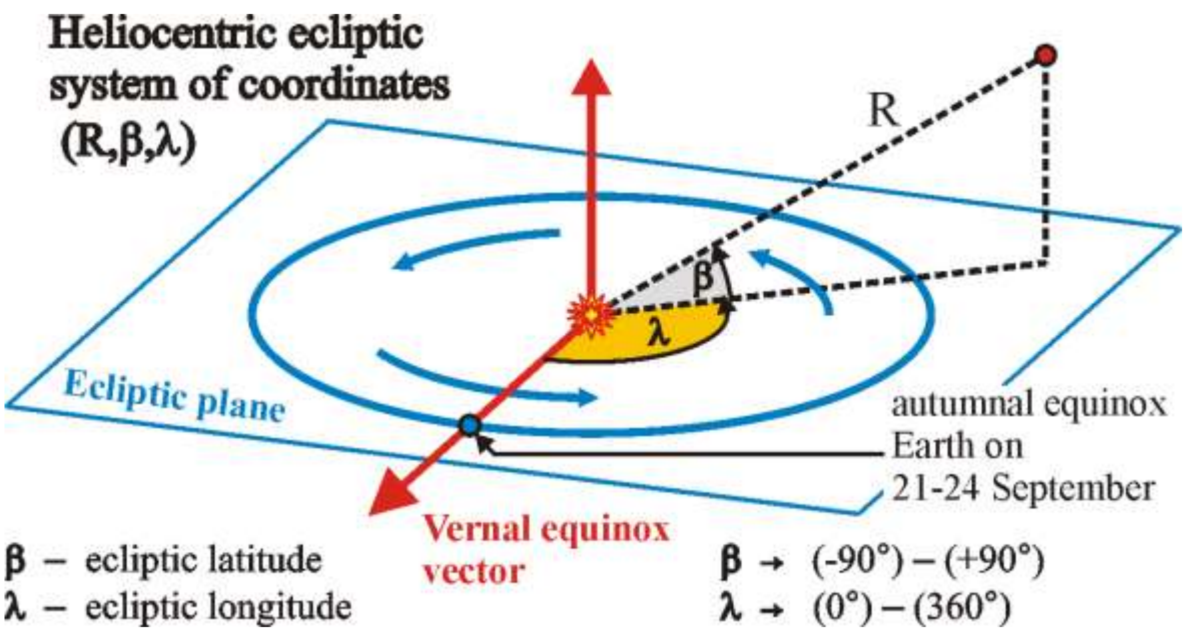
<u>Vernal Equinox vector</u>

- Earth to Sun on March 21st
- Planes intersect @ Equinox

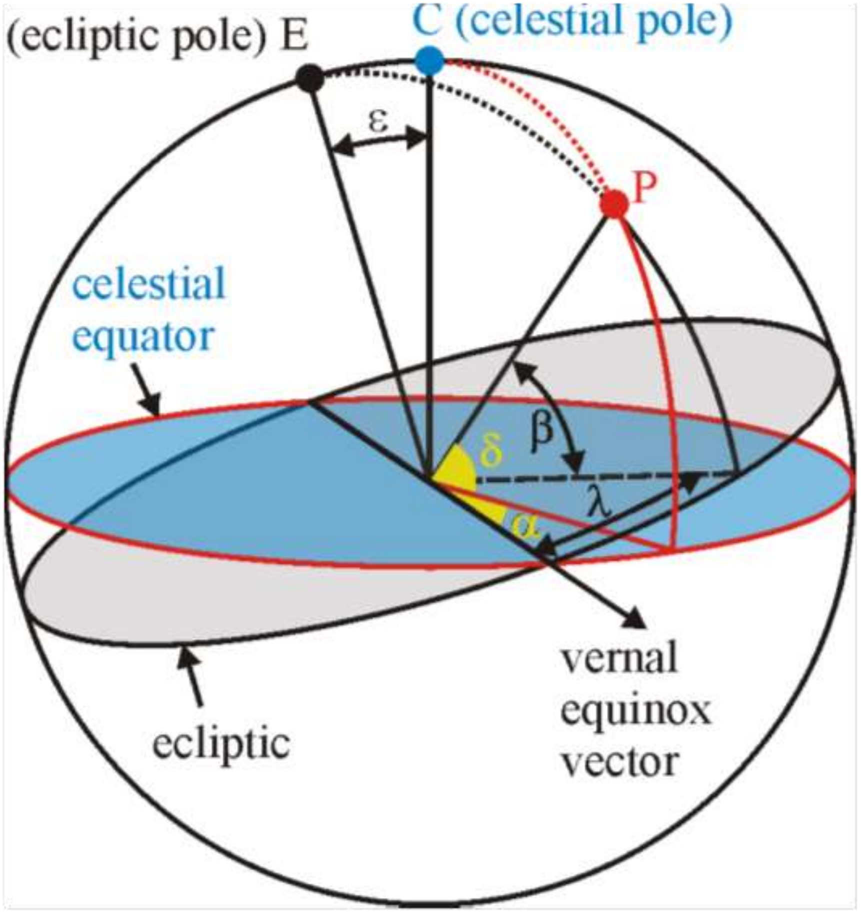


# Inertial Coordinates



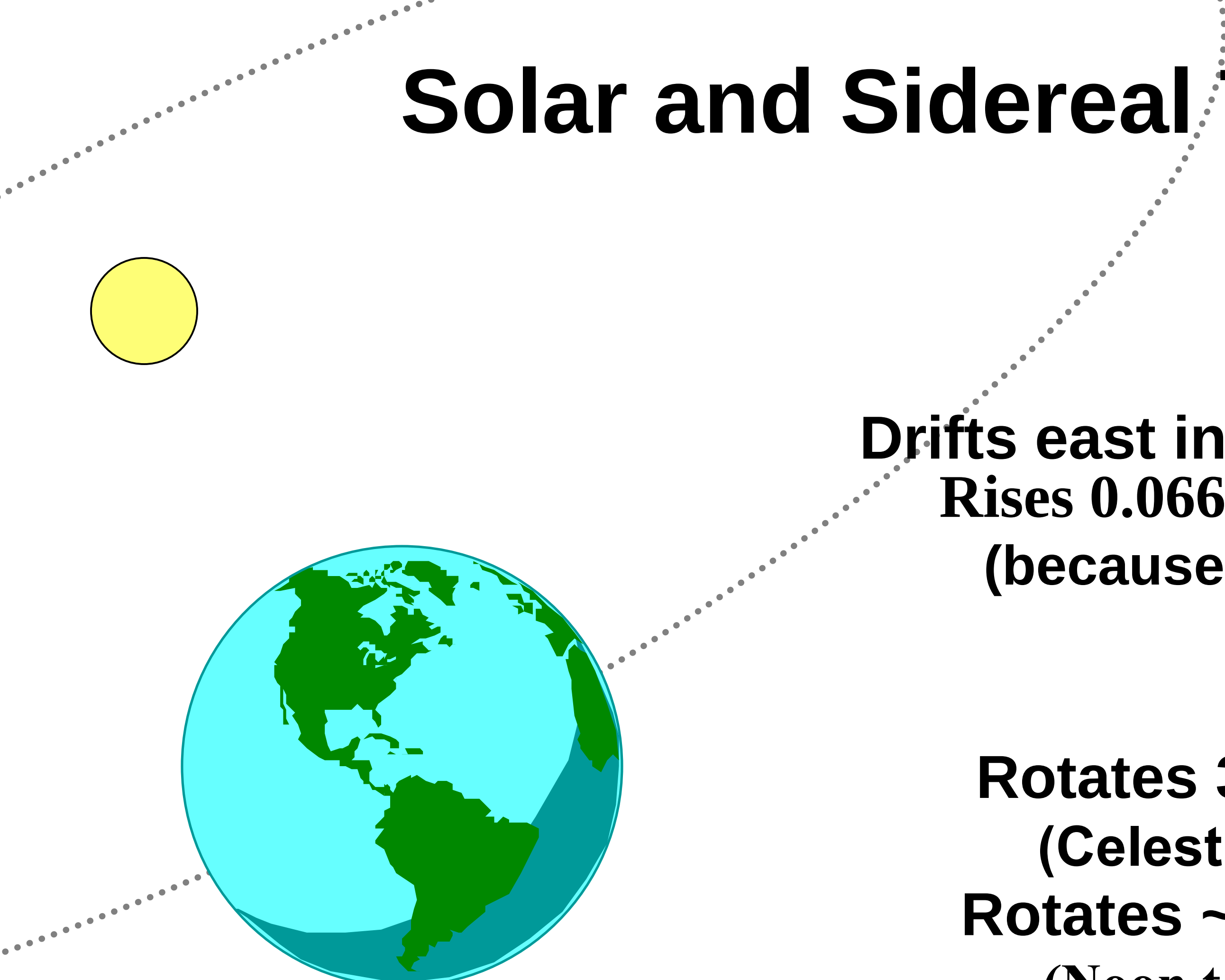
# Relationship between Coordinate Frames




$$\sin\beta = \sin\delta\,\cos\varepsilon - \cos\delta\,\sin\varepsilon\,\sin\alpha$$
$$\cos\beta\,\cos\lambda = \cos\delta\,\cos\alpha$$
$$\cos\beta\,\sin\lambda = \sin\delta\,\sin\varepsilon + \cos\delta\,\cos\varepsilon\,\sin\alpha$$

$$\sin\delta = \sin\beta\,\cos\varepsilon + \cos\beta\,\sin\varepsilon\,\sin\lambda$$
$$\cos\delta\,\cos\alpha = \cos\beta\,\cos\lambda$$
$$\cos\delta\,\sin\alpha = -\sin\beta\,\sin\varepsilon + \cos\beta\,\cos\varepsilon\,\sin\lambda$$

# Solar and Sidereal Time



## The Sun

**Drifts east in the sky ~1° per day.**
**Rises 0.066 hours later each day.**
**(because the earth is orbiting)**

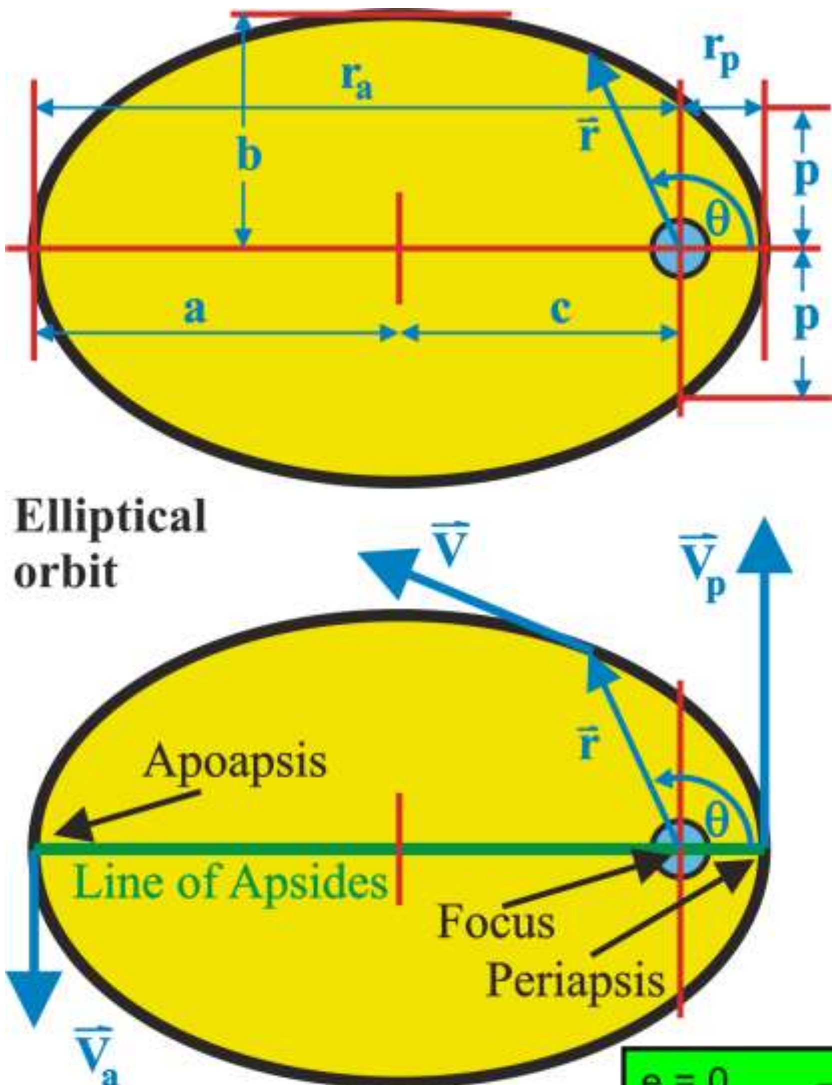
## The Earth…

**Rotates 360° in 23.934 hours**
**(Celestial or "Sidereal" Day)**
**Rotates ~361° in 24.000 hours**
**(Noon to Noon or "Solar" Day)**

**Satellites orbits are aligned to the Sidereal day – *not* the solar day**

# Astrodynamics
## Orbital Elements

# Properties of Orbits




- **a** is the semimajor axis;
- **b** is the semiminor axis;
- $\mathbf{r_{MAX} = r_a}$, $\mathbf{r_{MIN} = r_p}$ are the maximum and minimum radius-vectors;
- **c** is the distance between the focus and the center of the ellipse;
- **e = c/a** is eccentricity

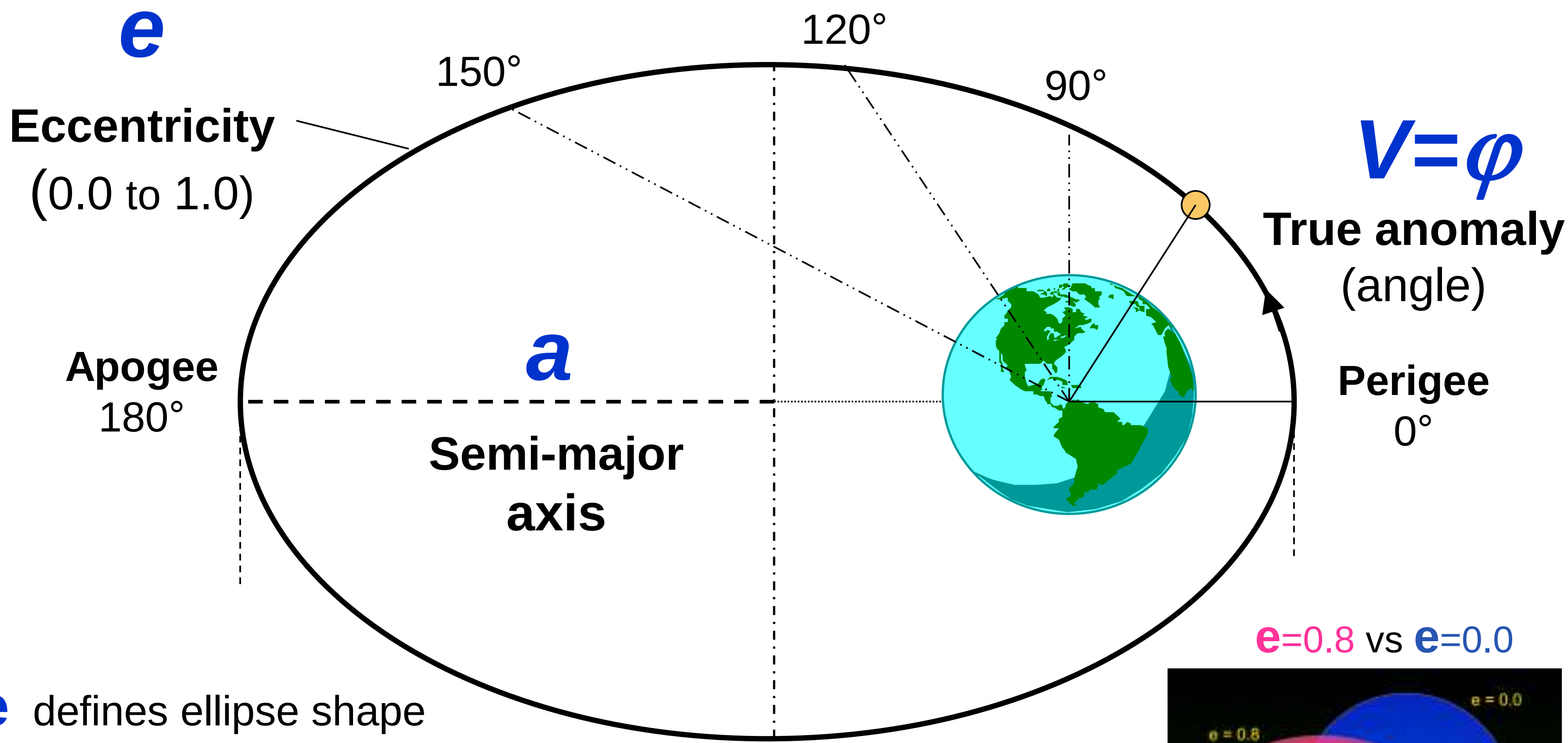
$$\left(\frac{b}{a}\right)^2 = 1 - e^2$$

- **2p** is the latus rectum
  (*latus = side, rectum = straight*)
  **p** — semilatus rectum or semiparameter
- **A = πab** is the area of the ellipse

| | | |
|---|---|---|
| e = 0 | → | circle |
| e < 1 | → | ellipse |
| e = 1 | → | parabola |
| e > 1 | → | hyperbola |

$$\frac{p}{r} = 1 + e\cos\theta$$

# Orbital Elements

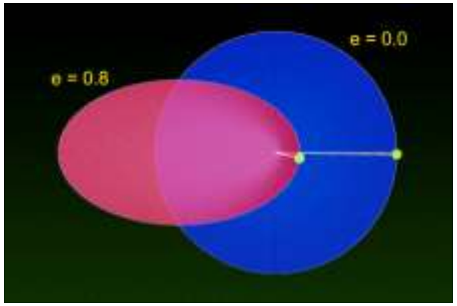



**e** defines ellipse shape

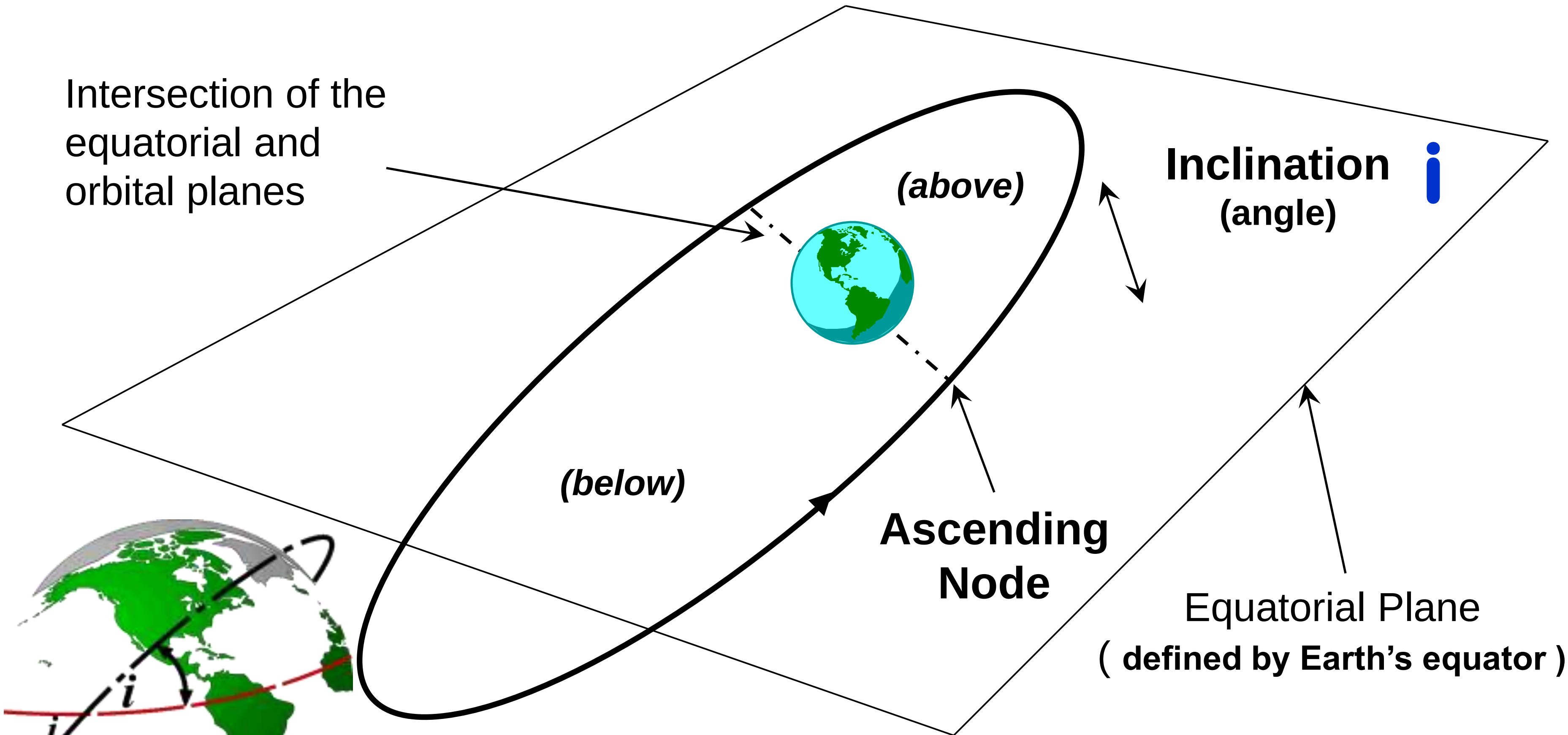
**a** defines ellipse size

**v/φ** defines satellite angle from perigee

**e**=0.8 vs **e**=0.0



# Inclination i

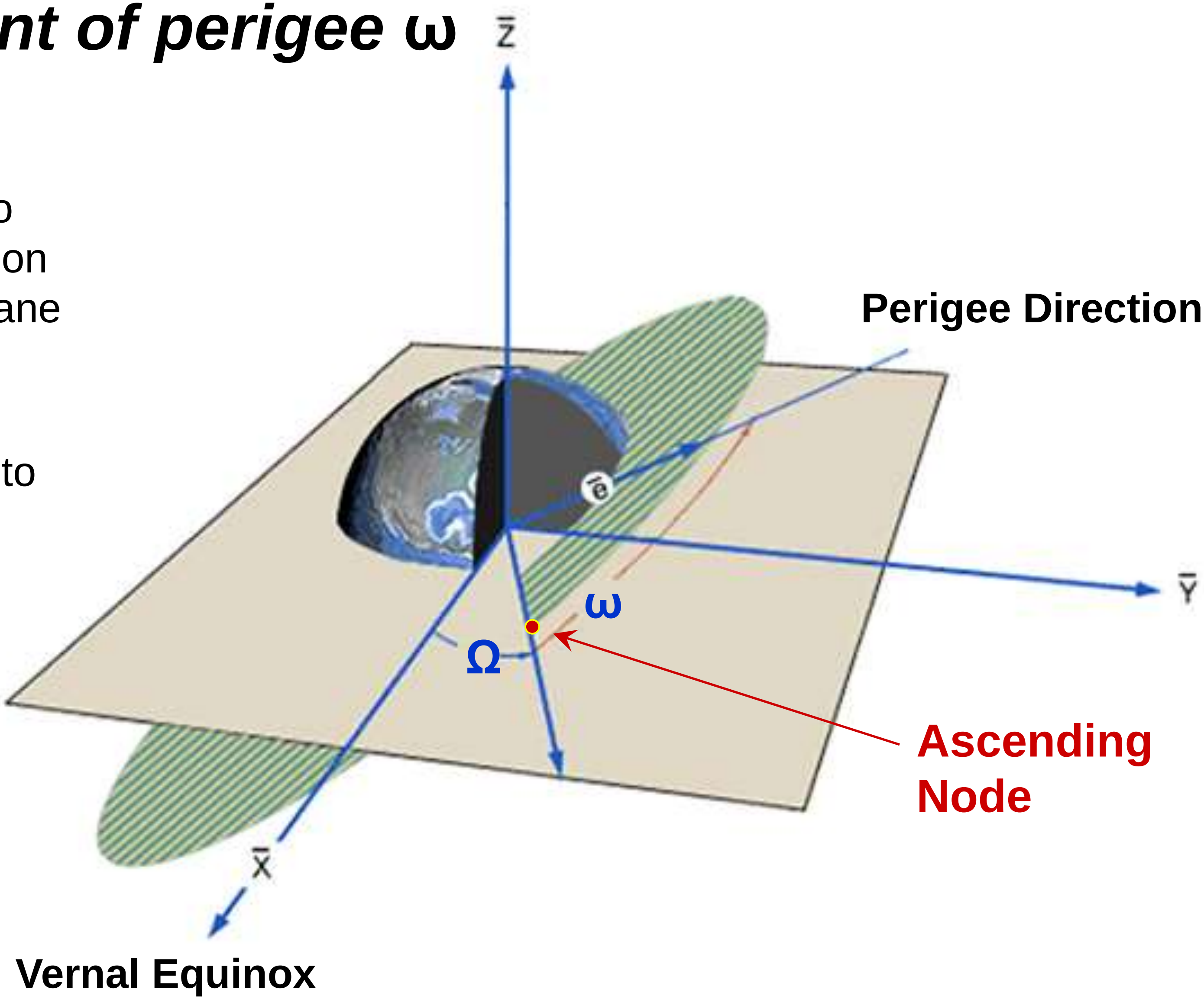



**Ascending Node** is where a satellite crosses the equatorial plane moving south to north

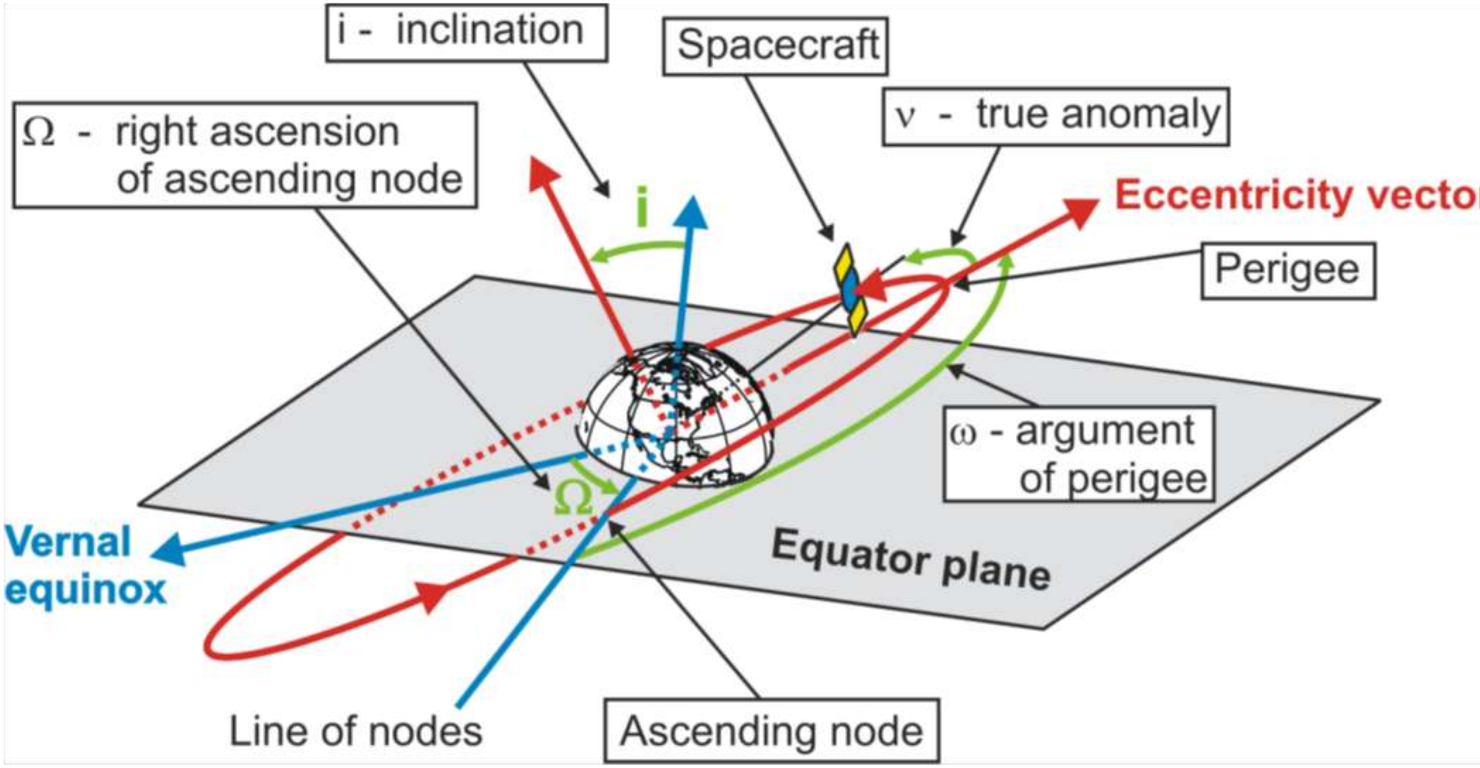
# *Right Ascension of the ascending node Ω and Argument of perigee ω*

**Ω** = angle from vernal equinox to ascending node on the equatorial plane

**ω** = angle from ascending node to perigee on the orbital plane



# Orbital Elements




a - semi-major axis Ω - right ascension of ascending node

e - eccentricity ω - argument of perigee

i - inclination ν - true anomaly (also φ)

# The Six Orbital Elements

**a** = **Semi-major axis** (usually in kilometers or nautical miles)

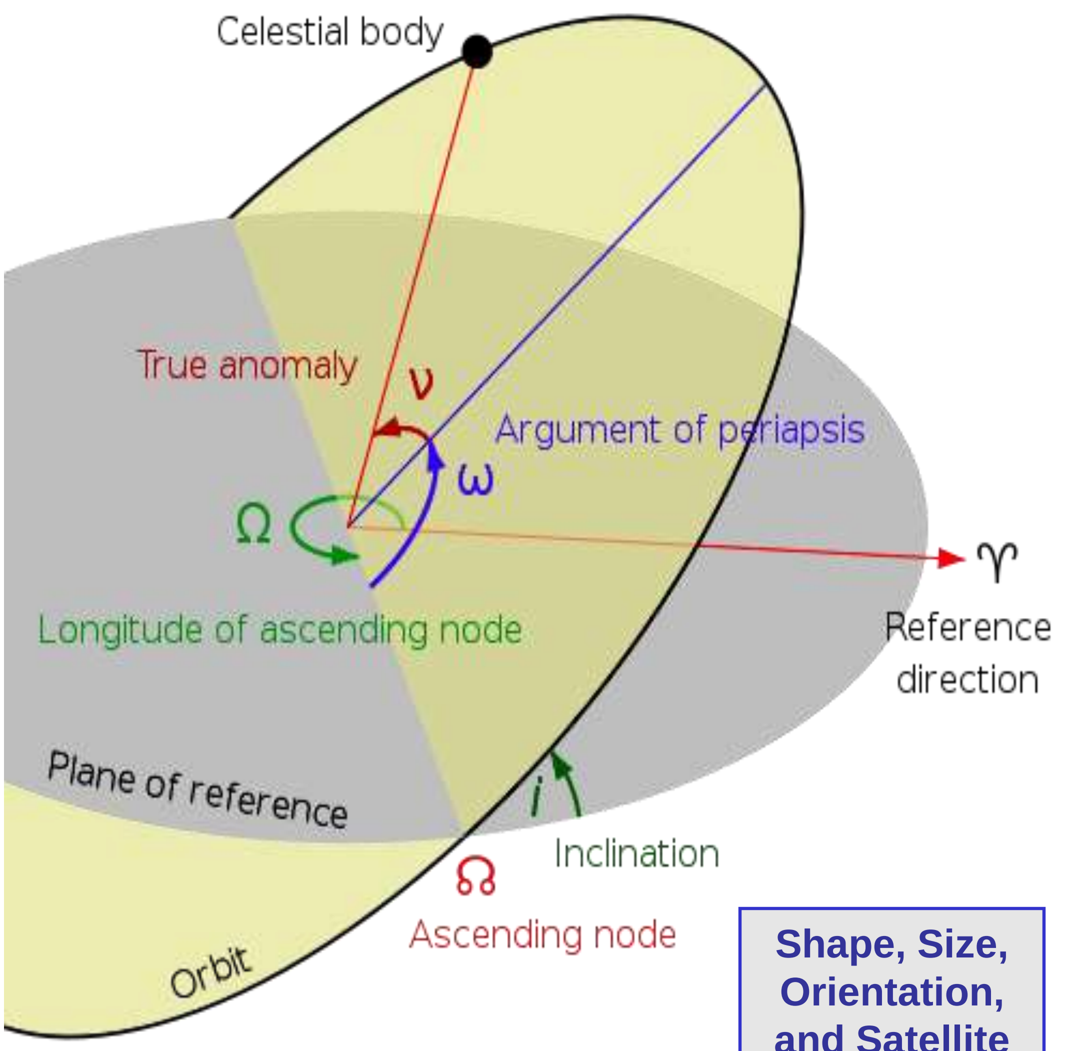
**e** = **Eccentricity** (of the elliptical orbit)

**ν/φ** = **True anomaly** The angle between perigee and satellite in the orbital plane at a specific time

**i** = **Inclination** The angle between the orbital and equatorial planes

**Ω** = **Right Ascension (longitude) of the ascending node** The angle from the Vernal Equinox vector to the ascending node on the equatorial plane

**ω** = **Argument of perigee** The angle measured between the ascending node and perigee




**Shape, Size, Orientation, and Satellite Location.**

# Two Line Orbital Elements

N ASA and NORAD Standard for specifying orbits of Earth-orbiting satellites

ISS (ZARYA)

1 25544U 98067A   08264.51782528 −.00002182  00000-0 -11606-4 0  2927

2 25544  51.6416 247.4627 0006703 130.5360 325.0288 15.72125391563537

| Field | Columns | Content | Example |
|---|---|---|---|
| 1 | 01–01 | Line number | 1 |
| 2 | 03–07 | Satellite number | 25544 |
| 3 | 08–08 | Classification (U=Unclassified) | U |
| 4 | 10–11 | International Designator (Last two digits of launch year) | 98 |
| 5 | 12–14 | International Designator (Launch number of the year) | 067 |
| 6 | 15–17 | International Designator (Piece of the launch) | A |
| 7 | 19–20 | Epoch Year (Last two digits of year) | 08 |
| 8 | 21–32 | Epoch (Day of the year and fractional portion of the day) | 264.51782528 |
| 9 | 34–43 | First Time Derivative of the Mean Motion divided by two [2] | −.00002182 |
| 10 | 45–52 | Second Time Derivative of Mean Motion divided by six (decimal point assumed) | 00000-0 |
| 11 | 54–61 | BSTAR drag term (decimal point assumed) [2] | -11606-4 |
| 12 | 63–63 | The number 0 (Originally this should have been "Ephemeris type") | 0 |
| 13 | 65–68 | Element set number. incremented when a new TLE is generated for this object. [2] | 292 |
| 14 | 69–69 | Checksum (Modulo 10) | 7 |

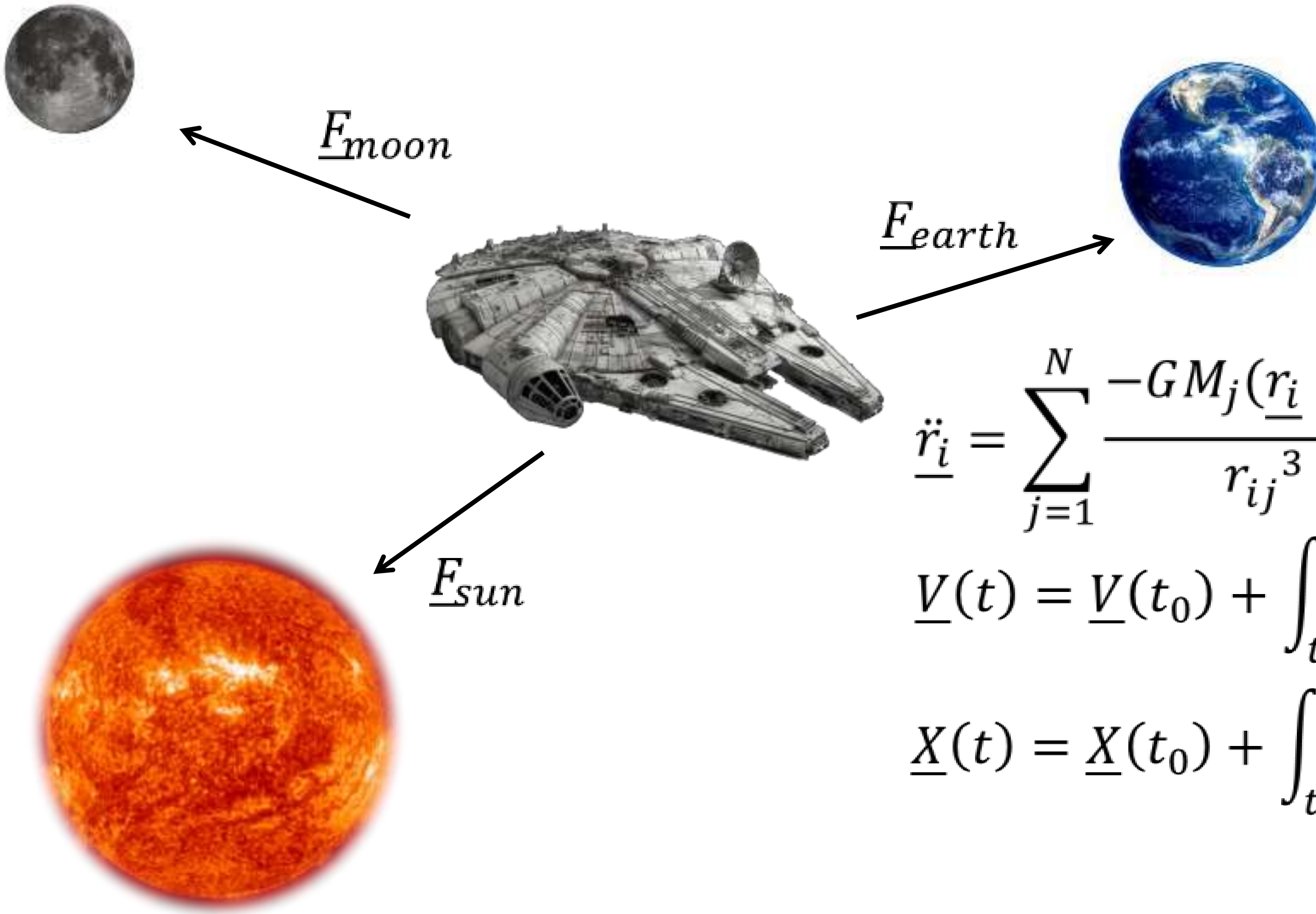
| Field | Columns | Content | Example |
|---|---|---|---|
| 1 | 01–01 | Line number | 2 |
| 2 | 03–07 | Satellite number | 25544 |
| 3 | 09–16 | Inclination [Degrees] | 51.6416 |
| 4 | 18–25 | Right Ascension of the Ascending Node [Degrees] | 247.4627 |
| 5 | 27–33 | Eccentricity (decimal point assumed) | 0006703 |
| 6 | 35–42 | Argument of Perigee [Degrees] | 130.5360 |
| 7 | 44–51 | Mean Anomaly [Degrees] | 325.0288 |
| 8 | 53–63 | Mean Motion [Revs per day] | 15.72125391 |
| 9 | 64–68 | Revolution number at epoch [Revs] | 56353 |
| 10 | 69–69 | Checksum (Modulo 10) | 7 |

Ref:  http://en.wikipedia.org/wiki/Two-line_element_set

# Astrodynamics

## Equations of Motion

# Integrating Multi-Body Dynamics




$$\underline{\ddot{r}_i} = \sum_{j=1}^{N} \frac{-GM_j(\underline{r_i} - \underline{r_j})}{{r_{ij}}^3}$$

$$\underline{V}(t) = \underline{V}(t_0) + \int_{t_0}^{t} \underline{\ddot{r}_i}(t)dt$$

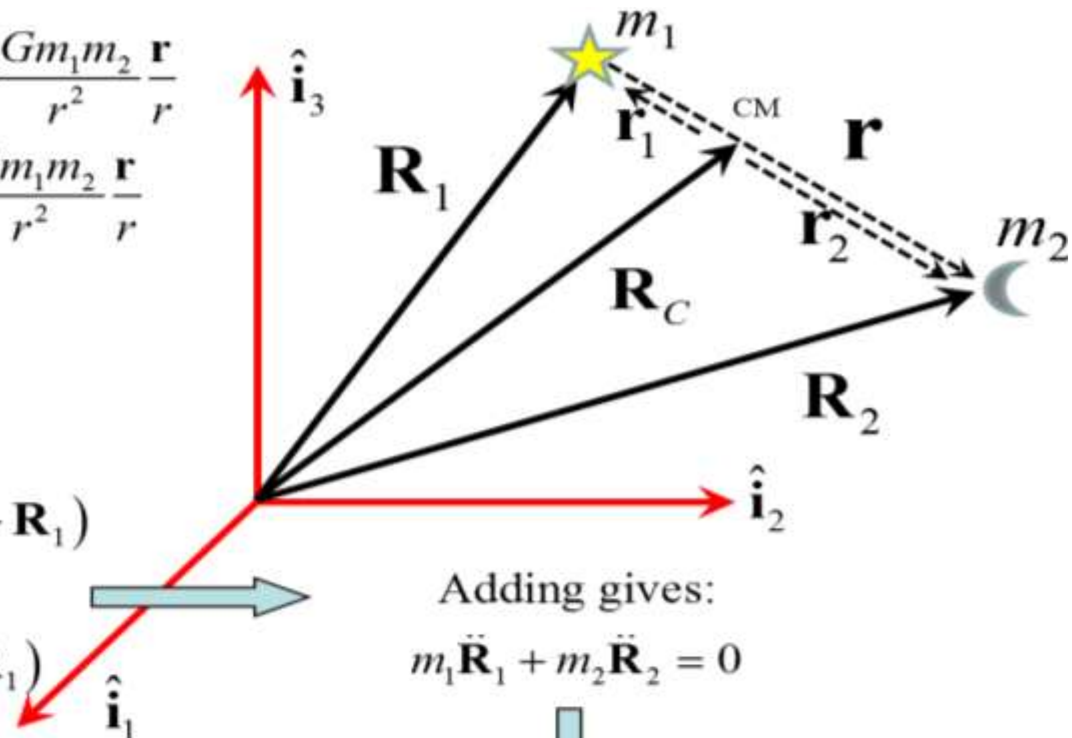
$$\underline{X}(t) = \underline{X}(t_0) + \int_{t_0}^{t} \underline{V}(t)dt$$

# Equations of Motion – The 2-Body Problem

Force on $m_2$ due to $m_1$ is $\mathbf{F}_{21} = -\frac{Gm_1m_2}{r^2}\frac{\mathbf{r}}{r}$

Force on $m_1$ due to $m_2$ is $\mathbf{F}_{12} = \frac{Gm_1m_2}{r^2}\frac{\mathbf{r}}{r}$

where $\mathbf{r} = \mathbf{R}_2 - \mathbf{R}_1$




$$\begin{cases} \mathbf{F}_{21} = m_2\ddot{\mathbf{R}}_2 = -\dfrac{Gm_1m_2}{|\mathbf{R}_2 - \mathbf{R}_1|^3}(\mathbf{R}_2 - \mathbf{R}_1) \\ \mathbf{F}_{12} = m_1\ddot{\mathbf{R}}_1 = \dfrac{Gm_1m_2}{|\mathbf{R}_2 - \mathbf{R}_1|^3}(\mathbf{R}_2 - \mathbf{R}_1) \end{cases}$$

Adding gives:

$$m_1\ddot{\mathbf{R}}_1 + m_2\ddot{\mathbf{R}}_2 = 0$$

Center of mass position

$$(m_1 + m_2)\mathbf{R}_C = m_1\mathbf{R}_1 + m_2\mathbf{R}_2$$

$$(m_1 + m_2)\ddot{\mathbf{R}}_C = m_1\ddot{\mathbf{R}}_1 + m_2\ddot{\mathbf{R}}_2 = 0$$

$$\downarrow$$

$$\ddot{\mathbf{R}}_C = \mathbf{0} \quad \dot{\mathbf{R}}_C = \mathbf{v}_{co} \quad \mathbf{R}_C = \mathbf{v}_{co}t + \mathbf{R}_{C0}$$

**Center of mass moves at constant velocity.**

# Equations of Motion (2)

$$\ddot{\mathbf{R}}_2 - \ddot{\mathbf{R}}_1 = -\frac{G(m_1 + m_2)}{|\mathbf{R}_2 - \mathbf{R}_1|^3}(\mathbf{R}_2 - \mathbf{R}_1) = -\frac{\mu}{|\mathbf{R}_2 - \mathbf{R}_1|^3}(\mathbf{R}_2 - \mathbf{R}_1)$$

$$\mu = G(m_1 + m_2)$$

for a S/C-Earth two body problem

$$m_1 = m_E >> m_2 \quad \rightarrow \quad \boxed{\mu_E = Gm_1 = 398{,}600.441 \text{ km}^3 / \text{s}^2}$$

$$\mathbf{r} = \mathbf{R}_2 - \mathbf{R}_1 \quad \rightarrow \quad \boxed{\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r}}$$

**Fundamental Equation (#1)**

The motion of each body can be broken out as,

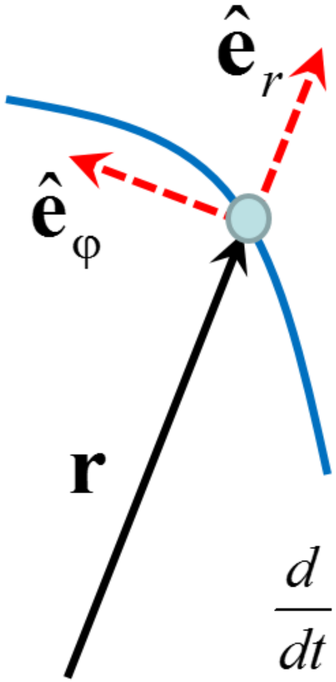
$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 \quad \rightarrow \quad \mathbf{r}_1 = -\frac{m_2}{(m_1 + m_2)}\mathbf{r}, \qquad \mathbf{r}_2 = \frac{m_1}{(m_1 + m_2)}\mathbf{r}$$

Differentiating each gives:

$$\ddot{\mathbf{r}}_1 = -\frac{\mu_1}{r_1^3}\mathbf{r}_1, \qquad \ddot{\mathbf{r}}_2 = -\frac{\mu_2}{r_2^3}\mathbf{r}_2$$

Relative to the Center of Mass, each body behaves similarly.

# Equations of Motion - Energy



$$\mathbf{r} = r\hat{\mathbf{e}}_r \qquad \dot{\mathbf{r}} = \dot{r}\hat{\mathbf{e}}_r + \boldsymbol{\omega} \times r\hat{\mathbf{e}}_r = \dot{r}\hat{\mathbf{e}}_r + r\dot{\varphi}\hat{\mathbf{e}}_\varphi$$

$$\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r} \cdot \dot{\mathbf{r}} = -\frac{\mu}{r^3}(r\dot{r}) = -\frac{\mu\dot{r}}{r^2} = \frac{d}{dt}\left(\frac{\mu}{r}\right)$$

Also note that, $\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}} = \frac{d}{dt}\left(\frac{1}{2}\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}\right) = \frac{d}{dt}\left(\frac{1}{2}v^2\right)$

$$\frac{d}{dt}\left(\frac{1}{2}v^2 - \frac{\mu}{r}\right) = 0 \quad \rightarrow \quad \boxed{\frac{1}{2}v^2 - \frac{\mu}{r} = \mathcal{E}}$$

**Fundamental equation (#2)**

(Energy equation)

Kinetic Energy

Potential Energy

Example: Escape velocity allows you to reach $r = \infty$ with $v = 0$

$$\frac{1}{2}v^2 - \frac{\mu}{r} = \mathcal{E} = 0 \quad \rightarrow \quad v_{esc} = \sqrt{\frac{2\mu}{r}}$$

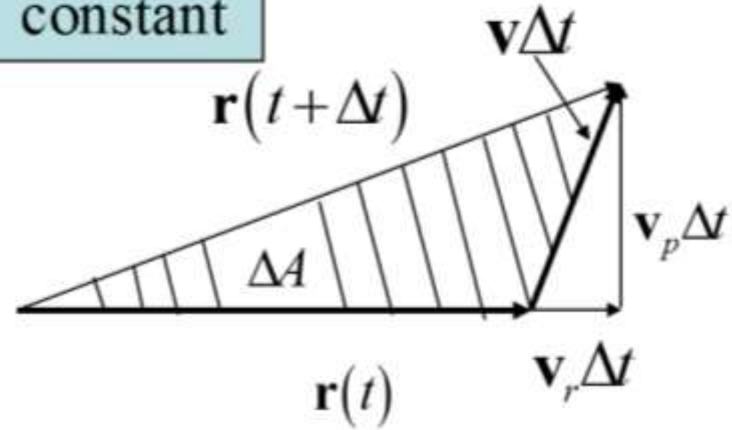
# Equations of Motion – Angular Momentum

**Fundamental equation (#3)**

$$\frac{d}{dt}(\mathbf{r}\times\dot{\mathbf{r}}) = \underbrace{\dot{\mathbf{r}}\times\dot{\mathbf{r}}}_{0} + \mathbf{r}\times\ddot{\mathbf{r}} = -\mathbf{r}\times\left(\frac{\mu}{r^3}\mathbf{r}\right) = 0$$

$$\boxed{\mathbf{r}\times\dot{\mathbf{r}} = \mathbf{h} = \text{ angular momentum vector} = \text{ constant}}$$

(it provides three constants)




$$\Delta A = \frac{1}{2}(r + v_r\Delta t)v_p\Delta t - \frac{1}{2}(v_r\Delta t)(v_p\Delta t)$$

$$\frac{dA}{dt} = \lim_{\Delta t\to 0}\frac{\Delta A}{\Delta t} = \lim_{\Delta t\to 0}\left[\frac{1}{2}(r+v_r\Delta t)v_p - \frac{1}{2}v_r v_p\Delta t\right] = \frac{1}{2}rv_p = \frac{1}{2}|\mathbf{r}\times\mathbf{v}|$$

$$2\frac{dA}{dt} = |\mathbf{r}\times\mathbf{v}| = |\mathbf{r}\times\dot{\mathbf{r}}| = h \quad \leftrightarrow \quad \text{Kepler's } 2^{\text{nd}} \text{ law!}$$

(Sweeps out equal areas in equal times!)

