Grade Seven

As students enter seventh grade, they have an understanding of variables and how to apply properties of operations to write and solve simple one-step equations. They are fluent in all positive rational number operations. Students have been introduced to ratio concepts and applications, concepts of negative rational numbers, absolute value, and all four quadrants of the coordinate plane. Students have a solid foundation for understanding area, surface area, and volume of geometric figures and have been introduced to statistical variability and distributions (Adapted from The Charles A. Dana Center Mathematics Common Core Toolbox 2012).

WHAT STUDENTS LEARN IN GRADE SEVEN

[Note: Sidebar]

Grade Seven Critical Areas of Instruction

In grade seven instructional time should focus on four critical areas: (1) developing understanding of and applying proportional relationships, including percentages; (2) developing understanding of operations with rational numbers and working with expressions and linear equations; (3) solving problems involving scale drawings and informal geometric constructions and working with two- and three-dimensional shapes to solve problems involving area, surface area, and volume; and (4) drawing inferences about populations based on samples. (CCSSO 2010, Grade 7 Introduction).

Students also work towards fluently solving equations of the form $px + q = r$ and $p(x + q) = r$.

Grade Seven Standards for Mathematical Content

The Standards for Mathematical Content emphasize key content, skills, and practices at each grade level and support three major principles:

- **Focus:** Instruction is focused on grade level standards.
- **Coherence:** Instruction should be attentive to learning across grades and should link major topics within grades.
- **Rigor:** Instruction should develop conceptual understanding, procedural skill and fluency, and application.

Grade level examples of focus, coherence and rigor will be indicated throughout the chapter.

Not all of the content in a given grade is emphasized equally in the standards. Cluster headings can be viewed as the most effective way to communicate the **focus and coherence** of the standards. Some clusters of standards require a greater instructional emphasis than the others based on the depth of the ideas, the time that they take to master, and/or their importance to future mathematics or the later demands of college and career readiness.

The following Grade 7 Cluster-Level Emphases chart highlights the content emphases in the standards at the cluster level for this grade. The bulk of instructional time should be given to "Major" clusters and the standards within them. However, standards in the "Supporting" and "Additional" clusters should not be neglected. To do so will result in gaps in students' learning, including skills and understandings they may need in later grades. Instruction should reinforce topics in major clusters by utilizing topics in the supporting and additional clusters. Instruction should include problems and activities that support natural connections between clusters.

Teachers and administrators alike should note that the standards are not topics to be checked off a list during isolated units of instruction, but rather content to be developed throughout the school year through rich instructional experiences and presented in a coherent manner. (Adapted from the Partnership for Assessment of Readiness for College and Careers [PARCC] 2012).

[**Note**: The Emphases chart should be a graphic inserted in the grade level section. The explanation "key" needs to accompany it.]

Grade 7 Cluster-Level Emphases

Ratios and Proportional Relationships

- **[m]:** Analyze proportional relationships and use them to solve real-world and mathematical problems. (**7.RP.1-3▲**)

The Number System

- [**m**]: Apply and extend previous understandings of operations with fractions to add, subtract, multiply, and divide rational numbers. (**7.NS.1-3▲**)

Expressions and Equations

- [**m**]: Use properties of operations to generate equivalent expressions. **(7.EE.1-2▲)**
- [**m**]: Solve real-life and mathematical problems using numerical and algebraic expressions and equations. **(7.EE.3-4▲)**

Geometry

- [**a/s**]: Draw, construct and describe geometrical figures and describe the relationships between them. **(7.G.1-3)**
- [**a/s**]: Solve real-life and mathematical problems involving angle measure, area, surface area, and volume. **(7.G.4-6)**

Statistics and Probability

- [**a/s**]: Use random sampling to draw inferences about a population[1]. **(7.SP.1-2)**
- [**a/s**]: Draw informal comparative inferences about two populations[2]. **(7.SP.3-4)**
- [**a/s**]: Investigate chance processes and develop, use, and evaluate probability models. **(7.SP.5-8)**

Explanations of Major, Additional and Supporting Cluster-Level Emphases
Major[3] [m] (▲) clusters – areas of intensive focus where students need fluent understanding and application of the core concepts. These clusters require greater emphasis than the others based on the depth of the ideas, the time that they take to master, and/or their importance to future mathematics or the demands of college and career readiness.
Additional [a] clusters – expose students to other subjects; may not connect tightly or explicitly to the major work of the grade **Supporting [s]** clusters – rethinking and linking; areas where some material is being covered, but in a way that applies core understanding; designed to support and strengthen areas of major emphasis. *A Note of Caution: Neglecting material will leave gaps in students' skills and understanding and will leave students unprepared for the challenges of a later grade.

(Adapted from Smarter Balanced Assessment Consortia [Smarter Balanced], DRAFT Content Specifications 2012).

Connecting Mathematical Practices and Content

[1] *The standards in this cluster represent opportunities to apply percentages and proportional reasoning. In order to make inferences about a population, one needs to apply such reasoning to the sample and the entire population.*

[2] *Probability models draw on proportional reasoning and should be connected to the major work in those standards.*

[3] The ▲symbol will indicate standards in a Major Cluster in the narrative.

The Standards for Mathematical Practice (MP) are developed throughout each grade and, together with the content standards, prescribe that students experience mathematics as a coherent, useful, and logical subject that makes use of their ability to make sense of mathematics. The MP standards represent a picture of what it looks like for students to understand and do mathematics in the classroom and should be integrated into every mathematics lesson for all students.

Although the description of the MP standards remains the same at all grades, the way these standards look as students engage with and master new and more advanced mathematical ideas does change. Below are some examples of how the MP standards may be integrated into tasks appropriate for Grade 7 students. (Refer to pages 9–13 in the Overview of the Standards Chapters for a complete description of the MP standards.)

Standards for Mathematical Practice (MP)

Explanations and Examples for Grade Seven

Standards for Mathematical Practice	Explanation and Examples
MP.1 Make sense of problems and persevere in solving them.	In grade seven, students solve problems involving ratios and rates and discuss how they solved them. Students solve real-world problems through the application of algebraic and geometric concepts. Students seek the meaning of a problem and look for efficient ways to represent and solve it. They may check their thinking by asking themselves, “Does this make sense?” or “Can I solve the problem in a different way?” When students compare arithmetic and algebraic solutions to the same problem (7.EE.4a), they are identifying correspondences between different approaches.
MP.2 Reason abstractly and quantitatively.	Students represent a wide variety of real-world contexts through the use of real numbers and variables in mathematical expressions, equations, and inequalities. Students contextualize to understand the meaning of the number or variable as related to the problem and decontextualize to manipulate symbolic representations by applying properties of operations.
MP.3 Construct viable arguments and critique the reasoning of others.	Students construct arguments using verbal or written explanations accompanied by expressions, equations, inequalities, models, and graphs, and tables. They further refine their mathematical communication skills through mathematical discussions in which they critically evaluate their own thinking and the thinking of other students. For example, as students notice when given geometric conditions determine a unique triangle, more than one triangle or no triangle (7.G.2), they have an opportunity to construct viable arguments and critique the reasoning of others. Students should be encouraged to answer questions, such as “How did you get that?”, “Why is that true?” and “Does that always work?”

MP.4 Model with mathematics.	Seventh-grade students model real-world situations symbolically, graphically, in tables, and contextually. Students form expressions, equations, or inequalities from real-world contexts and connect symbolic and graphical representations. Students use experiments or simulations to generate data sets and create probability models. Proportional relationships present opportunities for modeling. For example, the number of people who live in an apartment building might be taken as proportional to the number of stories in the building for modeling purposes. Students should be encouraged to answer questions, such as “What are some ways to represent the quantities?” or “How might it help to create a table, chart, graph…?”
MP.5 Use appropriate tools strategically.	Students consider available tools (including estimation and technology) when solving a mathematical problem and decide when certain tools might be helpful. For instance, students in grade seven may decide to represent similar data sets using dot plots with the same scale to visually compare the center and variability of the data. Students might use physical objects, spreadsheets, or applets to generate probability data and use graphing calculators or spreadsheets to manage and represent data in different forms. Teachers might ask, “What approach are you considering trying first?” or “Why was it helpful to use…?”
MP.6 Attend to precision.	Students continue to refine their mathematical communication skills by using clear and precise language in their discussions with others and in their own reasoning. Students define variables, specify units of measure, and label axes accurately. Students use appropriate terminology when referring to rates, ratios, probability models, geometric figures, data displays, and components of expressions, equations, or inequalities. Teachers might ask “What mathematical language, definitions, properties…can you use to explain…?
MP.7 Look for and make use of structure.	Students routinely seek patterns or structures to model and solve problems. For instance, students recognize patterns that exist in ratio tables making connections between the constant of proportionality in a table with the slope of a graph. Students apply properties to generate equivalent expressions and solve equations. Students compose and decompose two- and three-dimensional figures to solve real-world problems involving scale drawings, surface area, and volume. Students examine tree diagrams or systematic lists to determine the sample space for compound events and verify that they have listed all possibilities. Solving an equation such as $8 = 4\left(x - \frac{1}{2}\right)$ is easier if students can see and make use of structure, temporarily viewing $\left(x - \frac{1}{2}\right)$ as a single entity.
MP.8 Look for and express regularity in repeated reasoning.	In grade seven, students use repeated reasoning to understand algorithms and make generalizations about patterns. After multiple opportunities to solve and model problems, they may notice that $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$ and construct other examples and models that confirm their generalization. Students should be encouraged to answer questions, such as “How would we prove that…?” or “How is this situation like and different from other situations using this operations?”

(Adapted from Arizona Department of Education [Arizona] 2012, North Carolina Department of Public Instruction [N. Carolina] 2012, and Georgia Department of Education [Georgia] 2011)

Standards-based Learning at Grade Seven

The following narrative is organized by the domains in the Standards for Mathematical Content and highlights some necessary foundational skills from previous grades. It provides exemplars to explain the content standards, highlight connections to the various Standards for Mathematical Practice (**MP**), and demonstrate the importance of developing conceptual understanding, procedural skill and fluency, and application. A triangle symbol (**▲)** indicates standards in the major clusters (refer to the Grade 7 Cluster-Level Emphases table on page 2).

Domain: Ratio and Proportional Relationships

A critical area of instruction in grade seven is developing an understanding and application of proportional relationships, including percentages. In grade seven, students extend their reasoning about ratios and proportional relationships in several ways. Students use ratios in cases that involve pairs of rational number entries, and they compute associated rates. They identify unit rates in representations of proportional relationships. They work with equations in two variables to represent and analyze proportional relationships. They also solve multi-step ratio and percent problems, such as problems involving percent increase and decrease.

(The University of Arizona Progressions Documents for the Common Core Math Standards [Progressions] 6-7 Ratios and Proportional Relationships [RP] 2011).

Ratios and Proportional Relationships	7.RP

Analyze proportional relationships and use them to solve real-world and mathematical problems.

1. Compute unit rates associated with ratios of fractions, including ratios of lengths, areas and other quantities measured in like or different units. *For example, if a person walks 1/2 mile in each 1/4 hour, compute the unit rate as the complex fraction* $^{1/2}/_{1/4}$ *miles per hour, equivalently 2 miles per hour.*
2. Recognize and represent proportional relationships between quantities.
   a. Decide whether two quantities are in a proportional relationship, e.g., by testing for equivalent ratios in a table or graphing on a coordinate plane and observing whether the graph is a straight line through the origin.
   b. Identify the constant of proportionality (unit rate) in tables, graphs, equations, diagrams, and verbal descriptions of proportional relationships.
   c. Represent proportional relationships by equations. *For example, if total cost t is proportional to the number n of items purchased at a constant price p, the relationship*

between the total cost and the number of items can be expressed as t = pn.

d. Explain what a point *(x, y)* on the graph of a proportional relationship means in terms of the situation, with special attention to the points (0, 0) and (1, *r)* where *r* is the unit rate.

Important in grade seven is the concept of the unit rate associated with a ratio. For a ratio $a:b$ with $a, b \neq 0$,[4] the *unit rate* is the *number* $\frac{a}{b}$. In sixth grade, students worked primarily with ratios involving whole number quantities. In addition, they discovered what it meant to have *equivalent ratios*. In grade seven, students will find unit rates in ratios involving fractional quantities (**7.RP.1▲**). For example, when a recipe calls for $1\frac{1}{2}$ cups of sugar for 3 cups of flour, this results in a unit rate of $\frac{1\frac{1}{2}}{3} = \frac{3}{6}$. The fact that any pair of quantities in a proportional relationship can be divided to find the unit rate will be useful when solving problems involving proportional relationships, as this will allow students to set up an equation with equivalent fractions and use reasoning about equivalent fractions to solve them. For a simple example, if we made a recipe with the same ratio as given above using 12 cups of flour and wanted to know how much sugar to use, we could set up an equation that sets *unit rates* equal to each other, such as $\frac{1\frac{1}{2}}{3} = \frac{S}{12}$, where S represents the number of cups needed in the recipe.

In grade six, students worked with many examples of proportional relationships and represented them numerically, pictorially, graphically, and with equations in simple cases. In grade seven, students continue this work, but they examine more closely the characteristics of proportional relationships. In particular, they begin to identify:

- When proportional quantities are represented in a table, pairs of entries represent equivalent ratios.
- The graph of a proportional relationship lies on a straight line that passes through the point (0,0) (indicating that when one quantity is 0, so is the other).[5]

[4] While it is possible to define *ratio* so that a can be zero, this will rarely happen in context, and so the discussion proceeds assuming both a and b are non-zero.

