# Statistics of Critical Points in Kähler Geometry and String Theory

by

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# Abstract

We examine the expected number $\mathcal{N}^{\text{crit}}_{N,h}$ of critical points of random holomorphic sections of positive line bundles over compact Kähler manifolds. We show that, on average, the critical points of minimal Morse index are the most plentiful for holomorphic sections of $\mathcal{O}(N) \to \mathbb{CP}^m$ and, in an asymptotic sense, for those of line bundles over general compact Kähler manifolds. We calculate the expected number of these critical points in both cases and use these to obtain growth rates and asymptotic bounds for the total expected number of critical points. We also show that the asymptotic expansion of $\mathcal{N}^{\text{crit}}_{N,h}$ is non-topological in all dimensions and that Calabi extremal metrics asymptotically minimize $\mathcal{N}^{\text{crit}}_{N,h}$, whenever they exist.

READERS: Steve Zelditch (Advisor) and Bernie Shiffman.

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# Contents

# Chapter 1

# Introduction

In this dissertation we study the asymptotic behavior of the expected number $\mathcal{N}^{\mathrm{crit}}_{N,h}$ of critical points of random holomorphic sections of the $N$th tensor power of a positive line bundle over a compact Kähler manifold. We obtain the growth rate as the dimension of the manifold goes to infinity for the special case, $\mathcal{O}(N) \to \mathbb{CP}^m$. We then use the asymptotic expansion of $\mathcal{N}^{\mathrm{crit}}_{N,h}$ in $N$ and obtain upper and lower bounds on the leading coefficient of the expansion for the general case. In both of these cases we work with formulas for the expected number $\mathcal{N}^{\mathrm{crit}}_{N,q,h}$ of critical points of Morse index $q$, obtaining exact formulas for the expected number of critical points of minimal Morse index and showing that these are the most plentiful. We also use the asymptotic expansion to show that $\mathcal{N}^{\mathrm{crit}}_{N,h}$ is non-topological and that it is asymptotically minimized by Calabi extremal metrics whenever they exist.

While research into the statistics of random zeros goes back to at least 1943 ([Kac]), research into the statistics of random critical points has only recently appeared in the mathematics literature. It does, however, have a longer history in the physics literature. These statistics are important in the study of the complicated landscapes that show up in areas such as spin glasses, extremal black holes, and string/M theory. In an attempt to get a handle on the “vacuum selection problem” in string/M theory, M. Douglas and his co-workers proposed a statistical approach in [AD, DD, Dou] to count the number of possible vacua and determine the distributions of various physically relevant quantities within the set. This led to the series of articles [DSZ1, DSZ2, DSZ3] with B. Shiffman and S. Zelditch, where the rigorous mathematical foundation for the study of critical points of random holomorphic sections was presented.

In [DSZ1] the authors laid out the basics of the study in the most general case, that of the critical points of a random section within some subset $S$ of the set of holomorphic sections of a Hermitian holomorphic line bundle over a complex manifold relative to the Chern connection on the bundle and a Gaussian measure on $S$. They derived the formulas for the expected density and number of these critical points and obtained more explicit versions in the case of Riemann surfaces. In the second paper, they turned their attention to the purely geometric considerations of studying the metric dependence and asymptotic minimization of the expected number. In this paper they restricted their attention to the case of a positive Hermitian line bundle over a compact Kähler manifold. They derived the asymptotic expansion of the expected number of critical points for this case as well as integral formulas for the universal constants in the leading term and the first non-topological term. In this paper they also derived a more explicit formula for the expected number of critical points in the case of tensor powers of the hyperplane section line bundle over complex projective space. Then, in [DSZ3], they turned to the physically relevant case for string/M theory which is a discrete ensemble of sections (known as flux superpotentials) forming a lattice of full rank in a real subspace of the set of holomorphic sections of a negative line bundle over an incomplete Kähler manifold. They showed that the statistics of critical points for this discrete lattice of sections is well approximated by those for the

Gaussian measures studied in the previous two papers and were thus able to present the first rigorous results on counting the number of vacua in string theory with remainder estimates. They also presented heuristic estimates of the growth rate of the number of vacua.

In this paper we will be working with the same setting as [DSZ2], a positive Hermitian line bundle over a compact Kähler manifold. Our main results are proofs of a series of conjectures from this paper.

We first present a brief discussion of the background material and define the necessary notation in Chapter 2. We devote the next three chapters to proving some technical lemmas. In Chapter 3 we derive an altenate formula for the leading coefficient of the asymptotic expansion of $\mathcal{N}^{\mathrm{crit}}_{N,q,h}$ using an argument based on the Itzykson-Zuber integral formula. This argument was used in [DSZ2] to simplify two other formulas that we will make use of. In the next chapter we use an iterated residues technique to evaluate an integral which we will need later on to further simplify two of our formulas. Then, in Chapter 5, we discuss Selberg's integral formula and some extensions of it. We extend Aomoto's integral formula and use it to derive an extension of the exponential Selberg integral formula. This formula will play a key role in the proofs of our main results.

The final three chapters in the paper are devoted to the proofs of our main theorems. In Chapter 6 we prove the following result which shows that the expected number of critical points grows exponentially with the dimension for the case of the hyperplane section line bundle over complex projective space, $\mathcal{O}(N) \to \mathbb{CP}^m$.

**Main Theorem 1.** *Let $\mathcal{N}^{\mathrm{crit}}_{N,q,h}(\mathbb{CP}^m)$ denote the expected number of critical points of Morse index $q$ for random sections $s \in H^0(\mathbb{CP}^m, \mathcal{O}(N))$ so that $\mathcal{N}^{\mathrm{crit}}_{N,h}(\mathbb{CP}^m) =$*

$\sum_{q=m}^{2m} \mathcal{N}^{\rm crit}_{N,q,h}(\mathbb{CP}^m)$, *then*

$$\mathcal{N}^{\rm crit}_{N,m,h}(\mathbb{CP}^m) = \frac{2(m+1)(N-1)^{m+1}}{(m+2)N-2},$$

*and when* $N > 2$

$$\mathcal{N}^{\rm crit}_{N,q+1,h}(\mathbb{CP}^m) < \mathcal{N}^{\rm crit}_{N,q,h}(\mathbb{CP}^m).$$

*Therefore,*

$$\frac{2(m+1)(N-1)^{m+1}}{(m+2)N-2} < \mathcal{N}^{\rm crit}_{N,h}(\mathbb{CP}^m) < \frac{2(m+1)^2(N-1)^{m+1}}{(m+2)N-2}.$$

This result is of particular interest in string/M theory because, as was mentioned in [DSZ3], the formula for the critical point density in this case is very similar to the one given there for the density of vacua. In fact, the conjectured growth rate of $\mathcal{N}^{\rm crit}_{N,q,h}(\mathbb{CP}^m)$ with the dimension, given in [DSZ2], which we have now proved, was used as a basis for the heuristic estimate of the growth rate of the number of vacua in [DSZ3].

In Chapter 7 we return to the more general setting of a positive Hermitian line bundle over a compact Kähler manifold and consider the asymptotic expansions of $\mathcal{N}^{\rm crit}_{N,h}$ and $\mathcal{N}^{\rm crit}_{N,q,h}$. We prove our second result, which shows that critical points of minimal Morse index are the most plentiful in an asymptotic sense and gives upper and lower bounds on the leading coefficient of the asymptotic expansion of $\mathcal{N}^{\rm crit}_{N,h}$ in $N$.

**Main Theorem 2.** *Let* $n_q(m)$ *denote the universal constant in the leading order term of the asymptotic expansion of* $\mathcal{N}^{\rm crit}_{N,q,h}$, *and let* $n(m) = \sum_{q=m}^{2m} n_q(m)$, *so that*

$$\mathcal{N}^{\rm crit}_{N,q,h} \sim n_q(m)\, c_1(L)^m\, N^m \qquad \textit{and} \qquad \mathcal{N}^{\rm crit}_{N,h} \sim n(m)\, c_1(L)^m\, N^m.$$

*Then*

$$n_m(m) = 2\,\frac{m+1}{m+2} \qquad \textit{and} \qquad 0 < n_{q+1}(m) < \left(\frac{2m-q}{2m-q+1}\right)^2 n_q(m)\ ,$$

*and thus*

$$2\,\frac{m+1}{m+2} < n(m) < \frac{2m+3}{3}\ .$$

In the final chapter we continue our analysis of the asymptotic expansion of $\mathcal{N}^{\mathrm{crit}}_{N,h}$, this time focusing on the term of order $m-2$. This term is the sum of a topological invariant and the Calabi functional multiplied by the universal constant, $\beta_2(m)$. Since the first two terms in the expansion are topological invariants of the bundle, it is easy to see that the metric dependence of the expansion depends on the value of $\beta_2(m)$. We prove that:

**Main Theorem 3.** *The universal constant $\beta_2(m)$ is strictly positive in all dimensions. Therefore, $\mathcal{N}^{\mathrm{crit}}_{N,h}$ is non-topological, having a metric dependence in the term of order $m-2$ in its asymptotic expansion, in all dimensions. In addition, Calabi extremal metrics asymptotically minimize $\mathcal{N}^{\mathrm{crit}}_{N,h}$, whenever they exist.*

These results have been submitted for publication in [Ba1] and [Ba2].

# Chapter 2

# Critical Point Formulas

The setting throughout this paper will be a positive Hermitian line bundle $(L, h) \to (M^m, \omega_h)$ over a compact Kähler manifold of dimension $m$. Here, $h$ denotes the hermitian metric on $L$ and $\omega_h$ is the Kähler form on $M$ given by $\omega_h = \frac{i}{2}\Theta_h$, where $\Theta_h = -\partial\bar{\partial}\log h$ is the curvature form of the metric. This Kähler form always exists for positive line bundles. In order to study the asymptotics we will need to take tensor powers of these line bundles. We will let $L^N$ denote the $N$th tensor power of the bundle $L$. The metric on $L^N$ is just $h^N$. The connection on the bundle is always taken to be the Chern connection $\nabla$ associated to $h$. The Chern connection is the unique connection of type (1,0) on the bundle that is compatible with both the metric and the complex structure. Relative to the connection, the critical points of a holomorphic section $s \in H^0(M, L^N)$ are given by $\nabla s(z) = 0$, and the set of critical points of $s$ will be denoted by $Crit(s, h^N)$.

It is important to note that, in general, the critical point equation is not holomorphic, and thus the cardinality of $Crit(s, h^N)$ is a non-constant random variable on the space $H^0(M, L^N)$. Indeed, in a local frame $e$, we can write $s = fe$ and then $\nabla s = (\partial f - f\partial K) \otimes e_L$, where $K = -\log \|e\|^2_{h^N}$ is the Kähler potential. From this we see that the critical point equation in the local frame is $\partial f - f\partial K = 0$, which is holomorphic only when $K$ is.

In order to be able to make use of some of the results from [DSZ1] and [DSZ2], we will need our bundle to satisfy a technical condition known as the *2-jet spanning property.* Although a formal definition would require a discussion beyond the scope of this paper, what this requirement boils down to is that the global sections in $H^0(M, L^N)$ must attain all possible values and derivatives of order $\leq 2$ at each point of the manifold.

Next, we need a probability measure on the space of holomorphic sections in order to have a notion of a random section. To this end, we endow the space $H^0(M, L^N)$ with the Gaussian measure $\gamma_N$ given by

$$d\gamma_N(s) = \frac{1}{\pi^d} e^{-\|c\|^2} dc\ , \qquad s = \sum_{j=1}^{d} c_j e_j.$$

Here $dc$ is Lebesgue measure and $\{e_j\}$ is an orthonormal basis of $H^0(M, L^N)$ relative to the inner product

$$\langle s_1, s_2 \rangle = \frac{1}{m!} \int_M h^N(s_1(z), s_2(z))\ \omega_h^m\ ,$$

which is induced by $h^N$ on $H^0(M, L^N)$.

Now we are ready to make the following definitions.

**Definition 2.1.** *The expected distribution of critical points of* $s \in H^0(M, L^N)$ *with respect to* $\gamma_N$*, is defined to be*

$$\mathbf{K}^{\text{crit}}_{N,h} := \int_{H^0(M,L)} \left[ \sum_{z \in Crit(s,\, h^N)} \delta_z \right] d\gamma_N(s),$$

*where* $\delta_z$ *is the Dirac point mass at* $z$*. This is an un-normalized measure on* $M$*.*

**Definition 2.2.** *The density of* $\mathbf{K}^{\text{crit}}_{N,h}$ *with respect to the volume form* $dV_h := \frac{1}{m!}\omega_h^m$ *on* $M$ *will be denoted by* $\mathcal{K}^{\text{crit}}_{N,h}$ *and is defined by the equation,* $\mathbf{K}^{\text{crit}}_{N,h} = \mathcal{K}^{\text{crit}}_{N,h} dV_h$*.*

**Definition 2.3.** *The expected number of critical points of* $s \in H^0(M, L^N)$ *with respect to* $\gamma_N$*, is defined to be*

$$\mathcal{N}^{\mathrm{crit}}_{N,h} := \mathbf{K}^{\mathrm{crit}}_{N,h}(M) = \int_M \mathcal{K}^{\mathrm{crit}}_{N,h}\, dV_h.$$

We recall that the Morse index $q$ of a critical point of a real-valued function is given by the number of negative eigenvalues of its Hessian at the point and that for a positive line bundle $m \le q \le 2m$ [Bo]. Since the critical points of $s$ with respect to $\nabla$ are the same as those of the real-valued function $\log \|s\|^2_{h^N}$, we can consider the Morse indices of the critical points in $Crit(s, h^N)$ by viewing them as critical points of this function. To this end, we let $\mathbf{K}^{\mathrm{crit}}_{N,q,h}$ denote the expected distribution of critical points of Morse index $q$, $\mathcal{K}^{\mathrm{crit}}_{N,q,h}$ denote the density of $\mathbf{K}^{\mathrm{crit}}_{N,q,h}$ with respect to $dV_h$, and $\mathcal{N}^{\mathrm{crit}}_{N,q,h}$ denote the expected number of these critical points. It follows that

$$\mathbf{K}^{\mathrm{crit}}_{N,h} = \sum_{q=m}^{2m} \mathbf{K}^{\mathrm{crit}}_{N,q,h}\,, \quad \mathcal{K}^{\mathrm{crit}}_{N,h} = \sum_{q=m}^{2m} \mathcal{K}^{\mathrm{crit}}_{N,q,h}, \quad \text{and} \quad \mathcal{N}^{\mathrm{crit}}_{N,h} = \sum_{q=m}^{2m} \mathcal{N}^{\mathrm{crit}}_{N,q,h}\,.$$

Since the analysis is simplified by considering the contributions from critical points of different Morse indices separately, we will be working mainly with formulas that depend on the Morse indices of the critical points and will take the sum over the Morse index at the end to obtain our main results, which are independent of $q$.

A critical point of $s$ is a zero of $\nabla s$, which is a $C^\infty$ section of the bundle $T^*M \otimes L$. Therefore, in [DSZ1], the authors were able to apply the previous work by two of the authors in [SZ, BSZ1, BSZ2] on the statistics of zeros of random $C^\infty$ sections of complex vector bundles to the zeros of $\nabla s$ and derive formulas for the expected density of critical points under varying assumptions. One of the formulas they derived was for the special case of critical points of sections of a positive line bundle over a complex manifold with the Morse indices of the critical points taken into account. This formula was modified slightly in [DSZ2] in order to apply it to the asymptotic case where tensor powers of the bundle are considered. This is the formula that is relevant for us.

**Theorem 2.4** (**[DSZ2]**). *Let $(L^N, h^N) \to M$ denote the $N$th tensor power of a positive holomorphic line bundle with the 2-jet spanning property. Give $M$ the volume form $dV_h = \frac{1}{m!}\left(\frac{i}{2}\Theta_h\right)^m$ induced from the curvature of $L$. Then the density relative to $dV_h$ of the expected distribution $\mathbf{K}^{\mathrm{crit}}_{N,q,h}$ of critical points of Morse index $q$ of $\log \|s_N\|_h$ for random sections $s \in H^0(M, L^N)$ is given by*

$$\mathcal{K}^{\mathrm{crit}}_{N,q,h}(z) = \frac{\pi^{-\binom{m+2}{2}}}{\det A_N(z) \det \Lambda_N(z)} \times \int_{\mathbf{S}_{m,q-m}} \left|\det(HH^* - |x|^2 I)\right| e^{-\langle \Lambda_N(z)^{-1}(H,x),(H,x)\rangle}\, dH\, dx\,. \quad (2.1)$$

*where*

$$\mathbf{S}_{m,k} = \{(H,x) \in \mathrm{Sym}(m,\mathbb{C}) \times \mathbb{C} : \mathrm{index}(HH^* - |x|^2 I) = k\}$$

*and*

$$\begin{aligned}
\Lambda_N(z_0) &= C_N(z_0) - B_N(z_0)^* A_N(z_0)^{-1} B_N(z_0)\,,\\
A_N(z_0) &= \Big[\big(\nabla_{z_j}\nabla_{\bar{w}_{j'}}\Pi_N\big)\Big]\,,\\
B_N(z_0) &= \Big[\big(\tau_{j'q'}\,\nabla_{z_j}\nabla_{\bar{w}_{q'}}\nabla_{\bar{w}_{j'}}\Pi_N\big) \quad \big(N\,\nabla_{z_j}\Pi_N\big)\Big]\,,\\
C_N(z_0) &= \begin{bmatrix} \big(\tau_{jq}\tau_{j'q'}\,\nabla_{z_q}\nabla_{z_j}\nabla_{\bar{w}_{q'}}\nabla_{\bar{w}_{j'}}\Pi_N\big) & \big(\tau_{jq}N\,\nabla_{z_q}\nabla_{z_j}\Pi_N\big) \\ \big(\tau_{j'q'}N\,\nabla_{\bar{w}_{q'}}\nabla_{\bar{w}_{j'}}\Pi_N\big) & N^2\,\Pi_N \end{bmatrix}\,,
\end{aligned}$$

$$\tau_{jq} = \sqrt{2} \ \ \text{if} \ \ j<q\,,\ \tau_{jj}=1\,,\ 1\le j\le m\,,\ 1\le j\le q\le m\,,\ 1\le j'\le q'\le m\,.$$

*Here $\Pi_N$, the Szegö kernel of $H^0(M, L^N)$, and its covariant derivatives are evaluated at $(z_0, 0; z_0, 0)$.*

By differentiating the Tian-Yau-Zelditch asymptotic expansion of the Szegö kernel (see [Ti, Ya, Ze]), the authors were able to derive the asymptotic expansion of $\mathcal{N}^{\mathrm{crit}}_{N,q,h}$

in [DSZ2]. After showing that the coefficients in the expansion are universal, they also derived integral formulas for the leading coefficient and the coefficient of the first non-topological term in the expansion by directly computing them in the case where $M$ is the tensor product of $\mathbb{CP}^1$ with $m-1$ copies of an elliptic curve and $L$ is the tensor product of degree one line bundles on the two factors.

