University of Washington

STATISTICS

STAT 498 B

Statistical Tolerancing

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Objective of Statistical Tolerancing

Concerns itself with mass production, not custom made items.

Dimensions and properties of parts are not exactly what they should be.

Worst case tolerancing can be quite costly.

Manage variation in mechanical assemblies or systems.

Take advantage of statistical independence in variation cancelation.

Also known as statistical error propagation.

Useful when errors and system sensitivities are small.

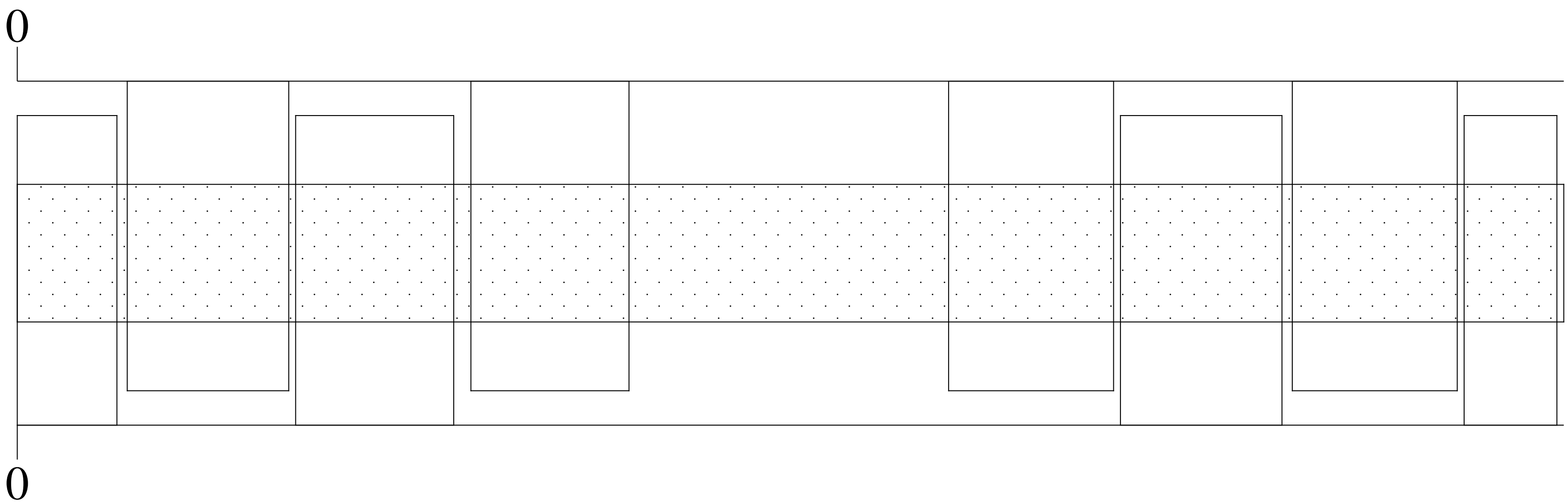
It is more in the realm of probability than statistics (no inference).

Exchangeability of 757 Cargo Doors

At issue were the tolerances of gaps and lugs of hinges and their placement on the hinge lines of aircraft body and door.

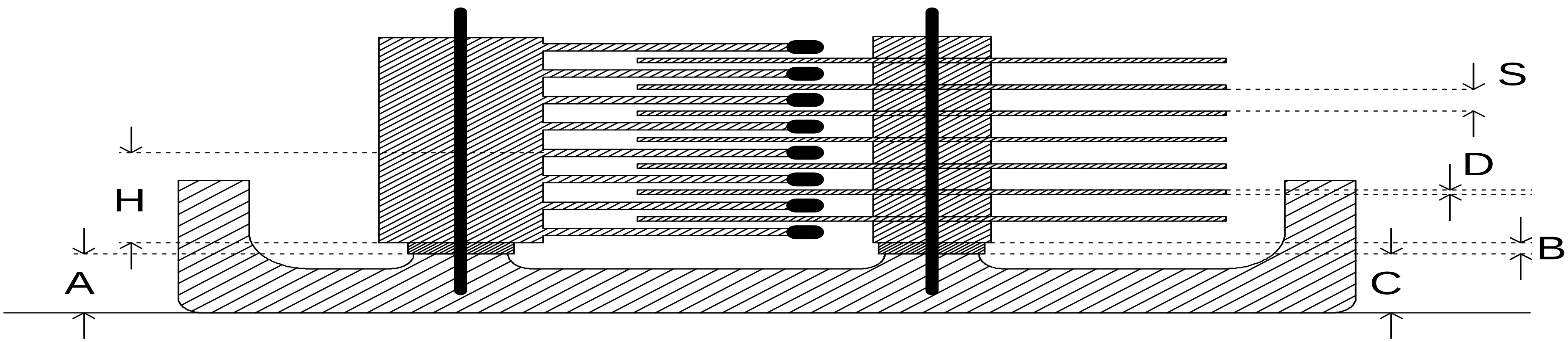
10 hinges with 12 lugs/gaps each.

That means that a lot of dimensions have to fit just about right.

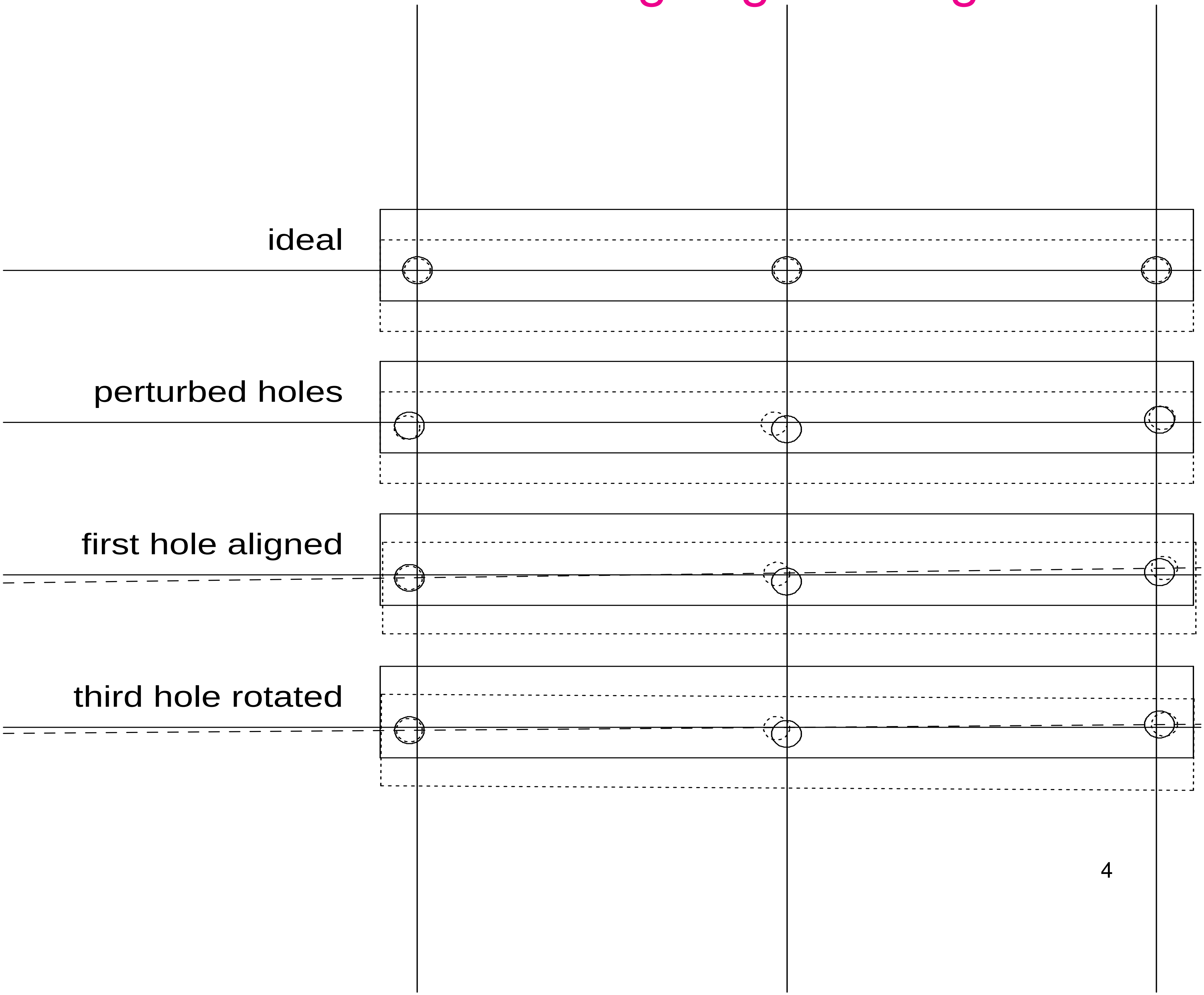


The Root Sum Square (RSS) paradigm does not work here!

IBM Collaboration: Disk Drive Tolerances



Coordination Holes for Aligning Fuselage Panels



Main Ingredients: Mean, Variance & Standard Deviation

The dimension or property of interest, X, is treated as a random variable.

$X \sim f(x)$ (density), CDF $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\, dt$.

Mean: $\mu = \mu_X = \mathrm{E}(X) = \int_{-\infty}^{x} t\, f(t)\, dt$

Variance: $\sigma^2 = \sigma_X^2 = \mathrm{var}(X) = \mathrm{E}((X-\mu)^2) = \mathrm{E}(X^2) - \mu^2 = \int_{-\infty}^{x} (t-\mu)^2 f(t)\, dt$

Standard Deviation: $\sigma = \sqrt{\mathrm{var}(X)}$

Rules for $\mathrm{E}(X)$ and $\mathrm{var}(X)$

For constants $a_1, \ldots, a_k$ and random variables $X_1, \ldots, X_k$

we have for $Y = a_1 X_1 + \ldots + a_k X_k$

$$\mathrm{E}(Y) = \mathrm{E}(a_1 X_1 + \ldots + a_k X_k) = a_1 \mathrm{E}(X_1) + \ldots + a_k \mathrm{E}(X_k)$$

For constants $a_1, \ldots, a_k$ and independent random variables $X_1, \ldots, X_k$ we have

$$\sigma_Y^2 = \mathrm{var}(Y) = \mathrm{var}(a_1 X_1 + \ldots + a_k X_k) = a_1^2 \mathrm{var}(X_1) + \ldots + a_k^2 \mathrm{var}(X_k)$$

It is this latter property that justifies the existence of the variance concept.

$$\sigma_Y = \sqrt{a_1^2 \mathrm{var}(X_1) + \ldots + a_k^2 \mathrm{var}(X_k)}$$

Central Limit Theorem (CLT) I

- Suppose we randomly and independently draw random variables $X_1, \ldots, X_n$ from n possibly different populations with respective means and standard deviations $\mu_1, \ldots, \mu_n$ and $\sigma_1, \ldots, \sigma_n$

- Suppose further that

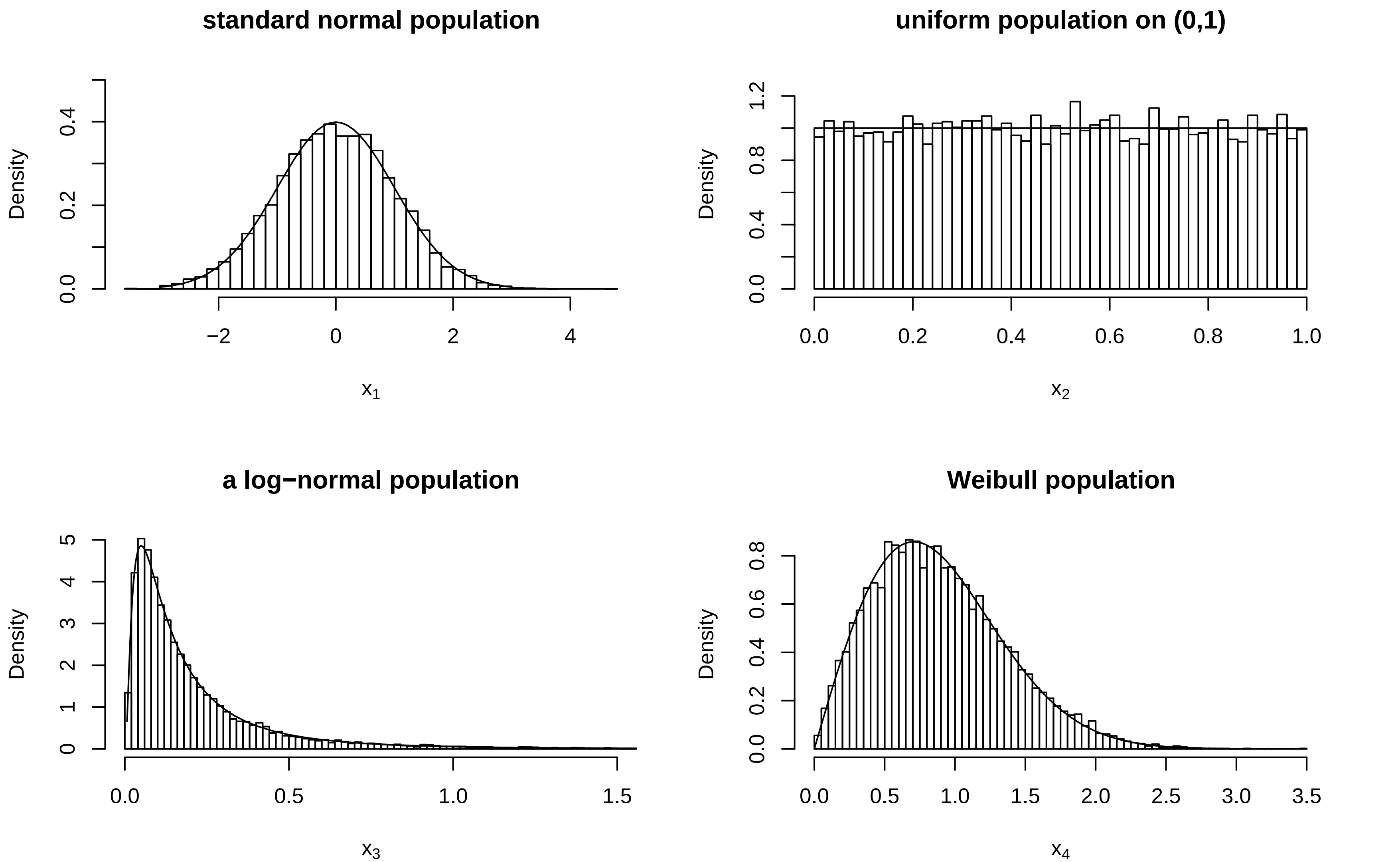
$$\frac{\max\left(\sigma_1^2, \ldots, \sigma_n^2\right)}{\sigma_1^2 + \ldots + \sigma_n^2} \to 0\,, \quad \text{as} \quad n \to \infty$$

i.e., none of the variances dominates among all variances

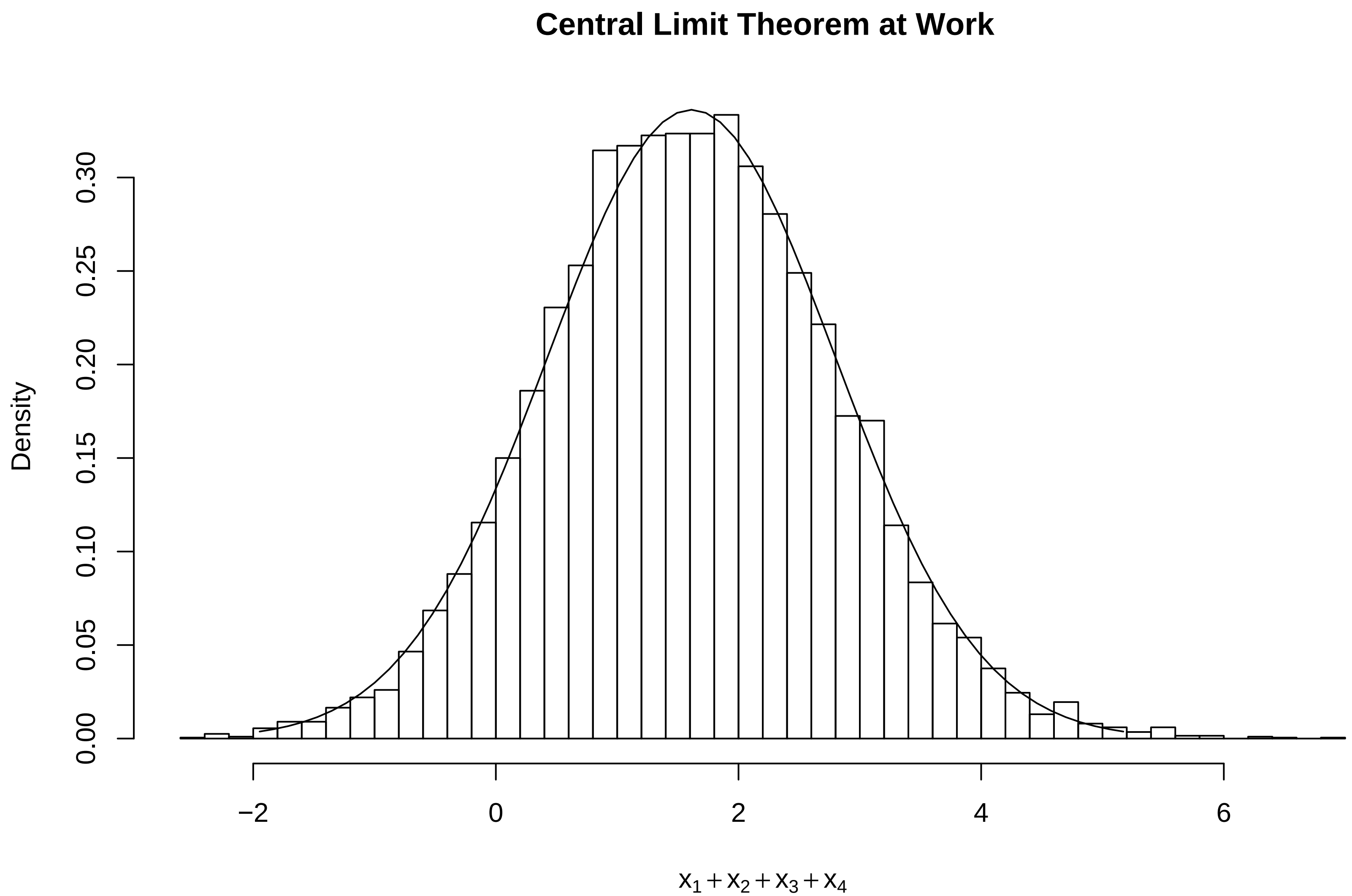
- Then $Y_n = X_1 + \ldots + X_n$ has an approximate normal distribution with mean and variance given by

$$\mu_Y = \mu_1 + \ldots + \mu_n \qquad \text{and} \qquad \sigma_Y^2 = \sigma_1^2 + \ldots + \sigma_n^2\,.$$