[5] The formal reasoning behind this principle and the next one is not expected until grade eight (see 8.EE.B). But students will notice and informally use both principles in grade seven.

- Equations of proportional relationships in a ratio of $a: b$ always take the form $y = k \cdot x$, where k is the constant $\frac{b}{a}$ if the variables x and y are defined so that the ratio $x: y$ is equivalent to $a: b$. (The number k is also known as the *constant of proportionality*). (**7.RP.2▲**).

Thus a first, and often overlooked, step is for students to decide when and why two quantities are actually in a proportional relationship (7.RP.2a▲). They can do this by checking the characteristics listed above, or by using reasoning (e.g., a runner's heart rate might increase steadily as he runs faster, but his heart rate when he is standing still is not 0 beats per minute, hence running speed and heart rate are not proportional).

[Note: Sidebar]

A ***ratio*** is a pair of non-negative numbers, $A: B$, which are not both 0. When there are A units of one quantity for every B units of another quantity, a ***rate*** associated with the ratio $A: B$ is $\frac{A}{B}$ units of the first quantity per 1 unit of the second quantity. (Note that the two quantities may have different units.)

The associated ***unit rate*** is the number $\frac{A}{B}$. The term *unit rate* refers to the numerical part of the rate; the "unit" is used to highlight the 1 in "per 1 unit of the second quantity." Unit rates should not be confused with unit fractions (which have a 1 in the numerator).

A ***proportional relationship*** is a collection of pairs of numbers that are in equivalent ratios. A ratio $A: B$ with $B \neq 0$ determines a proportional relationship, namely the collection of pairs $(cA; cB)$, for c positive. A proportional relationship is described by an equation of the form $y = kx$, where k is a positive constant, often called a *constant of proportionality*. The constant of proportionality, k, is equal to the value $\frac{B}{A}$. The graph of a proportional relationship lies on a ray with endpoint at the origin.

(Adapted from Progressions 6-7 RP 2011).

Examples: Determining Proportional Relationships.

1. If Josh is 20 and his niece Reina is 10, how old will Reina be when Josh is 40?

Solution: If students erroneously think that this is a proportional relationship, they may decide that Reina will be 20 when Josh is 40. However, it is not true that their ages change in a ratio of $20: 10$ (or $2: 1$). As Josh ages 20 years so does Reina. She would be 30. Students might further investigate this situation by graphing ordered pairs (a, b) where a is Josh's age and b is Reina's age at the same time. How does the graph differ from a graph of a proportional relationship?

2. If it takes 2 people 5 hours to paint a fence, does it take 4 people 10 hours to paint a fence of the same size?

Solution: No. (At least, not so if we assume they all work at the same rate.) If *more* people are contributing to the work, then they can paint the fence faster. This is not a proportional relationship.

3. If it costs $4.50 for 2 pounds of melon at the grocery store, would 7 pounds cost $15.75?

Solution: Since typically at a grocery store, price per pound is constant, it stands to reason that there is a proportional relationship. Since

$$\frac{\$4.50}{2 \text{ pounds}} = \frac{7 \times (\$4.50)}{7 \times (2 \text{ pounds})} = \frac{\$31.50}{14 \text{ pounds}} = \frac{(\$31.50) \div 2}{(14 \text{ pounds}) \div 2} = \frac{\$15.75}{7 \text{ pounds}},$$

it makes sense that 7 pounds would cost $15.75. (Alternatively, the unit rate is $\frac{\$4.50}{2} = \$2.25$, for a rate of $2.25 per pound. At that rate, 7 pounds costs $7 \times \$2.25 = 7 \times \$2 + 7 \times \$0.25$ (thinking mentally) = $14 + (4 quarters) + (3 quarters) = $14 + $1 + $0.75, or $15.75.)

4. The table gives the price for different numbers of books. Do the numbers in the table represent a proportional relationship?

No. of Books	Price ($)
1	4
2	7
3	10
4	13

Solution: If there were a proportional relationship, we should be able to make equivalent ratios using entries from the table. But since the ratios $1: 4$ and $2: 7$ are not equivalent, the table does not represent a proportional relationship. (Also, the value of the first ratio is $\frac{1}{4}$ or 0.25, and the value of the second ratio is $\frac{2}{7}$ or about 0.28.)

(Adapted from Arizona 2012 and North Carolina Department of Public Instruction [N. Carolina] 2012)

[Note: Sidebar]

Focus, Coherence, and Rigor:

Proportional relationship problems support mathematical practices as students make sense of problems (**MP.1**), reason abstractly and quantitatively (**MP.2**), and model proportional relationships (**MP.4**). For example, the number of people who live in an apartment building might be taken as proportional to the number of stories in the building for modeling purposes. (Adapted from PARCC 2012).

As students work with proportional relationships, they write equations of the form $y = kx$, where k is a constant of proportionality (i.e., a unit rate.). They see this unit rate as the amount of increase in y as x increases by 1 unit in a ratio table, and they recognize the unit rate as the vertical increase in a “unit rate triangle” (or “slope triangle”) with a horizontal side of length 1 for a graph of a proportional relationship.

Example: Representing Proportional Relationships. To contrast between grade six and grade seven work with proportional reasoning, the same example from grade 6 is presented here but from a grade seven perspective.

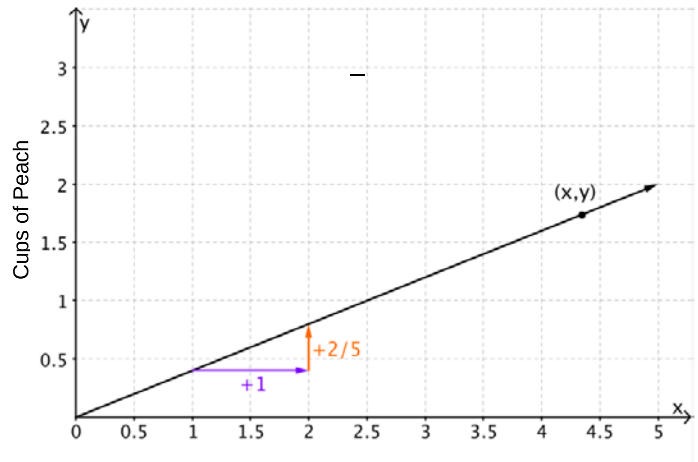
A juice mixture calls for 5 cups of grape juice for every 2 cups of peach juice. Use a table to represent several different batches of juice that could be made according to this recipe. Graph the data in your table on a coordinate plane. Finally, write an equation to represent the relationship between cups of grape juice and cups of peach juice in any batch of juice made according to the recipe. Identify the unit rate in each of the three representations of the proportional relationship.

Using a Table: In grade seven, students identify pairs of values that include fractions as well as whole numbers. Thus, they include fractional amounts between 5 cups of grape juice and 2 cups of peach juice in their tables. They see that as amounts of cups of grape juice increase by 1 unit, the corresponding amounts of cups of peach juice increase by $\frac{2}{5}$ unit, so that if we add x cups of grape juice, then we would add $x \cdot \frac{2}{5}$ cups of peach juice. Seeing this relationship will help students see the resulting equation $y = \frac{2}{5}x$. Another way to derive the equation is by seeing $\frac{y}{x} = \frac{2}{5}$, and so multiplying each side by x would yield

	x cups of grape	**y cups of peach**
Batch A	(0)	(0)
Batch B	5	2
Batch C	1	$\frac{2}{5}$
Batch D	2	$2 \cdot \frac{2}{5}$
Batch E	3	$3 \cdot \frac{2}{5}$
Batch F	4	$4 \cdot \frac{2}{5}$
Any batch made according to the recipe	x	$x \cdot \frac{2}{5}$

$$x \cdot \left(\frac{y}{x}\right) = \left(\frac{2}{5}\right) \cdot x \quad \text{which results in} \quad y = \frac{2}{5}x.$$

Using a Graph: Students create a graph, realizing that even non-whole number points represent possible combinations of grape and peach juice mixtures. They are learning to identify key features of the graph, in particular, that the resulting graph is a ray (i.e., is contained in a straight line) that emanates from the origin, representing the fact that the point $(0,0)$ is also part of the data. They see the point $\left(1,\frac{2}{5}\right)$ as the point that corresponds to the unit rate, and they see that every positive horizontal movement of 1 unit (e.g., adding 1 cup of grape juice), results in a positive vertical movement of $\frac{2}{5}$ of a unit (e.g., adding $\frac{2}{5}$ cup of peach juice). The connection between this rate of change as seen in the graph and the equation $y=\frac{2}{5}x$ should be made explicit for students, and they should test that indeed, every point on the graph is of the form $\left(x,\frac{2}{5}x\right)$.



Deriving an Equation: Both the table and the graph show that for every 1 cup of grape juice added, $\frac{2}{5}$ cup of peach juice is added. Thus, starting with an empty bowl, when x cups of grape juice are added, $\frac{2}{5}x$ cup of peach juice must be added. On the graph, this corresponds to the fact that starting from $(0,0)$, every movement horizontally of x units results in a vertical movement of $\frac{2}{5}x$ units. The equation in either case becomes $y=\frac{2}{5}x$.

(Adapted from Progressions 6-7 RP 2011)

Students also solve problems involving proportional relationships, using a variety of methods. They should have opportunities to solve proportional relationship problems using a variety of strategies such as making tape diagrams and double number lines, using tables, using rates, and by relating proportional relationships to equivalent fractions as described above.

Examples of Proportional Reasoning in Grade 7. Janet can sew 35 costumes in 2 hours. At this rate, how many costumes can she sew in 5 hours?

Solution Strategies:

(a) Using the Rate. Since she can sew 35 costumes in 2 hours, this means she can sew at a rate of $35 \div 2 = 17.5$ costumes per hour. Since she'll be working for 5 hours, we find she can sew

$$17.5 \text{ costumes per hour } \times 5 \text{ hours} = 87.5 \text{ costumes},$$

which we interpret as meaning she can sew 87 costumes in 5 hours.

(b) Setting Unit Rates Equal. The unit rate in this case is $\frac{35}{2} = 17.5$. If we let C be the number of costumes she can sew in 5 hours, then we can set up the equation

$$\frac{\text{number of costumes}}{\text{number of hours}} = \frac{35}{2} = \frac{C}{5},$$

since $\frac{C}{5}$ also represents the unit rate. To solve this, we can reason that since $2 \times 2.5 = 5$, it must be true that $35 \times 2.5 = C$, giving $C = 87.5$, which we interpret as saying that she can sew 87 costumes in 5 hours. Alternately, one can see the equation above as being of the form $b = ax$, where a, b are rational numbers. In that case, $C = \frac{35}{2} \div \frac{1}{5}$.

(c) Recognizing an Equation. Students can reason that an equation that relates the number of costumes C and the number of hours h takes the form $C = 17.5h$, so that C can be found by $C = 17.5(5) = 87.5$. Again, we interpret the answer as saying that she can sew 87 costumes in 5 hours.

Typically, solving proportional reasoning problems has been simplified to "setting up a proportion and cross-multiplying." The standards move away from this strategy, and towards strategies in which students reason about solutions and why they work. Setting up an equation to solve a proportional relationship problem makes perfect sense if students understand that they are setting unit rates equal to each other. However, introducing the term "proportion" (or "proportion equation") can needlessly clutter up the curriculum; rather, students should simply see this as setting up an equation in a single variable. On the other hand, if after solving multiple problems by reasoning with equivalent fractions (as in strategy (b) above) students begin to see the pattern that $\frac{a}{b} = \frac{c}{d}$ precisely when $ad = bc$, then this is something to be examined (rather than avoided) and used as a general strategy provided students can justify why they use it. Below are some more examples of multiple ways to solve proportional reasoning problems.

More Examples of Proportional Reasoning in Grade 7.
1. A truck driver averaged about 300 miles in the last 5.5 hours he drove. About how much more driving time does it take him to drive the remaining 1000 miles on his route? **Solution**: Students might see the unit rate as $\frac{300}{5.5}$, and set up the following equation: $$\frac{300}{5.5} = \frac{1000}{h},$$ where h represents the number of driving hours to go the remaining 1000 miles. Students might see that $1000 \div 300 = \frac{10}{3}$, so that it must also be true that $h \div 5.5 = \frac{10}{3}$. This means that $$h = \frac{10}{3} \times 5.5 = \frac{10}{3} \times \frac{11}{2} = \frac{110}{6} = 18\frac{1}{3}.$$ This means the truck driver has around 18 hours and 20 minutes of driving time remaining.
2. If $\frac{1}{2}$ gallon of paint covers $\frac{1}{6}$ of a wall, then how much paint is needed to cover the entire wall? **Solution:** Students may see this as asking for the rate, i.e., how much paint is needed per 1 wall. In that case, students would divide: $$\frac{1}{2} \div \frac{1}{6} = \frac{1}{2} \cdot \frac{6}{1} = 3,$$ so that 3 gallons of paint covers the entire wall.
3. The recipe for Perfect Purple paint mix calls for mixing $\frac{1}{2}$ cup blue paint with $\frac{1}{3}$ cup red paint. If you wanted to mix blue and red paint in the same ratio to make 20 cups of Perfect Purple paint, how many cups of blue paint and how many cups of red paint will you need? **Solution: (Strategy 1)** "If I make 6 batches of purple, then that means I use 6 times as much blue and red paint. That means I used $6 \cdot \frac{1}{2} = 3$ cups of blue and $6 \cdot \frac{1}{3} = 2$ cups of red. This makes a total of 5 cups of purple paint (i.e., 6 batches yields 5 cups). So to make 20 cups, I can multiply these amounts of blue and red by 4, to get 12 cups of blue and 8 cups of red." **(Strategy 2)** One batch of Perfect Purple is $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$ cup in volume. The fraction of one batch that is blue is then $\frac{1}{2} \Big/ \frac{5}{6} = \frac{1}{2} \div \frac{5}{6} = \frac{1}{2} \cdot \frac{6}{5} = \frac{6}{10}$. The fraction of one batch that is red is $\frac{1}{3} \Big/ \frac{5}{6} = \frac{1}{3} \div \frac{5}{6} = \frac{1}{3} \cdot \frac{6}{5} = \frac{6}{15}$. If I find these fractions of 20, that gives me how much blue and red to use: $$\frac{6}{10} \cdot 20 = 12 \text{ and } \frac{6}{15} \cdot 20 = 8.$$ This means I need 12 cups of blue and 8 cups of red.