**Theorem 2.5** (**[DSZ2]**). *Let $(L,h) \to (M,\omega_h)$ be a positive holomorphic line bundle on a compact Kähler manifold, with $\omega_h = \frac{i}{2}\Theta_h$. Then the expected number of critical points of Morse index $q$ ($m \le q \le 2m$) of random sections in $H^0(M,L^N)$ has the asymptotic expansion*

$$\begin{aligned}\mathcal{N}^{\rm crit}_{N,q,h} \;\sim\; & \left[\frac{\pi^m b_{0q}}{m!}\,c_1(L)^m\right] N^m + \left[\frac{\pi^m \beta_{1q}}{(m-1)!}\,c_1(M)\cdot c_1(L)^{m-1}\right] N^{m-1} \\ & + \Bigg[\beta_{2q}\int_M \rho^2 d\mathrm{Vol}_h + \beta'_{2q}\,c_1(M)^2\cdot c_1(L)^{m-2} \\ & \qquad\qquad + \beta''_{2q}\,c_2(M)\cdot c_1(L)^{m-2}\Bigg] N^{m-2} + \cdots\,,\end{aligned}$$

*where $b_{0q}, \beta_{1q}, \beta_{2q}, \beta'_{2q}, \beta''_{2q}$ are universal constants depending only on the dimension $m$. In particular, we have the formulas*

$$b_{0q} = \pi^{-\binom{m+2}{2}} \int_{\mathbf{S}_{m,q-m}} \left|\det(2HH^* - |x|^2 I)\right| \, e^{-\langle (H,x),(H,x)\rangle}\, dH\, dx\,, \tag{2.2}$$

*and*

$$\beta_{2q}(m) = \frac{1}{4\,\pi^{\binom{m+2}{2}}} \int_{\mathbf{S}_{m,q-m}} \gamma(H)\, \left|\det(2HH^* - |x|^2 I)\right| \, e^{-\langle (H,x),(H,x)\rangle}\, dH\, dx\;, \tag{2.3}$$

*where*

$$\gamma(H) = \frac{1}{2}|H_{11}|^4 - 2|H_{11}|^2 + 1\,.$$

For the remainder of this paper we will focus on the three formulas: (2.1), (2.2), and (2.3). These formulas will be simplified by first applying an argument based on the Itzykson-Zuber integral formula and then computing a subset of the resulting integrals using an iterated residues technique. After this, one of several variants of the Selberg integral formula will be applied to get an exact formula for critical points of minimal Morse index in each case. We will then utilize various change of variable and symmetry arguments to manipulate the formulas for the other Morse index cases to derive our main results. We will apply (2.1) to the special case, $\mathcal{O}(N) \to \mathbb{CP}^m$, to obtain the growth rate of the expected number of critical points of sections of this bundle as the dimension of the manifold goes to infinity. In the general case, we will use (2.2) and (2.3) to derive the asymptotic growth rate, metric dependence, and asymptotic minimization of the expected number of critical points.

As the reader has probably already noticed, these three formulas are very similar, the only differences (other than the coefficients) being the presence of the function $\gamma(H)$ in the integrand in (2.3) and the presence of the operator $\Lambda_N(z)^{-1}$ in the exponential in (2.1). Thus, initially, the simplification steps are very similar. In addition, in [DSZ2], the Itzykson-Zuber calculation was done for (2.1) and (2.3) and the iterated residues calculation was done for (2.1). Hence, our approach will be to reproduce the results derived in [DSZ2], carry out the calculations for the cases that were not done in that paper, and note any important differences between our proofs and theirs. In the next chapter we will present alternate formulas for each of the three cases, which were derived using the Itzykson-Zuber method, and present the proof of the $b_{0q}$ formula. The following two chapters will be devoted to proving two results that will be applied later to each of the three formulas.

# Chapter 3

# Itzykson-Zuber Method

The absolute value that appears in the integrands of (2.1), (2.2), and (2.3), prevent us from applying Wick's formula, making the formulas difficult to evaluate directly. Therefore, in [DSZ2], the authors used an argument based on the Itzykson-Zuber integral formula and Gaussian integration to derive the following alternate formulas for $\mathcal{K}^{\text{crit}}_{N,q,h}(z)$ and $\beta_{2q}(m)$.

**Theorem 3.1** (**[DSZ2]**). *Under the same assumptions as before, the density of the expected distribution of critical points of Morse index $q$ of $\log \|s_N\|_h$ is also given by:*

$$\mathcal{K}^{\text{crit}}_{N,q,h}(z) = \frac{(-i)^{m(m-1)/2}}{2^m\,\pi^{2m}\,\prod_{j=1}^{m-1} j!\,\det A_N} \times \lim_{\varepsilon'\to 0^+} \int_{Y_{2m-q}} \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^m} \int_{\mathrm{U}(m)} \frac{\Delta(\xi)\,\Delta(\lambda)\,|\prod_j \lambda_j|\,e^{i\langle\xi,\lambda\rangle} e^{-\epsilon|\xi|^2-\epsilon'|\lambda|^2}}{\det\left[i\widehat{D}(\xi)\rho(g)\Lambda_N(z)\rho(g)^*+I\right]}\,dg\,d\xi\,d\lambda,$$

*where*

- $\Delta(\lambda) = \Pi_{i<j}(\lambda_i - \lambda_j)$,
- $Y_p = \{\lambda \in \mathbb{R}^m : \lambda_1 > \cdots > \lambda_p > 0 > \lambda_{p+1} > \cdots > \lambda_m\}$ ,
- *$dg$ is unit mass Haar measure on $\mathrm{U}(m)$,*

- $\widehat{D}(\xi)$ *is the Hermitian operator on* $\mathrm{Sym}(m,\mathbb{C})\oplus\mathbb{C}$ *given by*

$$\widehat{D}(\xi)\big((H_{jk}),x\big)=\left(\left(\frac{\xi_j+\xi_k}{2}\,H_{jk}\right),\ -\left(\textstyle\sum_{q=1}^m \xi_q\right)x\right)\ ,$$

- $\rho$ *is the representation of* $\mathrm{U}(m)$ *on* $\mathrm{Sym}(m,\mathbb{C})\oplus\mathbb{C}$ *given by*

$$\rho(g)(H,x)=(gHg^t,x)\ .$$

**Lemma 3.2** (**[DSZ2]**).

$$\beta_{2q}(m)=\frac{(-i)^{m(m-1)/2}}{4\,\pi^{2m}\prod_{j=1}^{m-1}j!}\int_{Y_{2m-q}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}\Delta(\lambda)\,\Delta(\xi)\prod_{j=1}^m|\lambda_j|\,e^{i\langle\lambda,\xi\rangle}\,\mathcal{I}(\lambda,\xi)\,d\xi_1\cdots d\xi_m\,d\lambda\ ,$$

*where*

$$\mathcal{I}(\lambda,\xi)=\frac{F(D(\lambda))+\left[\frac{4\sum_{j=1}^m\lambda_j}{m(m+1)(m+3)}-\frac{2}{m+1}\right]\frac{1}{\left(1-\frac{i}{2}\sum_j\xi_j\right)}+\frac{2}{(m+1)(m+3)\left(1-\frac{i}{2}\sum_j\xi_j\right)^2}}{\left(1-\frac{i}{2}\sum_j\xi_j\right)\prod_{j\le k}\left[1+\frac{i}{2}(\xi_j+\xi_k)\right]}\ .$$

*Here,* $D(\lambda)$ *is the diagonal matrix with diagonal entries* $\lambda=(\lambda_1,\dots,\lambda_m)$*,*

$$F(P)=1-\frac{4\,\mathrm{Tr}\,P}{m(m+1)}+\frac{4(\mathrm{Tr}\,P)^2+8\,\mathrm{Tr}(P^2)}{m(m+1)(m+2)(m+3)}\ , \tag{3.1}$$

*for (Hermitian)* $m\times m$ *matrices* $P$*, and* $\Delta(\lambda)$ *and* $Y_p$ *are the same as in Theorem 3.1. The iterated* $d\xi_j$ *integrals are defined in the distribution sense.*

In the $\beta_{2q}(m)$ case an additional argument was needed to show that $\gamma(H)$ could be replaced by $F(HH^*)$ in the integrand of (2.3) prior to applying the Itzkson-Zuber method.

While these new alternate formulas certainly look more complicated, they are now in a form which can be calculated by computer programs in low dimensions. These low dimensional calculations were important in formulating the conjectures that we will prove in this paper. As we will see later, these alternate formulas will also yield to further simplification in certain cases, by the evaluation of the $\xi$ integrals using an iterated residues technique.

We will now apply the method, which was used in [DSZ2] to obtain the above formulas, to (2.2) with very few modifications in order to obtain an alternate formula for $b_{0q}(m)$ as well. The proof in this case differs somewhat towards the end from the one given for Theorem 3.1 due to the difference in the phase, but is more or less identical to the proof of Lemma 3.2.

**Lemma 3.3.**

$$b_{0q}(m) = \frac{(-i)^{m(m-1)/2}}{\pi^{2m}\prod_{j=1}^{m-1} j!} \times \int_{Y_{2m-q}} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \frac{\Delta(\lambda)\,\Delta(\xi)\,\prod_{j=1}^{m}|\lambda_j|\,e^{i\langle\lambda,\xi\rangle}}{\left(1-\frac{i}{2}\sum_j \xi_j\right)\prod_{j\le k}\left[1+\frac{i}{2}(\xi_j+\xi_k)\right]}\,d\xi_1\cdots d\xi_m\,d\lambda\ .$$

*Here, $\Delta(\lambda)$ and $Y_p$ are as in Theorem 3.1, and the iterated $d\xi_j$ integrals are defined in the distribution sense.*

*Proof.* First, we let

$$\mathcal{I}_{\varepsilon,\varepsilon'} = \frac{1}{\pi^{d_m}} \int_{\mathcal{H}_m}\int_{\mathcal{H}_m(m-q)}\int_{\mathrm{Sym}(m,\mathbb{C})\times\mathbb{C}} |\det(2P)|\,e^{i\langle\Xi, P-HH^*+\frac{1}{2}|x|^2 I\rangle} \times\ e^{-\mathrm{Tr}HH^*-|x|^2}\,e^{-\epsilon\mathrm{Tr}\,\Xi\Xi^*-\varepsilon'\mathrm{Tr}PP^*}\,dH\,dx\,dP\,d\Xi\ , \quad (3.2)$$

where $\mathcal{H}_m$ is the space of $m\times m$ Hermitian matrices, $\mathcal{H}_m(m-q) = \{P\in\mathcal{H}_m : \mathrm{index}\,P = m-q\}$, and $d_m = \dim_{\mathbb{C}}(\mathrm{Sym}(m,\mathbb{C})\times\mathbb{C}) = \frac{1}{2}(m^2+m+2)$. We note that absolute convergence in the above integral is guaranteed by the Gaussian factors in each of the variables $(H, x, P, \Xi)$. It then follows that

$$b_{0q}(m) = \frac{1}{\pi^m\,(2\pi)^{m^2}}\ \lim_{\varepsilon'\to 0}\ \lim_{\varepsilon\to 0}\mathcal{I}_{\varepsilon,\varepsilon'}\ . \quad (3.3)$$

To see this, evaluate $\int e^{i\langle\Xi,P-HH^*+\frac{1}{2}|x|^2\rangle}e^{-\varepsilon\mathrm{Tr}\,\Xi\Xi^*}d\Xi$ first to obtain a dual Gaussian, which approximates the delta function $\delta_{HH^*-\frac{1}{2}|x|^2}(P)$. As $\epsilon\to 0$, the $dP$ integral then yields the integrand at $P = HH^* - \frac{1}{2}|x|^2 I$; then we let $\varepsilon'\to 0$ to obtain the original integral.

Next, we conjugate $P$ to a diagonal matrix $D(\lambda)$ with $\lambda = (\lambda_1, \dots, \lambda_m)$ by an element $h \in \mathrm{U}(m)$. We recall that

$$\int_{\mathcal{H}_m} \phi(P)\, dP = c'_m \int_{\mathbb{R}^m} \int_{\mathrm{U}(m)} \phi(hD(\lambda)h^*) \Delta(\lambda)^2\, dh\, d\lambda\,, \quad c'_m = \frac{(2\pi)^{\binom{m}{2}}}{\prod_{j=1}^m j!}, \tag{3.4}$$

where $dh$ is unit mass Haar measure on $\mathrm{U}(m)$ (see for example [ZZ, (1.9)]), and use this to obtain

$$\begin{aligned} \mathcal{I}_{\varepsilon,\varepsilon'} &= \frac{2^m\, c'_m}{\pi^{d_m}} \int_{\mathrm{U}(m)} \int_{\mathcal{H}_m} \int_{Y_{2m-q}} \int_{\mathrm{Sym}(m,\mathbb{C})\times\mathbb{C}} \Delta(\lambda)^2 \prod_{j=1}^m |\lambda_j|\; e^{-\mathrm{Tr} HH^* - |x|^2} \\ &\qquad \times\; e^{i\langle \Xi,\, hD(\lambda)h^* + \frac12 |x|^2 I - HH^*\rangle} e^{-\left[\epsilon \mathrm{Tr}\Xi\Xi^* + \epsilon' \sum \lambda_j^2\right]}\, dH\, dx\, d\lambda\, d\Xi\, dh\,. \end{aligned}$$

Again using (3.4), applied this time to $\Xi$, we obtain

$$\begin{aligned} \mathcal{I}_{\varepsilon,\varepsilon'} &= \frac{2^m (c'_m)^2}{\pi^{d_m}} \int_{\mathrm{U}(m)} \int_{\mathrm{U}(m)} \int_{\mathbb{R}^m} \int_{Y_{2m-q}} \int_{\mathrm{Sym}(m,\mathbb{C})\times\mathbb{C}} \Delta(\lambda)^2\, \Delta(\xi)^2 \prod_{j=1}^m |\lambda_j| \\ &\qquad \times\; e^{i\langle gD(\xi)g^*,\, hD(\lambda)h^* + \frac12 |x|^2 I - HH^*\rangle} \\ &\qquad \times\; e^{-\mathrm{Tr} HH^* - |x|^2 - \sum(\varepsilon \xi_j^2 + \varepsilon' \lambda_j^2)}\, dH\, dx\, d\lambda\, d\xi\, dh\, dg\,. \end{aligned}$$

We then transfer the conjugation by $g$ to the right side of the inner product in the first exponent and make the change of variables $h \mapsto gh$, $H \mapsto gHg^t$ to eliminate $g$ from the integrand:

$$\begin{aligned} \mathcal{I}_{\varepsilon,\varepsilon'} &= \frac{2^m (c'_m)^2}{\pi^{d_m}} \int_{\mathrm{U}(m)} \int_{\mathbb{R}^m} \int_{Y_{2m-q}} \int_{\mathrm{Sym}(m,\mathbb{C})\times\mathbb{C}} \Delta(\lambda)^2\, \Delta(\xi)^2 \prod_{j=1}^m |\lambda_j| \\ &\qquad \times\; e^{i\langle D(\xi),\, hD(\lambda)h^* + \frac12 |x|^2 I - HH^*\rangle} e^{-\mathrm{Tr} HH^* - |x|^2 - \sum(\varepsilon \xi_j^2 + \varepsilon' \lambda_j^2)}\, dH\, dx\, d\lambda\, d\xi\, dh\,. \end{aligned}$$

Next, we recognize the integral $\int_{\mathrm{U}(m)} e^{i\langle D(\xi), hD(\lambda)h^*\rangle} dh$ as the well-known Itzykson-Zuber-Harish-Chandra integral [Ha] (cf., [ZZ]):

$$J(D(\lambda), D(\xi)) = (-i)^{m(m-1)/2} \left(\prod_{j=1}^{m-1} j!\right) \frac{\det[e^{i\lambda_j \xi_k}]_{j,k}}{\Delta(\lambda)\Delta(\xi)}\,. \tag{3.5}$$