# Equations of Motion - Eccentricity Vector

$$\frac{d}{dt}(\dot{\mathbf{r}}\times\mathbf{h}) = \ddot{\mathbf{r}}\times\mathbf{h} + \dot{\mathbf{r}}\times\underbrace{\dot{\mathbf{h}}}_{0} = \ddot{\mathbf{r}}\times\mathbf{h} =$$

$$= -\frac{\mu}{r^3}(\mathbf{r}\times\mathbf{h}) = -\frac{\mu}{r^3}\left[\mathbf{r}\times(\mathbf{r}\times\dot{\mathbf{r}})\right] = -\frac{\mu}{r^3}\left[(\mathbf{r}\cdot\dot{\mathbf{r}})\mathbf{r} - (\mathbf{r}\cdot\mathbf{r})\dot{\mathbf{r}}\right] =$$

$$= -\frac{\mu}{r^3}\left[r\dot{r}\mathbf{r} - r^2\dot{\mathbf{r}}\right] = -\frac{\mu}{r^2}\dot{r}\mathbf{r} + \frac{\mu}{r}\dot{\mathbf{r}} = \frac{d}{dt}\left(\frac{\mu}{r}\mathbf{r}\right)$$

**Fundamental equation (#4)**

$$\frac{d}{dt}\left(\dot{\mathbf{r}}\times\mathbf{h} - \frac{\mu}{r}\mathbf{r}\right) = \mathbf{0} \quad \rightarrow \quad \dot{\mathbf{r}}\times\mathbf{h} - \frac{\mu}{r}\mathbf{r} = \boldsymbol{C} = \text{ constant vector}$$

Rearranging this gives, $\dfrac{\dot{\mathbf{r}}\times\mathbf{h}}{\mu} - \dfrac{\mathbf{r}}{r} = \dfrac{\boldsymbol{C}}{\mu}$, and define $\dfrac{\boldsymbol{C}}{\mu} \triangleq \boldsymbol{e}$

$\boldsymbol{e}$ is the dimensionless eccentricity vector

which lies in the orbit plane

# Equations of Motion – Conic Section

$$\mathbf{r}\cdot\left(\dot{\mathbf{r}}\times\mathbf{h}-\frac{\mu}{r}\mathbf{r}\right)=\mathbf{r}\cdot C=\mathbf{r}\cdot\mu\mathbf{e} \quad\rightarrow\quad \mathbf{r}\cdot(\dot{\mathbf{r}}\times\mathbf{h})-\frac{\mu}{r}\mathbf{r}\cdot\mathbf{r}=\mathbf{r}\cdot\mu\mathbf{e}$$

Using $A\cdot(B\times C)=(A\times B)\cdot C,\ \rightarrow\ (\mathbf{r}\times\dot{\mathbf{r}})\cdot\mathbf{h}=\mu r+\mu\mathbf{r}\cdot\mathbf{e}$

Since $(\mathbf{r}\times\dot{\mathbf{r}})=\mathbf{h},\ \dfrac{h^2}{\mu}=r+re\cos\varphi$

$$\boxed{r=\frac{h^2/\mu}{1+e\cos\varphi}=\frac{p}{1+e\cos\varphi}}$$

Polar form of conic section with the origin at one focus

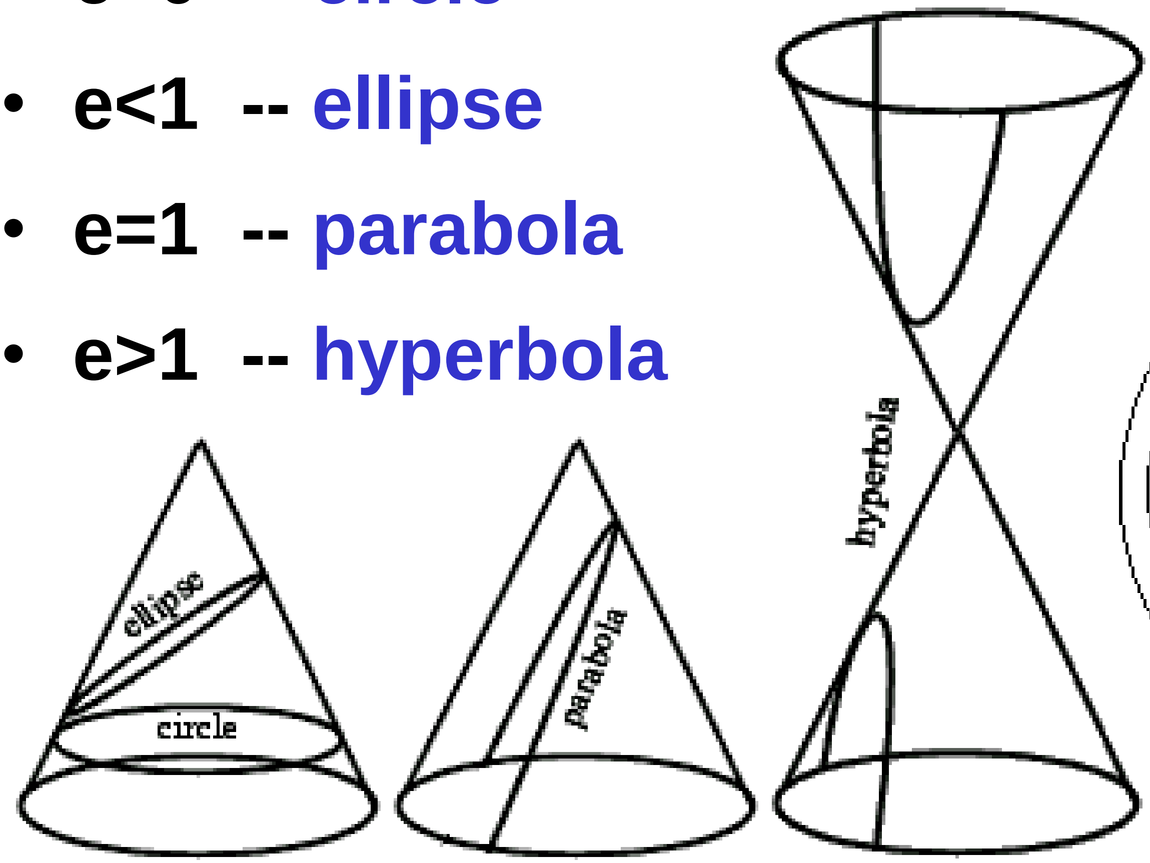
$p=h^2/\mu=a(1-e^2)\equiv$ "semilatus rectum" = const

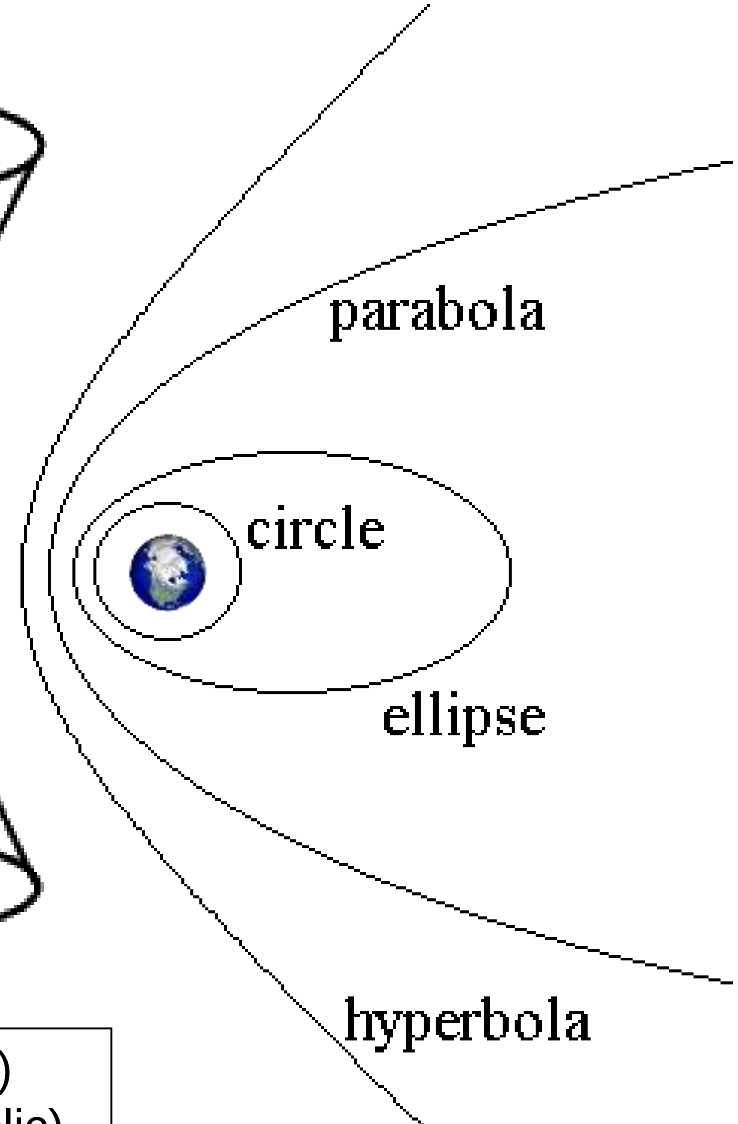
from Latin *semi*="half," *latus* = "side," and *rectum* = "straight"

$$\text{Kepler's 1}^{\text{st}}\text{ law:}\quad\begin{cases} e=0 & r=a & \text{circle}\\ e<1 & a>0 & \text{ellipse}\\ e=1 & a=\infty & \text{parabola}\\ e>1 & a<0 & \text{hyperbola}\end{cases}$$

# Possible Orbital Trajectories

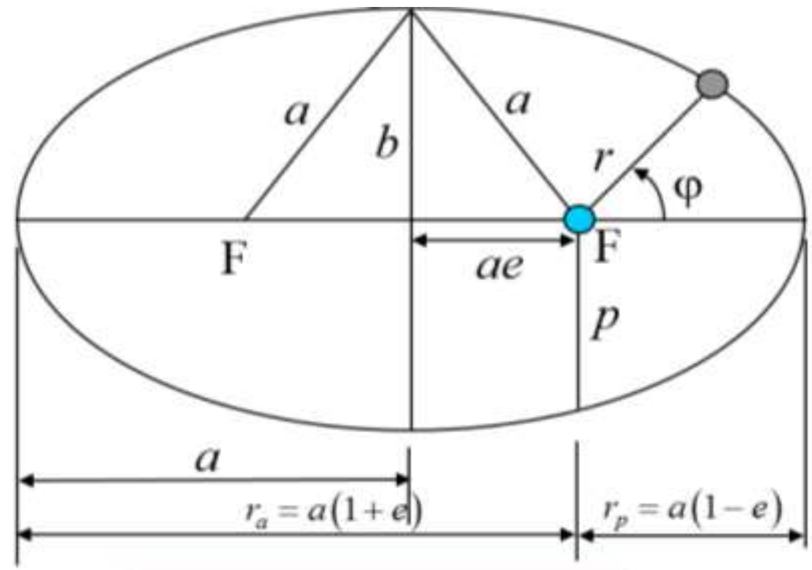
- **e=0 -- circle**
- **e<1 -- ellipse**
- **e=1 -- parabola**
- **e>1 -- hyperbola**




| | |
|---|---|
| e < 1 | Orbit is 'closed' – recurring path (elliptical) |
| e > 1 | Not an orbit – passing trajectory (hyperbolic) |

# Conic Section Geometry




$$r_p = a(1-e) \quad \text{and} \quad r_a = a(1+e)$$

$$\frac{r_a - r_p}{r_a + r_p} = \frac{a(1+e) - a(1-e)}{a(1+e) + a(1-e)} = \frac{2e}{2} = e$$

$$b^2 + (ae)^2 = a^2 \rightarrow \boxed{b = a\sqrt{1-e^2}}$$

Using $r = \dfrac{p}{1 + e\cos\varphi}$, at φ=0, $r_p = \dfrac{p}{1+e}$

So $p = r_p(1+e) = a(1-e)(1+e) = a(1-e^2)$ $\boxed{p = h^2/\mu = a(1-e^2)}$

Now examine momentum and energy at periapsis:

$$|\mathbf{h}| = h = \sqrt{\mu p} == \sqrt{\mu a(1-e^2)} = r_p v_p$$

$$v_p^2 = \frac{\mu a(1-e^2)}{r_p^2} = \frac{\mu a(1-e)(1+e)}{a^2(1-e)^2} = \frac{\mu(1+e)}{a(1-e)}$$

$$E = \frac{1}{2}v_p^2 - \frac{\mu}{r_p}$$

$$= \frac{\mu(1+e)}{2a(1-e)} - \frac{\mu}{a(1-e)} = -\frac{\mu}{2a}$$

So the general Energy Equation is:
(Also called the Vis Viva Equation)

$$\boxed{\frac{1}{2}v^2 - \frac{\mu}{r} = -\frac{\mu}{2a} \quad \text{or} \quad \frac{v^2}{\mu} = \frac{2}{r} - \frac{1}{a}}$$

# Other Useful Properties

For a Circular Orbit,

$$e = 0 \quad \rightarrow \quad r = r_p = r_a = p = a = b$$

$$v_p^2 = \frac{\mu(1+e)}{a(1-e)} \quad \rightarrow \quad \boxed{v_c = v_p = v_a = \sqrt{\frac{\mu}{r}}}$$

For Escape Velocity we would have,

$$\frac{1}{2}v^2 - \frac{\mu}{r} = \mathcal{E} = 0 \quad \rightarrow \quad v_{esc} = \sqrt{\frac{2\mu}{r}} = \sqrt{2}\sqrt{\frac{\mu}{r}} = \sqrt{2}v_c$$

Matching momentum at apoapsis and periapsis,

$$|\mathbf{h}| = r_p v_p = r_a v_a \quad \rightarrow \quad \frac{r_p}{r_a} = \frac{v_a}{v_p}$$

# Equations of Motion – Kepler's 3rd Law

$$2\frac{dA}{dt} = |\mathbf{r}\times\mathbf{v}| = |\mathbf{h}| = \sqrt{\mu p} = \sqrt{\mu a\left(1-e^2\right)} \quad \leftrightarrow \quad \text{Kepler's } 2^{\text{nd}} \text{ law!}$$

For the area of a full orbit we would have,

$$\frac{dA}{dt}T = |\mathbf{h}|\frac{T}{2} = \frac{\sqrt{\mu p}}{2}T = \pi ab = \pi a^2\sqrt{1-e^2}$$

$$p = a\left(1-e^2\right)$$

$$\text{Period:}\quad T = \frac{2\pi a^2\sqrt{1-e^2}}{\sqrt{\mu a\left(1-e^2\right)}} = 2\pi\sqrt{\frac{a^3}{\mu}} \quad \rightarrow \quad \underbrace{\boxed{T^2 = \frac{(2\pi)^2}{\mu}a^3}}_{\text{Kepler's 3rd law!}}$$

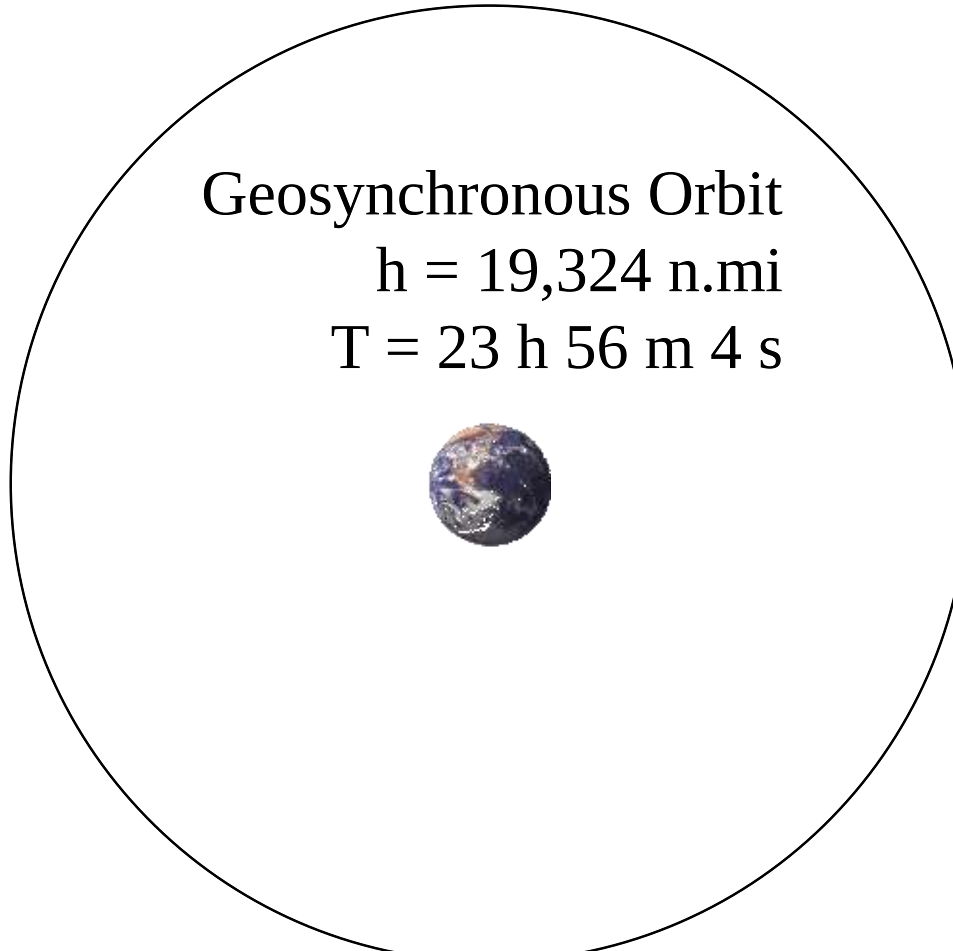
$$\text{Mean Motion:}\quad n = \frac{2\pi}{T} = \sqrt{\frac{\mu}{a^3}}$$

# Orbital Period vs. Altitude

$$T = 2\pi\sqrt{\frac{a^3}{\mu}}$$

h = 160 n.mi
T = 90 minutes

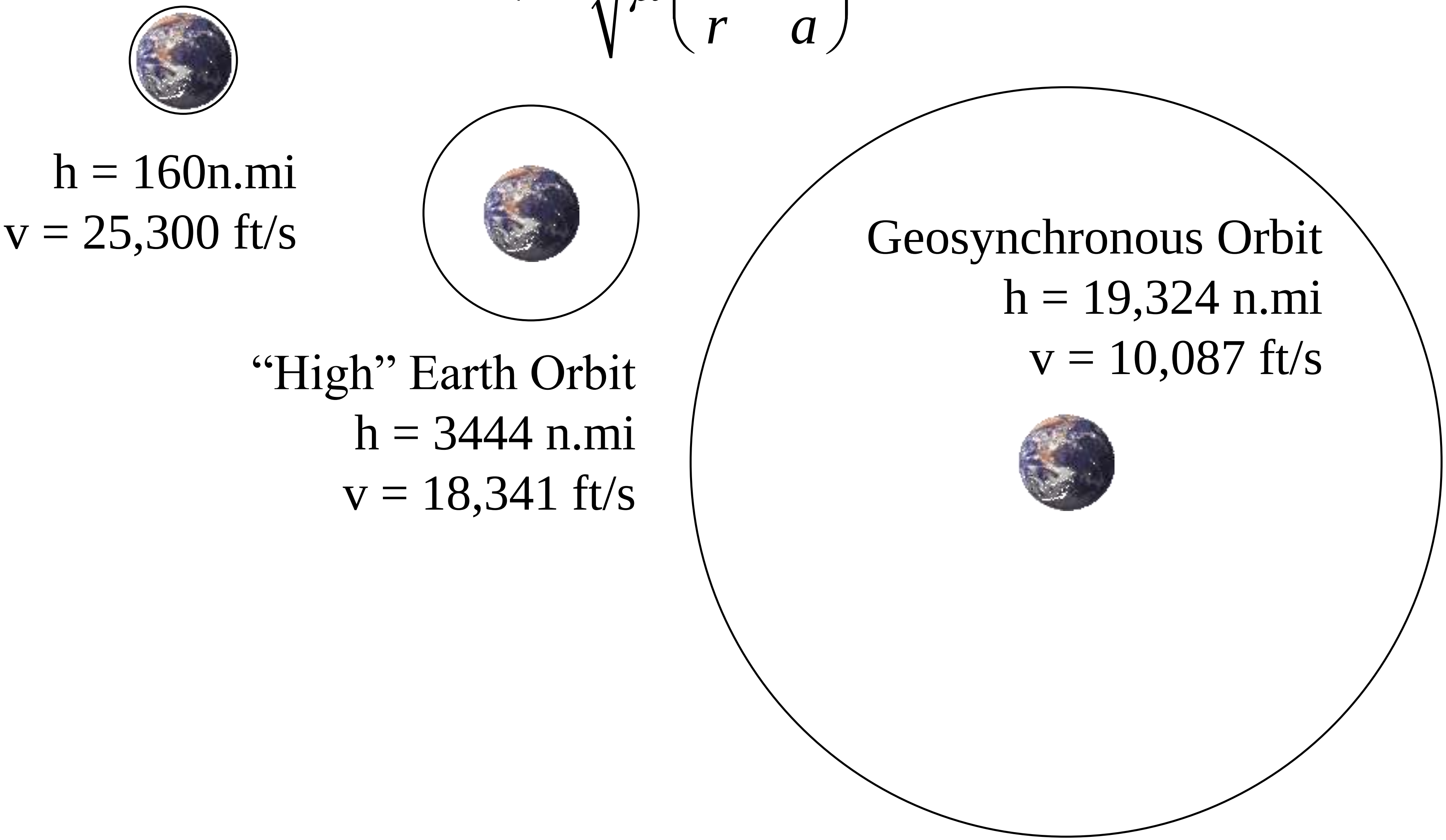
"High" Earth Orbit
h = 3444 n.mi
T = 4 hours



Geosynchronous Orbit
h = 19,324 n.mi
T = 23 h 56 m 4 s

# Orbital Velocity vs. Altitude

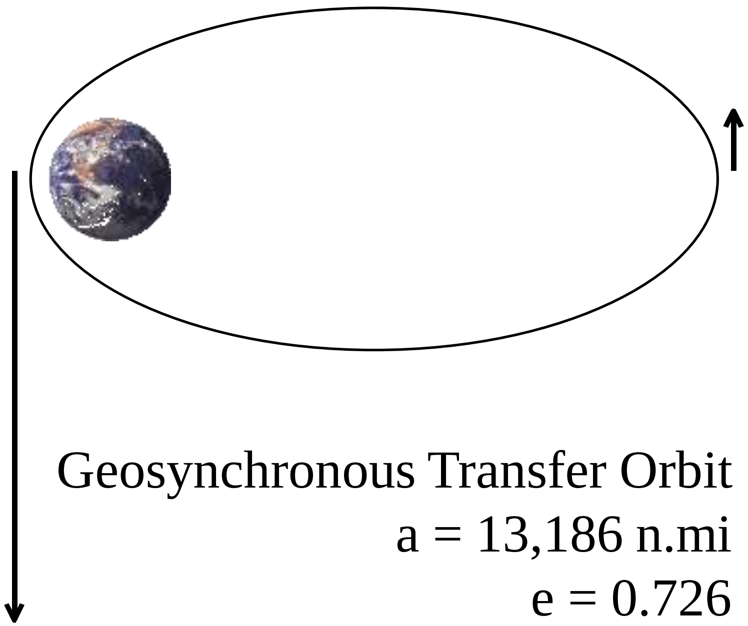
$$v = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)}$$



# Orbital Velocity vs. Altitude (2)
# (Elliptical Orbits)

$$v = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)}$$

h = 160 n.mi
v = 33,320 ft/s



h = 19,324 n.mi
v = 5,273 ft/s

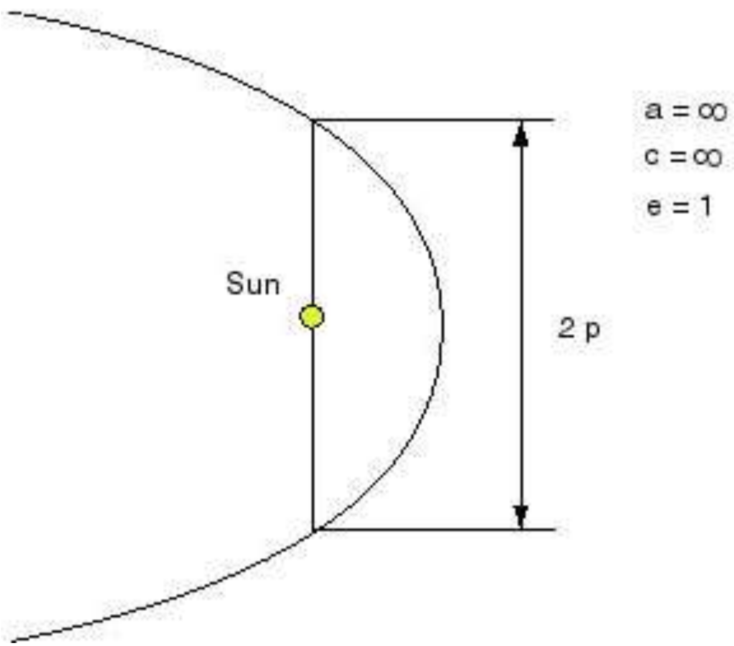
Geosynchronous Transfer Orbit
a = 13,186 n.mi
e = 0.726

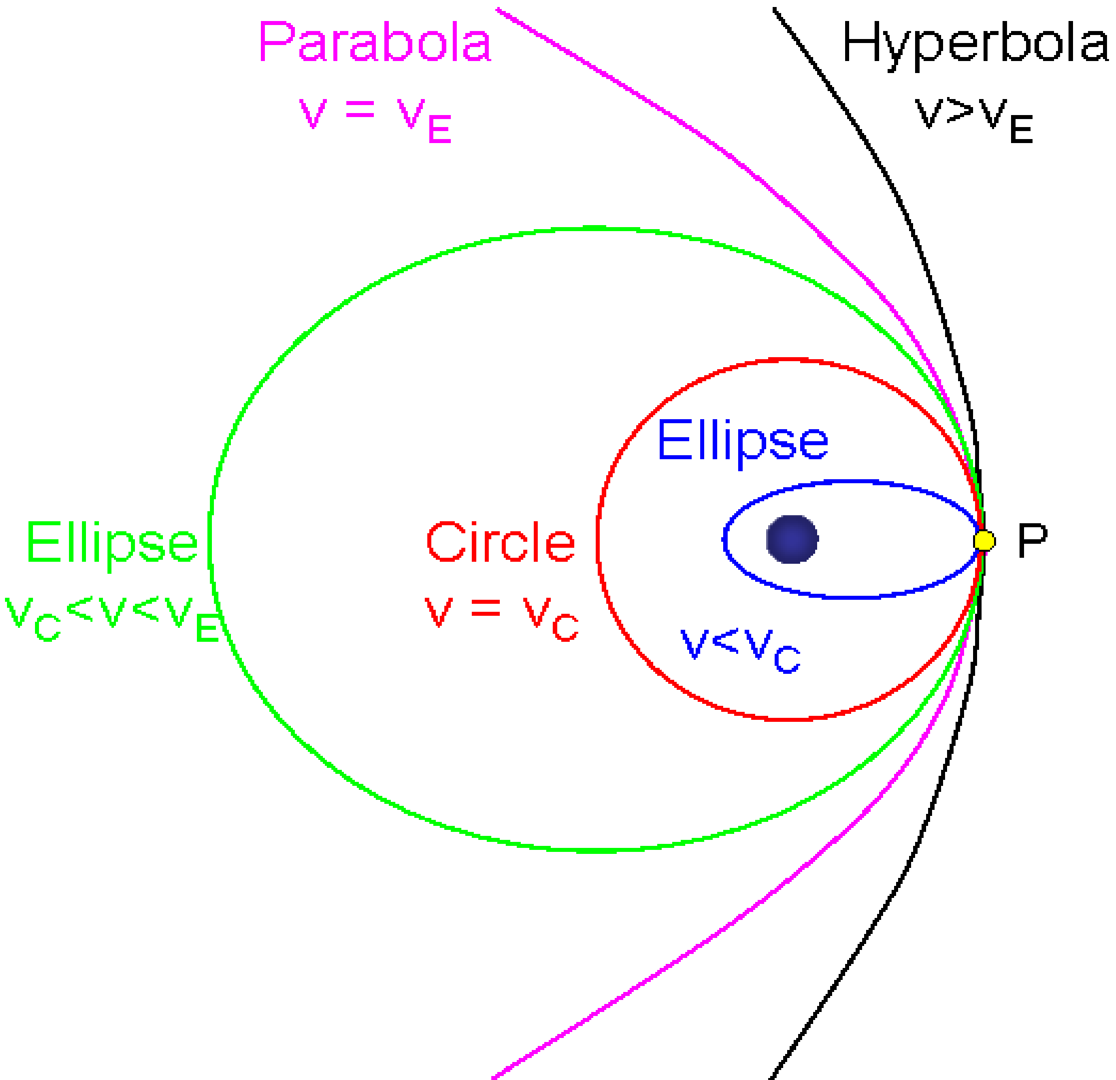
# Parabolic Trajectories

Total Energy = 0

$$\epsilon = 0 = \frac{v^2}{2} - \frac{\mu}{r}$$

$$v_{escape} = \sqrt{\frac{2\mu}{r}} = \sqrt{2}\sqrt{\frac{\mu}{r}} = \sqrt{2}\, v_{circular}$$



# Hyperbolic Trajectories

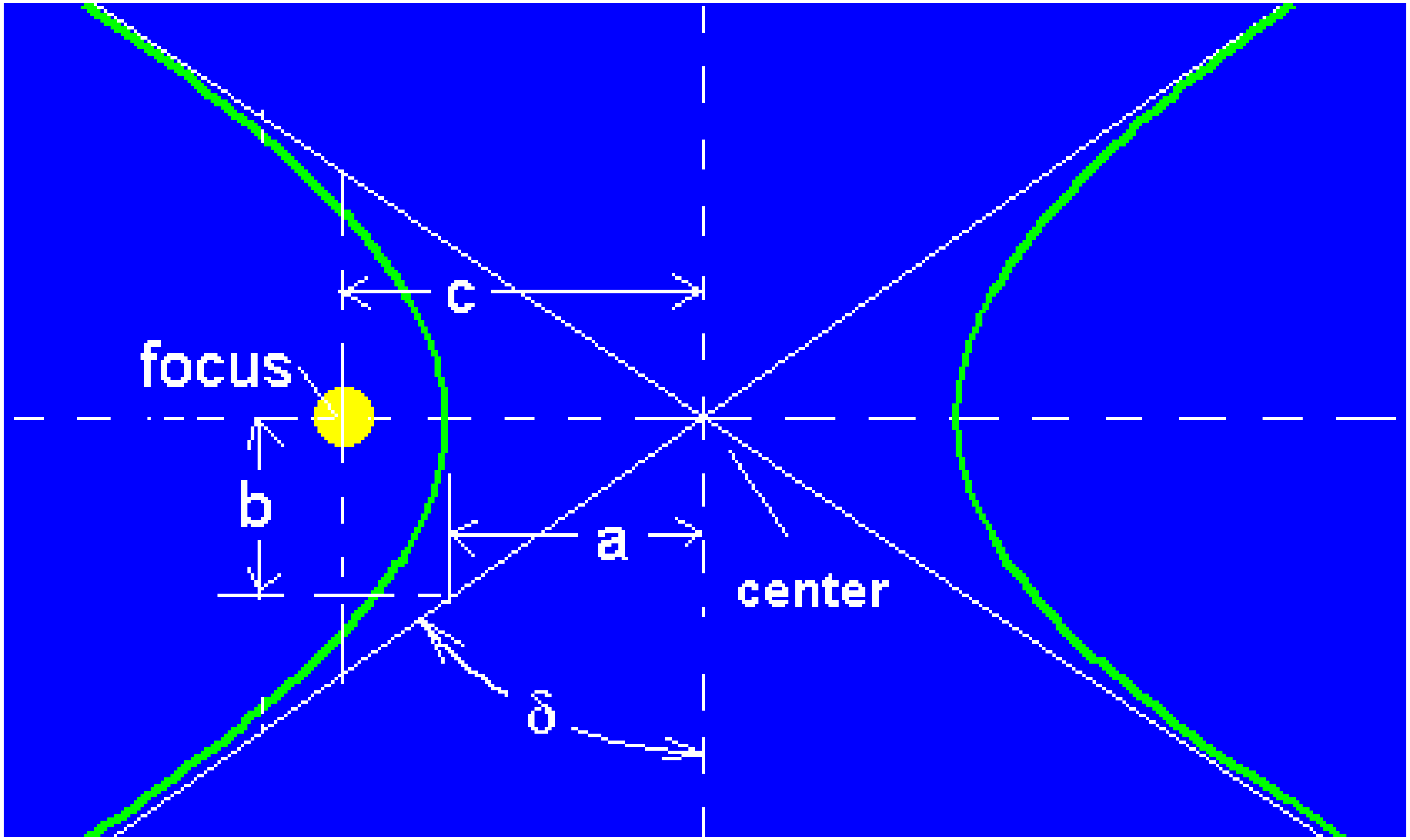
Total Energy > 0

$$\epsilon_{hyper} = -\frac{\mu}{2a}$$

$$a < 0$$

$$v = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)}$$

As $r \to \infty$, $\quad V_\infty \to \sqrt{\frac{\mu}{-a}}$



# Example 1 – Circular Orbit

A satellite in a polar circular orbit with an altitude of 274.6 km passes over USYD at time t=0, when is the next fly-over?

Assumptions:

$$\mathrm{R}_E = 6{,}378 \text{ km},\ \mu_E = 398{,}600.441 \text{ km}^3/\text{s}^2,\ \omega_E = 360^\circ / 24 \text{ hrs}$$

# Example 1 – Circular Orbit

A satellite in a polar circular orbit with an altitude of 274.6 km passes over USYD at time t=0, when is the next fly-over?

Orbit Period: $h = 274.6 \text{ km},\ \mathrm{R}_E \approx 6,378 \text{ km} \ \rightarrow \ r = a \approx 6,652.6$

$$T = 2\pi\sqrt{\frac{a^3}{\mu_E}} = 2\pi\sqrt{\frac{6652.6^3 \text{ km}^3}{398,600.441 \text{ km}^3/\text{s}^2}} = 5400 \text{ sec} = 90 \text{ min}$$

When does USYD cross the orbit plane?

$$\omega_E \approx \frac{360^\circ}{24 \text{ hrs}} \approx 15^\circ/\text{hr} \quad (\text{not exactly right as we shall see})$$

It happens twice - near the ascending and decending nodes.

$$\frac{180^\circ}{15^\circ/\text{hr}} = 12 \text{ hrs},\ \frac{360^\circ}{15^\circ/\text{hr}} = 24 \text{ hrs}$$

Where is the satellite in 12 and 24 hours?

Since $T = 90 \text{ min},\ n = \dfrac{360^\circ}{T} = 4^\circ/\text{min}$

In 12 hrs, $\Delta\varphi = 12 \times 60 \times 4^\circ/\text{min} = 2880^\circ = 8.0 \text{ orbits}$

The satellite is back near the ascending node, but USYD is on the descending node side...

In 24 hrs, $\Delta\varphi = 24 \times 60 \times 4^\circ/\text{min} = 5760^\circ = 16.0 \text{ orbits}$

The satellite and USYD meet again near the ascending node!

# Example 2 – Elliptical Orbit

A giant space station is placed in a permanent elliptical orbit for transporting crew and cargo from the Earth to the Moon. This ‘Cycler’ space station has an apogee at the same radius as the Moon’s orbit. Describe/draw the orbit that cycles twice per month (i.e. has a lunar rendezvous every other cycle)? What is the semi-major axis and the eccentricity? For a vehicle launching from the surface of the Earth, what is the rendezvous altitude to dock with the cycler spacecraft? How fast does a vehicle need to be going to rendezvous?

Assumptions:

$$\mu_E = 398{,}600.441\ \mathrm{km}^3/\mathrm{s}^2,\ \text{Lunar orbit is circular with } T = 27.3 \text{ days}$$

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$$T = \frac{27.3}{2} * 24 * 3600 = 2\pi\sqrt{\frac{a^3}{\mu_E}} = 2\pi\sqrt{\frac{a^3 \text{ km}^3}{398{,}600.441 \text{ km}^3/\text{s}^2}}$$

$$\rightarrow \quad \text{a=241,263 km}$$

$$r_a = a(1+e) = r_{Moon} = 384{,}403 \text{ km} \rightarrow e = 0.5933$$

$$r_p = a(1-e) = 98{,}122 \text{ km (rendezvous altitude)}$$

$$v_a r_a = v_p r_p, \text{ and } v_a = \mu_E\sqrt{\frac{2}{r_a} - \frac{1}{a}} = 410 \text{ km/s}$$

$$v_p = \frac{v_a r_a}{r_p} = 1606 \text{ km/s}$$

# Example 3 – Parabolic Trajectory

A comet from the Oort Cloud has entered the inner solar system and has a 20% probability of crashing into the Earth. If it misses the Earth, it will pass within 10 million kilometres of the Sun and then crash into Mercury with 100% probability on it's way back out. In order to avert disaster, and for scientific exploration, we would like to capture this comet and place it into Earth orbit. What change in velocity is required to capture the comet?

Assumptions:

$\mu_E = 398{,}600.441 \text{ km}^3/\text{s}^2, \ \mu_{Sun} = 132{,}712{,}440{,}018 \text{ km}^3/\text{s}^2$

Planned Earth orbit is circular with $h = 1{,}000$ km

$R_E = 6{,}378$ km, Distance to the Sun = $r_{\oplus} = 149{,}597{,}870.7$ km

# Example 3 – Parabolic Trajectory

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$$E = 0 = \frac{1}{2}v^2 - \frac{\mu_{Sun}}{r} \quad \rightarrow \quad v = \sqrt{\frac{2\mu_{Sun}}{r}}$$

At Earth's orbit intersection, $r_{\oplus} = 149{,}597{,}870.7 \text{ km}$

The comet's velocity would be,

$$v_{comet} = \sqrt{\frac{2\mu_{Sun}}{r_{\oplus}}}$$

A satellite in circular Earth orbit at $h = 1000 \text{ km}$

has velocity $v_{orbit} = \sqrt{\dfrac{\mu_E}{R_E + 1000}}$

The change in velocity is

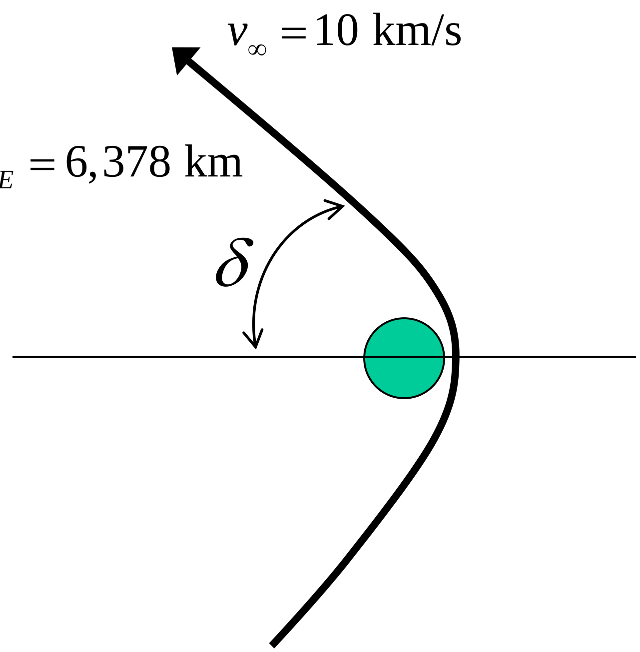
$$\Delta v = v_{orbit} - v_{comet} = \sqrt{\frac{\mu_E}{R_E + 1000}} - \sqrt{\frac{2\mu_{Sun}}{r_{\oplus}}} = -34.77 \text{ km/s}$$

# Example 4 – Hyperbolic Trajectory

An interplanetary probe needs to depart Earth with a velocity of 10 km/s (relative to the Earth). The last engine firing occurs at the perigee point with an altitude of 1000 km. Find the velocity needed at perigee and the parameters of the orbit (a, e)? What is the asymptotic departure angle $\delta$?

Assumptions:

$\mu_E = 398,600.441 \text{ km}^3/\text{s}^2, \ R_E = 6,378 \text{ km}$



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$$\text{Using, } v = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)}, \text{ for } r \to \infty,\ v_\infty = 10 \text{ km/s} = \sqrt{\frac{-\mu_E}{a}}$$

$$\to\ a = -3986 \text{ km}$$

$$\text{Therefore at perigee, } v_p = \sqrt{\mu_E\left(\frac{2}{R_E + 1000} - \frac{1}{a}\right)} = 14.42 \text{ km/s}$$

$$\text{Now } h = r_p v_p \quad \text{and} \quad h^2 / \mu = a(1 - e^2)$$

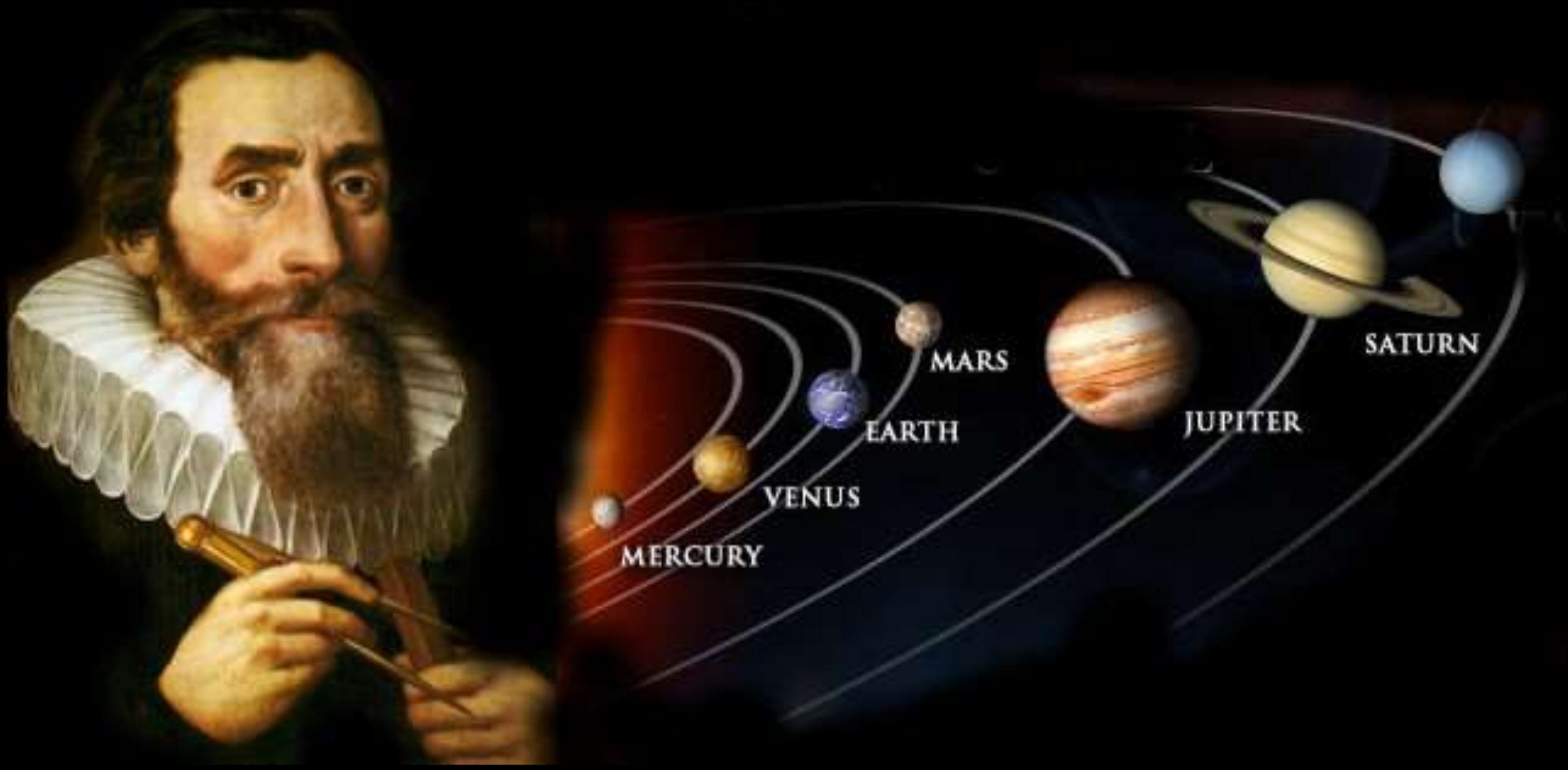
$$\text{So solving for } e = \sqrt{1 - \frac{h^2}{a\mu_E}} = \sqrt{1 - \frac{\left((R_E + 1000)v_p\right)^2}{a\mu_E}} = 2.85$$

$$\text{From } r = \frac{p}{1 + e\cos\varphi},\ r \to \infty \text{ as } 1 + e\cos\varphi \to 0$$

$$\text{This occurs at } \varphi = \cos^{-1}\left(\frac{-1}{2.85}\right) = 110.5^\circ,\quad \delta = 180 - \varphi = 69.5^\circ$$

# Astrodynamics

## Kepler’s Equation



# When Will You Be Where?

What do we know...

Given $T$, or $a$, we know the energy of an orbit $\mathrm{T} = 2\pi\sqrt{\frac{a^3}{\mu}},\ E = -\frac{\mu}{2a}$

Given $e$, we also know the angular momentum $\frac{h^2}{\mu} = p = a(1-e^2)$

and everything else about the orbit's shape... $b = a\sqrt{1-e^2},\ r_p = a(1-e),\ r_a = a(1+e)$

Given a true anomaly we can compute position $r = \frac{h^2/\mu}{1+e\cos\varphi}$

and with position we know velocity $v = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)}$

What don't we know?

- WHEN?
- All of this is in the orbit plane – we need 3D?