(Adapted from Progressions 6-7 RP 2011).

Ratios and Proportional Relationships 7.RP

Analyze proportional relationships and use them to solve real-world and mathematical problems.

3. Use proportional relationships to solve multistep ratio and percent problems. *Examples: simple interest, tax, markups and markdowns, gratuities and commissions, fees, percent increase and decrease, percent error.*

In grade six, students used ratio tables and unit rates to solve percent problems. In grade seven, students extend their work to solve multi-step ratio and percent problems (**7.RP.3▲**). They explain or show their work using a representation (e.g., numbers, words, pictures, physical objects, or equations) and verify that their answers are reasonable. Models help students identify parts of the problem and how values are related **(MP. 1, MP.3 and MP.4).** For percentage increase and decrease, students identify the original value, determine the difference, and compare the difference in the two values to the starting value.

Examples: Multi-Step Percent Problems.

1. A sweater is marked down 30%. Its original price was \$37.50. What is the price of the water after it is marked down?

Solution: A simple diagram like the one shown can help students see the relationship between the original price, the amount taken off, and the sale price of the sweater. In this case, students can solve the problem either by finding 70% of \$37.50, or by finding 30% of \$37.50 and subtracting it.

37.50 Original price of sweater	
30% of 37.50	70% of 37.50 Sale price of sweater

Seeing many examples of problems such as this one can allow students to see discount problems as taking the form $(100\% - r\%) \cdot p = d$, where r is the amount of reduction, p is the original price and d is the discounted price.

2. A shirt is on sale for 40% off. The sale price is \$12. What was the original price?

Solution: Again, a simple diagram can show the relationship between the sale price and the original price. In this case, what is known is the sale price, \$12, which

Discount 40% of original price	Sale Price - \$12 60% of original price
Original Price (p)	

represents 60% of the original price. In that case, we can set up a simple equation $0.6p = 12$, and solve for p: $p = 12 \div .6 = 20$. The original price was $20.

3. Your bill at a restaurant before tax is $52.60. The sales tax is 8%. You decide to leave a tip of 20% on the pre-tax amount. How much is the tip you'll leave? What is the total cost of dinner, including tax and tip?

Solution: To calculate the tip, students find $52.60 \cdot .2 = 10.52$, so the tip is $\$10.52$. The tax is found similarly: $52.60 \cdot .08 \approx 4.21$. This means the total bill is $\$52.60 + \$10.52 + \$4.21 = \$67.33$. Alternately, students can realize they are finding 128% of the pre-tax bill, and compute $\$52.60 \cdot 1.28 \approx \$67.33$.

(Adapted from Arizona 2012 and N. Carolina 2012)

Problems involving percentage increase or percentage decrease require careful attention to the referent whole. For example, consider the difference in these two problems:

> Skateboard problem 1. After a 20% discount, Eduardo paid $140 for a SuperSick skateboard. What was the price before the discount?

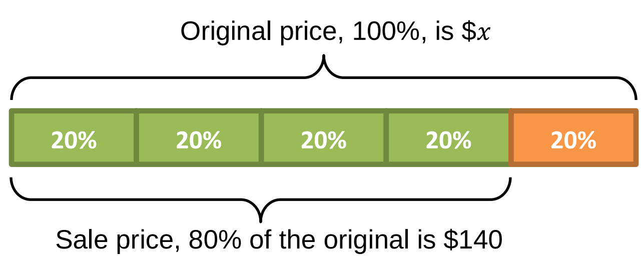
> Skateboard problem 2. A SuperSick skateboard costs $140 now, but its price will go up by 20%. What will the new price be after the increase?

The following solutions to these two problems are different because the 20% refers to different wholes (or 100% amounts). In the first problem, the 20% is 20% of the larger pre-discount amount, whereas in the second problem, the 20% is 20% of the smaller pre-increase amount.

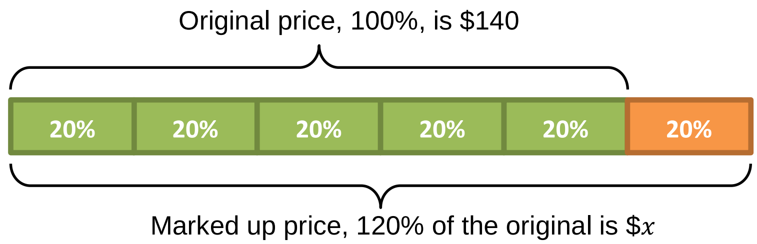
Solutions to Skateboard Problems:

Skateboard Problem 1. We can represent the problem with a tape diagram. Students can reason that since 80% is $140, 20% is $140 ÷ 4 = $35, so that 100% is then 5 × $35 = $175.

Equivalently, $.80 \cdot x = 140$, so that $x = 140 \div .8$, or $x = 140 \div \frac{4}{5} = 140 \times \frac{5}{4} = 35 \times 5 = 175$.



Skateboard Problem 2. We can represent the problem with a tape diagram. Students can reason that since 100% is $140, 20% is $140 ÷ 5 = $28, so that 120% is then 6 × $28 = $168. Equivalently, $x = (1.2)(140)$, so that $x = 168$.



(Adapted from Progressions 6-7 RP 2011).

A detailed discussion of ratios and proportional relationships is provided online at Draft 6–7 Progression on Ratios and Proportional Relationships (Progressions 6-7 RP 2011).

Following is an example of a classroom activity/task that connects the Standards for Mathematical Content to the Standards for Mathematical Practice.

Connecting to the Standards for Mathematical Practice – Grade 7

Standards	***Explanations and Examples***
Students are expected to: **7.EE.3**: Solve multi-step real-life and mathematical problems posed with positive and negative rational numbers in any form (whole numbers, fractions, and decimals), using tools strategically. Apply properties of operations to calculate with numbers in any form; convert between forms as appropriate; and assess the reasonableness of answers using mental computation and estimation strategies. *For example: If a woman making $25 an hour gets a 10% raise, she will make an additional* $^1/_{10}$ *of her salary an hour, or $2.50, for a new salary of $27.50. If you want to place a towel bar 9* $^3/_4$ *inches long in the center of a door that is 27* $^1/_2$ *inches wide, you will need to place the bar about 9 inches from each edge; this estimate can be used as a check on the exact computation.*	**Sample Problem:** Julie sees a jacket that costs $32 before a sale. During the sale, prices on all items are reduced by 25%. 1. What is the cost of the jacket during the sale? In the second week of the sale, all prices are reduced by 25% of the previous week's price. In the third week of the sale, prices are again reduced by 25% of the previous week's price. In the fourth week of the sale, prices are again reduced by 25% of the previous week's price. 2. Julie thinks that this means her jacket will be reduced to $0 at the end of the fourth week. Why might she think this and why is she wrong? 3. If Julie decides to buy the jacket at the end of the fourth week, then how much will she pay for it? **Classroom Connections:** Teachers can assess students' basic understanding of percentages and percent-off concepts with Question 1 above. However, when students are asked to reason why Julie is incorrect in thinking that the jacket will cost $0, since 4 × 25% = 100%, they are required to understand that the number that we are taking 25% of changes each week. The concept of what is **the whole** comes into play here, where in each situation involving percentages, ratios, or fractions, what constitutes the whole, unit, 1, 100% is important. Finally, the third question challenges students to compute the correct cost of the jacket, by either successively subtracting 0.25 times the new price, or even by multiplying successively by 0.75. The equivalence of these two methods can be explained in this problem situation. **Connecting to the Standards for Mathematical Practice:** (MP.2) Students must reason quantitatively with regards to percentages and should be able to flexibly compute with the given numbers in various forms. (MP.3) Students can argue with their peers in a class discussion environment why four reductions of 25% do not constitute a total reduction of the original price by 100%. Moreover, students can explain to each other how to do the problem correctly and the teacher can bring out student misconceptions about percentages. (MP.5) Students apply percentages correctly and use percentage reductions correctly.

Domain: The Number System

In grade six, students completed their understanding of division of fractions and achieved fluency with multi-digit division and multi-digit decimal operations. They also worked with concepts of positive and negative rational numbers. They learned about signed numbers and what kinds of quantities they can be used to represent. They located signed numbers on a number line. As a result of this study, students should have come away thinking of the negative side of the number line as being the mirror reflection of the positive side. For example, by reasoning that the reflection of a reflection is the thing itself, they will have learned that $-(-a) = a$. (Here a may be positive, negative, or zero.) Grade six students also learned about absolute value and ordering of rational numbers, including in real-world contexts. In grade seven, a critical area of instruction is developing an understanding of operations with rational numbers. Seventh grade students extend addition, subtraction, multiplication, and division to all rational numbers by applying these operations to both positive and negative numbers.

Adding, subtracting, multiplying, and dividing rational numbers is the culmination of numerical work with the four basic operations. The number system will continue to develop in grade eight, expanding to become the real numbers by the introduction of irrational numbers. Because there are no specific standards for rational number arithmetic in later grades and because so much other work in grade seven depends on rational number arithmetic, fluency with rational number arithmetic should be the goal in grade seven (Adapted from PARCC 2012).

[Note: Sidebar]

The rational numbers are an arithmetic system that includes 0 as well as positive and negative whole numbers and fractions. Wherever the term "rational numbers" is used, numbers of all types are implied, including fractions in decimal notation.

The Number System 7.NS

Apply and extend previous understandings of operations with fractions to add, subtract, multiply, and divide rational numbers.

1. Apply and extend previous understandings of addition and subtraction to add and subtract rational numbers; represent addition and subtraction on a horizontal or vertical number line diagram.
   a. Describe situations in which opposite quantities combine to make 0. *For example, a hydrogen atom has 0 charge because its two constituents are oppositely charged.*
   b. Understand $p + q$ as the number located a distance $|q|$ from p, in the positive or negative direction depending on whether q is positive or negative. Show that a number and its opposite have a sum of 0 (are additive inverses). Interpret sums of rational numbers by describing real-world contexts.
   c. Understand subtraction of rational numbers as adding the additive inverse, $p – q = p + (–q)$. Show that the distance between two rational numbers on the number line is the absolute value of their difference, and apply this principle in real-world contexts.
   d. Apply properties of operations as strategies to add and subtract rational numbers.

Previously in grade six, students learned that the absolute value of a rational number is its distance from zero on the number line. In grade seven, students represent addition and subtraction with positive and negative rational numbers on a horizontal or vertical number line diagram **(7.NS.1 a-c▲)**. Students add and subtract, understanding $p + q$ as the number located a distance $|q|$ from p on a number line, in the positive or negative direction, depending on whether q is positive or negative. They demonstrate that a number and its opposite have a sum of 0 (i.e. they are additive inverses), and they understand subtraction of rational numbers as adding the additive inverse. (**MP.2, MP.4,** and **MP.7**)

Students' work with signed numbers began in grade six, where they experienced situations in which positive and negative numbers represented (for example) credits or debits to an account, positive or negative charges, or increases or decreases, all relative to a 0. Now, students realize that in each of these situations, a positive quantity and negative quantity of the same absolute value add to make 0 **(7.NS.1 a▲)**. For instance, the positive charge of 5 protons would neutralize the negative charge of 5 electrons, and we represent this as: [6]

$$(+5) + (-5) = 0.$$

Students recognize that $+5$ and -5 are "opposites" as described in grade six, located the same distance from 0 on a number line. But they reason further here that opposites have the relationship that a number a and its opposite $-a$ always combine to make 0:

$$a + (-a) = 0.$$

This crucial fact lays the foundation for understanding addition and subtraction of signed numbers.

While for simplicity many of the examples to follow involve integers, students' work with rational numbers should include rational numbers in different forms—positive and negative fractions, decimals, and whole numbers (including combinations). Integers might be used to introduce the ideas of signed number operations, but student work and practice should not be limited to integer operations. If students learn to compute $4 + (-8)$ but not $4 + \left(-\frac{1}{3}\right)$, then they are not learning the rational number system.