We substitute (3.5) into the above integral and expand

$$\det[e^{i\xi_j\lambda_k}]_{jk} = \sum_{\sigma\in S_m} (-1)^\sigma\, e^{i\langle \xi, \sigma(\lambda)\rangle},$$

obtaining a sum of $m!$ integrals. However, by making the change of variables $\sigma(\lambda) \to \lambda'$ and noting that $\Delta(\lambda') = (-1)^\sigma \Delta(\lambda)$, we see that the integrals of all these terms are equal, and so we obtain

$$\begin{aligned} \mathcal{I}_{\varepsilon,\varepsilon'} &= (-i)^{m(m-1)/2} \frac{c''_m}{\pi^{d_m}} \int_{\mathbb{R}^m} \int_{Y_{2m-q}} \int_{\mathrm{Sym}(m,\mathbb{C})\times\mathbb{C}} \Delta(\lambda)\,\Delta(\xi) \prod_{j=1}^m |\lambda_j|\; e^{i\langle\lambda,\xi\rangle} \\ &\quad \times\; \exp\left(i \left\langle D(\xi),\, \tfrac{1}{2}|x|^2 I - HH^* \right\rangle - \mathrm{Tr} HH^* - |x|^2\right) \\ &\quad \times\; \exp\left(-\varepsilon \sum \xi_j^2 - \varepsilon' \sum \lambda_j^2\right)\, dH\, dx\, d\lambda\, d\xi\,, \end{aligned}$$

where

$$c''_m = \frac{2^{m^2}\, \pi^{m(m-1)}}{\prod_{j=1}^m j!}\,.$$

The phase

$$\begin{aligned} \Phi(H,x;\xi) &:= i \left\langle D(\xi), \frac{1}{2}|x|^2 I - HH^* \right\rangle - \mathrm{Tr} HH^* - |x|^2 \\ &= -\left[ \|H\|_{\mathrm{HS}}^2 + i \sum_{j,k=1}^m \xi_j |H_{jk}|^2 + \left(1 - \frac{i}{2} \sum_j \xi_j\right) |x|^2 \right] \\ &= -\left[ \sum_{j\le k} \left(1 + \frac{i}{2}(\xi_j + \xi_k)\right) |\widehat{H}_{jk}|^2 + \left(1 - \frac{i}{2} \sum_j \xi_j\right) |x|^2 \right], \end{aligned}$$

where

$$\widehat{H}_{jk} = \begin{cases} \sqrt{2}\, H_{jk} & \text{for } j < k \\ H_{jk} & \text{for } j = k \end{cases}.$$

Thus,

$$\begin{aligned} \mathcal{I}_{\varepsilon,\varepsilon'} &= (-i)^{m(m-1)/2} c''_m \\ &\times \int_{Y_{2m-q}} \int_{\mathbb{R}^m} \Delta(\lambda)\,\Delta(\xi) \prod_{j=1}^m |\lambda_j|\; e^{i\langle\lambda,\xi\rangle}\, \mathcal{I}(\lambda,\xi)\, e^{-\varepsilon \sum \xi_j^2 - \varepsilon' \sum \lambda_j^2}\, d\xi\, d\lambda\,, \end{aligned} \tag{3.6}$$

where

$$
\begin{aligned}
\mathcal{I}(\lambda,\xi) &= \frac{1}{\pi^{d_m}} \int_{\mathbb{C}} \int_{\mathrm{Sym}(m,\mathbb{C})} e^{\Phi(H,x;\xi)}\, dH\, dx \\
&= \frac{1}{\prod_{j\le k}\left(1+\frac{i}{2}(\xi_j+\xi_k)\right)} \int_{\mathbb{C}} e^{-\left(1-\frac{i}{2}\sum_j \xi_j\right)|x|^2}\, dx \\
&= \frac{\pi}{\left(1-\frac{i}{2}\sum_j \xi_j\right)\prod_{j\le k}\left(1+\frac{i}{2}(\xi_j+\xi_k)\right)}.
\end{aligned}
$$

To evaluate $\lim_{\varepsilon,\varepsilon'\to 0+} \mathcal{I}_{\varepsilon,\varepsilon'}$, we first observe that the map

$$
(\varepsilon_1,\ldots,\varepsilon_m) \mapsto \int_{\mathbb{R}^m} \Delta(\xi)\ e^{i\langle\lambda,\xi\rangle}\, \mathcal{I}(\lambda,\xi)\, e^{-\sum \varepsilon_j \xi_j^2}\, d\xi
$$

is a continuous map from $[0,+\infty)^m$ to the tempered distributions. In addition, since the integrand in (3.6) is invariant under identical simultaneous permutations of the $\xi_j$ and the $\lambda_j$, it follows that the integral equals $m!$ times the corresponding integral over $Y_{m-k}$. Hence, by (3.3) and (3.6), we have

$$
\begin{aligned}
b_{0q}(m) &= \frac{(-i)^{m(m-1)/2}}{\pi^{2m}\prod_{j=1}^{m-1} j!} \lim_{\varepsilon'\to 0^+} \lim_{\varepsilon_1,\ldots,\varepsilon_m\to 0^+} \int_{Y_{2m-q}} d\lambda \\
&\quad \times \int_{\mathbb{R}^m} \Delta(\lambda)\,\Delta(\xi) \prod_{j=1}^{m} |\lambda_j|\; e^{i\langle\lambda,\xi\rangle}\, \mathcal{I}(\lambda,\xi)\, e^{-\sum \varepsilon_j\xi_j^2 - \varepsilon'\sum \lambda_j^2}\, d\xi\ .
\end{aligned}
$$

We obtain the desired result by letting $\varepsilon_1 \to 0, \ldots, \varepsilon_m \to 0, \varepsilon' \to 0$ sequentially.

$\square$

# Chapter 4

# Iterated Residues

In this chapter we compute the value of a fairly complicated integral over $(\mathbb{R} - i)^m$ using an iterated residues technique. The lemma we prove will be applied, after a change of variables, to the above formulas for $\beta_{2q}$ and $b_{0q}$ in subsequent chapters. The calculation of the integral below for $s = 1$ was carried out in [DSZ2] in order to apply it to the expected density formula for the $\mathbb{CP}^m$ case. Although the $s = 1$ calculation is sufficient to simplify the $b_{0q}$ formula, in order to simplify the $\beta_{2q}$ formula we need to calculate the $s = 2$ and $s = 3$ cases as well. For the sake of completeness, we will present the proof for all three values of $s$.

**Lemma 4.1.** *Let $0 \le p \le m$, $c > 0$, $s \in \{1, 2, 3\}$, and*

$$\mathcal{I}_{\lambda,s,c} = \int_{(\mathbb{R}-i)^m} \frac{\Delta(t)\, e^{i\langle \lambda, t\rangle}}{(\sum t_j + ic)^s \prod_{1\le j\le k\le m}(t_j + t_k)}\, dt\,.$$

*Then for*

$$\lambda_1 > \cdots > \lambda_p > 0 > \lambda_{p+1} > \cdots > \lambda_m\,,$$

*we have*

$$\mathcal{I}_{\lambda,s,c} = \begin{cases} i^{m^2-s}\,\dfrac{\pi^m}{c^s}\, f_s(\lambda_m)\, e^{c\lambda_m} & \textit{for}\ \ p < m \\[2ex] i^{m^2-s}\,\dfrac{\pi^m}{c^s} & \textit{for}\ \ p = m \end{cases},$$

*where*

$$f_s(\lambda_m) = \begin{cases} 1 & \textit{for} \;\; s=1 \\ \\ 1 - c\lambda_m & \textit{for} \;\; s=2 \\ \dfrac{c^2\lambda_m^2 - 2c\lambda_m + 2}{2} & \textit{for} \;\; s=3 \end{cases} .$$

*Proof.* Let $s \in \{1,2,3\}$ and

$$\mathcal{I}(\lambda, t; s, c) = \frac{\Delta(t)\, e^{i\langle \lambda, t\rangle}}{(\sum t_j + ic)^s \prod_{1\le j\le k\le m}(t_j + t_k)} \; ,$$

so that

$$\mathcal{I}_{\lambda,s,c} = \int_{(\mathbb{R}-i)^m} \mathcal{I}(\lambda, t; s, c)\, dt.$$

When $p > 0$, we start by evaluating the $t_1$ integral. We close the contour of integration in the upper half plane and pick up the poles at $t_1 = 0$, and at $t_1 = -t_j$ for $j \neq 1$. The pole at $t_1 = -ic - \sum_{j\neq 1} t_j$ is below the contour.

The $t_1 = -t_j$ poles do not contribute to the integral. To see why, we compute the residue at the pole $t_1 = -t_2$ to obtain

$$\frac{(-1)^{m-2} e^{i[(\lambda_2-\lambda_1)t_2 + \lambda_3 t_3 + \cdots \lambda_m t_m]} 2t_2(t_2+t_3)\cdots(t_2+t_m)\Delta(t_2,\ldots,t_m)}{(t_3+\cdots+t_m+ci)^s\, 2t_2(-t_2+t_3)\cdots(-t_2+t_m)\prod_{2\le j\le k\le m}(t_j+t_k)}$$
$$= \frac{e^{i(\lambda_2-\lambda_1)t_2}}{2t_2}\mathcal{I}(\lambda_3,\ldots,\lambda_m, t_3,\ldots,t_m; s, c) \; .$$

It is easy to see that the integral of the above formula is zero, since to calculate the $t_2$ integral we would need to close the contour in the lower half plane ($\lambda_2 - \lambda_1 < 0$), and then the lone pole at $t_2 = 0$ would be above the contour. By the symmetry in $\mathcal{I}(\lambda, t; s, c)$ we could have replaced $t_2$ in the above argument with any of the other $t_j$'s and obtained the same result.

This leaves only the pole at $t_1 = 0$, and the residue of $\mathcal{I}(\lambda, t; s, c)$ at this pole is

$$\frac{(-1)^{m-1}}{2}\mathcal{I}(\lambda_2,\ldots,\lambda_m, t_2,\ldots,t_m; s, c) \; . \tag{4.1}$$

If we apply (4.1) recursively, we see that the integral with $0 < p < m$ is reduced to the case with all $\lambda_j$ 's negative:

$$\mathcal{I}_{\lambda,s,c} = (-1)^{(m-1)+(m-2)+\cdots+(m-p)}(\pi i)^p \int_{(\mathbb{R}-i)^{m-p}} \mathcal{I}(\lambda_{p+1},\ldots,\lambda_m,t_{p+1},\ldots,t_m;s,c)\,dt\,. \tag{4.2}$$

When $p = m$, we compute $\mathrm{Res}\,|_{t_m=0}\,\mathcal{I}(\lambda_m,t_m;s,c)$ to obtain

$$\mathcal{I}_{\lambda,s,c} = \frac{(-1)^{m(m-1)/2}(\pi i)^m}{(ic)^s}\,.$$

To calculate the integral in (4.2), we start this time with the $t_m$ integral and close the contour in the lower half plane, picking up the pole of order $s$ at $t_m = -ic - \sum_{k<m} t_k$. These residues are

$$\mathcal{R}(\lambda_1,\ldots,\lambda_{m-1},t_1,\ldots,t_{m-1};1,c) := \\ \frac{\Delta(t_1,\ldots,t_{m-1})\prod_{k<m}(ic+\sum_{l<m}t_l+t_k)e^{c\lambda_m+i\sum_j(\lambda_j-\lambda_m)t_j}}{2(-ic-\sum_{l<m}t_l)\prod_{1\le j\le k\le m-1}(t_j+t_k)\prod_{k<m}(-ic-\sum_{l<m,l\neq k}t_l)}\,,$$

$$\mathcal{R}(\lambda_{p+1},\ldots,\lambda_m,t_{p+1},\ldots,t_{m-1};2,c) := \\ \frac{\Delta(t_{p+1},\ldots,t_{m-1})\prod_{k<m}(ic+t_k+\sum_{l<m}t_l)e^{c\lambda_m+i\sum_{j<m}(\lambda_j-\lambda_m)t_j}}{2(-ic-\sum_{l<m}t_l)\prod_{j\le k<m}(t_j+t_k)\prod_{k<m}(-ic-\sum_{l<m,\,l\neq k}t_l)} \\ \times\left(i\lambda_m - \sum_{k<m}\frac{1}{ic+t_k+\sum_{l<m}t_l} + \sum_k \frac{1}{ic+\sum_{l<m,\,l\neq k}t_l}\right),$$

and

$$\mathcal{R}(\lambda_{p+1},\ldots,\lambda_m,t_{p+1},\ldots,t_{m-1};3,c) := \\ \frac{\Delta(t_{p+1},\ldots,t_{m-1})\prod_{k<m}(ic+t_k+\sum_{l<m}t_l)e^{c\lambda_m+i\sum_{j<m}(\lambda_j-\lambda_m)t_j}}{2(-ic-\sum_{l<m}t_l)\prod_{j\le k<m}(t_j+t_k)\prod_{k<m}(-ic-\sum_{l<m,\,l\neq k}t_l)} \\ \times\left[\left(i\lambda_m - \sum_{k<m}\frac{1}{ic+t_k+\sum_{l<m}t_l} + \sum_k \frac{1}{ic+\sum_{l<m,\,l\neq k}t_l}\right)^2 \right. \\ \left. - \sum_{k<m}\frac{1}{(ic+t_k+\sum_{l<m}t_l)^2} + \sum_k\frac{1}{(ic+\sum_{l<m,\,l\neq k}t_l)^2}\right],$$

for $s = 1,\ 2$, and 3, respectively.

Next, we evaluate the $t_{p+1}$ integral. We close the contour in the upper half plane and see that all of the denominatorial factors in which the summand $ic$ appears either cancel out or give poles below the contour.

It can then be verified by a straightforward (if somewhat tedious) calculation that the poles $t_{p+1} = -t_j$ do not contribute to the value of the integral. Indeed, after computing the residue at the pole $t_{p+1} = -t_j$, consider the $t_j$ integral. The coefficient of $t_j$ in the exponential will be $\lambda_j - \lambda_1$, which is always negative, and thus the contour can be closed in the lower half plane. All of the poles will be above the contour, since all of the denominatorial factors with an $ic$ will have canceled out, and therefore the integral will be zero.

This leaves only the pole at $t_{p+1} = 0$, and we calculate that

$$\begin{aligned}\operatorname{Res}|_{t_{p+1}=0}\, &\mathcal{R}(\lambda_{p+1}, \ldots, \lambda_m, t_{p+1}, \ldots, t_{m-1}; s, c) \\ &= \frac{(-1)^{m-p-1}}{2}\mathcal{R}(\lambda_{p+2}, \ldots, \lambda_m, t_{p+2}, \ldots, t_{m-1}; s, c)\end{aligned}$$

for $s \in \{1, 2, 3\}$. We apply this argument recursively and then compute the residue of $\mathcal{R}(\lambda_{m-1}, \lambda_m, t_{m-1}; s, c)$ at $t_{m-1} = 0$ to obtain

$$\int_{(\mathbb{R}-i)^{m-p}} \mathcal{I}(\lambda_{p+1}, \ldots, \lambda_m, t_{p+1}, \ldots, t_m; 1, c)\, dt = (-1)^{m(m-1)/2}\, (\pi i)^m \left(\frac{-i}{c}\right) e^{c\lambda_m}\ ,$$

$$\begin{aligned}\int_{(\mathbb{R}-i)^{m-p}} &\mathcal{I}(\lambda_{p+1}, \ldots, \lambda_m, t_{p+1}, \ldots, t_m; 2, c)\, dt \\ &= (-1)^{(m-p)(m-p-1)/2}(\pi i)^{m-p}\left(\frac{1-c\lambda_m}{(ic)^2}\right) e^{c\lambda_m}\ ,\end{aligned}$$

and

$$\begin{aligned}\int_{(\mathbb{R}-i)^{m-p}} &\mathcal{I}(\lambda_{p+1}, \ldots, \lambda_m, t_{p+1}, \ldots, t_m; 3, c)\, dt \\ &= (-1)^{(m-p)(m-p-1)/2}(\pi i)^{m-p}\left(\frac{c^2\lambda_m^2 - 2c\lambda_m + 2}{2(ic)^3}\right) e^{c\lambda_m}\ ,\end{aligned}$$

for $s = 1,\ 2$, and 3, respectively. Substituting these formulas into (4.2) and simplifying gives the desired result.

$\square$

# Chapter 5

# Selberg Integral Formulas

In this chapter we will derive an extension of Aomoto's integral formula (which is itself an extension of Selberg's integral formula) and use it to derive an extension of the exponential Selberg integral formula. We will use this formula in the following chapters to derive exact formulas in the case of critical points of minimal Morse index.

We start with Selberg's well-known integral formula, a generalization of Euler's beta integral [Se].

**Theorem 5.1** (**Selberg's Integral Formula**). *For any positive integer $n$, let*

$$\Phi(x) := \Phi(x_1, \cdots, x_n) = |\Delta(x)|^{2\gamma} \prod_{j=1}^{n} x_j^{\alpha-1}(1-x_j)^{\beta-1}.$$

*Then*

$$\int_0^1 \cdots \int_0^1 \Phi(x)dx = \prod_{j=0}^{n-1} \frac{\Gamma(1+\gamma+j\gamma)\Gamma(\alpha+j\gamma)\Gamma(\beta+j\gamma)}{\Gamma(1+\gamma)\Gamma(\alpha+\beta+\gamma(n+j-1))}, \tag{5.1}$$

*when $\alpha, \beta, \gamma \in \mathbb{C}$ with $Re\ \alpha > 0$, $Re\ \beta > 0$, $Re\ \gamma >$ -min $\left(\frac{1}{n}, \frac{Re\alpha}{(n-1)}, \frac{Re\beta}{(n-1)}\right)$.*

The exponential Selberg integral formula is a limiting case of the above formula (see [As]).