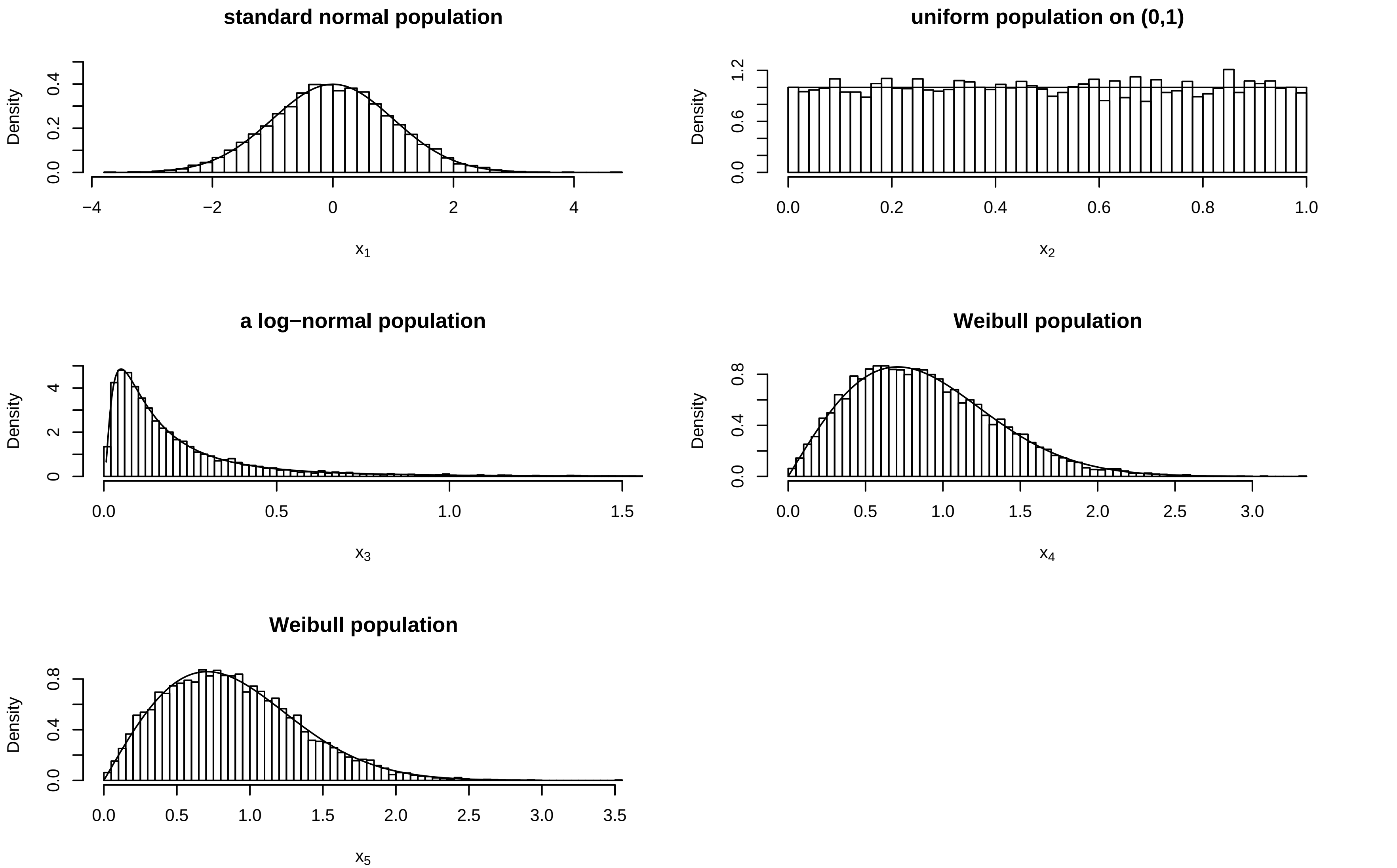
CLT: Example 1



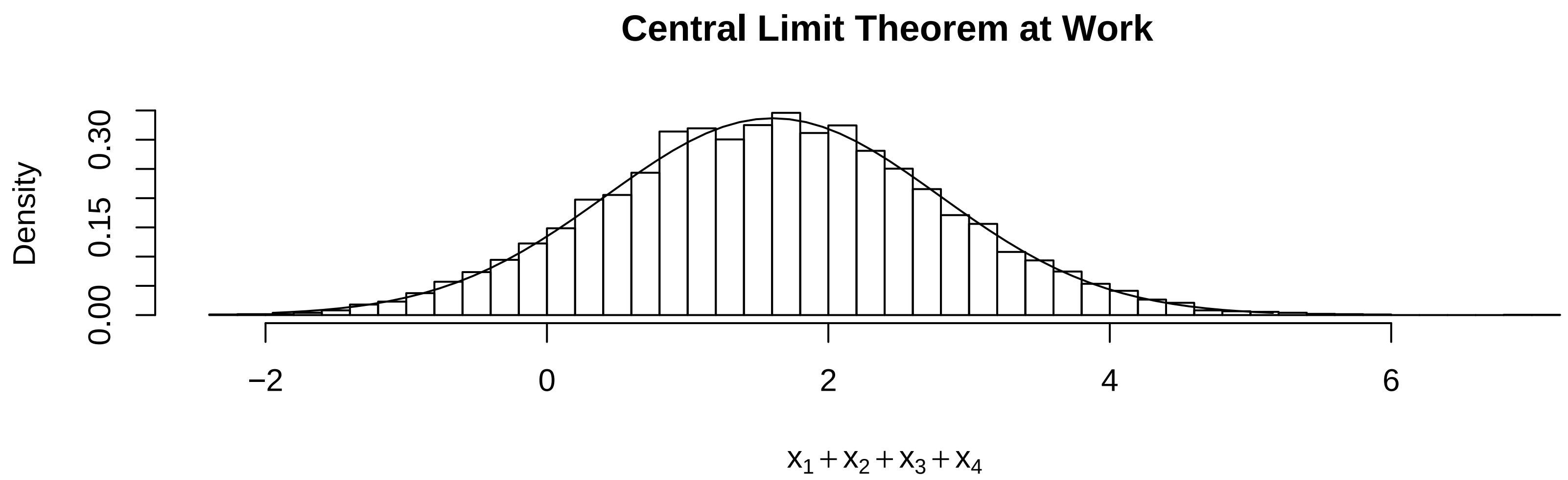
CLT: Example 2

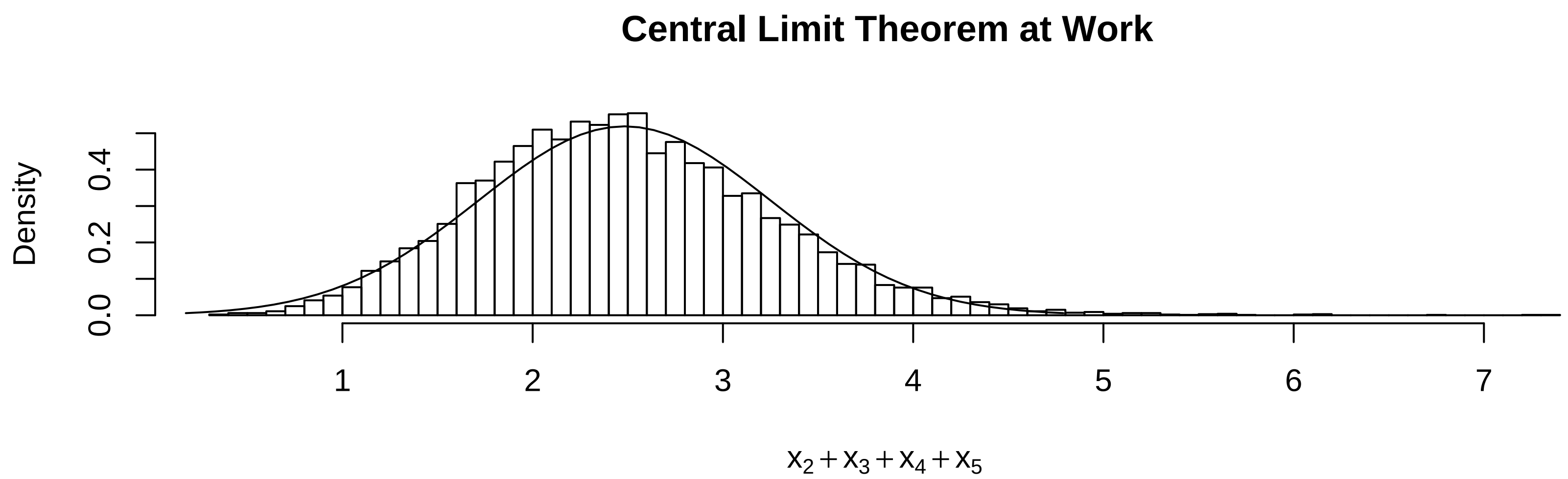


CLT: Example 3

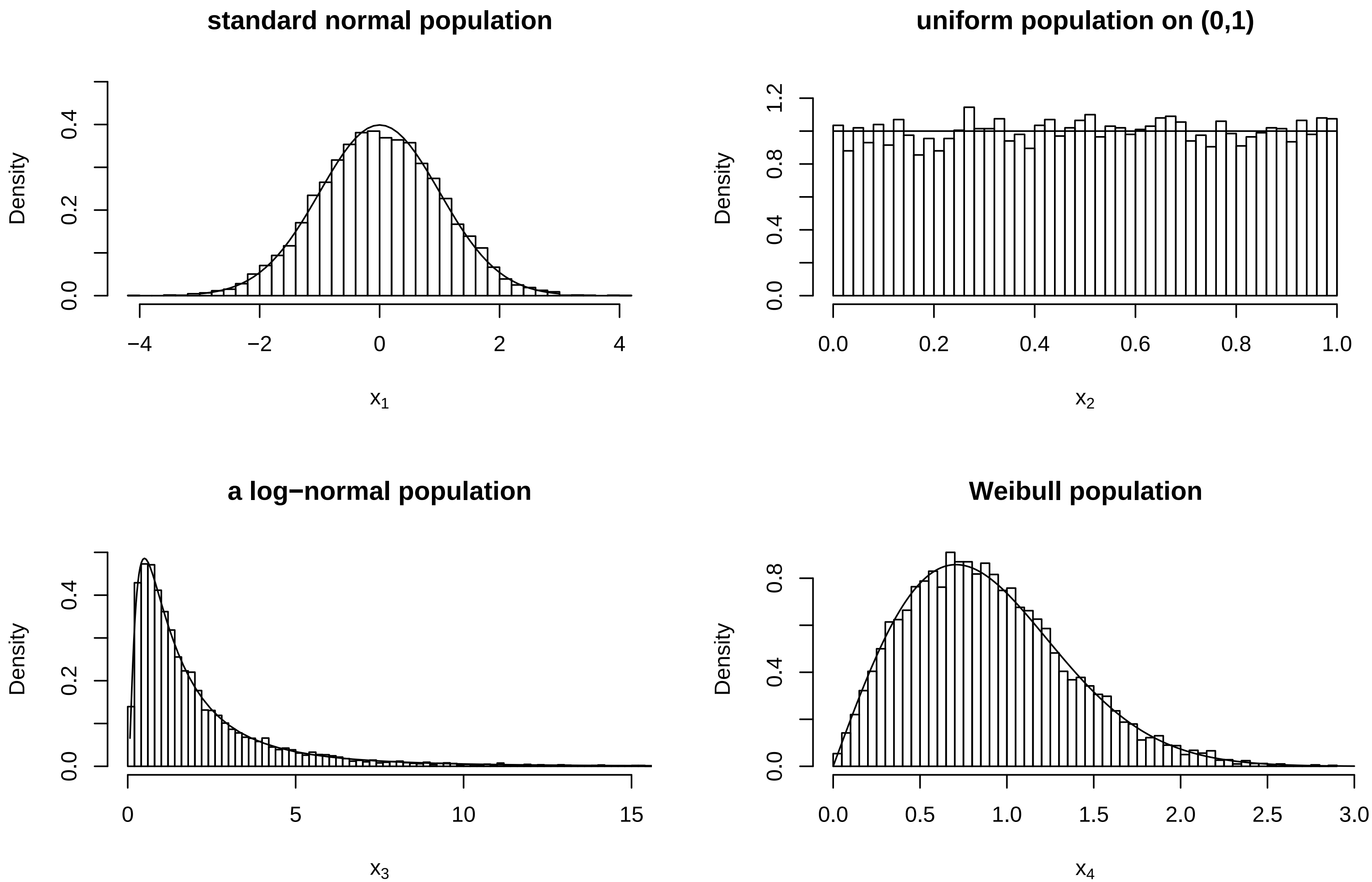


CLT: Example 4

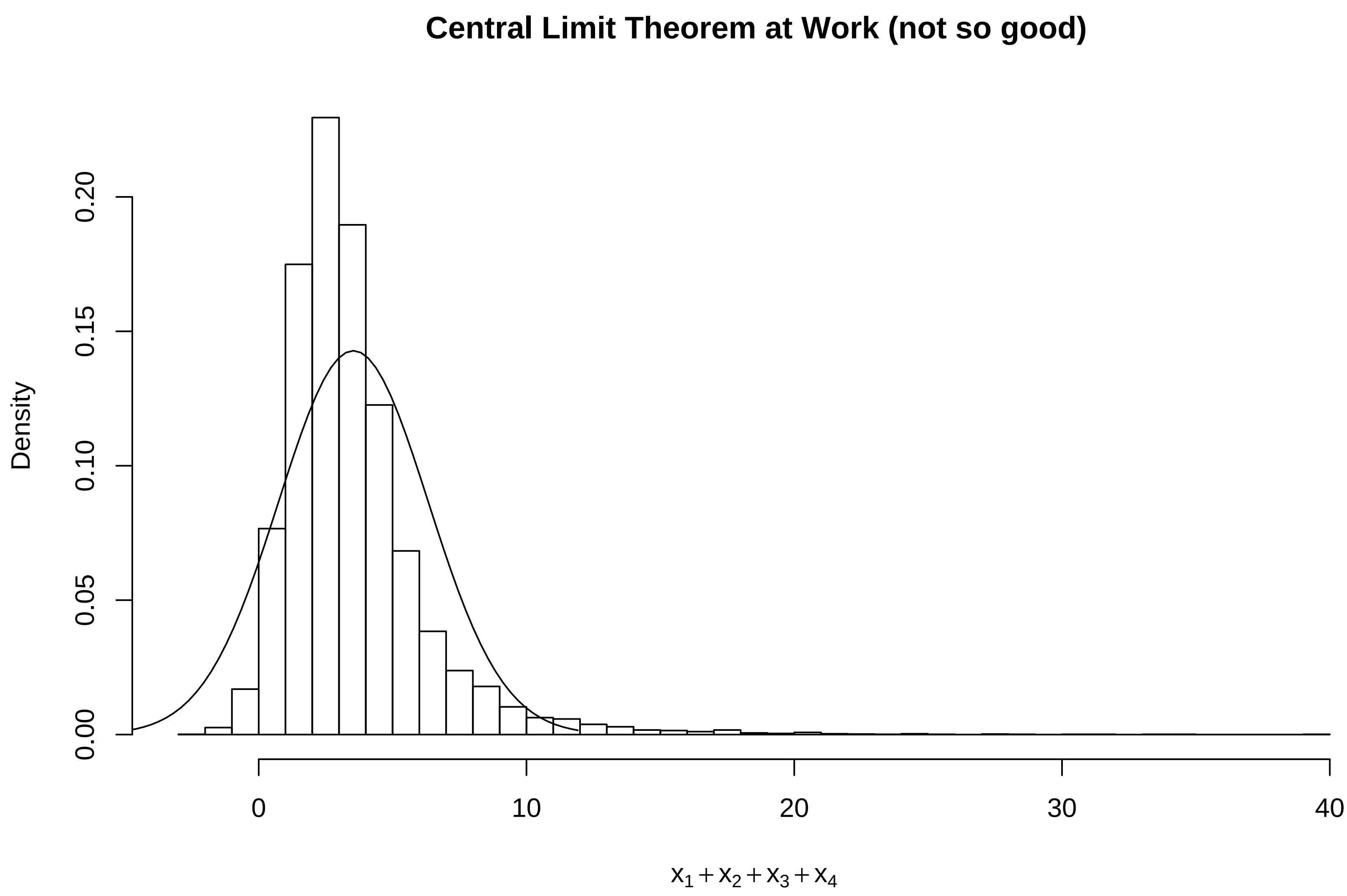




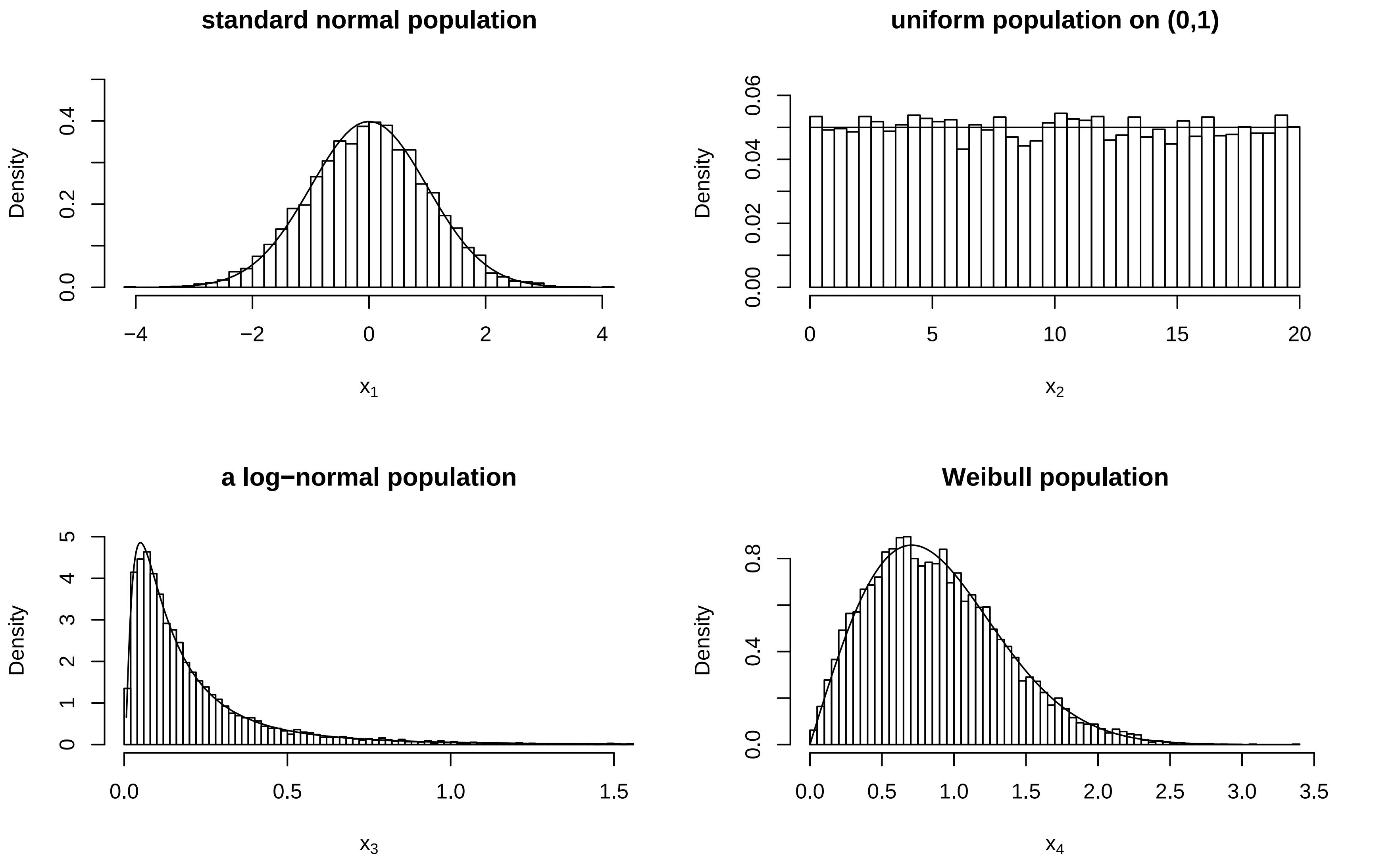
CLT: Example 5



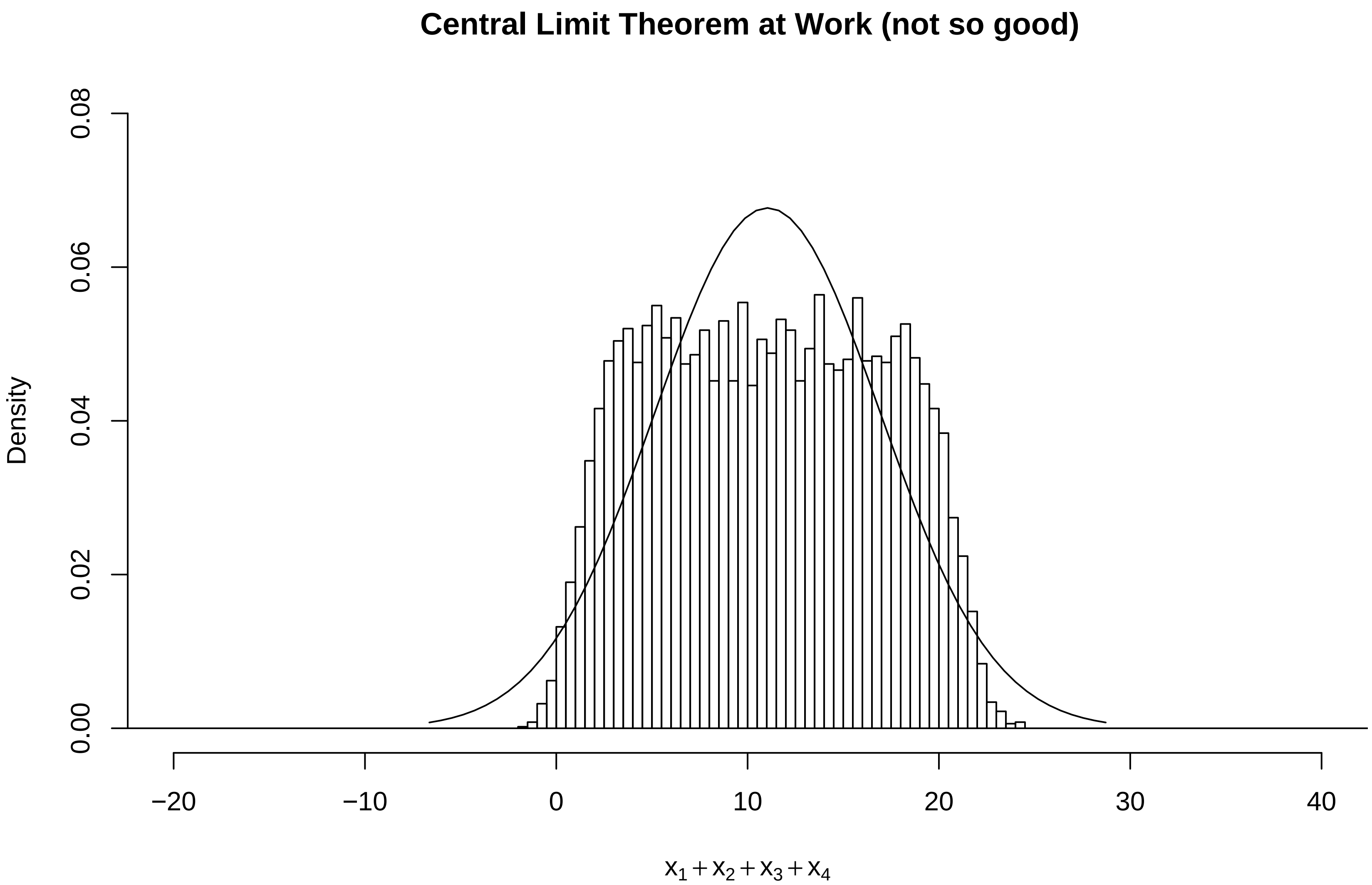
CLT: Example 6



CLT: Example 7



CLT: Example 8



What is a Tolerance?

- Tolerances recognize that **part dimensions are not what they should be.**
  "should be" $=$ nominal or exact according to engineering design
  Exact dimensions allow mass production assembly using interchangeable parts

- **Variations** around nominal are controlled by tolerances.

- Typical two-sided specification: $[\text{Nominal} - \text{Tolerance}, \ \ \text{Nominal} + \text{Tolerance}]$

- Specifications can be one-sided:
  $[\text{Nominal}, \ \ \text{Nominal} + \text{Tolerance}]$ or $[\text{Nominal} - \text{Tolerance}, \ \ \text{Nominal}]$

- Specifications can be asymmetric: $[\text{Nominal} - \text{Tolerance}_1, \ \ \text{Nominal} + \text{Tolerance}_2]$

Simple Examples

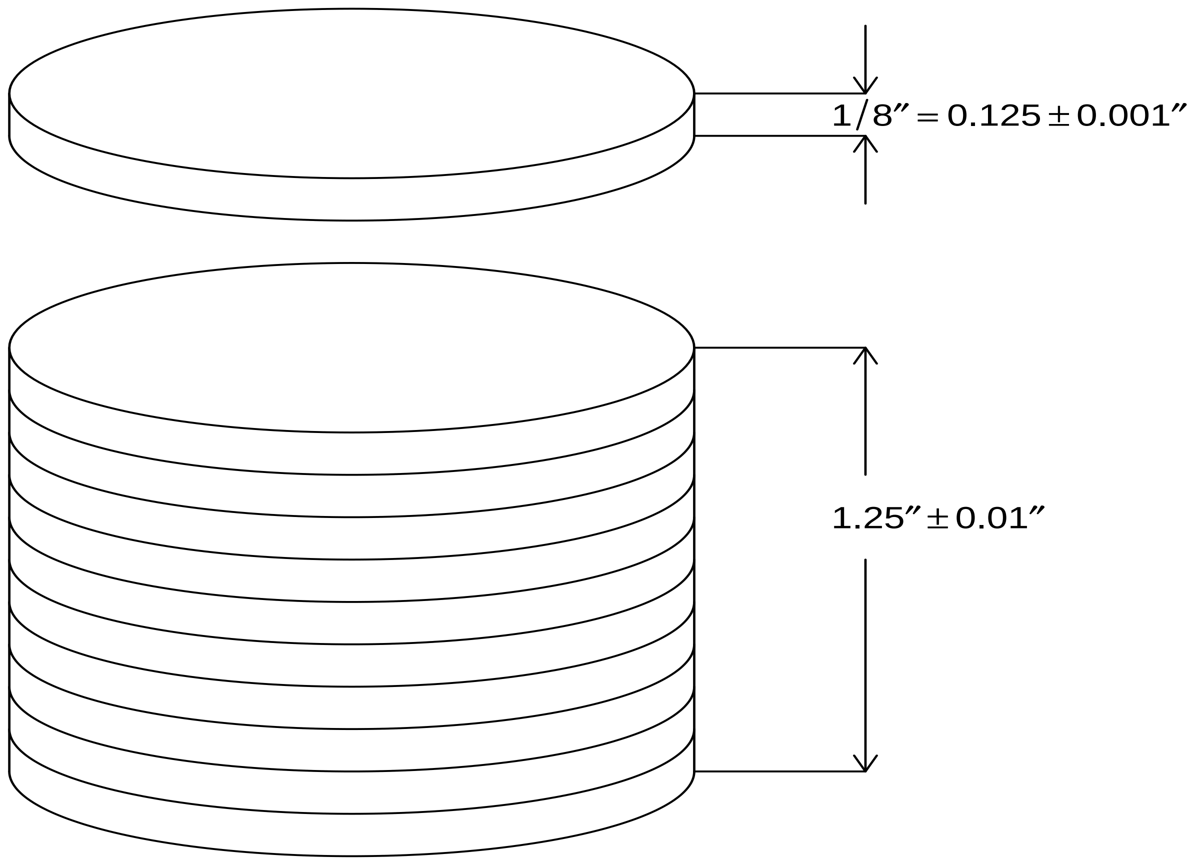
- **Example 1:** A disk should have thickness $1/8''$ with $\pm\ .001''$ tolerance, i.e., the disk thickness should be in the range

$$[.125'' - .001'',\ .125'' + .001''] = [.124'',\ .126''] \ .$$

- **Example 2:** A stack of **ten disks** should be $1.25''$ high with $\pm\ .01''$ tolerance, i.e., the stack height should be in the range

$$[1.25'' - .01'',\ 1.25'' + .01''] = [1.24'',\ 1.26''] \ .$$

Disk Stack



Worst Case or Arithmetic Tolerancing

- The tolerance specification in Example 1, if adhered to, **guarantees** the tolerance specification in Example 2.

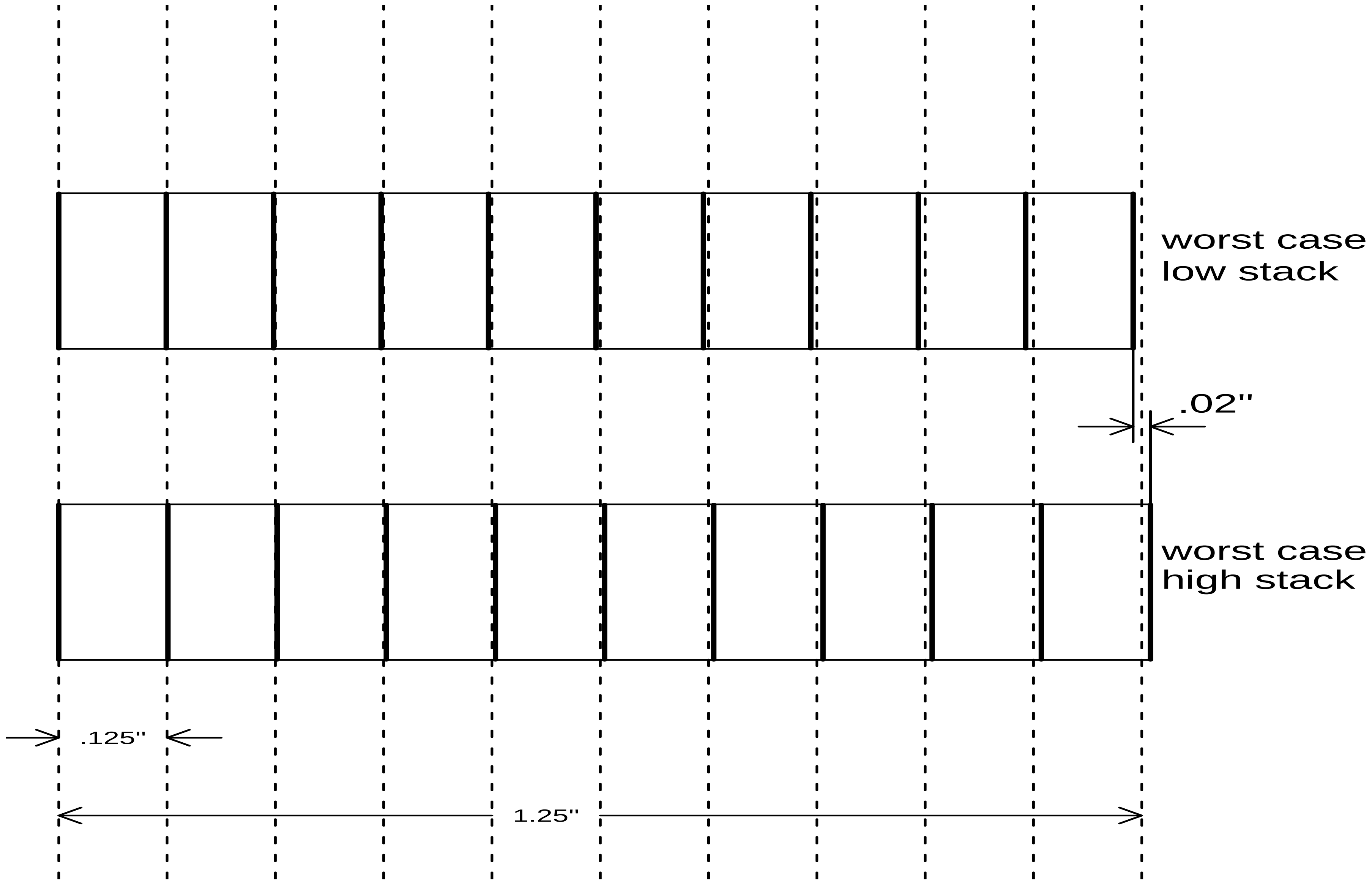
- The reasoning is based on **worst case or arithmetic tolerancing**

- The stack is highest when all disks are as thick as possible.
  $.126''$ per disk $\Longrightarrow$ stack height of $10 \times .126'' = 1.26''$.

- The stack is lowest when all disks are as thin as possible.
  $.124''$ per disk $\Longrightarrow$ stack height of $10 \times .124'' = 1.24''$.