Addition of Rational Numbers

Through experiences starting with whole numbers and their opposites (i.e., starting with integers only), students can develop the understanding that "like" quantities can be combined. That is, two positive quantities combine to become a "more positive" quantity, as in, $(+5) + (+7) = +12$, while two negative quantities combine to become a "more negative" quantity, as in, $(-2) + (-10) = -12$. When addition problems have mixed signs, students see that positive and negative quantities combine as necessary to partially make zeros (i.e., they "cancel" each other), and the appropriate amount of positive or negative charge remains:

Examples: Adding Signed Rational Numbers. (*Note*: The "neutral pair" approach in these examples is

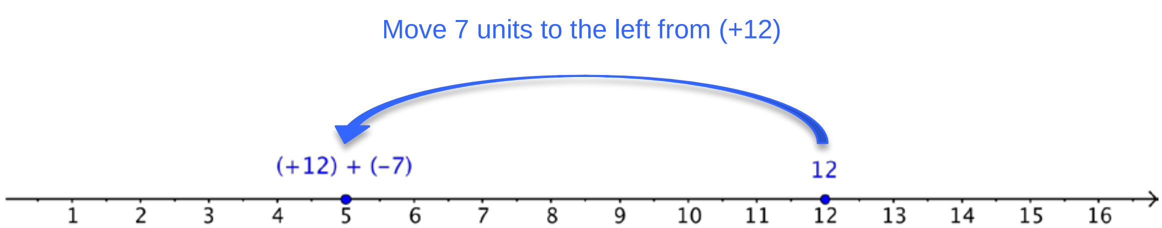
[6] Teachers may wish to temporarily include the plus sign (+) to indicate positive numbers and distinguish them clearly in problems. They should eventually be dropped, as they are not commonly used.

meant to show where the answer comes from; it is not meant to be an efficient algorithm for adding rational numbers.

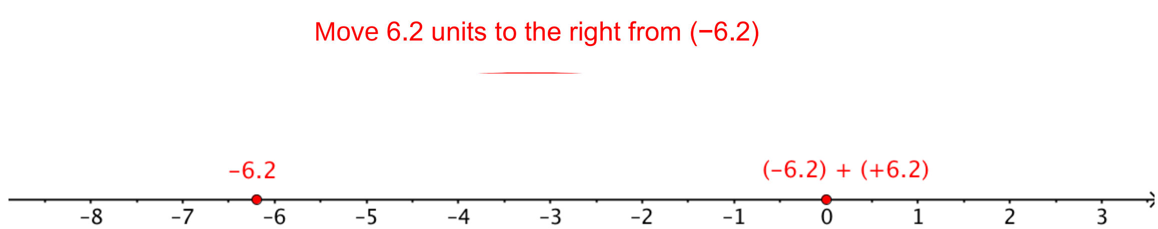
1. $(+12) + (-7) = (+5) + (+7) + (-7) = (+5) + (0) = +5$
2. $(-13.55) + (+10.50) = (-2.05) + (-10.50) + (+10.50) = (-2.05) + (0) = -2.05$
3. $\left(+\frac{17}{2}\right) + \left(-\frac{9}{2}\right) = \left(+\frac{8}{2}\right) + \left(+\frac{9}{2}\right) + \left(-\frac{9}{2}\right) = \left(+\frac{8}{2}\right) + (0) = +\frac{8}{2} = +4$

Eventually, students come to realize that when adding two numbers with different signs, the sum is equal to the number with absolute value equal to the positive difference of the two numbers and with the same sign as the number with the larger absolute value. This understanding eventually replaces the kinds of calculations shown above, which are meant to show students the concepts, not to serve as a practical computation method.

When students represent adding integers on a number line, they can develop a general understanding that the sum $p + q$ is the number found when moving a total of $|q|$ units from p to the right if q is positive, and to the left if q is negative **(7.NS.1 b▲)**. For example, for $(+12) + (-7)$:



This is particularly transparent for quantities that combine to become 0, e.g., with $(-6.2) + (+6.2) = 0$:



Subtraction of Rational Numbers

The key idea for students when it comes to subtracting rational numbers is to come to understand $p - q$ as giving the same result as $p + (-q)$ (i.e., subtracting q is equivalent to adding the opposite of q). Students have most likely already noticed that with sums such as $10 + (-2)$, the result was the same as finding the difference, $10 - 2$. In addition, where helpful, teachers can employ typical understandings of subtraction as "taking away" or comparing when it comes to subtracting quantities with the same sign, as in $(-12) - (-7)$ meaning to "take away -7 from -12," and compare this with $(-12) + 7$; with an understanding of these numbers as representing negative charges, the answer of -5 is arrived at fairly easily. However, by comparing this subtraction expression with the addition expression $(-12) + 7$, students see that both result in -5. Through many examples, students can generalize these results to understand that $p - q = p + (-q)$ **(7.NS.1.c▲)**.

Examples: Subtracting Signed Rational Numbers.
1. Students interpret $15 - 9$ as taking away 9 positive units from 15 positive units. Students should compare with $15 + (-9)$ to see that they both result in 6. 2. Students interpret $20.5 + (-17.5)$ as a credit and debit example. They compare this with $20.5 - 17.5$, and see they get the same result. 3. Students can use the relationship between addition and subtraction, familiar from previous grades, namely that $a - b = c$ if and only if $c + b = a$. For example, they can use this to reason that since $10 + (-9) = 1$, it must be true that $1 - (-9) = 10$. They compare this with $1 + 9$, and realize both give the same result. 4. Students can see subtraction as comparison, particularly visible on a horizontal or vertical number line. For example, they interpret $9 - (-13)$ as, "how any degrees warmer is a temperature of $9℃$ compared to a temperature of $-13℃$?

Common Concrete Models for Addition and Subtraction of Rational Numbers

There are several concrete models that can be used to represent rational numbers and operations with them. It is important for teachers to understand that all such concrete models have advantages and disadvantages, and so care should be taken when introducing them to students. Not every model will lend itself well to representing every aspect of operations with rational numbers. Some common concrete models are described briefly below.

Common Concrete Models for Representing Signed Rational Numbers (MP.5)

1. **Number Line Model (Vector Model).** In this model, a number line is used to represent the set of all rational numbers, and directed line segments (i.e., vectors, which simply look like arrows) are used to represent numbers. The length of the arrow is the absolute value of the number. The direction of the arrow tells the sign of the number. Thus, the arrow emanating from 0 to -3.5 on the number line represents the number -3.5. Addition is then represented by placing arrows head-to-tail and looking at the number to which the final arrow points. Subtraction is equivalent to adding the opposite, so we can represent $a - b$ by reversing the arrow for b and then adding it to a. This model has the advantage of interpreting multiplication as scaling, e.g., the product $\frac{1}{3}(-3)$ can be interpreted as a vector one-third the length of the vector (-3) in the same direction, that is, $\frac{1}{3}(-3) = -1$.
2. **Colored Chip Model.** In the colored chip model, white or yellow chips are used to represent positive units while red chips are used to represent negative units (sometimes physical plus and minus signs are used). These tools have the advantage that combining units is easy to represent, and are especially illustrative when positive and negative units are combined to create "zero pairs" (sometimes "neutral pairs"), representing that $a + (-a) = 0$. A disadvantage of this model is that multiplication and division are more difficult to represent, and chip models are typically only used to represent integer quantities (i.e., it is difficult to extend them to fractional quantities). Also, some imagination is required in order to view a pile of white and red chips as representing "nothing."
3. **Money Account Models.** Money account balance models can be used for representing addition and subtraction of rational numbers, though such numbers typically take the form of decimal dollar amounts. Positive amounts contribute to the balance while negative amounts subtract from it. Subtracting negatives must be delicately interpreted here, as in thinking of "$-(-\$35.00)$" as, "the bank forgave the negative balance of $\$35.00$," which one would interpret as receiving a credit of $\$35.00$.

[Note: Sidebar]

Focus, Coherence, and Rigor:

Teachers are encouraged to logically build up the rules for operations with rational numbers (**7.NS.1▲**), as modeled in the narratives on addition and subtraction, making use of the structure of the number system **(MP.7)**. Students should engage in class or small group discussions about the meaning of operations until a consensus understanding is reached **(MP.3)**. Building a foundation in using the structure of numbers with addition and subtraction will also help students understand the operations of multiplication and division of signed numbers (**7.NS.2▲**). Sufficient practice is required so that students can compute sums and products of rational numbers in all cases, and also apply these concepts reliably to real-world situations.

The Number System	7.NS

Apply and extend previous understandings of operations with fractions to add, subtract, multiply, and divide rational numbers.

2. Apply and extend previous understandings of multiplication and division and of fractions to multiply and divide rational numbers.
    a. Understand that multiplication is extended from fractions to rational numbers by requiring that operations continue to satisfy the properties of operations, particularly the distributive property, leading to products such as (–1)(–1) = 1 and the rules for multiplying signed numbers. Interpret products of rational numbers by describing real-world contexts.
    b. Understand that integers can be divided, provided that the divisor is not zero, and every quotient of integers (with non-zero divisor) is a rational number. If *p* and *q* are integers, then $-(p/q) = (-p)/q = p/(-q)$. Interpret quotients of rational numbers by describing real world contexts.
    c. Apply properties of operations as strategies to multiply and divide rational numbers.
    d. Convert a rational number to a decimal using long division; know that the decimal form of a rational number terminates in 0s or eventually repeats.
3. Solve real-world and mathematical problems involving the four operations with rational numbers.[1]

Students continue to develop their understanding of operations with rational numbers by seeing that multiplication and division can be extended to signed rational numbers (**7.NS.2▲**). For instance, students can understand that in an account balance model, $(-3)(\$40.00)$ can be thought of as a record of 3 groups of debits (indicated by the negative sign) of $\$40.00$ each, resulting in a total contribution to the balance of $-\$120.00$. In a vector model, students can interpret the expression $(2.5)(-7.5)$ as the vector that points in the same direction as the vector representing -7.5, but is 2.5 times as long. Interpreting multiplication of two negatives in everyday terms can be troublesome, since negative money cannot be withdrawn from a bank. In a vector model, multiplying by a negative number reverses the direction of the vector (in addition to any stretching or compressing of the vector). Division is often difficult to interpret in

[1] Computations with rational numbers extend the rules for manipulating fractions to complex fractions.

everyday terms as well, but can always be understood mathematically in terms of multiplication, specifically as multiplying by the reciprocal.

Multiplication of Signed Rational Numbers

In general, multiplication of signed rational numbers is performed as with fractions and whole numbers, but according to the following rules for determining the sign of the product:

$$\text{(i)} \quad (-a) \times b = -ab,$$

$$\text{(ii)} \quad (-a) \times (-b) = ab.$$

In these equations, both a and b can be positive, negative, or zero. Of particular importance is that $-1 \cdot a = -a$, that is, multiplying a number by negative one gives the opposite of the number. The first of these rules can be understood in terms of models as mentioned above. The second can be understood as being a result of properties of operations (refer to "A Derivation of the Fact that $(-1)(-1) = 1$" below). Students can also become comfortable with rule (ii) by examining patterns in products of signed numbers, such as in the table below, though this does not constitute a valid mathematical proof.

Example: Using Patterns to Investigate Products of Signed Rational Numbers.

Students can investigate a table like the one below. It is natural to conjecture that the missing numbers in the table should be $5, 10, 15,$ and 20 (reading them from left to right).

5×4	5×3	5×2	5×1	5×0	$5 \times (-1)$	$5 \times (-2)$	$5 \times (-3)$	$5 \times (-4)$
20	15	10	5	0	-5	-10	-15	-20
$(-5) \times 4$	$(-5) \times 3$	$(-5) \times 2$	$(-5) \times 1$	$(-5) \times 0$	$(-5) \times (-1)$	$(-5) \times (-2)$	$(-5) \times (-3)$	$(-5) \times (-4)$
-20	-15	-10	-5	0	??	??	??	??

Ultimately, if students come to an understanding that $(-1)(-1) = 1$, then the fact that $(-a)(-b) = ab$ follows immediately using the associative and commutative properties of multiplication:

$$(-a)(-b) = (-1 \cdot a)(-1 \cdot b) = (-1)a(-1)b = (-1)(-1)ab = 1 \cdot ab = ab.$$

After arriving at a general understanding of these two rules for multiplying signed numbers, students can multiply any rational numbers by finding the product of the absolute values of the numbers and then determining the sign according to the rules.

[Note: Sidebar]

A Derivation of the Fact that $(-1)(-1) = 1$.

Students are reminded that addition and multiplication are related by a very important algebraic property, the *distributive property of multiplication over addition:*

$$a(b + c) = ab + ac,$$

valid for all numbers a, b and c. This property plays an important role in the derivation here as it does in all of mathematics. The basis of this derivation is that the *additive inverse* of the number -1 (that is, the number you add to -1 to obtain 0) is equal to 1. We observe that if we add $(-1)(-1)$ and (-1), the distributive property reveals something interesting:

$$(-1)(-1) + (-1) = (-1)(-1) + (-1)(1) \quad \text{(since } (-1) = (-1)(1)\text{)}$$

$$= (-1)[(-1) + 1] \quad \text{(by the distributive property)}$$

$$= (-1) \cdot 0 = 0 \quad \text{(since } (-1) + 1 = 0.\text{)}$$

Thus, when adding the quantity $(-1)(-1)$ to -1, the result is 0. This implies that $(-1)(-1)$ is none other than the additive inverse of -1, or in other words, 1. This completes the derivation.

Division of Rational Numbers

The relationship between multiplication and division allows students to infer the sign of the quotient of two rational numbers. Otherwise, division is performed as usual with whole numbers and fractions, with the sign to be determined.

Examples: Determining the Sign of a Quotient.

