STAT 6201 - Mathematical Statistics

Recall: permutations

$$P_{n,k} = n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

↳ # of ways of picking k elements from a set of n without replacement when order matters

Counting

Suppose that a club consists of 25 members and that a president and a secretary are to be chosen from the membership. Determine the total possible number of ways in which these two positions can be filled.

$P_{25,2} = 25(24) = 600$

Suppose that six different books are to be arranged on a shelf. What is the number of possible permutations of the books ?

$6! = P_{6,6} = 720$

Counting

$\nearrow$ $(1)\left(\frac{5}{6}\right)\left(\frac{4}{6}\right)\left(\frac{3}{6}\right)$

If four dice are rolled, what is the probability that each of the four numbers that appear will be different?

How many total outcomes are there? $6^4 = 1296$

How many outcomes correspond to the event of interest (here, 4 distinct #s)?

$P_{6,4} = 6(5)(4)(3) = 360$

prob $= 360/1296 = 0.2778$

Suppose that three runners from team A and three runners from team B participate in a race. If all six runners have equal ability and there are no ties, what is the probability that the three runners from team A will finish first, second, and third, and the three runners from team B will finish fourth, fifth, and sixth?

How many possible orders are there for the finishers? $6! = 720$

How many ways are there for the first 3 from team A + the 2nd 3 from team B? $3!3!$

prob $= \frac{3!3!}{6!} = \frac{36}{720} = \frac{1}{20}$

Counting

of distinct sets of size k selected from n elements

Combination of n elements taken k at a time:

"n choose k" $$C_{n,k} = \boxed{\binom{n}{k}} = \frac{n!}{(n-k)!k!} = \frac{P_{n,k}}{k!}$$

This number is also called *the binomial coefficient*.

Divide by the # of orderings of the k elements.

But here, order doesn't matter

Ex. (a, b) + (b, a)

↳ these are 2 different permutations, but they are the same set of elements.

The binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Note:

$x=1, y=1 \Rightarrow 2^n = \sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n}$

$x=-1, y=1 \Rightarrow 0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k (1)^{n-k} = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} \cdots (-1)^n \binom{n}{n}$

Counting

Suppose that a fair coin is to be tossed 10 times. Determine

- the probability p of obtaining exactly three heads and
- the probability p' of obtaining three or fewer heads.

Total # of possible outcomes $= 2^{10}$
that correspond to exactly 3 heads $= \binom{10}{3}$

$$p = \frac{\binom{10}{3}}{2^{10}} = 0.1172 = \frac{\frac{10!}{7!\,3!}}{2^{10}}$$

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$$P(3 \text{ or fewer } H) = P(0\,H) + P(1\,H) + P(2\,H) + P(3\,H)$$
$$= \frac{\binom{10}{0}}{2^{10}} + \frac{\binom{10}{1}}{2^{10}} + \frac{\binom{10}{2}}{2^{10}} + \frac{\binom{10}{3}}{2^{10}}$$
$$= 0.1719 = p'$$

Counting

Suppose that a class contains 15 boys and 30 girls, and that 10 students are to be selected at random for a special assignment. Determine the probability p that exactly three boys will be selected.

How many ways can we pick 10 students from 45? $\binom{45}{10}$

How many ways can we pick 3 boys + 7 girls? $\binom{15}{3}\binom{30}{7}$

$$p = \frac{\binom{15}{3}\binom{30}{7}}{\binom{45}{10}} = 0.2904$$

Counting

Suppose that a deck of 52 cards containing four aces is shuffled thoroughly and the cards are then distributed among four players so that each player receives 13 cards. Determine the probability that each player will receive one ace.

Method 1:

# of ways to place 4 aces in the deck of 52 cards = $\binom{52}{4}$

$\overline{1}\ \overline{2}\ \overline{3}\ \cdots\ \overline{52}$

# of ways to place the aces so that each player gets one = 13^4

prob = $\dfrac{13^4}{\binom{52}{4}} = 0.1055$

# of ways to order 52 cards = $52!$

How many orderings give one ace per person? $13^4(4!)(48!)$

prob = $\dfrac{13^4(4!)(48!)}{52!}$

4 aces, 48 non-aces = $\dfrac{\binom{4}{1}\binom{48}{12}}{\binom{52}{13}}$

Multinomial coefficients

Suppose that 20 members of an organization are to be divided into three committees A, B, and C in such a way that each of the committees A and B is to have eight members and committee C is to have four members. Determine the number of different ways in which members can be assigned to these committees. Notice that each of the 20 members gets assigned to one and only one committee.

$$\underbrace{\binom{20}{8}}_{\text{\# of ways of picking the 8 members of Committee A}}\underbrace{\binom{12}{8}}_{\text{\# of ways of picking B}}\binom{4}{4} = \frac{20!}{8!\cancel{12!}}\frac{\cancel{12!}}{8!\cancel{4!}}\frac{\cancel{4!}}{0!4!} = \frac{20!}{8!8!4!}$$

Multinomial coefficients

If n distinct elements are to be divided into k different groups $k \geq 2$ in such a way that the jth group contains exactly n_j elements, where $n_1 + n_2 + \cdots + n_k = n$, the total number of different ways of doing that is

$$\binom{n}{n_1,\ n_2,\ n_3,\ \ldots,\ n_k} = \frac{n!}{n_1!n_2!\ldots n_k!}$$

This number is called the *multinomial coefficient*.

Multinomial Thoerem

$$(x_1 + x_2 + \cdots + x_k)^n = \sum \binom{n}{n_1,\ n_2,\ n_3,\ \ldots,\ n_k} x_1^{n_1} x_2^{n_2} \ldots x_k^{n_k}$$

DNA Sequences: A, C, G, T

Species 1: ACCGTT$\boxed{\text{A}}$AG
Species 2: ACCGTC$\boxed{\text{A}}$AC
Species 3: ACTGTC$\boxed{\text{T}}$TC

<u>Q</u> How many patterns for 4 species are there with the same letter? 4

Sp 1: A
Sp 2: A
Sp 3: A
Sp 4: A

$(xxxx)$, $x \in \{A, C, G, T\}$

How many with pattern $xxyz$, $x \neq y \neq z$, $x, y, z \in \{A, C, G, T\}$ $\underline{\underline{24}}$

$xxyy$? 12

Now, suppose I get DNA sequences with 3 $xxxx$, 4 $xxyy$, 2 $xxyz$

<u>Q</u> How many different DNA sequences are there like this?

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$$\binom{9}{3}\binom{6}{4}\binom{2}{2} = \frac{9!}{3!\,4!\,2!}$$