- This gives the total possible stack height range as $[1.24'', 1.26'']$.

disk stack/tolerance stack



Worst Case or Arithmetic Tolerancing in Reverse

- This reasoning can be reversed.
- If the stack height has specified end tolerance $\pm .01''$,

  and if the disk tolerances are to be the same for all disks (exchangeable),

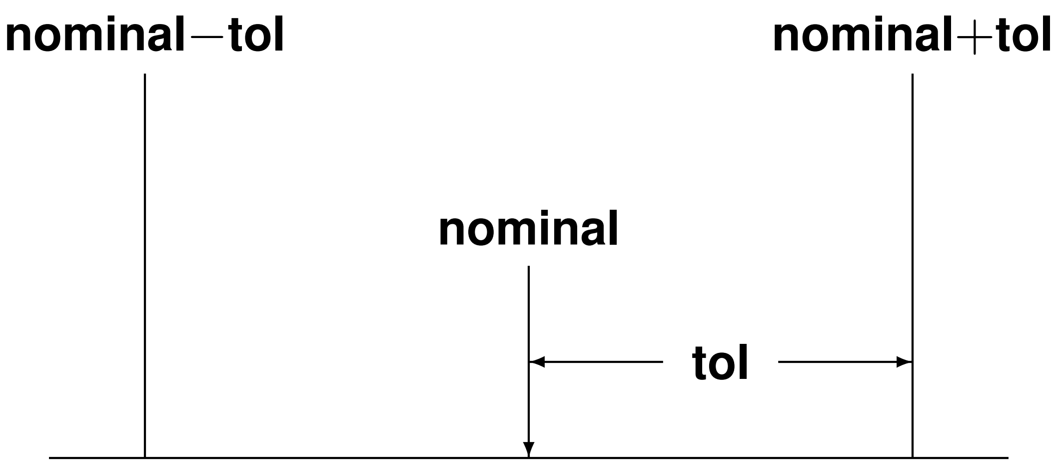
  then we should, by the worst case tolerancing reasoning, assign

$$\pm .01''/10 = \pm .001''$$

  tolerances to the individual disks (item tolerances).

- End tolerances can create very tight and unrealistic item tolerances. Costly!

Worst Case Analysis or Goal Post Mentality



Add some structure, aim for the middle

$\Longrightarrow$ Statistical Tolerancing

Statistical Tolerancing Assumption

- **Statistical tolerancing** assumes that disks are chosen **at random**, not deliberately to make a worst possible stack, one way or the other.

- The disk thickness variation within tolerances is described by a distribution.

- The histogram, summarizing these thicknesses, is often assumed to be $\approx$ **normal** or **Gaussian** with center μ_D at the middle of the tolerance range and with standard deviation such that

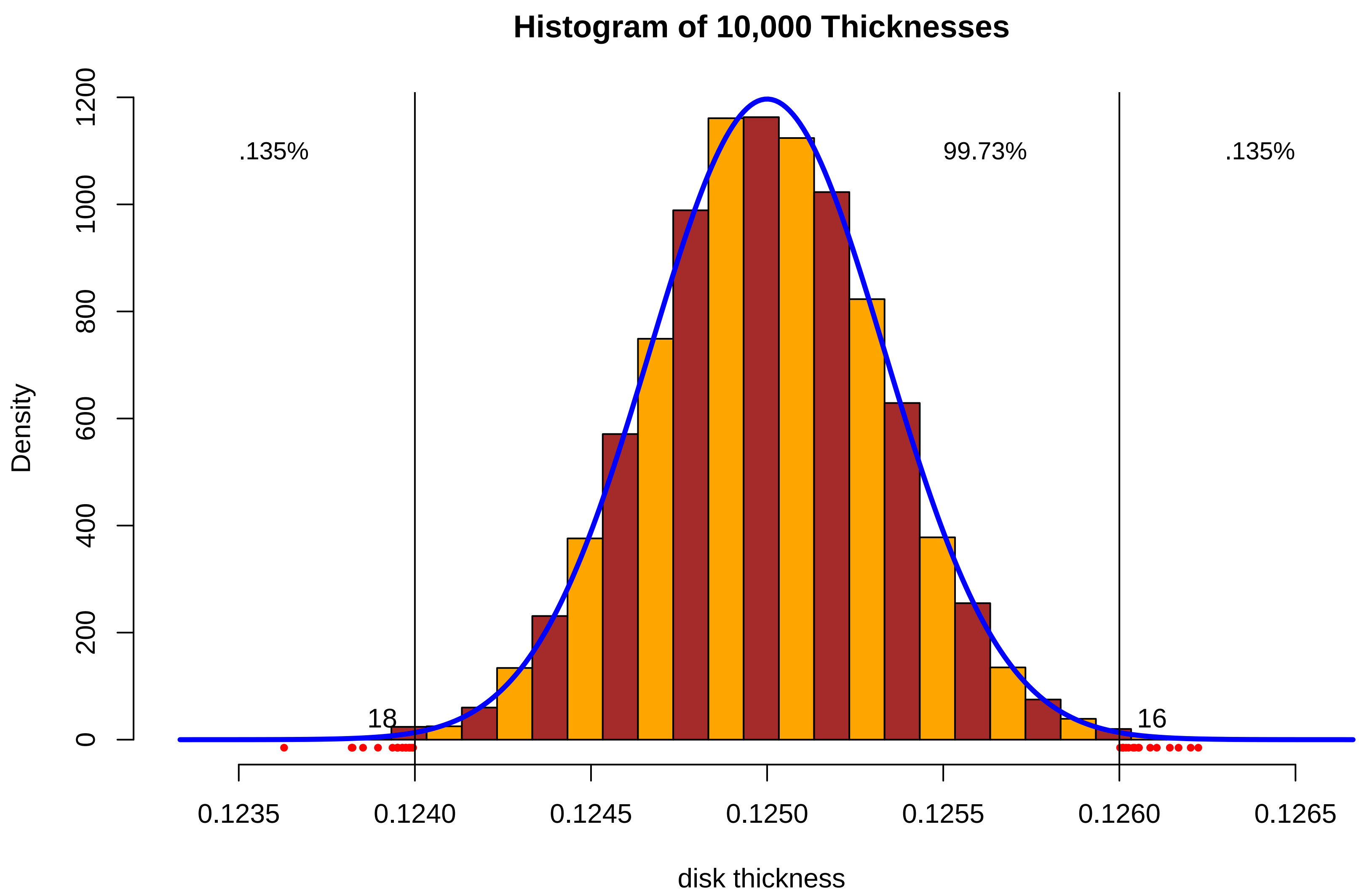
  $$\pm 3 \text{ standard deviations} \quad = \quad \pm \text{ tolerance} \, .$$

  or

  $$\sigma_D = \frac{1}{3} \times \text{TOL}_D \quad \text{so that} \quad [\mu_D - 3\sigma_D, \, \mu_D + 3\sigma_D] = \text{tolerance interval}$$

- The normality assumption is a simplification, but is not essential.

Normal Histogram/Distribution of Disk Thicknesses



Why Does Statistical Tolerancing Work

- Under the normal population model $\Longrightarrow$ we will see about 13.5 out of $10,000$ disks with thickness $\geq .126''$.

- The chance of randomly selecting such a fat or fatter disk is $.00135 = 13.5/10,000$

- $\Rightarrow$ The chance of having such bad (thick) luck **ten times in a row** is

$$.00135 \times \ldots \times .00135 = (.00135)^{10} = 2.01 \times 10^{-29} \; (!!!)$$

- Choosing thicknesses at random from this normal population we (justifiably) hope that thick and thin will **average** out to some extent.

- Make **independent** variation work for you, not against you!

  If life gives you lemons, make lemonade! Turn a negative into a positive!

The Insurance Principle of Averaging

- We look forward to the day when everyone will receive
  more than the average wage.
  Australian Minister of Labour, 1973

- The etymology of “average” derives from the Arabic: awārīyah meaning shipwreck, damaged goods, and linking it to the custom of averaging the losses of damaged cargo across all merchants

- You get the good with the bad

- “Havarie” in German means: shipwreck

- “Awerij” in Dutch/Afrikaans means: average, damage to ship or cargo

Distribution of Stack Heights

- Choosing many stacks $S = D_1 + \ldots + D_{10}$ of ten disks each we get a normal population of stack heights,

- with mean $\mathrm{E}(S) = \mathrm{E}(D_1) + \ldots + \mathrm{E}(D_{10}) = 10 \times .125'' = 1.25''$,

- and standard deviation

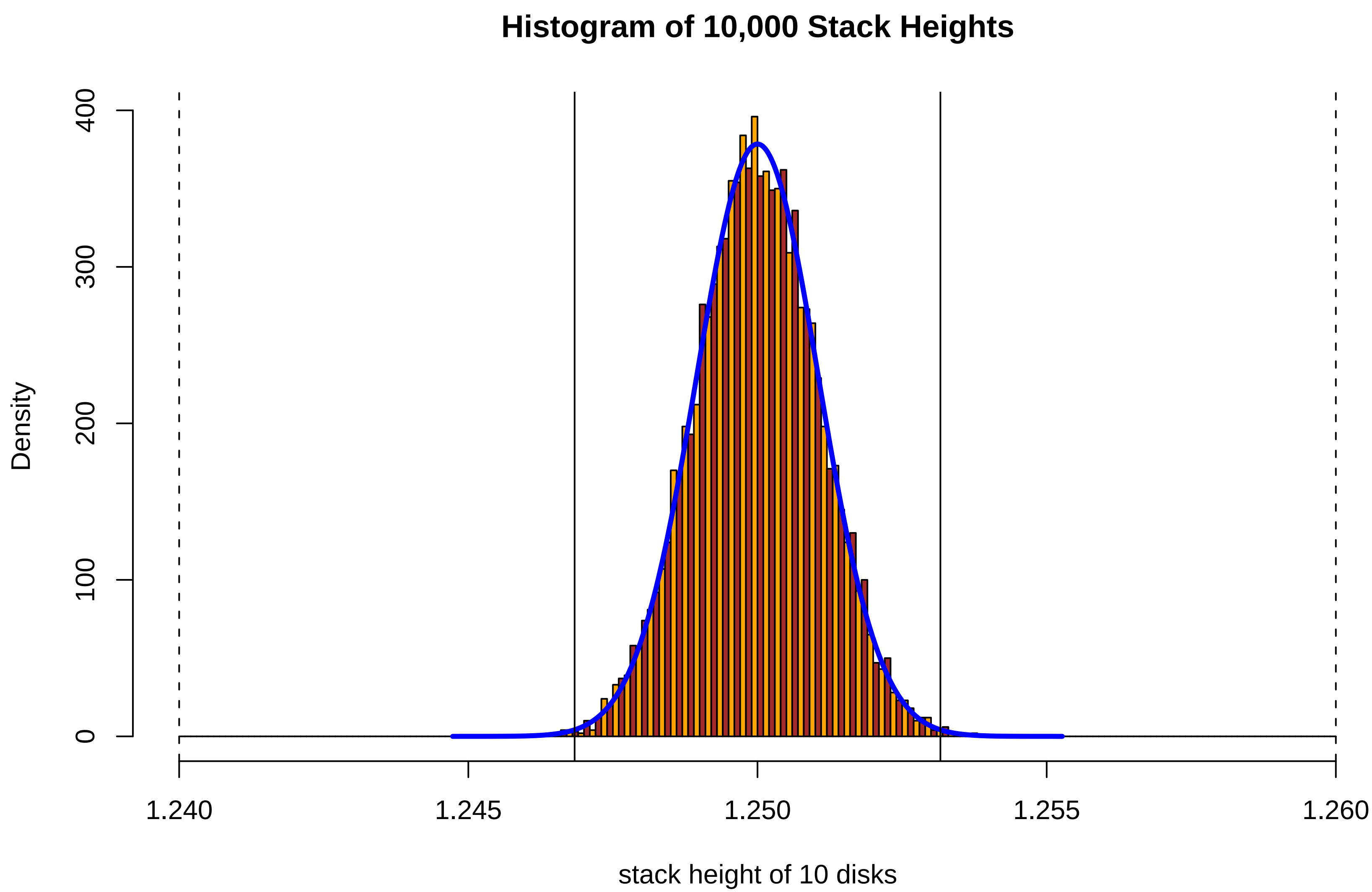
$$\sigma_S = \sqrt{\sigma^2_{D_1} + \ldots + \sigma^2_{D_{10}}} = \sqrt{10} \times \sigma_D = \sqrt{10} \times .001''/3 = .00105''$$

- thus S ranges over

$$1.25'' \pm 3 \times \sqrt{10} \times .001''/3 = 1.25'' \pm \sqrt{10} \times .001'' = 1.25'' \pm .00316''$$

$$.00316'' = \sqrt{10} \times .001'' \ll 10 \times .001'' = .01''$$

Normal Histogram/Distribution of Stacks



Root Sum Square (RSS) Method

For $S = D_1 + \ldots + D_{10}$, with independent disk thicknesses D_i, we have

$$\sigma_S = \sqrt{\mathrm{var}(D_1 + \ldots + D_{10})} = \sqrt{\sigma^2_{D_1} + \ldots + \sigma^2_{D_{10}}}$$

Interpreting $\mathrm{TOL}_i = \mathrm{TOL}_{D_i} = 3\sigma_{D_i}$ and $\mathrm{TOL}_S = 3\sigma_S$ we have

$$\begin{aligned} \mathrm{TOL}_S = 3\sigma_S &= 3\sqrt{\sigma^2_{D_1} + \ldots + \sigma^2_{D_{10}}} = \sqrt{(3\sigma_{D_1})^2 + \ldots + (3\sigma_{D_{10}})^2} \\ &= \sqrt{\mathrm{TOL}_1^2 + \ldots + \mathrm{TOL}_{10}^2} = \sqrt{10} \times \mathrm{TOL}_D \end{aligned}$$

$\mu_S \pm 3\sigma_S$ contains 99.73% of the S values, because $S \sim \mathcal{N}(\mu_S, \sigma_S^2)$.

This is referred to as the Root Sum Square (RSS) Method of tolerance stacking.

Contrast with arithmetic or worst case tolerance stacking

$$\mathrm{TOL}_S^\star = \mathrm{TOL}_1^\star + \ldots + \mathrm{TOL}_{10}^\star = 10 \times \mathrm{TOL}_D^\star$$

Some Comments on $\star$ Notation

Numerically $\mathrm{TOL}_i = \mathrm{TOL}_i^\star$ are the same, they are just different in what they represent: statistical variation range versus worst case variation range.

Again, TOL_S and $\mathrm{TOL}_S^\star$ represent statistical and worst case variation ranges, but they are not the same since

$$\sqrt{\mathrm{TOL}_1^2 + \ldots + \mathrm{TOL}_{10}^2} = \sqrt{(\mathrm{TOL}_1^\star)^2 + \ldots + (\mathrm{TOL}_{10}^\star)^2} \leq \mathrm{TOL}_1^\star + \ldots + \mathrm{TOL}_{10}^\star$$

We get $=$ only in the trivial cases when $n = 1$ or when $n > 1$ and $\mathrm{TOL}_1 = \ldots = \mathrm{TOL}_n = 0$.

Statistical Tolerancing Benefits

- $\Longrightarrow$ stack height variation is much tighter than specified
- could try to relax the tolerances on the disks,
- relaxed tolerances $\Longrightarrow$ lower cost of part manufacture
- take advantage of tighter assembly tolerances $\Longrightarrow$ easier assembly

RSS for General n

When we stack n disks, replace 10 by n above:

$$\mathrm{TOL}_S = \sqrt{n} \times \mathrm{TOL}_D \quad \text{or} \quad \mathrm{TOL}_D = \frac{1}{\sqrt{n}} \times \mathrm{TOL}_S$$

As opposed to the worst case tolerancing relationships

$$\mathrm{TOL}_S^\star = n \times \mathrm{TOL}_D^\star \quad \text{or} \quad \mathrm{TOL}_D^\star = \frac{1}{n} \times \mathrm{TOL}_S^\star$$

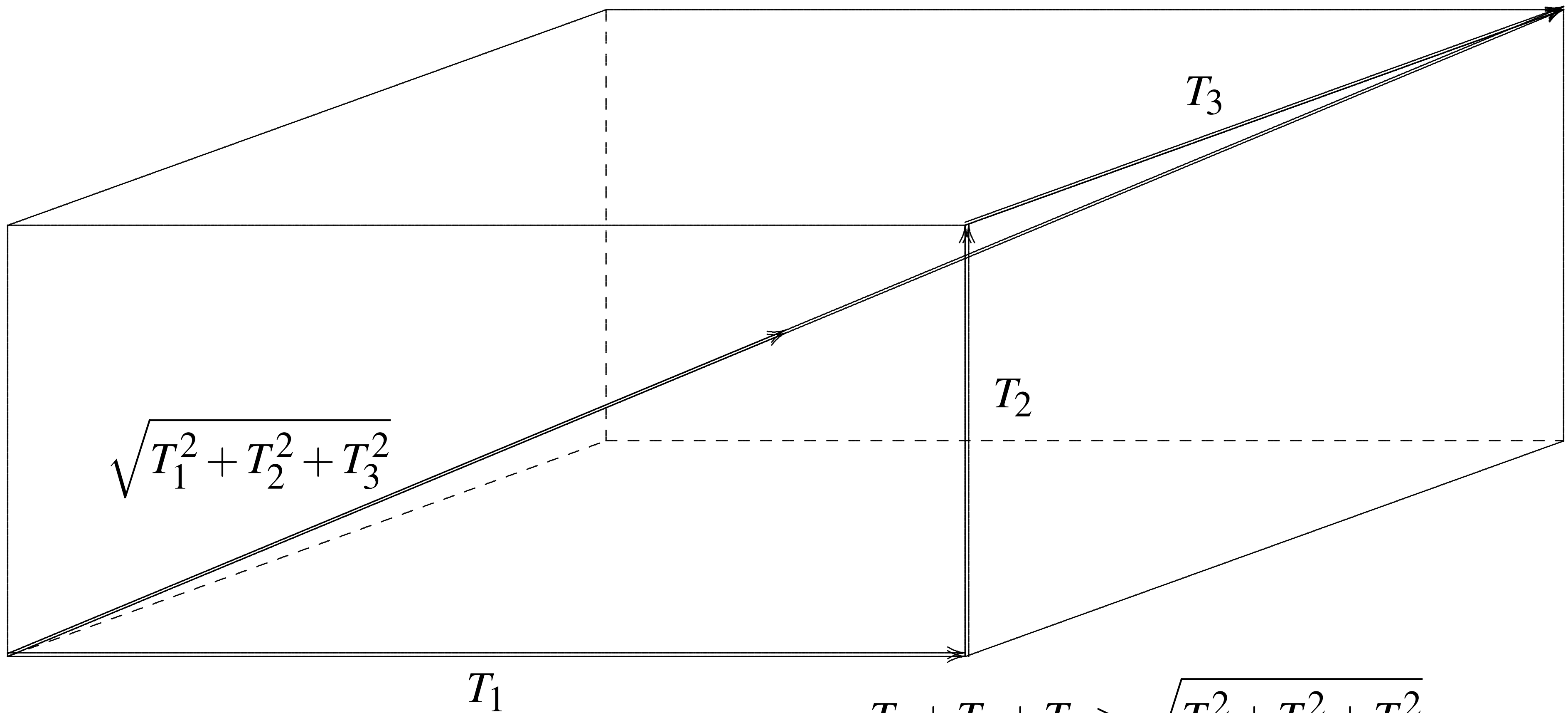
More generally when the TOL_{D_i} are not all the same

$$\mathrm{TOL}_S = \sqrt{\mathrm{TOL}_1^2 + \ldots + \mathrm{TOL}_n^2} \qquad \text{or} \qquad \mathrm{TOL}_S^\star = \mathrm{TOL}_{D_1}^\star + \ldots + \mathrm{TOL}_{D_n}^\star$$

Reverse engineering $\mathrm{TOL}_S \longrightarrow \mathrm{TOL}_{D_i}$ or $\mathrm{TOL}_S^\star \longrightarrow \mathrm{TOL}_{D_i}^\star$ not so obvious.

Reduce the largest TOL_{D_i} to get greatest impact on TOL_S. $\mathrm{TOL}_{D_i}^\star$???

RSS $=$ Pythagorean Shortcut



$$T_1 + T_2 + T_3 > \sqrt{T_1^2 + T_2^2 + T_3^2}$$

Benderizing

As much as RSS gives advantages over worst case or arithmetic tolerancing it was found that the RSS tolerance buildup was often optimistic in practice.

A simple remedy was proposed by Bender (1962) and it was called Benderizing.

It consists in multiplying the RSS expression by 1.5, i.e., use

$$\mathrm{TOL}_S = 1.5 \times \sqrt{\mathrm{TOL}_1^2 + \ldots + \mathrm{TOL}_n^2}$$

This still only grows on the order of $\sqrt{n}$, but provides a safety cushion.

The motivation? When shop mechanics were asked about the dimension accuracy they could maintain, they would respond based on experience memory.

It was reasoned that a mechanic's experience covers mainly a $\pm 2\sigma$ range.
To adjust $\mathrm{TOL}_i = 2\sigma_i$ to $\mathrm{TOL}_i = 3\sigma_i$ the factor $3/2 = 1.5$ was applied.

Uniform Part Variation

Suppose that the normal variation does not adequately represent the variation of the manufactured disks.

Assume that disk thicknesses vary uniformly over

$[\text{nominal} - \text{TOL}_D,\ \text{nominal} + \text{TOL}_D] = [\mu - \text{TOL}_D,\ \mu + \text{TOL}_D]$ due to tool wear.

$\Rightarrow \text{E}(D) = \mu$ and

$$\sigma_D^2 = \int_{\mu-\text{TOL}_D}^{\mu+\text{TOL}_D} \frac{1}{2\text{TOL}_D}(t-\mu)^2\,dt \qquad \text{substituting } (t-\mu)/\text{TOL}_D = x$$

$$= \text{TOL}_D^2 \int_{-1}^{1} \frac{1}{2}\,x^2\,dx \qquad \text{with } dt/\text{TOL}_D = dx$$

$$= \text{TOL}_D^2 \left[\frac{x^3}{6}\right]_{-1}^{1} = \text{TOL}_D^2 \left(\frac{1^3}{6} - \frac{(-1)^3}{6}\right) = \frac{\text{TOL}_D^2}{3}$$

$\Longrightarrow \sigma_D = \text{TOL}_D/\sqrt{3}$ or $3\sigma_D = \sqrt{3}\,\text{TOL}_D = c\,\text{TOL}_D,\quad c = \sqrt{3} = 1.732.$

Uniform Part Variation Impact on TOL_S

For $n \geq 3$ the distribution of S is approximately normal, i.e., $S \approx \mathcal{N}(\mu_S, \sigma_S^2)$

see next slide.

Thus most ($\approx 99.73\%$) of the S variation is within $\mu_S \pm 3\sigma_S$

$$\mathrm{TOL}_S = 3\sigma_S = \sqrt{(3\sigma_{D_1})^2 + \ldots + (3\sigma_{D_n})^2}$$

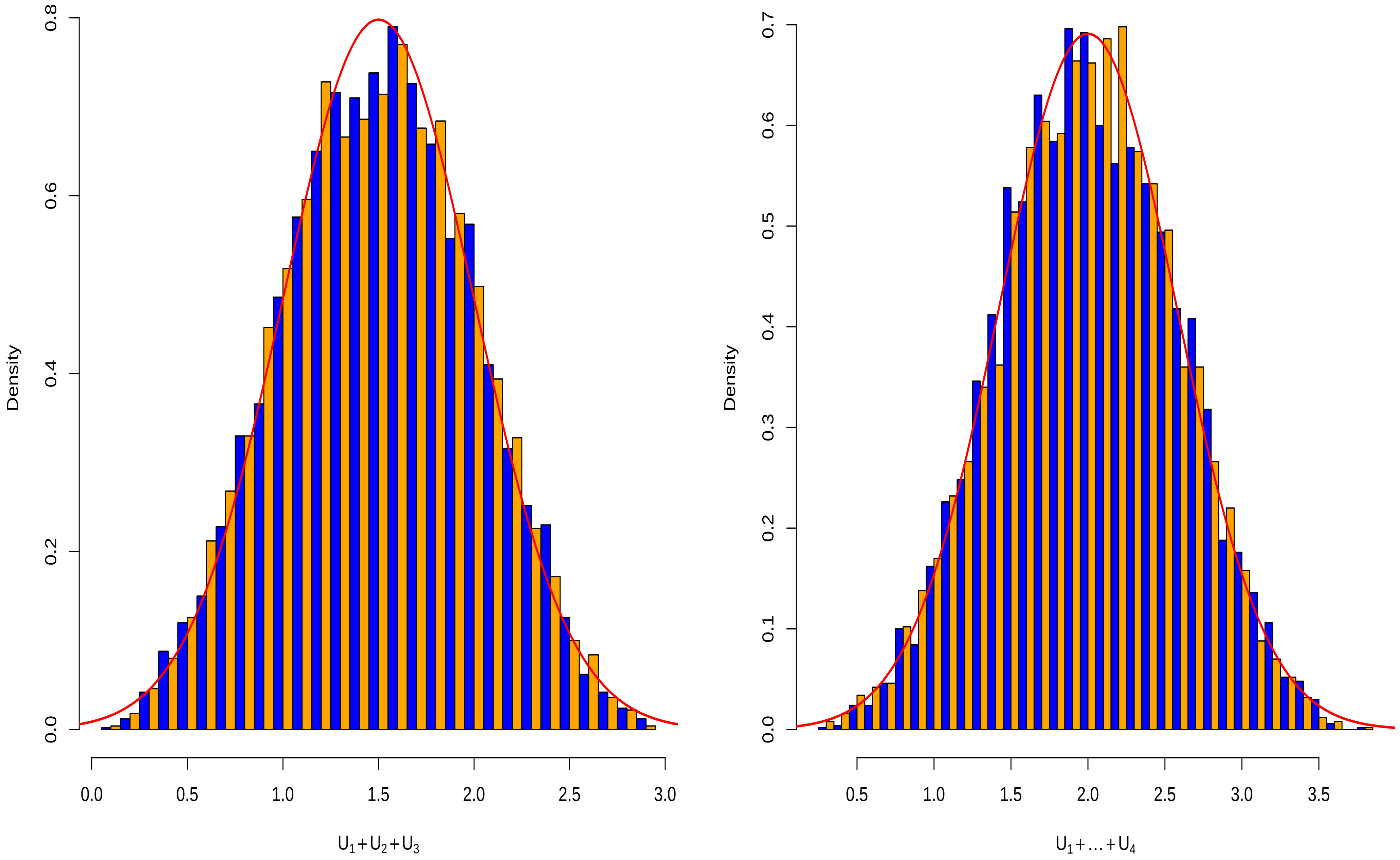
$$\sigma_S = \sqrt{n}\,\sigma_D \implies \mathrm{TOL}_S = 3\sigma_S = \sqrt{n}\,3\,\sigma_D = \sqrt{n}\,\sqrt{3}\,\mathrm{TOL}_D = \sqrt{n}\,c\,\mathrm{TOL}_D,$$

i.e., we have a uniform distribution penalty factor $c = \sqrt{3} = 1.732$.

