State Space Approach to Solving RLC circuits

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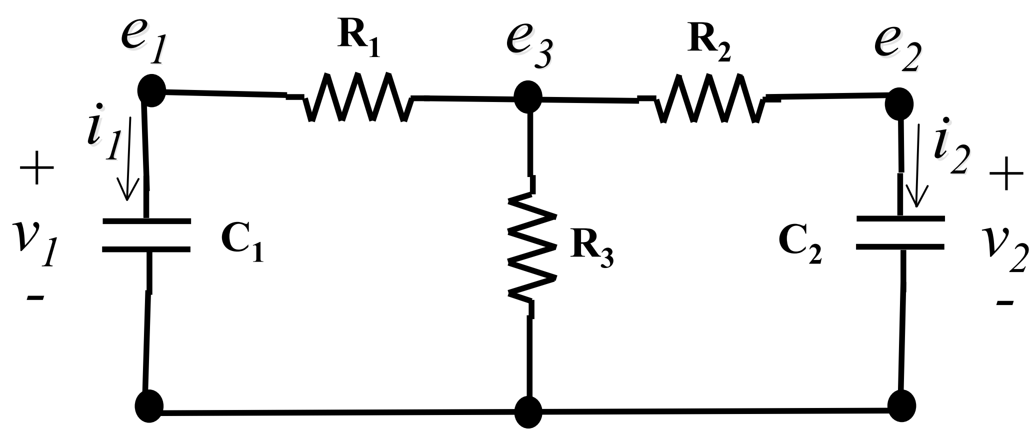
Learning Objectives

- **Analysis of basic circuit with capacitors and inductors, no inputs, using state-space methods**
  - **Identify the states of the system**
  - **Model the system using state vector representation**
  - **Obtain the state equations**

- **Solve a system of first order homogeneous differential equations using state-space method**
  - **Identify the exponential solution**
  - **Obtain the characteristic equation of the system**
  - **Obtain the natural response of the system using eigen-values and vectors**
  - **Solve for the complete solution using initial conditions**

Second order RC circuits



$R_1 = R_2 = R_3 = 1\Omega$

$C_1 = C_2 = 1F$

$$i_1 = C_1 \frac{dv_1}{dt} \qquad i_2 = C_2 \frac{dv_2}{dt}$$

Node equations :

$$e_1: \quad i_1 + (e_1 - e_3)/R_1 = 0$$

$$e_2: \quad (e_2 - e_3)/R_2 + i_2 = 0$$

$$e_3: \quad (e_3 - e_1)/R_1 + (e_3 - e_2)/R_2 + e_3/R_3 = 0$$

State of RLC circuits

- Voltages across capacitors ~ v(t)
- Currents through the inductors ~ i(t)

- Capacitors and inductors store energy
  - Memory in stored energy
  - State at time t depends on the state of the system prior to time t
  - Need initial conditions to solve for the system state at future times

E.g, given state at time 0, can obtain the system state at timest > 0

State at time 0 ~ $v_1(0)$, $v_2(0)$, etc.

State equations for RLC circuits

- **We want to obtain state equations of the form:**

$$\dot{\vec{x}}(t) = f(\vec{x}(t))$$

  - **Where f is a linear function of the states**

- **In our example,**

$$x(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}, \text{ and we need to find,}$$

$$\frac{d}{dt} v_1(t) = f_1(v_1(t), v_2(t)) \quad and \quad \frac{d}{dt} v_2(t) = f_2(v_1(t), v_2(t))$$

Obtaining the state equations

$$\text{We have,} \quad \frac{d}{dt}v_1(t) = \frac{i_1(t)}{C_1} \quad and \quad \frac{d}{dt}v_2(t) = \frac{i_2(t)}{C_2}$$

- **So we need to find $i_1(t)$ and $i_2(t)$ in terms of $v_1(t)$ and $v_2(t)$**
  - **Solve RLC circuit for $i_1(t)$ and $i_2(t)$ using the node or loop method**
- **We will use node method in our examples**

$$e_1: \quad i_1 + (e_1 - e_3)/R_1 = 0$$

$$e_2: \quad (e_2 - e_3)/R_2 + i_2 = 0$$

$$e_3: \quad (e_3 - e_1)/R_1 + (e_3 - e_2)/R_2 + e_3/R_3 = 0$$

- **Note that the equations at e_1 and e_2 give us i_1 and i_2 directly in terms of e_1, e_2, e_3**
  - **Also note that $v_1 = e_1$ and $v_2 = e_2$**
  - **Equation at e_3 gives e_3 in terms of e_1 and e_2**

$$e_3 = \frac{e_1 + e_2}{3}$$

Obtaining the state equations…

- **We now have,**

$$\frac{dv_1}{dt} = i_1 = \frac{-2}{3}v_1 + \frac{1}{3}v_2; \quad \frac{dv_2}{dt} = i_2 = \frac{1}{3}v_1 - \frac{2}{3}v_2$$

$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} -2/3 & 1/3 \\ 1/3 & -2/3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

- **Guessing an exponential solution to the above ODE's we get,**

$$v_1(t) = E_1 e^{st}, \; v_2(t) = E_2 e^{st}$$

$$E_1 s e^{st} + \frac{2E_1 e^{st}}{3} - \frac{E_2 e^{st}}{3} = 0 \Rightarrow E_1(s + 2/3) - E_2/3 = 0$$

$$E_2 s e^{st} - \frac{E_1 e^{st}}{3} + \frac{2E_2 e^{st}}{3} = 0 \Rightarrow -E_1/3 + E_2(s + 2/3) = 0$$