If $x = (-16) \div (-5)$, then $x \cdot (-5) = -16$. It follows that whatever the value of x is it must be a positive

number. In this case, $x = \frac{-16}{-5} = \frac{16}{5}$. This line of reasoning can be used to justify the general fact that for rational numbers p and q, with $q \neq 0$, $\frac{-p}{-q} = \frac{p}{q}$.
If $y = \frac{-0.2}{50}$, then $y \cdot 50 = -0.2$. This implies that y must be negative, and we have that $y = \frac{-0.2}{50} = -\frac{2}{500} = -\frac{4}{1000} = -0.004$.
If $z = \frac{0.2}{-50}$, then $z \cdot (-50) = 0.2$. This implies that z must be negative, and we have that $z = \frac{0.2}{-50} = -\frac{2}{500} = -\frac{4}{1000} = -0.004$.

The latter two examples above show that $\frac{-0.2}{50} = \frac{0.2}{-50}$. In general, it is true that $\frac{-p}{q} = \frac{p}{-q}$ for rational numbers (with $q \neq 0$). Students often have trouble interpreting the expression $-\left(\frac{p}{q}\right)$. To begin with, we should interpret this as meaning "the opposite of the number $\frac{p}{q}$." Considering a specific example, notice that since $-\left(\frac{5}{2}\right) = -(2.5)$ is a negative number, the product of 4 and $-\left(\frac{5}{2}\right)$ must also be a negative number. We determine that $4 \cdot \left(-\left(\frac{5}{2}\right)\right) = -10$. On the other hand, this equation implies that $-\left(\frac{5}{2}\right) = -10 \div 4$, in other words, that $-\left(\frac{5}{2}\right) = \frac{-10}{4} = \frac{-5}{2}$. A similar line of reasoning shows that $-\left(\frac{5}{2}\right) = \frac{5}{-2}$. Examples like these help justify that $-\left(\frac{p}{q}\right) = \frac{-p}{q} = \frac{p}{-q}$ **(7.NS.2.b▲)**.[7]

Students solve real-world and mathematical problems involving positive and negative rational numbers while learning to compute sums, differences, products, and quotients of rational numbers. They also come to understand that every rational number can be written as a decimal with an expansion that eventually repeats or terminates (i.e., eventually repeats with 0s). **(7.NS.2c-d, 7.NS.3▲)** (**MP.1, MP.2, MP.5, MP.6, MP.7, MP.8**)

Examples of Rational Number Problems.
1. During a business call, Marion was told of the most recent transactions in the business account.

[7] Incidentally, this also shows why it is unambiguous to write $-\frac{p}{q}$, and drop the parenthesis.

There were deposits of $\$1{,}250$ and $\$3040.57$, three withdrawals of $\$400$, and the bank removed two penalties to the account of $\$35$ that were the bank's errors. How much did the balance of the account change based on this report?

Solution: The deposits are considered positive changes to the account, the three withdrawals are considered negative changes, and the two removed penalties of $\$35$ can be considered as subtracting debits to the account. One might represent the total change to the balance as:

$$\$1{,}250.00 + \$3{,}040.57 - 3(\$400.00) - 2(-\$35.00) = \$3{,}160.57.$$

Thus, the balance of the account increased altogether by $\$3{,}160.57$

2. Find the product $(-373) \cdot 8$.

Solution: "I know that the first number has a factor of (-1) in it, so that the product will be negative. So now I just need to find $373 \cdot 8 = 2400 + 560 + 24 = 2984$. So $(-373) \cdot 8 = -2984$."

3. Find the quotient $\left(-\frac{25}{28}\right) \div \left(-\frac{5}{4}\right)$.

Solution: "I know that the result is a positive number. This looks like a problem where I can just divide numerator and denominator: $\frac{25}{28} \div \frac{5}{4} = \frac{25 \div 5}{28 \div 4} = \frac{5}{7}$. The quotient is $\frac{5}{7}$."

4. Represent each problem by a diagram, a number line, and an equation. Solve each problem. (a) A weather balloon is 100,000 feet above sea level, and a submarine is 3 miles below sea level. How far apart are the submarine and the weather balloon? (b) John was $3.75 in debt, and Mary was $0.50 ahead. John found an envelope with some money in it, and after that he had the same amount of money as Mary. How much was in the envelope?

Domain: Expressions and Equations

In grade six students began the study of equations and inequalities and methods for solving them. In grade seven students build on this understanding and use the arithmetic of rational numbers as they formulate expressions and equations in one variable and use these equations to solve problems. Students also work toward fluently solving equations of the form $px + q = r$ and $p(x + q) = r$.

Expressions and Equations	7.EE
Use properties of operations to generate equivalent expressions. 1. Apply properties of operations as strategies to add, subtract, factor, and expand linear expressions with rational coefficients.	

2. Understand that rewriting an expression in different forms in a problem context can shed light on the problem and how the quantities in it are related. *For example, a + 0.05a = 1.05a means that "increase by 5%" is the same as "multiply by 1.05."*

In this cluster of standards, students hone their skills working with linear expressions, where the distributive property plays a prominent role (**7.EE.1▲**). A fundamental understanding is that the distributive property works "on the right" as well as "on the left," in addition to "forwards" as well as "backwards." That is, students should have opportunities to see that for numbers $a, b,$ and c and $x, y,$ and z:

$$a(b+c) = ab + ac \quad \text{and} \quad ab + ac = a(b+c)$$
$$(x+y)z = xz + yz \quad \text{and} \quad xz + yz = (x+y)z$$

Students combine their understanding of parentheses as denoting single quantities with the standard order of operations, operations with rational numbers, and the properties above to rewrite expressions in different ways (**7.EE.2▲**).

Example: Working with the Distributive Property.
Students see expressions like $7 - 2(8 - 1.5x)$ and realize that the expression $(8 - 1.5x)$ is treated as a separate quantity in its own right, being multiplied by 2 and the result being subtracted from 7 **(MP.7)**. Students may mistakenly come up with the expressions below, and each case offers a chance for class discussion about why they are not equivalent to the original **(MP.3)**: • $5(8 - 1.5x)$, subtracting $7 - 2$, not realizing the multiplication must be done first, • $7 - 2(6.5x)$, performing the operation in parentheses first, though one cannot combine 8 and $-5x$ in this case, • $7 - 16 - 3x$, by misapplying the distributive property or not being attentive to the rules for multiplying negative numbers. Furthermore, students should have the opportunity to see this expression as equivalent to both $7 + (-2)(8 - 1.5x)$ and $7 - (2(8 - 1.5x))$, which can aid in seeing the correct way to handle the -2 part of the expression.

Note that the standards do not expressly refer to the "simplifying" of expressions. Simplifying an expression is a special case of generating equivalent expressions. This is not to say that simplifying is never important per se, but whether one expression is "simpler" than another to work with often depends on the context. For example, the expression $50 + (x - 500) \cdot 0.20$ represents the cost of a phone plan wherein the base cost is $\$50$ and any minutes over 500 cost $\$.20$ per minute (valid for $x \geq 500$). However, it is more difficult to see how the equivalent expression $0.20x - 50$ also represents the cost of this phone plan.

Focus, Coherence, and Rigor:

The work in standards **(7.EE.1-2▲)** is closely connected to standards **(7.EE.3-4▲)**, as well as the multi-step proportional reasoning problems in the domain Ratios and Proportional Relations (**7.RP.3▲**). Students' work with rational number arithmetic (**7.NS▲**) comes strongly into play as they write and solve equations (**7.EE▲**). Procedural fluency solving these types of equations is an explicit goal in **7.EE.4a**.

As students gain experience with multiple ways of writing an expression, they also learn that different ways of writing expressions can serve different purposes and provide different ways of seeing a problem. In Example 3 below, the connection between the expressions and the figure emphasizes they represent the same number, and the connection between the structure of each expression and a method of calculation emphasizes the fact that expressions are built up from operations on numbers.

Examples: Working with Expressions.

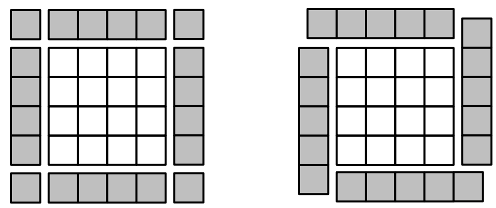
1. A rectangle is twice as long as it is wide. Find as many different ways as you can to write an expression for the perimeter of such a rectangle.

Solution: If W represents the width of the rectangle, and L represents the length, then we could express the perimeter as $L + W + L + W$. We could rewrite this as $2L + 2W$. Knowing that $L = 2W$, the perimeter could also be given by $W + W + 2W + 2W$, which we could rewrite as $6W$. Alternatively, we know that $W = \frac{L}{2}$, so the perimeter could be given in terms of the length as $L + L + \frac{L}{2} + \frac{L}{2}$, which we could rewrite as $3L$.

2. While Chris was driving a Canadian car, he figured out a way to mentally convert the outside temperature that the car displayed in degrees Celsius to degrees Fahrenheit. This was his method: "I take the temperature it shows and I double it, then I subtract one-tenth of that doubled amount. Then, I add 32 to get the Fahrenheit temperature." The standard expression for finding the temperature in degrees Fahrenheit when the degrees Celsius is known is given by $\frac{9}{5}C + 32$, where C is the temperature in degrees Celsius. Is Chris's method correct?

Solution: If C is the temperature in degrees Celsius, then the first step in Chris's calculation is to find $2C$. Then, he subtracts one-tenth of that quantity, which is $\frac{1}{10}(2C)$. Finally, he adds 32. The resulting expression is $2C - \frac{1}{10}(2C) + 32$. This can be rewritten as $2C - \frac{1}{5}C + 32$. Combining the first two terms, we get $2C - \frac{1}{5}C + 32 = \left(2 - \frac{1}{5}\right)C + 32 = \left(\frac{10}{5} - \frac{1}{5}\right)C + 32 = \frac{9}{5}C + 32$. Chris's calculation is correct.

3. In the well-known "Pool Border Problem," students are asked to determine the number of border tiles needed to construct a pool (or grid) of size $n \times n$ (note this is the size of the inner part that would be filled with water). They may first examine several examples and organize their counting of the border tiles, after which they can be asked to develop an expression for the number of border tiles, B **(MP.8)**. Many different expressions are correct, all equivalent to $4n + 4$. However, different expressions arise from different ways of seeing the construction of the border. A student who sees the border as four sides of length n plus four corners might develop the expression $4n + 4$, while a student who sees the border as four sides of length $n + 1$ may find the expression $4(n + 1)$. It is important for students to see many different representations and understand that they express the same quantity in different ways **(MP.7)**.



(Adapted from Progressions 6-8 Expressions and Equations [EE] 2011)

Expressions and Equations 7.EE

Solve real-life and mathematical problems using numerical and algebraic expressions and equations.

3. Solve multi-step real-life and mathematical problems posed with positive and negative rational numbers in any form (whole numbers, fractions, and decimals), using tools strategically. Apply properties of operations to calculate with numbers in any form; convert between forms as appropriate; and assess the reasonableness of answers using mental computation and estimation strategies. *For example: If a woman making $25 an hour gets a 10% raise, she will make an additional 1/10 of her salary an hour, or $2.50, for a new salary of $27.50. If you want to place a towel bar 9 3/4 inches long in the center of a door that is 27 1/2 inches wide, you will need to place the bar about 9 inches from each edge; this estimate can be used as a check on the exact computation.*
4. Use variables to represent quantities in a real-world or mathematical problem, and construct simple equations and inequalities to solve problems by reasoning about the quantities.
   a. Solve word problems leading to equations of the form $px + q = r$ and $p(x + q) = r$, where p, q, and r are specific rational numbers. Solve equations of these forms fluently. Compare an algebraic solution to an arithmetic solution, identifying the sequence of the operations used in each approach. *For example, the perimeter of a rectangle is 54 cm. Its length is 6 cm. What is its width?*
   b. Solve word problems leading to inequalities of the form $px + q > r$ or $px + q < r$, where p,

q, and r are specific rational numbers. Graph the solution set of the inequality and interpret it in the context of the problem. *For example: As a salesperson, you are paid $50 per week plus $3 per sale. This week you want your pay to be at least $100. Write an inequality for the number of sales you need to make, and describe the solutions.*

By grade seven students start to see whole numbers and their opposites, and positive and negative fractions as belonging to a single system of rational numbers. Students solve multi-step problems involving rational numbers presented in various forms (whole numbers, fractions, and decimals), assessing the reasonableness of their answers, and they solve problems that result in basic linear equations and inequalities **(7.EE.3-4▲).** This work is the culmination of many progressions of learning in arithmetic, problem solving, and mathematical practices.