Recall that under normal part variation we had: $\mathrm{TOL}_S = \sqrt{n}\,\mathrm{TOL}_D$.

Here the inflation factor is motivated differently from Benderizing.

CLT for Sums of Uniform Random Variables



Uniform Part Variation: Comparison with Worst Case

- Compare this to the worst case tolerancing

$$\mathrm{TOL}_D^\star = \frac{\mathrm{TOL}_S^\star}{n} \quad \text{or} \quad \mathrm{TOL}_S^\star = n \times \mathrm{TOL}_D^\star,$$

$$\mathrm{TOL}_S = \sqrt{3}\,\sqrt{n}\,\mathrm{TOL}_D \; < \; \mathrm{TOL}_S^\star = n\,\mathrm{TOL}_D^\star \qquad \text{when } 3 < n.$$

- The above σ_D calculation used calculus.

- What to do for other part variation models?

$\Longrightarrow$ More Calculus or Simulation!

Motivating the $3\sigma \leftrightarrow cT$ Link

Both T and σ capture the variability/scale of a distribution.

Increasing that scale by a factor ρ should increase σ and T by that same factor ρ.

$\mu \pm T$ captures (almost) all of the variation range.

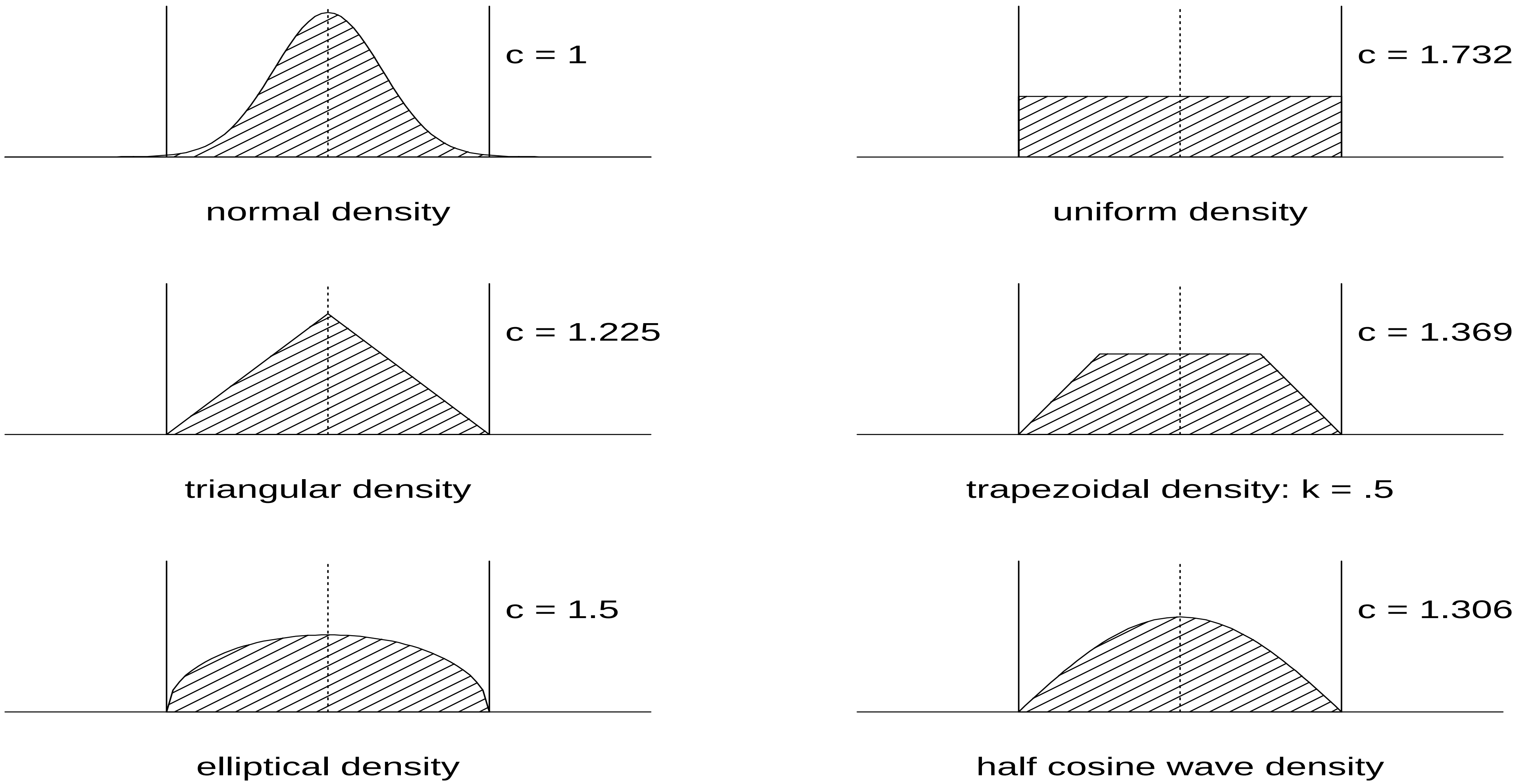
σ is a mathematically convenient scale measure, because of RSS rule.

For a normal distribution $\mu \pm 3\sigma$ captures almost all of the variation range.
There it makes sense to equate $T = 3\sigma$.

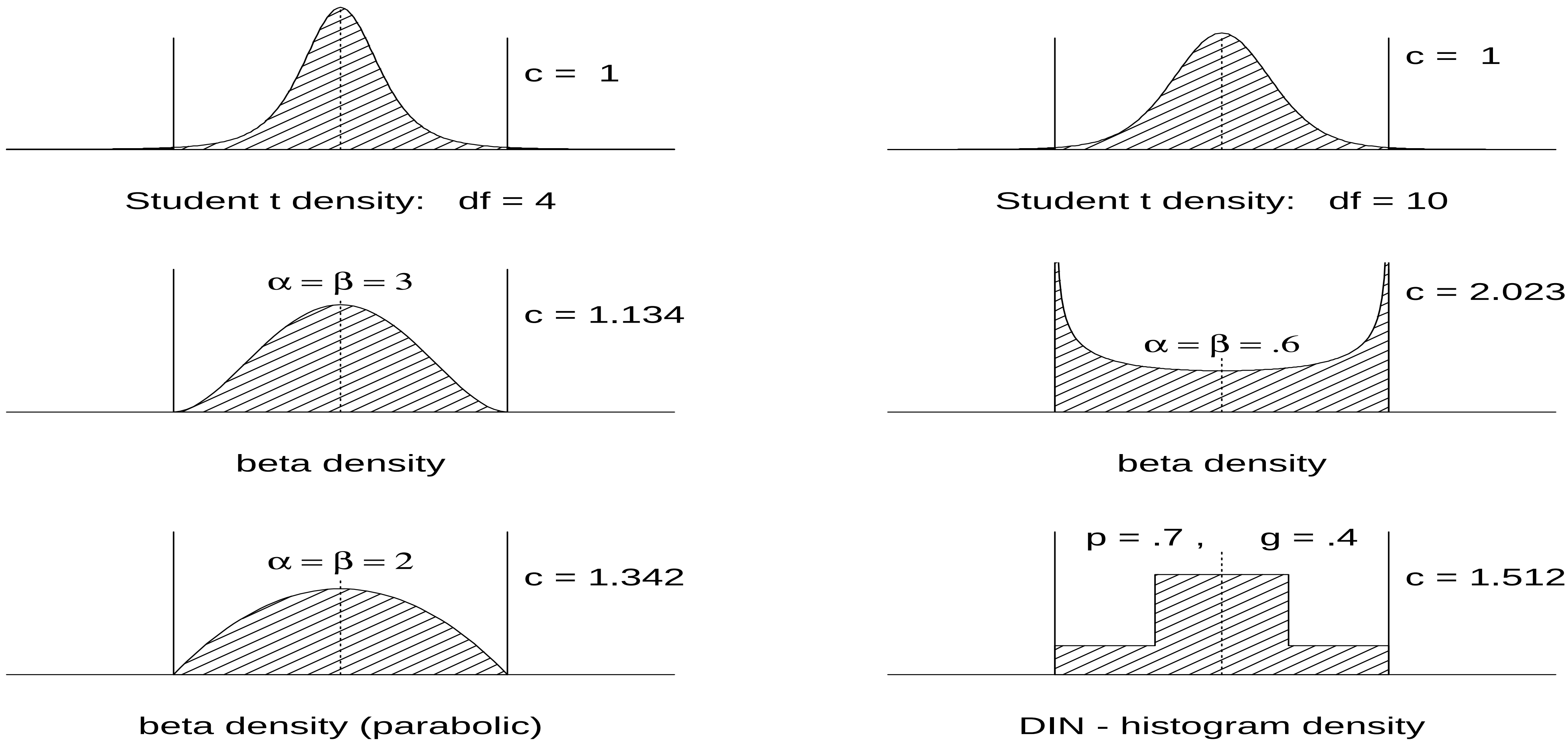
For other distributions we need a factor c to make that correspondence $T = 3\sigma/c$,
i.e., $\mu \pm 3\sigma/c$ captures (almost) all of the variation in the distribution.

$\Longrightarrow \quad 3\sigma = cT.$ The penalty or inflation factor c is found via calculus.

Distribution Inflation Factors 1



Distribution Inflation Factors 2



Details on Distribution Inflation Factors 1

- The factors c are chosen such that for finite range densities we have

$$3 \times \sigma_D = c \times \mathrm{TOL}_D$$

- $c_{\mathrm{normal}} = 1$

- Finite range densities can always be scaled to a range $[-1,1]$, except for beta where $[0,1]$ is the conventional standard interval.

- $c_{\mathrm{uniform}} = \sqrt{3}, \quad c_{\mathrm{triangular}} = \sqrt{1.5}, \quad c_{\mathrm{elliptical}} = 1.5, \quad c_{\cos} = 3\sqrt{1 - 8/\pi^2}$

- $c_{\mathrm{trapezoidal}} = \sqrt{3(1+k^2)/2}$ where $2k$ is the range of the middle flat part.

Details on Distribution Inflation Factors 2

- The beta density takes the following form:

$$f(z) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} z^{a-1}(1-z)^{b-1} \quad \text{for} \quad 0 \leq z \leq 1, \quad \text{and } g(z) = 0 \text{ else}$$

- For $a = b$ the beta density is symmetric around $.5$ $\Rightarrow c_{\text{beta}} = 3/\sqrt{2a+1}$.

- The histogram or DIN density takes the following form

$$f(z) = \begin{cases} \frac{p}{2g} & \text{for } |z| \leq g, \\ \frac{1-p}{2(1-g)} & \text{for } g < |z| \leq 1 \\ 0 & \text{else} \end{cases}$$

- $c_{\text{DIN}} = \sqrt{3[(1-p)(1+g)+g^2]}$

RSS with Mixed Distribution Inflation Factors

Assume that disk thicknesses D_i have different tolerance specifications $\mu_i \pm \mathrm{TOL}_i, \; i = 1, \ldots, n$ and with possibly different distribution factors $c_1, \ldots, c_n$

Again the stack dimension $S = D_1 + \ldots + D_n$ is approximately normally distributed with mean and standard deviation given by

$$\mu_S = \mu_1 + \ldots + \mu_n \quad \text{and} \quad \sigma_S = \sqrt{\sigma_1^2 + \ldots + \sigma_n^2}$$

By way of $3\sigma_i = c_i \mathrm{TOL}_i$ we get for S the tolerance range $\mu_S \pm \mathrm{TOL}_S$, where

$$\mathrm{TOL}_S = 3\sigma_S = \sqrt{(3\sigma_1)^2 + \ldots + (3\sigma_n)^2} = \sqrt{(c_1 \mathrm{TOL}_1)^2 + \ldots + (c_n \mathrm{TOL}_n)^2}$$

Statistical Tolerancing by Simulation

- Randomly generate part dimensions according to appropriate distributions over respective tolerance ranges
- Calculate the resulting critical assembly dimension, i.e., draw ten thicknesses from a distribution of thicknesses and compute the stack height (sum).
- Repeat the above many times, $N_{\text{sim}} = 1000$ (or $N_{\text{sim}} \geq 1000$) times.
- Form the histogram of the 1000 (or more) critical dimensions.
- Compare histogram with specified limits on the critical assembly dimension (stack height).

Statistical Tolerancing by Simulation & Iteration

- If histogram has lots of room within assembly specification or tolerance limits relax tolerances on the aggregating parts.

- If histogram violates assembly specification or tolerance limits significantly, tighten tolerances on the aggregating parts.

- Repeat process until satisfied. Opportunity for Experimental Design.

- Vectorize part dimension generation $\Longrightarrow$ critical dimension generation.

- All this can be done on a computer (e.g.,using R) in a matter of seconds and can save a lot of waste and rework.

- There are commercial tools, e.g., VSA

Is Linear Tolerance Stack Special?

- $\text{height} = \text{thickness}_1 + \ldots + \text{thickness}_n$ or $Y = X_1 + \ldots + X_n$

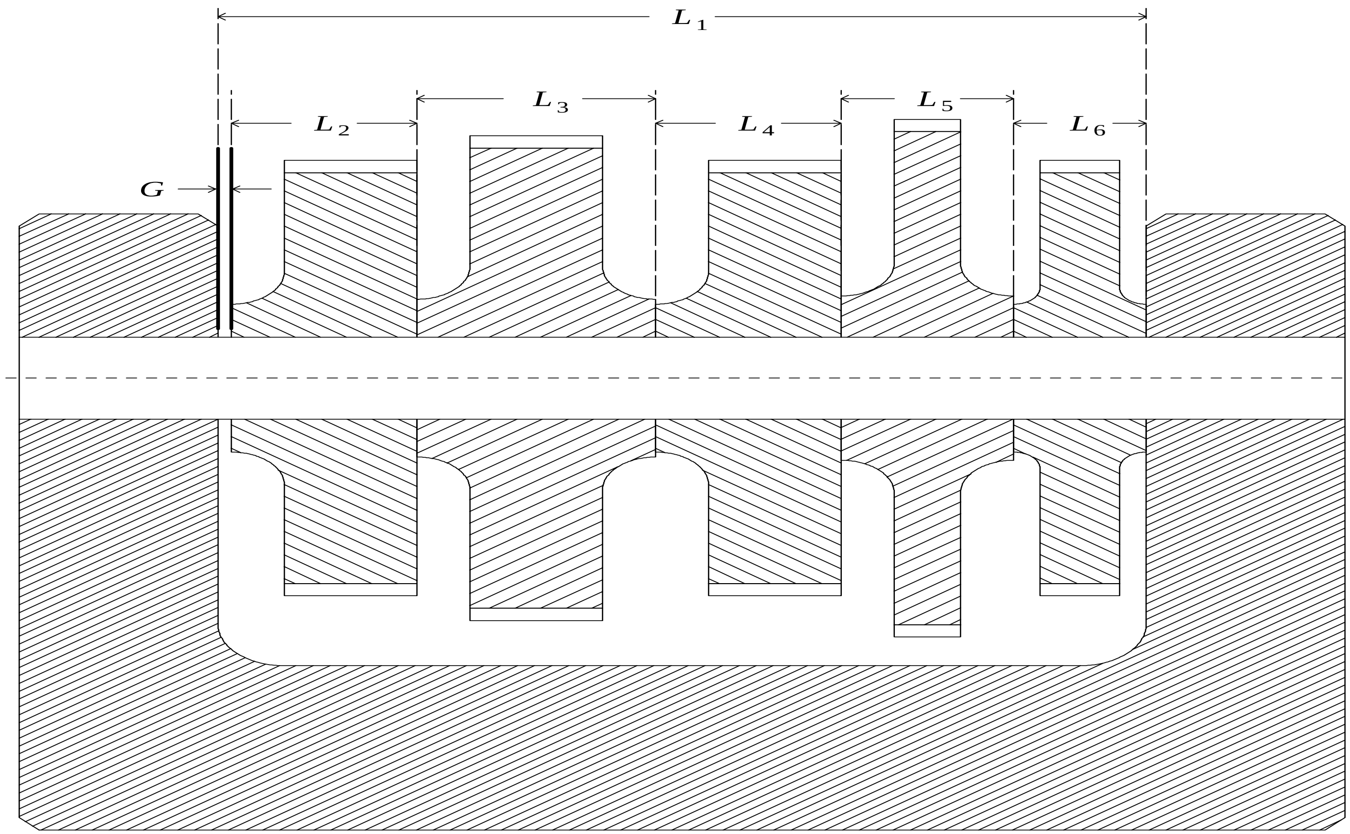
- From here it is a little step to $Y = a_0 + a_1 \times X_1 + \ldots + a_n \times X_n$, where $a_0,\ a_1, \ldots,\ a_n$ are known multipliers or coefficients.

- They are constant as opposed to the random quantities X_i.

- For example, $Y = 16 + 3 \times X_1 + 2 \times X_2 + 7 \times X_3 + (-2) \times X_4$

- Call $X_1, \ldots, X_n$ inputs or input dimensions and Y output dimension.

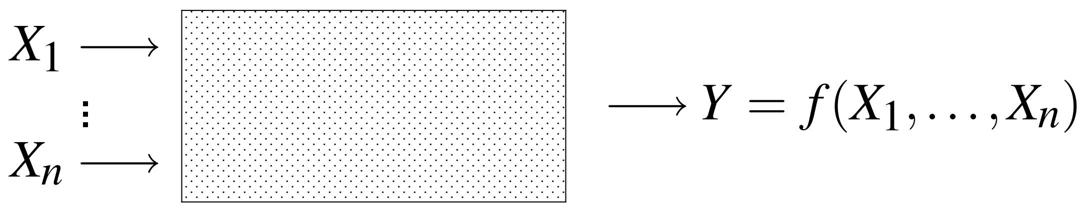
Crankcase Tolerance Chain



$$G = L_1 - L_2 - L_3 - L_4 - L_5 - L_6 = L_1 - (L_2 + \ldots + L_6)$$

Input/Output Black Box

- Of more general interest and applicability would be I/O relations of the following type $\quad Y = f(X_1, \ldots, X_n)$



Input/Output Black Box

- f describes what you have to do with the inputs X_i to arrive at an output Y.

- The propagation of variation in the X_i causes what variation in the output Y?

Smooth Functions f

- When the output Y varies smoothly with small changes in the X_i, then

$$Y \approx a_0 + a_1 \times X_1 + \ldots + a_n \times X_n$$

for all small perturbations in $X_1, \ldots, X_n$ around $\mu_1, \ldots, \mu_n$.

- The above approximation for $Y = f(X_1, \ldots, X_n)$ comes from

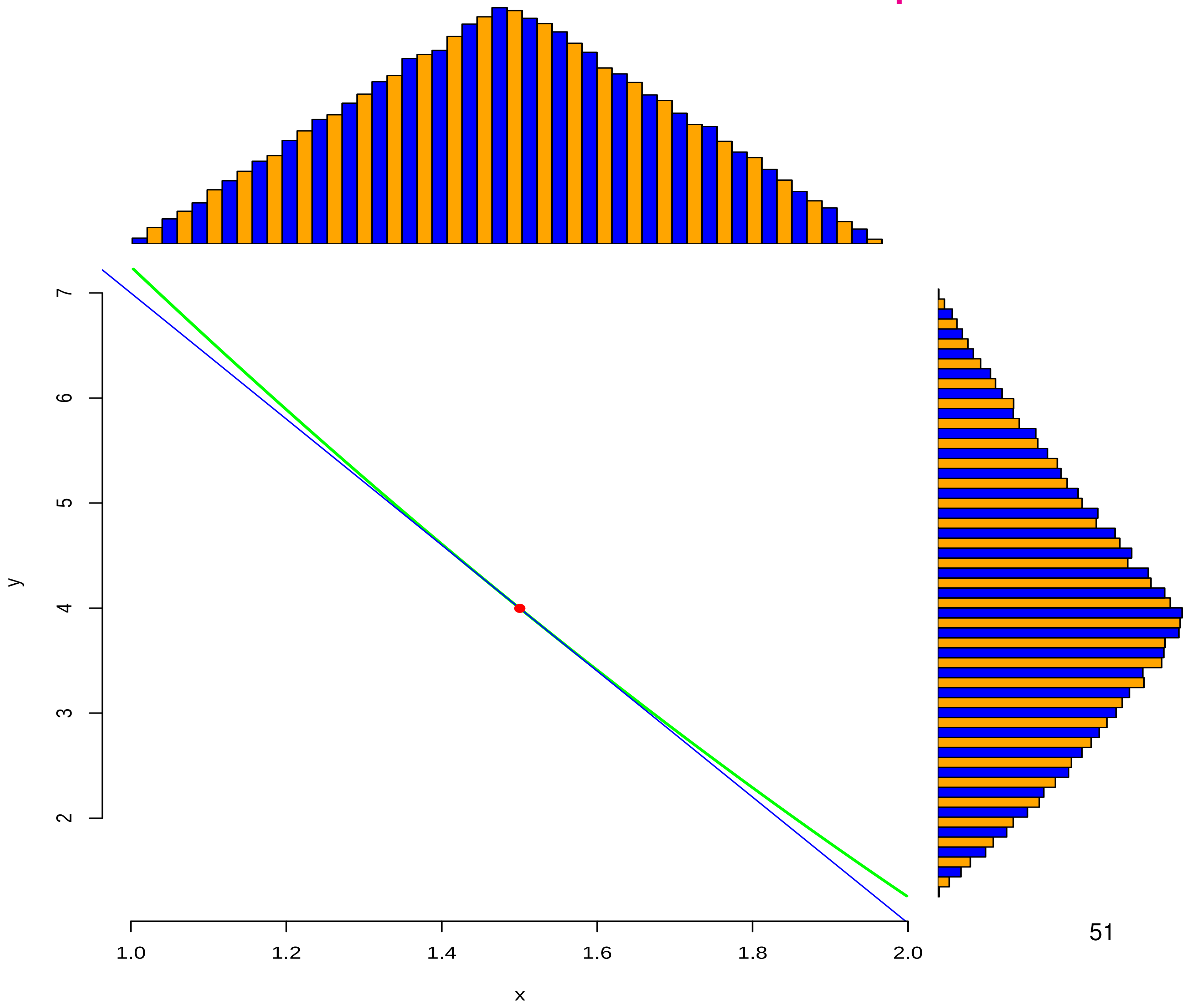
the one-term Taylor expansion of f around $\mu_1, \ldots, \mu_n$.

$$Y = f(X_1, \ldots, X_n) \approx f(\mu_1, \ldots, \mu_n) + \sum_{i=1}^{n} \frac{\partial f(\mu_1, \ldots, \mu_n)}{\partial \mu_i}(X_i - \mu_i)$$

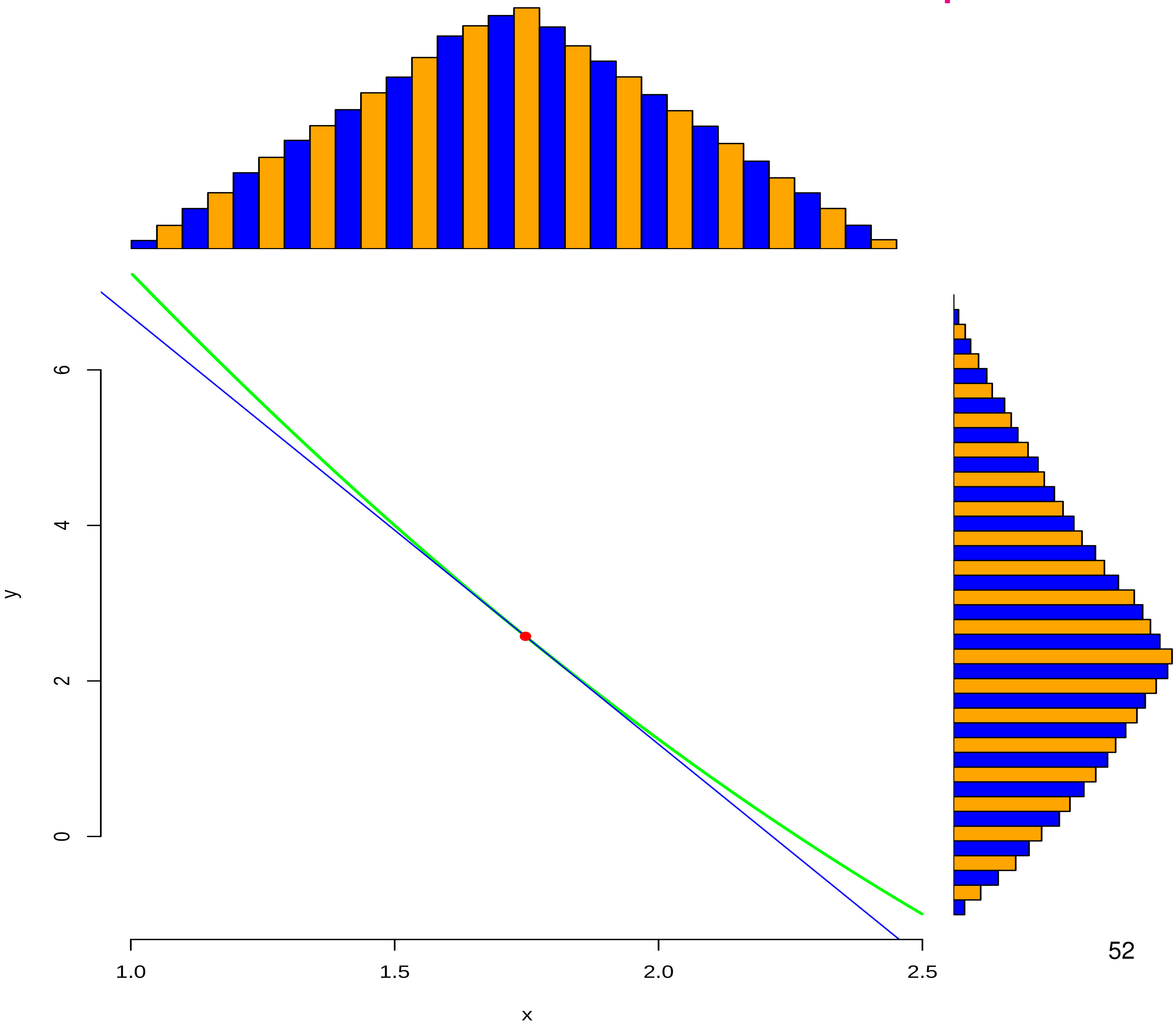
using

$$a_i = \frac{\partial f(\mu_1, \ldots, \mu_n)}{\partial \mu_i} \quad \text{and} \quad a_0 = f(\mu_1, \ldots, \mu_n) - \sum_{i=1}^{n} \frac{\partial f(\mu_1, \ldots, \mu_n)}{\partial \mu_i} \times \mu_i$$