**Corollary 5.2.** *For any positive integer $n$, let*

$$\Phi(x) := \Phi(x_1, \cdots, x_n) = |\Delta(x)|^{2\gamma} \prod_{j=1}^{n} x_j^{\alpha-1}\, e^{-x_j}.$$

*Then*

$$\int_0^\infty \cdots \int_0^\infty \Phi(x)dx = \prod_{j=0}^{n-1} \frac{\Gamma(1+\gamma+j\gamma)\Gamma(\alpha+j\gamma)}{\Gamma(1+\gamma)}, \tag{5.2}$$

*valid for complex $\alpha$, $\gamma$ with $Re\ \alpha > 0$, $Re\ \gamma >$ -min $\left(\frac{1}{n}, \frac{Re\,\alpha}{(n-1)}\right)$.*

This formula is obtained by making the change of variables $x_j \to \frac{x_j}{m}$, replacing $\beta$ with $\beta + m$, and then taking the limit as $m \to \infty$ in (5.1).

We will make use of the exponential Selberg integral formula in subsequent chapters in the proofs of two of our main results, but we will need an extension of it to prove the third. We could work directly with this formula to get a proof (see [Ba2]), however in the proof that we present below we will instead extend Aomoto's integral formula and then take a limit as described above to achieve the result. Not only does this method give an extension of Aomoto's formula as an auxiliary result, it also provides the reader with more insight into the important symmetries in these integrals which we will exploit more in later proofs. In addition, since we actually use a variant of Aomoto's argument to extend his integral formula, the reader will also see Aomoto's method of proving his extension of Selberg's integral formula.

In 1987 Aomoto found a simpler proof of a slightly more general integral which contains Selberg's integral as a subcase (see [Ao]).

**Theorem 5.3 (Aomoto's Integral Formula).** *For $1 \le \ell \le n$, let*

$$\Phi(x) := \Phi(x_1, \cdots, x_n) = |\,\Delta(x)\,|^{2\gamma} \prod_{j=1}^{n} x_j^{\alpha-1}(1-x_j)^{\beta-1}. \tag{5.3}$$

*Then*

$$\begin{aligned} I(\alpha,\beta,\gamma;\ell,n) &:= \int_0^1\cdots\int_0^1 x_1\cdots x_\ell\ \Phi(x)dx \\ &= \prod_{j=1}^{\ell}\frac{\alpha+\gamma(n-j)}{\alpha+\beta+\gamma(2n-j-1)} \\ &\quad\times\prod_{j=0}^{n-1}\frac{\Gamma(1+\gamma+j\gamma)\Gamma(\alpha+j\gamma)\Gamma(\beta+j\gamma)}{\Gamma(1+\gamma)\Gamma(\alpha+\beta+\gamma(n+j-1))}, \end{aligned} \tag{5.4}$$

*valid for complex* $\alpha$, $\beta$, $\gamma$ *with Re* $\alpha>0$, *Re* $\beta>0$, *Re* $\gamma>$ *-min* $\left(\frac{1}{n},\frac{Re\alpha}{(n-1)},\frac{Re\beta}{(n-1)}\right)$.

## 5.1 Extension of Aomoto's Integral Formula

In this section we prove an extension of Aomoto's integral by applying the same type of argument that he used in his proof.

**Lemma 5.4.** *Let*

$$\Phi(x) := \Phi(x_1,\cdots,x_n) = |\,\Delta(x)\,|^{2\gamma}\prod_{j=1}^{n}x_j^{\alpha-1}(1-x_j)^{\beta-1}. \tag{5.5}$$

*Then*

$$\begin{aligned} I(\alpha,\beta,\gamma;k,\ell,n) &:= \int_0^1\ldots\int_0^1\left(\prod_{i=1}^{k}x_i\right)\left(\prod_{i=1}^{\ell}x_i\right)\Phi(x)dx \\ &= \frac{I(\alpha,\beta,\gamma;\ell,n)}{\left(\frac{1+\alpha+\beta}{\gamma}+2n-k-1\right)_k} \\ &\quad\times\sum_{i=0}^{k}\binom{k}{i}\frac{(-n+\ell+i-1)_i\left(\frac{\alpha}{\gamma}+n-\ell-i\right)_i\left(\frac{1+\alpha}{\gamma}+2n-\ell-k\right)_{k-i}}{\left(\frac{\alpha+\beta}{\gamma}+2n-\ell-i-1\right)_i}, \end{aligned} \tag{5.6}$$

*where* $(a)_n = a(a+1)\ldots(a+n-1)$ *is the rising factorial and* $I(\alpha,\beta,\gamma;\ell,n)$ *is Aomoto's integral. This is valid for integer* $k,\ell,n$ *with* $1\le k\le\ell<n$ *and complex* $\alpha$, $\beta$, $\gamma$ *with Re* $\alpha>0$, *Re* $\beta>0$, *Re* $\gamma>$ *-min* $\left(\frac{1}{n},\frac{Re\alpha}{(n-1)},\frac{Re\beta}{(n-1)}\right)$.

*Proof.* First, we let

$$\mathcal{I}_{k,\ell} := \int_{C_n} \prod_{i=1}^{k} x_i w_\ell(x) dx = \int_{C_n} \prod_{i=1}^{k} x_i \prod_{i=1}^{\ell} x_i v(x) dx,$$

where

$$w_\ell(x) := w(x; \alpha, \beta, \gamma, \ell) = |\Delta(x)|^{2\gamma} \prod_{i=1}^{\ell} x_i \prod_{i+1}^{n} x_i^{\alpha-1}(1-x_i)^{\beta-1},$$

$$v(x) := v(x; \alpha, \beta, \gamma) = |\Delta(x)|^{2\gamma} \prod_{i=1}^{n} x_i^{\alpha-1}(1-x_i)^{\beta-1},$$

and

$$C_n = \underbrace{(0,1) \times \cdots \times (0,1)}_{n \text{ times}}.$$

Then we make the claim:

**Claim 5.5.** *With $\mathcal{I}_{k,\ell}$ and $w_\ell(x)$ defined by the above equations,*

$$\int_{C_n} \frac{\prod_{i=1}^{k} x_i \; w_\ell(x) dx}{x_1 - x_j} = \begin{cases} 0, & \text{if } 2 \le j \le k & (5.7a) \\ \frac{1}{2}\mathcal{I}_{k-1,\ell}, & \text{if } k < j \le \ell & (5.7b) \\ \mathcal{I}_{k-1,\ell}, & \text{if } \ell < j \le n & (5.7c) \end{cases}$$

*and*

$$\int_{C_n} \frac{x_1 \prod_{i=1}^{k} x_i \; w_\ell(x) dx}{x_1 - x_j} = \begin{cases} \frac{1}{2}\mathcal{I}_{k,\ell} & \text{if } 2 \le j \le k & (5.8a) \\ \mathcal{I}_{k,\ell} & \text{if } k < j \le \ell & (5.8b) \\ \mathcal{I}_{k,\ell} + \frac{1}{2}\mathcal{I}_{k-1,\ell+1} & \text{if } \ell < j \le n & (5.8c) \end{cases}$$

*valid for integer $j, k, l, n$ with $1 \le k \le l < n$.*

To prove the claim we will look at each of the six cases and consider the effect of making the transposition $x_1 \leftrightarrow x_j$ in each case. We will make use of the fact that $w_\ell(x)$ and $v(x)$ are symmetric under permutations of $\{x_1, \ldots, x_\ell\}$ and $\{x_1, \ldots, x_n\}$, respectively.

For (5.7a), we have $2 \le j \le k$, and we immediately see that the integral vanishes since we have

$$\int_{C_n} \frac{\prod_{i=1}^{k} x_i \; w_\ell(x) dx}{x_1 - x_j} = -\int_{C_n} \frac{\prod_{i=1}^{k} x_i \; w_\ell(x) dx}{x_1 - x_j}$$

after making the above transposition and utilizing the symmetry in $w_\ell(x)$. (The numerator is unchanged by the transposition due to the symmetry in $w_\ell(x)$, but the denominator changes sign.)

For (5.7b), $k < j \le \ell$, and the same transposition gives us the following

$$\int_{C_n} \frac{\prod_{i=1}^k x_i \; w_\ell(x)dx}{x_1 - x_j} = \int_{C_n} \frac{x_j \prod_{i=2}^k x_i \; w_\ell(x)dx}{x_j - x_1} = \int_{C_n} \left(1 - \frac{x_1}{x_1 - x_j}\right) \prod_{i=2}^k x_i \; w_\ell(x)dx,$$

and therefore

$$2\int_{C_n} \frac{\prod_{i=1}^k x_i \; w_\ell(x)dx}{x_1 - x_j} = \int_{C_n} \prod_{i=2}^k x_i \; w_\ell(x)dx = \int_{C_n} \prod_{i=1}^{k-1} x_i \; w_\ell(x)dx = \mathcal{I}_{k-1,\ell} \;.$$

For the second equality we merely made the transposition $x_1 \leftrightarrow x_k$.

In the case of (5.7c), we first note that

$$\int_{C_n} \left(\frac{x_1^2}{x_1 - x_j}\right) \prod_{i=2}^k x_i \prod_{i=2}^\ell x_i \; v(x)dx = \int_{C_n} \left(x_1 + \frac{x_1 x_j}{x_1 - x_j}\right) \prod_{i=2}^k x_i \prod_{i=2}^\ell x_i \; v(x)dx.$$

Then we make the transposition $x_1 \leftrightarrow x_j$, where $\ell < j \le n$, in the second term on the RHS of the above equation to obtain

$$\int_{C_n} \left(\frac{x_1 x_j}{x_1 - x_j}\right) \prod_{i=2}^k x_i \prod_{i=2}^\ell x_i \; v(x)dx = -\int_{C_n} \left(\frac{x_j x_1}{x_1 - x_j}\right) \prod_{i=2}^k x_i \prod_{i=2}^\ell x_i \; v(x)dx$$

by the symmetry in $v(x)$. Hence this integral also vanishes. Therefore

$$\int_{C_n} \frac{x_1^2 \prod_{i=2}^k x_i \prod_{i=2}^\ell x_i \; v(x)dx}{x_1 - x_j} = \int_{C_n} \prod_{i=2}^k x_i \; w_\ell(x)dx = \int_{C_n} \prod_{i=1}^{k-1} x_i \; w_\ell(x)dx = \mathcal{I}_{k-1,\ell} \;.$$

Next, for (5.8a) we have $2 \le j \le k$ again, and we see that the transposition $x_1 \leftrightarrow x_j$ leads to

$$\begin{aligned} \int_{C_n} \frac{x_1 \prod_{i=1}^k x_i \; w_\ell(x)dx}{x_1 - x_j} &= \int_{C_n} \frac{x_j \prod_{i=1}^k x_i \; w_\ell(x)dx}{x_j - x_1} \\ &= \int_{C_n} \left(1 - \frac{x_1}{x_1 - x_j}\right) \prod_{i=1}^k x_i \; w_\ell(x)dx, \end{aligned}$$

and thus

$$2\int_{C_n} \frac{x_1 \prod_{i=1}^k x_i \; w_\ell(x)dx}{x_1 - x_j} = \int_{C_n} \prod_{i=1}^k x_i \; w_\ell(x)dx = \mathcal{I}_{k,\ell} \;.$$

Then, for (5.8b) we have $k < j \leq \ell$, and we apply the same argument used for (5.7c) to obtain

$$\begin{aligned}
\int_{C_n} \frac{x_1^2 \prod_{i=2}^{k} x_i \; w_\ell(x)dx}{x_1 - x_j} &= \int_{C_n} \left( x_1 + \frac{x_1 x_j}{x_j - x_1} \right) \prod_{i=2}^{k} x_i \; w_\ell(x)dx \\
&= \int_{C_n} \prod_{i=1}^{k} x_i \; w_\ell(x)dx = \mathcal{I}_{k,\ell} \; .
\end{aligned}$$

Finally, for (5.8c) we have $\ell < j \leq n$, and we note that

$$\begin{aligned}
\int_{C_n} \left( \frac{x_1^3}{x_1 - x_j} \right) \prod_{i=2}^{k} x_i \prod_{i=2}^{\ell} x_i \; v(x)dx &= \int_{C_n} \left( x_1^2 + \frac{x_1^2 x_j}{x_1 - x_j} \right) \prod_{i=2}^{k} x_i \prod_{i=2}^{\ell} x_i \; v(x)dx \\
&= \int_{C_n} \prod_{i=1}^{k} x_i \; w_\ell(x)dx + \int_{C_n} \frac{\prod_{i=1}^{k} x_i \prod_{i=1}^{\ell+1} x_i \; v(x)dx}{x_1 - x_{\ell+1}} .
\end{aligned}$$

In the second equality we made the transposition $x_j \leftrightarrow x_{\ell+1}$, which leaves the integral unchanged due to the symmetry. We then apply (5.7b) to the second integral on the RHS of the above equation to obtain the desired result for this last case, and the claim is proved.

Returning to the proof of the lemma, we calculate that

$$\begin{aligned}
\int_{C_n} \frac{\partial}{\partial x_1} \left[ (1 - x_1) \prod_{i=1}^{k} x_i \; w_\ell(x) \right] dx \quad &= \quad (\alpha + 1) \int_{C_n} (1 - x_1) \prod_{i=2}^{k} x_i \; w_\ell(x)dx \qquad (5.9) \\
&\quad - \beta \int_{C_n} \prod_{i=1}^{k} x_i \; w_\ell(x)dx \\
&\quad + 2\gamma \sum_{i=2}^{n} \int_{C_n} (1 - x_1) \frac{x_1 \prod_{i=1}^{k} x_i \; w_\ell(x)}{x_1 - x_j} dx \, ,
\end{aligned}$$

but it is clear, by the Fundamental Theorem of Calculus, that the integral on the

LHS vanishes. Therefore, by applying the above claim to (5.9) we see that

$$\begin{aligned}
0 \;&=\; (\alpha+1)\mathcal{I}_{k-1,\ell} - (\alpha+1)\mathcal{I}_{k,\ell} \;-\; \beta\mathcal{I}_{k,\ell} \;+\; 2\gamma\left(\frac{(\ell-k)I_{k-1,\ell}}{2} + (n-\ell)\mathcal{I}_{k-1,\ell}\right)\\
&\quad - 2\gamma\left(\frac{(k-1)I_{k,\ell}}{2} + (\ell-k)\mathcal{I}_{k,\ell} + (n-\ell)(\mathcal{I}_{k,\ell} + \frac{\mathcal{I}_{k-1,\ell+1}}{2})\right)\\
&= \Big(\alpha - 1 - \beta - \gamma(k-1) - 2\gamma(\ell-k) - 2\gamma(n-\ell)\Big)\mathcal{I}_{k,\ell}\\
&\quad + \Big(\alpha + 1 + \gamma(\ell-k) + 2\gamma(n-\ell)\Big)\mathcal{I}_{k-1,\ell} - \gamma(n-\ell)\mathcal{I}_{k-1,\ell+1}\\
&= (\alpha + 1 - \gamma\ell - \gamma k + 2\gamma n)\mathcal{I}_{k-1,\ell} - (1 + \alpha + \beta - \gamma - \gamma k + 2\gamma n)\mathcal{I}_{k,\ell}\\
&\quad - \gamma(n-\ell)\mathcal{I}_{k-1,\ell+1}\\
&= \Big(1 + \alpha + \gamma(2n - \ell - k)\Big)\mathcal{I}_{k-1,\ell} - \Big(1 + \alpha + \beta - \gamma(1 + k - 2n)\Big)\mathcal{I}_{k,\ell}\\
&\quad - \gamma(n-\ell)\mathcal{I}_{k-1,\ell+1},
\end{aligned}$$

and thus

$$\mathcal{I}_{k,\ell} \;=\; \frac{\big(1 + \alpha + \gamma(2n - \ell - k)\big)\mathcal{I}_{k-1,\ell} - \gamma(n-\ell)\mathcal{I}_{k-1,\ell+1}}{1 + \alpha + \beta - \gamma(1 + k - 2n)}. \tag{5.10}$$