Examples: Solving Equations and Inequalities. (MP.2, MP.4, MP.7)
1. The youth group is going on a trip to the state fair. The trip costs \$52.50. Included in that price is \$11.25 for a concert ticket and the cost of 3 passes, two for rides and one for the game booths. Each of the passes costs the same price. Write an equation representing the cost of the trip and determine the price of one pass. **Solution:** Students can represent the situation with a tape-like diagram, showing that $3p + 11.25$ represents the total cost of the trip, if p represents the price each of the passes. Students find the equation $3p + 11.25 = 52.50$. They see the expression on the left side of the equation as some quantity ($3p$) plus 11.25 equaling 52.50. In that case, $52.50 - 11.25 = 41.25$ represents that quantity, by the relationship between addition and subtraction. So now, $3p = 41.25$, which means that $p = 41.25 \div 3 = 13.75$. Thus, each pass costs $\$13.75$.
2. The student body government initiates a campaign to change the school mascot. The school principal agrees to the change, but only if two-thirds of the student body plus one additional student vote for the change. The required number of votes is 255. How many students attend the school? **Solution:** If we let S represent the number of students that attend the school, then $\frac{2}{3}S$ represents two-thirds of the vote, while $\frac{2}{3}S + 1$ is one more than this. Since the required number of votes is 255, we can write $\frac{2}{3}S + 1 = 255$. Since the quantity $\frac{2}{3}S$ plus one gives 255, it follows that $\frac{2}{3}S = 254$. To solve this, we can find $S = 254 \div \frac{2}{3} = \frac{254}{1} \cdot \frac{3}{2} = \frac{762}{2} = 381$. Thus, there are 381 total students. Alternately, students can solve the equation $\frac{2}{3}S = 254$ by multiplying each side by $\frac{3}{2}$, giving $\frac{3}{2} \cdot \frac{2}{3}S =$

$254 \cdot \frac{3}{2}$, since $\frac{3}{2} \cdot \frac{2}{3} \cdot S = 1 \cdot S$, we get $S = 254 \cdot \frac{3}{2} = \frac{762}{2} = 381$.
3. Florencia can spend at most \$60 on clothes. She wants to buy a pair of jeans for \$22 and spend the rest on t-shirts. Each t-shirt costs \$8. Write an inequality for the number of t-shirts she can purchase. **Solution:** If t represents the number of t-shirts she buys, then an expression for the total amount she spends on clothes is $8t + 22$, since each t-shirt costs $\$8$. The term "at most" might be new to students, but it indicates that what she spends must be less than this amount. The inequality that results is $8t + 22 \leq 60$. Note that the symbol "≤" is used here to denote that the amount she spends can be less than *or* equal to $\$60$. This symbol should be introduced at this grade level.

[Note: Sidebar]

Students can use estimation strategies to assess the reasonableness of their answers, such as: (**MP.1, MP.5**),
• Front-end estimation with adjusting (using the highest place value and estimating from the front end making adjustments to the estimate by taking into account the remaining amounts) (**MP.5**), • Clustering around an average (when the values are close together an average value is selected and multiplied by the number of values to determine an estimate), • Rounding and adjusting (students round down or round up and then adjust their estimate depending on how much the rounding affected the original values), • Using friendly or compatible numbers such as factors (students seek to fit numbers together, e.g., rounding to factors and grouping numbers together that have round sums like 100 or 1000), and • Using benchmark numbers that are easy to compute (students select close whole numbers for fractions or decimals to determine an estimate).

Domain: Geometry

In grade seven, a critical area of instruction is for students to extend their study of geometry as they solve problems involving scale drawings and informal geometric constructions, and they work with two- and three-dimensional shapes to solve problems involving area, surface area, and volume.

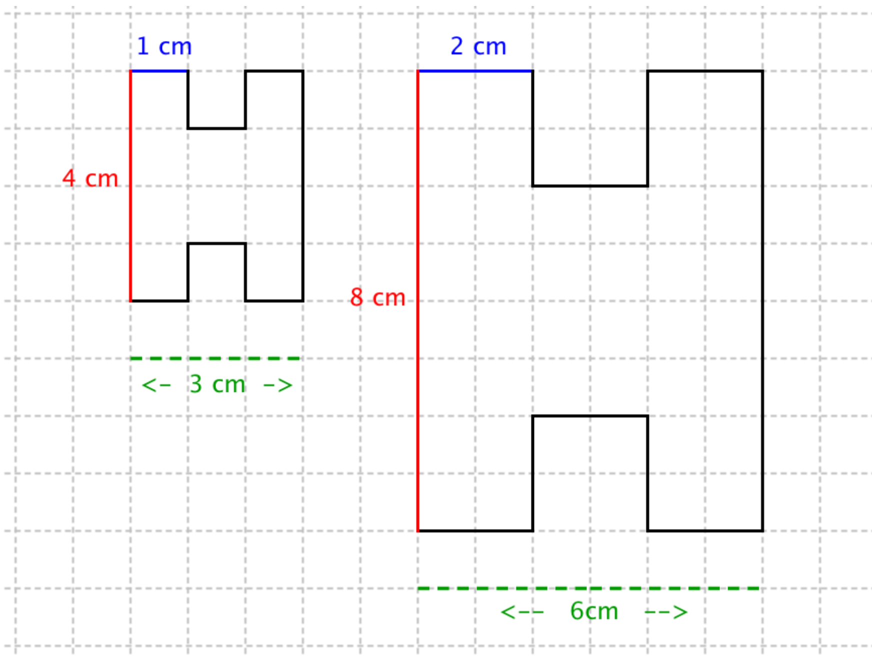
Geometry	7.G
Draw, construct, and describe geometrical figures and describe the relationships between them.	

1. Solve problems involving scale drawings of geometric figures, including computing actual lengths and areas from a scale drawing and reproducing a scale drawing at a different scale.

In standard **(7.G.1)**, students lay the foundation for understanding dilations as geometric transformations. This will lead to a definition of the concept of similar shapes in eighth grade as shapes that can be obtained from one another through the use of a dilation. These ideas are crucial for student understanding of the derivation of the equations $y = mx$ and $y = mx + b$ by using similar triangles and the relationships between them. Thus, standard **(7.G.1)** should be given significant attention in grade seven. Students solve problems involving scale drawings by applying their understanding of ratios and proportions, which started in grade six and has also continued in the domain Ratios and Proportional Relations (**7.RP.1-3▲).**

Teachers should note that the notion of *similarity* has not yet been addressed. Attempts to define similar shapes as those that have the "same shape but not necessarily the same size" should be avoided. Similarity will be precisely defined in grade eight and such imprecise notions of similarity may detract from student understanding of this important concept. Shapes drawn to scale are indeed similar to each other, but could safely be referred to as "scale drawings of each other" at this grade level.

The concept of a scale drawing can be effectively introduced by allowing students to blow-up or shrink pictures on grid paper. For example, students can be asked to recreate the image on the left below on the same sheet of grid paper but using 2 units of length for every 1 unit on the original picture:



By recording measurements in many examples, students come to see there are two important ratios with scale drawings: the ratios between two figures and the ratios within a single figure. For instance, students notice that the ratio of the blue segments and the ratio of the red segments are equal (ratios "between" figures are equal):

$$\frac{2\text{ cm}}{1\text{ cm}} = \frac{8\text{ cm}}{4\text{ cm}}.$$

Moreover, students see that the ratios of the blue to red or blue to green in each shape separately are equal (ratios "within" figures are equal):

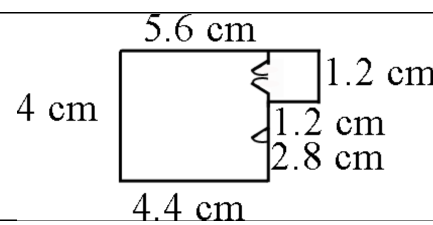
$$\frac{2\text{ cm}}{8\text{ cm}} = \frac{1\text{ cm}}{4\text{ cm}} \quad \text{and} \quad \frac{2\text{ cm}}{6\text{ cm}} = \frac{1\text{ cm}}{3\text{ cm}}.$$

These relationships should be exploited when solving problems involving scale drawings, including problems where students justify mathematically when drawings are *not* to scale.

Examples: Scale Drawing Problems.

1. Julie showed you the scale drawing of her room. If each 2 cm on the scale drawing equals 5 ft, what are the actual dimensions of Julie's room?

Solution: Since each 2 cm in the drawing represents 5 ft, we

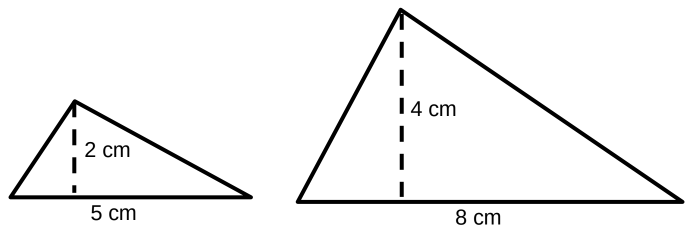


have a conversion rate of $\frac{5}{2}$ ft/cm. In that case, we multiply each measurement in centimeters to obtain the true measurement of the room in feet. Thus,

$$5.6 \text{ cm} \to 5.6 \cdot \frac{5}{2} = 14 \text{ ft}, \quad 1.2 \text{ cm} \to 1.2 \cdot \frac{5}{2} = 3 \text{ ft}, \quad 2.8 \text{ cm} \to 2.8 \cdot \frac{5}{2} = 7 \text{ ft, etc.}$$

2. Explain why the two triangles shown are *not* scale drawings of one another.

Solution: Since the ratios of the heights to bases of the triangles are different, one cannot be a scale drawing of the other: $\frac{2}{5} \neq \frac{4}{8}$.



(Adapted from Arizona 2012 and N. Carolina 2012)

Geometry	7.G

Draw, construct, and describe geometrical figures and describe the relationships between them.

2. Draw (freehand, with ruler and protractor, and with technology) geometric shapes with given conditions. Focus on constructing triangles from three measures of angles or sides, noticing when the conditions determine a unique triangle, more than one triangle, or no triangle.
3. Describe the two-dimensional figures that result from slicing three-dimensional figures, as in plane sections of right rectangular prisms and right rectangular pyramids.

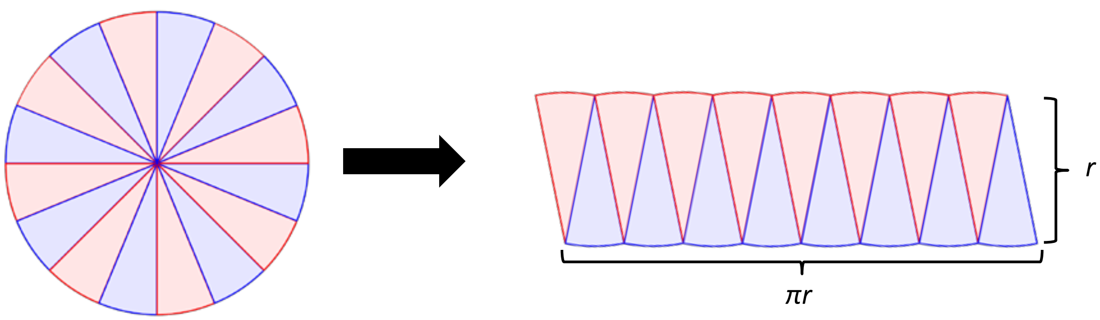
Students draw (freehand, with ruler and protractor, and with technology) geometric shapes with given conditions, focusing on triangles (**7.G.2**). They work with three-dimensional figures and relate them to two-dimensional figures by examining cross-sections that result when three-dimensional figures are split (**7.G.3**). Students also describe how two or more objects are related in space (e.g., skewed lines and the possible ways three planes might intersect).

Geometry	7.G

Solve real-life and mathematical problems involving angle measure, area, surface area, and volume.

4. Know the formulas for the area and circumference of a circle and use them to solve problems; give an informal derivation of the relationship between the circumference and area of a circle.
5. Use facts about supplementary, complementary, vertical, and adjacent angles in a multi-step problem to write and solve simple equations for an unknown angle in a figure.
6. Solve real-world and mathematical problems involving area, volume and surface area of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms.

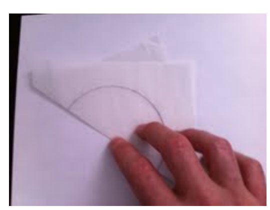
In grade seven students know the formulas for the area and circumference of a circle and use them to solve problems (**7.G.4**). To "know the formula" means to have an understanding of why the formula works and how the formula relates to the measure (area and circumference) and the figure. For instance, students can cut circles into finer and finer pie pieces (sectors) and arrange them into a shape that begins to approximate a parallelogram. Due to the way the shape was created, it has a length of approximately πr and a height of approximately r. Therefore, the approximate area of this shape is πr^2, which informally justifies the formula for the area of a circle.

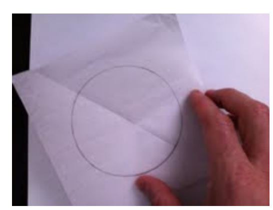


Another detailed discussion of the second part of standard **(7.G.4)**, is available at http://www.illustrativemathematics.org/illustrations/1553 (Illustrative Mathematics 2013).

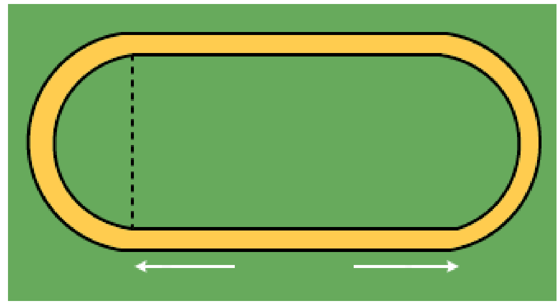
Examples: Working with the Circumference and Area of a Circle.

1. Students can explore the relationship between the circumference of a circle and its diameter (or radius). For example, by tracing the circumference of a cylindrical can of beans or some other cylinder on patty paper and finding the diameter by folding the patty paper appropriately, students can find the approximate diameter of the base of the cylinder. If they measure a piece of string the same length as the diameter, they will find that the string can wrap around the can approximately three and one-sixth times. That is, they find that $C \approx 3\frac{1}{6} \cdot d \approx 3.16$. When students do this for a variety of objects, they start to see that the ratio of the circumference of a circle to its diameter is always approximately the same number. (It is, of course, π.)