# Position and Time

Beginning with angular momentum $\boldsymbol{h} = \boldsymbol{r} \times \dot{\boldsymbol{r}}$, $h = r \cdot v_{\perp} = r^2 \dot{\varphi}$

Using $r = \dfrac{h^2 / \mu}{1 + e\cos\varphi}$, $\quad h = \left( \dfrac{h^2 / \mu}{1 + e\cos\varphi} \right)^2 \dfrac{d\varphi}{dt}$

Or

$$\frac{\mu^2}{h^3} dt = \frac{d\varphi}{\left(1 + e\cos\varphi\right)^2}$$

Integrating both sides,

$$\frac{\mu^2}{h^3}\left(t - t_0\right) = \int_0^{\varphi} \frac{d\varphi}{\left(1 + e\cos\varphi\right)^2}$$

Defining $t_0 \equiv 0$ at periapsis $\left(\varphi = 0\right)$,

The following equation relates time to position for the two-body problem:

$$t = \frac{h^3}{\mu^2} \int_0^{\varphi} \frac{d\varphi}{\left(1 + e\cos\varphi\right)^2}$$

# Kepler's Equation

For an elliptical orbit $(0 < e < 1)$, the solution becomes,

$$t = \frac{h^3}{\mu^2}\int_0^{\varphi}\frac{d\varphi}{(1+e\cos\varphi)^2} = \frac{(h^3/\mu^2)}{(1-e^2)^{3/2}}\underbrace{\left[2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\varphi}{2}\right) - \frac{e\sqrt{1-e^2}\sin\varphi}{1+e\cos\varphi}\right]}_{M}$$

$$M = \left[\frac{\mu^2}{h^3}(1-e^2)^{3/2}\right]t = \left[\frac{\mu}{h^2}\frac{\mu}{h}\sqrt{(1-e^2)^2(1-e^2)}\right]t = \left[\frac{\mu}{h^2}(1-e^2)\frac{\mu}{h}\sqrt{(1-e^2)}\right]t$$

Now recall, $p = h^2/\mu = a(1-e^2)$,

So, $$M = \left[\frac{1}{a}\frac{\mu}{h}\cdot\sqrt{\frac{h^2}{a\mu}}\right]t = \left[\sqrt{\frac{\mu^2}{a^2h^2}\frac{h^2}{a\mu}}\right]t = \left[\sqrt{\frac{\mu}{a^3}}\right]t$$

And since, $T = 2\pi\sqrt{\frac{a^3}{\mu}}$, $\qquad M = \left(\frac{2\pi}{T}\right)t = nt$

$M$ is called the 'Mean Anomaly' and $n$ is the 'Mean Motion'. $M$ corresponds to the angular position of a satellite on an equivalent circular orbit with the same period. $n$ is the average angular velocity of the orbit.

# Kepler's Equation (2)

From $$M = \underbrace{2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\varphi}{2}\right)}_{\Updownarrow} - e\underbrace{\frac{\sqrt{1-e^2}\sin\varphi}{1+e\cos\varphi}}_{\Updownarrow} = E - e\sin E$$

It is convenient to define, $E$ in which case, $\sin E$

$$\underbrace{\tan\frac{E}{2} = \frac{\sin E}{1+\cos E}}_{\text{Trig Identity}} = \frac{\dfrac{\sqrt{1-e^2}\sin\varphi}{1+e\cos\varphi}}{1+\sqrt{1-\left(\dfrac{\sqrt{1-e^2}\sin\varphi}{1+e\cos\varphi}\right)^2}} = \frac{\sqrt{1-e^2}\sin\varphi}{\left(1+e\cos\varphi\right)+\sqrt{\left(1+e\cos\varphi\right)^2-\left(\sqrt{1-e^2}\sin\varphi\right)^2}}$$

$$= \frac{\sqrt{1-e^2}\sin\varphi}{\left(1+e\cos\varphi\right)+\sqrt{1+2e\cos\varphi+e^2\cos^2\varphi-\left(1-e^2\right)\left(1-\cos^2\varphi\right)}}$$

$$= \frac{\sqrt{(1+e)(1-e)}\sin\varphi}{\left(1+e\cos\varphi\right)+\sqrt{1+2e\cos\varphi+e^2\cos^2\varphi-1+e^2-e^2\cos^2\varphi+\cos^2\varphi}}$$

$$= \frac{\sqrt{(1+e)(1-e)}\sin\varphi}{\left(1+e\cos\varphi\right)+\sqrt{2e\cos\varphi+e^2+\cos^2\varphi}} = \frac{\sqrt{(1+e)(1-e)}\sin\varphi}{\left(1+e\cos\varphi\right)+\sqrt{(e+\cos\varphi)^2}} = \frac{\sqrt{(1+e)(1-e)}\sin\varphi}{1+e\cos\varphi+e+\cos\varphi}$$

$$= \frac{\sqrt{(1+e)(1-e)}\sin\varphi}{(1+e)(1+\cos\varphi)} = \sqrt{\frac{(1+e)(1-e)}{(1+e)^2}}\frac{\sin\varphi}{1+\cos\varphi} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\varphi}{2}$$

Therefore, $\tan\dfrac{E}{2} = \sqrt{\dfrac{1-e}{1+e}}\tan\dfrac{\varphi}{2}$ and $E = 2\tan^{-1}\left(\sqrt{\dfrac{1-e}{1+e}}\tan\dfrac{\varphi}{2}\right)$

# Kepler's Equation (3)

What does this mean?

Now we can relate position and time!

For a given orbit $(a, e)$ and position (i.e. true anomaly) $\varphi$,

Compute the 'eccentric anomaly' $E = 2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\varphi}{2}\right)$

Compute the mean anomaly $M = E - e\sin E$

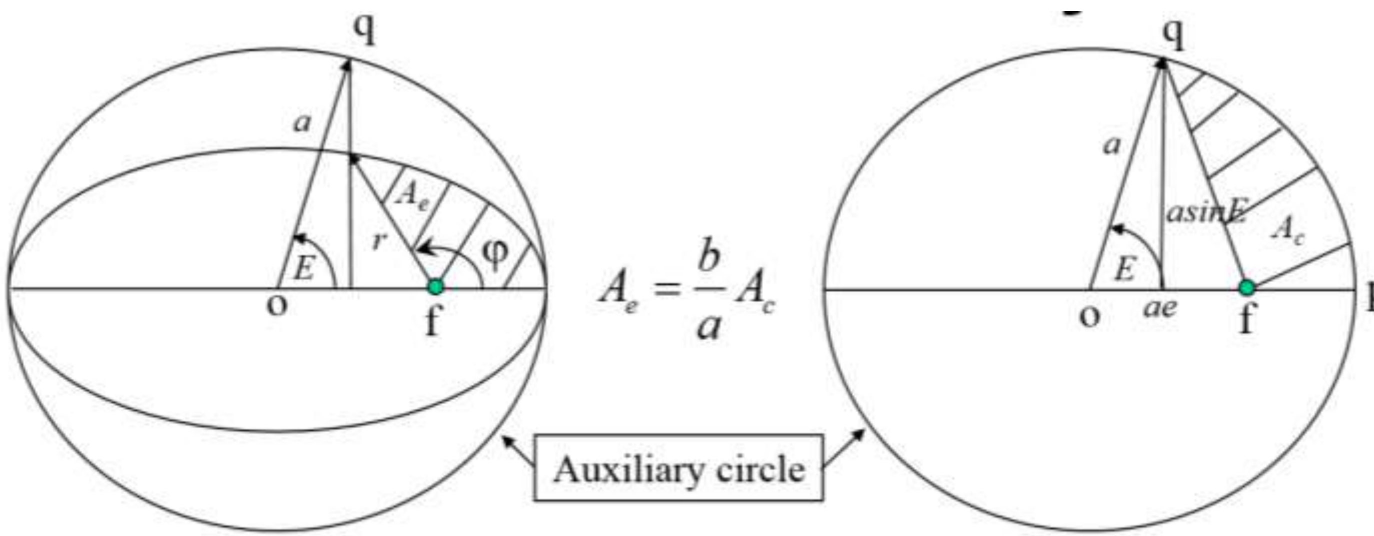
Then using the period $T$, the

time since periapsis passage is $t - t_0 = \frac{M}{2\pi}T$

*Note*: From $\varphi \to t$ is easy!

From $t \to \varphi$ is not so easy (cannot write $E = f(M, e)$)

# Physical Interpretation of the Eccentric Anomaly *E*




$$A_e = \frac{b}{a} A_c$$

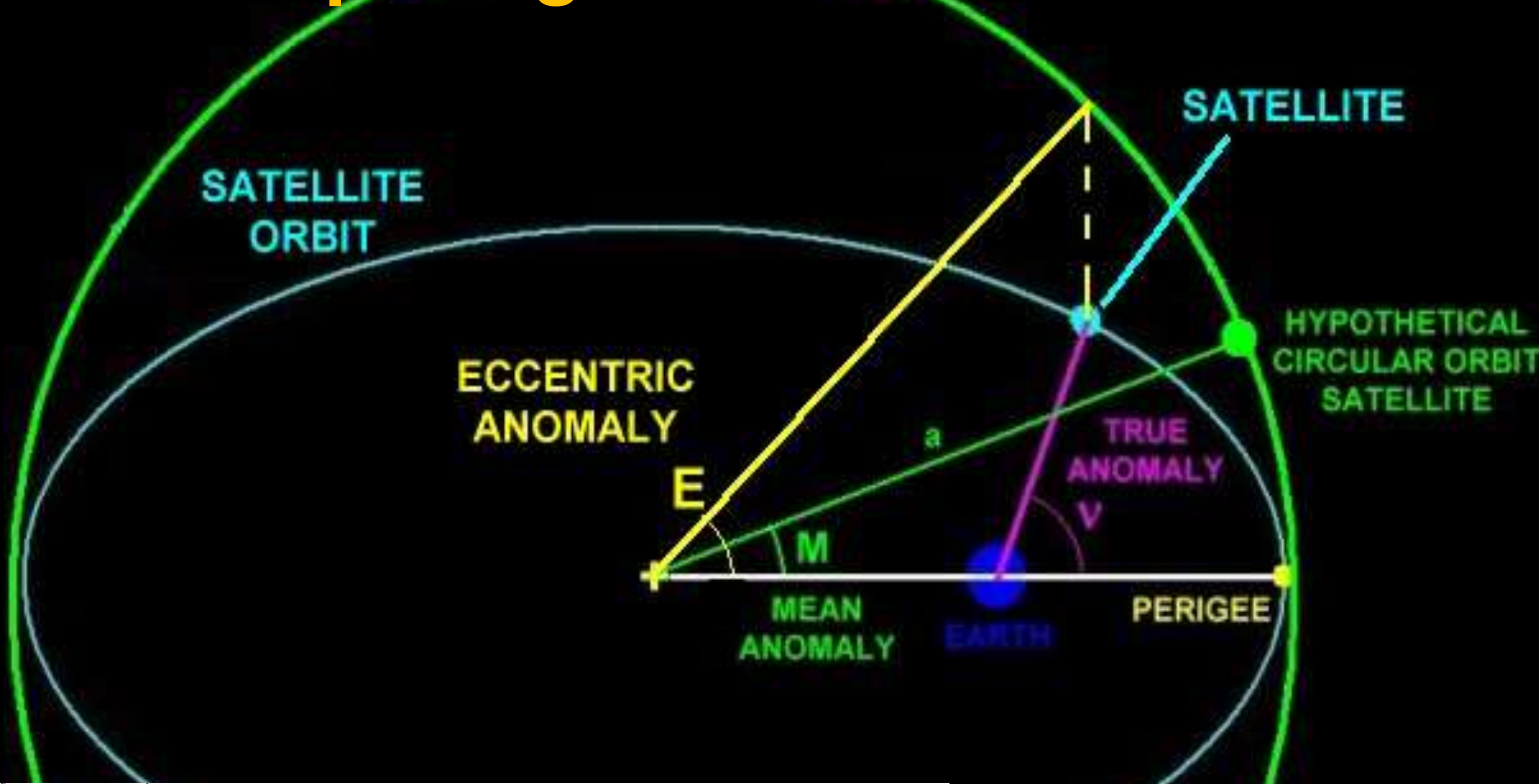
For an arbitray true anomaly φ,

$$A_c = \overbrace{\text{opq}}^{\text{area}} - \overbrace{\text{ofq}}^{\text{area}} = \left(\frac{E}{2\pi}\right)\pi a^2 - \frac{(ae)(a\sin E)}{2} = \frac{a^2}{2}(E - e\sin E)$$

$$A_e = \frac{b}{a}\frac{a^2}{2}(E - e\sin E) = \frac{dA}{dt}(t-t_0) = \frac{\pi ab}{T}(t-t_0) = \frac{ab}{2}\sqrt{\frac{\mu}{a^3}}(t-t_0) = \frac{nab}{2}(t-t_0)$$

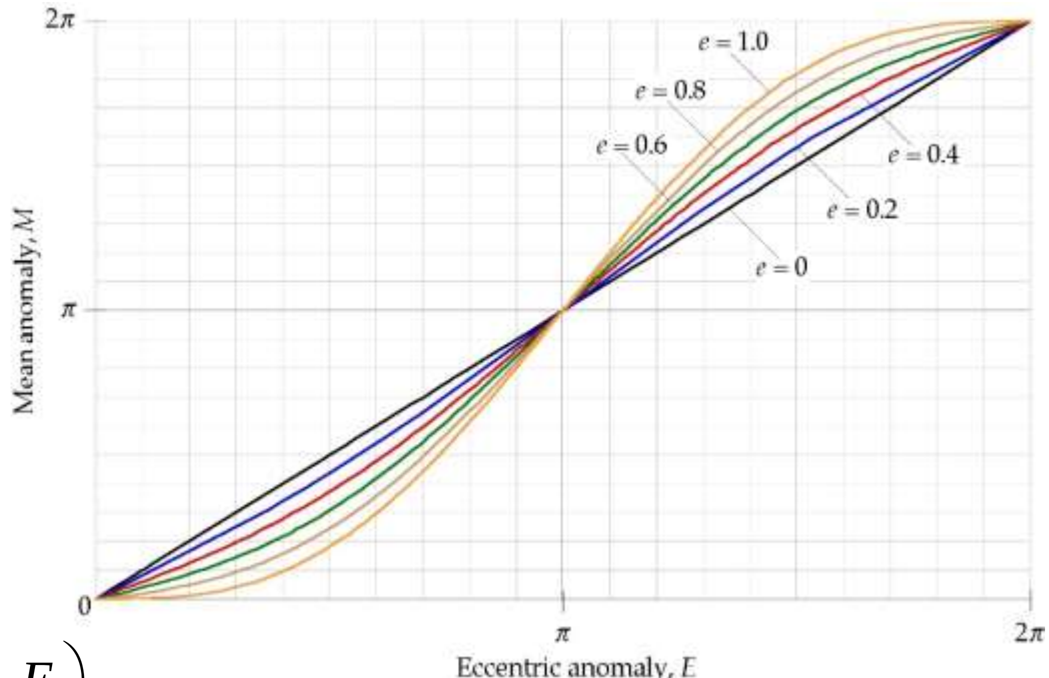
$$n(t-t_0) = E - e\sin E = M$$

# Computing Orbital Position




| | |
|---|---|
| $M(t) = n(t - T_0)$ | Mean Anomaly |
| $E(t) = M(t) + e \sin E(t)$ | Eccentric Anomaly |
| $V(t) = 2 \arctan\left[\sqrt{\frac{1+e}{1-e}} \tan \frac{E(t)}{2}\right]$ | True Anomaly |
| $n = \frac{2\pi}{P} = \sqrt{\frac{\mu}{a^3}}$ | Mean Motion |

# Solving Kepler's Equation




In the case of finding position as a function of time $\varphi(t), r(t)$

From,

$$t \to M: \quad M = \frac{2\pi}{T}(t - t_0)$$

$$M \to E: \quad E - e\sin E = M$$

$$E \to \varphi: \quad \varphi = 2\tan^{-1}\left(\sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}\right)$$

$$E/\varphi \to r: \quad r = a(1 - e\cos E) \text{ or } r = a(1-e^2)/(1+e\cos\varphi)$$

So the only problem is finding $E$ from $M$

Fortunately, the function $M(E) = E - \sin E$ is monotonic ($if\ E\uparrow\ then\ M\uparrow$)

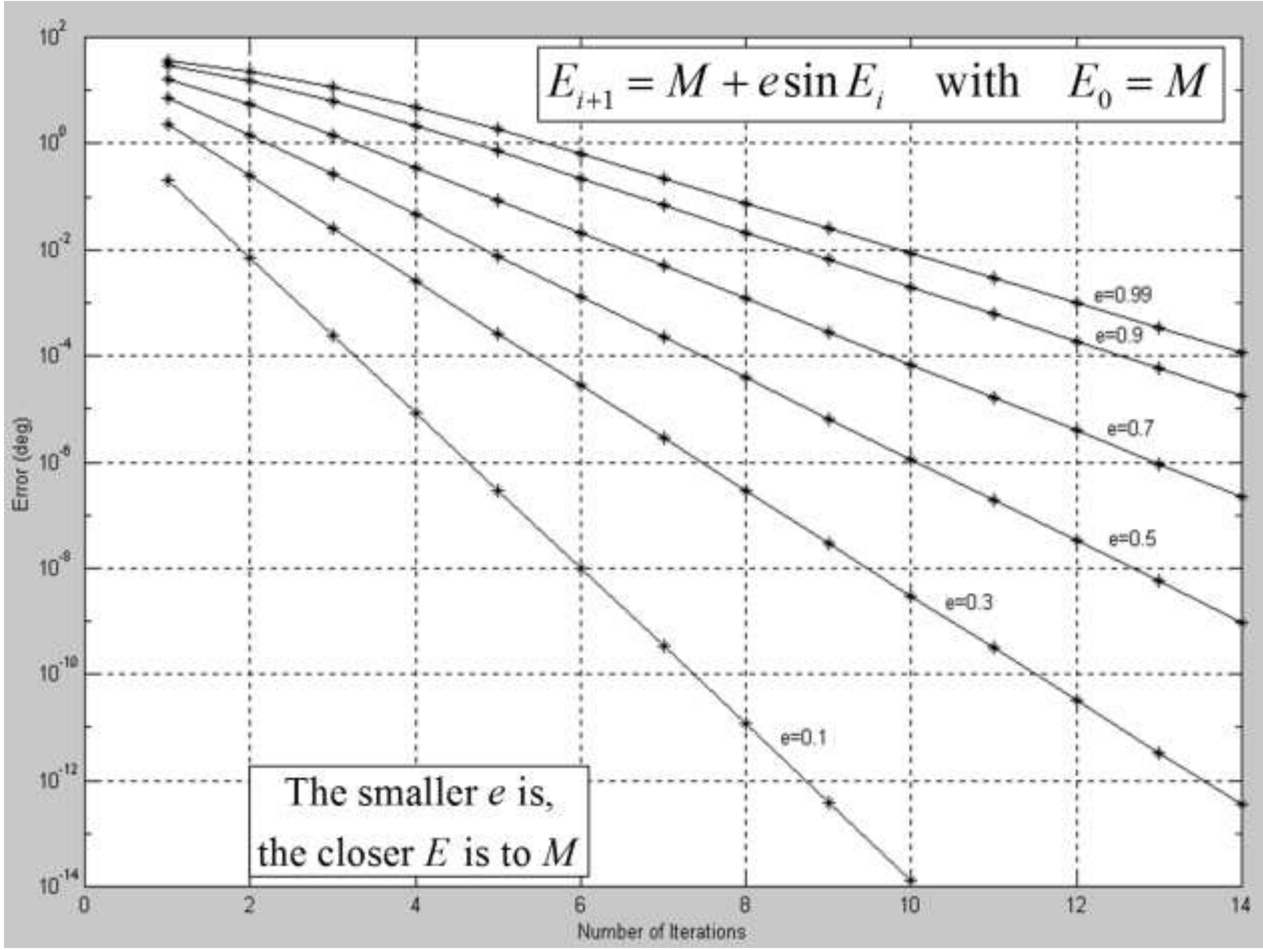
# Solving Kepler's Equation (2)

Method 1: Simple Iteration

For small eccentricity $e$, the values of $M$ and $E$ are close

Initially guess that $E = M$

Then calculate a new estimate $E$ using $E_{i+1} = M + e\sin E_i$



# Solving Kepler's Equation (3)

Method 2: Newton's Method

To find the root of $f(x) = 0$, start with an estimate $x_i$ and update the estimate based on a simple slope calculation:

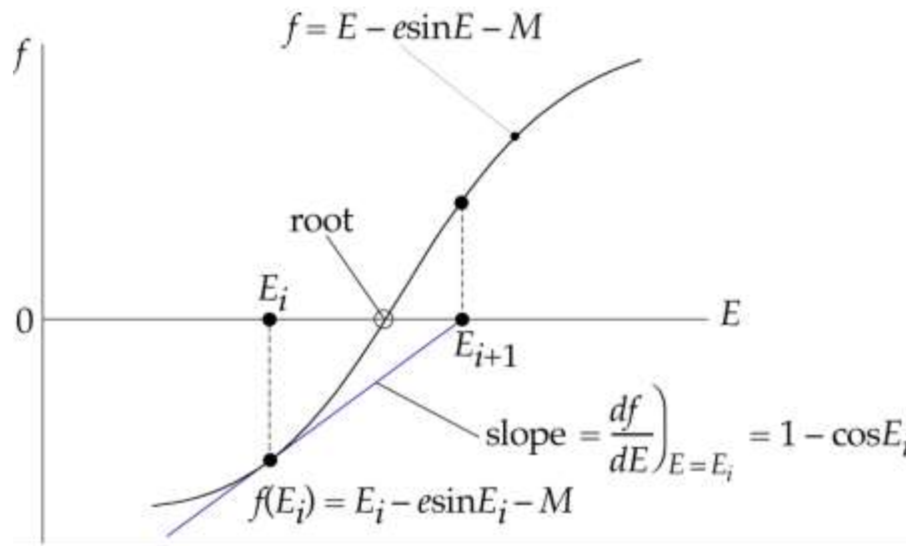
$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$

Choose $x_{i+1}$ such that $f(x_{i+1}) = 0$,

that is... $$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

In this case, $f(E) = E - e\sin E - M$

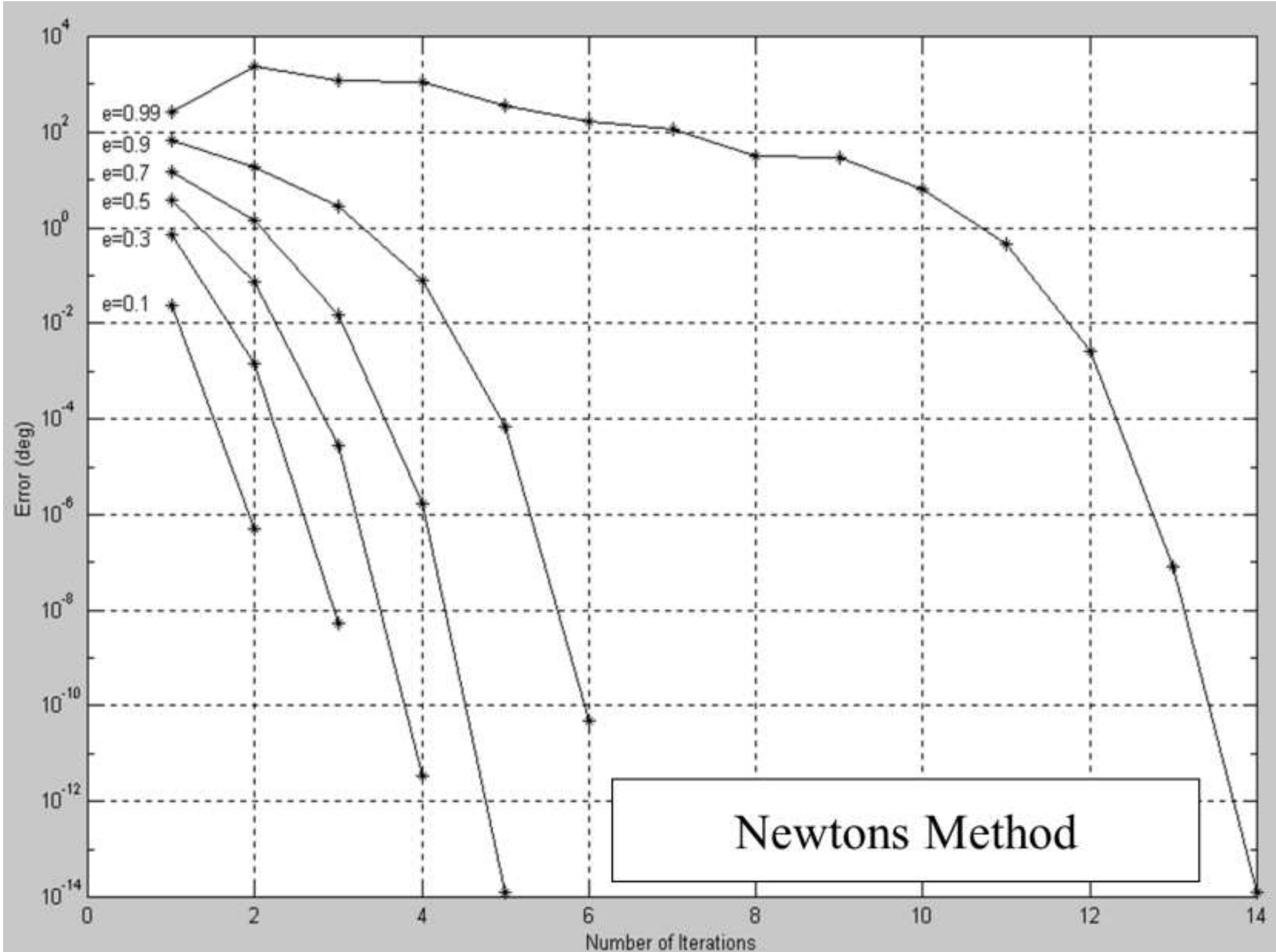
$$f'(E) = 1 - e\cos E$$




So the iteration would be... $$E_{i+1} = E_i - \frac{E_i - e\sin E_i - M}{1 - e\cos E_i}$$

Stop when the update is smaller than a desired tolerance $$\left|\frac{f(x_i)}{f'(x_i)}\right| < error_{desired}$$

# Solving Kepler's Equation (4)



# 350 years of searching for the fastest method (least number of computations)

**Newton (1686)**

$$f(x)=0$$

$$x_{k+1}=x_k-\frac{f(x_k)}{f'(x_k)}$$

**Euler (1740),** $E_0=M$

$$E_{k+1}=M+e\sin E_k$$

**Lagrange (1771)**

$$E=M+\sum_{k=1}^{\infty}\frac{1}{k!}\frac{d^{k-1}(\sin^k M)}{dM^{k-1}}e^k$$

**Bessel (1817)**

$$E=M+\sum_{k=1}^{\infty}\frac{2}{k}J_k(ke)\sin(kM)$$

$$J_k(x)=\frac{1}{\pi}\int_0^{\pi}\cos\ \ x\sin\vartheta-k\vartheta\ \ d\vartheta$$

**Levi-Civita (1904)** $E=\sum_{k=1}^{\infty}L_k(M)\,z^k$

$$\begin{cases} z(e)=\dfrac{e\exp\ \sqrt{1-e^2}}{1+\sqrt{1-e^2}} \\ L_k(M)=p\ \ c_i\cos^i M, s_j\sin^j M \end{cases}$$

**Halley (1742)**

$$E_{k+1}=E_k-\frac{2f(E_k)f'(E_k)}{2f'(E_k)^2-f(E_k)f''(E_k)}$$

$$\begin{cases} f(E_k)=E_k-e\sin E_k-M \\ f'(E_k)=1-e\cos E_k \\ f''(E_k)=e\sin E_k \end{cases}$$

# Position and Time for Parabolic Trajectories

Beginning again with $t = \frac{h^3}{\mu^2}\int_0^{\varphi}\frac{d\varphi}{(1+e\cos\varphi)^2}$

For $e=1$, the result is less complicated than for the ellipse,

$$t = \frac{h^3}{\mu^2}\int_0^{\varphi}\frac{d\varphi}{(1+\cos\varphi)^2} = \frac{h^3}{\mu^2}\left(\frac{1}{2}\tan\frac{\varphi}{2}+\frac{1}{6}\tan^3\frac{\varphi}{2}\right) = \frac{h^3}{\mu^2}M_p$$

From $\varphi \to t$: $\quad M_p = \frac{1}{2}\tan\frac{\varphi}{2}+\frac{1}{6}\tan^3\frac{\varphi}{2}, \quad t = \frac{h^3}{\mu^2}M_p$

From $t \to \varphi$: $\quad M_p = \frac{\mu^2}{h^3}t$

$$\varphi = \text{root of } \frac{1}{6}\left(\tan\frac{\varphi}{2}\right)^3+\frac{1}{2}\tan\frac{\varphi}{2}-M_p=0$$

The only real root is,

$$\varphi = 2\tan^{-1}\left[\left(3M_p+\sqrt{9M_p^{\ 2}+1}\right)^{1/3}-\left(3M_p+\sqrt{9M_p^{\ 2}+1}\right)^{-1/3}\right]$$

# Position and Time for Hyperbolic Trajectories

Analogous to the Elliptical Solution, but with hyperbolic functions...

Recall: $\sinh x = \left(e^{x} - e^{-x}\right)/2, \ \cosh x = \left(e^{x} + e^{-x}\right)/2$

Kepler's Equations becomes: $M_h = e\sinh F - F$ (hyperbolic eccentric anomaly)

The relation between $\varphi$ and $F$ becomes,

$$\tanh\frac{F}{2} = \sqrt{\frac{e-1}{e+1}}\tan\frac{\varphi}{2}$$

From $\varphi \rightarrow t$:

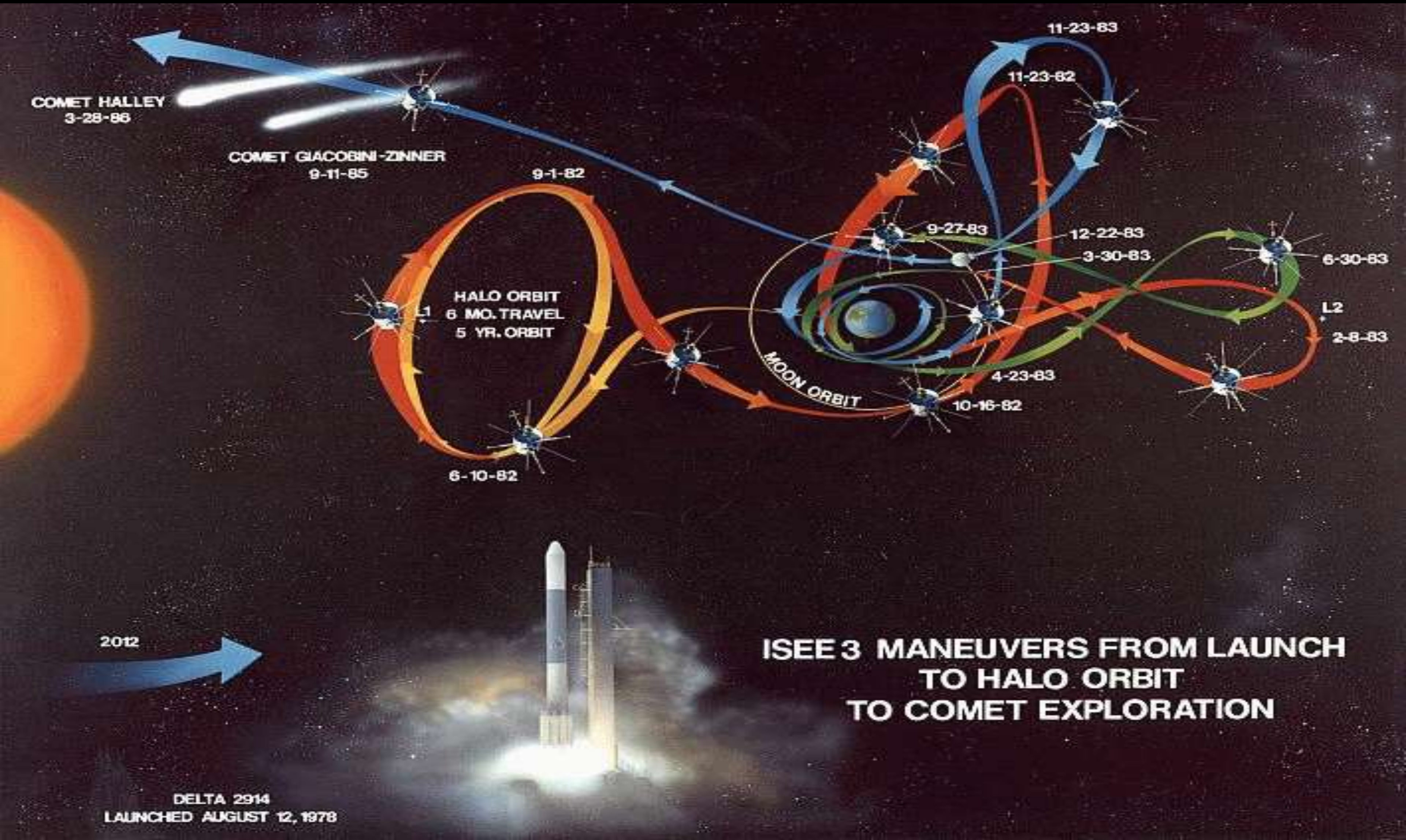
$$F = 2\tanh^{-1}\left(\sqrt{\frac{e-1}{e+1}}\tan\frac{\varphi}{2}\right), \quad M_h = e\sinh F - F, \quad t = \frac{h^3}{\mu^2}M_h\left(e^2-1\right)^{2/3}$$

From $t \rightarrow \varphi$:

$$M_h = \frac{\mu^2}{h^3}\left(e^2-1\right)^{3/2}t, \quad e\sinh F - F = M_h \text{ (Iteration)}, \quad \varphi = 2\tan^{-1}\left(\sqrt{\frac{e+1}{e-1}}\tanh\frac{F}{2}\right)$$

# Astrodynamics

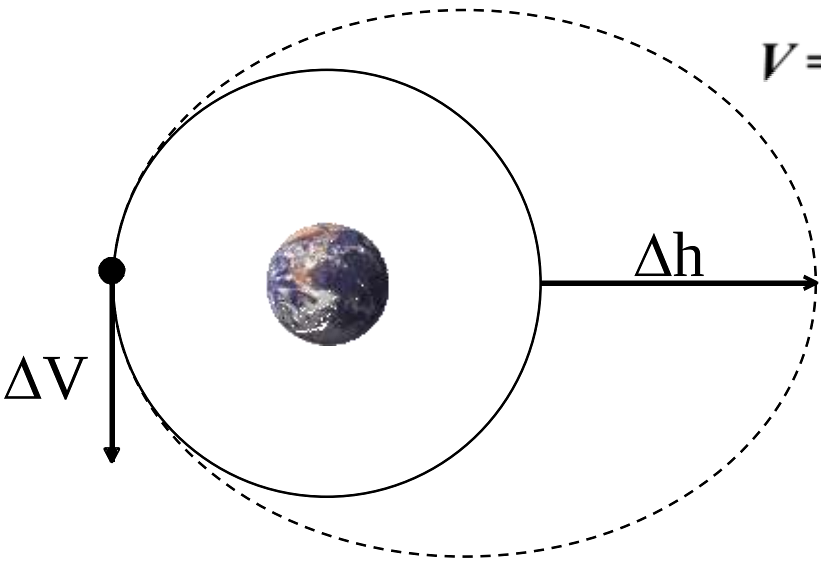
## Orbital Maneuvers

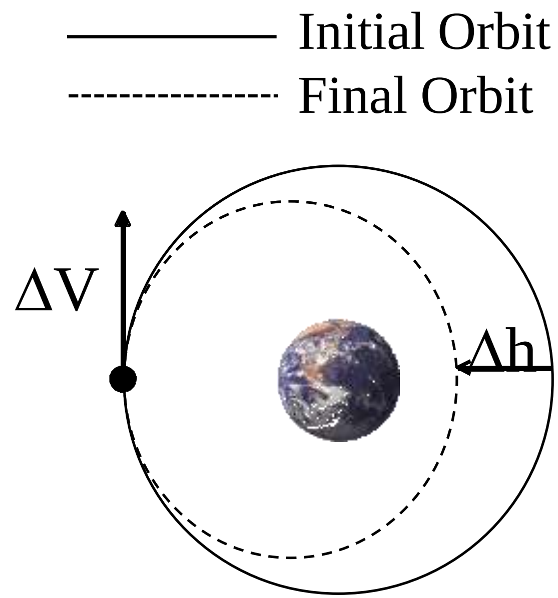


# Changing Orbits - The Effects of Burns

## Posigrade & Retrograde

$$V = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)}$$




Typical for low Earth orbit

$\Delta v\ of\ 1\ ft/s \rightarrow \Delta h \approx 0.5\ nmi$

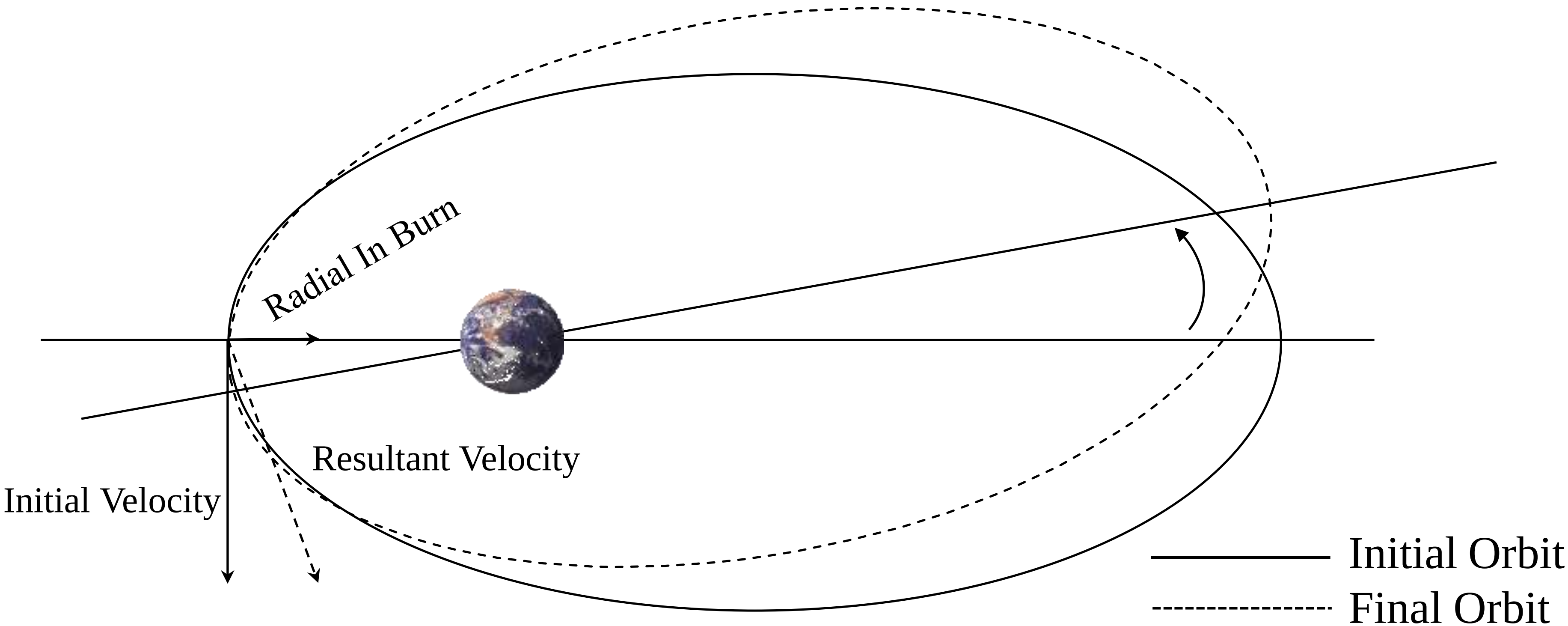
A posi-grade burn will RAISE orbital altitude.
A retro-grade burn will LOWER orbital altitude.
Note – max effect is at 180° from the burn point.

# Changing Orbits - The Effect of Burns

## Radial In & Radial Out

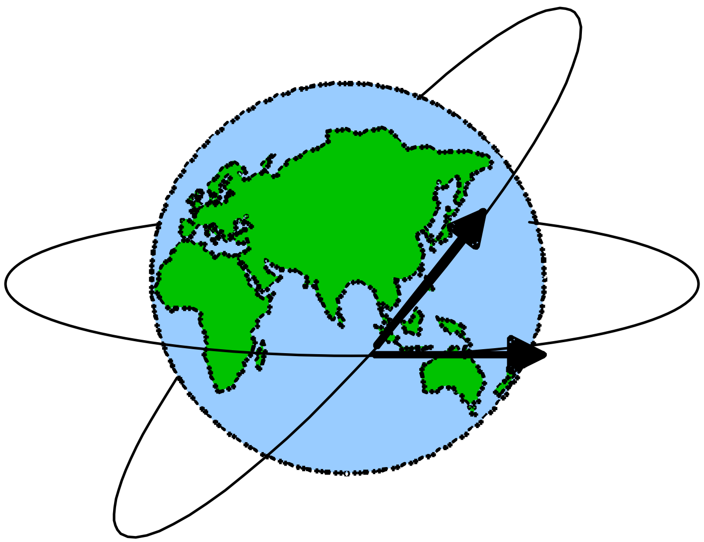
*EXAMPLE: Radial In Burn at Perigee*

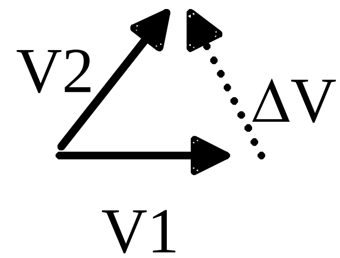
$$V = \sqrt{\mu \left(\frac{2}{r} - \frac{1}{a}\right)}$$




Radial burns shift the argument of perigee without significantly altering other orbital parameters

# Orbital Transfers - Changing Planes






•Burn point must be intersection of two orbits (“nodal crossings”)

•Extremely expensive energy-wise:

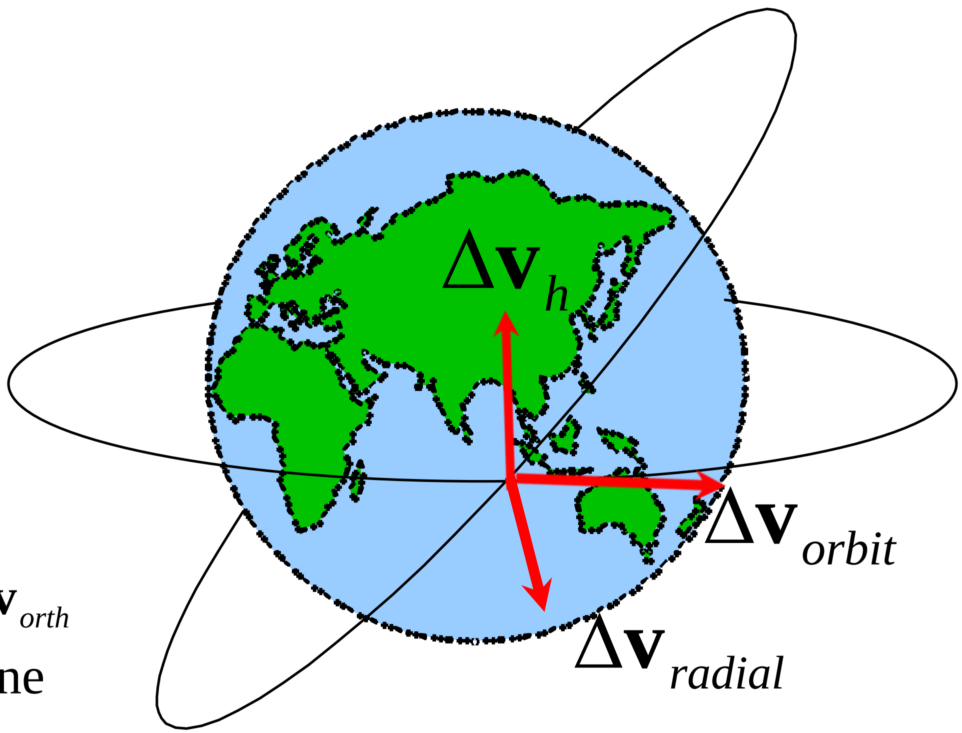
For 160 nmi circular orbit, V≈25,600 ft/sec.

A plane change of 1° requires a ΔV of 470 ft/sec.

# Plane Change Maneuver

The components of a $\Delta v$ determine how the orbit is affected.

- In-plane $\Delta v$ can change the parameters $(a, e, \omega, \varphi)$
- Out-of-plane $\Delta v$ can change the parameters $(\Omega, i)$.




$\Delta \mathbf{v}$ can be expressed as:

$$\Delta \mathbf{v} = \Delta \mathbf{v}_{radial} + \Delta \mathbf{v}_{orth} = \Delta v_{radial}\, \hat{\mathbf{r}} + \Delta \mathbf{v}_{orth}$$

$\Delta \mathbf{v}_{radial}$ cannot change the orbit plane

$$\Delta \mathbf{h} = \mathbf{r} \times \Delta \mathbf{v}_{radial} = \mathbf{r} \times \left(\Delta v_{radial}\, \hat{\mathbf{r}}\right) = \mathbf{0}$$

$\Delta \mathbf{v}_{orth}$ can have components in and out of plane

$$\Delta \mathbf{v}_{orth} = \Delta \mathbf{v}_{orbit} + \Delta \mathbf{v}_h = \left(\Delta v_{orbit}\, \hat{\boldsymbol{\varphi}}\right) + \Delta \mathbf{v}_h$$

where $\hat{\boldsymbol{\varphi}} \perp \hat{\mathbf{r}}$ and $\hat{\boldsymbol{\varphi}} \perp \mathbf{h}$

$\Delta \mathbf{v}_{orbit}$ can only change to magnitude of $\mathbf{h}$ (not direction)

$\Delta \mathbf{v}_h$ is the only component that can change the orbit plane

# Plane Change Maneuver (2)

There are two kinds of plane changes $\Delta i, \Delta\Omega$:

For $\Delta i$, Burn at the equatorial intersection (node)

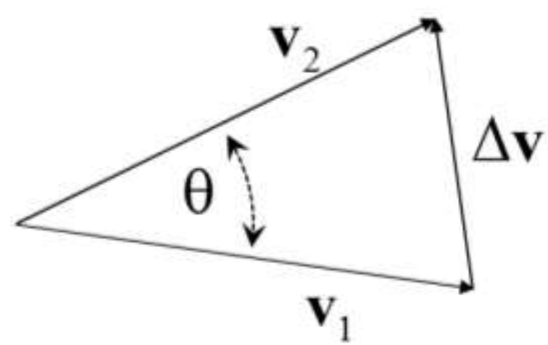
For $\Delta\Omega$, Burn at the maximum latitude (anti-node)

For a pure plane change:

$$\text{Energy} = \text{const. and } h = |\mathbf{r} \times \mathbf{v}| = \text{const.} \quad \rightarrow \begin{cases} a = \text{const.} \\ e = \text{const.} \end{cases}$$

If θ is small,

$$\Delta v = 2v\sin\left(\frac{\theta}{2}\right) \approx 2v\frac{\theta}{2} = v\theta$$



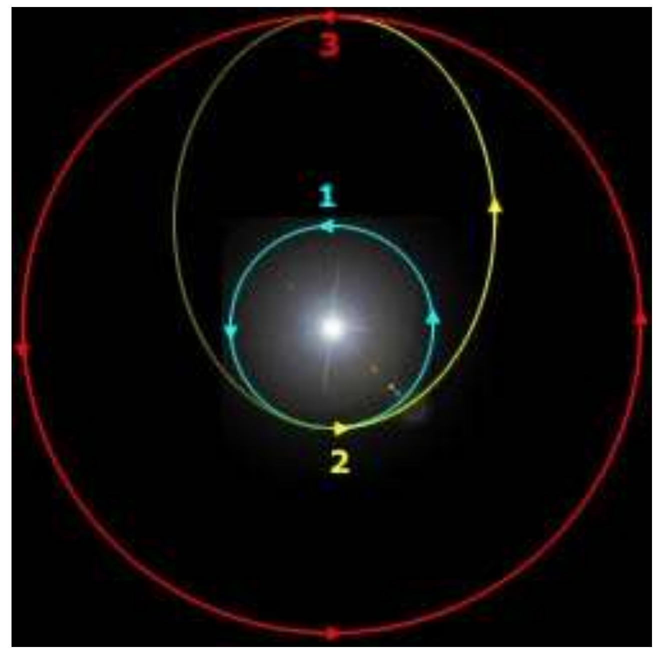

In general,

$$\Delta v^2 = \mathbf{v}_1 \cdot \mathbf{v}_1 + \mathbf{v}_2 \cdot \mathbf{v}_2 - 2\mathbf{v}_1 \cdot \mathbf{v}_2 = v_1^2 + v_2^2 - 2v_1 v_2 \cos\theta$$

# Hohmann Transfer

A Hohmann Transfer is an orbital maneuver that transfers a satellite from one circular orbit to another. It was invented by Walter Hohmann, a German scientist, in 1925.

A Hohmann Transfer is the most fuel efficient way to get from one circular orbit to another circular orbit.