The non-trivial solution

$$\begin{bmatrix} s+2/3 & -1/3 \\ -1/3 & s+2/3 \end{bmatrix}\begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = 0$$

- **The above equations have a non-trivial (non-zero) solution if equations are linearly dependent. From linear algebra we know this implies:**

$$\det\begin{bmatrix} s+2/3 & -1/3 \\ -1/3 & s+2/3 \end{bmatrix} = 0 \Rightarrow (s+2/3)^2 - 1/9 = 0$$

$$s^2 + \frac{4}{3}s + 1/3 = 0 \Rightarrow s_1 = -1,\ s_2 = -1/3$$

$$s_1 = -1 \Rightarrow v_1(t) = E_1^{s_1} e^{-t},\ \ v_2(t) = E_2^{s_1} e^{-t}$$

$$s_2 = -1/3 \Rightarrow v_1(t) = E_1^{s_2} e^{-t/3},\ \ v_2(t) = E_2^{s_2} e^{-t/3}$$

Obtaining the complete solution

- **We must solve for the values of E_1 and E_2**

$$s_1 = -1 \Rightarrow \begin{bmatrix} -1/3 & -1/3 \\ -1/3 & -1/3 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = 0 \Rightarrow E_1 = -E_2, \;\; choose\, E^{s_1} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$s_2 = -1/3 \Rightarrow \begin{bmatrix} 1/3 & -1/3 \\ -1/3 & 1/3 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = 0 \Rightarrow E_1 = E_2, \;\; choose\, E^{s_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- **Any multiple of the above is also a valid choice for E_1,E_2**
  - **Notice that with this choice for s_i, the equations are linearly dependant, which implies solution is not unique**

- **With this choice for E_i's and s_i's we get the following solutions:**

$$s_1 = -1 \Rightarrow v_1(t) = -1e^{-t}, \;\; v_2(t) = 1e^{-t}$$

$$s_2 = -1/3 \Rightarrow v_1(t) = 1e^{-t/3}, \;\; v_2(t) = 1e^{-t/3}$$

The complete solution

- Since the system is linear (and homogeneous - I.e., no inputs) any linear combination of the above solution is also a solution. I.e.,

$$v_1(t) = -Ae^{-t} + \ Be^{-t/3}$$
$$v_2(t) = Ae^{-t} + Be^{-t/3}$$

- In the above, A and B are constants
  - Can solve for A and B using initial conditions $V_1(0)$ and $V_2(0)$

Solving the state equations using Eigen-values and Eigen-Vectors

$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} -2/3 & 1/3 \\ 1/3 & -2/3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Rightarrow A = \begin{bmatrix} -2/3 & 1/3 \\ 1/3 & -2/3 \end{bmatrix}$$

- **Any linear homogeneous system can be expressed in the form:**

$$\dot{x} = Ax, \ \text{ where A is an nxn matrix}$$

- **Guess a solution of the form:** $\vec{x} = \vec{V}e^{st}$

$$\left.\begin{aligned} \dot{\vec{x}} &= \vec{V}se^{st} \\ \dot{\vec{x}} &= Ax \\ \vec{x} &= \vec{V}e^{st} \end{aligned}\right\} \Rightarrow \vec{V}se^{st} = A\vec{V}e^{st}$$

$$s\vec{V} = A\vec{V} \Rightarrow (sI - A)\vec{V} = 0; \ (I : nxn \ identity \ matrix)$$

The characteristic equation

- A non-trivial solution to $(sI - A)\vec{V} = 0$ exists only if:

$$\det[sI - A] = 0 \ \text{(characterstic equation for A)}$$

- Values of s satisfying the characteristic equation are Eigen-values of the matrix A
  - Corresponding solution for V is an eigen-vector

- General solution to $\dot{x} = Ax$ is:

$$x(t) = \sum_i a_i \vec{V}_i e^{-\lambda_i t}$$

$$\lambda_i \sim \text{are eigen-values of A}$$

$$\vec{V}_i \sim \text{corresponding eigen-vector}$$

$$a_i \sim \text{constant that depends on initial conditions}$$

Example

$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} -2/3 & 1/3 \\ 1/3 & -2/3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Rightarrow A = \begin{bmatrix} -2/3 & 1/3 \\ 1/3 & -2/3 \end{bmatrix}$$

$$sI - A = \begin{bmatrix} s+2/3 & -1/3 \\ -1/3 & s+2/3 \end{bmatrix}$$

$$\det \begin{bmatrix} s+2/3 & -1/3 \\ -1/3 & s+2/3 \end{bmatrix} = 0 \Rightarrow (s+2/3)^2 - 1/9 = 0$$

$$s^2 + \frac{4}{3}s + 1/3 = 0 \Rightarrow s_1 = -1,\ s_2 = -1/3 \quad "eigen-values"$$

$$eigen-vectors:\ V_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, V_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Example, cont.

$$\vec{v}(t) = a_1\vec{V}_1 e^{-t} + a_2\vec{V}_2 e^{-t/3}$$

$$v_1(t) = -a_1 e^{-t} + a_2 e^{-t/3}$$

$$v_2(t) = a_1 e^{-t} + a_2 e^{-t/3}$$

$$Initial\ \ conditions : v_1(0) = 10,\ v_1(0) = 0 \Rightarrow$$

$$\left.\begin{aligned} -a_1 + a_2 &= 10 \\ a_1 + a_2 &= 0 \end{aligned}\right\} \Rightarrow a_1 = -5,\ a_2 = 5$$

$$Solution : v_1(t) = 5e^{-t} + 5e^{-t/3},\ v_2(t) = -5e^{-t} + 5e^{-t/3}$$