2. The straight sides of a standard track that is 400 meters around measure 84.39 meters. Assuming the rounded sides of the track are half-circles, find the distance from one side of the track to the other.

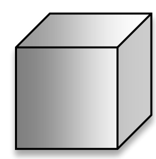


Solution: The two rounded portions of the track together make one circle, the circumference of which is $400 - 2(84.39) = 231.22$ m. The length across the track is represented by the diameter of this circle. If we label the diameter d, then we obtain the equation $231.22 = \pi d$. Using a calculator and an approximation for $\pi \approx 3.14$, we obtain $d = 231.22 \div \pi \approx 231.22 \div 3.14 \approx 73.64$ meters.

Students continue work from grades five and six to solve problems involving area, surface area, and volume of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms **(7.G.6).**

Example: Surface Area and Volume.

The surface area of a cube is 96 square inches. What is the volume of the cube?



Solution: Students understand from working with nets in grade six that the cube has six faces, all with equal area. Thus, the area of one face of the cube is $96 \div 6 = 16$ square inches. Since each face is a square, the length of one side of the cube is 4 inches. This makes the volume $V = 4^3 = 64$ cubic inches.

Domain: Statistics and Probability

Students were introduced to statistics in sixth grade. In seventh grade they extend their work with single-data distributions to compare two data distributions and address questions about differences between populations. They also begin informal work with random sampling.

Statistics and Probability	7.SP

Use random sampling to draw inferences about a population.

1. Understand that statistics can be used to gain information about a population by examining a sample of the population; generalizations about a population from a sample are valid only if the sample is representative of that population. Understand that random sampling tends to produce representative samples and support valid inferences.

2. Use data from a random sample to draw inferences about a population with an unknown characteristic of interest. Generate multiple samples (or simulated samples) of the same size to gauge the variation in estimates or predictions. *For example, estimate the mean word length in a book by randomly sampling words from the book; predict the winner of a school election based on randomly sampled survey data. Gauge how far off the estimate or prediction might be.*

Seventh-grade students begin informal work with random sampling. They use data from a random sample to draw inferences about a population with an unknown characteristic. (**7.SP.1-2**) For example, they predict the winner of a school election based on randomly sampled survey data.

Students recognize that it is difficult to gather statistics on an entire population. Instead a random sample can be representative of the total population, and it will generate valid predictions. Students use this information to draw inferences from data. (**MP.1, MP.2, MP.3, MP.4, MP.5, MP.6, MP.7**) The standards in this cluster represent opportunities to apply percentages and proportional reasoning. In order to make inferences about a population, one applies such reasoning to the sample and the entire population.

Example: Random Sampling (7.SP.1)

Shown is the data collected from two random samples of 100 students regarding students' school lunch preferences. Make at least two inferences based on the results.

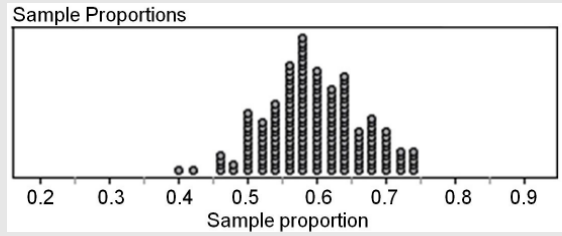
Student Sample	Ham-burgers	Tacos	Pizza	Total
#1	12	14	74	100
#2	12	11	77	100

Possible solutions: Since the sample sizes are quite large, and a vast majority in both samples prefer pizza, it would be safe to say both:

- Most students prefer pizza.
- More people prefer pizza than hamburgers and tacos combined.

Variability in samples can be studied using simulation **(7.SP.2)**. Web-based software and spreadsheet programs can be used to run samples. For example, suppose students are using random sampling to determine the proportion of students who prefer football as their favorite sport, and suppose that 60% is the true proportion of the population. Then students can simulate the sampling by doing an experiment as simple as placing a collection of red and blue colored chips in a container in a ratio of 60:40, randomly selecting a chip 50 separate times with replacement, and recording the proportion that came out red. If this experiment is repeated 200 times, students might obtain a distribution of the sample proportions that looks like the picture shown. This way, students gain an understanding that the sample proportion can vary quite a bit, from as low as 45% to as high as 75%. Students can conjecture whether this variability will increase or decrease when the sample size increases, or if this variability is at all dependent on the true population proportion. **(MP.3)**. (Adapted from Progressions on 6-8 Statistics and Probability)

Results of simulations



Proportions of red chips in 200 random samples of size 50 from a population in which 60% of the chips are red.

Statistics and Probability	7.SP

Draw informal comparative inferences about two populations.

3. Informally assess the degree of visual overlap of two numerical data distributions with similar variabilities, measuring the difference between the centers by expressing it as a multiple of a measure of variability. *For example, the mean height of players on the basketball team is 10 cm greater than the mean height of players on the soccer team, about twice the variability (mean absolute deviation) on either team; on a dot plot, the separation between the two distributions of heights is noticeable.*
4. Use measures of center and measures of variability for numerical data from random samples to draw informal comparative inferences about two populations. *For example, decide whether the words in a chapter of a seventh-grade science book are generally longer than the words in a chapter of a fourth-grade science book.*

Comparing two data sets is a new concept for students (**7.SP.3-4**). Students build on their understanding of graphs, mean, median, mean absolute deviation (MAD), and interquartile range from sixth grade. They know:

- Understanding data requires consideration of the measures of variability as well as mean or median,
- Variability is responsible for the overlap of two data sets and that an increase in variability can increase the overlap, and
- Median is paired with the interquartile range and mean is paired with the mean absolute deviation.

Example: Comparing Two Populations. (Adapted from illustrativemath.org "Offensive Linemen." http://www.illustrativemathematics.org/illustrations/1341)

College football teams are grouped with similar teams into "divisions" based on many factors. Schools from the Football Bowl Subdivision (FBS) are typically much larger than schools of other divisions in terms of enrollment and revenue. "Division III" is a division of schools typically with a smaller enrollment and more limited resources.

It is generally believed that the offensive linemen of FBS schools are heavier on average than those of Division III schools.

For the 2012 season, the University of Mount Union Purple Raiders football team won the Division III National Championship, while the University of Alabama Crimson Tide football team won the FBS National Championship. Below are the weights of the offensive lineman for both teams from that season.[8]

Ala.	277	265	292	303	303	320	300	313	267	288	311	280	302	335	310	290	312	340	292
Mt. U	250	250	290	260	270	270	310	290	280	315	280	295	300	300	260	255	300		

A combined dot plot for both teams is also shown.

Here are some examples of conclusions that can be drawn from the data and the dot plot:

a. Based on a visual inspection of the dot plots, the mean of the Alabama group seems to be higher than the mean of the Mount Union group. However, the overall spread of each distribution appears

[8] Accessed at http://athletics.mountunion.edu/sports/fball/2012-13/roster, http://www.rolltide.com/sports/m-footbl/mtt/alab-m-footbl-mtt.html on 1/14/13

similar, so we might expect the *variability* to be similar as well.

b. The Alabama mean is 300 pounds, with a MAD of 15.68 pounds. The Mount Union mean is 280.88 pounds, with a MAD of 17.99 pounds.

c. So it appears that on average, an Alabama lineman's weight is about 20 pounds heavier than that of a Mount Union lineman. We also notice that the difference in the average weights of each team is greater than 1 MAD for either team. This could be interpreted as saying that for Mount Union, on average a lineman's weight is not greater than 1 MAD above 280.88 pounds, while the average Alabama lineman's weight is already above this amount!

d. If we were to assume that the players from Alabama represent a random sample of players from their division (the FBS) and likewise for Mount Union with respect to Division III, then it is plausible that on average offensive linemen from FBS schools are heavier than offensive linemen from Division III schools.

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Focus, Coherence, and Rigor: Probability models draw on proportional reasoning and should be connected to major work at this grade in the cluster "Analyze proportional relationships and use them to solve real-world and mathematical problems" **(7.RP.1-3▲).**

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Statistics and Probability 7.SP

Investigate chance processes and develop, use, and evaluate probability models.

5. Understand that the probability of a chance event is a number between 0 and 1 that expresses the likelihood of the event occurring. Larger numbers indicate greater likelihood. A probability near 0 indicates an unlikely event, a probability around 1/2 indicates an event that is neither unlikely nor likely, and a probability near 1 indicates a likely event.
6. Approximate the probability of a chance event by collecting data on the chance process that produces it and observing its long-run relative frequency, and predict the approximate relative frequency given the probability. *For example, when rolling a number cube 600 times, predict that a 3 or 6 would be rolled roughly 200 times, but probably not exactly 200 times.*
7. Develop a probability model and use it to find probabilities of events. Compare probabilities from a model to observed frequencies; if the agreement is not good, explain possible sources of the discrepancy.
   a. Develop a uniform probability model by assigning equal probability to all outcomes, and use the model to determine probabilities of events. *For example, if a student is selected at random from a class, find the probability that Jane will be selected and the probability that a girl will be selected.*
   b. Develop a probability model (which may not be uniform) by observing frequencies in data generated from a chance process. *For example, find the approximate probability that a spinning penny will land heads up or that a tossed paper cup will land open-end down. Do the outcomes for the spinning penny appear to be equally likely based on the observed frequencies?*

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Probability is formally introduced in seventh grade, as students interpret probability as indicating the long-run relative frequency of the occurrence of an event. Students can use simulations to support their understanding. (Marble Mania at http://www.sciencenetlinks.com/interactives/marble/marblemania.html and Random Drawing Tool at http://illuminations.nctm.org/activitydetail.aspx?id=67 are two examples of online resources for using simulations.

Students develop and use *probability models* to find the probabilities of events, and investigate both *empirical probabilities* (i.e., probabilities based on observing outcomes of a simulated random process) and *theoretical probabilities* (i.e., probabilities based on the structure of the *sample space* of an event) (**7.SP.7**).

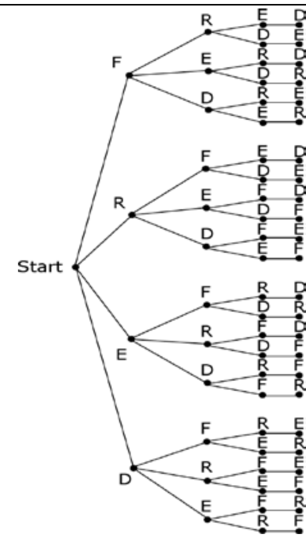
Example: A Simple Probability Model.
A box contains 10 red chips and 10 black chips. Each student reaches into the box without looking and pulls out a chip. If each of the first five students pulls out (and keeps) a red chip, what is the probability that the sixth student will pull a red chip? **Solution:** The events in question, pulling out a red or black chip, should be considered equally likely. Furthermore, though students new to probability may believe in the “gambler’s fallacy”—that since 5 red chips have already been chosen, there is a very large chance that a black chip will be chosen next—students must still compute the probabilities of events as equally likely. Since there are 15 chips left in the box, five red and ten black, the probability that the sixth student will select a red chip is simply $\frac{5}{15} = \frac{1}{3}$.

Statistics and Probability	7.SP
Investigate chance processes and develop, use, and evaluate probability models. 8. Find probabilities of compound events using organized lists, tables, tree diagrams, and simulation. a. Understand that, just as with simple events, the probability of a compound event is the fraction of outcomes in the sample space for which the compound event occurs. b. Represent sample spaces for compound events using methods such as organized lists, tables and tree diagrams. For an event described in everyday language (e.g., “rolling double sixes”), identify the outcomes in the sample space which compose the event. c. Design and use a simulation to generate frequencies for compound events. *For example, use random digits as a simulation tool to approximate the answer to the question: If 40% of donors have type A blood, what is the probability that it will take at least 4 donors to find one with type A blood?*	

Students in grade seven also examine *compound events* (such as tossing a coin and rolling a standard number cube) and use basic counting ideas for finding the total number of equally likely outcomes for such an event (e.g. 2 outcomes for the coin and 6 outcomes for the number cube result in 12 total outcomes). There is no need at this grade level to introduce formal methods of finding permutations and combinations. Students also use various means of organizing the outcomes of an event, such as two-way tables or tree diagrams **(7.SP.8a-b)**.

Example: Tree Diagrams. Show all possible arrangements of the letters in the word FRED using a tree diagram. If each of the letters is on a tile and drawn at random, what is the probability that you will draw the letters F-R-E-D in that order? What is the probability that your "word" will have an F as the first letter?

Solution: A tree diagram reveals that there is only one outcome of the letters F-R-E-D appearing in that order out of 24 total outcomes, so that the probability of the event occurring is $\frac{1}{24}$. On the other hand, the entire top branch (6 outcomes) represent the outcomes where the first letter is F, and so the probability of this occurring is $\frac{6}{24} = \frac{1}{4}$.



Finally, grade seven students use simulations to determine probabilities (frequencies) for compound events **(7.SP.8.c)**. (For a more complete discussion of the Statistics and Probability domain, see ~~the~~ "Progressions for the Common Core State Standards in Mathematics: 6-8 Statistics and Probability," available at: http://ime.math.arizona.edu/progressions/)

Example: If 40 percent of donors have type B blood, what is the probability that it will take at least four donors to find one with type B blood?