# Hohmann Transfer (2)

- For Example, if we want to move a spacecraft from LEO → GEO and assuming both orbits are in the same plane

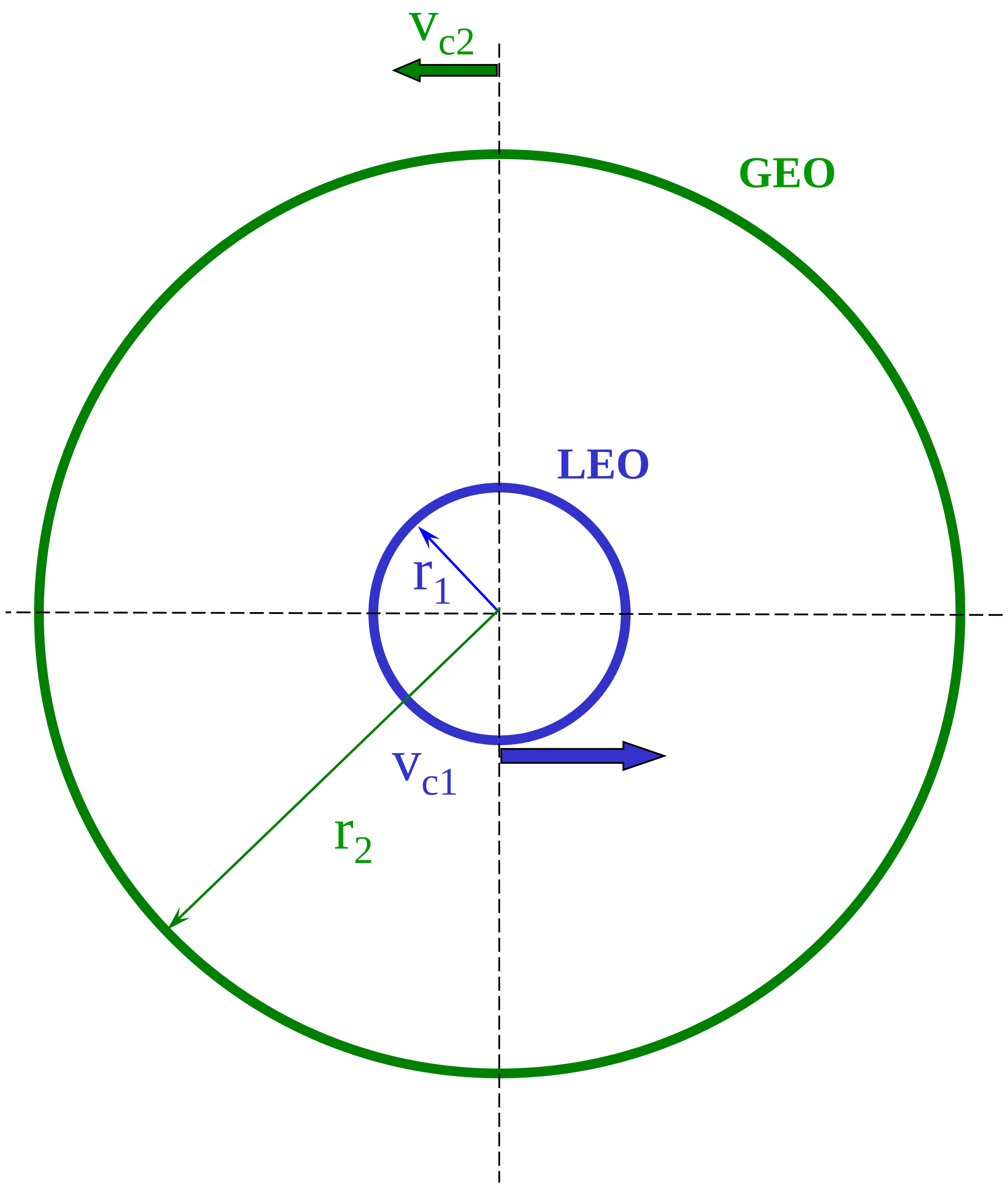
- Initial LEO orbit has radius $r_1$ and velocity $v_{c1}$

$$v_{c1} = \sqrt{\frac{\mu}{r_1}}$$

- Desired GEO orbit has radius $r_2$ and velocity $v_{c2}$

- At LEO ($r_1$), $v_{c1}$ = 7,724 m/s
- At GEO ($r_2$), $v_{c2}$ = 3,074 m/s

- Could accomplish this in many ways



# Hohmann Transfer (3)

- For Example, if we want to move a spacecraft from LEO → GEO and assuming both orbits are in the same plane

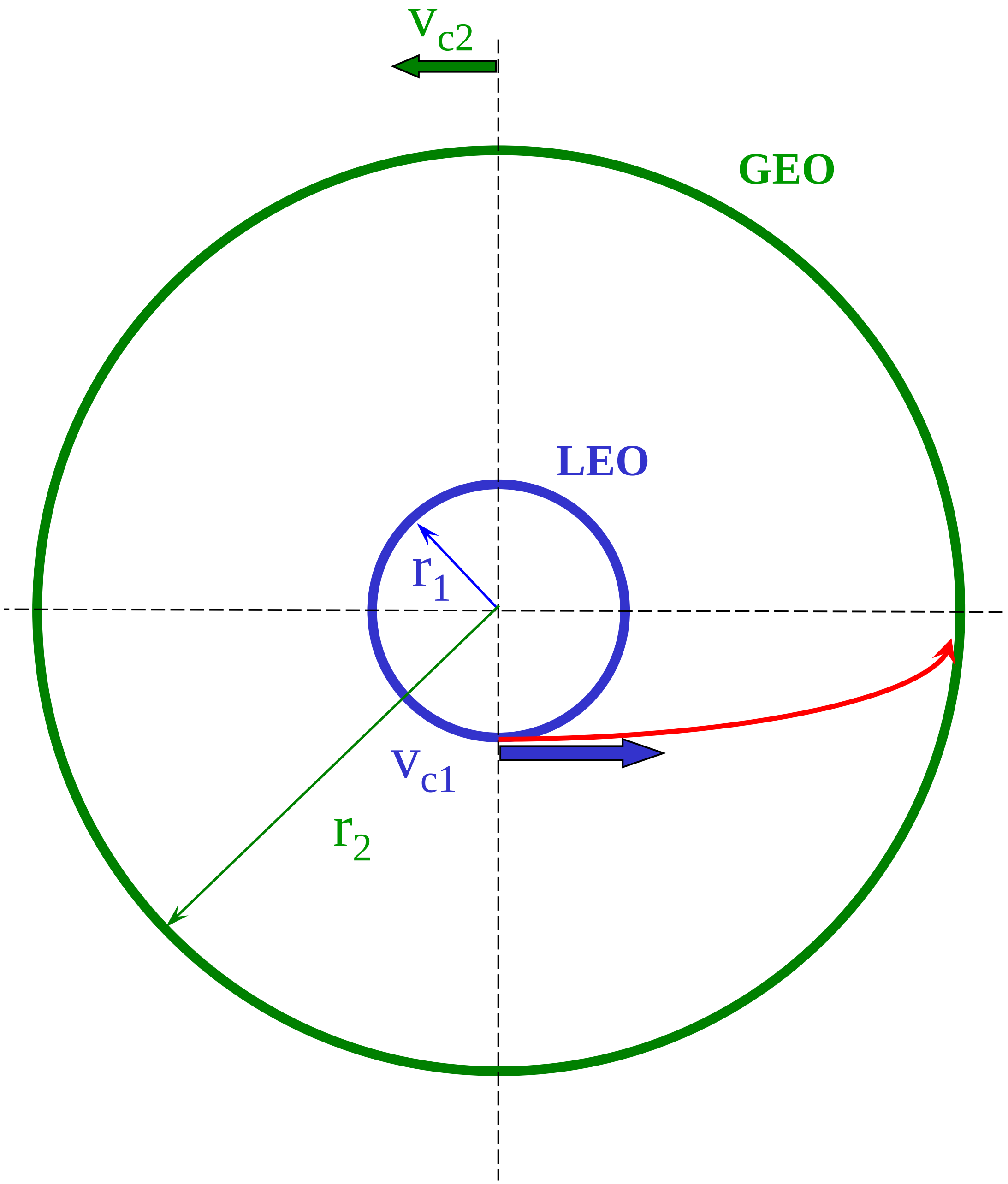
- Initial LEO orbit has radius $r_1$ and velocity $v_{c1}$

$$v_{c1} = \sqrt{\frac{\mu}{r_1}}$$

- Desired GEO orbit has radius $r_2$ and velocity $v_{c2}$

- At LEO ($r_1$), $v_{c1}$ = 7,724 m/s
- At GEO ($r_2$), $v_{c2}$ = 3,074 m/s

- Could accomplish this in many ways



# Hohmann Transfer (4)

- For Example, if we want to move a spacecraft from LEO → GEO and assuming both orbits are in the same plane

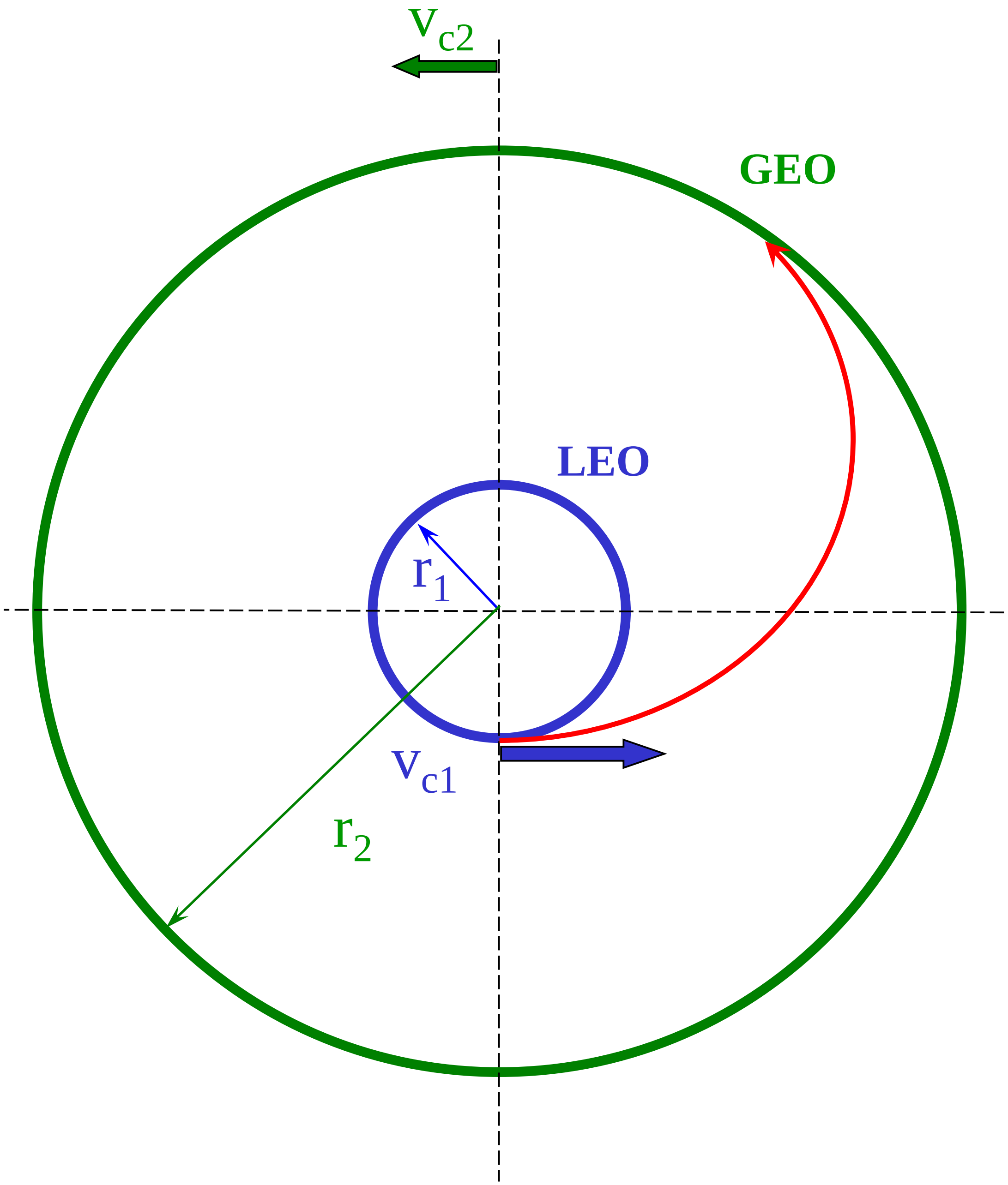
- Initial LEO orbit has radius $r_1$ and velocity $v_{c1}$

$$v_{c1} = \sqrt{\frac{\mu}{r_1}}$$

- Desired GEO orbit has radius $r_2$ and velocity $v_{c2}$

- At LEO ($r_1$), $v_{c1} = 7{,}724$ m/s
- At GEO ($r_2$), $v_{c2} = 3{,}074$ m/s

- Could accomplish this in many ways



# Hohmann Transfer (5)

- For Example, if we want to move a spacecraft from LEO → GEO and assuming both orbits are in the same plane

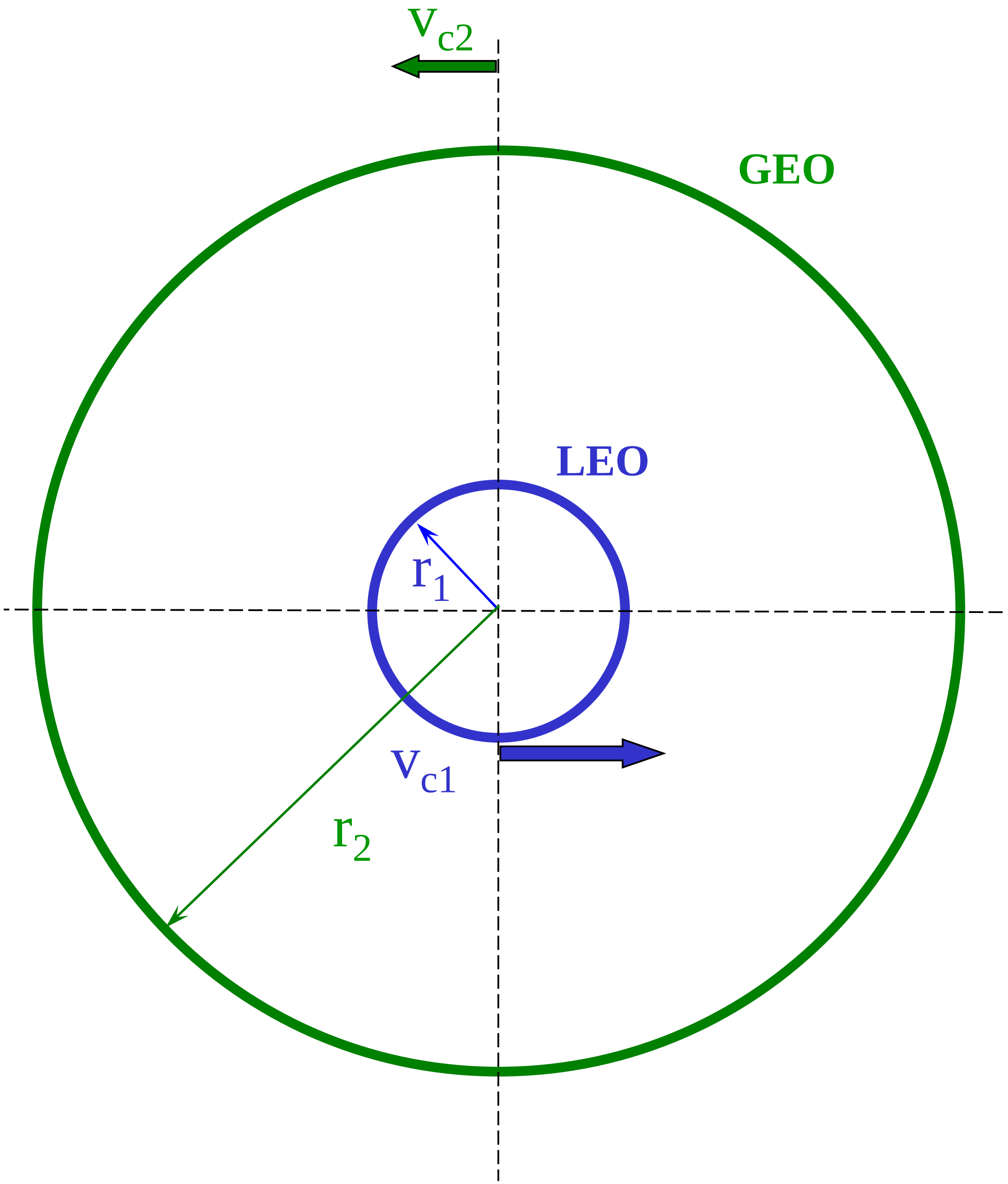
- Initial LEO orbit has radius $r_1$ and velocity $v_{c1}$

$$v_{c1} = \sqrt{\frac{\mu}{r_1}}$$

- Desired GEO orbit has radius $r_2$ and velocity $v_{c2}$

- At LEO ($r_1$), $v_{c1}$ = 7,724 m/s
- At GEO ($r_2$), $v_{c2}$ = 3,074 m/s

- The Hohmann transfer is the most efficient path



# Hohmann Transfer (7)

$$V = \sqrt{\mu \left(\frac{2}{r} - \frac{1}{a}\right)}$$

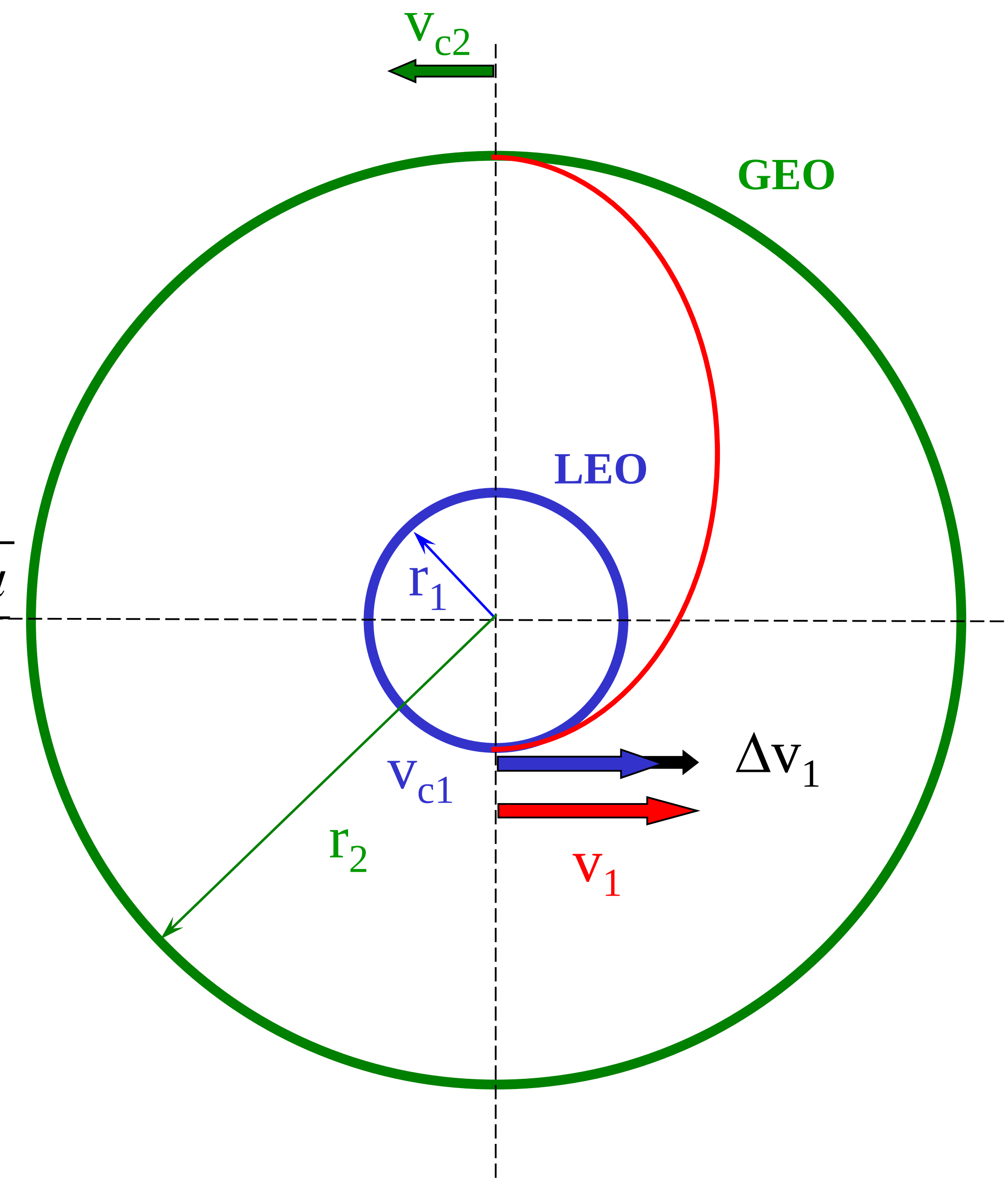
- Impulsive $\Delta_{V1}$ is applied to get on Hohmann transfer orbit at perigee:

$$\Delta v_1 = v_{transfer} - v_{LEO}$$

$$= \sqrt{\mu\left(\frac{2}{r_{transfer}} - \frac{1}{a_{transfer}}\right)} - \sqrt{\frac{\mu}{r_1}}$$

$$\Delta v_1 = \sqrt{\mu\left(\frac{2}{r_1} - \frac{1}{\frac{1}{2}(r_1 + r_2)}\right)} - \sqrt{\frac{\mu}{r_1}}$$

- Leave LEO ($r_1$) with a total velocity of $v_1$

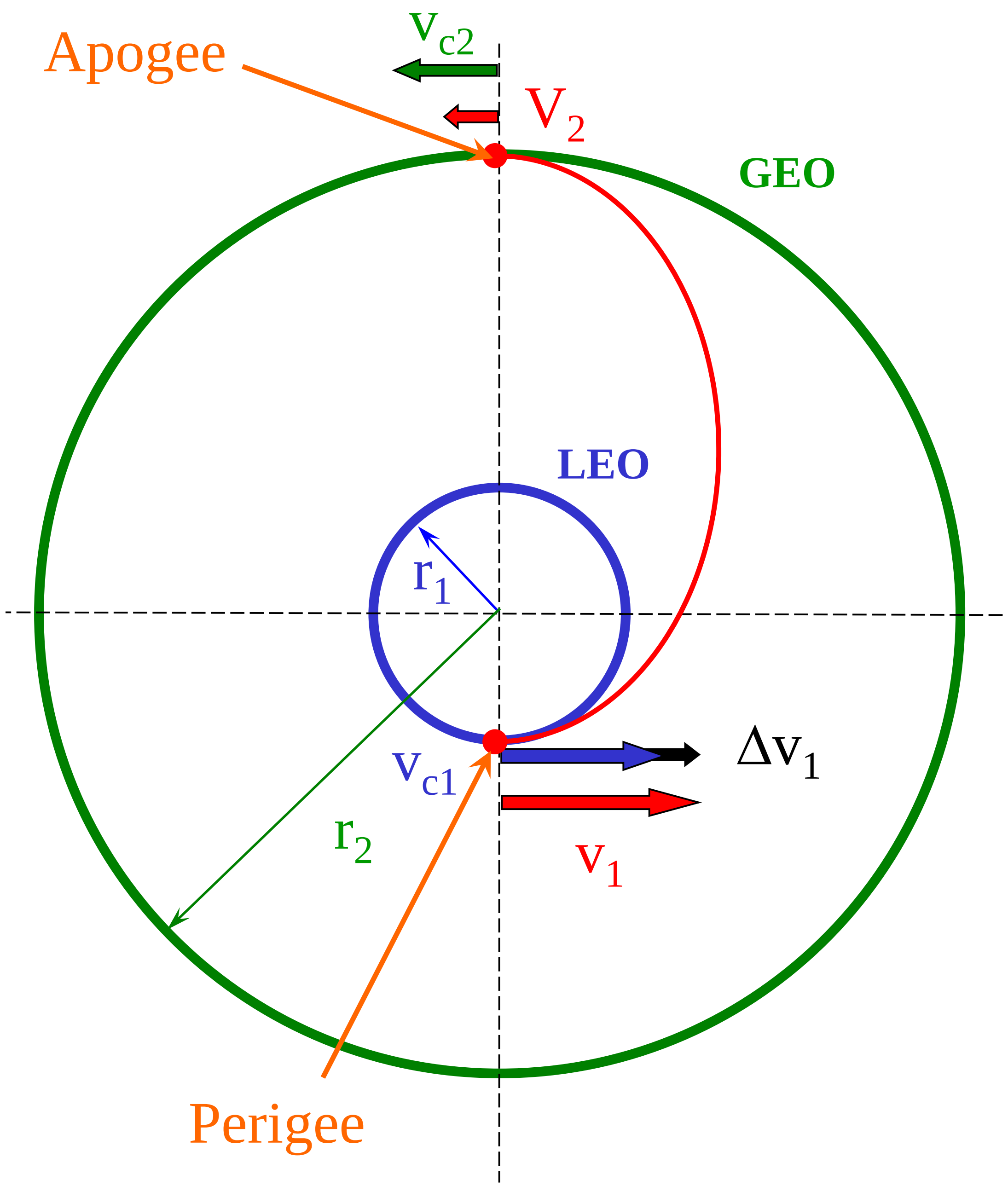


# Hohmann Transfer (8)

- Transfer orbit is an ellipse with
  - Perigee located at $r_1$
  - Apogee located at $r_2$

- Arrive at GEO (apogee) with $v_2$

$$v_2 = \sqrt{\mu\left(\frac{2}{r_2} - \frac{1}{a_{transfer}}\right)}$$



# Hohmann Transfer (9)

- When arriving at GEO, which is at apogee of elliptical transfer orbit, must apply some $\Delta v_2$ in order to circularize:

$$\Delta v_2 = v_{GEO} - v_{transfer}$$

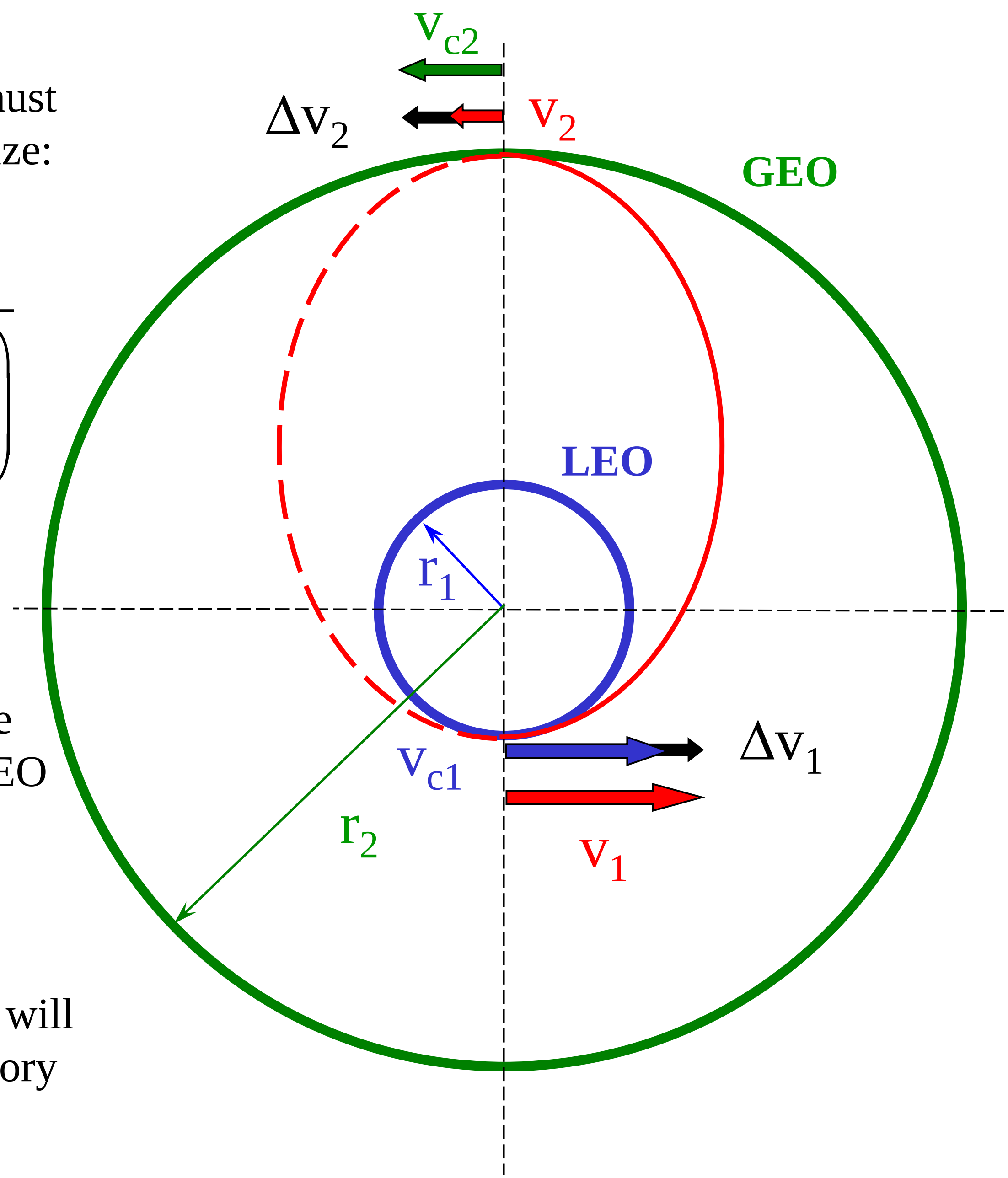
$$= \sqrt{\frac{\mu}{r_2}} - \sqrt{\mu\left(\frac{2}{r_2} - \frac{1}{a_{transfer}}\right)}$$

$$= \sqrt{\frac{\mu}{r_2}} - \sqrt{\frac{2\mu}{r_2} - \frac{2\mu}{r_1 + r_2}}$$

- This is exactly the $\Delta v$ that should be applied to circularize the orbit at GEO ($r_2$)

$$v_{C2} = v_2 + \Delta v_2$$

- If this $\Delta V$ is not applied, spacecraft will continue on dashed elliptical trajectory



# Hohmann Transfer (10)

Hohmann Transfer Summary

- Initial LEO orbit has radius $r_1$ and velocity $v_{c1}$

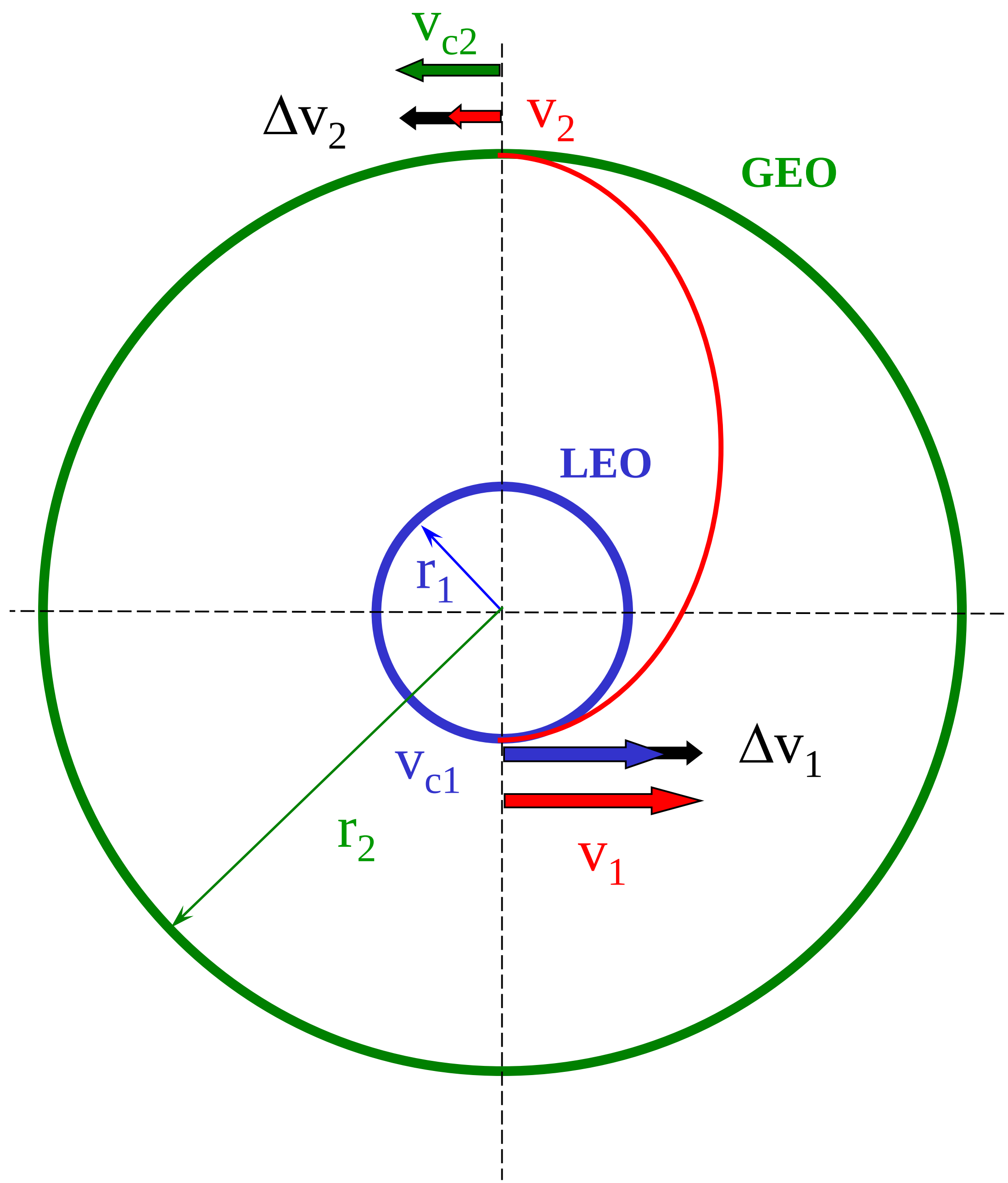
$$v_{c1} = \sqrt{\frac{\mu}{r}}$$

- Desired GEO orbit has radius $r_2$ and velocity $v_{c2}$
- Impulsive $\Delta_{V1}$ is applied to get on Hohmann transfer orbit at perigee:

$$\Delta v_1 = \sqrt{\frac{2\mu}{r_1} - \frac{2\mu}{r_1 + r_2}} - \sqrt{\frac{\mu}{r_1}}$$

- Coast to apogee and apply impulsive $\Delta_{V2}$:

$$\Delta v_2 = \sqrt{\frac{\mu}{r_2}} - \sqrt{\frac{2\mu}{r_2} - \frac{2\mu}{r_1 + r_2}}$$



# Example 1: Planar Hohmann Transfer

From an initial orbit with radius $r_1 = 14,000\ km$ compute the Hohmann transfer orbit to achieve an orbit with $r_2 = 28,000\ km.$ What are the initial, intermediate and final speeds? What is the total $\Delta v$?

Useful information:

$$\mu_E = 398,600.441\ km^3 / s^2$$

$$v_{c1} = \sqrt{\frac{\mu}{r_1}} \qquad v_{c2} = \sqrt{\frac{\mu}{r_2}} \qquad v = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)}$$

# Example 1: Planar Hohmann Transfer(2)

$$v_{c1} = \sqrt{\frac{\mu}{r_1}} = 5.336 \qquad v_{c2} = \sqrt{\frac{\mu}{r_2}} = 3.773$$

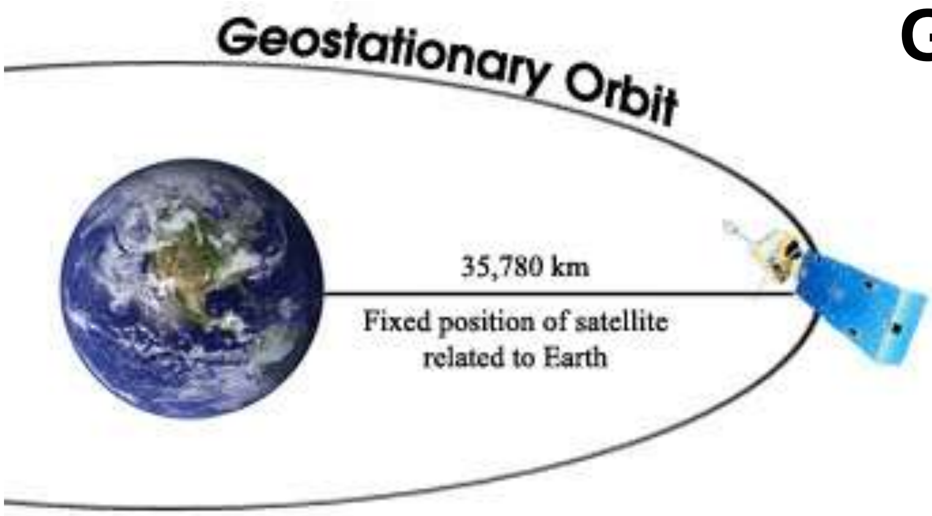
$$\Delta v_1 = \sqrt{2\mu\left(\frac{1}{r_1} - \frac{1}{r_1 + r_2}\right)} - \sqrt{\frac{\mu}{r_1}} \approx 0.825, \qquad v_1 = v_{C1} + \Delta v_1 = 6.161$$

$$v_2 = \sqrt{2\mu\left(\frac{1}{r_2} - \frac{1}{r_1 + r_2}\right)} = 3.08$$

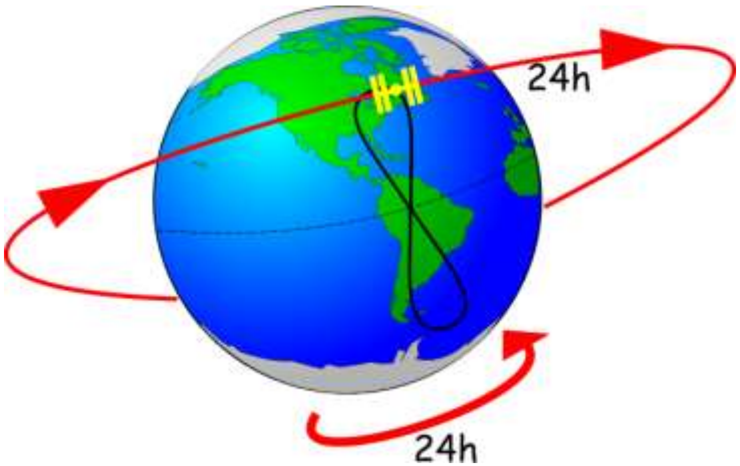
$$\Delta v_2 = v_{C2} - v_2 = \sqrt{\frac{\mu}{r_2}} - \sqrt{\frac{2\mu}{r_2} - \frac{2\mu}{r_1 + r_2}} = 0.693$$

$$\Delta v_{tot} = \Delta v_1 + \Delta v_2 = 1.5197 \; km/\sec$$

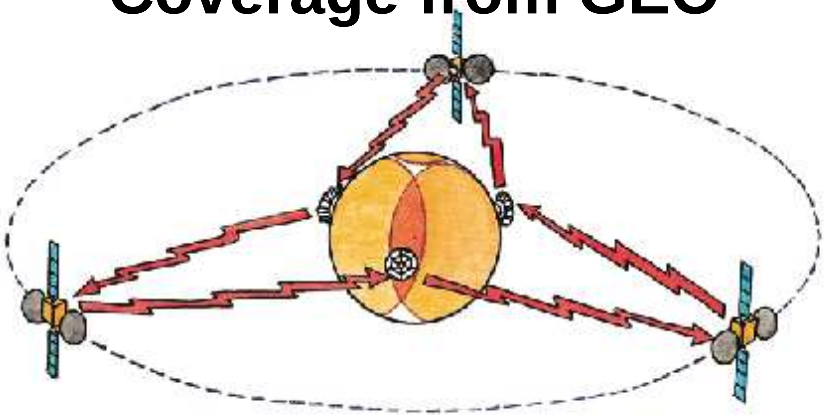
# Geosynchronous Orbit




## Geosynchronous Orbit




## Coverage from GEO



**TDRSS Comm Coverage**

# Geosynchronous Transfer

So, how can we get a spacecraft from a non-equatorial orbit into a geosynchronous one?

- Launching into a LEO orbit will have an inclination greater than or equal to the latitude of the launch site.
- Need to do a plane change as well as raising the orbital altitude.
- Solution - Do the transfer orbit first and do the plane change and circularization burn at apogee!
- Rationale – For the same plane change angle, $\Delta v$ is less where $v$ is less.

# Example 2: Geosynchronous Transfer

For launch from Cape Canaveral the initial orbit has an inclinition of $i = 28$ deg and an altitude of $h_1 = 300\ km$. We are targeting a geosynchronous orbit with $r_2 = 42{,}186\ km$, design the two $\Delta v$ burns, taking into account the plane change. Is it possible that there is a more efficient maneuver?

Useful information:

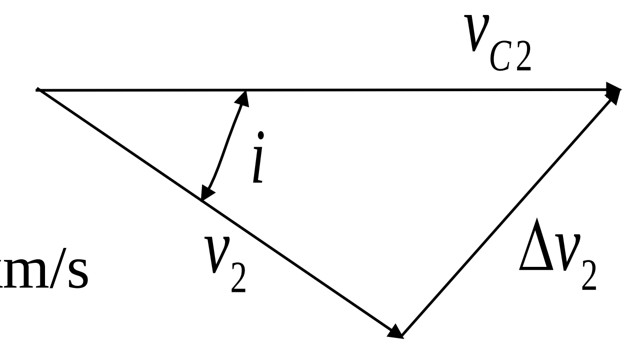
$$\mu_E = 398{,}600.441\ km^3 / s^2$$

$$v_{c1} = \sqrt{\frac{\mu}{r_1}} \qquad v_{c2} = \sqrt{\frac{\mu}{r_2}} \qquad v = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)}$$

$$\Delta v^2 = \mathbf{v}_1 \cdot \mathbf{v}_1 + \mathbf{v}_2 \cdot \mathbf{v}_2 - 2\mathbf{v}_1 \cdot \mathbf{v}_2 = v_1^2 + v_2^2 - 2 v_1 v_2 \cos\theta$$

# Example 2: Geosynchronous Transfer (2)

$$v_{C1} = \sqrt{\frac{\mu}{r_1}}, \quad a = \frac{(r_1 + r_2)}{2}, \quad v_1 = \sqrt{2\mu\left(\frac{1}{r_1} - \frac{1}{r_1 + r_2}\right)}$$




$$\Delta v_1 = v_1 - v_{C1} = \sqrt{2\mu\left(\frac{1}{r_1} - \frac{1}{r_1 + r_2}\right)} - \sqrt{\frac{\mu}{r_1}} = 2.426 \text{ km/s}$$

Determine velocity needed at apogee of transfer orbit

$$v_{C2} = \sqrt{\frac{\mu}{r_2}} = 3.074 \text{ km/s}, \quad v_2 = \sqrt{2\mu\left(\frac{1}{r_2} - \frac{1}{r_1 + r_2}\right)} = 1.607 \text{ km/s}$$

$$\Delta v_2^2 = v_{C2}^2 + v_2^2 - 2v_{C2}v_2 \cos i \quad \rightarrow \quad \Delta v_2 = 1.819 \text{ km/s}$$

$$\Delta v_{\text{tot}} = |\Delta v_1| + |\Delta v_2| = 4.245 \text{ km/s}$$

# Bi-Elliptic Transfer

The Hohmann transfer is the most efficient 2-burn solution. The Bi-Elliptic transfer can be more efficient with 3-burns when the final orbit has a much greater radius than the initial orbit.