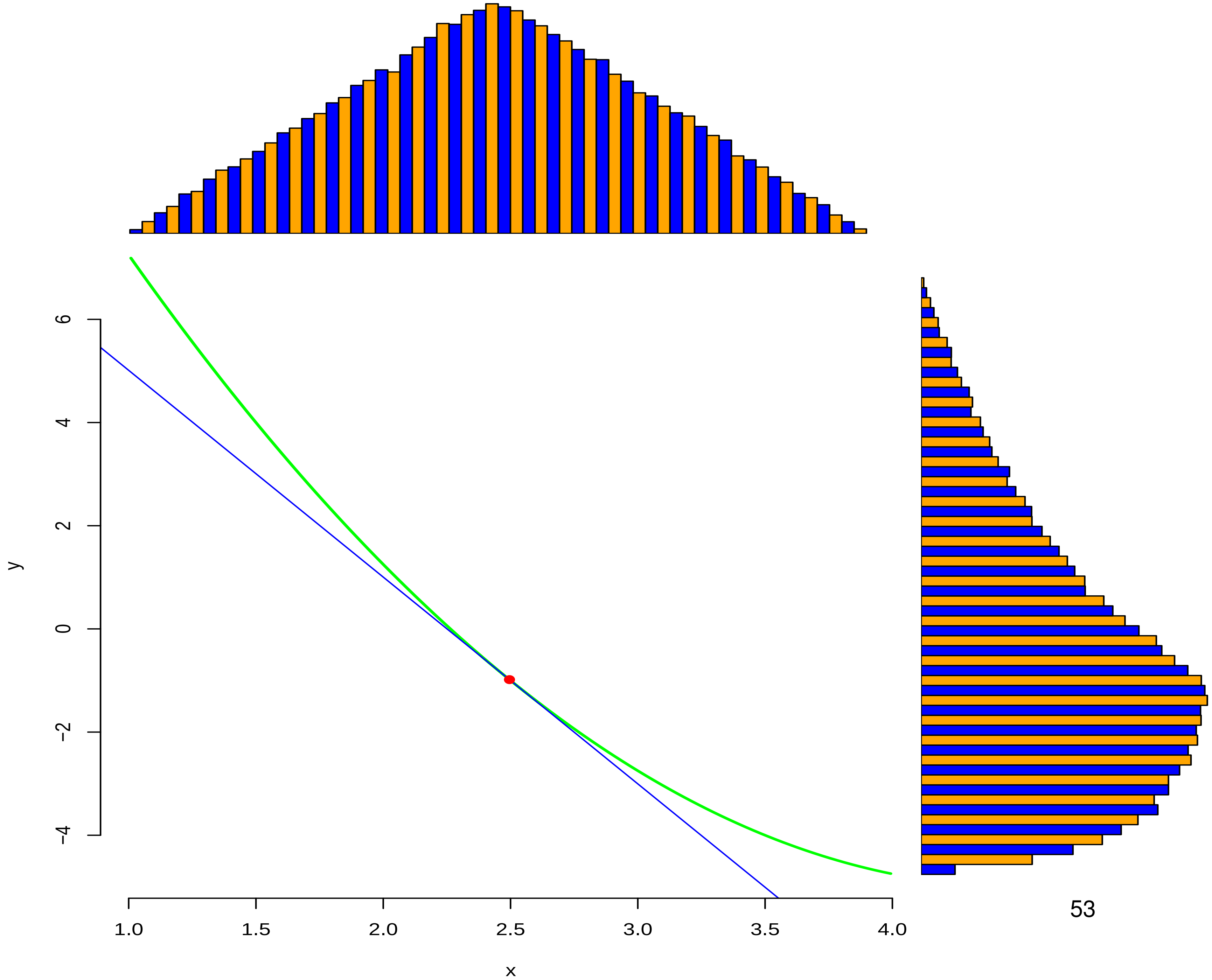
Good Linearization Example



Medium Linearization Example



Poor Linearization Example



The Sensitivity Coefficients or Derivatives

- The sensitivity coefficient a_i can then be determined by calculus or

- numerically by experimenting with the black box, making small changes in X_i near μ_i while holding the other X's fixed at their μ's and assessing the rate of change in Y in each case, i.e., for each $i = 1, \ldots, n$.

- The previous analysis can proceed, once we realize that

$$\sigma^2_{a_i \times X_i} = a_i^2 \times \sigma^2_{X_i} = (a_i \sigma_i)^2 \qquad \text{and} \qquad \sigma^2_{a_0} = 0 \, .$$

$$\begin{aligned} \sigma^2_Y &= \sigma^2_{a_0 + a_1 \times X_1 + \ldots + a_n \times X_n} \\ &= \sigma^2_{a_0} + \sigma^2_{a_1 \times X_1} + \ldots + \sigma^2_{a_n \times X_n} = \left(a_1 \sigma_{X_1}\right)^2 + \ldots + \left(a_n \sigma_{X_n}\right)^2 \end{aligned}$$

The General Tolerance Stack Formula

and by $3\,\sigma_{X_i} = c_i\, T_{X_i}$

$$
\begin{aligned}
(3\sigma_Y)^2 &= \left(3\,a_1\,\sigma_{X_1}\right)^2 + \ldots + \left(3\,a_n\,\sigma_{X_n}\right)^2 \\
&= \left(a_1 c_1 T_{X_1}\right)^2 + \ldots + \left(a_n c_n T_{X_n}\right)^2
\end{aligned}
$$

CLT $\Longrightarrow Y \approx \mathcal{N}(\mu_Y, \sigma_Y^2)$, i.e., most variation of Y is within $\mu_Y \pm 3\sigma_Y$

$$
\mathrm{TOL}_Y = 3\sigma_Y = \sqrt{\left(a_1 c_1 \mathsf{TOL}_{X_1}\right)^2 + \ldots + \left(a_n c_n \mathsf{TOL}_{X_n}\right)^2}
$$

I have seen engineers applying

$$
\mathrm{TOL}_Y = 3\sigma_Y = \sqrt{\mathrm{TOL}_1^2 + \ldots + \mathrm{TOL}_n^2}
$$

regardless of the a_i and c_i. RSS was a magic bullet they did not understand.

Simulation for General f

- Simulation of $Y = f(X_1, \ldots, X_n)$ is an option as well.
- A normal distribution for the inputs X_i is not essential.
- The CLT still gives us $\approx$ normal outputs, most of the time.
- The latter depends on the sensitivities/derivatives of f

  and the relative variations of the inputs.

Sensitivities and CLT

Recall the crucial condition for $Y = X_1 + \ldots + X_n \approx \mathcal{N}(\mu_Y, \sigma_Y^2)$

$$\frac{\max\left(\sigma_1^2, \ldots, \sigma_n^2\right)}{\sigma_1^2 + \ldots + \sigma_n^2} \longrightarrow 0\,, \quad \text{as} \quad n \to \infty$$

For $Y = a_0 + a_1 X_1 + \ldots + a_n X_n \approx \mathcal{N}(\mu_Y, \sigma_Y^2)$ this translates to

$$\frac{\max\left(a_1^2\sigma_1^2, \ldots, a_n^2\sigma_n^2\right)}{a_1^2\sigma_1^2 + \ldots + a_n^2\sigma_n^2} \longrightarrow 0\,, \quad \text{as} \quad n \to \infty$$

A large a_i can mess things up, i.e., make $a_i^2\sigma_i^2$ dominant.

A small a_i can dampen the effect of a large or otherwise dominant σ_i^2.

Mean Shifts

So far we have assumed that the distributions of part dimensions were centered on the middle of the tolerance interval.

Why should there be that much precision in centering when the actual inputs or part dimensions can be quite variable?

It makes sense to allow for some kind of mean shift or targeting error while still insisting on having all or most part dimensions within specified tolerance ranges.

Two Strategies of Dealing with Mean Shifts

Two strategies of dealing with mean shifts:

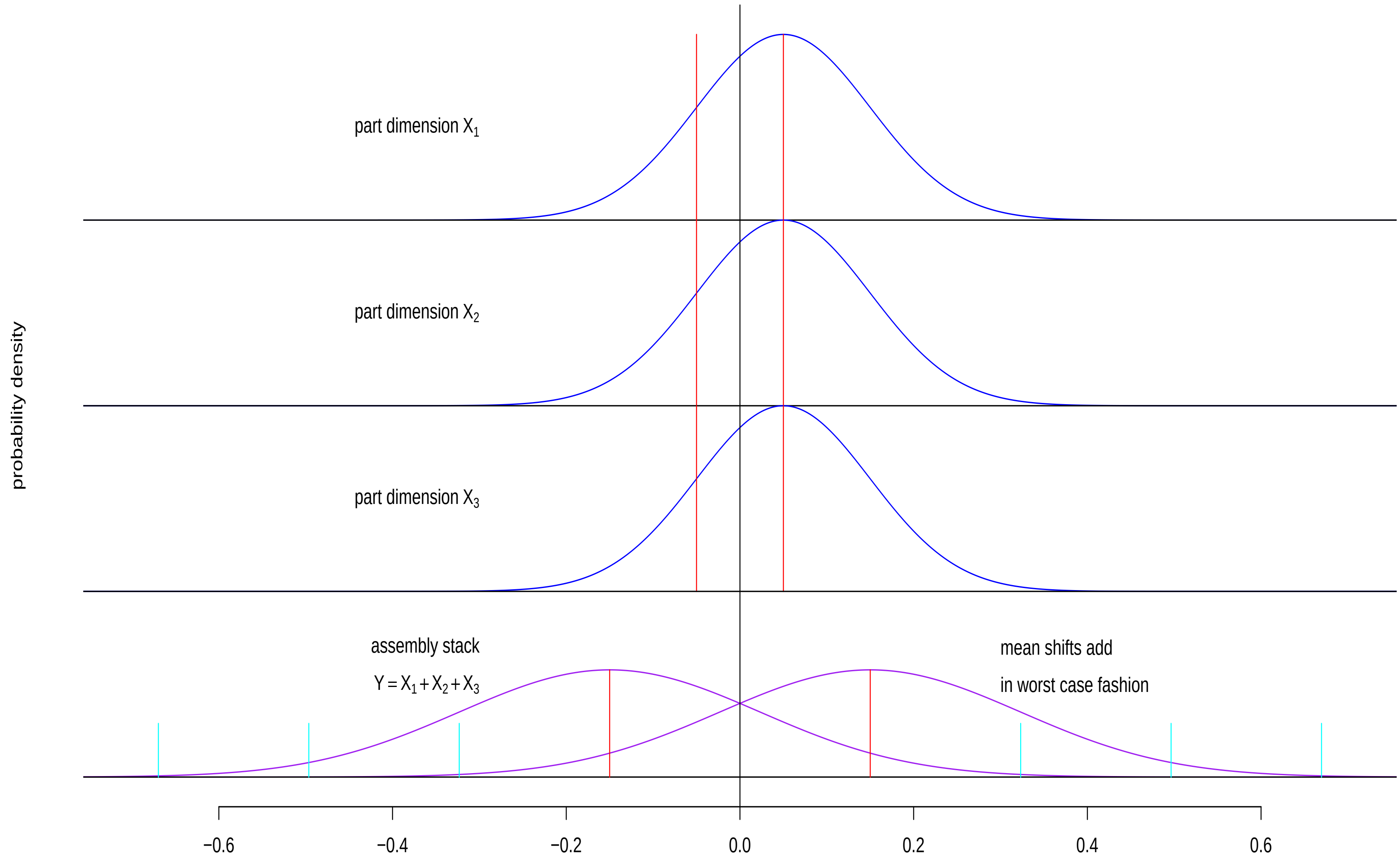
1. stack these shifts in worst case fashion arithmetically

2. stack these shifts statistically via RSS

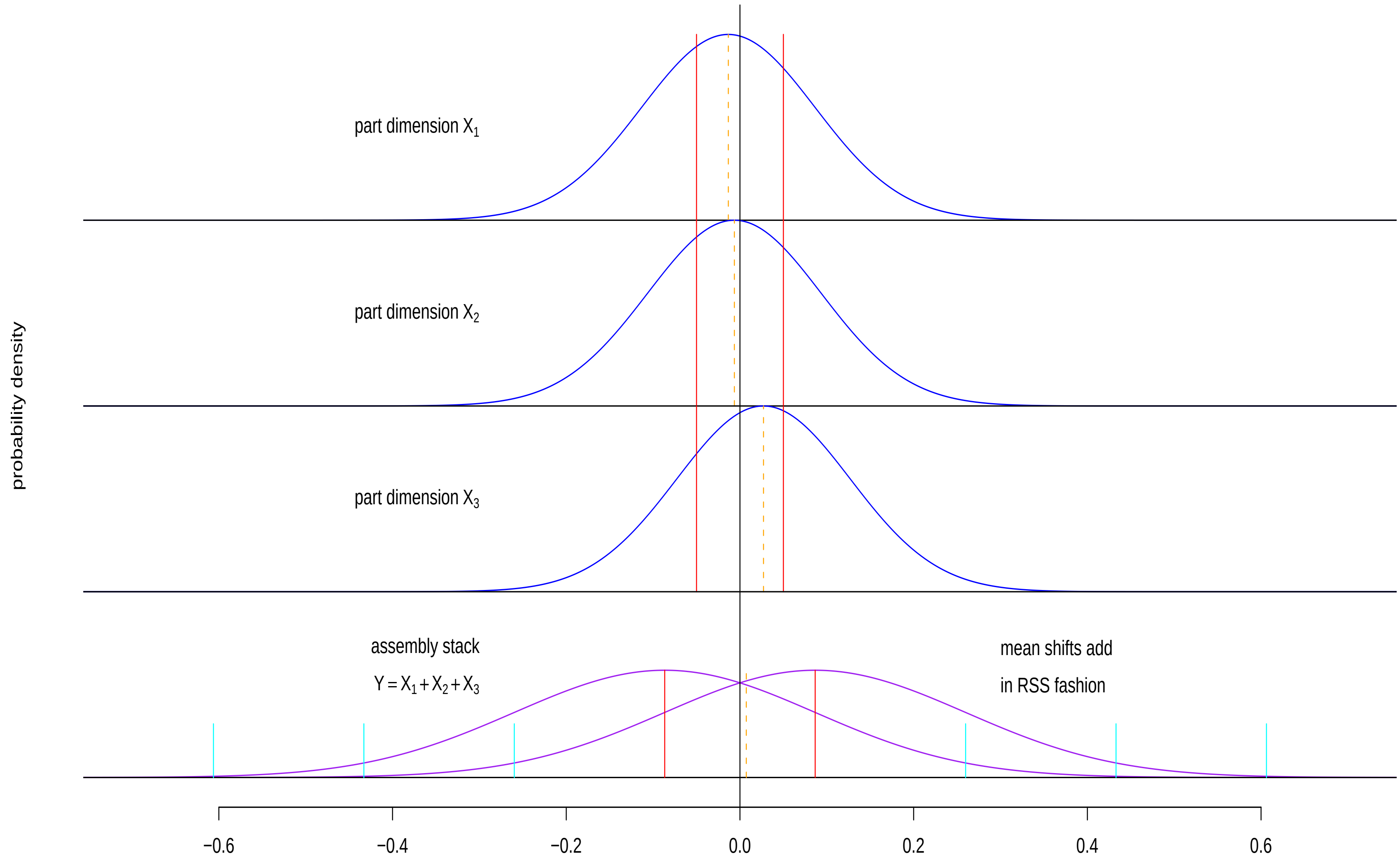
In either case combine this in worst case fashion or arithmetically with the RSS part variation stack.

The reason for the last worst case stacking step is that the mean shifts represent persistent effects that do not get played out independently and repeatedly for each produced part dimension.

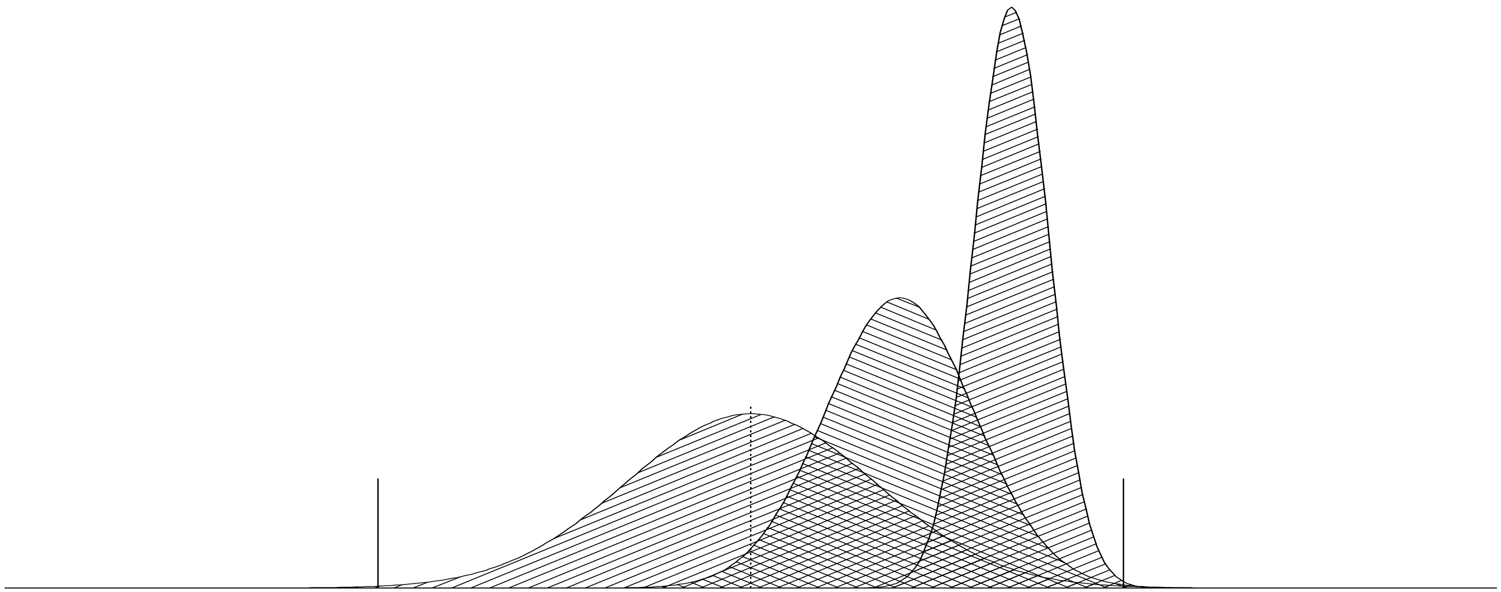
Mean Shifts Stacked Arithmetically



Mean Shifts Stacked via RSS



Mean Shifts within Tolerance Interval



For the part variation to stay within tolerance there has to be a tradeoff between variability and mean shift.

Mean Shifts, Variability & C_{pk}

- The capability index C_{pk} measures the distance of the mean μ to the closest tolerance limit in relation to 3σ.

- If the tolerance interval is given by $[L, U]$ then

$$C_{pk} = \min\left(\frac{U-\mu}{3\sigma}, \frac{\mu-L}{3\sigma}\right)$$

- $C_{pk} = 1$ means that we have somewhere between $.135\%$ to $.27\%$ of part dimensions falling out of tolerance.

- However, this does not control the mean shift. We could have $\mu \approx U$ and $C_{pk} = 1$. Then all part dimensions would be near $U \Longrightarrow$ worst case stacking.

Bounded Mean Shifts

- Bound the mean shift Δ_i, typically as a fraction of the tolerance T_i:

$$\Delta_i = \eta_i T_i \qquad 0 \leq \eta_i \leq 1 \ .$$

- But maintain $C_{pk} \geq 1$

$$\eta_i T_i + 3\sigma_i \leq T_i \implies 3\sigma_i \leq (1 - \eta_i) T_i$$

Arithmetically Stacking Mean Shifts

- $\Longrightarrow$ Hybrid tolerance stacking formula

  arithmetically combining arithmetically combined mean shifts and

  statistical tolerancing

$$\mathrm{TOL}_Y = \eta_1|a_1|\mathrm{TOL}_{X_1} + \ldots + \eta_n|a_n|\mathrm{TOL}_{X_n}$$

$$+\sqrt{(1-\eta_1)^2 a_1^2 c_1^2 \mathrm{TOL}_{X_1}^2 + \ldots + (1-\eta_n)^2 a_n^2 c_n^2 \mathrm{TOL}_{X_n}^2}$$

- This grows on the order of n and not $\sqrt{n}$, but with a reduction factor.

- $\eta_1 = \ldots = \eta_n = 0 \quad \Longrightarrow \quad$ RSS stacking.

- $\eta_1 = \ldots = \eta_n = 1 \quad \Longrightarrow \quad$ Worst case arithmetical stacking.

RSS Stacking of Mean Shifts

- $\Longrightarrow$ Hybrid tolerance stacking formula

  arithmetically combining RSS combined mean shifts and

  statistical tolerancing

$$\mathrm{TOL}_Y = \sqrt{\eta_1^2 \tilde{c}_1^2 a_1^2 \mathrm{TOL}_{X_1}^2 + \ldots + \eta_n^2 \tilde{c}_n^2 a_n^2 \mathrm{TOL}_{X_n}{}^2}$$

$$+ \sqrt{(1-\eta_1)^2 a_1^2 c_1^2 \mathrm{TOL}_{X_1}^2 + \ldots + (1-\eta_n)^2 a_n^2 c_n^2 \mathrm{TOL}_{X_n}^2}$$

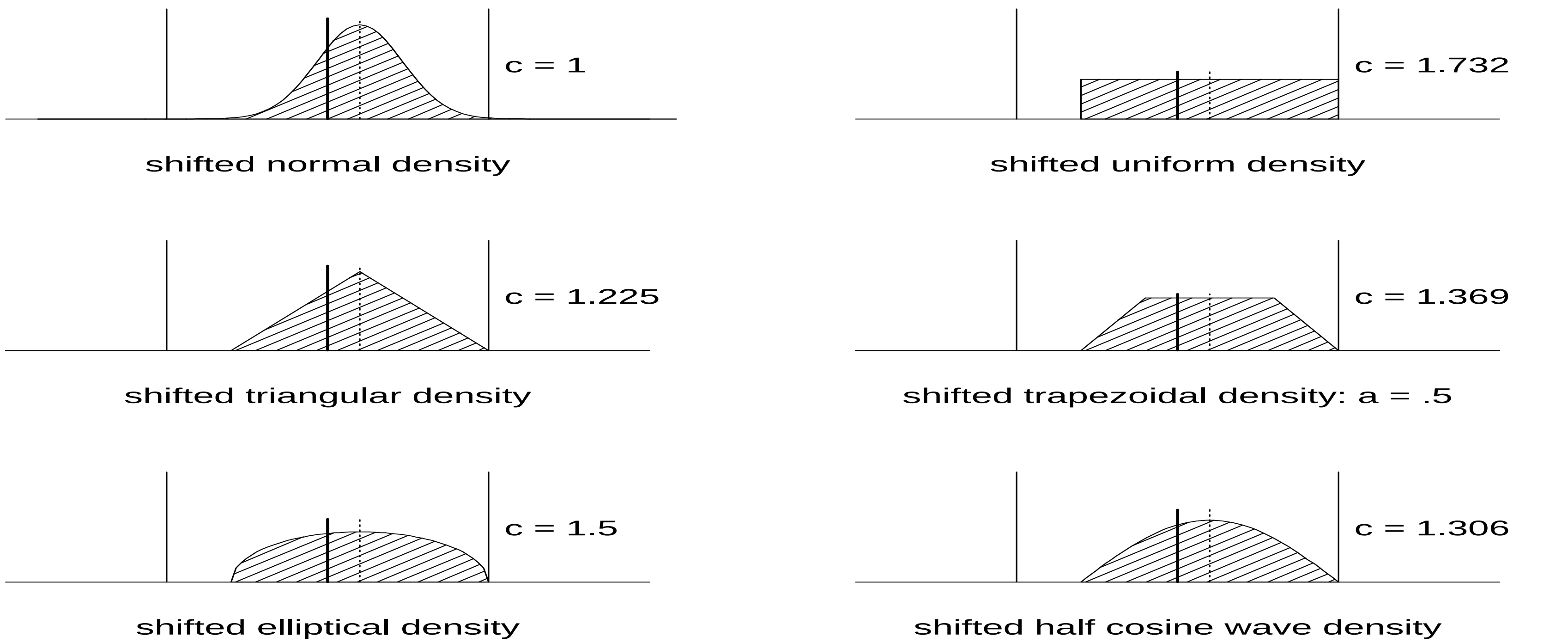
- The $\tilde{c}_i$ are the penalty factors for the distributions governing the mean shifts.

  The c_i are the penalty factors for the distributions governing part variation.