Our next step is to show that

$$\mathcal{I}_{k,\ell} = \frac{\mathcal{I}_{0,\ell}}{\prod_{i=1}^{k} C_i}\left(\sum_{i=0}^{k}\left(\binom{k}{i}\prod_{a=i+1}^{k} A_{a,\ell}\prod_{b=1}^{i} E_{\ell+b}\right)\right) \tag{5.11}$$

where

$$A_{k,\ell} = 1 + \alpha + \gamma(2n - \ell - k), \tag{5.12}$$

$$B_\ell = \gamma(n-\ell), \tag{5.13}$$

$$C_k = 1 + \alpha + \beta + \gamma(2n - k - 1), \tag{5.14}$$

$$D_\ell = \frac{\alpha + \gamma(n-\ell)}{\alpha + \beta + \gamma(2n - \ell - 1)}, \tag{5.15}$$

and

$$E_\ell = -B_{\ell-1}D_\ell. \tag{5.16}$$

We will prove this by induction. First, we compute that

$$\mathcal{I}_{1,\ell} = \frac{A_{1,\ell}\mathcal{I}_{0,\ell} - B_\ell\mathcal{I}_{0,\ell+1}}{C_1} \;=\; \frac{A_{1,\ell}\mathcal{I}_{0,\ell} - B_\ell D_{\ell+1}\mathcal{I}_{0,\ell}}{C_1},$$

where the first equality follows from (5.10) and the second from Theorem 5.1. Thus

$$\mathcal{I}_{1,\ell} = \frac{\mathcal{I}_{0,\ell}}{C_1} \left(A_{1,\ell} + E_{\ell+1}\right)$$

and the basis step is proved. Next we assume that

$$\mathcal{I}_{m,\ell} = \frac{\mathcal{I}_{0,\ell}}{\prod_{i-1}^{m} C_i} \left( \sum_{i-0}^{m} \left( \binom{m}{i} \prod_{a=i+1}^{m} A_{a,\ell} \prod_{b=1}^{i} E_{\ell+b} \right) \right) \tag{5.17}$$

and then use (5.10), (5.17), and Theorem 5.1 to calculate that

$$\begin{aligned}
\mathcal{I}_{m+1,\ell} &= \frac{A_{m+1,\ell}\mathcal{I}_{m,\ell} - B_\ell \mathcal{I}_{m,\ell+1}}{C_{m+1}} \\
&= \frac{A_{m+1,\ell}}{C_{m+1}} \left[ \frac{\mathcal{I}_{0,\ell}}{\prod_{i=1}^{m} C_i} \sum_{i=0}^{m} \left( \binom{m}{i} \prod_{a=i+1}^{m} A_{a,\ell} \prod_{b=1}^{i} E_{\ell+b} \right) \right] \\
&\quad - \frac{B_\ell}{C_{m+1}} \left[ \frac{\mathcal{I}_{0,\ell+1}}{\prod_{i=1}^{m} C_i} \sum_{i=0}^{m} \left( \binom{m}{i} \prod_{a=i+1}^{m} A_{a,\ell+1} \prod_{b=1}^{i} E_{\ell+b+1} \right) \right] \\
&= \frac{\mathcal{I}_{0,\ell}}{\prod_{i=1}^{m+1} C_i} \left[ A_{m+1,\ell} \sum_{i=0}^{m} \left( \binom{m}{i} \prod_{a=i+1}^{m} A_{a,\ell} \prod_{b=1}^{i} E_{\ell+b} \right) \right. \\
&\qquad \left. + E_{\ell+1} \sum_{i=0}^{m} \left( \binom{m}{i} \prod_{a=i+1}^{m} A_{a+1,\ell} \prod_{b=1}^{i} E_{\ell+b+1} \right) \right] \\
&= \frac{\mathcal{I}_{0,\ell}}{\prod_{i=1}^{m+1} C_i} \sum_{i=0}^{m} \left[ \left( A_{m+1,\ell} \binom{m}{i} \prod_{a=i+1}^{m} A_{a,\ell} \prod_{b=1}^{i} E_{\ell+b} \right) \right. \\
&\qquad \left. + \left( E_{\ell+1} \binom{m}{i} \prod_{a=i+1}^{m} A_{a+1,\ell} \prod_{b=1}^{i} E_{\ell+b+1} \right) \right] \\
&= \frac{\mathcal{I}_{0,\ell}}{\prod_{i=1}^{m+1} C_i} \sum_{i=0}^{m} \left( \binom{m}{i} \prod_{a=i+1}^{m+1} A_{a,\ell} \prod_{b=1}^{i} E_{\ell+b} + \binom{m}{i} \prod_{a=i+2}^{m+1} A_{a,\ell} \prod_{b=1}^{i+1} E_{\ell+b} \right) \\
&= \frac{\mathcal{I}_{0,\ell}}{\prod_{i=1}^{m+1} C_i} \left[ \binom{m}{0} \prod_{a=1}^{m+1} A_{a,\ell} + \left( \binom{m}{0} + \binom{m}{1} \right) \prod_{a=2}^{m+1} A_{a,\ell} \prod_{b=1}^{1} E_{\ell+b} \right. \\
&\qquad + \left( \binom{m}{1} + \binom{m}{2} \right) \prod_{a=3}^{m+1} A_{a,\ell} \prod_{b=1}^{2} E_{\ell+b} + \cdots + \\
&\qquad \left. + \left( \binom{m}{m-1} + \binom{m}{m} \right) \prod_{a=m+1}^{m+1} A_{a,\ell} \prod_{b=1}^{m} E_{\ell+b} + \binom{m}{m} \prod_{b=1}^{m+1} E_{\ell+b} \right] \\
&= \frac{\mathcal{I}_{0,\ell}}{\prod_{i=1}^{m+1} C_i} \sum_{i=0}^{m+1} \left( \binom{m+1}{i} \prod_{a=i+1}^{m+1} A_{a,\ell} \prod_{b=1}^{i} E_{\ell+b} \right).
\end{aligned}$$

This completes the induction step, thus proving equality in (5.11).

Finally, we note that $\mathcal{I}_{0,\ell}$ is Aomoto's integral, and we substitute (5.12) - (5.16) into (5.11) to obtain

$$\mathcal{I}_{k,\ell} = I(\alpha,\beta,\gamma;\ell,n)\prod_{i=1}^{k}\frac{1}{1+\alpha+\beta+\gamma(2n-i-1)}$$
$$\times\sum_{i=0}^{k}\left(\binom{k}{i}\prod_{g=i+1}^{k}(1+\alpha+\gamma(2n-\ell-g))\prod_{h=1}^{i}\frac{-\gamma(n-\ell-h+1)(\alpha+\gamma(n-\ell-h))}{\alpha+\beta+\gamma(2n-\ell-h-1)}\right),$$

which we then simplify to obtain the desired result. □

## 5.2 Extension of the Exponential Selberg Integral Formula

In this section we will use Lemma 5.4 to prove an extension of the exponential Selberg integral formula.

**Lemma 5.6.**

$$\int_0^\infty\cdots\int_0^\infty\left(\prod_{i=1}^{k}x_i\right)\left(\prod_{i=1}^{\ell}x_j\right)|\Delta(x)|^{2\gamma}\prod_{j=1}^{n}\left(x_j^{\alpha-1}e^{-x_j}\right)dx$$
$$=\gamma^{k+\ell}\left(\frac{1+\alpha}{\gamma}+2n-\ell-k\right)_k\left(\frac{\alpha}{\gamma}+n-\ell\right)_\ell\prod_{j=0}^{n-1}\frac{\Gamma(1+\gamma+j\gamma)\Gamma(\alpha+j\gamma)}{\Gamma(1+\gamma)},$$

*where $(a)_n = a(a+1)\dots(a+n-1)$ is the rising factorial. This is valid for integer $k,\ell,n$ with $0\le k\le\ell<n$ and complex $\alpha$ and $\gamma$ with Re $\alpha>0$, Re $\gamma>$ -min $\left(\frac{1}{n},\frac{Re\ \alpha}{(n-1)}\right)$.*

*Proof.* First, recall that by definition

$$I(\alpha,\beta+m,\gamma;k,\ell,n) = \int_0^1\cdots\int_0^1\left(\prod_{i=1}^{k}x_i\right)\left(\prod_{i=1}^{\ell}x_i\right)\Phi(x)dx\,,$$

where

$$\Phi(x) = |\,\Delta(x)\,|^{2\gamma}\prod_{j=1}^{n}x_j^{\alpha-1}(1-x_j)^{\beta+m-1}. \tag{5.18}$$

On the other hand, we just proved in Lemma 5.4 that

$$I(\alpha,\beta{+}m,\gamma;k,\ell,n)=\frac{I(\alpha,\beta{+}m,\gamma;\ell,n)}{\left(\frac{1+\alpha+\beta+m}{\gamma}+2n-k-1\right)_k}\times\sum_{i=0}^{k}\frac{\binom{k}{i}(-n+\ell+i-1)_i\left(\frac{\alpha}{\gamma}+n-\ell-i\right)_i\left(\frac{1+\alpha}{\gamma}+2n-\ell-k\right)_{k-i}}{\left(\frac{\alpha+\beta+m}{\gamma}+2n-\ell-i-1\right)_i},$$

which becomes

$$I(\alpha,\beta{+}m,\gamma;k,\ell,n)=\prod_{j=1}^{\ell}\frac{\alpha+\gamma(n-j)}{\alpha+\beta+m+\gamma(2n-j-1)}\times\prod_{j=0}^{n-1}\frac{\Gamma(1+\gamma+j\gamma)\Gamma(\alpha+j\gamma)\Gamma(\beta+m+j\gamma)}{\Gamma(1+\gamma)\Gamma(\alpha+\beta+m+\gamma(n+j-1))}\times\sum_{i=0}^{k}\frac{\binom{k}{i}(-n+\ell+i-1)_i\left(\frac{\alpha}{\gamma}+n-\ell-i\right)_i\left(\frac{1+\alpha}{\gamma}+2n-\ell-k\right)_{k-i}}{\left(\frac{1+\alpha+\beta+m}{\gamma}+2n-k-1\right)_k\left(\frac{\alpha+\beta+m}{\gamma}+2n-\ell-i-1\right)_i}\tag{5.19}$$

after we apply (5.4).

Next, we make the change of variables

$$x_j\to\frac{x_j}{m}$$

in the original formula for $I(\alpha,\beta{+}m,\gamma;k,\ell,n)$, and thus

$$I(\alpha,\beta{+}m,\gamma;k,\ell,n)=m^{-k-\ell-\alpha n-\gamma n(n-1)}\int_0^m\cdots\int_0^m\left(\prod_{i=1}^{k}x_i\right)\left(\prod_{i=1}^{\ell}x_j\right)\ \Phi'(x)dx,\tag{5.20}$$

where

$$\Phi'(x)=|\Delta(x)|^{2\gamma}\prod_{j=1}^{n}\left(x_j^{\alpha-1}\left(1-\frac{x_j}{m}\right)^{\beta+m-1}\right).$$

By combining (5.19) and (5.20) and then simplifying, we see that

$$\int_0^m \cdots \int_0^m \left(\prod_{i=1}^k x_i\right)\left(\prod_{i=1}^\ell x_j\right)\,\Phi'(x)dx$$
$$= \frac{1}{\left(\frac{1}{\gamma}+\frac{1+\alpha+\beta+2n-k-1}{m}\right)_k}$$
$$\times \sum_{i=0}^{k}\left(\frac{\binom{k}{i}(-n+\ell+i-1)_i\left(\frac{\alpha}{\gamma}+n-\ell-i\right)_i\left(\frac{1+\alpha}{\gamma}+2n-\ell-k\right)_{k-i}}{\left(\frac{\alpha+\beta+m}{\gamma}+2n-\ell-i-1\right)_i}\right)$$
$$\times \prod_{j=1}^{\ell}\frac{\alpha+\gamma(n-j)}{1+\frac{\alpha+\beta+\gamma(2n-j-1)}{m}}\prod_{j=0}^{n-1}\frac{\Gamma(1+\gamma+j\gamma)\Gamma(\alpha+j\gamma)\Gamma(\beta+m+j\gamma)m^{\alpha+\gamma(n-1)}}{\Gamma(1+\gamma)\Gamma(\alpha+\beta+m+\gamma(n+j-1))}.$$

Then, we take the limit as $m\to\infty$ on both sides of the equation and use the fact that $\frac{\Gamma(A+m)}{\Gamma(B+m)}m^{B-A}\to 1$ and $\left(1-\frac{x}{m}\right)^{A+m}\to e^{-x}$ as $m\to\infty$, to obtain

$$\int_0^\infty \cdots \int_0^\infty \left(\prod_{i=1}^k x_i\right)\left(\prod_{i=1}^\ell x_j\right)|\Delta(x)|^{2\gamma}\prod_{j=1}^{n}\left(x_j^{\alpha-1}e^{-x_j}\right)dx$$
$$=\gamma^{k+\ell}\left(\frac{1+\alpha}{\gamma}+2n-\ell-k\right)_k\left(\frac{\alpha}{\gamma}+n-\ell\right)_\ell\prod_{j=0}^{n-1}\frac{\Gamma(1+\gamma+j\gamma)\Gamma(\alpha+j\gamma)}{\Gamma(1+\gamma)}.$$

The limit on the LHS can be justified by using cutoff functions and applying the monotone convergence theorem.

$\square$

# Chapter 6

# Growth Rate of $\mathcal{N}^{\text{crit}}_{N,h}(\mathbb{CP}^m)$ in $m$

In this chapter we will show that the expected number $\mathcal{N}^{\text{crit}}_{N,h}(\mathbb{CP}^m)$ of critical points of random holomorphic sections of $\mathcal{O}(N) \to \mathbb{CP}^m$ grows exponentially with the dimension of the manifold. We will derive an exact formula for the expected number of critical points of minimal Morse index that holds in all dimensions and then show that the expected number $\mathcal{N}^{\text{crit}}_{N,q,h}(\mathbb{CP}^m)$ of critical points of Morse index $q$ decreases as $q$ increases. These two facts give an upper and lower bound on $\mathcal{N}^{\text{crit}}_{N,h}(\mathbb{CP}^m)$. The growth rate that we obtain was conjectured in [DSZ2, DSZ3] and was used as a basis in [DSZ3, Sec. 7.3] for the heuristic estimate of the growth rate for the expected density of vacua in string/M theory. This estimate was a means of tying the rigorous results obtained in this paper together with previous estimates of the number of vacua in the literature, such as in [BP]. This growth rate is also consistent with the analogous estimates of the growth rate of the number of metastable states of spin glasses [Fy].

## 6.1 Critical Point Formulas for $\mathcal{O}(N)\to\mathbb{CP}^m$

The setting under consideration in this chapter is the $N$th tensor power of the hyperplane section line bundle over $m$-dimensional complex projective space $\mathcal{O}(N) \to \mathbb{CP}^m$ with the Fubini-Study metric $h^N$ on the bundle. This metric and the Gaussian measure induced by it on $H^0(\mathbb{CP}^m, \mathcal{O}(N))$ are both invariant under the $\mathrm{SU}(m+1)$ action on $\mathbb{CP}^m$, and therefore the expected density is constant on $\mathbb{CP}^m$. By taking the formula for $\mathcal{K}^{crit}_{N,q,h}$ given in Theorem 3.1 and computing the derivatives of the normalized Szegö kernel of $H^0(\mathbb{CP}^m, \mathcal{O}(N))$ at the point $z = 0 \in \mathbb{C}^m \subset \mathbb{CP}^m$, the following formula for the expected density of critical points specific to this case was derived in [DSZ2].

**Lemma 6.1.** *The density of the expected distribution of critical points of Morse index $q$ for random sections $s \in H^0(\mathbb{CP}^m, \mathcal{O}(N))$ relative to $dV_h$ is given by*

$$\begin{aligned}\mathcal{K}^{\mathrm{crit}}_{N,q,h}(z) &= i^{m+1}\frac{m!\,|c_m|}{N^m}\lim_{\varepsilon'\to 0^+}\int_{Y_{2m-q}} d\lambda \left|\prod_j \lambda_j\right| \Delta(\lambda)\, e^{-\varepsilon'|\lambda|^2} \\ &\quad\times \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^m} \frac{\Delta(\xi)\, e^{i\langle\lambda,\xi\rangle} e^{-\epsilon|\xi|^2}\, d\xi}{(N^2\sum \xi_j + i)\prod_{1\le j\le k\le m}\{i - N(N-1)(\xi_j+\xi_k)\}}\,,\end{aligned}$$

*where $Y_p$ is as in Theorem 3.1 and*

$$c_m = \frac{(-i)^{m(m-1)/2}}{2^m\,\pi^{2m}\,\prod_{j=1}^m j!}\,.$$

Since the expected density is constant on $\mathbb{CP}^m$, it follows immediately from the definitions that

$$\mathcal{N}^{\mathrm{crit}}_{N,q,h}(\mathbb{CP}^m) = \frac{\pi^m}{m!}\mathcal{K}^{\mathrm{crit}}_{N,q,h}(z).$$

Therefore, by computing the $\xi$ integrals in the above formula using the iterated residue argument that was given in the proof of Lemma 4.1, the authors derived the following formula for $\mathcal{N}^{\mathrm{crit}}_{N,q,h}(\mathbb{CP}^m)$.

**Theorem 6.2.** *The expected number of critical points of Morse index $q$ for random sections $s \in H^0(\mathbb{CP}^m, \mathcal{O}(N))$ is given by*

$$\mathcal{N}^{\mathrm{crit}}_{N,q,h}(\mathbb{CP}^m) = \frac{2^{\frac{m^2+m+2}{2}}}{\prod_{j=1}^m j!}\ \frac{(N-1)^{m+1}}{(m+2)N-2}$$
$$\times \int_{Y_{2m-q}} d\lambda \left|\prod_{j=1}^m \lambda_j\right| \Delta(\lambda)\, e^{-\sum_{j=1}^m \lambda_j} \times \begin{cases} e^{(m+2-2/N)\lambda_m} & \text{for } q > m \\ 1 & \text{for } q = m \end{cases},$$

*for $N \geq 2$, where $Y_p$ is as in Theorem 3.1.*

We will be working with this formula for the remainder of the chapter. In §6.2 we will derive some intermediate lemmas, which we will then apply to the above formula in §6.3 to prove Main Theorem 1.

## 6.2 Intermediate Lemmas

In this section we will derive two lemmas which will be used in the section below and in the next chapter to prove two of our main theorems.

In our first lemma we evaluate an integral using the exponential Selberg integral formula.