Such questions are perfect opportunities for students to construct simulation models. The proportion of donors with blood type B being 40% can be modeled by blind drawings from a box with markers labeled B and Not-B. A box with 40 B-markers and 60-Not-B markers would be one option. How best to model the situation would depend on the size of the donor pool. If the donor pool was of size 25, one could model the situation by drawing markers (without replacement) from a box containing 10 B-makers and 15 Not B-markers until a type B marker is drawn. Reasonable estimates could be achieved in 20 trials. However, if the class were to decide that the donor pool was as large as 1000, and circumstances dictated using a box with only ten markers, then each marker drawn would represent a population of 100 potential donors. In such a situation, one would model the situation by making successive draws with replacement until a type B marker was drawn. In order to speed up the experiment, students might note that once 3 Not-B markers have been drawn the stated conditions have been met. An advanced class could be led to the observation that if the donor pool were very large, the probability of the first three donors having blood types A or O is approximated by $(1 - 0.4)^3 = (0.6)^3 = 0.216$. Of course, this task represents a nice opportunity for collaboration with science faculty.

Essential Learning for the Next Grade

In middle school, multiplication and division develop into powerful forms of ratio and proportional reasoning. The properties of operations take on prominence as arithmetic matures into algebra. The theme of quantitative relationships also becomes explicit in grades six through eight, developing into the formal notion of a function by grade eight. Meanwhile, the foundations of high school deductive geometry are laid in the middle grades. The gradual development of data representations in kindergarten through grade five leads to statistics in middle school: the study of shape, center, and spread of data distributions; possible associations between two variables; and the use of sampling in making statistical decisions (Adapted from PARCC 2012)..

To be prepared for grade eight mathematics students should be able to demonstrate they have acquired certain mathematical concepts and procedural skills by the end of grade seven and have met the fluency expectations for the grade seven.

Seventh grade students are expected to fluently solve equations of the form $px + q = r$ and $p(x + q) = r$ (**7.EE.4▲**), which also requires fluency with rational number arithmetic (**7.NS.1–3▲**), as well as fluency to some extent with applying properties operations to rewrite linear expressions with rational coefficients (**7.EE.1▲**). Also adding, subtracting, multiplying and dividing rational numbers (**7.NS.1–2▲**) is the culmination of numerical work with the four basic operations. The number system will continue to develop in grade eight, expanding to become the real numbers by the introduction of irrational numbers, and will develop further in high school, expanding to become the complex numbers with the introduction of imaginary numbers. Because there are no specific standards for rational number arithmetic in later grades and because so much other work in grade seven depends on rational number arithmetic, fluency with rational number arithmetic should also be the goal in grade seven. These fluencies and the conceptual understandings that support them are foundational for work in grade eight.

Of particular importance for students to attain in grade seven are skills and understandings to analyze proportional relationships and use them to solve real-world and mathematical problems (**7.RP.1-3▲**); apply and extend previous understanding of operations with fractions to add, subtract, multiply, and divide rational numbers (**7.NS.1-3▲**); use properties of operations to generate equivalent expressions (**7.EE.1-2▲**); and solve real-life and mathematical problems using numerical and algebraic expressions and equations (**7.EE.3-4▲**).

Guidance on Course Placement and Sequences

The Common Core standards for grades six, seven, and eight are comprehensive, rigorous, and non-redundant. Acceleration will require compaction not the former strategy of deletion. Therefore, careful consideration needs to be made before placing a student into higher mathematics coursework in middle grades. Acceleration may get students to advanced coursework but might create gaps in students' mathematical background. Careful consideration and systematic collection of multiple measures of individual student performance on both the content and practice standards will be required. For additional information and guidance on course placement, see Appendix A: Course Placement and Sequences in this framework.

Grade 7 Overview

Ratios and Proportional Relationships

- Analyze proportional relationships and use them to solve real-world and mathematical problems.

The Number System

- Apply and extend previous understandings of operations with fractions to add, subtract, multiply, and divide rational numbers.

Expressions and Equations

- Use properties of operations to generate equivalent expressions.
- Solve real-life and mathematical problems using numerical and algebraic expressions and equations.

Geometry

- Draw, construct and describe geometrical figures and describe the relationships between them.
- Solve real-life and mathematical problems involving angle measure, area, surface area, and volume.

Statistics and Probability

- Use random sampling to draw inferences about a population.
- Draw informal comparative inferences about two populations.
- Investigate chance processes and develop, use, and evaluate probability models.

Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Grade 7

Ratios and Proportional Relationships 7.RP

Analyze proportional relationships and use them to solve real-world and mathematical problems.

1. Compute unit rates associated with ratios of fractions, including ratios of lengths, areas and other quantities measured in like or different units. *For example, if a person walks 1/2 mile in each 1/4 hour, compute the unit rate as the complex fraction* $^{1/2}/_{1/4}$ *miles per hour, equivalently 2 miles per hour.*
2. Recognize and represent proportional relationships between quantities.
   a. Decide whether two quantities are in a proportional relationship, e.g., by testing for equivalent ratios in a table or graphing on a coordinate plane and observing whether the graph is a straight line through the origin.
   b. Identify the constant of proportionality (unit rate) in tables, graphs, equations, diagrams, and verbal descriptions of proportional relationships.
   c. Represent proportional relationships by equations. *For example, if total cost t is proportional to the number n of items purchased at a constant price p, the relationship between the total cost and the number of items can be expressed as t = pn*.
   d. Explain what a point *(x, y)* on the graph of a proportional relationship means in terms of the situation, with special attention to the points (0, 0) and (1, *r)* where *r* is the unit rate.
3. Use proportional relationships to solve multistep ratio and percent problems. *Examples: simple interest, tax, markups and markdowns, gratuities and commissions, fees, percent increase and decrease, percent error.*

The Number System 7.NS

Apply and extend previous understandings of operations with fractions to add, subtract, multiply, and divide rational numbers.

1. Apply and extend previous understandings of addition and subtraction to add and subtract rational numbers; represent addition and subtraction on a horizontal or vertical number line diagram.
   a. Describe situations in which opposite quantities combine to make 0. *For example, a hydrogen atom has 0 charge because its two constituents are oppositely charged.*
   b. Understand *p* + *q* as the number located a distance |*q*| from *p*, in the positive or negative direction depending on whether *q* is positive or negative. Show that a number and its opposite have a sum of 0 (are additive inverses). Interpret sums of rational numbers by describing real-world contexts.
   c. Understand subtraction of rational numbers as adding the additive inverse, *p* – *q* = *p* + (–*q*). Show that the distance between two rational numbers on the number line is the absolute value of their difference, and apply this principle in real-world contexts.
   d. Apply properties of operations as strategies to add and subtract rational numbers.
2. Apply and extend previous understandings of multiplication and division and of fractions to multiply and divide rational numbers.
   a. Understand that multiplication is extended from fractions to rational numbers by requiring that operations continue to satisfy the properties of operations, particularly the distributive property, leading to products such as (–1)(–1) = 1 and the rules for multiplying signed numbers. Interpret products of rational numbers by describing real-world contexts.
   b. Understand that integers can be divided, provided that the divisor is not zero, and every quotient of integers (with non-zero divisor) is a rational number. If *p* and *q* are integers, then –(*p*/*q*) = (–*p*)/*q* = *p*/(–*q*). Interpret quotients of rational numbers by describing real world contexts.
   c. Apply properties of operations as strategies to multiply and divide rational numbers.
   d. Convert a rational number to a decimal using long division; know that the decimal form of a rational number terminates in 0s or eventually repeats.
3. Solve real-world and mathematical problems involving the four operations with rational numbers.[1]

[1]Computations with rational numbers extend the rules for manipulating fractions to complex fractions.

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Expressions and Equations 7.EE

Use properties of operations to generate equivalent expressions.

1. Apply properties of operations as strategies to add, subtract, factor, and expand linear expressions with rational coefficients.
2. Understand that rewriting an expression in different forms in a problem context can shed light on the problem and how the quantities in it are related. *For example, $a + 0.05a = 1.05a$ means that "increase by 5%" is the same as "multiply by 1.05."*

Solve real-life and mathematical problems using numerical and algebraic expressions and equations.

3. Solve multi-step real-life and mathematical problems posed with positive and negative rational numbers in any form (whole numbers, fractions, and decimals), using tools strategically. Apply properties of operations to calculate with numbers in any form; convert between forms as appropriate; and assess the reasonableness of answers using mental computation and estimation strategies. *For example: If a woman making $25 an hour gets a 10% raise, she will make an additional 1/10 of her salary an hour, or $2.50, for a new salary of $27.50. If you want to place a towel bar 9 3/4 inches long in the center of a door that is 27 1/2 inches wide, you will need to place the bar about 9 inches from each edge; this estimate can be used as a check on the exact computation.*
4. Use variables to represent quantities in a real-world or mathematical problem, and construct simple equations and inequalities to solve problems by reasoning about the quantities.
   a. Solve word problems leading to equations of the form $px + q = r$ and $p(x + q) = r$, where p, q, and r are specific rational numbers. Solve equations of these forms fluently. Compare an algebraic solution to an arithmetic solution, identifying the sequence of the operations used in each approach. *For example, the perimeter of a rectangle is 54 cm. Its length is 6 cm. What is its width?*
   b. Solve word problems leading to inequalities of the form $px + q > r$ or $px + q < r$, where p, q, and r are specific rational numbers. Graph the solution set of the inequality and interpret it in the context of the problem. *For example: As a salesperson, you are paid $50 per week plus $3 per sale. This week you want your pay to be at least $100. Write an inequality for the number of sales you need to make, and describe the solutions.*

Geometry 7.G

Draw, construct, and describe geometrical figures and describe the relationships between them.

1. Solve problems involving scale drawings of geometric figures, including computing actual lengths and areas from a scale drawing and reproducing a scale drawing at a different scale.
2. Draw (freehand, with ruler and protractor, and with technology) geometric shapes with given conditions. Focus on constructing triangles from three measures of angles or sides, noticing when the conditions determine a unique triangle, more than one triangle, or no triangle.
3. Describe the two-dimensional figures that result from slicing three-dimensional figures, as in plane sections of right rectangular prisms and right rectangular pyramids.

Solve real-life and mathematical problems involving angle measure, area, surface area, and volume.

4. Know the formulas for the area and circumference of a circle and use them to solve problems; give an informal derivation of the relationship between the circumference and area of a circle.
5. Use facts about supplementary, complementary, vertical, and adjacent angles in a multi-step problem to write and solve simple equations for an unknown angle in a figure.
6. Solve real-world and mathematical problems involving area, volume and surface area of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms.

Statistics and Probability 7.SP

Use random sampling to draw inferences about a population.

1. Understand that statistics can be used to gain information about a population by examining a sample of the

population; generalizations about a population from a sample are valid only if the sample is representative of that population. Understand that random sampling tends to produce representative samples and support valid inferences.

2. Use data from a random sample to draw inferences about a population with an unknown characteristic of interest. Generate multiple samples (or simulated samples) of the same size to gauge the variation in estimates or predictions. *For example, estimate the mean word length in a book by randomly sampling words from the book; predict the winner of a school election based on randomly sampled survey data. Gauge how far off the estimate or prediction might be.*

Draw informal comparative inferences about two populations.

3. Informally assess the degree of visual overlap of two numerical data distributions with similar variabilities, measuring the difference between the centers by expressing it as a multiple of a measure of variability. *For example, the mean height of players on the basketball team is 10 cm greater than the mean height of players on the soccer team, about twice the variability (mean absolute deviation) on either team; on a dot plot, the separation between the two distributions of heights is noticeable.*
4. Use measures of center and measures of variability for numerical data from random samples to draw informal comparative inferences about two populations. *For example, decide whether the words in a chapter of a seventh-grade science book are generally longer than the words in a chapter of a fourth-grade science book.*

Investigate chance processes and develop, use, and evaluate probability models.

5. Understand that the probability of a chance event is a number between 0 and 1 that expresses the likelihood of the event occurring. Larger numbers indicate greater likelihood. A probability near 0 indicates an unlikely event, a probability around 1/2 indicates an event that is neither unlikely nor likely, and a probability near 1 indicates a likely event.
6. Approximate the probability of a chance event by collecting data on the chance process that produces it and observing its long-run relative frequency, and predict the approximate relative frequency given the probability. *For example, when rolling a number cube 600 times, predict that a 3 or 6 would be rolled roughly 200 times, but probably not exactly 200 times.*
7. Develop a probability model and use it to find probabilities of events. Compare probabilities from a model to observed frequencies; if the agreement is not good, explain possible sources of the discrepancy.
    a. Develop a uniform probability model by assigning equal probability to all outcomes, and use the model to determine probabilities of events. *For example, if a student is selected at random from a class, find the probability that Jane will be selected and the probability that a girl will be selected.*
    b. Develop a probability model (which may not be uniform) by observing frequencies in data generated from a chance process. *For example, find the approximate probability that a spinning penny will land heads up or that a tossed paper cup will land open-end down. Do the outcomes for the spinning penny appear to be equally likely based on the observed frequencies?*
8. Find probabilities of compound events using organized lists, tables, tree diagrams, and simulation.
    a. Understand that, just as with simple events, the probability of a compound event is the fraction of outcomes in the sample space for which the compound event occurs.
    b. Represent sample spaces for compound events using methods such as organized lists, tables and tree diagrams. For an event described in everyday language (e.g., “rolling double sixes”), identify the outcomes in the sample space which compose the event.
    c. Design and use a simulation to generate frequencies for compound events. *For example, use random digits as a simulation tool to approximate the answer to the question: If 40% of donors have type A blood, what is the probability that it will take at least 4 donors to find one with type A blood?*