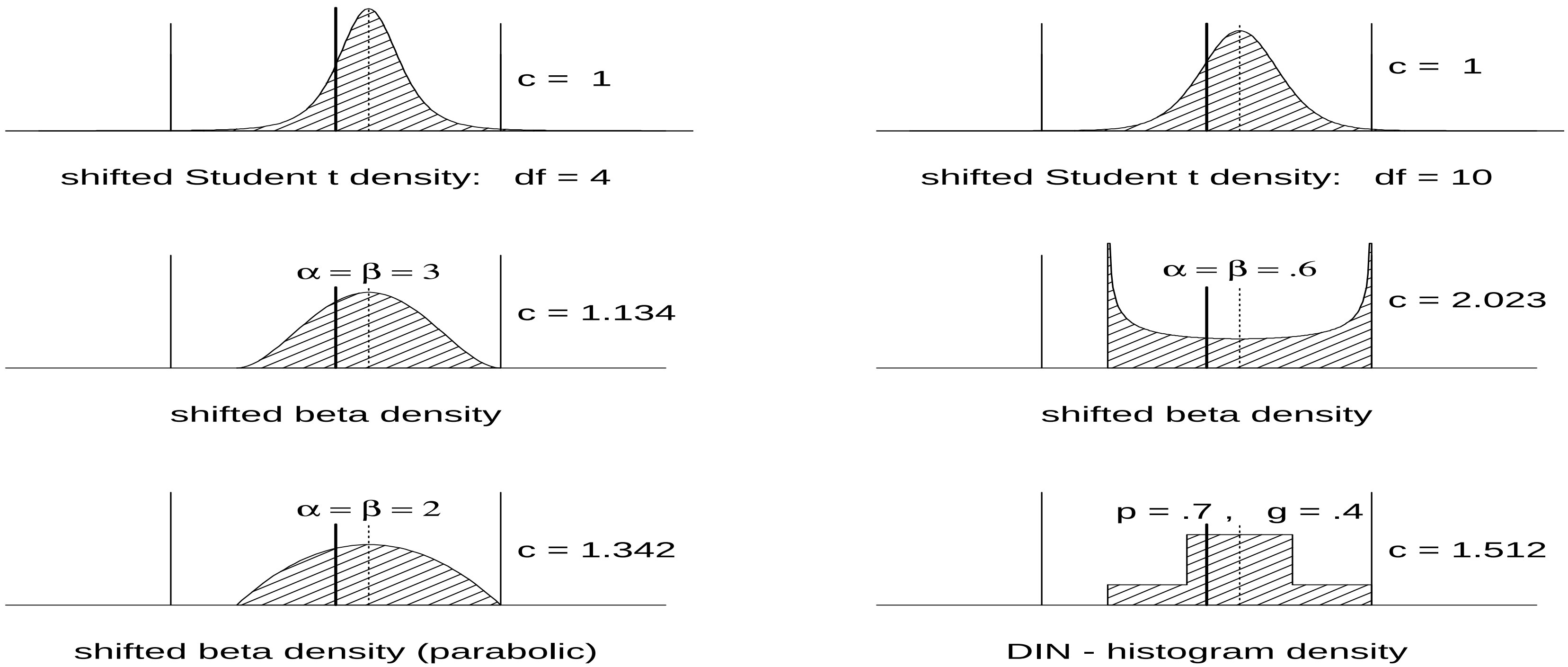
- What is the interpretation of $\eta_1 = \ldots = \eta_n = 1$?

  Consistent part dimensions with system output $Y = E(Y) \in \mu \pm \mathrm{TOL}_Y$.

Distributions with Mean Shift I



Distributions with Mean Shift II



Other Variants

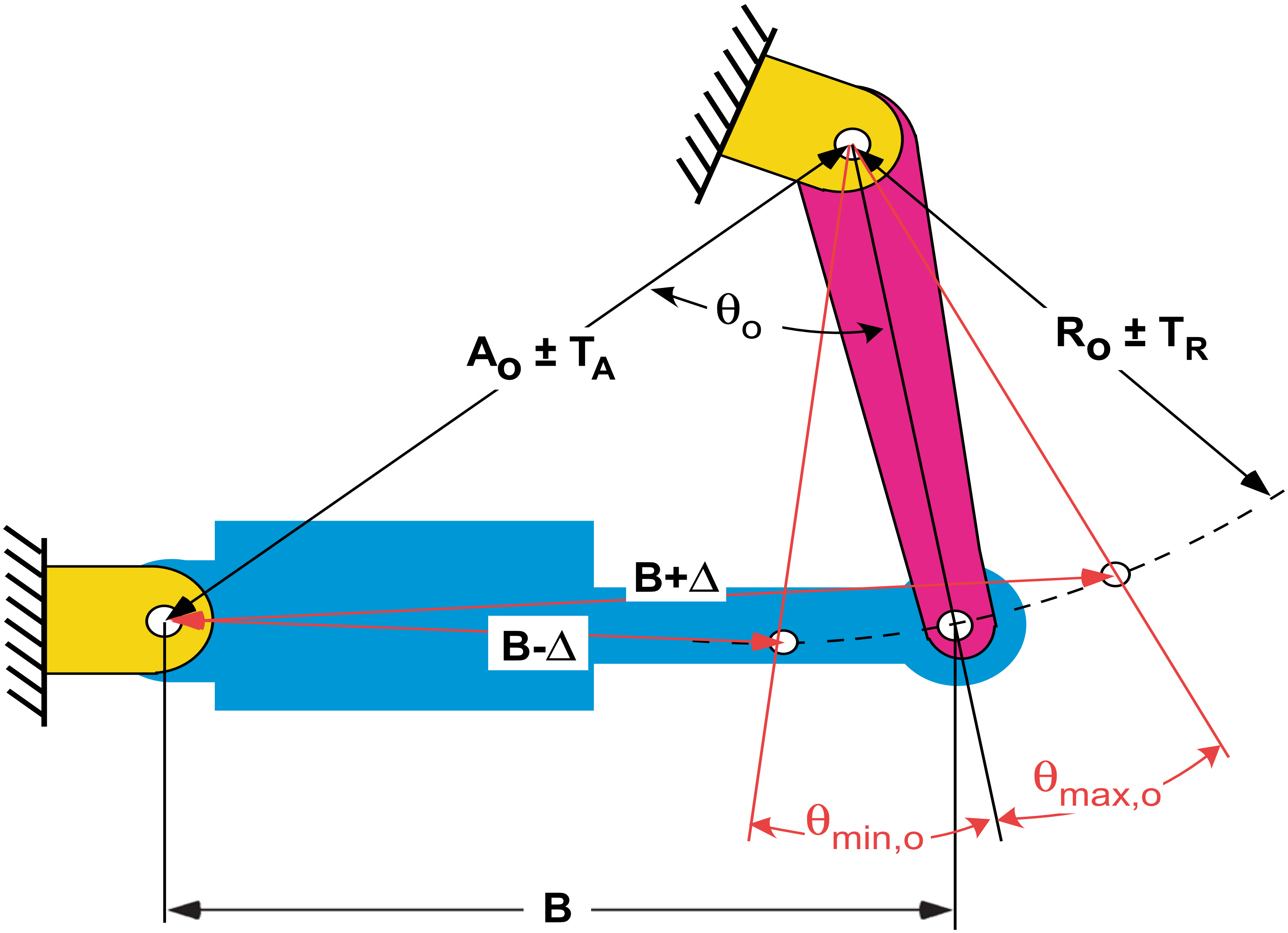
- So far we have accommodated mean shifts at the price of reduced part dimension variability in order to maintain $C_{pk} \geq 1$.

- Rather than dividing up TOL into mean shift and a 3σ range (by squeezing down 3σ to maintain $C_{pk} \geq 1$) we can increase TOL to the sum of the original $\mathrm{TOL}' = 3\sigma$ plus the mean shift represented as a fraction η of the increased TOL, i.e.,

$$\mathrm{TOL}_i = 3\sigma_i + \eta_i \mathrm{TOL}_i \quad \text{or} \quad \mathrm{TOL}_i = \frac{3\sigma_i}{1-\eta_i} = \frac{\mathrm{TOL}_i'}{1-\eta_i}\,.$$

- For details on how the stacking formulas change see the provided reports.

Actuator



Actuator Case Study

The following geometric problem arose in an actuator design situation.

In the abstract: we have a triangle with legs A, R and B.

The angle between A and R is denoted by θ.

We have the following tolerance specifications $A \in A_0 \pm T_A$ and $R \in R_0 \pm T_R$.

The leg B, representing the actuator, can be adjusted such that the angle θ agrees exactly with a specified value θ_0.

Once $\theta = \theta_0$ is achieved the actuator is in its neutral position.

From there B can extend or contract by an amount $\pm\Delta$ thus changing the angle θ to a maximum and minimum value $\theta_{\max}$ and $\theta_{\min}$, respectively.

The Question of Interest

$A = A_0$ and $R = R_0 \Longrightarrow$ nominal values for $\theta_{\max}$ and $\theta_{\min}$, denoted by $\theta_{\max,0}$ and $\theta_{\min,0}$, respectively.

The question of interest is:

How much variation of $\theta_{\max}$ and $\theta_{\min}$ around $\theta_{\max,0}$ and $\theta_{\min,0}$ can we expect due to the variations in A and R over their respective tolerance ranges $A_0 \pm T_A$ and $R_0 \pm T_R$?

Geometric Considerations

Given A, R and θ_0 the length of the (neutral position) actuator length is

$$B = B(A,R) = \sqrt{A^2 + R^2 - 2AR\cos(\theta_0)}\ .$$

Extending/contracting the actuator by $x = \pm\,\Delta$ from the neutral position

$$\Longrightarrow \qquad \theta_x = 2\arctan\left(\sqrt{\frac{(s_x - A)(s_x - R)}{s_x(s_x - B_x)}}\right)\ ,$$

where $B_x = B(A,R) + x$ and $s_x = (A + R + B_x)/2$.

Note that θ_Δ corresponds to $\theta_{\max}$ and $\theta_{-\Delta}$ corresponds to $\theta_{\min}$.

θ_x is affected by A and R in quite a variety of ways

$$\Longrightarrow \qquad \theta_{\max} = \theta_{\max}(A,R) \qquad \text{and} \qquad \theta_{\min} = \theta_{\min}(A,R)\ .$$

Statistical Tolerancing via Simulation

The simplest way of dealing with the variation behavior of $\theta_\Delta = \theta_{\max}$ and $\theta_{-\Delta} = \theta_{\min}$ due to variation in A and R is through simulation $\Longrightarrow$ R.

Get N-vectors of A and R values from $\mathcal{N}(\mu_A, (T_A/3)^2)$ and $\mathcal{N}(\mu_R, (T_R/3)^2)$.

Calculate the correspondingly adjusted $B = B(A,R)$ vector and from that the N-vectors of $\theta_{\max}$ and $\theta_{\min}$, respectively.

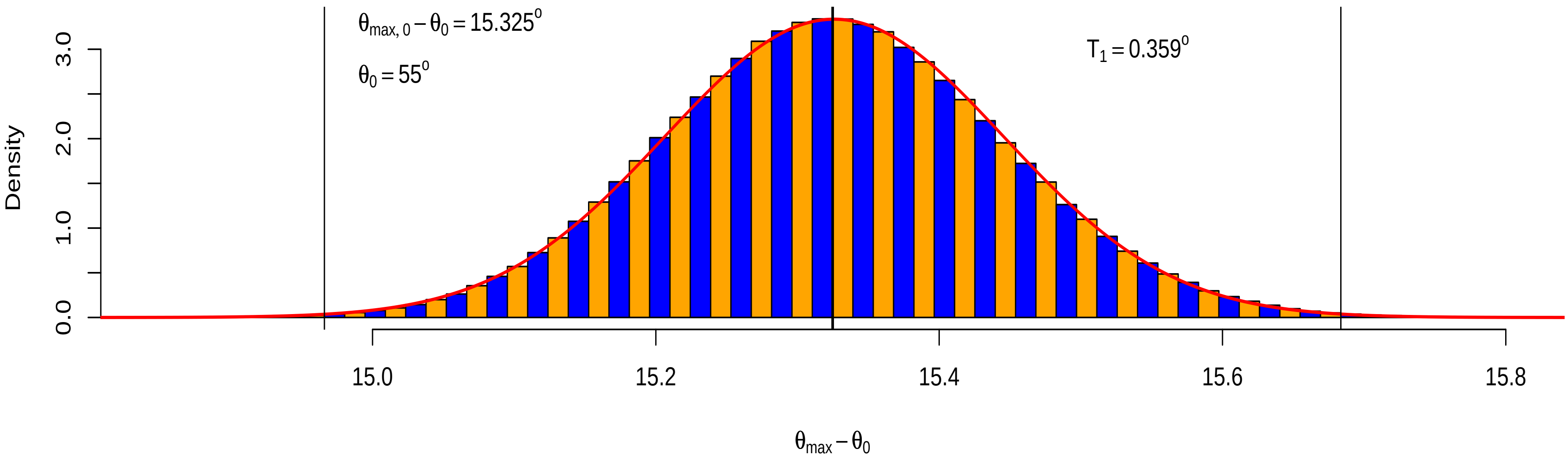
Here $\mu_A = A_0$, $\mu_R = R_0$ and $\sigma_A = T_A/3$, $\sigma_R = T_R/3$ normal distribution.

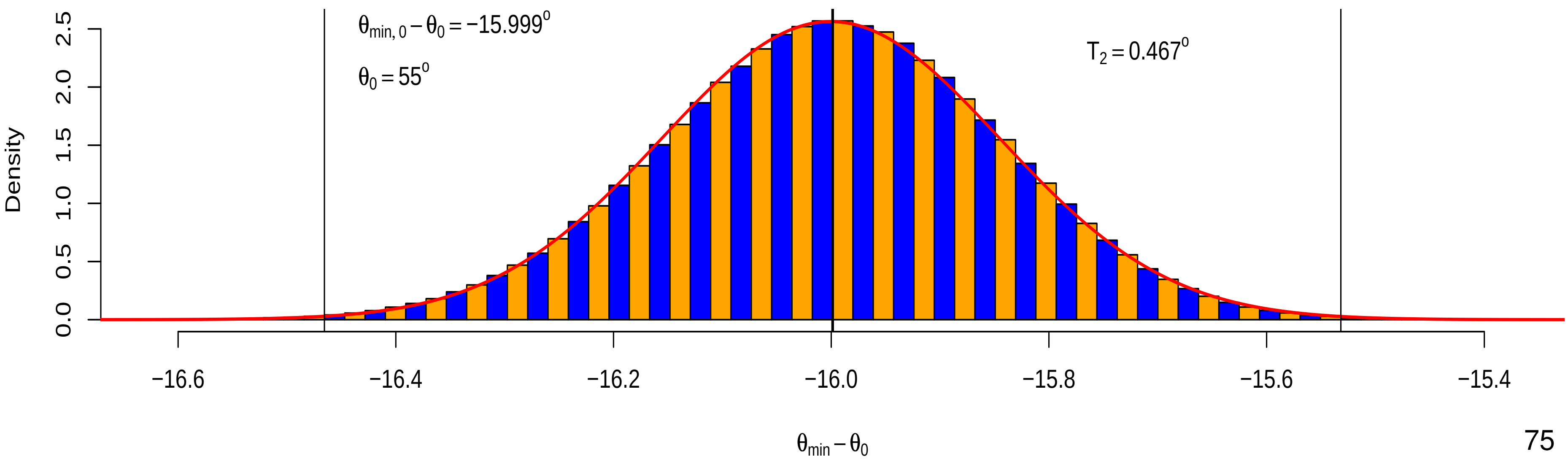
The results using $N = 1,000,000$ simulations is shown on the next slide.
It used `theta.simNN` and took just a few seconds to run.

Vertical bars on either side of the histograms $=$ estimated $\pm 3\sigma = \pm T$ limits.

It is easy to change the distributions describing the variation in A and R.

$(A,R) \sim (\mathcal{N},\mathcal{N})$ Simulation Output, $N_{\text{sim}} = 10^6$





Statistical Tolerancing via RSS

$$T_1 = \sqrt{a_{\max,A}^2 \times T_A^2 + a_{\max,R}^2 \times T_R^2} \qquad \text{and} \qquad T_2 = \sqrt{a_{\min,A}^2 \times T_A^2 + a_{\min,R}^2 \times T_R^2}\,,$$

where

$$a_{\max,A} = \frac{\partial\theta_{\max}}{\partial A}, \quad a_{\max,R} = \frac{\partial\theta_{\max}}{\partial R}, \quad a_{\min,A} = \frac{\partial\theta_{\min}}{\partial A}, \quad \text{and} \quad a_{\min,R} = \frac{\partial\theta_{\min}}{\partial R}$$

All derivatives are evaluated at the nominal values (A_0, R_0) of (A, R).

These RSS formulae come from the linearization of $\theta_x(A, R)$ near (A_0, R_0), i.e.,

$$\theta_x(A,R) = \theta_x(A_0,R_0) + (A - A_0) \times \left.\frac{\partial\theta_x}{\partial A}\right|_{A=A_0,R=R_0} + (R - R_0) \times \left.\frac{\partial\theta_x}{\partial R}\right|_{A=A_0,R=R_0}\,,$$

which is then taken as an approximation for $\theta_x(A, R)$ near $(A, R) = (A_0, R_0)$.

Approximation Quality

The approximation quality depends on the smoothness of the function θ_x with respect to A and R at (A_0, R_0).

The approximation quality also depends on the tolerances T_A and T_R.
T_A and T_R determine over what range θ_x is approximated.

When T_A or T_R get too large, quadratic terms may come into play $\Rightarrow$ normality???

All this assumes of course that θ_x is differentiable near $(A, R) = (A_0, R_0)$.

There are tolerance situation where differentiability is an issue and in that case the RSS paradigm does not work.

The Derivatives

$$\frac{\partial \theta_x}{\partial A} = \frac{1}{1 + \frac{(s_x - A)(s_x - R)}{s_x(s_x - B_x)}} \frac{\partial}{\partial A} \sqrt{\frac{(s_x - A)(s_x - R)}{s_x(s_x - B_x)}}$$

and

$$\frac{\partial \theta_x}{\partial R} = \frac{1}{1 + \frac{(s_x - A)(s_x - R)}{s_x(s_x - B_x)}} \frac{\partial}{\partial R} \sqrt{\frac{(s_x - A)(s_x - R)}{s_x(s_x - B_x)}} \, .$$

Next we have

$$\frac{\partial}{\partial A} \sqrt{\frac{(s_x - A)(s_x - R)}{s_x(s_x - B_x)}} = \left\{ 2 \sqrt{\frac{(s_x - A)(s_x - R)}{s_x(s_x - B_x)}} \right\}^{-1} \frac{\partial}{\partial A} \frac{(s_x - A)(s_x - R)}{s_x(s_x - B_x)}$$

and

$$\frac{\partial}{\partial R} \sqrt{\frac{(s_x - A)(s_x - R)}{s_x(s_x - B_x)}} = \left\{ 2 \sqrt{\frac{(s_x - A)(s_x - R)}{s_x(s_x - B_x)}} \right\}^{-1} \frac{\partial}{\partial R} \frac{(s_x - A)(s_x - R)}{s_x(s_x - B_x)} \, .$$

More Derivatives

We also have the following list of derivative expressions

$$\frac{\partial B_x}{\partial A} = \frac{A - R\cos(\theta_0)}{\sqrt{A^2 + R^2 - 2AR\cos(\theta_0)}} \quad \text{and} \quad \frac{\partial B_x}{\partial R} = \frac{R - A\cos(\theta_0)}{\sqrt{A^2 + R^2 - 2AR\cos(\theta_0)}}$$

$$\frac{\partial(s_x - A)}{\partial A} = \frac{1}{2}\left(\frac{A - R\cos(\theta_0)}{B} - 1\right) \quad \text{and} \quad \frac{\partial(s_x - R)}{\partial A} = \frac{1}{2}\left(\frac{A - R\cos(\theta_0)}{B} + 1\right)$$

$$\frac{\partial(s_x - A)}{\partial R} = \frac{1}{2}\left(\frac{R - A\cos(\theta_0)}{B} + 1\right) \quad \text{and} \quad \frac{\partial(s_x - R)}{\partial R} = \frac{1}{2}\left(\frac{R - A\cos(\theta_0)}{B} - 1\right)$$

$$\frac{\partial s_x}{\partial A} = \frac{1}{2}\left(\frac{A - R\cos(\theta_0)}{B} + 1\right) \quad \text{and} \quad \frac{\partial s_x}{\partial R} = \frac{1}{2}\left(\frac{R - A\cos(\theta_0)}{B} + 1\right)$$

$$\frac{\partial(s_x - B_x)}{\partial A} = \frac{1}{2}\left(1 - \frac{A - R\cos(\theta_0)}{B}\right) \quad \text{and} \quad \frac{\partial(s_x - B_x)}{\partial R} = \frac{1}{2}\left(1 - \frac{R - A\cos(\theta_0)}{B}\right) .$$

And More Derivatives

$$\frac{\partial}{\partial A}\frac{(s_x - A)(s_x - R)}{s_x(s_x - B_x)}$$

$$= \frac{1}{s_x^2(s_x - B_x)^2}\left\{\left[(s_x - R)\frac{\partial}{\partial A}(s_x - A) + (s_x - A)\frac{\partial}{\partial A}(s_x - R)\right] s_x(s_x - B_x)\right.$$

$$\left. -(s_x - A)(s_x - R)\left[(s_x - B_x)\frac{\partial}{\partial A}s_x + s_x\frac{\partial}{\partial A}(s_x - B_x)\right]\right\}$$

$$\frac{\partial}{\partial R}\frac{(s_x - A)(s_x - R)}{s_x(s_x - B_x)}$$

$$= \frac{1}{s_x^2(s_x - B_x)^2}\left\{\left[(s_x - R)\frac{\partial}{\partial R}(s_x - A) + (s_x - A)\frac{\partial}{\partial R}(s_x - R)\right] s_x(s_x - B_x)\right.$$

$$\left. -(s_x - A)(s_x - R)\left[(s_x - B_x)\frac{\partial}{\partial R}s_x + s_x\frac{\partial}{\partial R}(s_x - B_x)\right]\right\}.$$

And More Derivatives

Rather than just using these expressions as they are it is advisable to simplify them somewhat to avoid significance loss in the calculations.

Thus we obtained the following reduced expressions:

$$(s_x - R)\frac{\partial}{\partial A}(s_x - A) + (s_x - A)\frac{\partial}{\partial A}(s_x - R) = \frac{R}{2}[1 - \cos(\theta_0)] + \frac{x}{2B}[A - R\cos(\theta_0)]$$

$$(s_x - B_x)\frac{\partial}{\partial A}s_x + s_x\frac{\partial}{\partial A}(s_x - B_x) = \frac{R}{2}[1 + \cos(\theta_0)] - \frac{x}{2B}[A - R\cos(\theta_0)]$$

$$(s_x - R)\frac{\partial}{\partial R}(s_x - A) + (s_x - A)\frac{\partial}{\partial R}(s_x - R) = \frac{A}{2}[1 - \cos(\theta_0)] + \frac{x}{2B}[R - A\cos(\theta_0)]$$

$$(s_x - B_x)\frac{\partial}{\partial R}s_x + s_x\frac{\partial}{\partial R}(s_x - B_x) = \frac{A}{2}[1 + \cos(\theta_0)] - \frac{x}{2B}[R - A\cos(\theta_0)]\ .$$

RSS Calculations

The R function `deriv.theta` produced the following derivatives for $A_0 = 12.8$, $R_0 = 6$, $\theta_0 = 55^\circ$, and $\Delta = 1.6$

$$\frac{\partial\theta_{\text{max}}}{\partial A} = -.00006636499 \qquad \text{and} \qquad \frac{\partial\theta_{\text{min}}}{\partial A} = -.004038650$$

and

$$\frac{\partial\theta_{\text{max}}}{\partial R} = -0.04473785 \qquad \text{and} \qquad \frac{\partial\theta_{\text{min}}}{\partial R} = 0.05810921\ .$$

The RSS calculation using normal variation for A and R then gives the following values for T_1 and T_2 based on $T_A = .12$ and $T_R = .14$

$$T_1 = 0.3588609 \qquad \text{and} \qquad T_2 = 0.4669441\ ,$$

which agree remarkably well with the simulated quantities.

The derivatives of θ_{max} and θ_{min} with respect to A are smaller than

the derivatives with respect to R by at least an order of magnitude.

Important when considering other distributions governing the variation of A and R.

Numerical Differentiation

The derivation of the derivatives was quite laborious, but R code is compact.

Useful in understanding the variation propagation in the tolerance analysis.

An obvious alternative approach is numerical differentiation.

It requires the evaluation of the function θ_x, used in the simulation anyway.

The respective derivatives are approximated numerically at $(A,R) = (A_0,R_0)$ by difference quotients for very small values of δ

$$\left.\frac{\partial\theta_x}{\partial A}\right|_{A=A_0,R=R_0} \approx \frac{\theta_x(A_0+\delta,R_0)-\theta_x(A_0,R_0)}{\delta}$$

$$\left.\frac{\partial\theta_x}{\partial R}\right|_{A=A_0,R=R_0} \approx \frac{\theta_x(A_0,R_0+\delta)-\theta_x(A_0,R_0)}{\delta}\ .$$

Numerical Differentiation Example

For $\delta = .00001$ the R function `deriv.numeric` gives

$$\left.\frac{\partial\theta_{\max}}{\partial A}\right|_{A=A_0,R=R_0} \approx -.00006636269 \qquad \text{and} \qquad \left.\frac{\partial\theta_{\min}}{\partial A}\right|_{A=A_0,R=R_0} \approx -.004038651$$

and

$$\left.\frac{\partial\theta_{\max}}{\partial R}\right|_{A=A_0,R=R_0} \approx -0.04473777 \qquad \text{and} \qquad \left.\frac{\partial\theta_{\min}}{\partial R}\right|_{A=A_0,R=R_0} \approx 0.05810908\,.$$

These agree very well with the derivatives obtained previously via calculus.

Revisit RSS for Linear Combinations

A linear combination Y of independent, normal variation terms X_i

$$Y = a_0 + a_1 X_1 + \ldots + a_n X_n \quad \text{with known constants } a_0, a_1, \ldots, a_n,$$

is normally distributed.