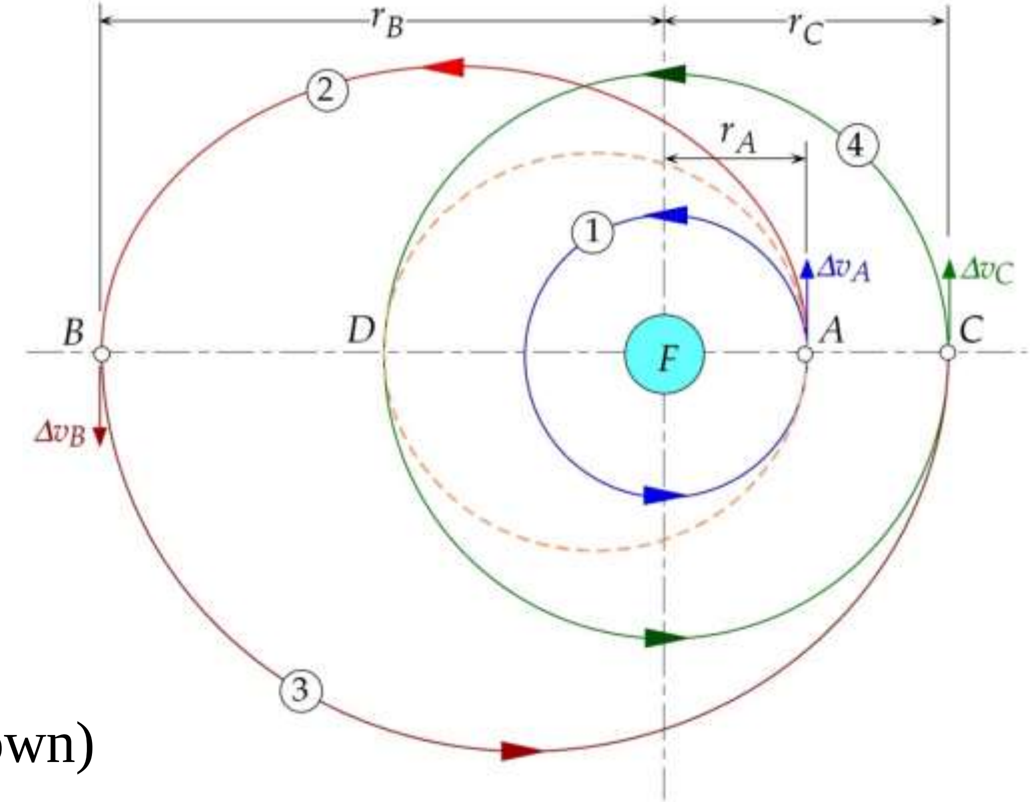
Bi-Elliptic sequence:

(1) Circular Orbit 1 with radius $r_A$

(2) $\Delta v_A$ to Orbit 2 with radii $r_A, r_B$

(3) $\Delta v_B$ to Orbit 3 with radii $r_B, r_C$

(4) $\Delta v_C$ to Circularize to Orbit 4




Notes:

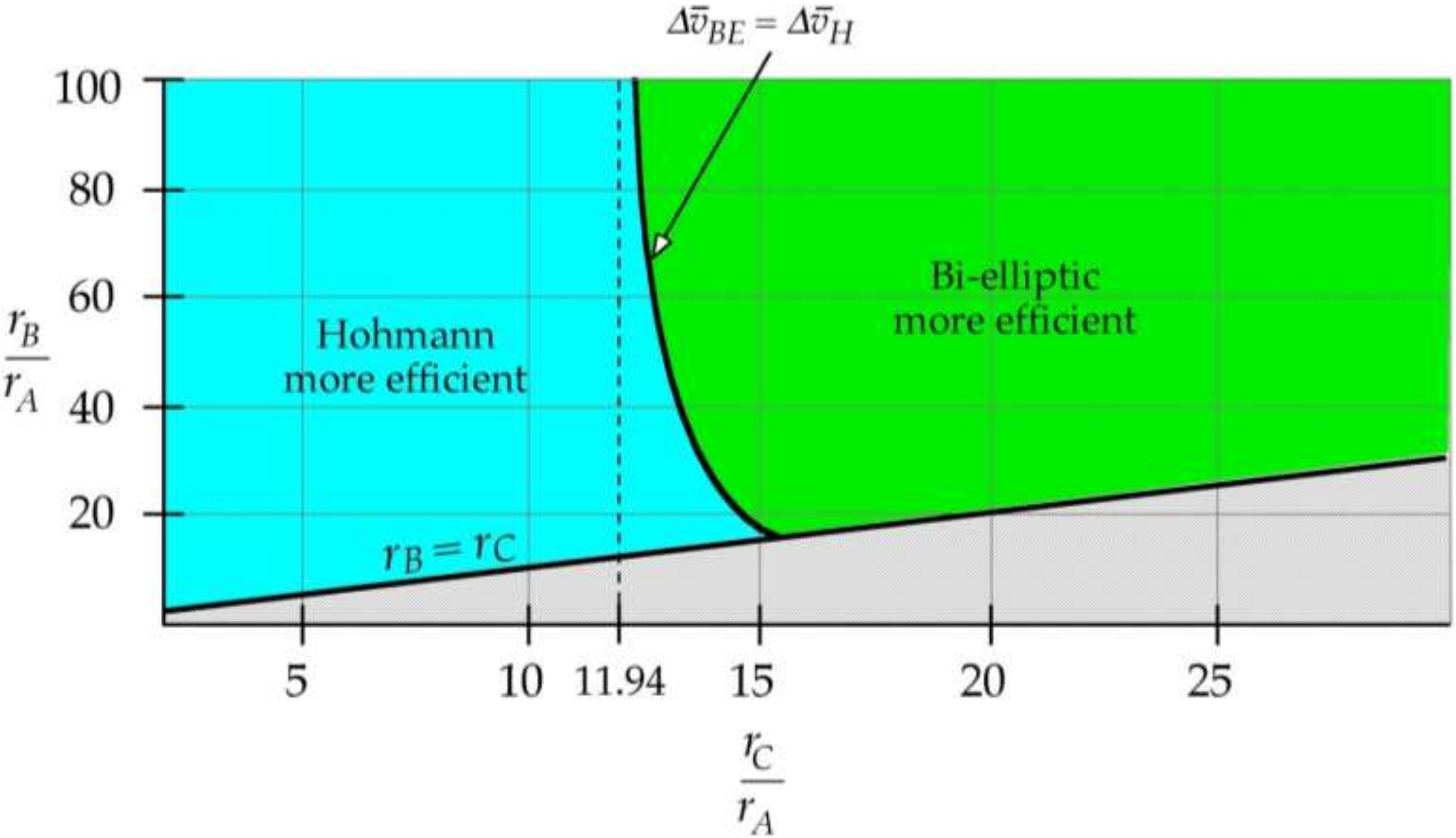
- As $r_B \rightarrow \infty,\ \Delta v_B \rightarrow 0$
- $\Delta v_C$ will be retrograde (slowdown)
- Long transfer time , $T_{Manueuver} = \frac{1}{2}T_2 + \frac{1}{2}T_3 = \frac{1}{2}\left(2\pi\sqrt{\frac{a_2^3}{\mu}} - 2\pi\sqrt{\frac{a_3^3}{\mu}}\right)$

# Bi-Elliptic Transfer (2)

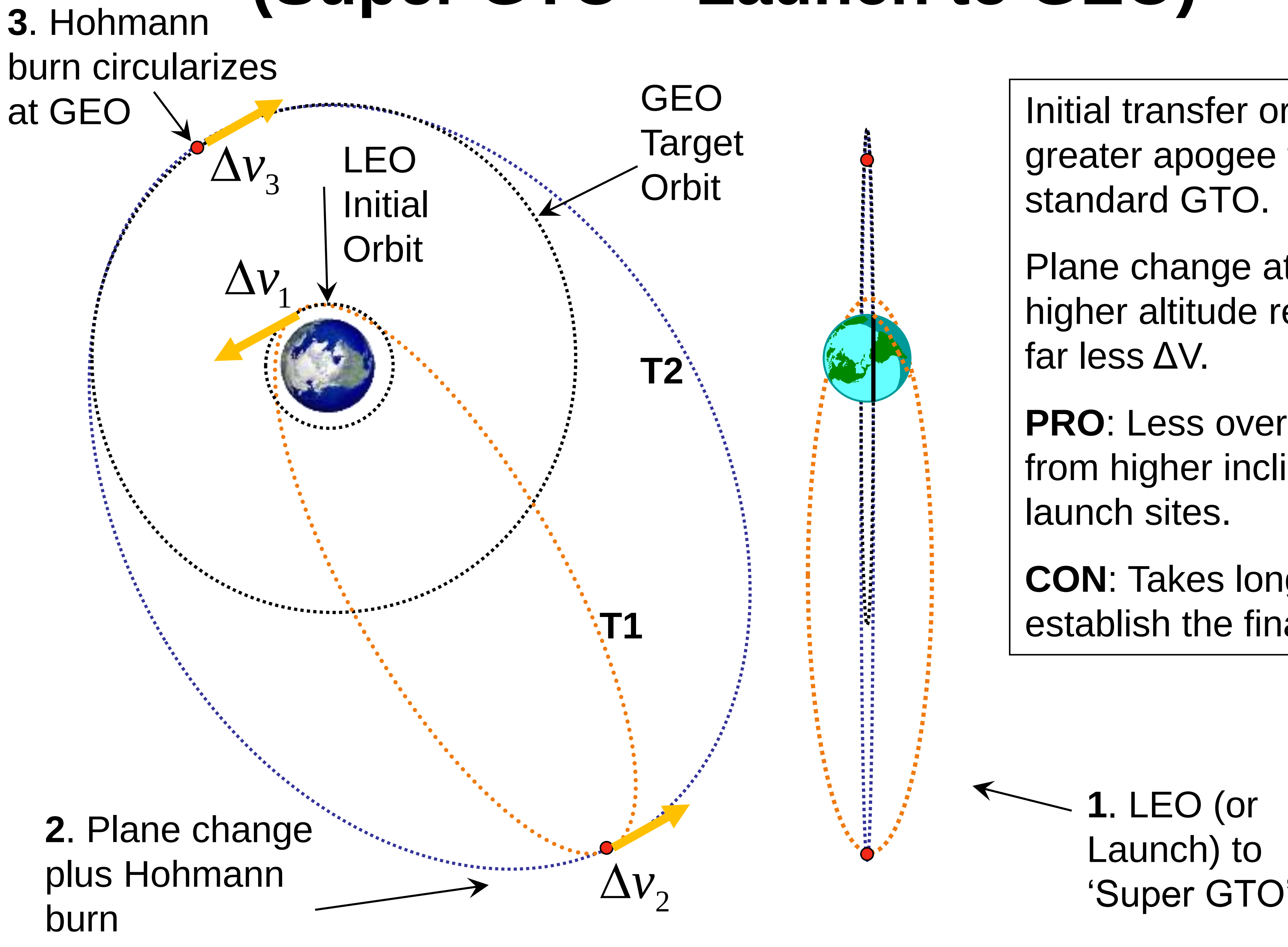
The Hohmann transfer is more efficient if $r_C/r_A < 11.94$

The Bi-Elliptic transfer is more efficient if $r_C/r_A > 15.58$

Otherwise, it depends on $r_B/r_A$ as shown below.



# Super Geosynchronous Transfer
## (Super GTO – Launch to GEO)




Initial transfer orbit has greater apogee than standard GTO.

Plane change at much higher altitude requires far less ΔV.

**PRO**: Less overall ΔV from higher inclination launch sites.

**CON**: Takes longer to establish the final orbit.

# Phasing Maneuvers

Phasing orbits are used to change a spacecraft's position in it's orbit (i.e. for rendezvous, targeting, timing).

Prograde Burn:

– Additional $\Delta v$

– New orbit is slower, fall back

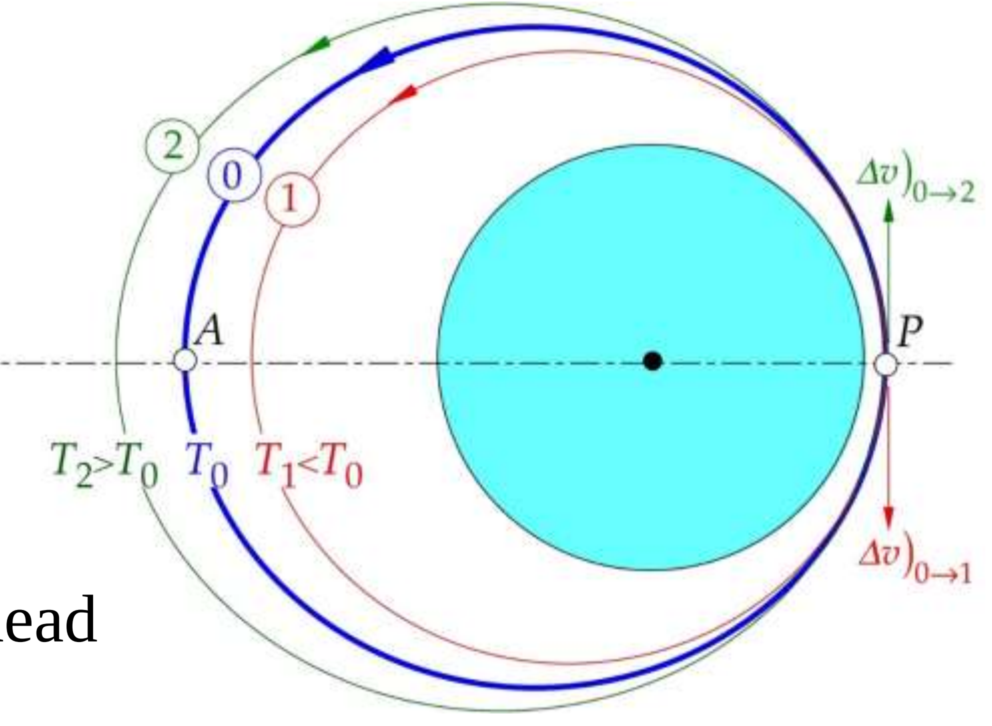
– "*Speed Up To Slow Down*"

Retrograde Burn:

– Negative $\Delta v$

– Smaller orbit is faster, move ahead

– "*Slow Down To Speed Up*"

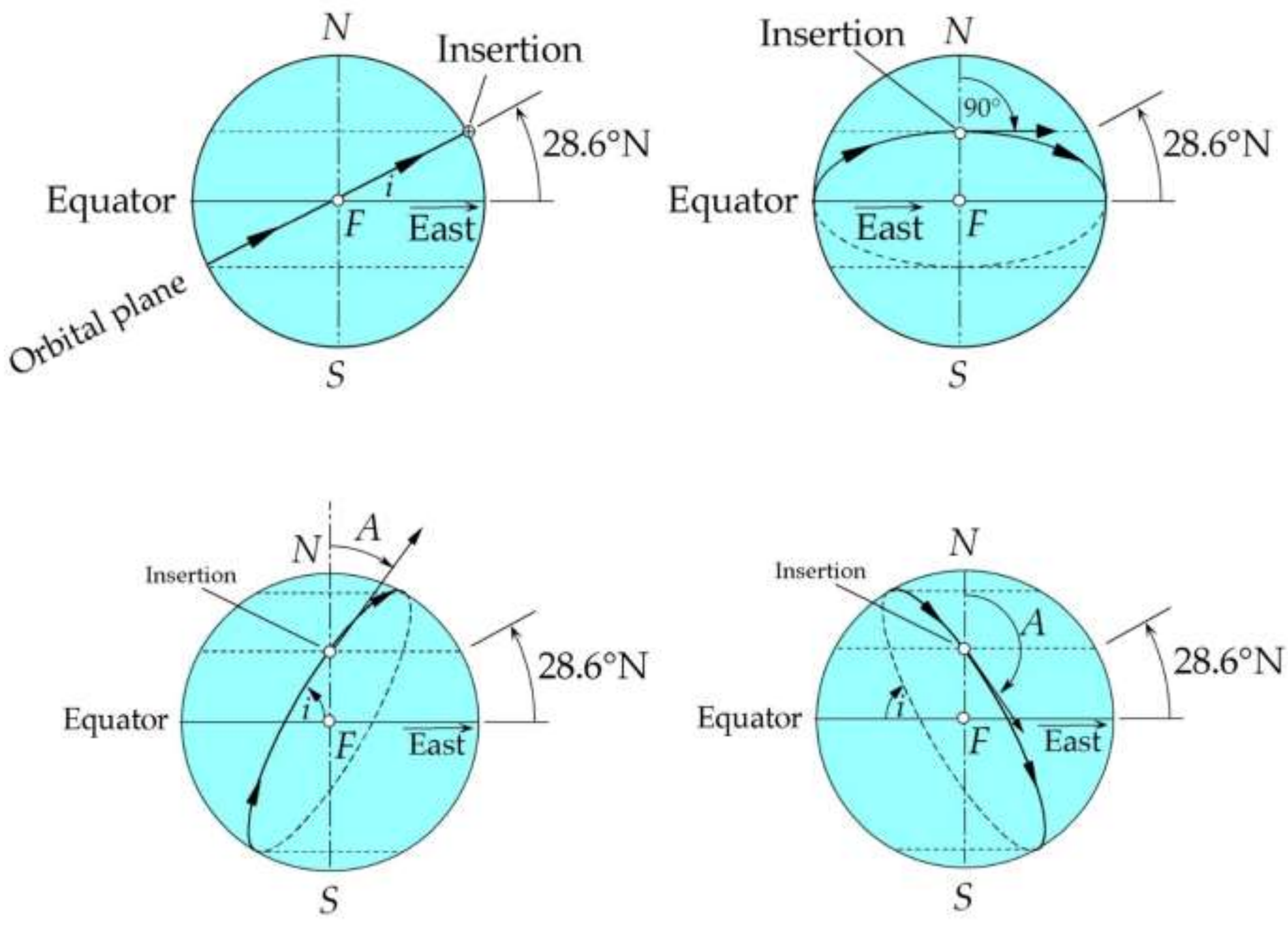



$$\Delta t = 2\pi N\left(\sqrt{\frac{a^3}{\mu}} - \sqrt{\frac{a^3_{Phase}}{\mu}}\right),\quad \text{where } N = \#\text{ of cycles on phasing orbit}$$

# Launch Azimuth Angle to Orbit Plane

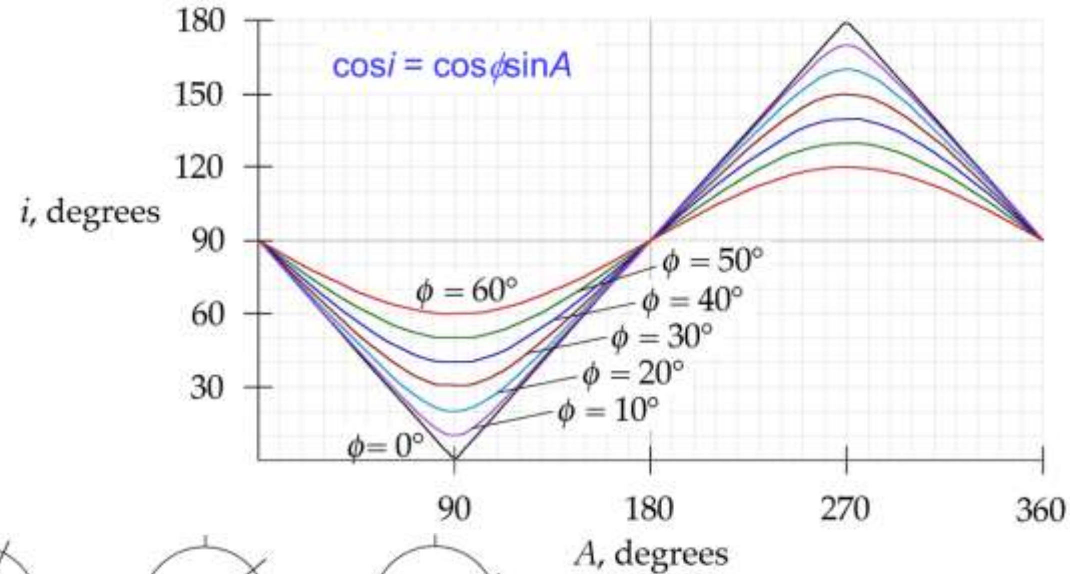
The lowest attainable orbit inclination

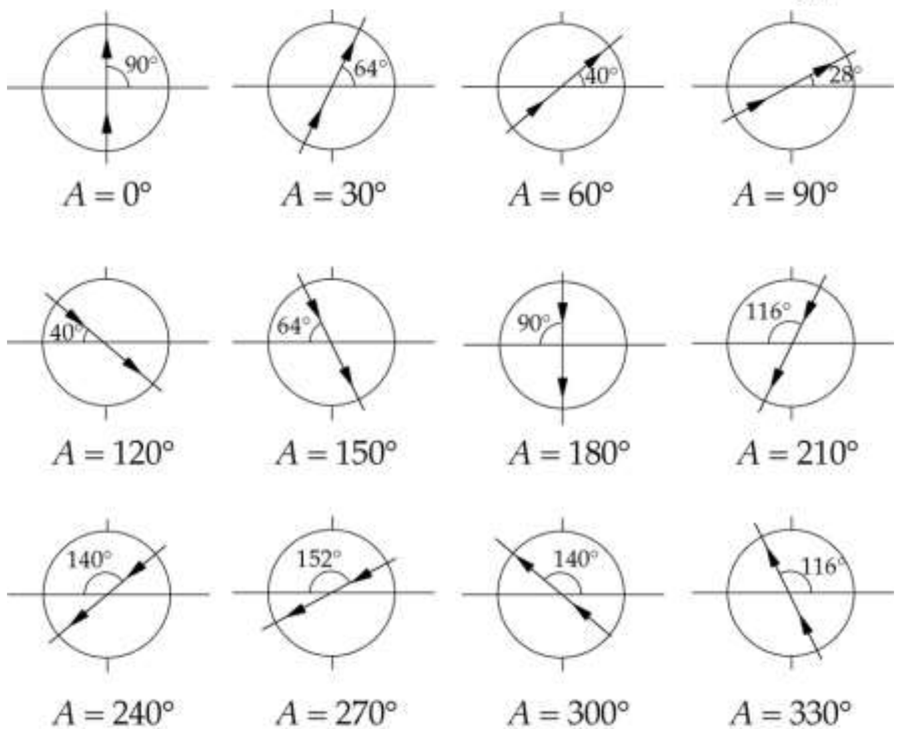
matches the latitude of the launch site.



# Launch to Orbit Plane (2)

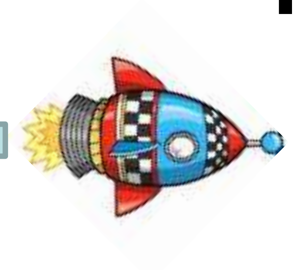
Relationship between Launch Azimuth Angle and Orbital Inclination as Function of Latitude



# Propulsion Requirements

## Balance of momentum



$\dot{m}v_e$ $\longleftarrow$ $\longrightarrow$ $m\dot{v}$


"Look, it's quite simple. It isn't rocket science you know... Er... Well, you know what I mean!"

$$\frac{d(mV)}{dt} = \dot{m}v + m\dot{v} = 0$$

$$\int_{v_i}^{v_f} dv = -v_e \int_{m_i}^{m_f} \frac{dm}{m}$$

$$\Delta v = v_f - v_i = -v_e\left(\ln m_f - \ln m_i\right) = v_e \ln\left(\frac{m_i}{m_f}\right)$$

Defining $I_{sp} = \dfrac{T}{\dot{m}g} = \dfrac{\dot{m}v_e}{\dot{m}g} = \dfrac{v_e}{g} \rightarrow v_e = gI_{sp}$

$$\Delta v = gI_{sp} \ln\left(\frac{m_i}{m_f}\right) \quad \rightarrow \quad \frac{m_i}{m_f} = e^{\left(\frac{\Delta v}{gI_{sp}}\right)}$$

Since $m_i - m_{prop} = m_f$

$$\frac{m_i}{m_i - m_{prop}} = e^{\left(\frac{\Delta v}{gI_{sp}}\right)}, \quad m_i e^{-\left(\frac{\Delta v}{gI_{sp}}\right)} = m_i - m_{prop}$$

Finally, $\quad m_{prop} = m_i\left[1 - e^{\left(\frac{-\Delta v}{gI_{sp}}\right)}\right]$

Example:

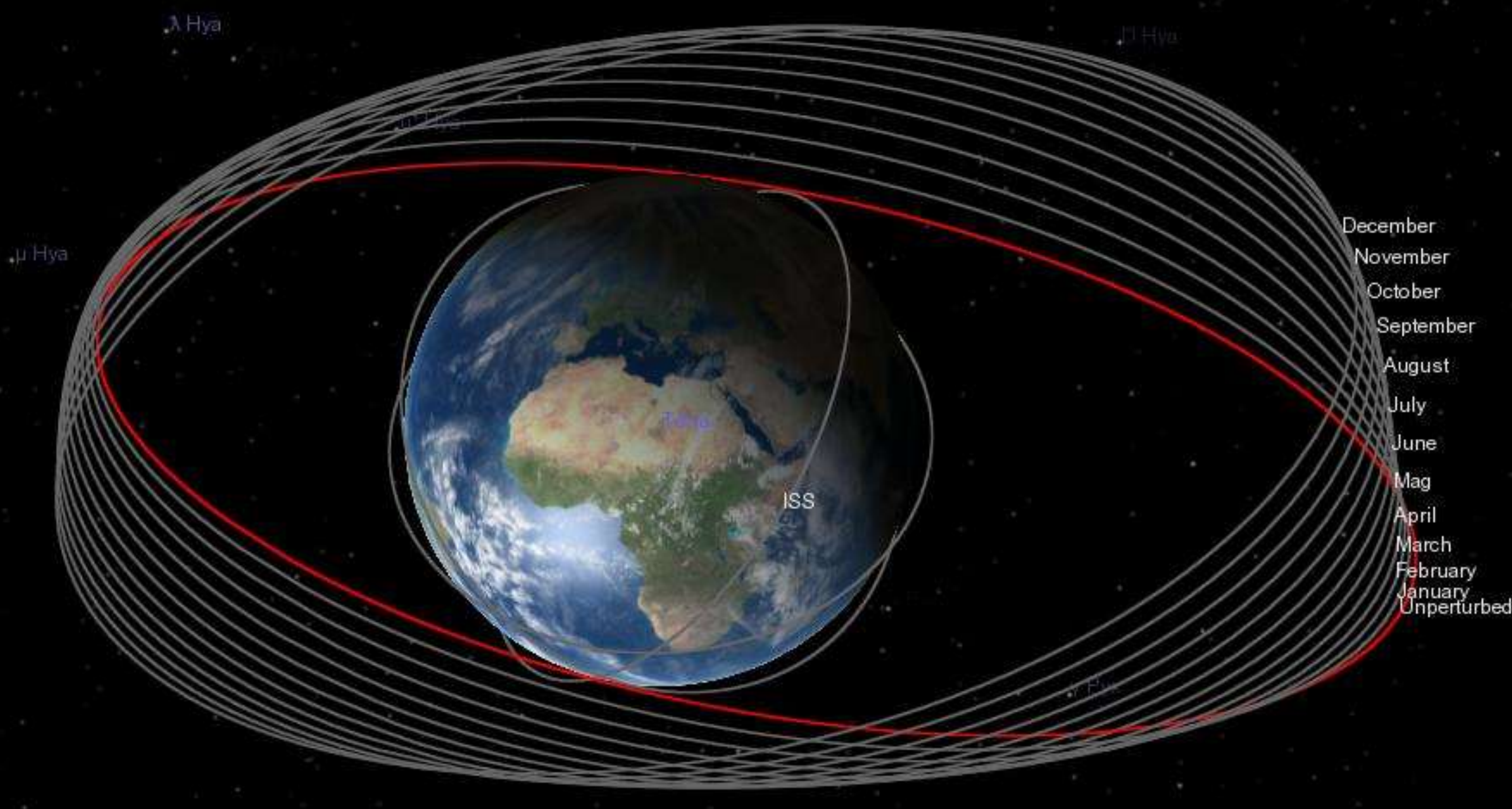
For an engine using $LH_2$ / $LO_2$ ($I_{sp}$=450), what is the mass fraction of propellant to obtain a $\Delta v = 10\ km/s$?

$$\frac{m_{prop}}{m_i} = 1 - e^{\left(\frac{-10{,}000}{9.8 \cdot 450}\right)} = 0.896$$

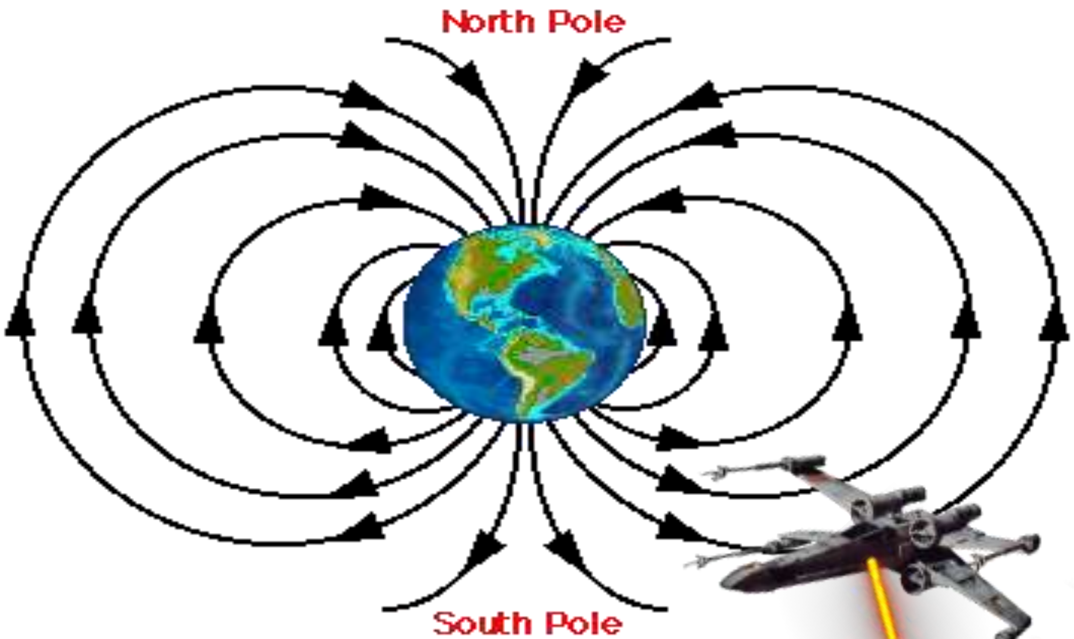
That is, 90% Propellant

# Astrodynamics

## Orbital Perturbations

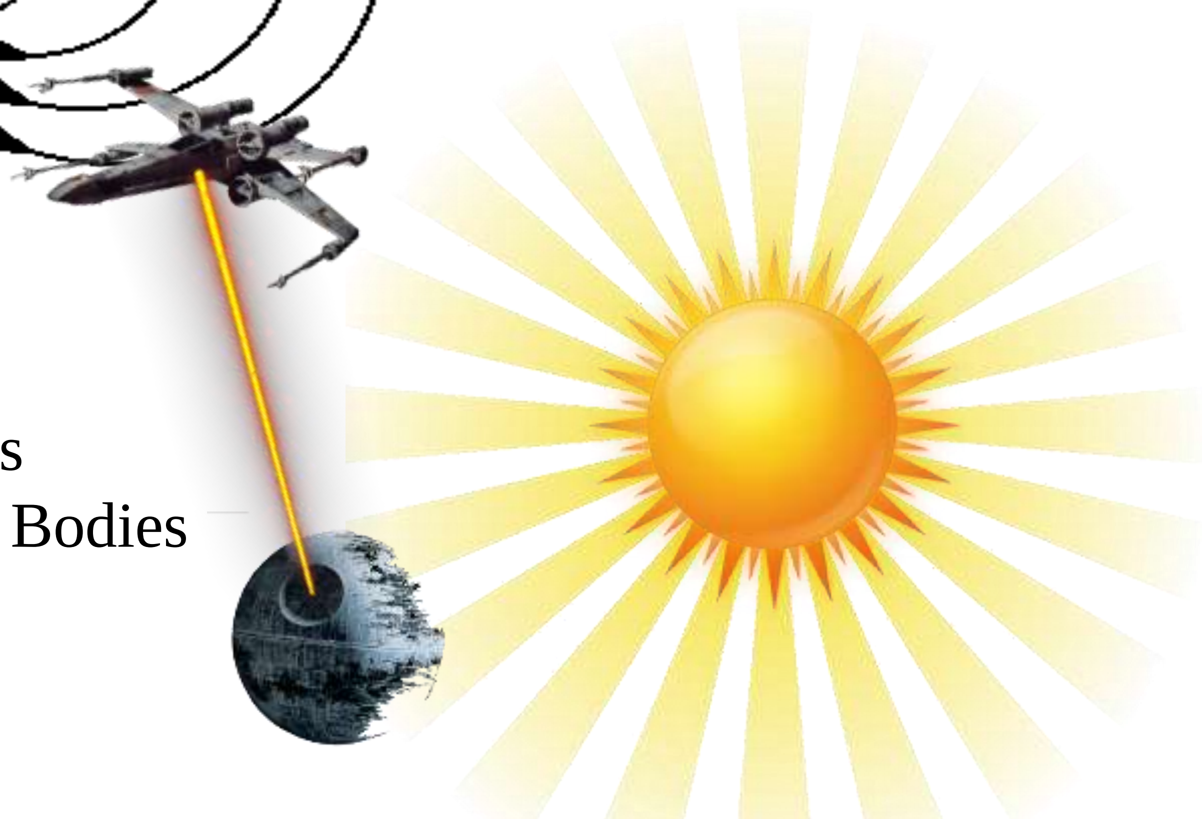


# Orbital Perturbations

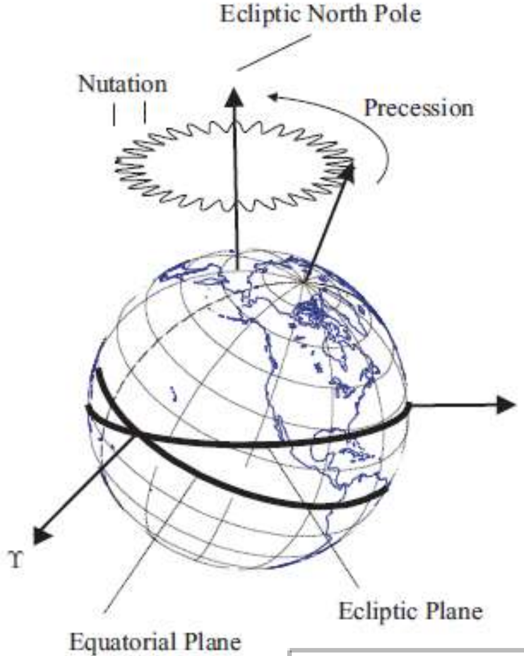


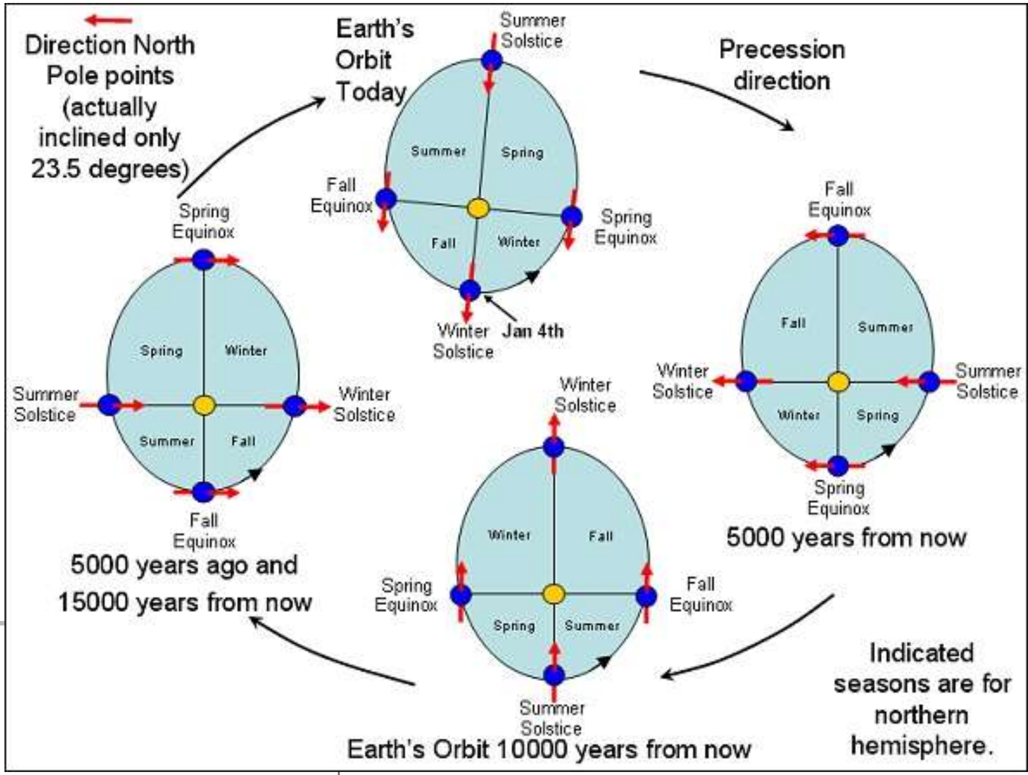

- Nonspherical Planet
- Gravitational Anomalies
- Atmospheric Density
- Magnetic Fields

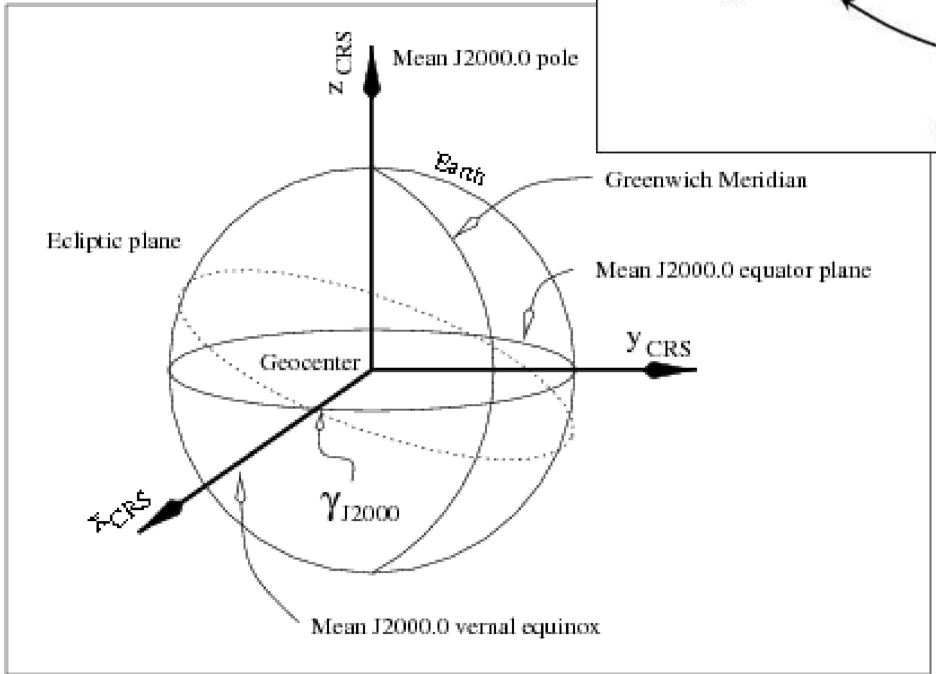
- Solar Pressure
- Thruster Firings
- Other Celestial Bodies
- Phaser Blasts



# Perturbations - J2000 Inertial Frame



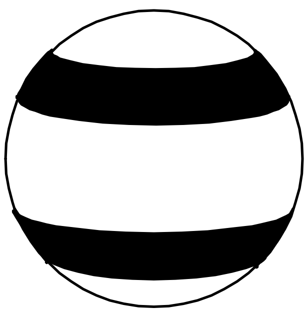



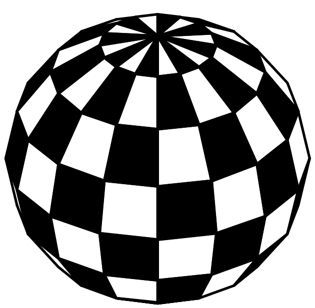
# The Earth's Gravitational Potential *U*

$$U = \frac{\mu}{r}\left\{1 - \sum_{n=2}^{\infty} J_n \left[\frac{a_e}{r}\right]^n P_n(\sin(\phi)) + \sum_{n=2}^{\infty}\sum_{m=1}^{\infty}\left[\frac{a_e}{r}\right]^n P_{nm}(\sin(\phi))\left[C_{nm}\cos(m\lambda) + S_{nm}\sin(m\lambda)\right]\right\}$$

Spherical Term



Zonal Harmonics



Tesseral Harmonics

$n \neq m$

Sectorial Harmonics

$n = m$

where

$\mu$ = Universal Gravitational Constant x Mass of Earth

$r$ = Spacecraft Radius Vector from Center of Earth

$a_e$ = Earth Equatorial Radius

P() = Legendre Polynomial Functions

$\phi$ = Spacecraft Latitude

$\lambda$ = Spacecraft Longitude

$J_n$ = Zonal Harmonic Constants

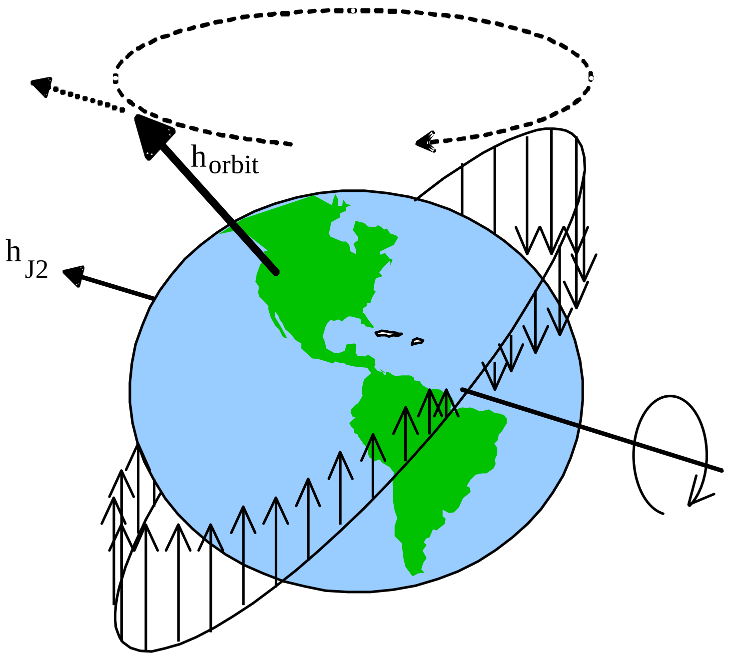
$C_{n,m}$, $S_{n,m}$ = Tesseral & Sectorial Harmonic Coefficients

Notes: $J_2$=0.001082635 has 1/1000th the effect of the spherical term.

All other terms start at 1/1000th of J2's effect.

Only the first 4x4 (n=4, m=4) Typically used for S/C software.

# The J2 Effect

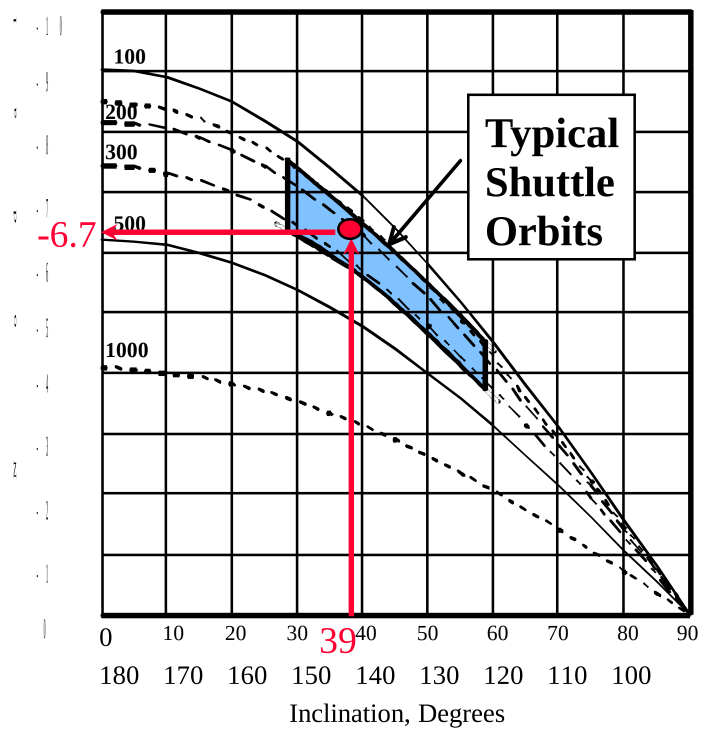



The Earth's oblateness causes the most significant perturbation of any of the nonspherical terms.

Right Ascension of the Ascending Node (Ω), Argument of Perigee (ω) and the Mean Motion (n) are affected.

Nodal Regression is the most important operationally.

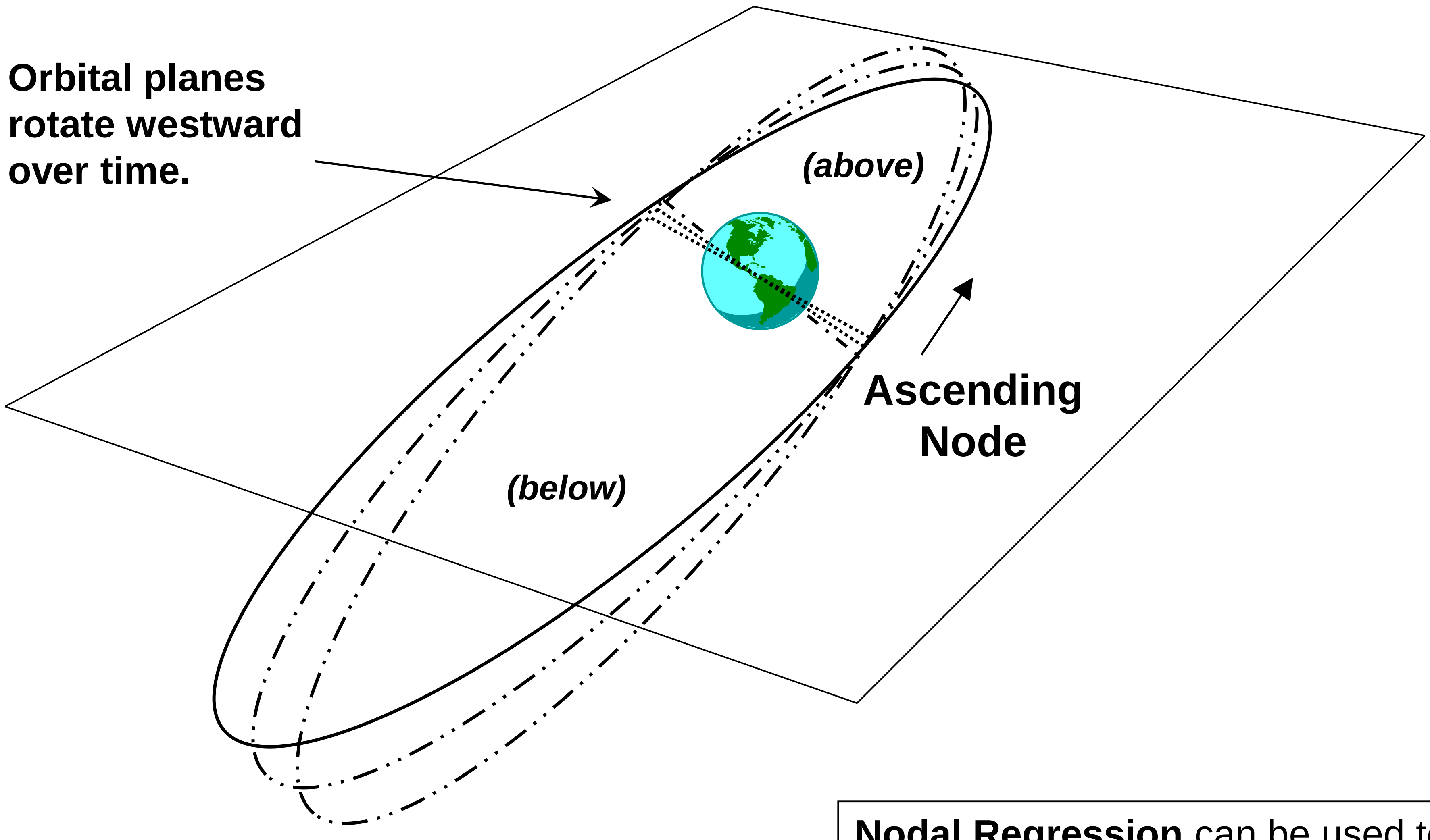
Magnitude depends on orbit size (a), shape (e) and inclination (i).