**Lemma 6.3.**

$$\frac{2^{\frac{m^2+m+2}{2}}}{\prod_{j=1}^m j!} \int_{0<\lambda_m<\cdots<\lambda_1<\infty} \prod_{j=1}^m \lambda_j\ \Delta(\lambda)\, e^{-\sum_{j=1}^m \lambda_j}\ d\lambda = 2(m+1) \tag{6.1}$$

*Proof.* In order to simplify the notation we will use $P(m)$ to denote the integral on the LHS of (6.1). We first see that we can rewrite this integral as

$$P(m) = \int_{0<\lambda_m<\cdots<\lambda_1<\infty} \prod_{j=1}^m \lambda_j\ |\Delta(\lambda)|\ e^{-\sum_{j=1}^m \lambda_j} d\lambda.$$

We then note that the integrand in the above equation is symmetric under permutations of $\lambda$. Therefore,

$$P(m) = \frac{1}{m!}\int_{\mathbb{R}^m_+} \prod_{j=1}^m \lambda_j\ |\Delta(\lambda)|\ e^{-\sum_{j=1}^m \lambda_j} d\lambda.$$

Now, we can apply Corollary 5.2 to the above formula with $\alpha = 2$, $\gamma = \frac{1}{2}$, and $n = m$ to obtain

$$P(m) = \frac{1}{m!} \prod_{j=0}^{m-1} \frac{\Gamma(\frac{3}{2} + \frac{j}{2})\Gamma(2 + \frac{j}{2})}{\Gamma(\frac{3}{2})} = (m+1) \prod_{j=1}^{m} 2^{-j} j!\,,$$

where the second equality follows from an application of Gauss's multiplication formula. The desired formula is then obtained by substituting $P(m)$ back into (6.1) and simplifying.

$\square$

In our next lemma, we make a change of variables which takes $Y_{2m-q} \to \mathbb{R}_{+}^{m}$ for each $q$ and then show that the value of a slightly more general form of the integral in Theorem 6.2 decreases as $q$ increases.

**Lemma 6.4.** *For $m \geq 1$ and $0 \leq p \leq m$, let*

$$P_{c,p}(m) = \int_{Y_p} d\lambda \left| \prod_{j=1}^{m} \lambda_j \right| \Delta(\lambda)\, e^{-\sum_{j=1}^{m-1} \lambda_j} \times \begin{cases} e^{(m+1+c)\lambda_m} & \textit{for } p < m \\ e^{-\lambda_m} & \textit{for } p = m \end{cases},$$

*where $Y_p$ is as in Theorem 3.1. Then*

$$P_{0,p}(m) = P_{0,q}(m)$$

*for $0 \leq p\,, q \leq m$, and for $c > 0$,*

$$P_{c,p-1}(m) < \left(\frac{p}{p+c}\right)^2 P_{c,p}(m).$$

*Proof.* In order to simplify the discussion, we will examine the case where $p = m$ separately from the others. In this case

$$P_{c,m}(m) = \int_{0<\lambda_m<\cdots<\lambda_1<\infty} \left(\prod_{j=1}^{m} \lambda_j\right) \left(\prod_{i=1}^{m-1} \prod_{j=i+1}^{m} (\lambda_i - \lambda_j)\right) e^{-\sum_{j=1}^{m} \lambda_j} d\lambda.$$

We make the change of variables

$$\lambda_i \to \sum_{j=i}^{m} \lambda_j,$$

to obtain

$$P_{c,m}(m) = \int_{\mathbb{R}_+^m} \left(\prod_{i=1}^{m}\sum_{j=i}^{m}\lambda_j\right)\left(\prod_{i=1}^{m-1}\prod_{j=i}^{m-1}\sum_{k=i}^{j}\lambda_k\right) e^{-\sum_{j=1}^{m} j\,\lambda_j} d\lambda\,.$$

Next, we see that

$$\begin{aligned}\left(\prod_{i=1}^{m}\sum_{j=i}^{m}\lambda_j\right)\left(\prod_{i=1}^{m-1}\prod_{j=i}^{m-1}\sum_{k=i}^{j}\lambda_k\right) &= \lambda_m\left(\prod_{i=1}^{m-1}\sum_{j=i}^{m}\lambda_j\right)\left(\prod_{i=1}^{m-1}\prod_{j=i}^{m-1}\sum_{k=i}^{j}\lambda_k\right)\\ &= \lambda_m\left(\prod_{i=1}^{m-1}\prod_{j=i}^{m}\sum_{k=i}^{j}\lambda_k\right) \qquad (6.2)\\ &= \prod_{i=1}^{m}\prod_{j=i}^{m}\sum_{k=i}^{j}\lambda_k\end{aligned}$$

and so

$$P_{c,m}(m) = \int_{\mathbb{R}_+^m}\left(\prod_{i=1}^{m}\prod_{j=i}^{m}\sum_{k=i}^{j}\lambda_k\right) e^{-\sum_{j=1}^{m} j\,\lambda_j} d\lambda\,. \qquad (6.3)$$

Now, we consider $P_{c,p}(m)$ when $0 \le p < m$. For these cases

$$P_{c,p}(m) = \int_{Y_p}\left|\prod_{j=1}^{m}\lambda_j\right|\left(\prod_{i=1}^{m-1}\prod_{j=i+1}^{m}(\lambda_i-\lambda_j)\right) e^{-\sum_{j=1}^{m-1}\lambda_j+(m+c)\lambda_m} d\lambda,$$

and we make the change of variables

$$\lambda_i \to \begin{cases}\sum_{j=i}^{p}\lambda_j, & \text{for } 1\le i\le p\\ -\sum_{j=p+1}^{i}\lambda_j, & \text{for } p<i\le m\end{cases},$$

to obtain

$$\begin{aligned}P_{c,p}(m) = \int_{\mathbb{R}_+^m} &\left(\prod_{i=1}^{p}\sum_{j=i}^{p}\lambda_j\right)\left(\prod_{i=p+1}^{m}\sum_{j=p+1}^{i}\lambda_j\right)\left(\prod_{i=1}^{p-1}\prod_{j=i}^{p-1}\sum_{k=i}^{j}\lambda_k\right)\\ &\times\left(\prod_{i=1}^{p}\prod_{j=p+1}^{m}\sum_{k=i}^{j}\lambda_k\right)\left(\prod_{i=p+2}^{m}\prod_{j=i}^{m}\sum_{k=i}^{j}\lambda_k\right) e^{-\sum_{j=1}^{p} j\,\lambda_j-\sum_{j=p+1}^{m}(j+c)\,\lambda_j} d\lambda\,.\end{aligned}$$

We can combine the first quantity with the third, and the second with the fifth, as we did in (6.2), and thus

$$\begin{aligned}P_{c,p}(m) = \int_{\mathbb{R}_+^m}&\left(\prod_{i=1}^{p}\prod_{j=i}^{p}\sum_{k=i}^{j}\lambda_k\right)\left(\prod_{i=p+1}^{m}\prod_{j=i}^{m}\sum_{k=i}^{j}\lambda_k\right)\\ &\times\left(\prod_{i=1}^{p}\prod_{j=p+1}^{m}\sum_{k=i}^{j}\lambda_k\right)e^{-\sum_{j=1}^{p} j\,\lambda_j-\sum_{j=p+1}^{m}(j+c)\,\lambda_j} d\lambda\,.\end{aligned}$$

Now, it is clear that

$$\left(\prod_{i=1}^{p}\prod_{j=i}^{p}\sum_{k=i}^{j}\lambda_k\right)\left(\prod_{i=1}^{p}\prod_{j=p+1}^{m}\sum_{k=i}^{j}\lambda_k\right)=\prod_{i=1}^{p}\prod_{j=i}^{m}\sum_{k=i}^{j}\lambda_k\,,$$

and then

$$\left(\prod_{i=1}^{p}\prod_{j=i}^{m}\sum_{k=i}^{j}\lambda_k\right)\left(\prod_{i=p+1}^{m}\prod_{j=i}^{m}\sum_{k=i}^{j}\lambda_k\right)=\prod_{i=1}^{m}\prod_{j=i}^{m}\sum_{k=i}^{j}\lambda_k\,.$$

Therefore,

$$P_{c,p}(m)=\int_{\mathbb{R}_+^m}\left(\prod_{i=1}^{m}\prod_{j=i}^{m}\sum_{k=i}^{j}\lambda_k\right)e^{-\sum_{j=1}^{p}j\,\lambda_j-\sum_{j=p+1}^{m}(j+c)\,\lambda_j}d\lambda. \tag{6.4}$$

We note that in this formula the only dependence on $p$ is in the exponential, and we see from (6.3) that this formula covers the $p=m$ case as well.

When $c=0$, the formula does not depend on $p$ at all, so we see that $P_{0,r}(m)=P_{0,s}(m)$ for $0\le r,s\le m$.

Next we let $c>0$ and rewrite (6.4) as follows,

$$P_{c,p}(m)=\int_{\mathbb{R}_+^m}\mathcal{I}(\lambda_1,\dots,\lambda_m)\left(\prod_{i=1}^{m}\lambda_i\right)e^{-\sum_{j=1}^{p}j\,\lambda_j-\sum_{j=p+1}^{m}(j+c)\,\lambda_j}d\lambda\,,$$

where

$$\mathcal{I}(\lambda_1,\dots,\lambda_m)=\prod_{i=1}^{m}\prod_{j=i+1}^{m}\sum_{k=i}^{j}\lambda_k.$$

Then we make the change of variable $\lambda_p\to\frac{p}{p+c}\lambda_p$ in the formula for $P_{c,p-1}$ to obtain

$$\begin{aligned}P_{c,p-1}(m)=\left(\frac{p}{p+c}\right)^2&\int_{\mathbb{R}_+^m}\mathcal{I}(\lambda_1,\dots,\tfrac{p}{p+c}\lambda_p,\dots,\lambda_m)\\&\qquad\times\left(\prod_{i=1}^{m}\lambda_i\right)e^{-\sum_{i=1}^{p}i\,\lambda_i-\sum_{j=p+1}^{m}(j+c)\,\lambda_j}d\lambda\\<\left(\frac{p}{p+c}\right)^2&P_{c,p}(m).\end{aligned}$$

□

## 6.3 Proof of Main Theorem 1

In this section we will use the two lemmas that were derived in the previous section to prove Main Theorem 1. In the proof, we will first work out the exact formula for the minimal Morse index case and then proceed to show that $\mathcal{N}^{\mathrm{crit}}_{N,q,h}(\mathbb{CP}^m) > \mathcal{N}^{\mathrm{crit}}_{N,q+1,h}(\mathbb{CP}^m)$ for $m \leq q \leq 2m$.

**Main Theorem 1.** *Let $\mathcal{N}^{\mathrm{crit}}_{N,q,h}(\mathbb{CP}^m)$ denote the expected number of critical points of Morse index $q$ for random sections $s \in H^0(\mathbb{CP}^m, \mathcal{O}(N))$ so that $\mathcal{N}^{\mathrm{crit}}_{N,h}(\mathbb{CP}^m) = \sum_{q=m}^{2m} \mathcal{N}^{\mathrm{crit}}_{N,q,h}(\mathbb{CP}^m)$, then*

$$\mathcal{N}^{\mathrm{crit}}_{N,m,h}(\mathbb{CP}^m) = \frac{2(m+1)(N-1)^{m+1}}{(m+2)N-2},$$

*and when $N > 2$*

$$\mathcal{N}^{\mathrm{crit}}_{N,q+1,h}(\mathbb{CP}^m) < \mathcal{N}^{\mathrm{crit}}_{N,q,h}(\mathbb{CP}^m).$$

*Therefore,*

$$\frac{2(m+1)(N-1)^{m+1}}{(m+2)N-2} < \mathcal{N}^{\mathrm{crit}}_{N,h}(\mathbb{CP}^m) < \frac{2(m+1)^2(N-1)^{m+1}}{(m+2)N-2}.$$

*Proof.* First, when $q = m$, we see from Theorem 6.2 that

$$\mathcal{N}^{\mathrm{crit}}_{N,m,h}(\mathbb{CP}^m) = \frac{2^{\frac{m^2+m+2}{2}}}{\prod_{j=1}^m j!} \, \frac{(N-1)^{m+1}}{(m+2)N-2} \int_{Y_m} \left| \prod_{j=1}^m \lambda_j \right| \Delta(\lambda) \, e^{-\sum_{j=1}^m \lambda_j} d\lambda.$$

We then apply Lemma 6.3 to obtain

$$\mathcal{N}^{\mathrm{crit}}_{N,m,h}(\mathbb{CP}^m) = \frac{2(m+1)(N-1)^{m+1}}{(m+2)N-2}\,. \tag{6.5}$$

For the general case, we recall, once again from Theorem 6.2, that

$$\mathcal{N}^{\mathrm{crit}}_{N,q,h}(\mathbb{CP}^m) = \frac{2^{\frac{m^2+m+2}{2}}}{\prod_{j=1}^m j!} \, \frac{(N-1)^{m+1}}{(m+2)N-2} \int_{Y_{2m-q}} d\lambda \left| \prod_{j=1}^m \lambda_j \right| \Delta(\lambda) \, e^{-\sum_{j=1}^m \lambda_j} \times \begin{cases} e^{(m+2-2/N)\lambda_m} & \text{for } q > m \\ 1 & \text{for } q = m \end{cases}. \tag{6.6}$$

First, we see that for $N=2$ we can apply Lemma 6.4 with $p=2m-q$ and $c=0$ to the integral in (6.6). Thus, $\mathcal{N}^{\rm crit}_{2,r,h}(\mathbb{CP}^m)=\mathcal{N}^{\rm crit}_{2,s,h}(\mathbb{CP}^m)$ for $m\le r,s\le 2m$. From (6.5) we calculate that $\mathcal{N}^{\rm crit}_{2,m,h}(\mathbb{CP}^m)=1$, and therefore, for $m\ge 1$,

$$\mathcal{N}^{\rm crit}_{2,h}(\mathbb{CP}^m):=\sum_{q=m}^{2m}\mathcal{N}^{\rm crit}_{2,q,h}(\mathbb{CP}^m)=m+1.$$

Then, when $N>2$, we apply Lemma 6.4 with $p=2m-q$ and $c=1-\frac{2}{N}$ to (6.6) and see that

$$\mathcal{N}^{\rm crit}_{N,q+1,h}(\mathbb{CP}^m)<\left(\frac{2m-q}{2m-q+1-\frac{2}{N}}\right)^2\mathcal{N}^{\rm crit}_{N,q,h}(\mathbb{CP}^m).$$

Therefore,

$$\frac{2(m+1)(N-1)^{m+1}}{(m+2)N-2}<\mathcal{N}^{\rm crit}_{N,h}(\mathbb{CP}^m)<\frac{2(m+1)^2(N-1)^{m+1}}{(m+2)N-2}.$$

□

**Remark.** The modulus of the spectral determinant shows up in the various integral formulas for the expected number of critical points ([AD], [BM], [DSZ2], [Fy]). As the modulus presents a serious technical challenge in evaluating the integral, it is often dropped from the calculation (see [AD] and [BM]), which results in counting the critical points with signs. In string theory this is known as computing the "supergravity index", while in spin glass theory there is some debate over the validity and implications of the calculation (see [ABM] and references therein). In our case, Morse theory tells us that the number of critical points of each $s\in H^0(\mathbb{CP}^m,\mathcal{O}(N))$ counted with signs is a topological invariant and is given by

$$\sum_{z:\nabla s(z)=0}(-1)^q=c_m(T^{*1,0}_{\mathbb{CP}^m}\otimes\mathcal{O}(N))$$
$$=\sum_{j=0}^{m}(-1)^j\binom{m+1}{j}N^{m-j}=\frac{(N-1)^{m+1}+(-1)^m}{N},$$

where $q$ is the Morse index of $z$. We see that this "index counting" provides a fairly good estimate of the total expected number of critical points.

# Chapter 7

# Asymptotic Bounds of $\mathcal{N}^{crit}_{N,h}$ as $N \to \infty$

In this chapter we provide the proof of Main Theorem 2. We will use Lemmas 6.3 and 6.4 to derive an exact formula for the leading coefficient of the asymptotic expansion in $N$ of $\mathcal{N}^{crit}_{N,q,h}$ when $q = m$ and show that the coefficients gets smaller as $q$ gets larger. We then use these facts to derive upper and lower asymptotic bounds on the leading coefficient of $\mathcal{N}^{crit}_{N,h}$.