Most of the Y variation falls within $\pm 3\sigma_Y$ of its mean $\mu_Y = a_0 + a_1 \mu_{X_1} + \ldots + a_n \mu_{X_n}$.

$$\sigma_Y^2 = \sigma_{a_1 X_1}^2 + \ldots + \sigma_{a_n X_n}^2 = a_1^2 \sigma_{X_1}^2 + \ldots + a_n^2 \sigma X_n{}^2 \ .$$

For $X_i \sim \mathcal{N}$ equate $3\sigma_{X_i} = T_i$, i.e., most of the X_i variation falls within $\mu_i \pm 3\sigma_{X_i}$

$\Longrightarrow$ general RSS tolerance stacking formula

$$T_Y = 3\sigma_Y = \sqrt{a_1^2 (3\sigma_{X_1})^2 + \ldots + a_n^2 (3\sigma_{X_n})^2} = \sqrt{a_1^2 T_1^2 + \ldots + a_n^2 T_n^2}$$

applicable for linear approximations to smooth functions of normal inputs.

CLT and Adjustment Factors

$$Y = a_0 + a_1 X_1 + \ldots + a_n X_n \quad \text{with known constants } a_0, a_1, \ldots, a_n,$$

is approximately normally distributed provided

$$\max\left\{\frac{a_1^2\sigma_{X_1}^2}{a_1^2\sigma_{X_1}^2 + \ldots + a_n^2\sigma_{X_n}^2}, \ldots, \frac{a_n^2\sigma_{X_n}^2}{a_1^2\sigma_{X_1}^2 + \ldots + a_n^2\sigma_{X_n}^2}\right\} \quad \text{is small,}$$

i.e., none of the $a_i^2\sigma_i^2$ terms dominates the others.

Making use of adjustment factors, chosen such that $3\sigma_i = c_i T_i$, get

$$T_Y = 3\sigma_Y = \sqrt{a_1^2(3\sigma_{X_1})^2 + \ldots + a_n^2(3\sigma_{X_n})^2} = \sqrt{c_1^2 a_1^2 T_1^2 + \ldots + c_n^2 a_n^2 T_n^2}\,.$$

applicable for linear approximations to smooth functions of any random inputs,

subject to above CLT condition.

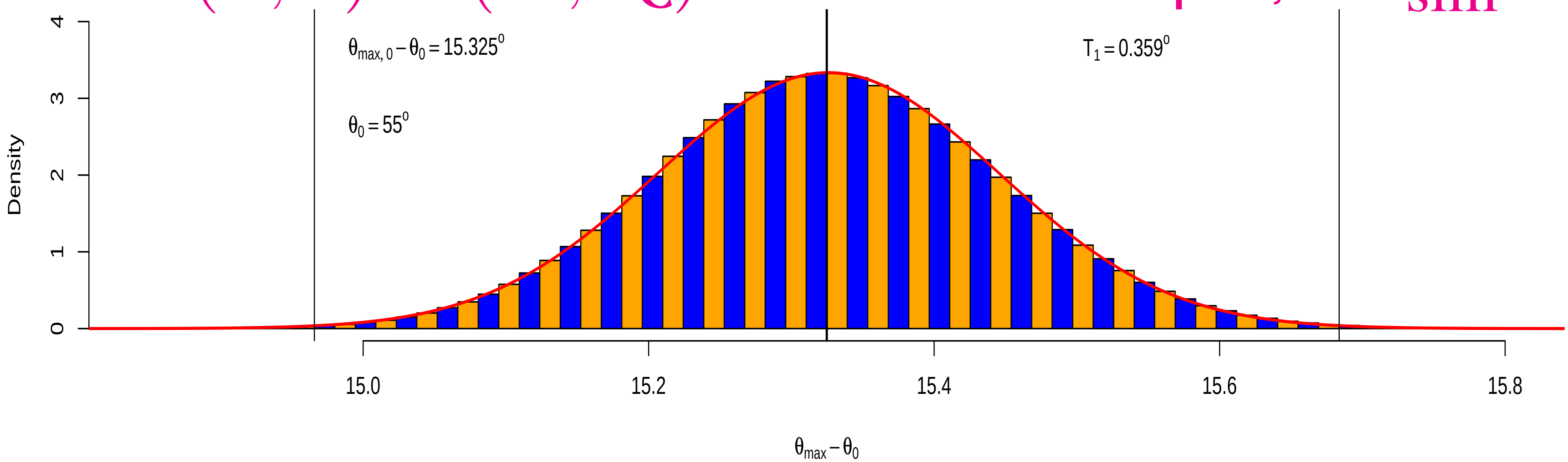
The T_i should be small for linearization to be reasonable.

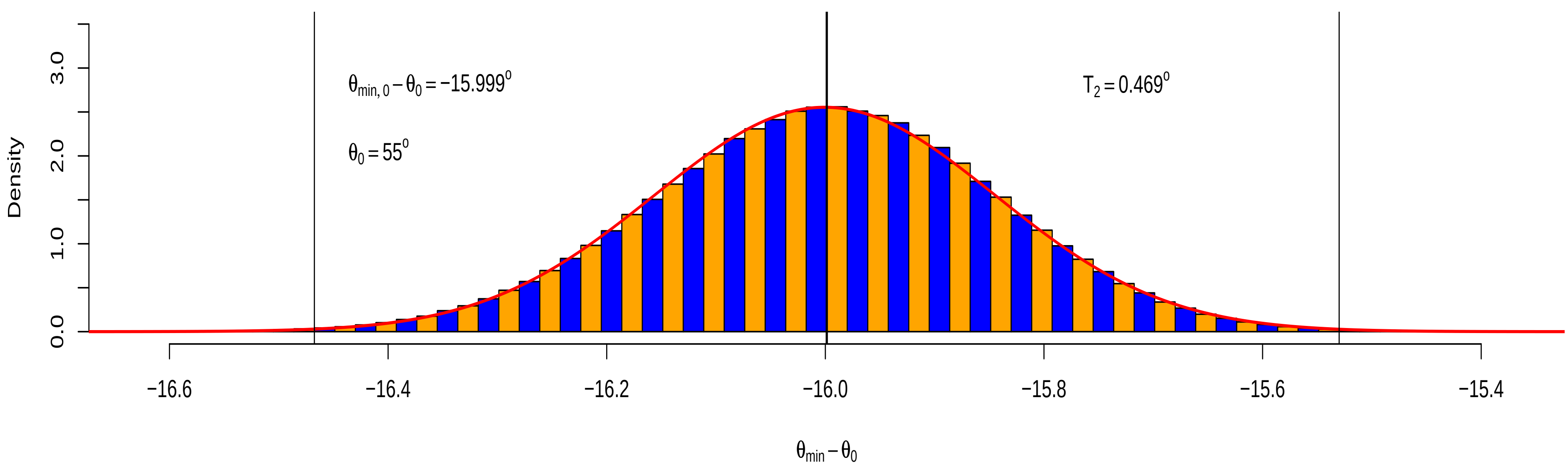
Simulations with Other Distributions for A and R

The next few slides show simulations with $\theta_0 = 55^\circ$ and $\Delta = 1.6$ and

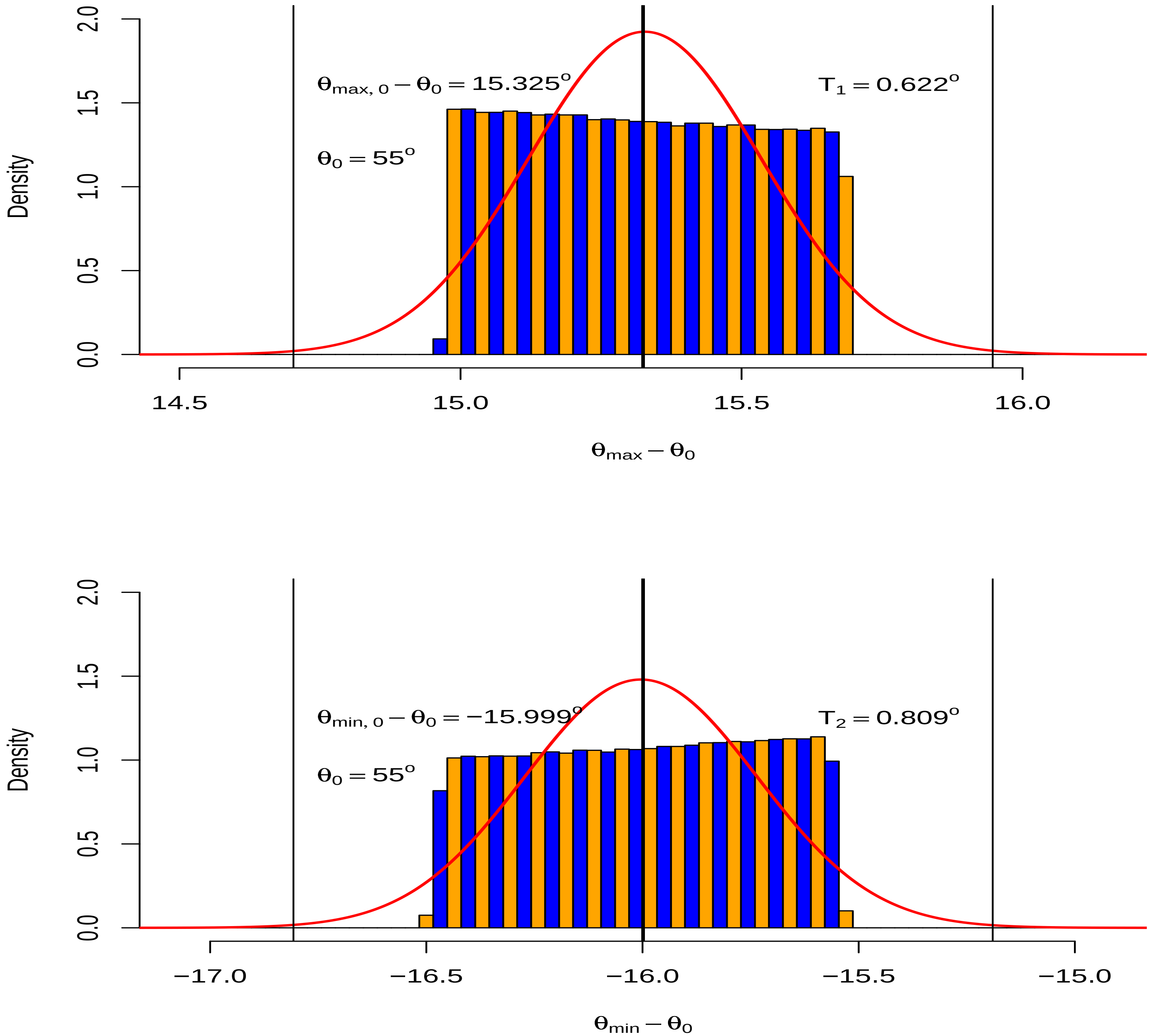
- $(A, R) \sim (\mathcal{U}(12.8 - .12, 12.8 + .12), \mathcal{N}(6, (.14/3)^2)$ using `sim.thetaUN`
- $(A, R) \sim (\mathcal{N}(12.8, (.12/3)^2), \mathcal{U}(6 - .14, 6 + .14))$ using `sim.thetaNU`
- $(A, R) \sim (\mathcal{U}(12.8 - .12, 12.8 + .12), \mathcal{U}(6 - .14, 6 + .14))$ using `sim.thetaUU`
- $(A, R) \sim (\mathcal{U}(12.8 - .012, 12.8 + .012), \mathcal{U}(6 - .014, 6 + .014))$ using `sim.thetaUU`

$(A,R) \sim (\mathcal{U},\mathcal{N})$ Simulation Output, $N_{\text{sim}} = 10^6$

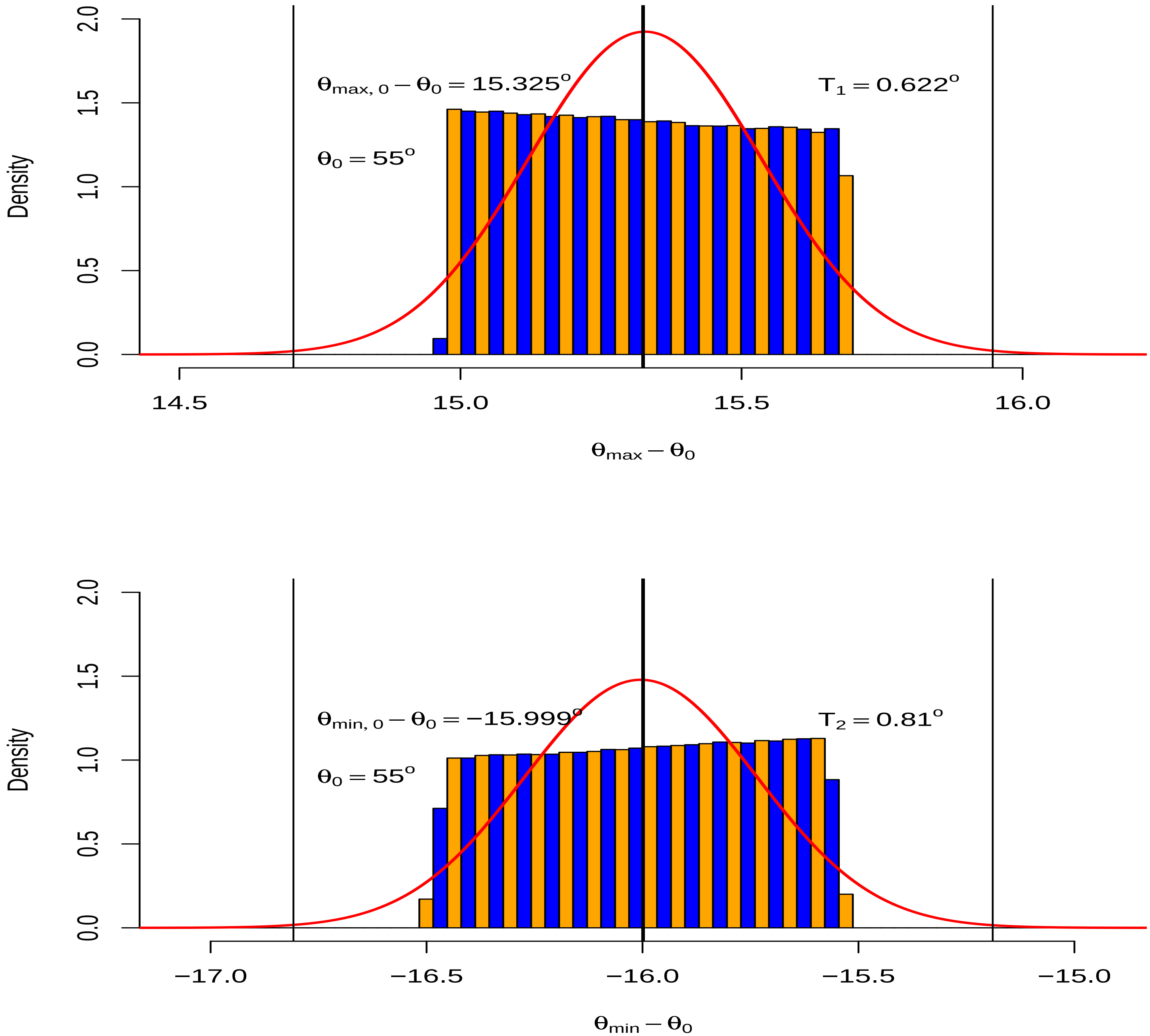




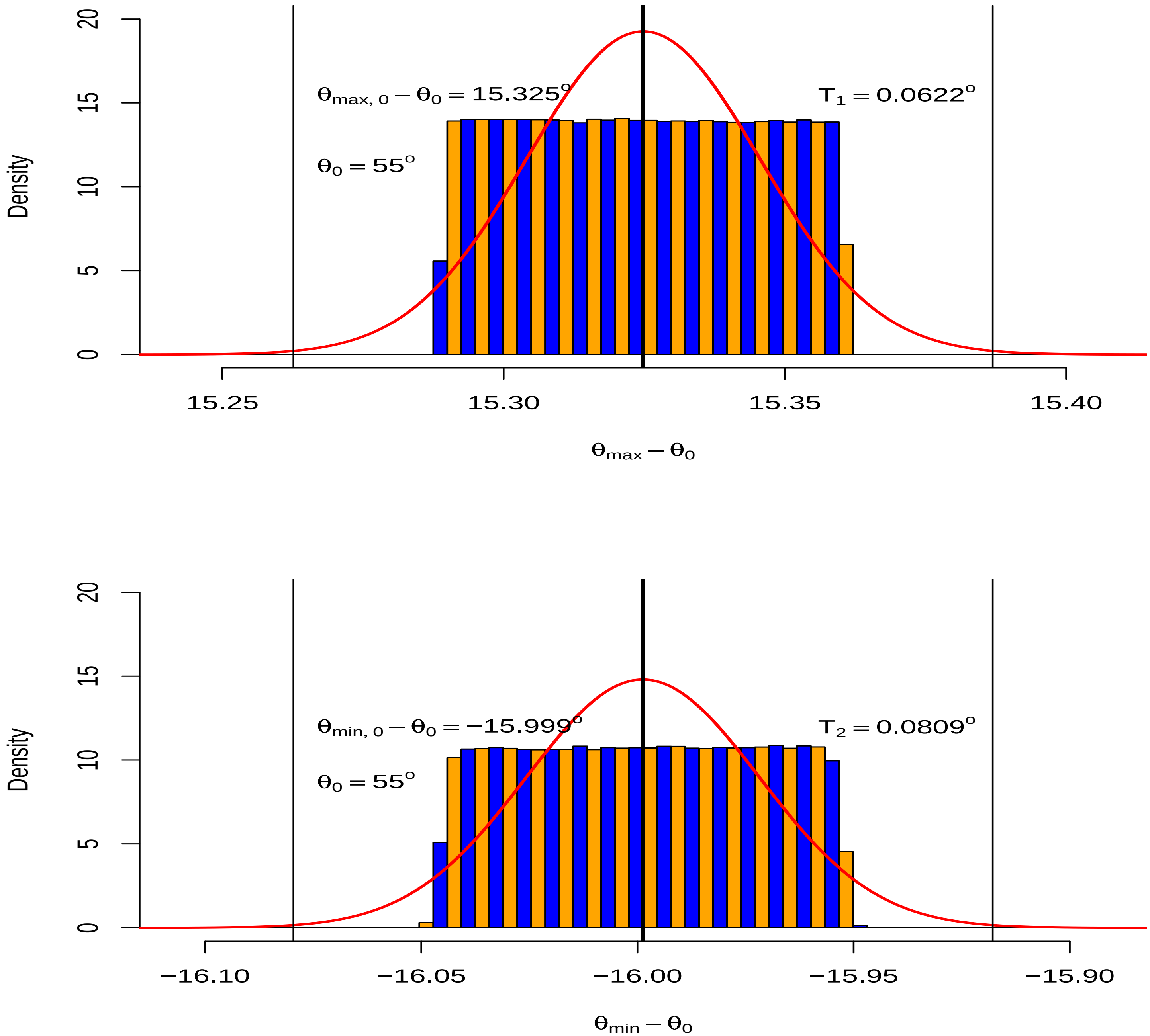
$(A,R) \sim (\mathcal{N},\mathcal{U})$ Simulation Output, $N_{\text{sim}} = 10^6$



$(A,R) \sim (\mathcal{U},\mathcal{U})$ Simulation Output, $N_{\text{sim}} = 10^6$



$(A,R) \sim (\mathcal{U},\mathcal{U})$ Simulation Output, $N_{\text{sim}} = 10^6$



RSS Calculation with Inflation Factors

Applying the RSS formula assuming a uniform distribution for both A and R we get

$$T_1 = \sqrt{(-.00006636269)^2 \cdot 3 \cdot .12^2 + (-.04473777)^2 \cdot 3 \cdot .14^2} \cdot \frac{360}{2\pi} = 0.6215642^\circ$$

and

$$T_2 = \sqrt{(-.004038651)^2 \cdot 3 \cdot .12^2 + (.05810908)^2 \cdot 3 \cdot .14^2} \cdot \frac{360}{2\pi} = 0.8087691^\circ$$

using the inflation factor $c = \sqrt{3}$ and the numerical derivatives in both cases.

Reasonable agreement with the values $.622^\circ$ and $.81^\circ$ from simulation.

Not surprising when linearization is good. We are simply using the variance rules.

However, T_1 and T_2 do not capture the variation range of θ_x, since the CLT fails.

Tightening the tolerances in last case $\Longrightarrow$ echoes the uniform distribution of R.

Linearity was not good with wider tolerances $\Longrightarrow$ “tilted uniform.”

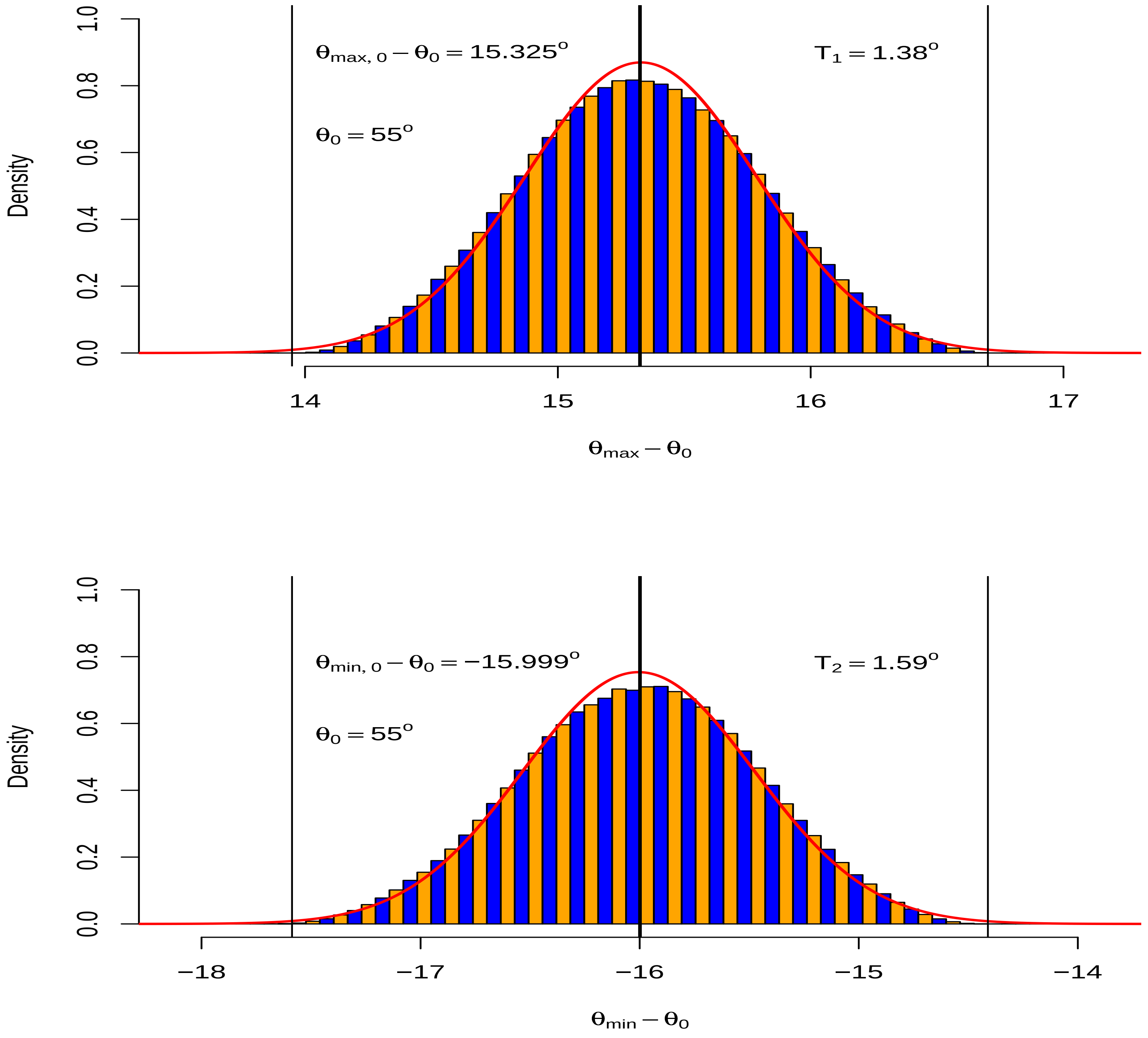
`theta.simUUUU`

Here we let 4 inputs vary with result shown on next slide.

- $A \sim \mathcal{U}(12.8 - .22, 12.8 + .22)$
- $R \sim \mathcal{U}(6 - .15, 6 + .15)$
- $\Delta \sim \mathcal{U}(1.6 - .05, 1.6 + .05)$
- $\theta_0^\star \sim \mathcal{U}(55 - .5, 55 + .5)$

Try other tolerances in these uniform distributions.