Posigrade orbits' nodes regress Westward (0° < i < 90°)

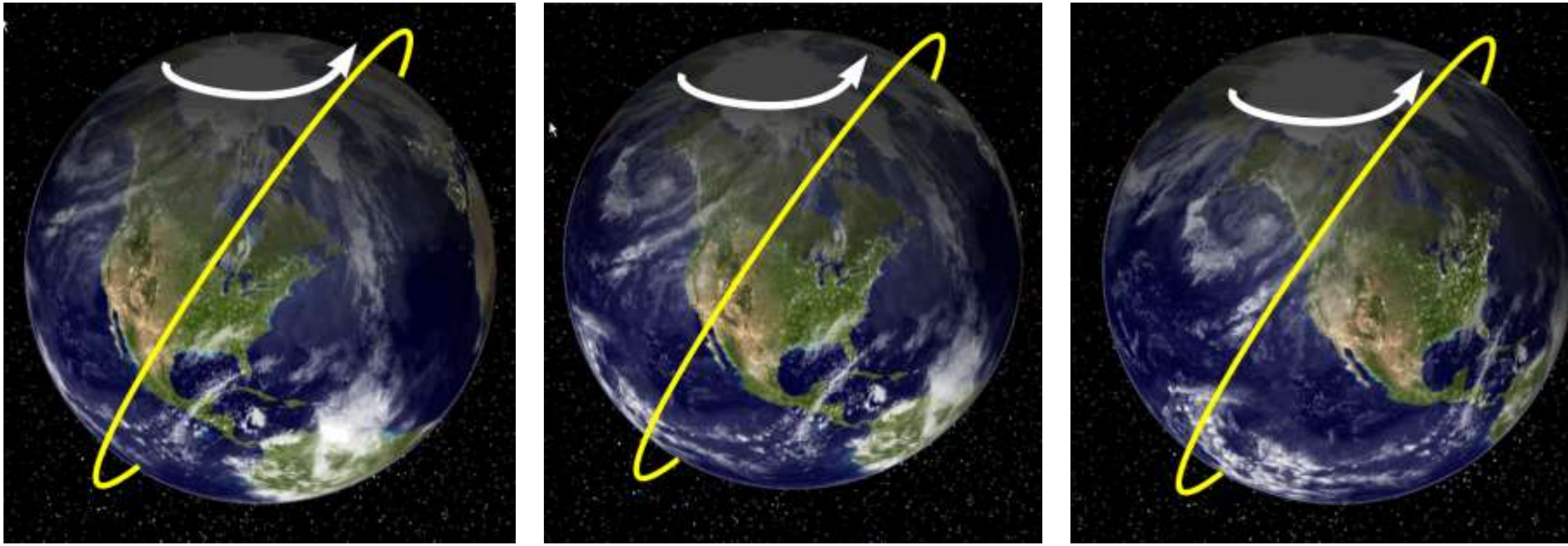
Retrograde orbits' nodes regress Eastward (90° < i <180°)

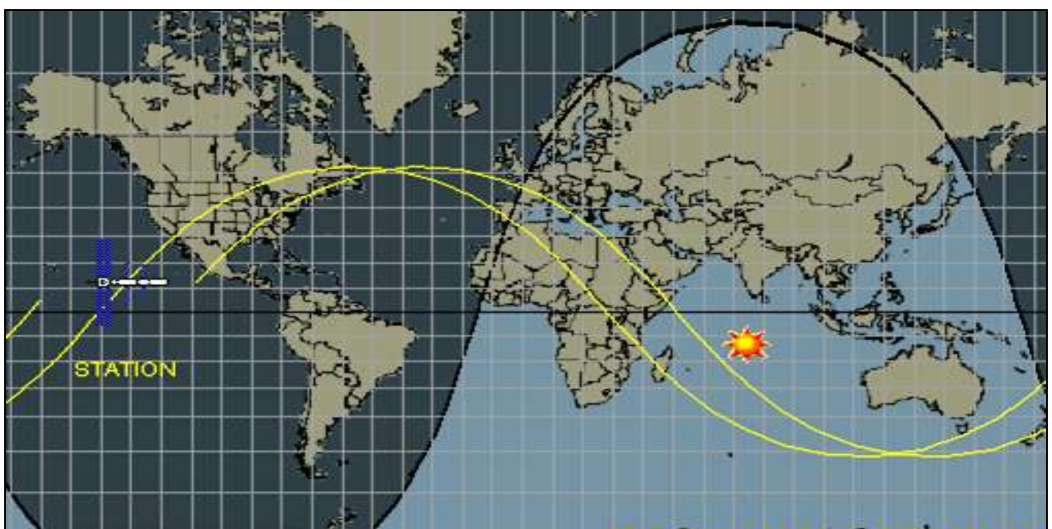
# Nodal Regression




**Nodal Regression** can be used to advantage (such as assuring desired lighting conditions)

# Ground Track





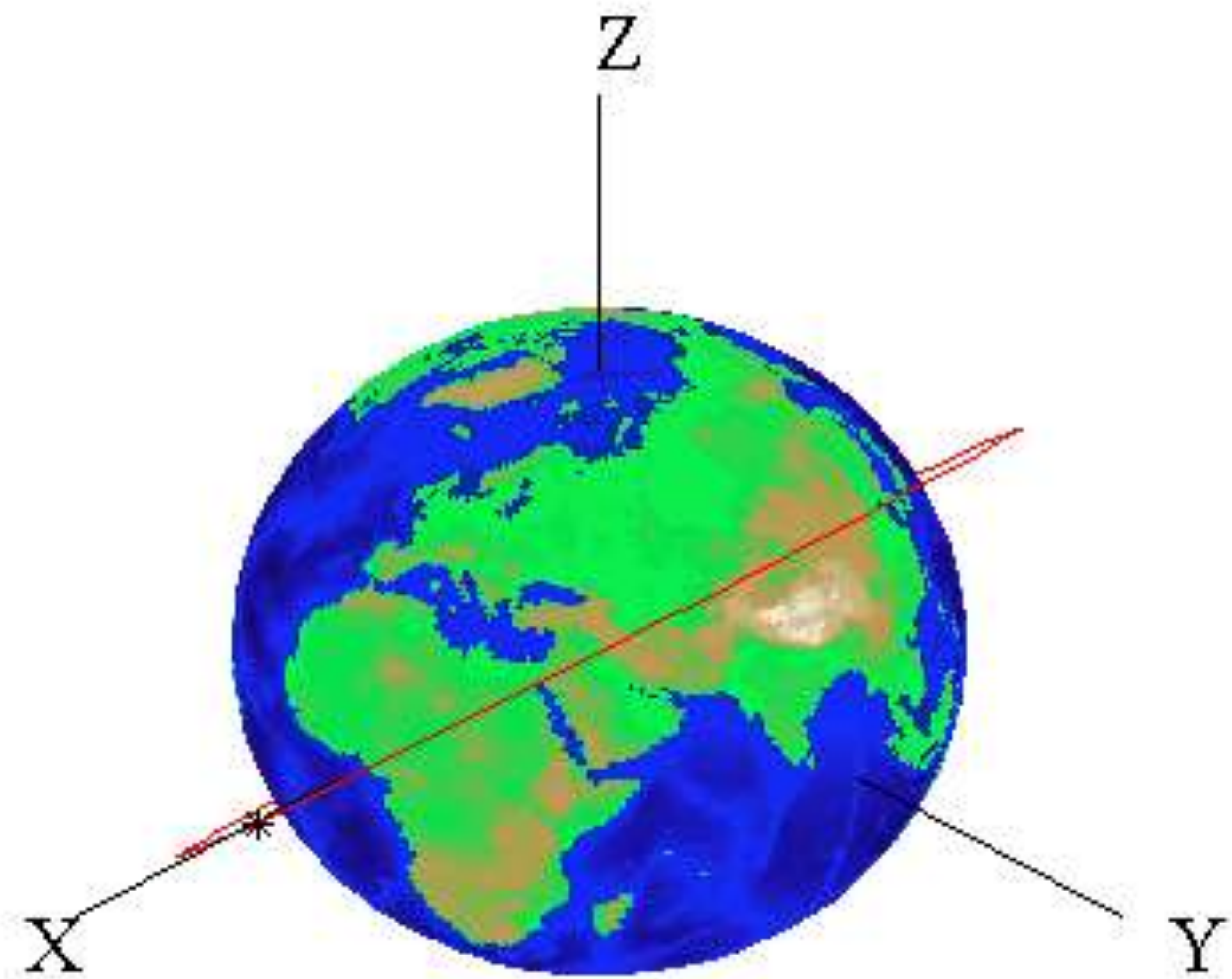

**Ground tracks drift westward as the Earth rotates below.**

**360 deg / 24 hrs**

**= 15 deg/hr**

**+ Nodal Regression**

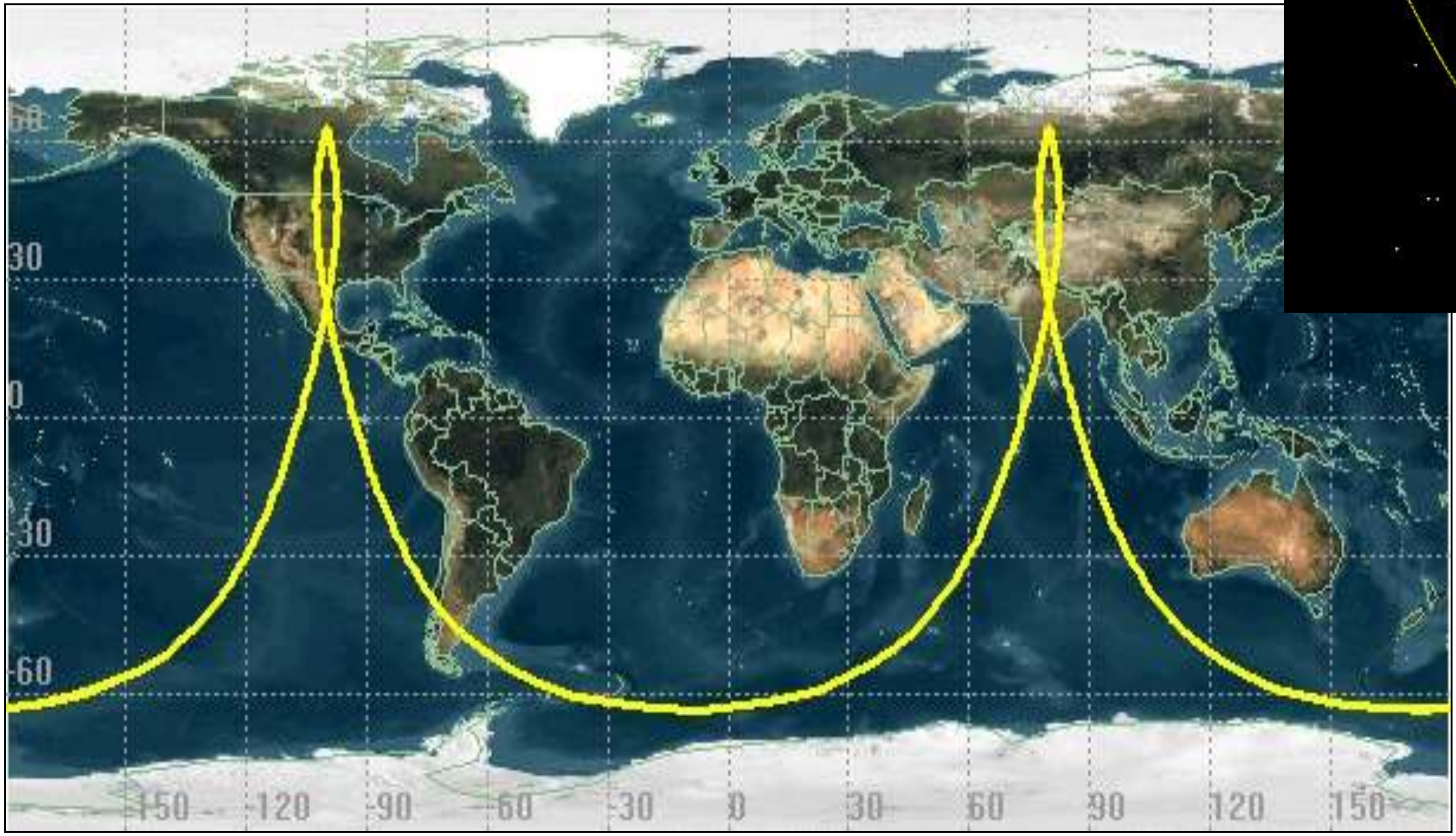
$N_{\Omega}=1$, $N_{\oplus}=130$, $i=136$, $a=8378$ km



# Molniya - 12hr Period

Sets inclination such that argument of perigee regression is zero. This enables a long loitering time over the apogee position.

Used by USA and USSR for spy satellites

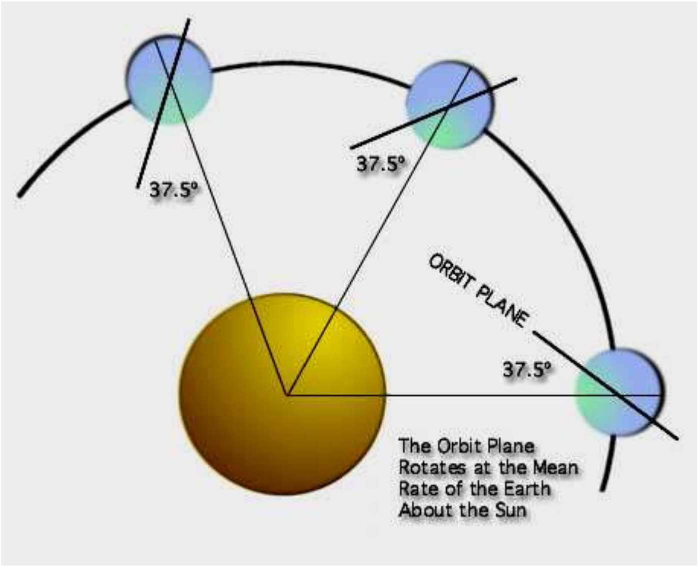
# Sun-Synchronous Orbits (2)

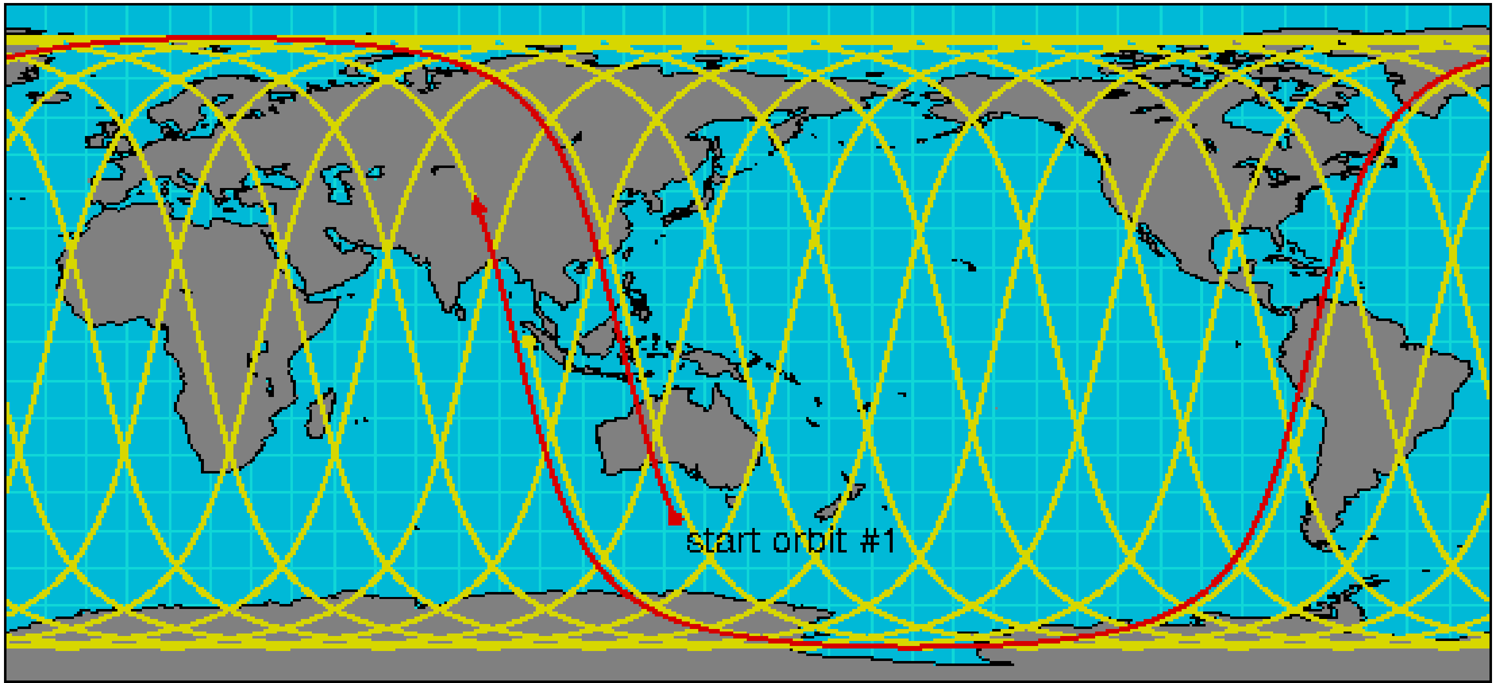
A Sun-Synchronous Orbit has a shift in ascending node ~1° per day.

Scans the same path under the same lighting conditions each day.

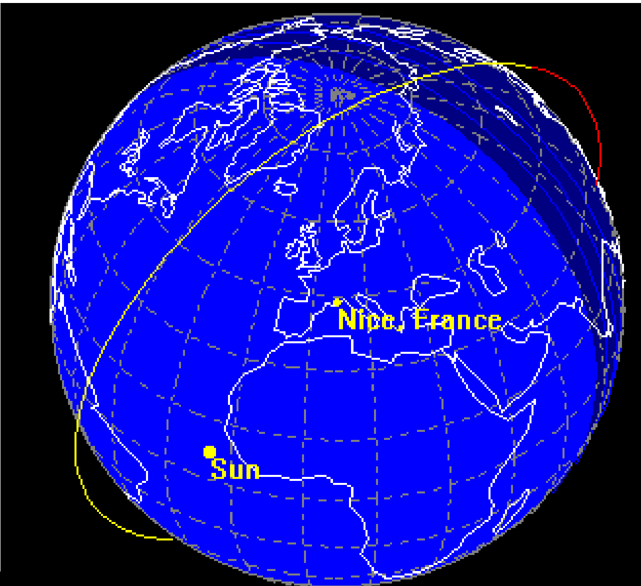
Requires a slightly retrograde orbit (For example: I = 97.56° for a 550km SSO).

Used for reconnaissance, terrain mapping, etc.



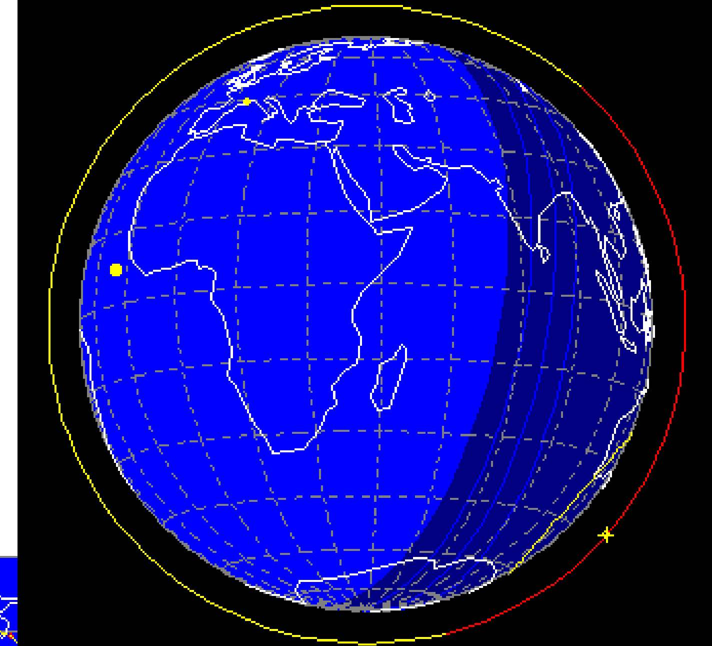
# Sunsynchronous (Landsat 7)

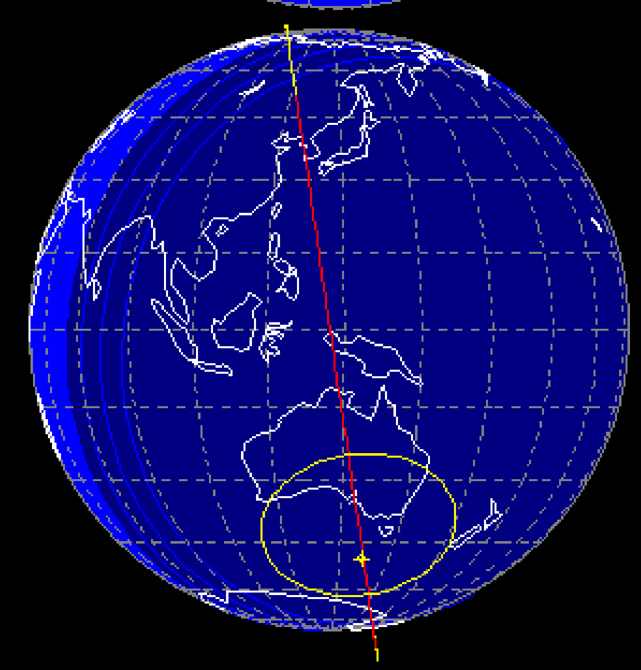


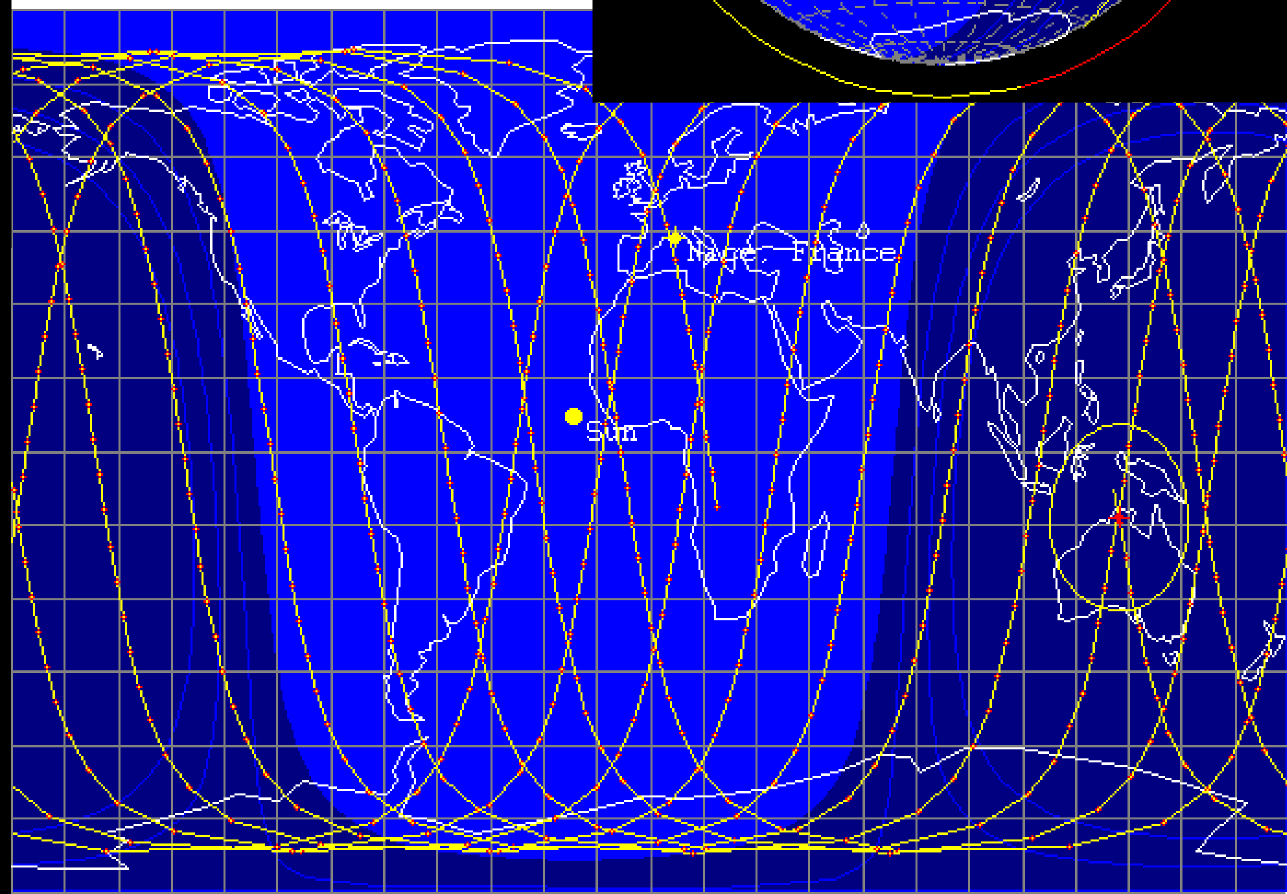

$i = 98.8°$

Period 98 min

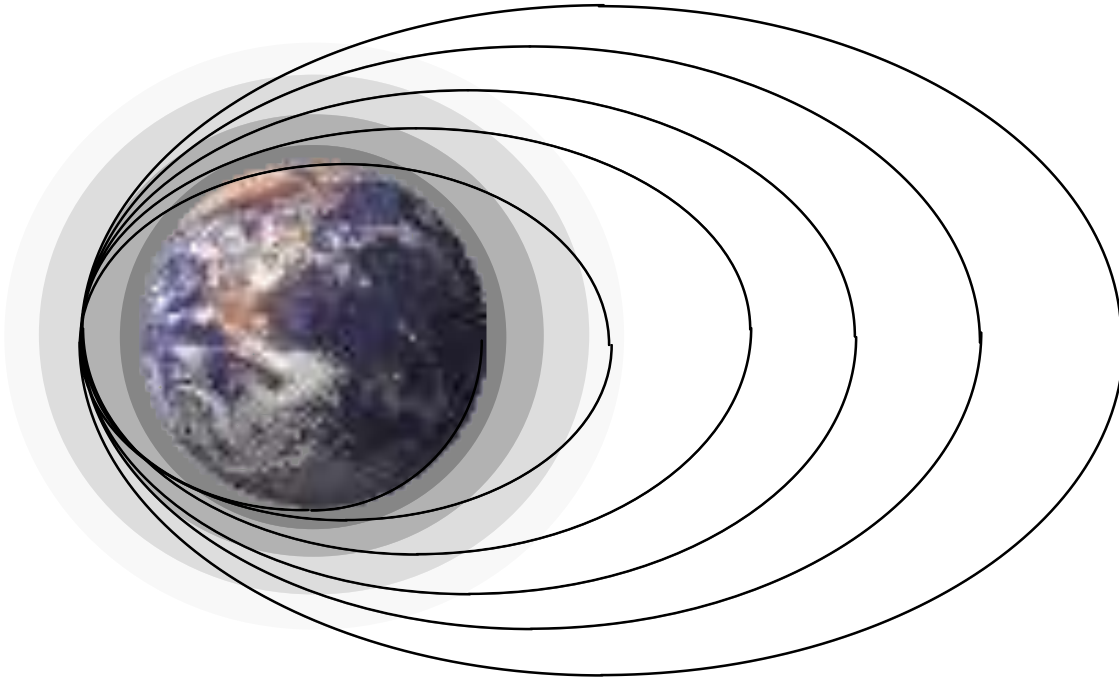
700 km







# Atmospheric Drag Perturbations



**Atmospheric density** is a function of altitude, latitude, solar heating, season, time of day, land mass vs water, etc.

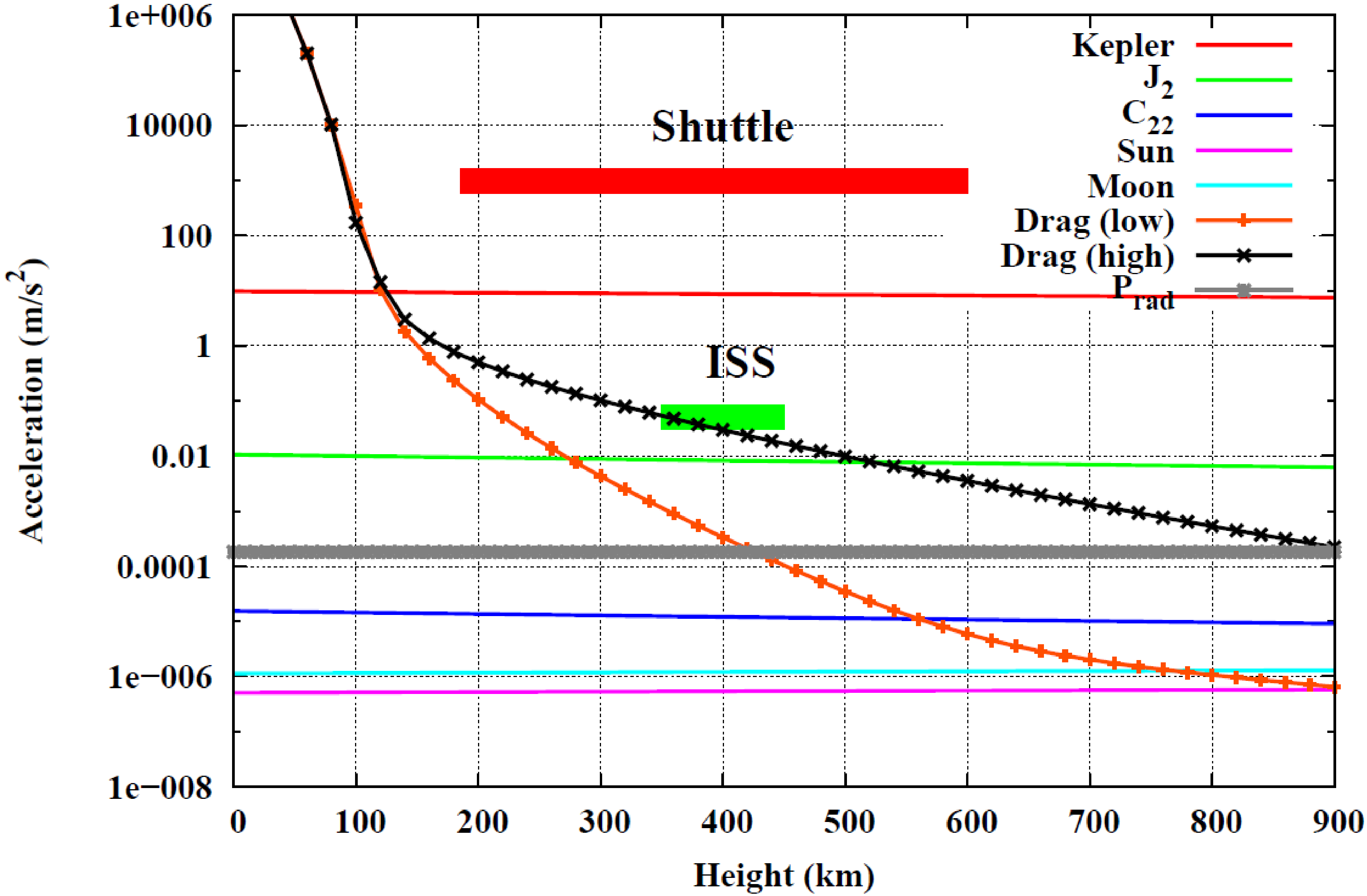
**Drag** depends on atmospheric density, but also spacecraft speed, attitude, frontal area, material properties, etc.

Effect is to lower the apogee of an elliptical orbit

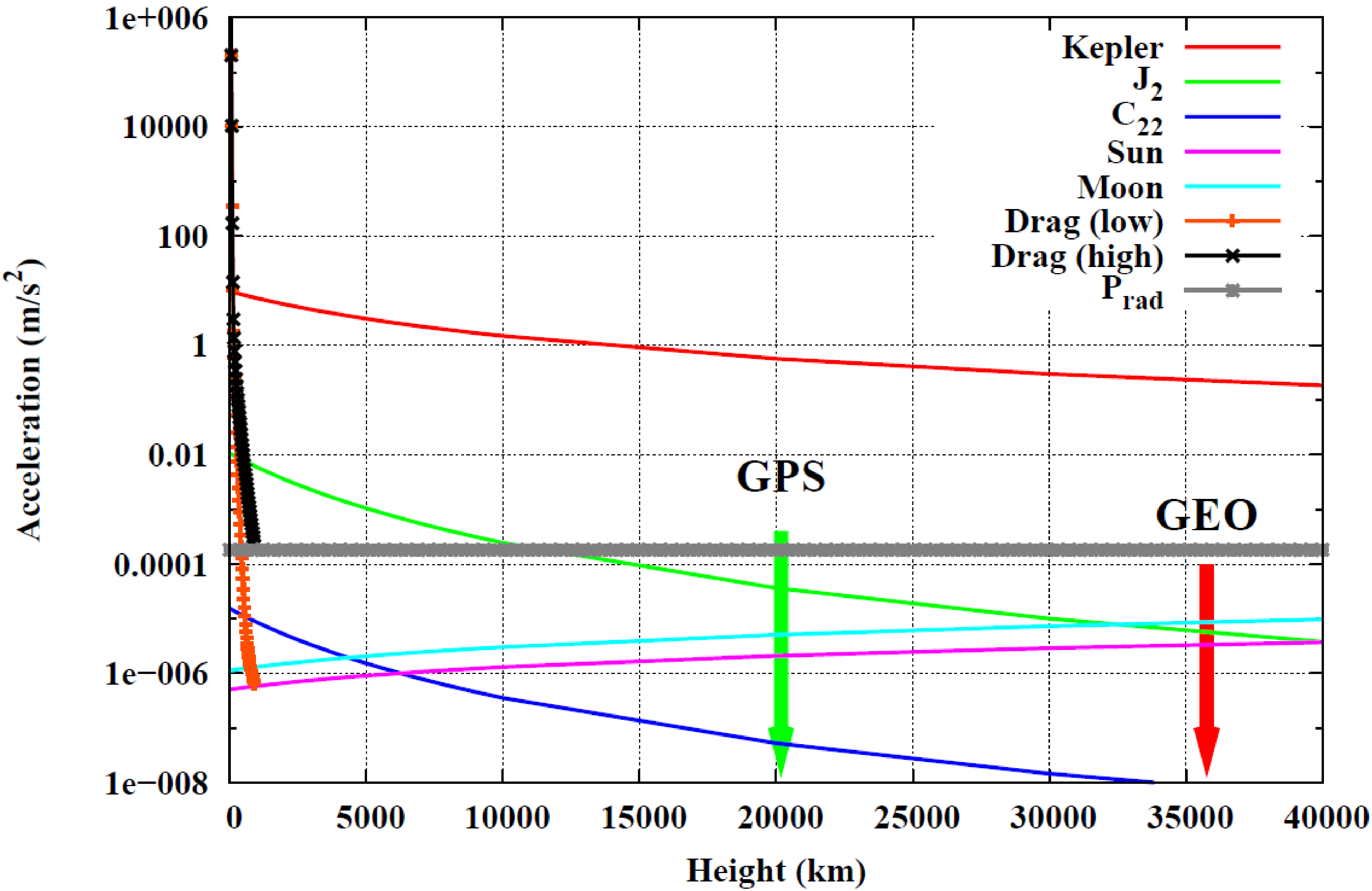
Perigee remains relatively constant

$$F_{Drag} = \frac{1}{2} C_D \rho V^2 A_{Frontal}$$

# Relative Magnitude of Perturbations



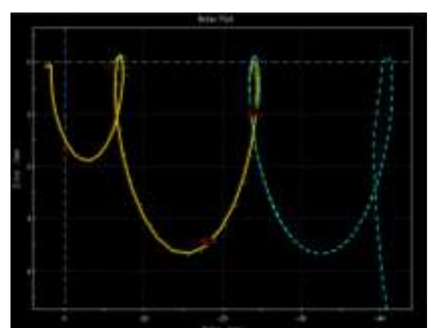
# Relative Magnitude of Perturbations (2)



# Astrodynamics

## Relative Motion

# Relative Motion, Rendezvous & Proximity Operations (Prox Ops)

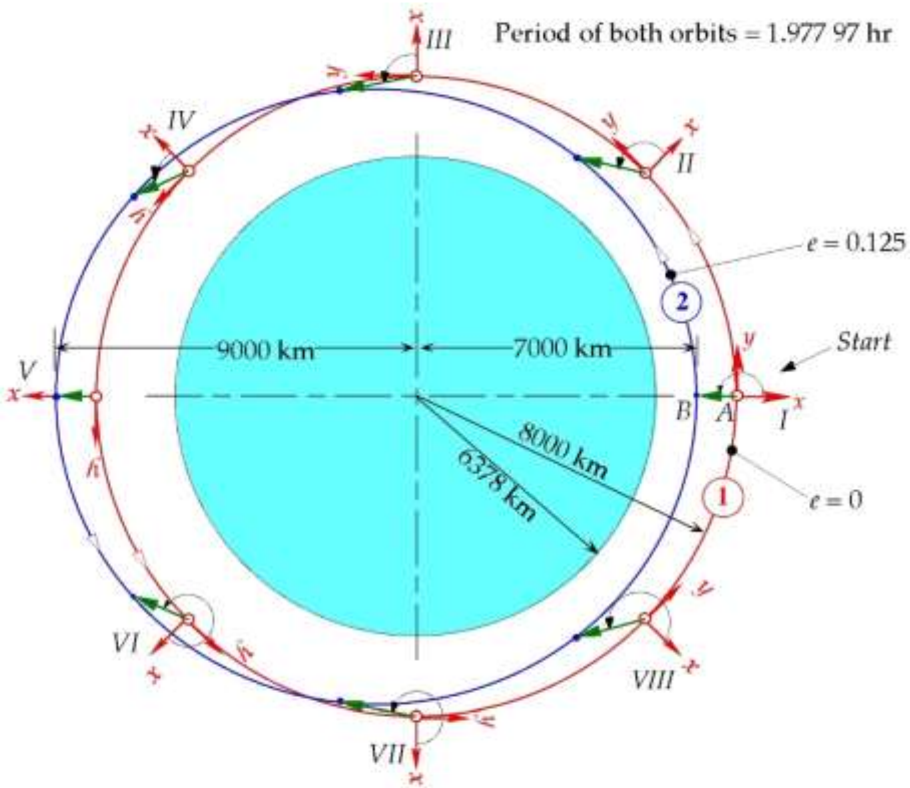
# Relative Motion

Consider two spacecraft flying in proximity to one another.

Target vehicle *A* is passive.   All maneuvers are performed by chase vehicle *B*.

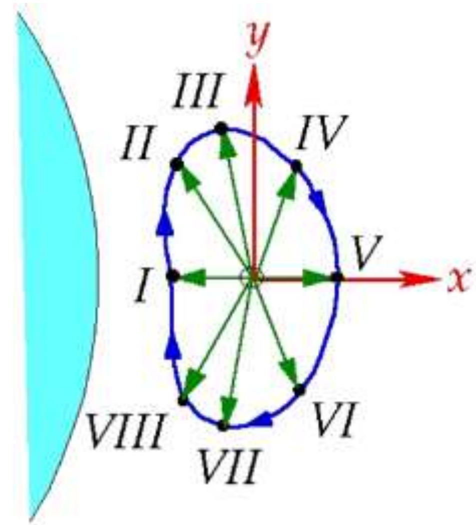
We are interested in the motion of *B* relative to *A* - as viewed from vehicle *A*.

*Inertial Perspective*

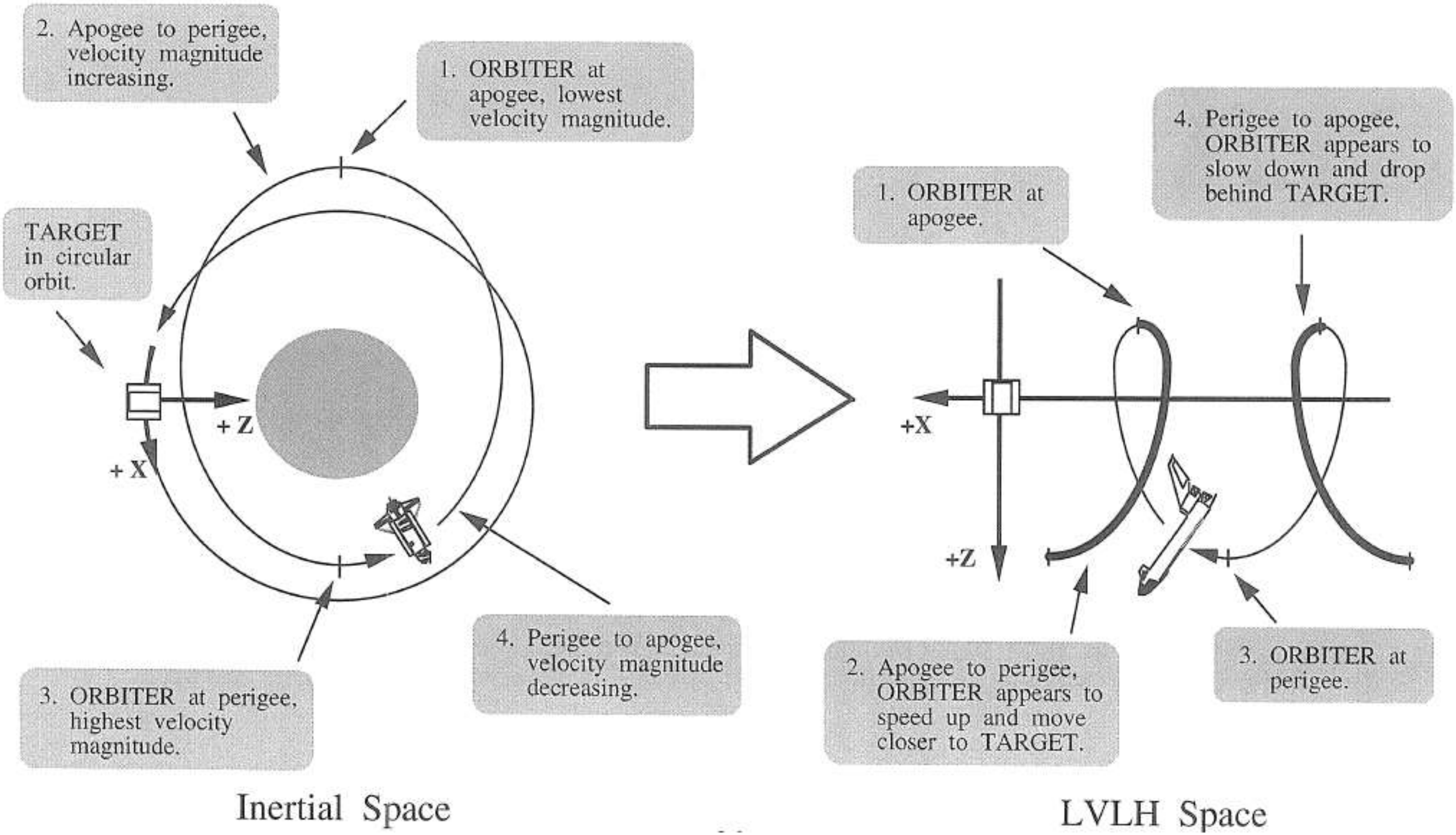



*Relative Perspective*

*from Vehicle A*



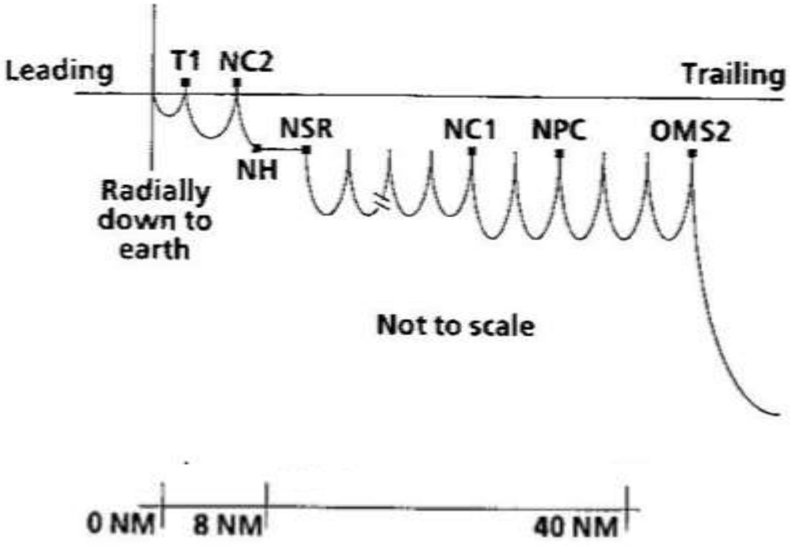
For circular orbits, the magnitude of the velocity vector is constant. The magnitude of the velocity vector varies for elliptical orbits. It has its greatest value at perigee and its lowest value at apogee. This results in the "loop de loop" (or "wifferdil") trajectory when an elliptical orbit is viewed in LVLH space.

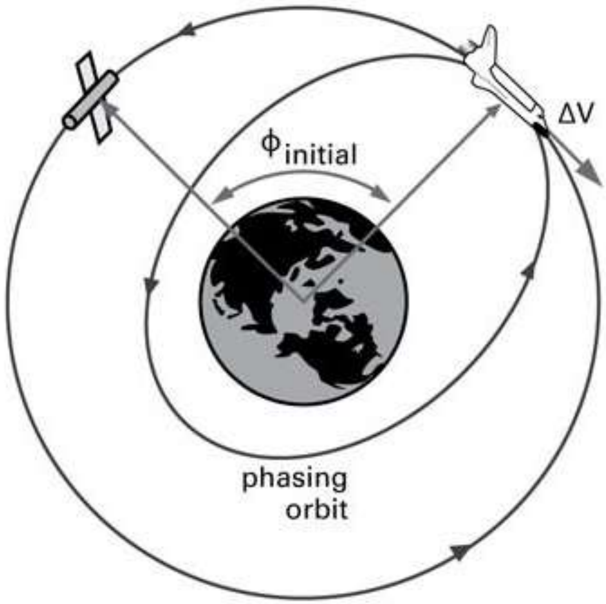


# Astrodynamics

## Targeting Maneuvers

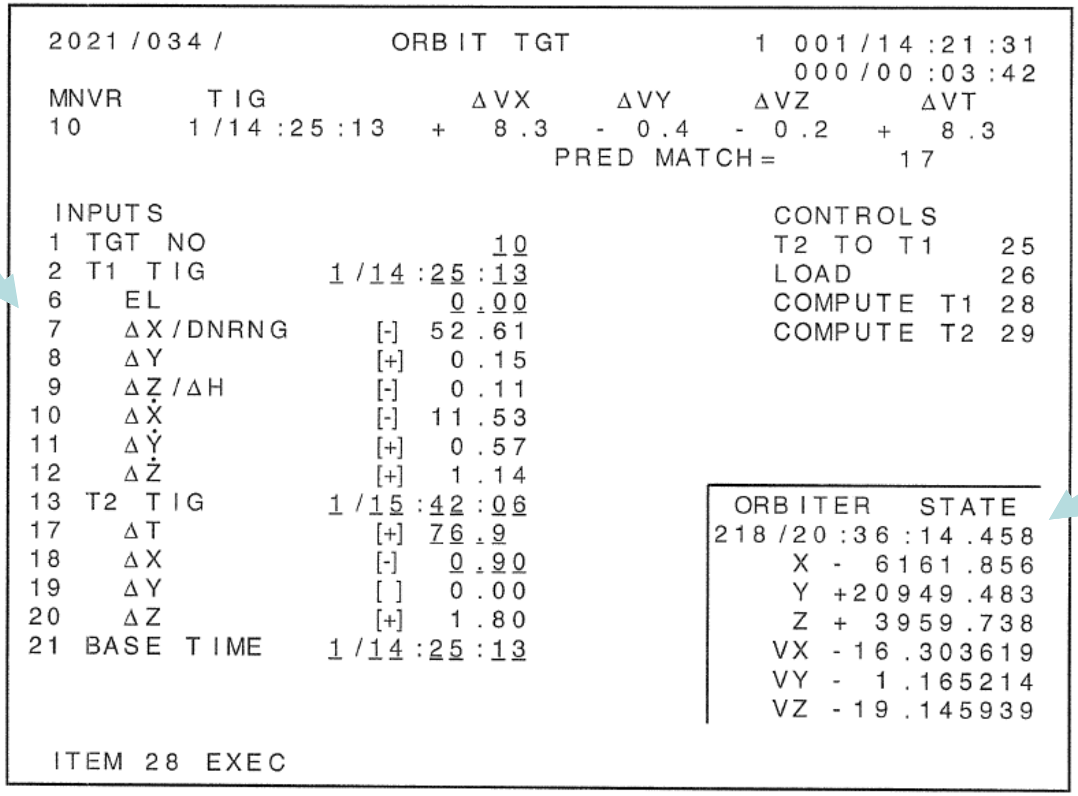
# Targeting Maneuvers



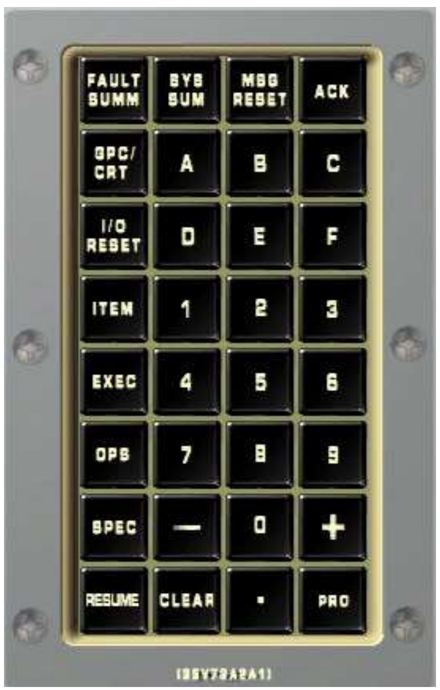



# Space Shuttle Orbital Targeting Display

Initial and desired position and times




Current State Vector




Computing the $\Delta v$ for targeted maneuvers

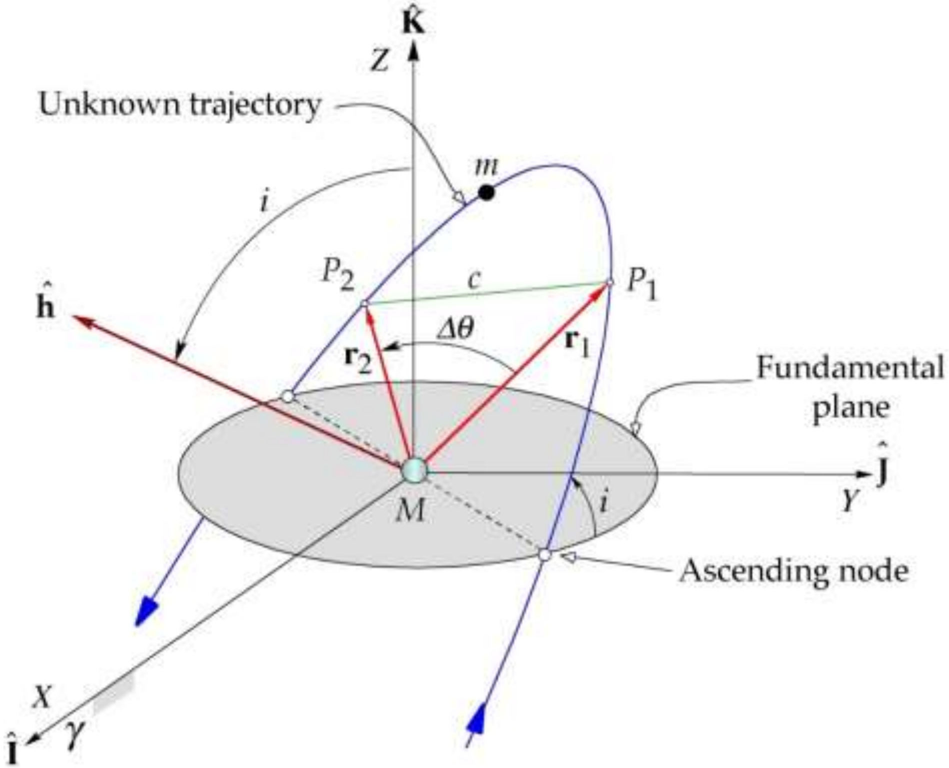
What's under the hood?

*Lambert Targeting*!

# Lambert's Problem

Given $r1$, $r2$ and the flight time $\Delta t$ from $P1$ to $P2$,

find the orbital trajectory from $P1$ to $P2$.




Note: Considered an orbit determination problem, but commonly used as a rendezvous and intercept technique.

The trajectory from $P1$ to $P2$ also determines the $\Delta v$ required at $P1$. A sequence of desired relative points can be accomplished with a series of Lambert targeted burns.