**Main Theorem 2.** *Let $n_q(m)$ denote the universal constant in the leading order term of the asymptotic expansion of $\mathcal{N}^{\mathrm{crit}}_{N,q,h}$, and let $n(m) = \sum_{q=m}^{2m} n_q(m)$, so that*

$$\mathcal{N}^{\mathrm{crit}}_{N,q,h} \sim n_q(m)\, c_1(L)^m\, N^m \qquad \textit{and} \qquad \mathcal{N}^{\mathrm{crit}}_{N,h} \sim n(m)\, c_1(L)^m\, N^m.$$

*Then*

$$n_m(m) = 2\,\frac{m+1}{m+2} \qquad \textit{and} \qquad 0 < n_{q+1}(m) < \left(\frac{2m-q}{2m-q+1}\right)^2 n_q(m)\ ,$$

*and thus*

$$2\,\frac{m+1}{m+2} < n(m) < \frac{2m+3}{3}\ .$$

*Proof.* We start with the formula for $b_{0q}$ given in Lemma 3.3 and re-write this as

$$b_{0q}(m) = \frac{(-i)^{m(m-1)/2}}{\pi^{2m}\prod_{j=1}^{m-1} j!}\int_{Y_{2m-q}} \prod_{j=1}^{m} |\lambda_j|\, \Delta(\lambda)\, \mathcal{I}_\lambda d\lambda\ , \tag{7.1}$$

where

$$\mathcal{I}_\lambda = \int_{\mathbb{R}^m} \frac{\Delta(\xi)\, e^{i\langle\lambda,\xi\rangle}\, d\xi}{\left(1-\frac{i}{2}\sum\xi_j\right)\prod_{j\le k}\left[1+\frac{i}{2}(\xi_j+\xi_k)\right]}.$$

In order to simplify the formula, we make the change of variables $\xi_j \to t_j + i$ to obtain

$$\mathcal{I}_\lambda = -(-2i)^{\frac{m^2+m+2}{2}}\, e^{-\sum\lambda_j}\, \mathcal{I}_{\lambda,1,m+2}\ ,$$

where

$$\mathcal{I}_{\lambda,s,c} = \int_{(\mathbb{R}-i)^m} \frac{\Delta(t)\, e^{i\langle\lambda,t\rangle}}{\left(\sum t_j + ic\right)^s \prod_{1\le j\le k\le m}(t_j+t_k)}\, dt\ .$$

Putting this together we have

$$b_{0q}(m) = \frac{(-i)^{m^2-1} 2^{\frac{m^2+m+2}{2}}}{\pi^{2m}\prod_{j=1}^{m-1} j!}\int_{Y_{2m-q}} d\lambda \prod_{j=1}^{m}|\lambda_j|\, \Delta(\lambda)\, e^{-\sum\lambda_j}\, \mathcal{I}_{\lambda,1,m+2}\ . \tag{7.2}$$

We apply Lemma 4.1 with $s=1$ and $c=m+2$ to obtain

$$b_{0q}(m) = \frac{2^{\frac{m^2+m+2}{2}}}{\pi^m (m+2)\prod_{j=1}^{m-1} j!} \times \int_{Y_{2m-q}} d\lambda \left|\prod_{j=1}^m \lambda_j\right| \Delta(\lambda)\, e^{-\sum_{j=1}^m\lambda_j} \times \begin{cases} e^{(m+2)\lambda_m} & \text{for } q>m \\ 1 & \text{for } q=m \end{cases}.$$

Then, from Theorem 2.5 we see that

$$n_q := \frac{\pi^m b_{0q}}{m!},$$

and thus

$$n_q(m) = \frac{2^{\frac{m^2+m+2}{2}}}{(m+2)\prod_{j=1}^{m} j!} \times \int_{Y_{2m-q}} d\lambda \left|\prod_{j=1}^m \lambda_j\right| \Delta(\lambda)\, e^{-\sum_{j=1}^m\lambda_j} \times \begin{cases} e^{(m+2)\lambda_m} & \text{for } q>m \\ 1 & \text{for } q=m \end{cases}. \tag{7.3}$$

When $q = m$, we can apply Lemma 6.3 directly to the above integral and simplify to obtain

$$n_m(m) = 2\,\frac{m+1}{m+2}. \tag{7.4}$$

Then, for $m \leq q \leq 2m$, we apply Lemma 6.4 with $p = 2m - q$ and $c = 1$ to the integral in (7.3) to obtain the relation

$$n_{q+1}(m) < \left(\frac{2m-q}{2m-q+1}\right)^2 n_q(m). \tag{7.5}$$

By definition $n(m) = \sum_{q=m}^{2m} n_q(m)$, and thus it follows from (7.4) and (7.5) that

$$\begin{aligned}
n(m) <\; & 2\frac{m+1}{m+2} + 2\frac{m+1}{m+2}\sum_{i=m}^{2m-1}\prod_{j=m}^{i}\left(\frac{2m-j}{2m-j+1}\right)^2 \\
&= 2\frac{m+1}{m+2}\left(1 + \sum_{i=m}^{2m-1}\left(\frac{2m-i}{m+1}\right)^2\right) \\
&= 2\frac{m+1}{m+2}\left(1 + \frac{m(2m+1)}{6(m+1)}\right) \\
&= \frac{2m+3}{3}\,.
\end{aligned}$$

□

# Chapter 8

# Metric Dependence of $\mathcal{N}_{N,h}^{crit}$

In this chapter we will prove Main Theorem 3, showing that the asymptotic expansion of $\mathcal{N}_{N,h}^{crit}$ has a metric dependence in the term of order $m-2$ in all dimensions and that $\mathcal{N}_{N,h}^{crit}$ is asymptotically minimized by Calabi extremal metrics, whenever they exist.

While it is clear that the number of critical points of a given holomorphic section will vary with the metric, it is not clear whether the same is true for the expected number of critical points of a random section. Therefore, we consider the asymptotic expansion of $\mathcal{N}_{N,h}^{\mathrm{crit}}$ in order to determine the metric dependence of this statistic.

We have the following obvious corollary of Theorem 2.5:

**Corollary 8.1.** *Under the same conditions as Theorem 2.5, the expected number of critical points of random sections in $H^0(M, L^N)$ has the asymptotic expansion*

$$\begin{aligned}\mathcal{N}_{N,h}^{\mathrm{crit}} \sim & \left[\frac{\pi^m b_0}{m!}\, c_1(L)^m\right] N^m + \left[\frac{\pi^m \beta_1}{(m-1)!}\, c_1(M)\cdot c_1(L)^{m-1}\right] N^{m-1} \\ & + \left[\beta_2 \int_M \rho_h^2 d\mathrm{Vol}_h + \beta_2'\, c_1(M)^2 \cdot c_1(L)^{m-2} + \beta_2''\, c_2(M)\cdot c_1(L)^{m-2}\right] N^{m-2} + \cdots ,\end{aligned}$$

*where $b_0, \beta_1, \beta_2, \beta_2', \beta_2''$ denote the sum over $q$ of $b_{0q}, \beta_{1q}, \beta_{2q}, \beta_{2q}', \beta_{2q}''$, respectively.*

It is easy to see from this corollary that the asymptotic expansion of $\mathcal{N}_{N,h}^{\mathrm{crit}}$ is topologically invariant to two orders in $N$, but that the third term in the expansion has a non-topological summand, the universal constant $\beta_2(m)$ times the Calabi functional $\int_M \rho_h^2 d\mathrm{Vol}_h$. Here, $\rho_h$ is the scalar curvature of the Kähler form $\omega_h := \frac{i}{2}\Theta_h$. Therefore, as long as $\beta_2(m)$ does not vanish, the expansion is non-topological; it will have a metric dependence due to the presence of the Calabi functional in the third term. In this chapter we will prove that $\beta_2(m) > 0$ for all $m$ and show that this implies that Calabi extremal metrics asymptotically minimize $\mathcal{N}^{\mathrm{crit}}_{N,h}$, whenever they exist.

As mentioned before, the analysis is simplified by treating the contributions from critical points of different Morse indices separately, and so we will start by working with the integral formula for $\beta_{2q}$ from Lemma 3.2.

Our first step is to use Lemma 4.1 to further simplify this formula.

**Lemma 8.2.**

$$\beta_{2q}(m) = \frac{2^{\frac{m^2+m}{2}}}{4\,\pi^m \prod_{j=1}^{m-1} j!} \int_{Y_{2m-q}} \mathcal{I}_q(\lambda)\,\Delta(\lambda) \prod_{j=1}^{m} |\lambda_j|\, e^{-\sum_{j=1}^m \lambda_j}\, d\lambda\,,$$

*where*

$$\mathcal{I}_m(\lambda) = \frac{2F(D(\lambda))}{m+2} + \frac{16(m+2)\left(\sum_{j=1}^m \lambda_j\right) + 16m}{m(m+1)(m+2)^3(m+3)} - \frac{8}{(m+1)(m+2)^2}$$

*and, for $m < q \leq 2m$,*

$$\mathcal{I}_q(\lambda) = \left(\frac{2F(D(\lambda))}{m+2} + \frac{8\left((m+2)^2\lambda_m^2 - 2(m+2)\lambda_m + 2\right)}{(m+1)(m+2)^3(m+3)} \right. \\ \left. - \frac{4((m+2)\lambda_m - 1)}{(m+2)^2}\left(\frac{4\sum_{j=1}^m \lambda_j}{m(m+1)(m+3)} - \frac{2}{m+1}\right)\right) e^{(m+2)\lambda_m}. \tag{8.1}$$

*Proof.* We rewrite $\beta_{2q}(m)$ as

$$\beta_{2q}(m) = \frac{(-i)^{m(m-1)/2}}{4\,\pi^{2m} \prod_{j=1}^{m-1} j!} \int_{Y_{2m-q}} d\lambda \prod_{j=1}^{m} |\lambda_j|\,\Delta(\lambda) \\ \times \left(F(D(\lambda))\mathcal{I}_{\lambda,1} + \left[\frac{4\sum_{j=1}^m \lambda_j}{m(m+1)(m+3)} - \frac{2}{m+1}\right]\mathcal{I}_{\lambda,2} + \frac{2\mathcal{I}_{\lambda,3}}{(m+1)(m+3)}\right)\,,$$

where

$$\mathcal{I}_{\lambda,s} = \int_{\mathbb{R}^m} \frac{\Delta(\xi)\, e^{i\langle\lambda,\xi\rangle}\, d\xi}{\left(1 - \frac{i}{2}\sum \xi_j\right)^s \prod_{j\le k}\left[1 + \frac{i}{2}(\xi_j + \xi_k)\right]}.$$

Then we make the change of variables $\xi_j \to t_j + i$ to obtain

$$\mathcal{I}_{\lambda,s} = (-1)^{\frac{m^2+m}{2}} (2i)^{\frac{m^2+m+2s}{2}}\, e^{-\sum \lambda_j}\, \mathcal{I}_{\lambda,s,m+2}\,,$$

where

$$\mathcal{I}_{\lambda,s,c} = \int_{(\mathbb{R}-i)^m} \frac{\Delta(t)\, e^{i\langle\lambda,t\rangle}}{\left(\sum t_j + ic\right)^s \prod_{1\le j\le k\le m}(t_j + t_k)}\, dt\,.$$

Putting this together we have

$$\begin{aligned}
\beta_{2q}(m) = \frac{2^{\frac{m^2+m}{2}}}{4\,\pi^{2m}\prod_{j=1}^{m-1} j!} \int_{Y_{2m-q}} & d\lambda \prod_{j=1}^{m} |\lambda_j|\, \Delta(\lambda)\, e^{-\sum_{j=1}^m \lambda_j} \\
& \times \left( F(D(\lambda)) \frac{2\mathcal{I}_{\lambda,1,m+2}}{i^{m^2-1}} + \frac{2^4 \mathcal{I}_{\lambda,3,m+2}}{i^{m^2-3}(m+1)(m+3)} \right. \qquad (8.2) \\
& \quad \left. + \left( \frac{4\sum_{j=1}^m \lambda_j}{m(m+1)(m+3)} - \frac{2}{m+1} \right) \frac{2^2 \mathcal{I}_{\lambda,2,m+2}}{i^{m^2-2}} \right).
\end{aligned}$$

To complete the proof we apply Lemma 4.1 with $p = 2m - q$ and $c = m + 2$ to (8.2) and simplify to obtain the desired formula.

□

In the next section we will derive an exact formula for the case $q = m$ by applying our extension of the exponential Selberg integral formula to the formula for $\beta_{2m}$ given in the lemma above. Then in §8.2 we will consider the other Morse indices and derive a formula for the sum over $q \neq m$, which we will then show is positive in all dimensions. In §8.3 we will complete the proof of Main Theorem 3.

## 8.1 Exact Formula when $q = m$

**Lemma 8.3.** *For $m \geq 1$ and $q = m$,*

$$\beta_{2m}(m) = \frac{4\,m!}{\pi^m (m+2)^3 (m+3)}.$$

*Proof.* We see from Lemma 8.2 that

$$\beta_{2m}(m) = \frac{2^{\frac{m^2+m-4}{2}}}{\pi^m \prod_{j=1}^{m-1} j!} \int_{Y_m} d\lambda\, \Delta(\lambda) \prod_{j=1}^{m} |\lambda_j|\, e^{-\sum_{j=1}^m \lambda_j}$$
$$\times \left( \frac{2F(D(\lambda))}{m+2} + \frac{16(m+2)\sum_{j=1}^m \lambda_j + 16m}{m(m+1)(m+2)^3(m+3)} - \frac{8}{(m+1)(m+2)^2} \right).$$

Using (3.1), we can rewrite this as

$$\beta_{2m}(m) = c \int_{Y_m} d\lambda\, \Delta(\lambda) \prod_{j=1}^{m} |\lambda_j|\, e^{-\sum_{j=1}^m \lambda_j} \Bigg( m(m+1)(m(m+3)(m+4)-4)$$
$$+ 4(m+2)\left( -(m+1)(m+4)\sum_{j=1}^m \lambda_j + \left(\sum_{j=1}^m \lambda_j\right)^2 + 2\sum_{j=1}^m \lambda_j^2 \right)\Bigg),$$

where

$$c = \frac{2^{\frac{m^2+m-2}{2}}}{m(m+1)(m+2)^3(m+3)\pi^m \prod_{j=1}^{m-1} j!}.$$

We then see that making the change $\Delta(\lambda)\prod_{j=1}^m |\lambda_j| \to |\Delta(\lambda)| \prod_{j=1}^m \lambda_j$ in the integrand above does not change its value on the region over which we are integrating. After doing this, we notice that the integrand is now symmetric under permutations of $\lambda$, allowing us to take the integral over $\mathbb{R}_+^m$ and replace each of the sums with a multiple of one of the summands. Thus,

$$\beta_{2m}(m) = \frac{c}{m!} \int_{\mathbb{R}_+^m} d\lambda\, |\Delta(\lambda)| \prod_{j=1}^{m} \lambda_j\, e^{-\sum_{j=1}^m \lambda_j} \Bigg( m(m+1)(m(m+3)(m+4)-4)$$
$$+ 4(m+2)\left( -m(m+1)(m+4)\lambda_1 + m(m-1)\lambda_1\lambda_2 + 3m\lambda_1^2 \right)\Bigg). \quad (8.3)$$

Next, we apply Lemma 5.6 to (8.3), with $n = m$, $\alpha = 2$, and $\gamma = \frac{1}{2}$, for each of the four cases: $(k,l) = (0,0)$, $(k,l) = (0,1)$, $(k,l) = (0,2)$, and $(k,l) = (1,1)$, to obtain

$$\beta_{2m}(m) = \frac{c}{m!} \prod_{j=0}^{m-1} \frac{\Gamma(\frac{3}{2}+\frac{j}{2})\Gamma(2+\frac{j}{2})}{\Gamma(\frac{3}{2})} \Bigg( m(m+1)(m(m+3)(m+4)-4)$$
$$+ 4m(m+2)(m+3)\left( -\frac{(m+1)(m+4)}{2} + \frac{(m-1)(m+2)}{4} + \frac{3(2m+4)}{4} \right)\Bigg).$$

After applying Gauss's multiplication formula and simplifying, we have

$$\beta_{2m}(m) = \frac{m(m+1)\prod_{j=1}^{m} j!}{2^{\frac{m^2+m-6}{2}}}\, c\,.$$

Substituting in for $c$ gives the desired result.

□

## 8.2 Positivity of $\beta_2'(m)$

Unfortunately, due to the presence of the additional exponential term in $\mathcal{I}_q(\lambda)$ when $m < q \le 2m$, direct application of a variant of the Selberg integral formula is not possible for the other cases. Although it would be possible to calculate the exact formulas for one or two more cases, these would be extremely complicated and we would be no closer to our goal of showing that $\beta_2 > 0$ for all $m$. Instead, we will consider the sum over the other indices and will show that this quantity is positive.

**Lemma 8.4.** *Let $\beta_2'(m) := \sum_{q=m+1}^{2m} \beta_{2q}(m)$, then $\beta_2'(m) > 0$ for $m \ge 1$.*

*Proof.* From Lemma 8.2 we immediately see that

$$\beta_2'(m) = \frac{2^{\frac{m^2+m}{2}}}{4\,\pi^m \prod_{j=1}^{m-1} j!} \left( \sum_{q=m+1}^{2m} \int_{Y_{2m-q}} \mathcal{I}_q(\lambda)\, \Delta(\lambda) \prod_{j=1}^{m} |\lambda_j|\, e^{-\sum_{j=1}^{m} \lambda_j}\, d\lambda \right),$$

where $\mathcal{I}_q(\lambda)$ is given by (8.1).

Now we need the following lemma, which we will prove in §8.2.1.

**Lemma 8.5.**

$$\beta_2'(1) = 32c \int_{\mathbb{R}_+} \lambda_1\, e^{-2\lambda_1} d\lambda\,, \tag{8.4}$$

$$\beta_2'(2) = 48c \int_{\mathbb{R}_+^2} \left(\lambda_1^2 - 6\lambda_1 + 7\right) \lambda_1\lambda_2 |(\lambda_1 - \lambda_2)|\, e^{-\lambda_1 - 2\lambda_2} d\lambda\,, \tag{8.5}$$

*and for $m \geq 3$,*

$$\beta_2'(m) = c \int_{\mathbb{R}_+^m} d\lambda \, |\Delta(\lambda)| \prod_{j=1}^{m} \lambda_j \, e^{-\sum_{j=1}^m \lambda_j} \, e^{-\lambda_m} \Bigg( m(m+1)(m(m+3)(m+4)-4)$$

$$+ 4(m-1)(m+2) \left(-(m+1)(m+4)\lambda_1 + (m-2)\lambda_1\lambda_2 + 3\lambda_1^2\right) \Bigg) , \quad (8.6)$$

*where*

$$c = \frac{2^{\frac{m^2+m-2}{2}}}{\pi^m (m+1)(m+2)^3(m+3) \prod_{j=1}^m j!} .$$

When $m = 1$, there is nothing to prove since the integrand in (8.4) is clearly positive. For the other cases we need another lemma, which we will prove in §8.2.2.