Varying A, R, $\theta_0^\star$ and Δ Uniformly

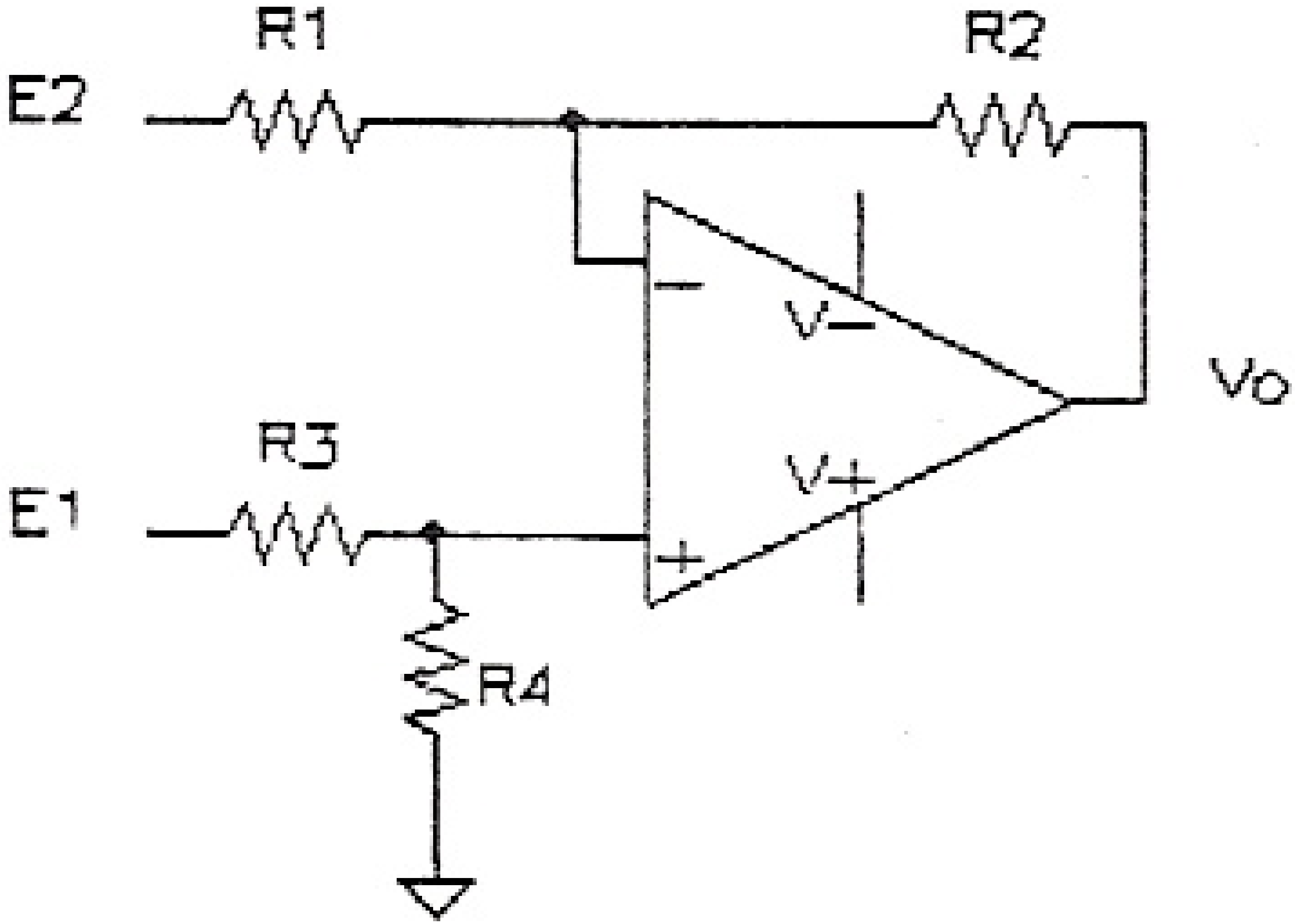


Final Comments

This actuator example has been very instructive. It showed

- the importance of dominant variability by a single input
- the effect of the CLT when sufficiently many contributing inputs are involved
- the importance of simulation
- the importance of derivatives
- the effect of the variability ranges on the linearization approximation quality.

Voltage Amplifier



Output Voltage V_0

The amplified output voltage is a function of 6 variables,

2 input voltages E_1, and E_2 and 4 resistances $R_1, \ldots, R_4$

$$V_0 = f(E_1, E_2, R_1, R_2, R_3, R_4) = \frac{E_1 \cdot \left(1 + \frac{R_2}{R_1}\right)}{1 + \frac{R_3}{R_4}} - \frac{E_2 \cdot R_2}{R_1}$$

Nominal values:

$E_1 = 1V$, $E_2 = -1V$, $R_1 = 10\Omega$, $R_2 = 100\Omega$, $R_3 = 10\Omega$, and $R_4 = 100\Omega$.

$$\Longrightarrow \quad V_0 = 20V.$$

The Derivatives

$$\frac{\partial V_0}{\partial E_1} = \frac{1+\frac{R_2}{R_1}}{1+\frac{R_3}{R_4}}\,, \qquad \frac{\partial V_0}{\partial E_2} = -\frac{R_2}{R_1}$$

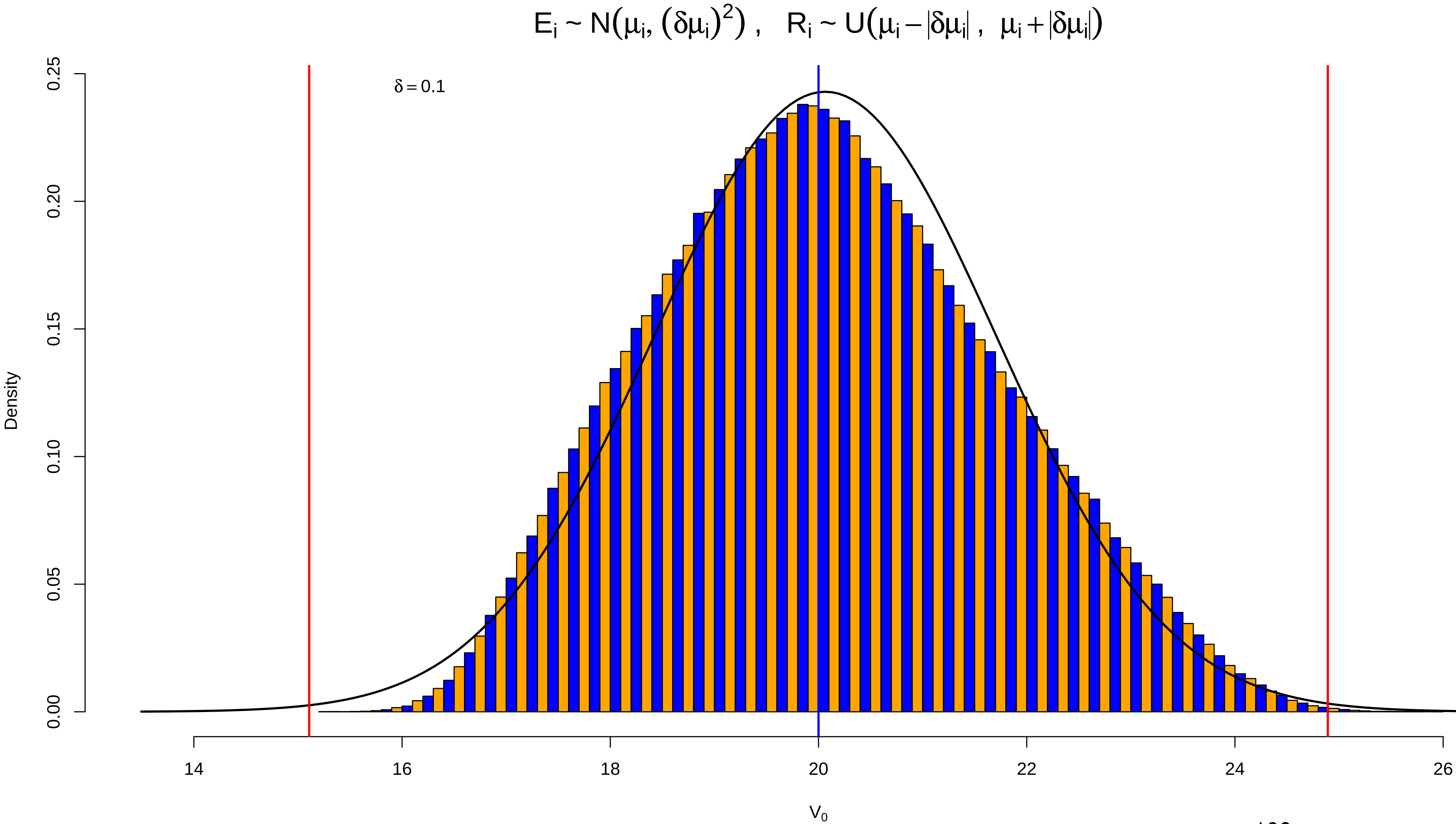
$$\frac{\partial V_0}{\partial R_1} = -\frac{E_1}{1+\frac{R_3}{R_4}} \cdot \frac{R_2}{R_1^2} + \frac{E_2 \cdot R_2}{R_1^2}\,, \qquad \frac{\partial V_0}{\partial R_2} = \frac{\frac{E_1}{R_1}}{1+\frac{R_3}{R_4}} - \frac{E_2}{R_1}$$

$$\frac{\partial V_0}{\partial R_3} = -\frac{E_1 \cdot \left(1+\frac{R_2}{R_1}\right)}{\left(1+\frac{R_3}{R_4}\right)^2} \cdot \frac{1}{R_4}\,, \qquad \frac{\partial V_0}{\partial R_4} = \frac{E_1 \cdot \left(1+\frac{R_2}{R_1}\right)}{\left(1+\frac{R_3}{R_4}\right)^2} \cdot \frac{R_3}{R_4^2}$$

V.amp.simN2U4(del=.1)

```
> V.amp.simN2U4(del=.1)
$V0
[1] 20

$delta
[1] 0.1

$derivatives
[1]  10.000000000 -10.000000000  -1.909090909   0.190909091
+ -0.090909091   0.009090909

$sigmas
[1] 0.33314890 0.33326565 1.10195837 1.10243208
+ 0.05248074 0.05246410

$nominals
[1]   1  -1  10 100  10 100
```

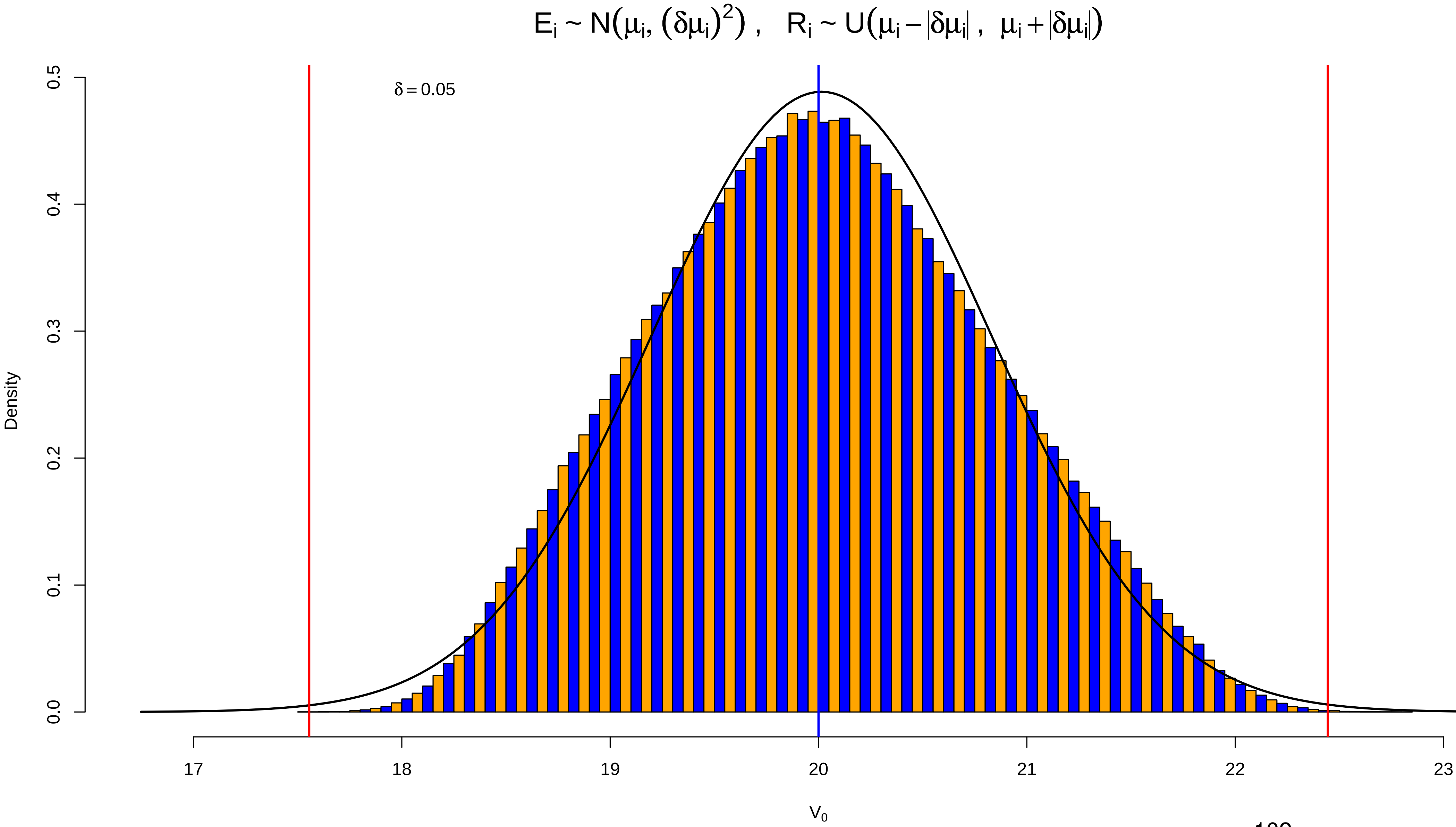
V.amp.simN2U4(del=.1)



V.amp.simN2U4(del=.05)

```
> V.amp.simN2U4(del=.05)
$V0
[1] 20

$delta
[1] 0.05

$derivatives
[1]  10.000000000 -10.000000000  -1.909090909   0.190909091
+ -0.090909091   0.009090909

$sigmas
[1] 0.16657056 0.16676230 0.55079156 0.55108759
+ 0.02627634 0.02624854

$nominals
[1]   1  -1  10 100  10 100
```

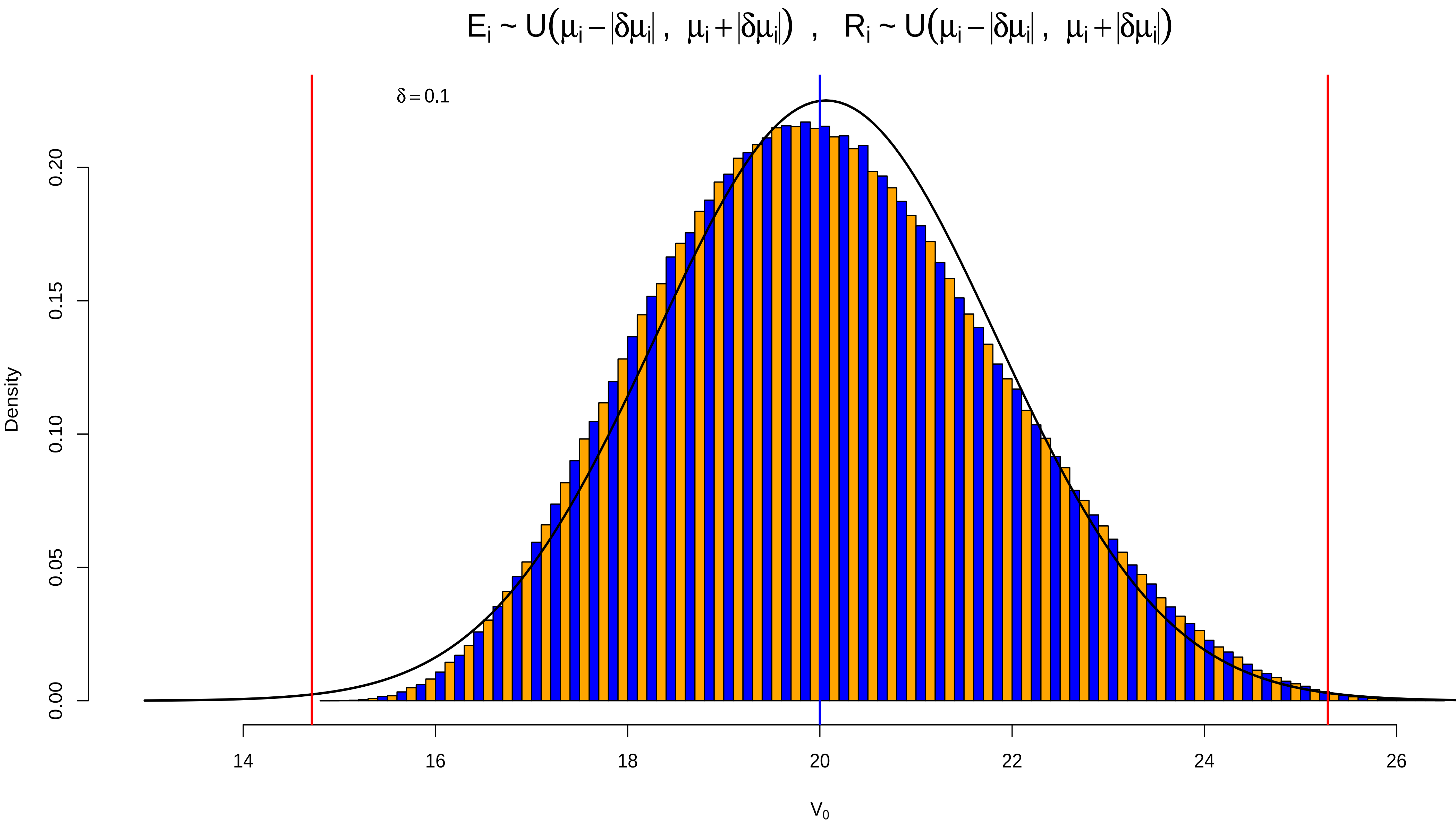
V.amp.simN2U4(del=.05)



V.amp.simU6(del=.1)

```
> V.amp.simU6(del=.1)
$V0
[1] 20

$delta
[1] 0.1

$derivatives
[1]  10.000000000 -10.000000000  -1.909090909   0.190909091
+ -0.090909091   0.009090909

$sigmas
[1] 0.57739282 0.57698360 1.10221137 1.10199967
+ 0.05251682 0.05253420

$nominals
[1]   1  -1  10 100  10 100
```

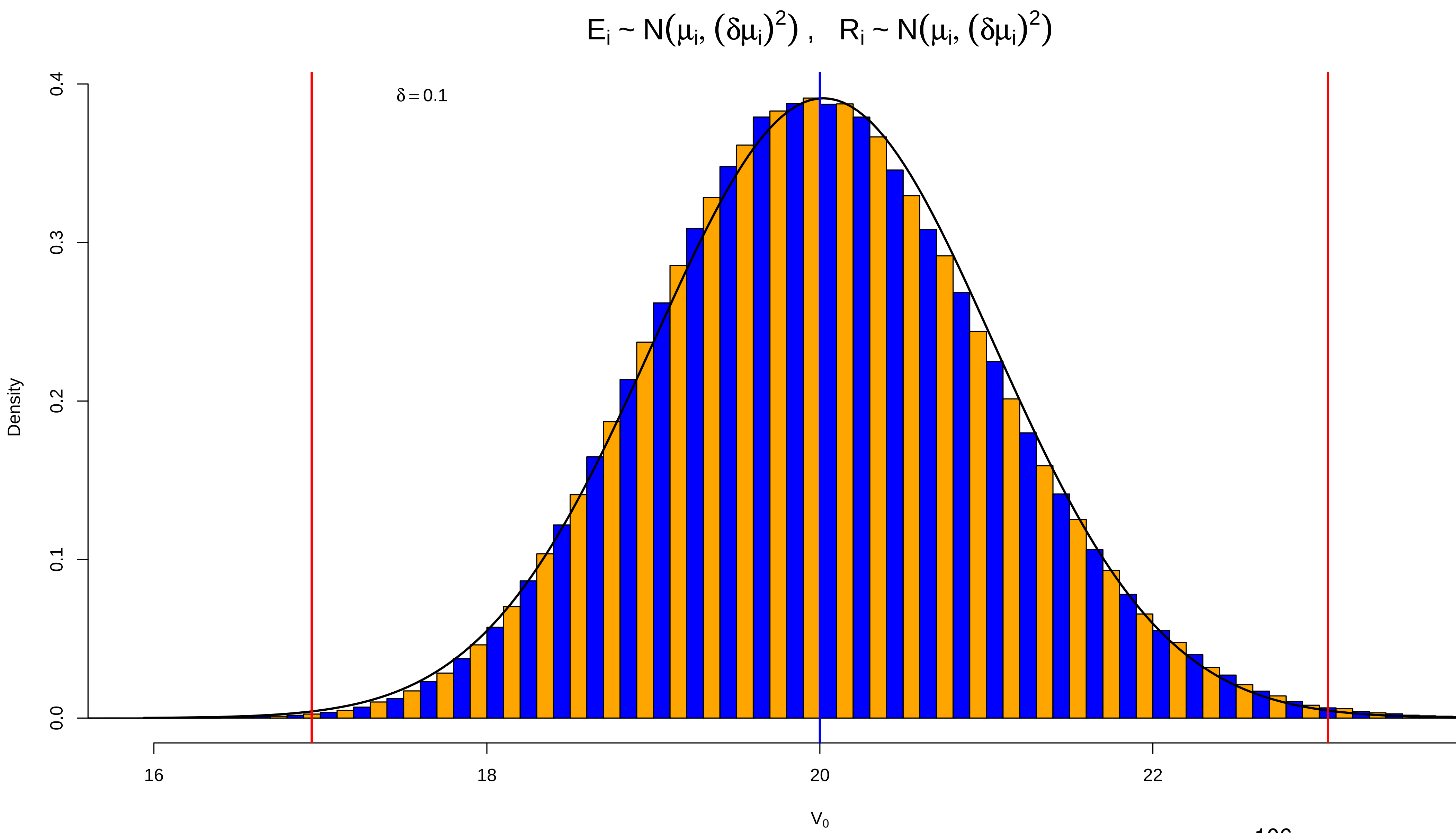
V.amp.simU6(del=.1)



V.amp.simN6(del=.1)

```
> V.amp.simN6(del=.1)
$V0
[1] 20

$delta
[1] 0.1

$derivatives
[1]  10.000000000 -10.000000000  -1.909090909   0.190909091
+ -0.090909091   0.009090909

$sigmas
[1] 0.33348276 0.33332256 0.63653780 0.63714909
+ 0.03031808 0.03029352

$nominals
[1]   1  -1  10 100  10 100
```

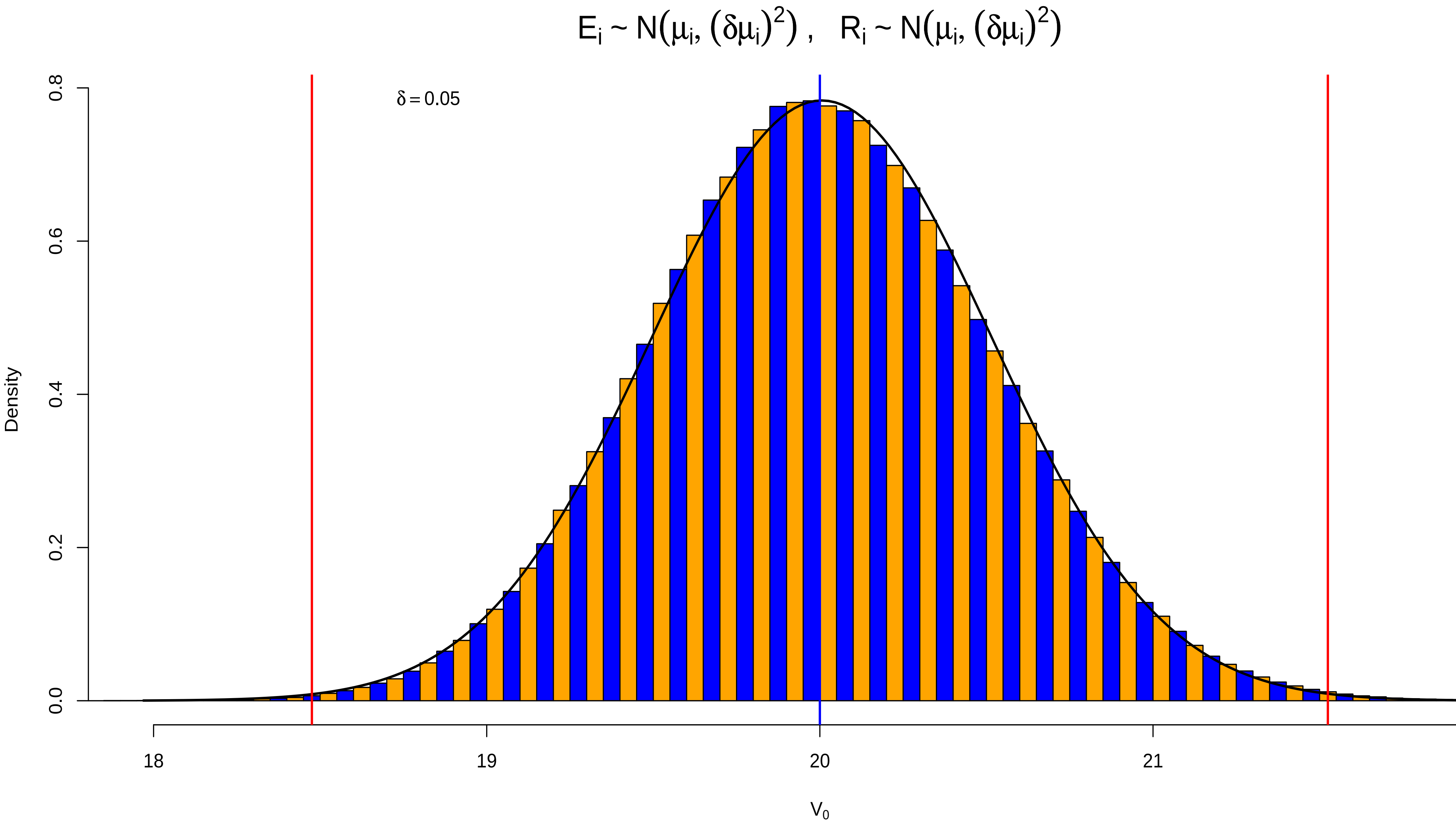
V.amp.simN6(del=.1)



V.amp.simN6(del=.05)

```
> V.amp.simN6(del=.05)
$V0
[1] 20

$delta
[1] 0.05

$derivatives
[1]  10.000000000 -10.000000000  -1.909090909   0.190909091
+ -0.090909091  0.009090909

$sigmas
[1] 0.16656830 0.16669687 0.31840687 0.31774622
+ 0.01514106 0.01513453

$nominals
[1]   1  -1  10 100  10 100
```

V.amp.simN6(del=.05)



Some Final Comments

R_3 and R_4 appear to have negligible effect.

Normal variations on all 6 inputs produce approximately normal V_0 distributions.

The linearizations appears to be a mild issue here.

$E_i \sim \mathcal{N}$ and $R_i \sim \mathcal{U}$ show much stronger deviations from normality,

but not too bad as far as the $\pm T_{V_0} = \pm 3\sigma_{V_0}$ range is concerned.

Distributions appear nearly triangular, because of dominance of R_1 and R_2.

For $E_i \sim \mathcal{U}$ and $R_i \sim \mathcal{U}$ the distribution seems similar to previous case.

The main terms R_1 and R_2 are not as dominant compared to E_1 and E_2.

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