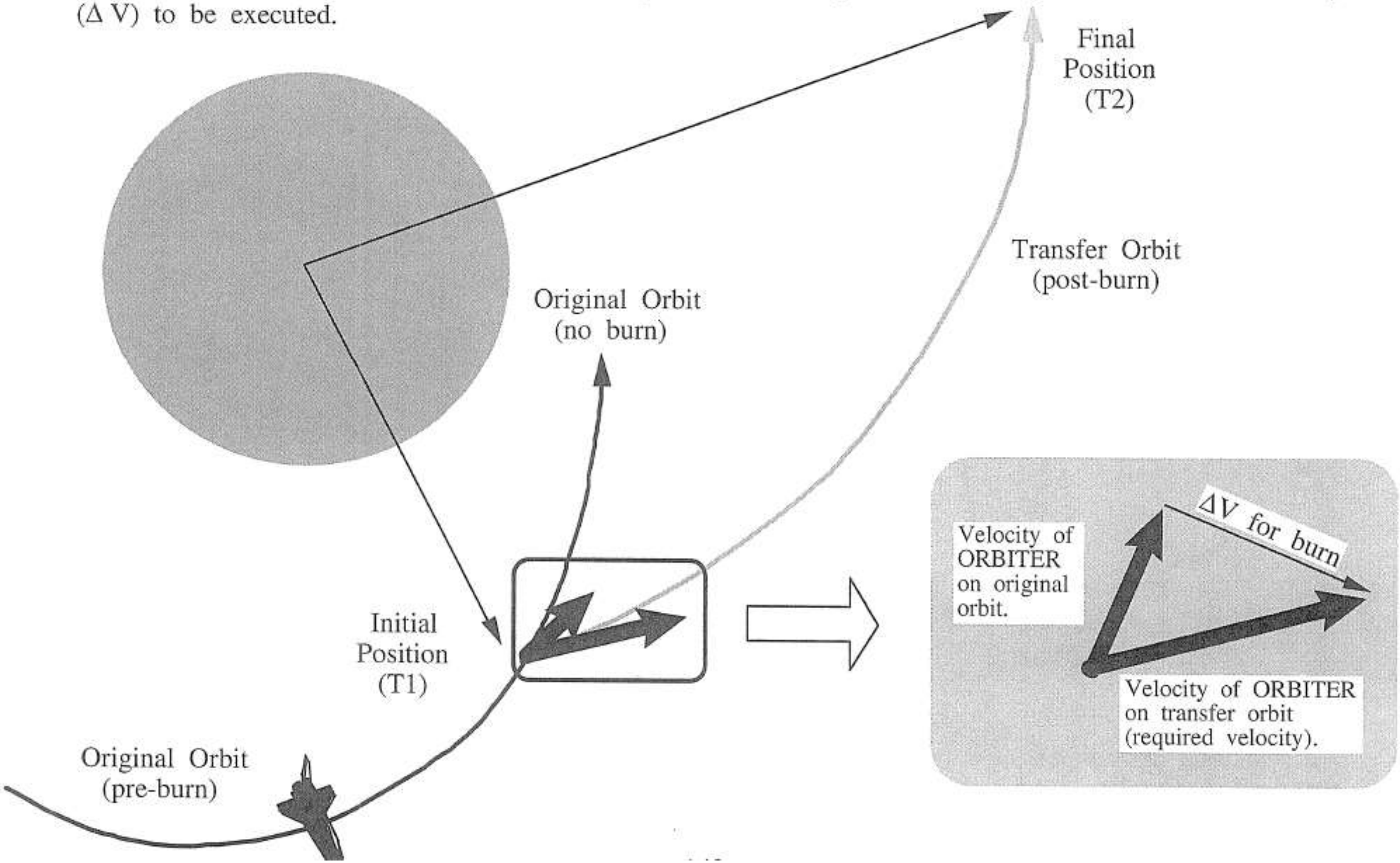
**To Solve this we're going to need Lagrange Coefficients and Universal Variables!**

# Astrodynamics

## Shuttle/ISS Rendezvous & Prox Ops

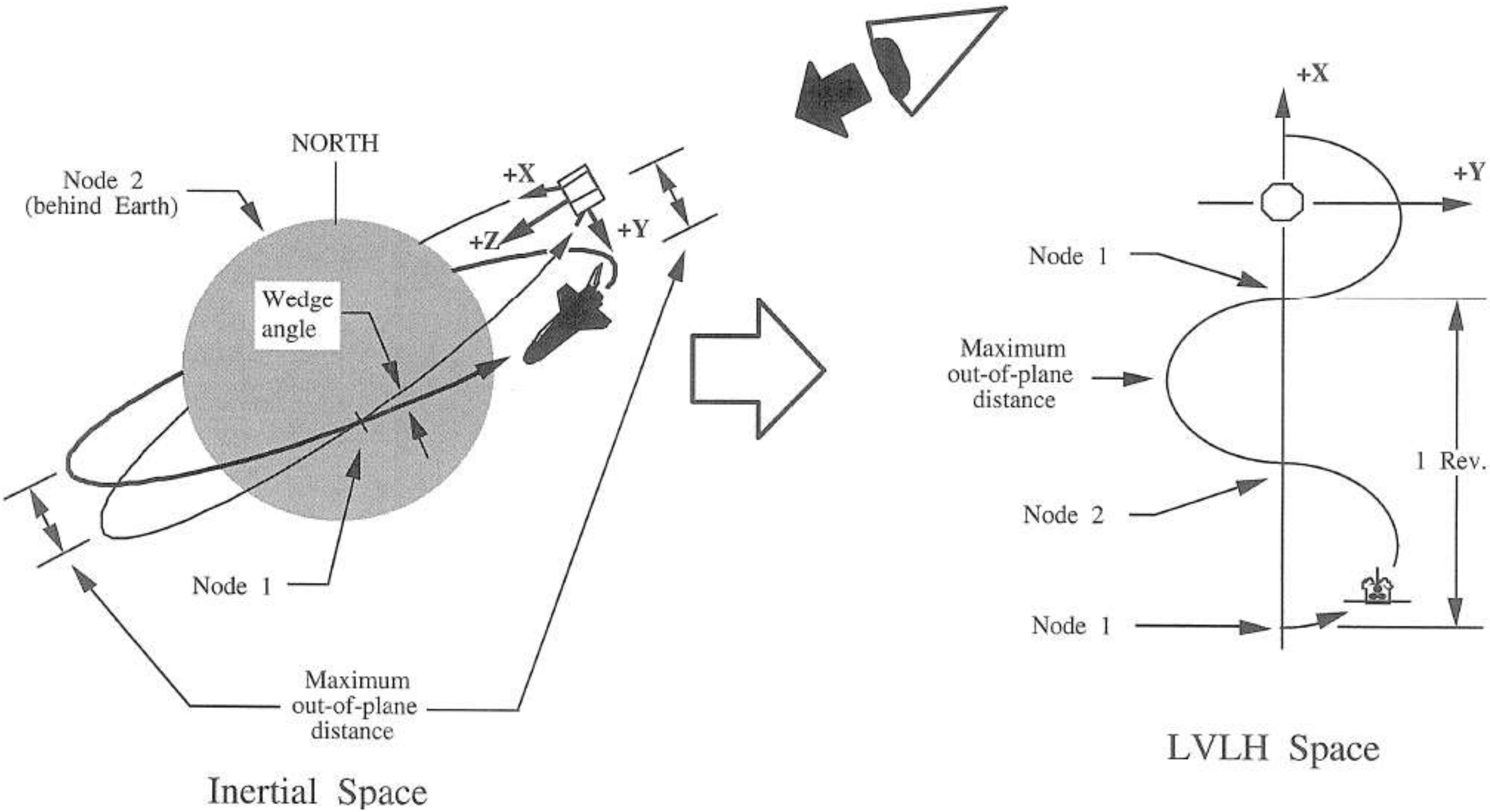
# Space Shuttle Maneuver Targeting

A burn must be executed at the initial position to place the ORBITER on the transfer orbit. LAMBERT computes the ORBITER's required velocity at the initial point to achieve the transfer. The difference between the required velocity and the actual (pre-burn) velocity at the initial point is the delta velocity (Δ V) to be executed.



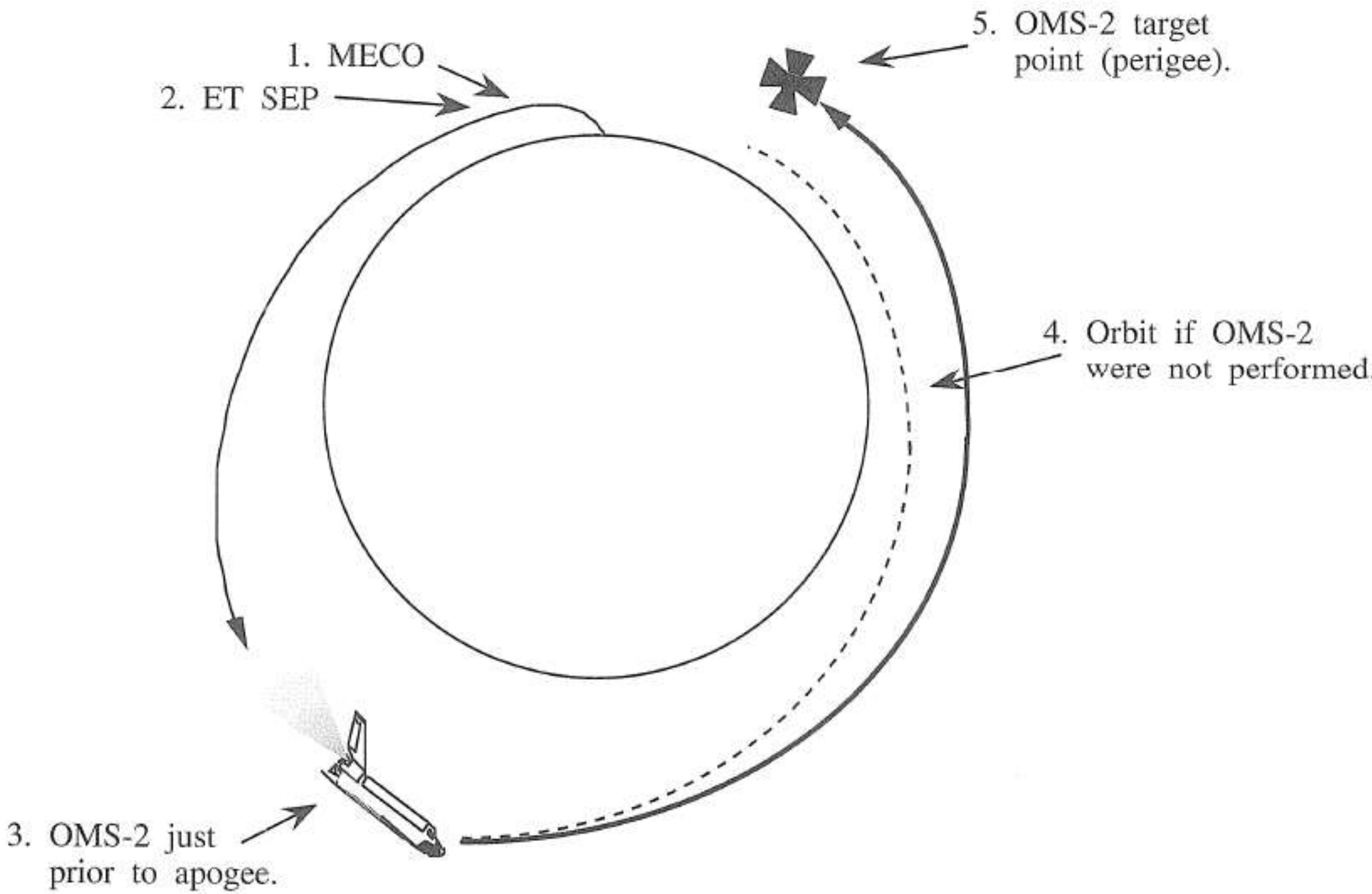
# 3.6 Relative Motion Projected Into the Local Horizontal Plane

When viewed in the local horizontal plane, orbiter out-of-plane motion appears to be sinusoidal due to the wedge angle between the orbiter and target orbits.



# 5.2 OMS-2 Burn

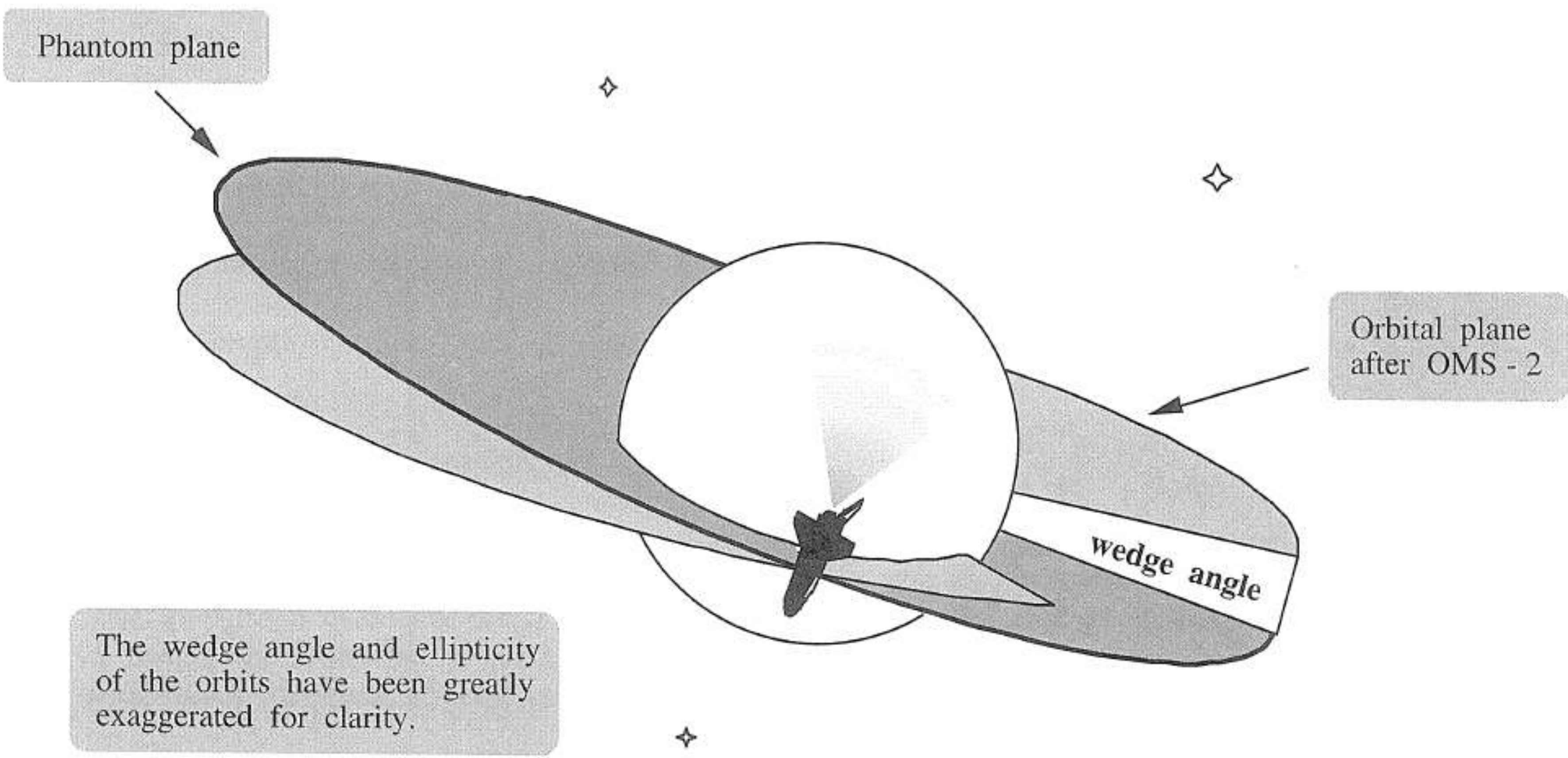
After MECO, the ORBITER is in a highly elliptical orbit. Just prior to apogee, the OMS-2 maneuver is executed to raise the ORBITER's perigee. The post OMS-2 orbit could be either circular or elliptical.



# 5.4 Plane Change Burn (NPC)

For a ground - up rendezvous, yaw steering is performed to place the ORBITER in the phantom plane. Dispersions during ascent may cause the actual orbital plane to be different than the desired phantom plane. The difference is measured in terms of wedge angle.

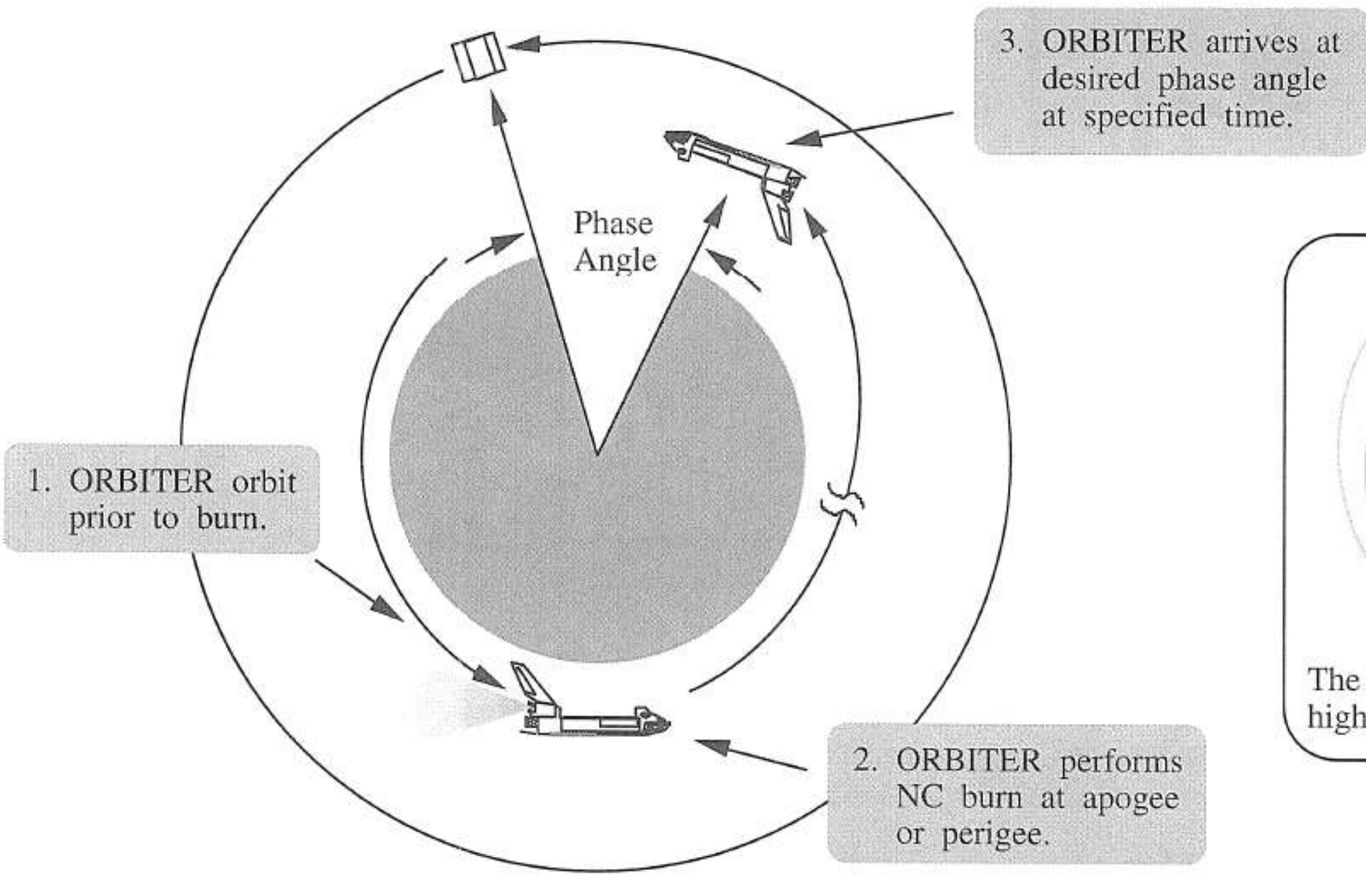
This maneuver corrects for the ascent planar dispersions by placing the ORBITER in the phantom plane. NPC is done at the point where the actual and phantom planes intersect (node). Typically only one NPC maneuver is performed.

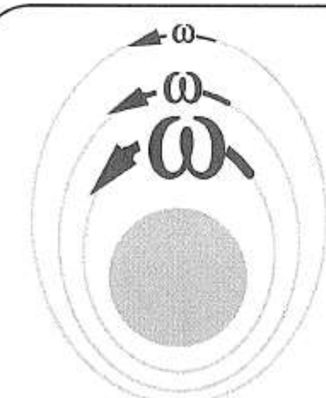


# 5.5 Phasing Burn (NC)

NC burns are used to control how quickly the ORBITER is approaching the TARGET. They may be executed either at apogee or perigee. By changing the altitude of apogee or perigee, the ORBITER can control the rate at which it orbits the Earth.

The NC burn is designed so that the orbital rate of the ORBITER will place the ORBITER at some desired down - range position (phase angle) relative to the TARGET at a designated time.

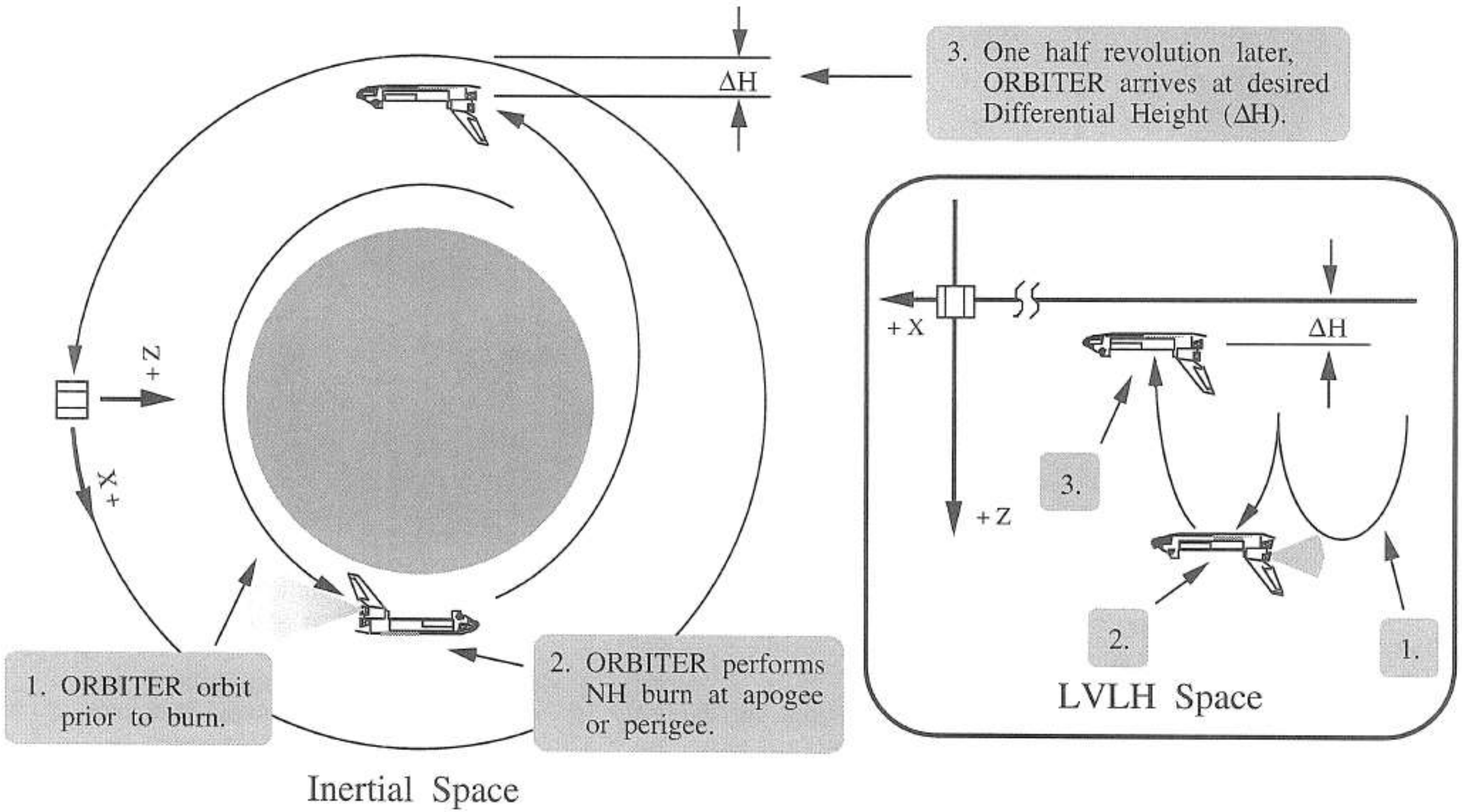



The lower the orbit, the higher the orbital rate, ω.

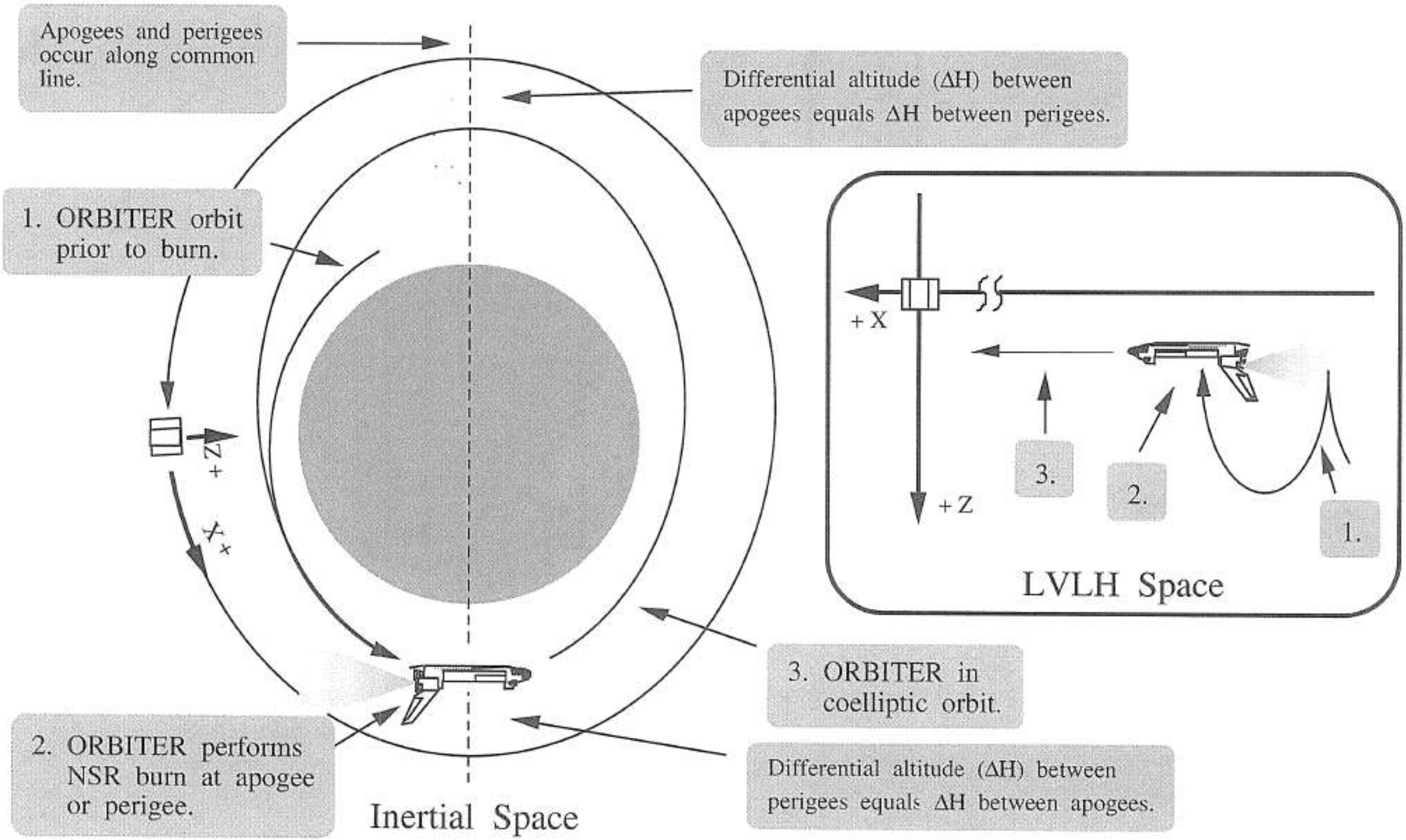
# 5.6 Altitude Burn (NH)

The NH burn controls the differential height (ΔH) between the ORBITER's orbit and the TARGET's orbit. It is executed at either apogee or perigee. NH is designed so that the ΔH condition is met after half a revolution (180 degrees of orbit travel).



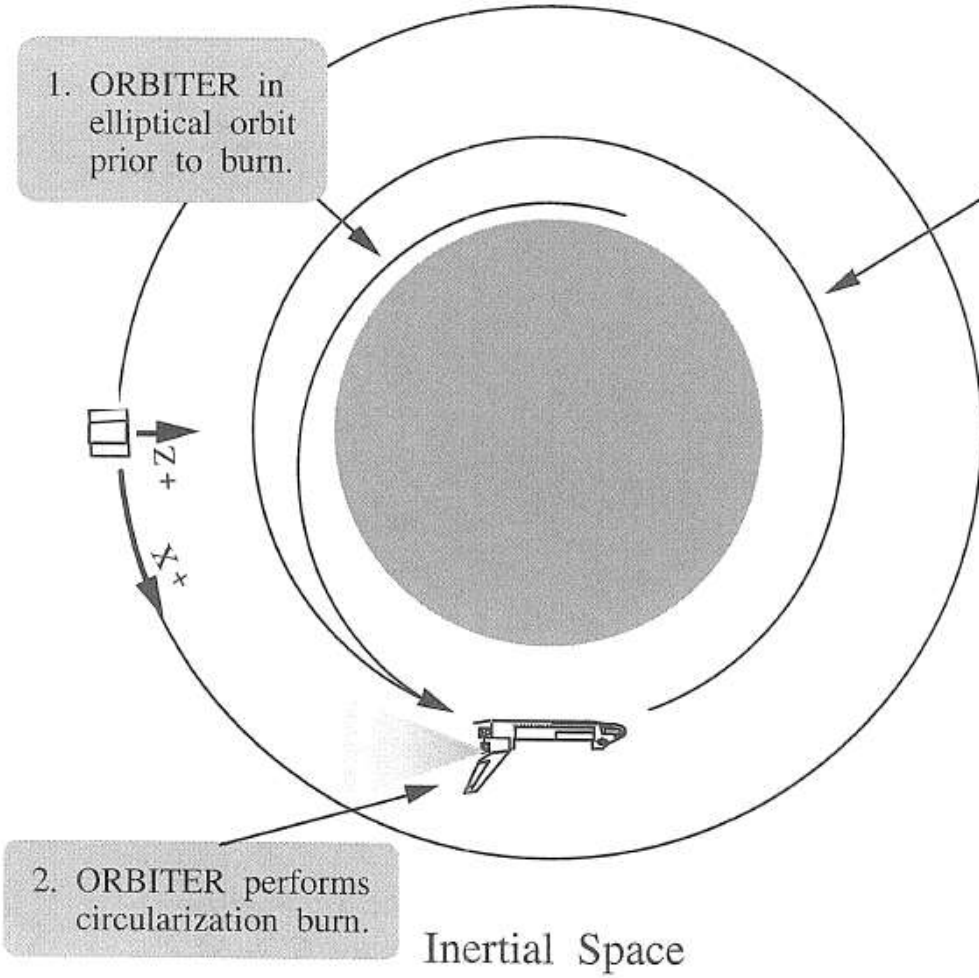
# 5.7 Coelliptic Burn (NSR)

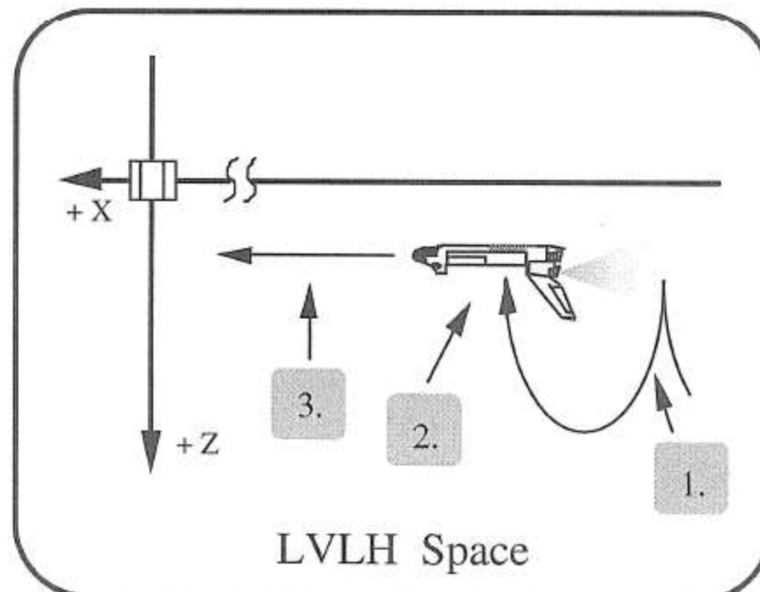
An NSR (Slow Rate) burn places the ORBITER in a coelliptic orbit with the TARGET. NSR burns are used to meet lighting requirements on the day of rendezvous.



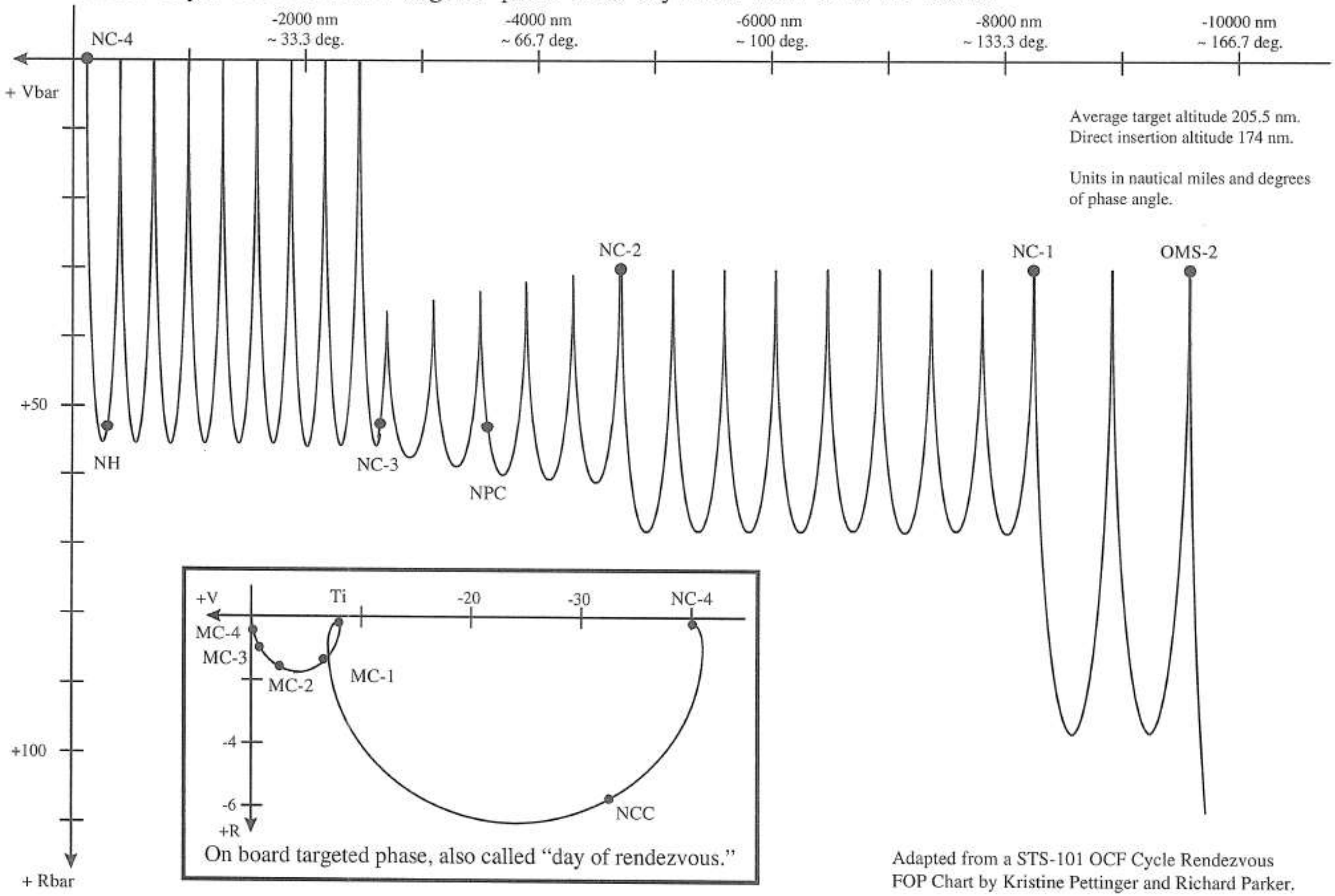
# 5.8 Circularization Burn (CIRC)

A fifth type of burn that may be executed is the circularization (CIRC) burn. It is performed at either apogee or perigee and changes the orbit from elliptical to circular. For a circular TARGET orbit, CIRC is equivalent to NSR.



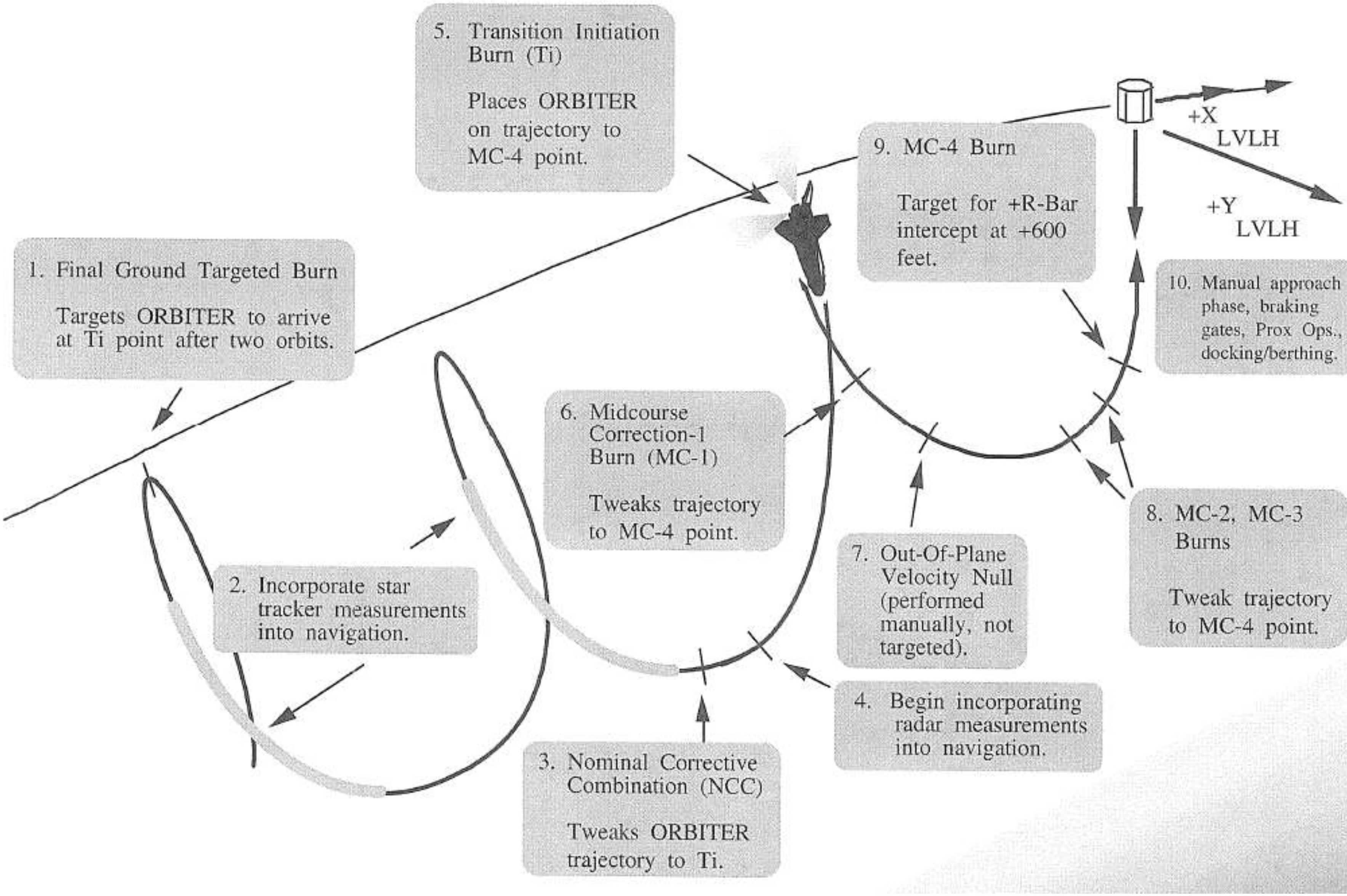



The ground targeted phase (from post OMS-2 to the final NC burn, NC-4 in this example) may last several days. The on-board targeted phase lasts anywhere from 3 to 4.5 hours.



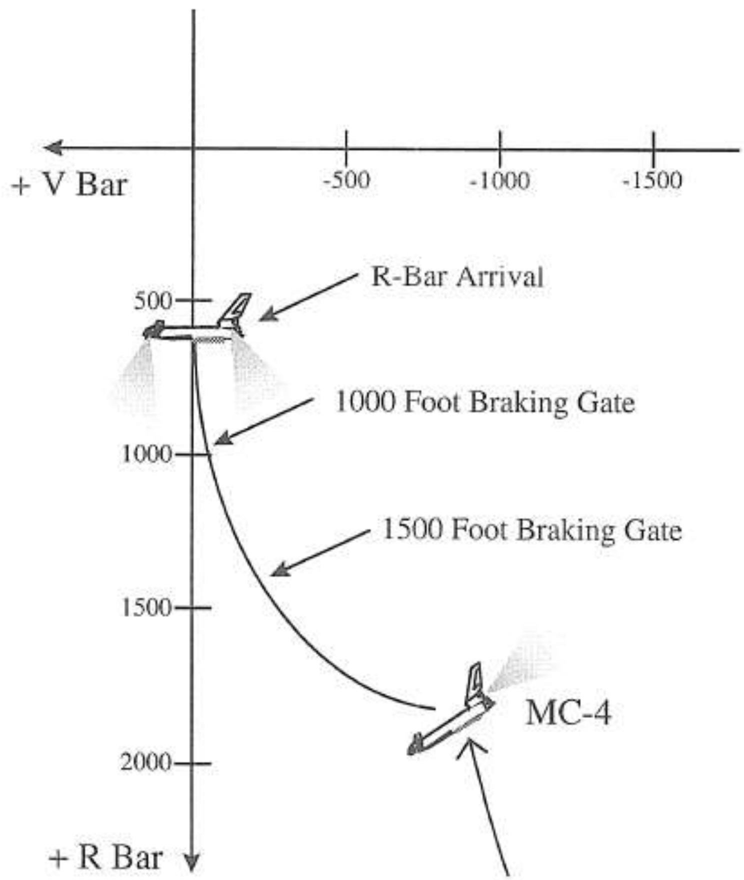

Adapted from a STS-101 OCF Cycle Rendezvous FOP Chart by Kristine Pettinger and Richard Parker.

# 6.16 Summary Of ORBT On-Board Targeted Phase Events



# 6.15 Final Approach

The ORBT profile design takes advantage of orbital mechanics effects to perform most of the braking inside 600 feet, rather than using propellant to do most of it. MC-4 is the start of the manual phase. The commander will keep the TARGET centered in the COAS and perform braking gates. After R-Bar is established at 600 feet, procedures related to proximity operations, station keeping, grapple or docking are executed. These procedures are often mission dependent. ISS flights 4A and 5A are planned to use +Rbar approaches, as was done with Mir.



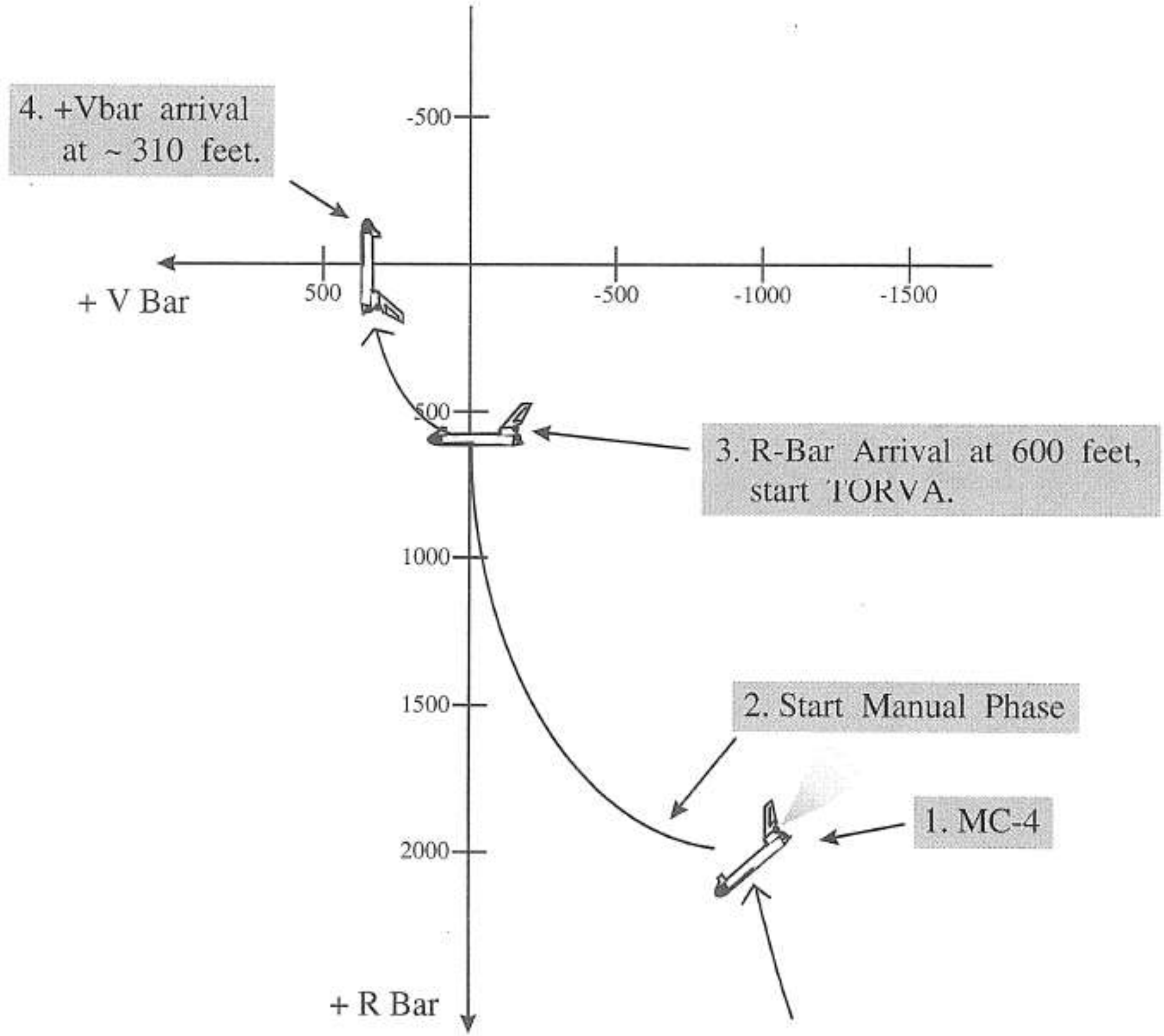

MC-4 @ -900, 0, +1800 feet LVLH

Establish R-Bar @ 0, 0, +600 feet LVLH

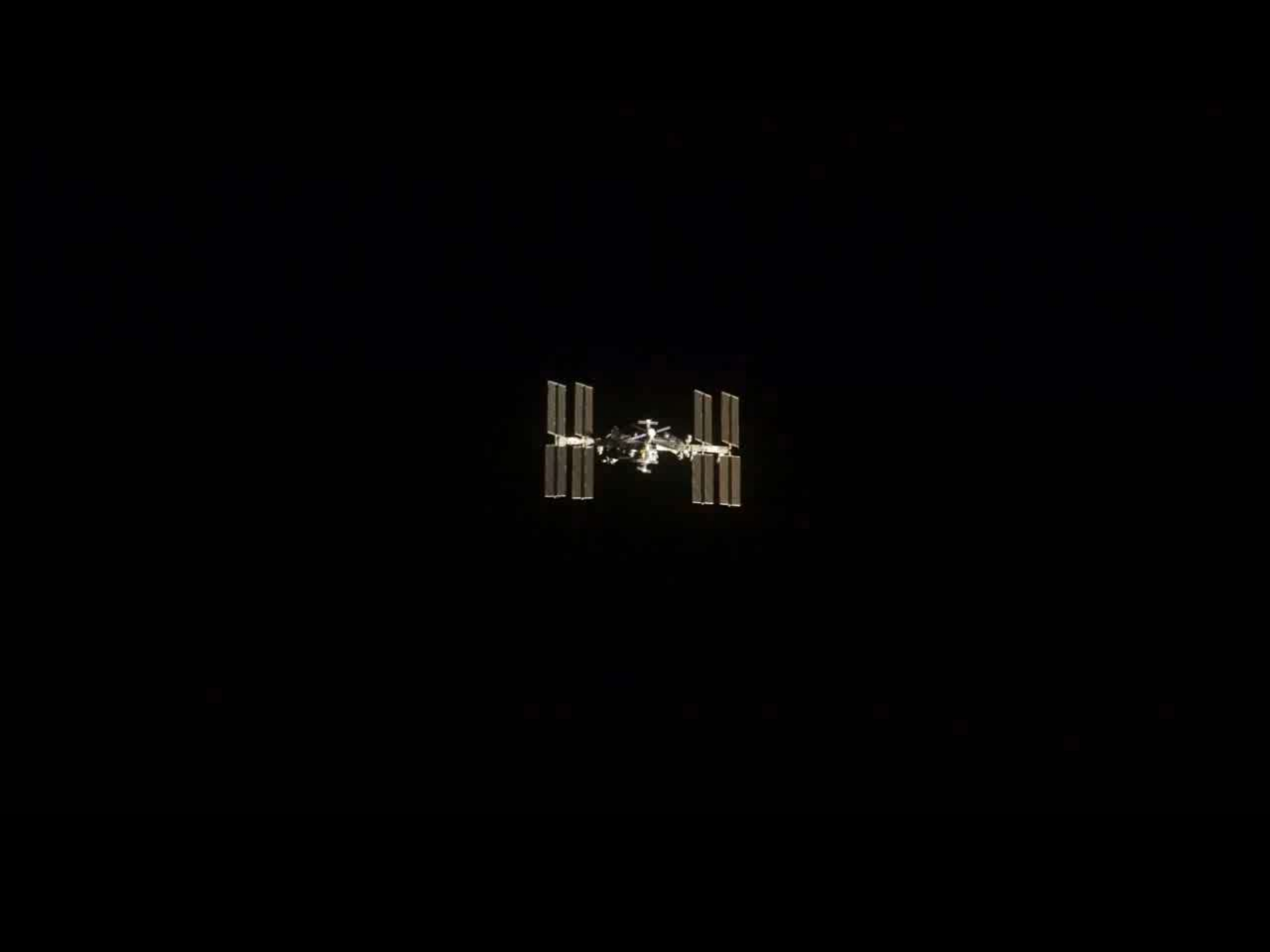
| Range (kilo-feet) | Desired Rdot (ft/sec) |
|---|---|
| 2.0 | -3.0 |
| 1.5 | -2.3 |
| 1.0 | -1.5 |
| 0.6 | -0.8 |
| 0.5 | -0.5 |
| 0.4 | -0.4 |
| 0.3 | -0.3 |

The nominal trajectory will have rates within a few tenths of a foot per second of the rates in the table.

All ISS flights after 5A are planned to fly to the +Rbar intercept point, then transition to the +Vbar using the Twice Orbital Rate Rbar To Vbar Approach (TORVA).

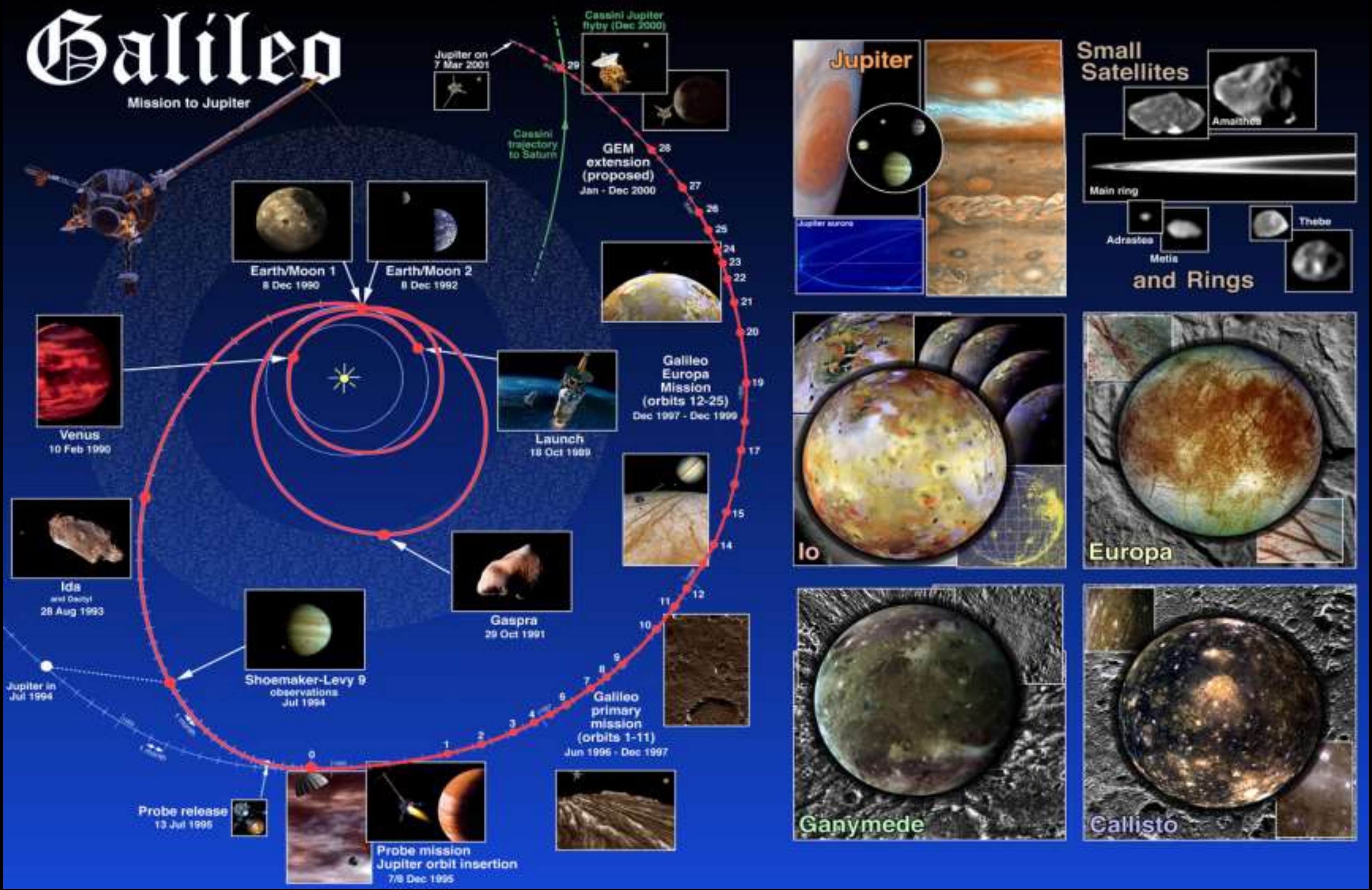


Prop Age 3
R 220 Ṙ -0.21
X 220 Ẋ -0.22
Y 3 Ẏ -0.02
Z -5 Ż -0.03
Orb DP to Tgt DP
Raw TCS 2(CW)
Refl 1
Age 0
Rng 226
Rdot -0.20
Elv -0.71
Azi 0.11
MET: 1/20:57:06
Pitch 181
Alt 187
100
100
Rng Rdot
HML/dt 222 -0.26
HMLRaw 222 -0.24
RESID RATIO ACPT REJ
RNG -0.0 0.01 389 0
RDOT 0.01 0.07 387 0
ELV -0.04 0.09 389 0
AZI -0.04 0.09 389 0
Range: 225.4 ft R-Dot -0.20 ft/s Data Good Mode Track

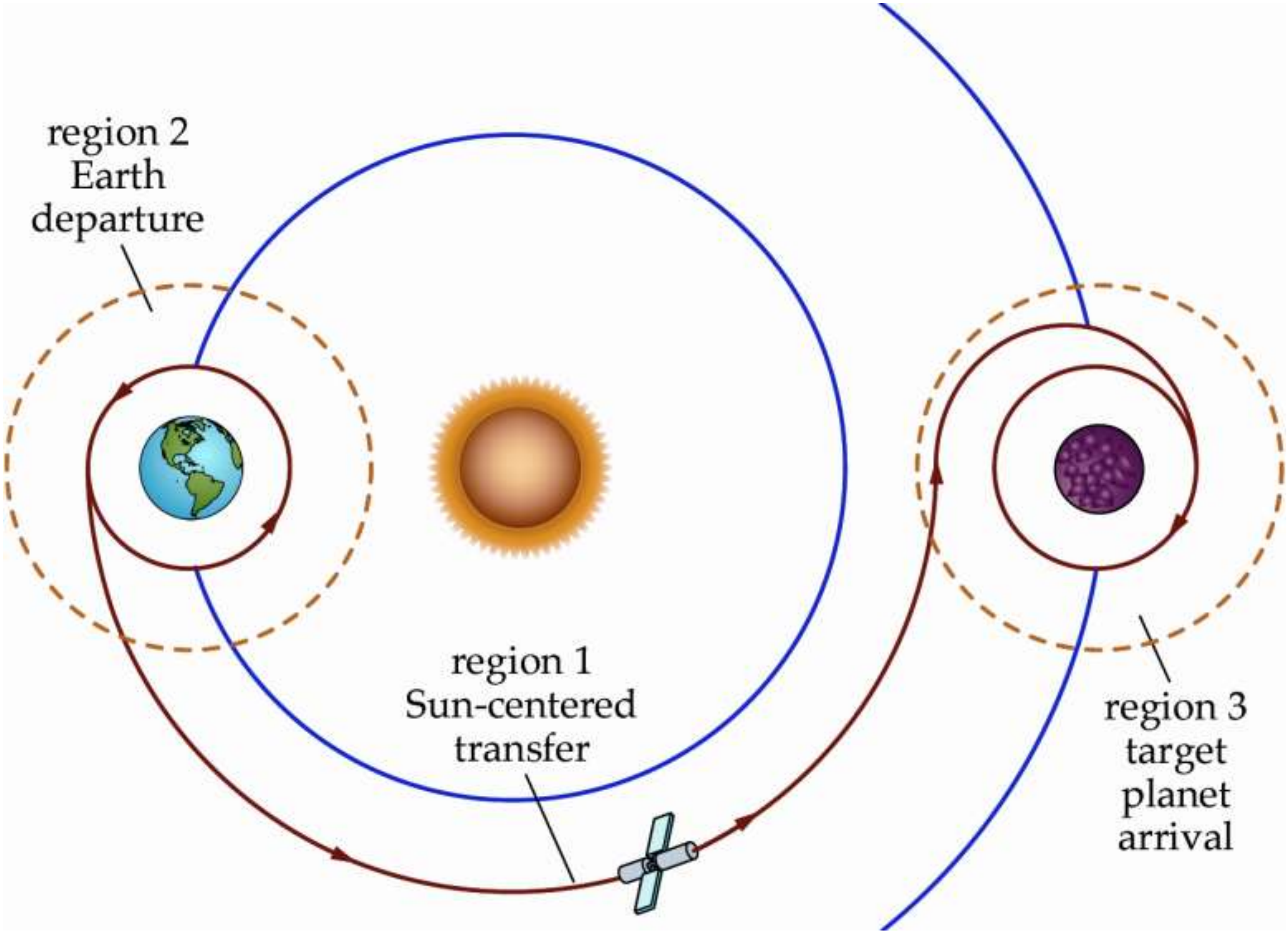


# Astrodynamics

## Interplanetary Trajectories



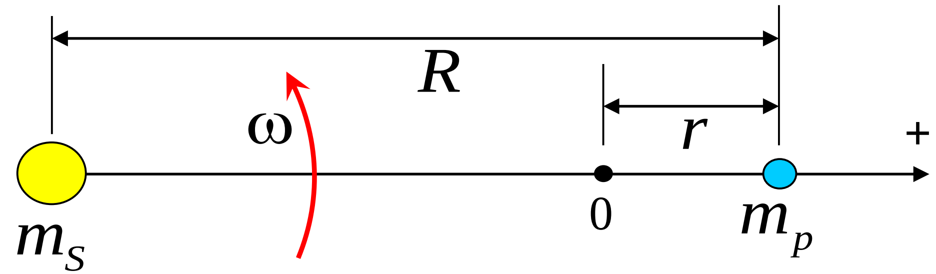
# Interplanetary Trajectories
## (Patched Conic Approximation)



# Sphere of Influence

Interplanetary Trajectory has 3 regimes:

- 1st Planet's gravity field is dominant
- Sun's gravity field is dominant
- 2nd Planet's gravity field is dominant




**Sphere of influence** - The sphere about a body in which its gravity field is dominant. Approximately coincides with the definition of the $L_1$ Lagrangian point.

Consider a spacecraft in orbit about planet with mass $m_p$ at radius $r$, and the planet is in orbit about the Sun (mass $m_S$) at radius $R$. $L_1$ is located where forces balance.

$$-\frac{Gm_S}{|R-r|^2}+\frac{Gm_p}{r^2}+\omega^2(R-r)=-\frac{Gm_S}{|R-r|^2}+\frac{Gm_p}{r^2}+\frac{Gm_S}{R^3}(R-r)=0 \quad \text{since} \quad \omega^2=\frac{Gm_S}{R^3}$$

$$\text{Generally } \boxed{r \ll R}, \quad \rightarrow \quad |R-r|^{-2} \simeq R^{-2}\left(1+2\frac{r}{R}+\cdots\right) \quad \text{therefore,}$$

$$-\frac{m_S}{R^2}\left(1+2\frac{r}{R}\right)+\frac{m_p}{r^2}+\frac{m_S}{R^2}\left(1-\frac{r}{R}\right)=0 \quad \rightarrow \quad 3\frac{m_S}{R^3}r=\frac{m_p}{r^2} \quad \rightarrow \quad \frac{r}{R}\sim\left(\frac{m_p}{3m_S}\right)^{1/3}$$

(Lagrange - more precise analysis - see Curtis) $\frac{r_{SOI}}{R}\sim\left(\frac{m_p}{m_S}\right)^{2/5}$

# Interplanetary Hohmann Transfer

The most efficient inter-planetary trajectory occurs when the departure and arrival velocities are tangent to the planetary orbits.

Assumptions:

- Both planets orbit the Sun in the same plane
- The planetary orbits are circular
- The transfer ellipse is only affected by the Sun's gravity

$$V_D = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)} = \sqrt{\mu_{Sun}\left(\frac{2}{R_1} - \frac{1}{(R_1 + R_2)/2}\right)}$$

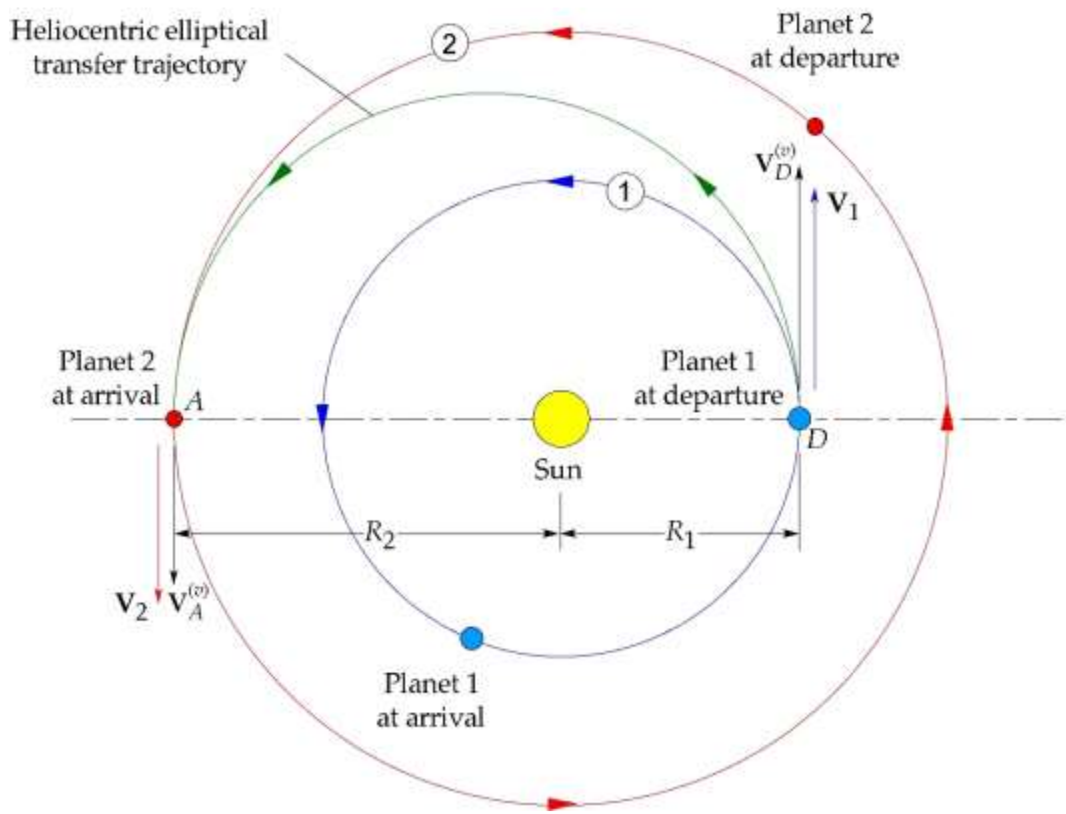
$$V_D = \sqrt{\frac{\mu_{Sun}}{R_1}}\sqrt{\frac{2R_2}{R_1 + R_2}}$$

$$V_1 = \sqrt{\frac{\mu_{Sun}}{R_1}}$$

$$\Delta V_D = V_D - V_1 = \sqrt{\frac{\mu_{Sun}}{R_1}}\sqrt{\frac{2R_2}{R_1 + R_2}} - \sqrt{\frac{\mu_{Sun}}{R_1}}$$

$$\Delta V_D = \sqrt{\frac{\mu_{Sun}}{R_1}}\left(\sqrt{\frac{2R_2}{R_1 + R_2}} - 1\right)$$

Likewise, $\Delta V_A = \sqrt{\frac{\mu_{Sun}}{R_2}}\left(1 - \sqrt{\frac{2R_1}{R_1 + R_2}}\right)$



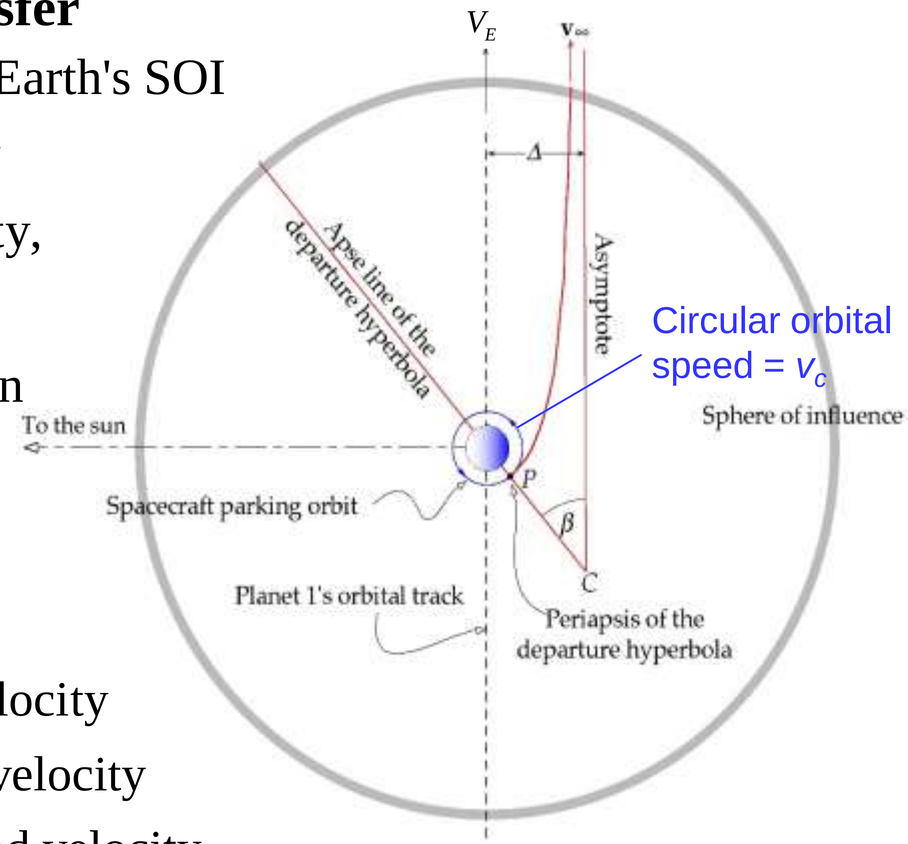
# Hyperbolic Departure

**Example: Earth-to-Mars Hohmann Transfer**

- Departure is a hyperbolic trajectory in the Earth's SOI
- Departure is parallel to the Earth's velocity

  Define $V_P \equiv$ Required Perihelion Velocity,

  Then $V_P = V_E + v_\infty$
- Transfer to Planet 2 is Elliptical around Sun
- Arrival is hyperbolic relative to Planet 2
- Trajectory begins in a circular Earth orbit




Define the following terms:

$\mathbf{R}_E, \mathbf{V}_E$ - Heliocentric Earth radius and velocity

$\mathbf{r}$, $\mathbf{v}$ - Geocentric spacecraft position and velocity

$\mathbf{R}$, $\mathbf{V}$ - Heliocentric spacecraft position and velocity

$\boldsymbol{v}_\infty$ - Hyperbolic excess velocity at infinity from Earth

$r_0$ - radius of circular Earth parking orbit

$v_c$ - speed in Earth circular orbit of radius $r_0$

$v_1$ - speed at perigee of Earth hyperbolic trajectory (at $r_0$)

$V_A, V_P$ - speed at aphelion and perihelion of heliocentric transfer ellipse

# Hyperbolic Departure (2)

First compute the transfer orbit velocity at perihelion:

The Earth to Mars transfer orbit has $a_{\text{tr}} = \dfrac{R_E + R_M}{2}$

The equation of energy applied to perihelion

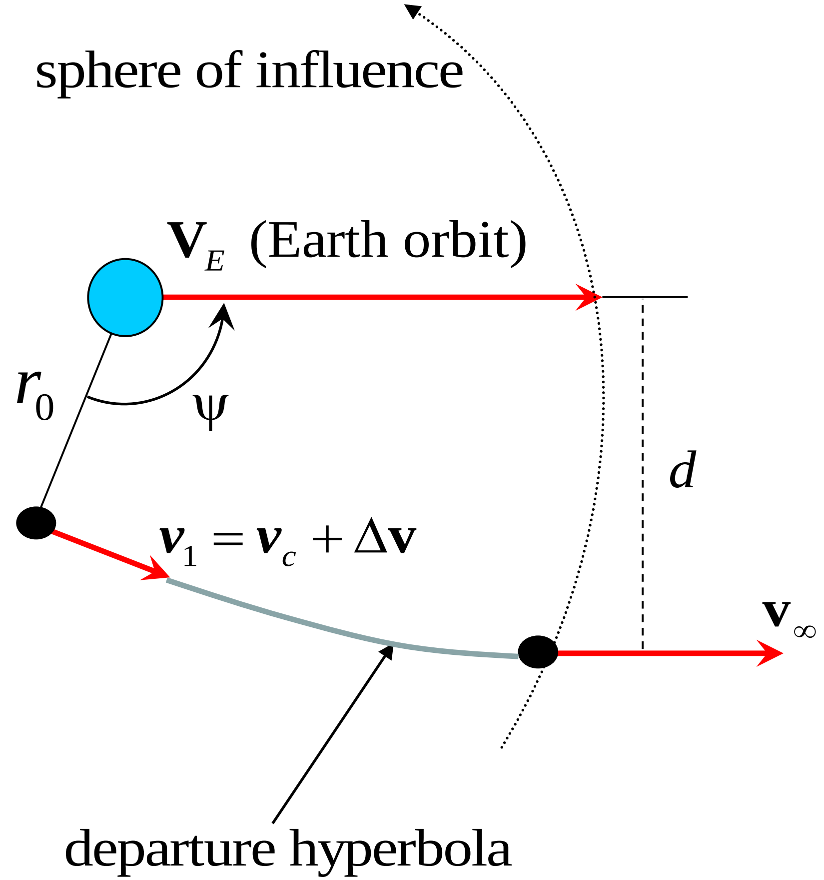
$$\mathcal{E} = \frac{V_P^2}{2} - \frac{\mu_S}{R_E} = -\frac{\mu_S}{2a_{\text{tr}}} = -\frac{\mu_S}{R_E + R_M}$$

So,

$$V_P^2 = \frac{2\mu_S R_M}{R_E\left(R_E + R_M\right)}$$

At sphere of influence (Patch conditions)

$$\begin{cases} \mathbf{R} = \mathbf{R}_E + \mathbf{r} \\ \mathbf{V}_P = \mathbf{V}_E + \mathbf{v}_\infty \end{cases}$$



# Hyperbolic Departure (3)

At the SOI the velocity is $\boldsymbol{v} = \mathbf{V}_P - \mathbf{V}_E = \boldsymbol{v}_\infty$

Comparing energy between perigee and the SOI,

$$\mathcal{E} = \frac{v_1^2}{2} - \frac{\mu_E}{r_0} = \frac{v_{\text{SOI}}^2}{2} - \frac{\mu_E}{r_{\text{SOI}}} \simeq \frac{v_\infty^2}{2}$$

which gives the velocity required at perigee,

$$v_1 = \sqrt{v_\infty^2 + 2\frac{\mu_E}{r_0}}$$

Therefore, the required $\Delta v_1$ is

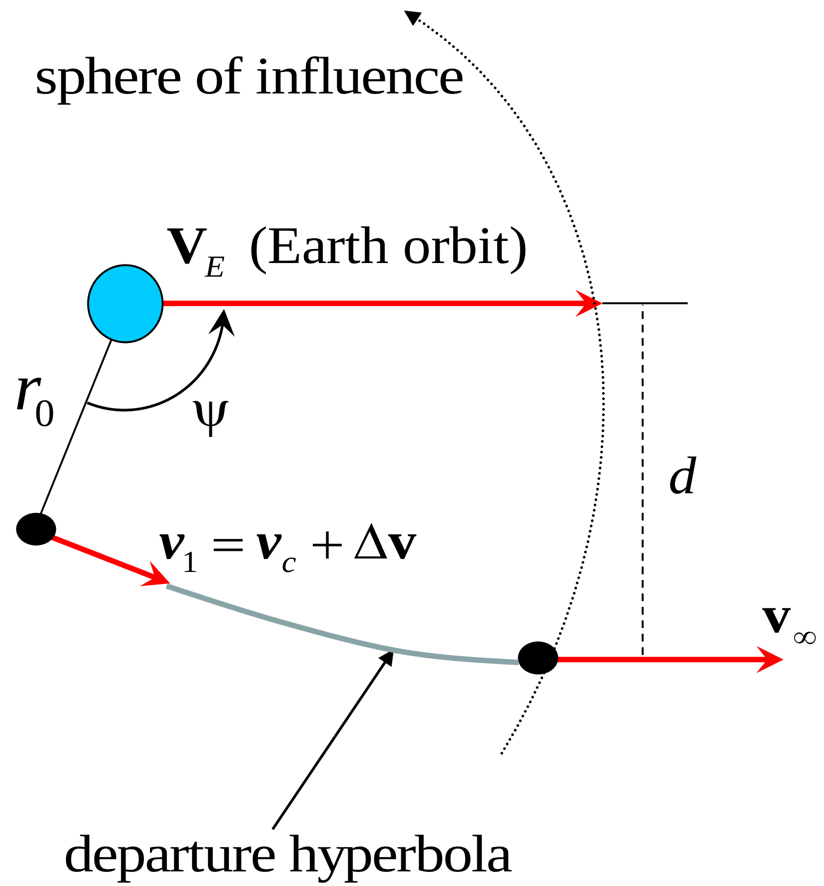
$$\Delta v_1 = v_1 - v_c = \sqrt{v_\infty^2 + 2\frac{\mu_E}{r_0}} - \sqrt{\frac{\mu_E}{r_0}}$$

To determine the position for the burn,

use, $a_{hyperbola} = -\dfrac{\mu_E}{2\varepsilon}$ and $r_0 = a(1-e)$,

compute, $e = 1 - \dfrac{r_0}{a_{hyperbola}} > 1$,

then from $r = \dfrac{p}{1 + e\cos\varphi} = \dfrac{a(1-e^2)}{1 + e\cos\varphi}$

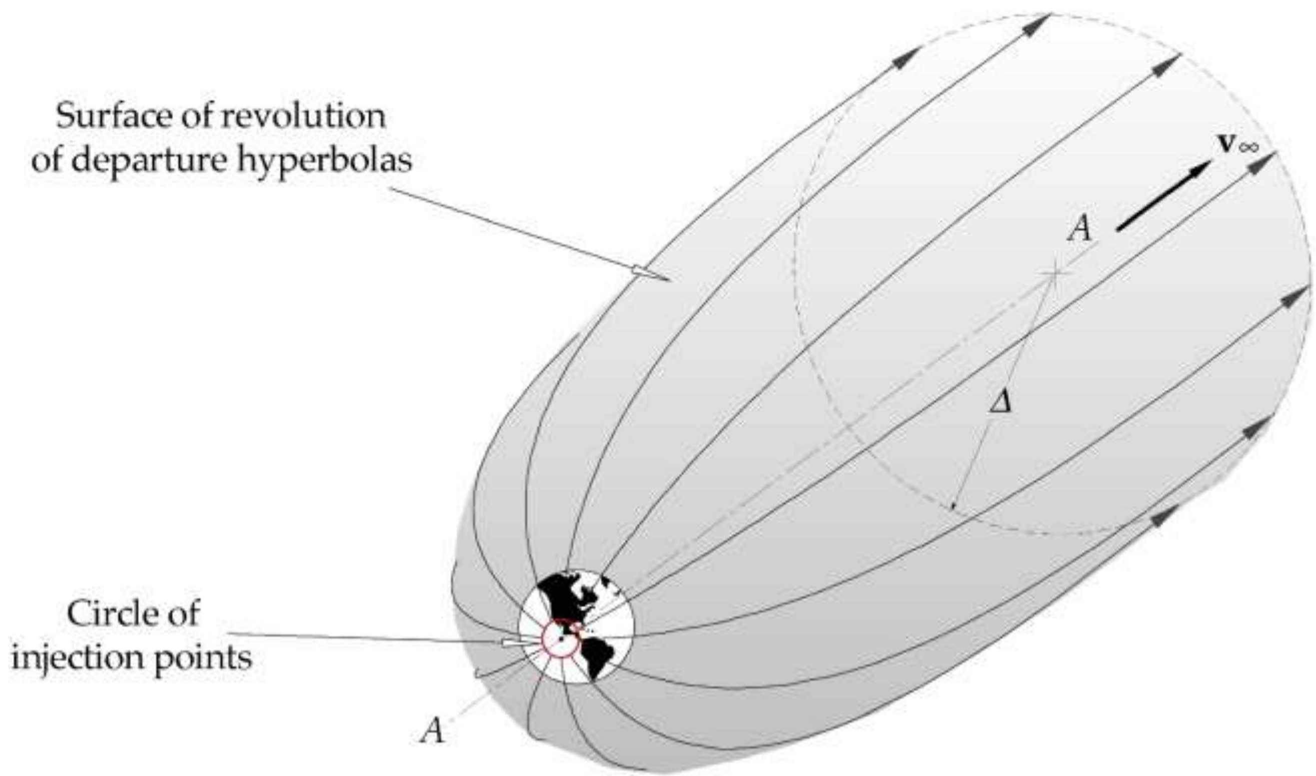



Note that as $r \to r_{\text{SOI}} \simeq \infty$, $(1 + e\cos\varphi) \to 0$ and $\varphi = \psi \to \cos^{-1}\left(-\dfrac{1}{e}\right)$

# Hyperbolic Departure (4)

Any departure velocity parallel to $\boldsymbol{V}_E$ with magnitude $v_\infty$ is valid, so this results in a surface of possible departure hyperbolas.

- Each hyperbolic plane includes the Earth center and has a perigee $r_0$, while prescribes a cone of departure points.

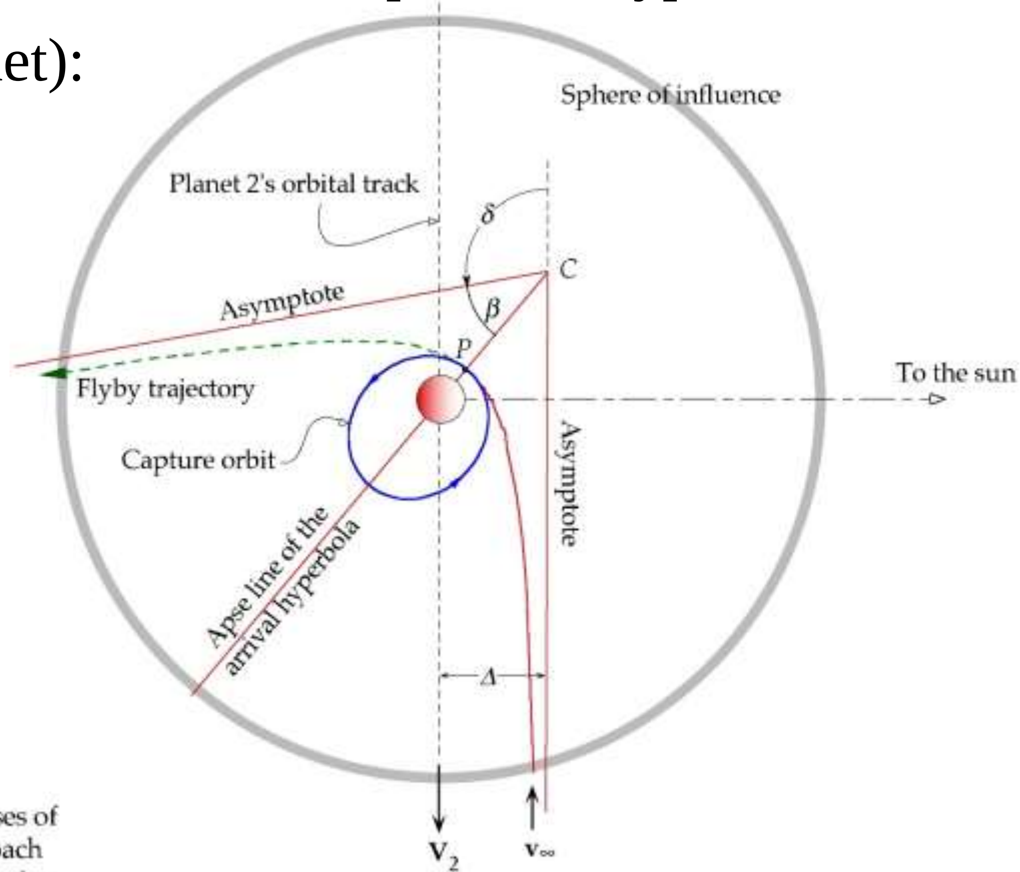


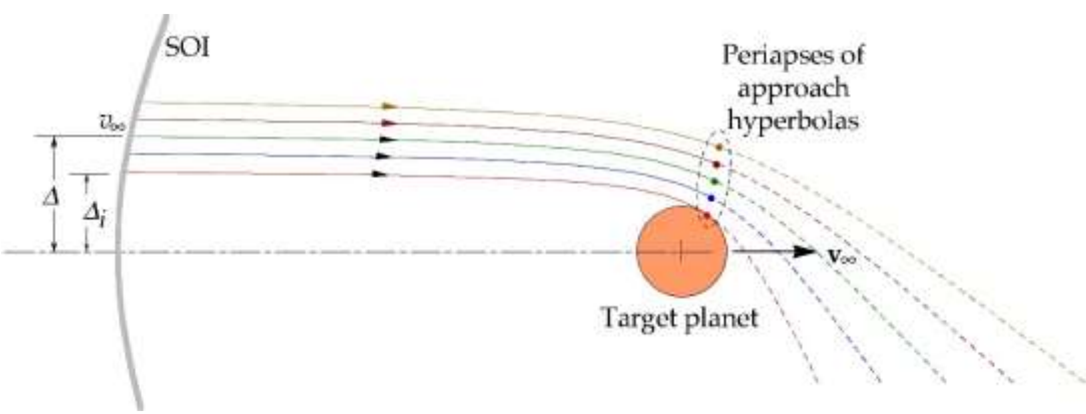
# Hyperbolic Arrival

Arrival trajectories similarly fall on a surface of possible hyperbolas.

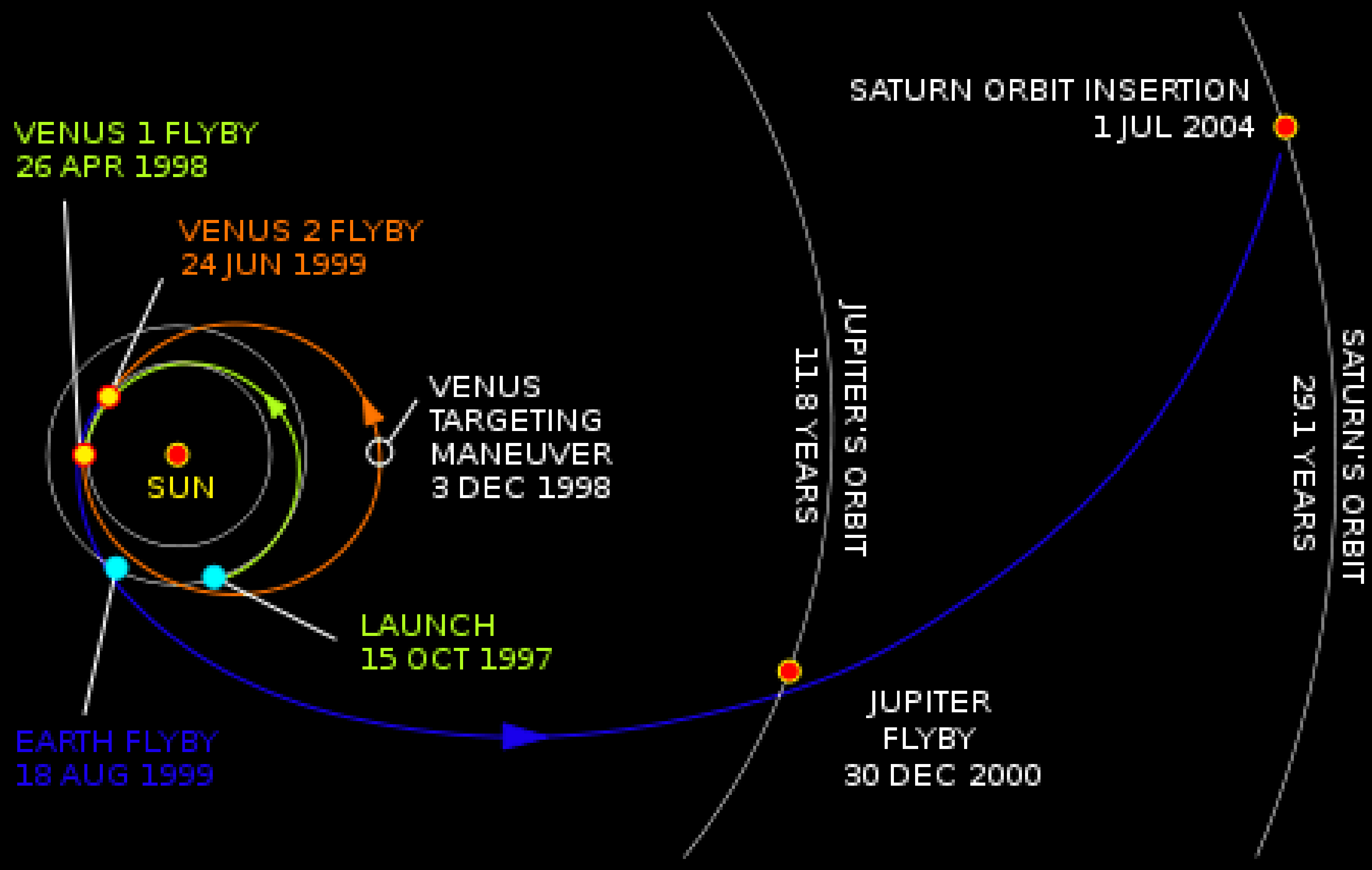
For planetary capture (outer planet):

- a positive $\Delta\boldsymbol{v}$ is required
- rendevous must occur from ahead of the planet!
- a burn is required at periapsis for capture
- The $\Delta\boldsymbol{v}$ required depends on the periapsis radius $r_p$

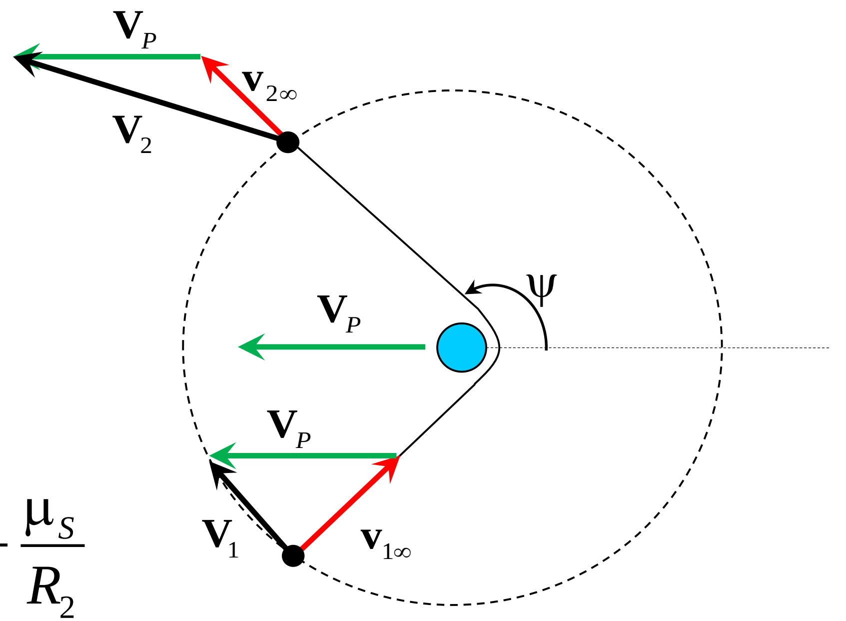


# Gravity Assist



# Gravity Assist Maneuvers




Trailing side flyby increases velocity

Leading side flyby decreases velocity

$$\mathbf{V}_1 = \mathbf{V}_P + \boldsymbol{v}_{1\infty} \quad \text{and} \quad \mathbf{V}_2 = \mathbf{V}_P + \boldsymbol{v}_{2\infty}$$

$$|\boldsymbol{v}_{1\infty}| = |\boldsymbol{v}_{2\infty}|$$

Consider the energy gain

$$\text{(before)} \;\; \mathcal{E}_1 = \frac{1}{2}V_1^2 - \frac{\mu_S}{R_1}, \quad \text{(after)} \;\; \mathcal{E}_2 = \frac{1}{2}V_2^2 - \frac{\mu_S}{R_2}$$

$$R_1 \approx R_2$$

$$\Delta\mathcal{E} = \frac{1}{2}\left(V_2^2 - V_1^2\right) = \frac{1}{2}[(V_p^2 + 2\mathbf{V}_p \cdot \boldsymbol{v}_{2\infty} + v_{2\infty}^2) - (V_p^2 + 2\mathbf{V}_p \cdot \boldsymbol{v}_{1\infty} + v_{1\infty}^2)]$$

$$|\boldsymbol{v}_{1\infty}| = |\boldsymbol{v}_{2\infty}| \quad \rightarrow \quad v_{1\infty}^2 = v_{2\infty}^2$$

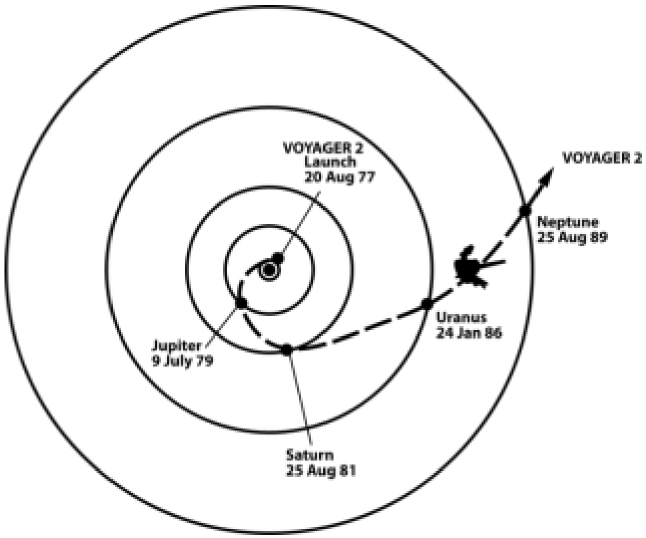
The energy gained through gravity assist is:

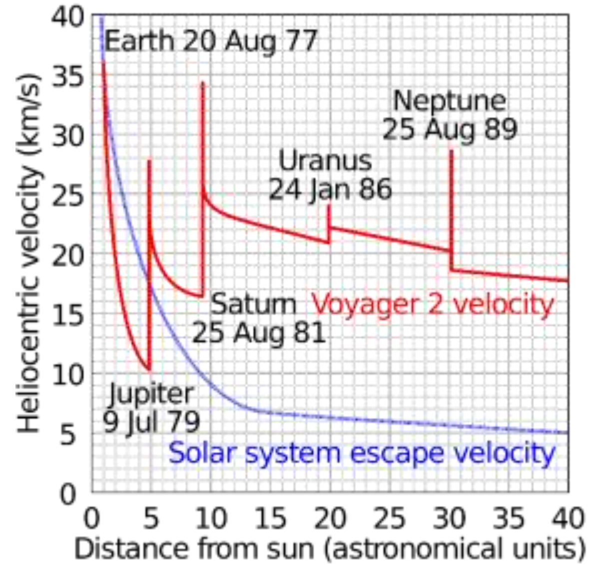
$$\Delta\mathcal{E} = \mathbf{V}_p \cdot \left(\boldsymbol{v}_{2\infty} - \boldsymbol{v}_{1\infty}\right) = 2V_p v_\infty \cos(\pi - \psi)$$

$\psi$ = true anomaly at SOI (or at $\infty$), as $\psi \uparrow, \Delta\mathcal{E} \uparrow$

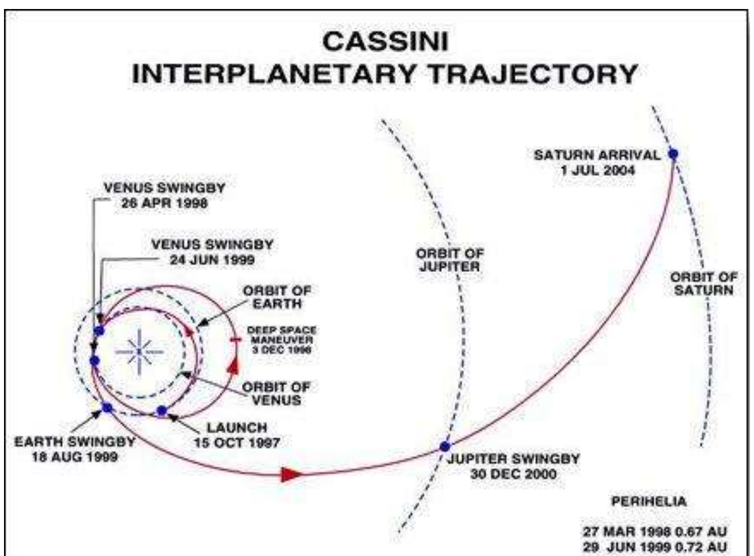
# Gravity Assist Maneuvers (2)

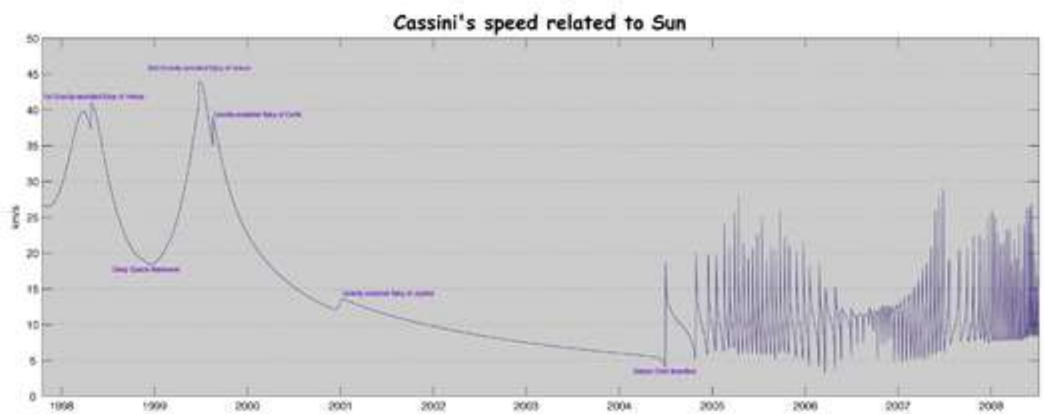
## Voyager 2




## Cassini



# Farewell