**Lemma 8.6.** *Let $\mathcal{I}(\lambda) = |\Delta(\lambda)| \prod_{j=1}^m \left(\lambda_j \, e^{\lambda_j}\right)$, then we have the following identities:*

$$\int_{\mathbb{R}_+^m} \lambda_1 \, e^{-\lambda_m} \mathcal{I}(\lambda) d\lambda = \int_{\mathbb{R}_+^m} \left( \frac{m+2}{2} + \frac{\lambda_1}{\lambda_1 - \lambda_m} \right) e^{-\lambda_m} \mathcal{I}(\lambda) d\lambda \, , \quad (8.7)$$

$$\int_{\mathbb{R}_+^m} \lambda_1^2 \, e^{-\lambda_m} \mathcal{I}(\lambda) d\lambda$$

$$= \int_{\mathbb{R}_+^m} \left( \frac{(m+1)(m+2)}{2} + (m+1) \frac{\lambda_1}{\lambda_1 - \lambda_m} + \frac{\lambda_1^2}{\lambda_1 - \lambda_m} \right) e^{-\lambda_m} \mathcal{I}(\lambda) d\lambda \, , \quad (8.8)$$

$$\int_{\mathbb{R}_+^m} \frac{\lambda_1^2}{\lambda_1 - \lambda_m} \, e^{-\lambda_m} \mathcal{I}(\lambda) d\lambda$$

$$= \int_{\mathbb{R}_+^m} \left( \frac{3\lambda_1}{\lambda_1 - \lambda_m} + \frac{(m-2)\lambda_1^2}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_m)} + \frac{2\lambda_1^2 \, \delta(\lambda_1 - \lambda_m)}{|\lambda_1 - \lambda_m|} \right) e^{-\lambda_m} \mathcal{I}(\lambda) d\lambda \, , \quad (8.9)$$

*where $\delta(x)$ is the Dirac delta function,*

$$\int_{\mathbb{R}_+^m} \lambda_1 \lambda_2 \, e^{-\lambda_m} \mathcal{I}(\lambda) d\lambda$$

$$= \int_{\mathbb{R}_+^m} \left( \frac{(m+1)(m+2)}{4} + \left( \frac{m+1}{2} \right) \frac{\lambda_1}{\lambda_1 - \lambda_m} + \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_m} \right) e^{-\lambda_m} \mathcal{I}(\lambda) d\lambda \, , \quad (8.10)$$

$$\int_{\mathbb{R}_+^m} \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_m} \, e^{-\lambda_m} \mathcal{I}(\lambda) d\lambda$$

$$= \int_{\mathbb{R}_+^m} \left( \left( \frac{m+1}{2} \right) \frac{\lambda_1}{\lambda_1 - \lambda_m} + \frac{\lambda_1 \lambda_2}{(\lambda_1 - \lambda_m)(\lambda_2 - \lambda_m)} - \frac{\lambda_1 \lambda_2}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_m)} \right) e^{-\lambda_m} \mathcal{I}(\lambda) d\lambda \, ,$$

(8.11)

$$\int_{\mathbb{R}_+^m} \frac{\lambda_1\lambda_2^2 + \lambda_1^2\lambda_2 - 2\lambda_1\lambda_2\lambda_m}{(\lambda_1-\lambda_2)\,(\lambda_1-\lambda_m)\,(\lambda_2-\lambda_m)}\, e^{-\lambda_m}\mathcal{I}(\lambda)d\lambda = 0\,. \tag{8.12}$$

*The first three identities hold for $m \geq 2$ and the last three for $m \geq 3$.*

When $m = 2$, we can apply (8.7) and (8.8) to (8.5) to obtain

$$\beta_2'(2) = 48c\int_{\mathbb{R}_+^2}\left(\frac{\lambda_1^2}{\lambda_1-\lambda_2} - 3\frac{\lambda_1}{\lambda_1-\lambda_2} + 1\right)\lambda_1\lambda_2|(\lambda_1-\lambda_2)|\,e^{-\lambda_1-2\lambda_2}d\lambda\,.$$

Applying (8.9) to this gives

$$\beta_2'(2) = 48c\int_{\mathbb{R}_+^2}\big(2\lambda_1^2\,\delta(\lambda_1-\lambda_2) + 1\big)\,\lambda_1\lambda_2\,e^{-\lambda_1-2\lambda_2}d\lambda\,. \tag{8.13}$$

When $m \geq 3$, we apply (8.7), (8.8), and (8.10) to (8.6) to obtain

$$\beta_2'(m) = c\int_{\mathbb{R}_+^m} d\lambda\,|\Delta(\lambda)|\prod_{j=1}^m\left(\lambda_j\,e^{\lambda_j}\right)\,e^{-\lambda_m}\Bigg(16(m+1) + 2(m-1)(m+2)\left(\frac{-(m+1)(m+4)\lambda_1 + 2(m-2)\lambda_1\lambda_2 + 6\lambda_1^2}{\lambda_1-\lambda_m}\right)\Bigg).$$

We then apply (8.9) and (8.11) to this, and we have

$$\beta_2'(m) = c\int_{\mathbb{R}_+^m} d\lambda\,|\Delta(\lambda)|\prod_{j=1}^m\left(\lambda_j\,e^{\lambda_j}\right)\,e^{-\lambda_m}\Bigg(16(m+1) + 4(m-1)(m+2)\left(\frac{(m-2)\,(\lambda_1\lambda_2^2 + \lambda_1^2\lambda_2 - 2\lambda_1\lambda_2\lambda_m)}{(\lambda_1-\lambda_2)\,(\lambda_1-\lambda_m)\,(\lambda_2-\lambda_m)} + \frac{6\lambda_1^2\,\delta(\lambda_1-\lambda_m)}{|\lambda_1-\lambda_m|}\right)\Bigg).$$

By (8.12), the middle term vanishes and therefore

$$\beta_2'(m) = c\int_{\mathbb{R}_+^m} d\lambda\,|\Delta(\lambda)|\prod_{j=1}^m\left(\lambda_j\,e^{\lambda_j}\right)\,e^{-\lambda_m}\left(16(m+1) + 24(m-1)(m+2)\frac{\lambda_1^2\,\delta(\lambda_1-\lambda_m)}{|\lambda_1-\lambda_m|}\right). \tag{8.14}$$

It is now clear that $\beta_2'(m)$ is positive. Indeed, by computing the $\lambda_1$ integral we see that

$$\int_{\mathbb{R}_+^m}\frac{\lambda_1^2\,\delta(\lambda_1-\lambda_m)}{|\lambda_1-\lambda_m|}|\Delta(\lambda)|\prod_{j=1}^m\left(\lambda_j\,e^{\lambda_j}\right)\,e^{-\lambda_m}d\lambda$$
$$= \int_{\mathbb{R}_+^{m-1}}\lambda_m^3\,|\Delta(\lambda_2,\ldots,\lambda_{m-1})|\prod_{j=2}^m\lambda_j\prod_{j=2}^{m-1}(\lambda_m-\lambda_j)^2\,e^{-\sum_{j=2}^m\lambda_j}\,e^{-2\lambda_m}d\lambda_2\ldots d\lambda_m.$$

□

### 8.2.1 Proof of Lemma 8.5

We recall that from Lemma 8.2 we have

$$\beta_2'(m) = \frac{2^{\frac{m^2+m}{2}}}{4\,\pi^m \prod_{j=1}^{m-1} j!} \left( \sum_{q=m+1}^{2m} \int_{Y_{2m-q}} \mathcal{I}_q(\lambda)\, \Delta(\lambda) \prod_{j=1}^{m} |\lambda_j|\, e^{-\sum_{j=1}^m \lambda_j}\, d\lambda \right) ,$$

where

$$\mathcal{I}_q(\lambda) = \Bigg( \frac{1}{m+2} + \frac{4\left(\sum_{j=1}^m \lambda_j\right)^2 + 8\sum_{j=1}^m \lambda_j^2}{m(m+1)(m+2)^2(m+3)} + \frac{8\left((m+2)^2\lambda_m^2 - 2(m+2)\lambda_m + 2\right)}{(m+1)(m+2)^3(m+3)}$$
$$- \frac{4\sum_{j=1}^m \lambda_j}{m(m+1)(m+2)} - \frac{4((m+2)\lambda_m - 1)}{(m+2)^2}\left(\frac{4\sum_{j=1}^m \lambda_j}{m(m+1)(m+3)} - \frac{2}{m+1}\right)\Bigg) e^{(m+2)\lambda_m}.$$

For each $q$ we make the change of variables

$$\lambda_i \to \begin{cases} \lambda_i - \lambda_{2m-q+1}, & \text{for } \ 1 \le i \le 2m-q \\ \lambda_{i+1} - \lambda_{2m-q+1}, & \text{for } \ 2m-q < i < m \\ -\lambda_{2m-q+1}, & \text{for } \ i = m \end{cases}$$

in the integral over $Y_{2m-q}$ above. This change of variables is a composition of the following two changes of variables:

$$\lambda_i \to \begin{cases} \sum_{j=i}^{2m-q} \lambda_j, & \text{for } \ 1 \le i \le 2m-q \\ -\sum_{j=2m-q+1}^{i} \lambda_j, & \text{for } \ 2m-q < i \le m \end{cases}$$

and

$$\lambda_i \to \lambda_i - \lambda_{i+1}.$$

These changes take $Y_{2m-q} \to \mathbb{R}_+^m$ and $\mathbb{R}_+^m \to Y_m$, respectively, and thus all of the integrals will now be over a common region of integration, $Y_m$.

Under this change of variables we see that $\Delta(\lambda) \prod_{j=1}^m |\lambda_j|$ is unchanged and that

$$\sum_{j=1}^m \lambda_j \ \to \ \sum_{j=1}^m \lambda_j - (m+1)\lambda_p\,,$$

$$\left(\sum_{j=1}^m \lambda_j\right)^2 \;\rightarrow\; \left(\sum_{j=1}^m \lambda_j\right)^2 - 2(m+1)\lambda_p\left(\sum_{j=1}^m \lambda_j\right) + (m+1)^2\lambda_p^2\,,$$

and

$$\sum_{j=1}^m \lambda_j^2 \;\rightarrow\; \sum_{j=1}^m \lambda_j^2 - 2\lambda_p\left(\sum_{j=1}^m \lambda_j\right) + (m+1)\lambda_p^2\,.$$

Here we have let $p = 2m-q+1$ to simplify the notation. Therefore, since the absolute value of the Jacobian is 1, we have

$$\beta_2'(m) = c\int_{Y_m} \Delta(\lambda)\prod_{j=1}^m |\lambda_j|\, e^{-\sum_{j=1}^m \lambda_j}\,\sum_{p=1}^m P_p(\lambda)\,d\lambda\,, \tag{8.15}$$

where

$$c = \frac{2^{\frac{m^2+m-2}{2}}}{\pi^m m(m+1)(m+2)^3(m+3)\prod_{j=1}^{m-1} j!}$$

and

$$\begin{aligned}
P_p(\lambda) = \Bigg(&-4(m{+}2)\lambda_p^2 + 4(m{+}2)\left((m{+}1)(m{+}4) - 2\sum_{i=1}^m \lambda_i\right)\lambda_p \\
&+ m(m{+}1)(m(m{+}3)(m{+}4)-4) - 4(m{+}1)(m{+}2)(m{+}4)\sum_{i=1}^m \lambda_i \\
&\qquad + 4(m{+}2)\left(\sum_{i=1}^m \lambda_i\right)^2 + 8(m{+}2)\sum_{i=1}^m \lambda_i^2\Bigg)e^{-\lambda_p} \\
= \Bigg(&m(m{+}1)(m(m{+}3)(m{+}4)-4) \\
&-4(m{+}1)(m{+}2)(m{+}4)\sum_{\substack{i=1\\ i\neq p}}^m \lambda_i + 4(m{+}2)\left(\sum_{\substack{i=1\\ i\neq p}}^m \lambda_i\right)^2 + 8(m{+}2)\sum_{\substack{i=1\\ i\neq p}}^m \lambda_i^2\Bigg)e^{-\lambda_p}\,.
\end{aligned}$$

Next we see that making the change $\Delta(\lambda)\prod_{j=1}^m |\lambda_j| \rightarrow |\Delta(\lambda)|\prod_{j=1}^m \lambda_j$ in (8.15) does not change the value of the integrand on the region over which we are integrating. We make this change and now the integrand is symmetric under permutations of $\lambda$, so we can take the integral over $\mathbb{R}_+^m$. Thus,

$$\beta_2'(m) = \frac{c}{m!}\int_{\mathbb{R}_+^m} |\Delta(\lambda)|\prod_{j=1}^m \left(\lambda_j\, e^{\lambda_j}\right)\,\sum_{p=1}^m P_p(\lambda)\,d\lambda\,.$$

The symmetry also allows us to replace any symmetric sum with a multiple of one of its summands. Therefore,

$$\beta_2'(m) = \frac{c}{(m-1)!}\int_{\mathbb{R}_+^m} |\Delta(\lambda)| \prod_{j=1}^{m} \left(\lambda_j\, e^{\lambda_j}\right)\; P_m(\lambda)\, d\lambda\,.$$

To obtain the desired formulas, we see that

$$P_m(\lambda) = (m(m+1)(m(m+3)(m+4)-4))\, e^{-\lambda_1} = 32 e^{-\lambda_1}$$

for $m = 1$,

$$\begin{aligned} P_m(\lambda) &= \left(m(m{+}1)\left(m(m{+}3)(m{+}4){-}4\right){+}4(m{+}2)\left(-(m{+}1)(m{+}4)\lambda_1{+}3\lambda_1^2\right)\right) e^{-\lambda_2} \\ &= 48\left(\lambda_1^2 - 6\lambda_1 + 7\right) e^{-\lambda_2} \end{aligned}$$

for $m = 2$, and finally for $m \geq 3$, we compute that

$$\begin{aligned} \beta_2'(m) = \frac{c}{(m{-}1)!}&\int_{\mathbb{R}_+^m} d\lambda\, |\Delta(\lambda)| \prod_{j=1}^{m}\left(\lambda_j\, e^{\lambda_j}\right)\; e^{-\lambda_m} \Bigg( m(m{+}1)(m(m{+}3)(m{+}4){-}4) \\ &{-}4(m{+}1)(m{+}2)(m{+}4)\sum_{i=1}^{m-1}\lambda_i {+} 4(m{+}2)\left(\sum_{i=1}^{m-1}\lambda_i\right)^2 {+} 8(m{+}2)\sum_{i=1}^{m-1}\lambda_i^2\Bigg) \\ = \frac{c}{(m{-}1)!}&\int_{\mathbb{R}_+^m} d\lambda\, |\Delta(\lambda)| \prod_{j=1}^{m}\left(\lambda_j\, e^{\lambda_j}\right)\; e^{-\lambda_m} \Bigg( m(m{+}1)(m(m{+}3)(m{+}4){-}4) \\ &{+}4(m{-}1)(m{+}2)\left(-(m{+}1)(m{+}4)\lambda_1 {+} (m{-}2)\lambda_1\lambda_2 {+} 3\lambda_1^2\right)\Bigg) \end{aligned}$$

where once again we have replaced symmetric sums with multiples of one of their summands.

### 8.2.2 Proof of Lemma 8.6

For each of the identities, the proof will consist of taking a partial derivative with respect to some $\lambda_i$ inside of the integral on the LHS of the equation. By the Fundamental Theorem of Calculus this will integrate to zero. We compute the derivative and manipulate the result, utilizing the symmetry in $\mathcal{I}(\lambda)$, to achieve the desired formula. We will also make use of the fact that

$$\frac{\partial}{\partial x}|x-y| = \frac{|x-y|}{x-y} \qquad \text{and} \qquad \frac{\partial^2}{\partial x^2}|x-y| = 2\,\delta\,(x-y).$$

First, for (8.7) we have

$$\begin{aligned} 0 &= \int_{\mathbb{R}_+^m} \frac{\partial}{\partial \lambda_1}\left(\lambda_1\, e^{-\lambda_m}\mathcal{I}(\lambda)\right) d\lambda \\ &= \int_{\mathbb{R}_+^m} \left(2-\lambda_1+\sum_{i=2}^{m}\frac{\lambda_1}{\lambda_1-\lambda_i}\right) e^{-\lambda_m}\mathcal{I}(\lambda)d\lambda\,. \end{aligned} \tag{8.16}$$

Since $\mathcal{I}(\lambda)$ is symmetric under permutations of $\lambda$, for $1<i<m$, we make the transposition $\lambda_1 \leftrightarrow \lambda_i$ and see that

$$\int_{\mathbb{R}_+^m} \frac{\lambda_1\, e^{-\lambda_m}\mathcal{I}(\lambda)d\lambda}{\lambda_1-\lambda_i} = \int_{\mathbb{R}_+^m} \frac{\lambda_i\, e^{-\lambda_m}\mathcal{I}(\lambda)d\lambda}{\lambda_i-\lambda_1} = \int_{\mathbb{R}_+^m}\left(1-\frac{\lambda_1}{\lambda_1-\lambda_i}\right) e^{-\lambda_m}\mathcal{I}(\lambda)d\lambda,$$

and therefore

$$\int_{\mathbb{R}_+^m} \frac{\lambda_1\, e^{-\lambda_m}\mathcal{I}(\lambda)d\lambda}{\lambda_1-\lambda_i} = \frac{1}{2}\int_{\mathbb{R}_+^m} e^{-\lambda_m}\mathcal{I}(\lambda)d\lambda. \tag{8.17}$$

Combining (8.16) and (8.17) gives the desired result.

Next we have

$$\begin{aligned} 0 &= \int_{\mathbb{R}_+^m} \frac{\partial}{\partial \lambda_1}\left(\lambda_1^2\, e^{-\lambda_m}\mathcal{I}(\lambda)\right) d\lambda \\ &= \int_{\mathbb{R}_+^m} \left(3\lambda_1-\lambda_1^2+\sum_{i=2}^{m}\frac{\lambda_1^2}{\lambda_1-\lambda_i}\right) e^{-\lambda_m}\mathcal{I}(\lambda)d\lambda \\ &= \int_{\mathbb{R}_+^m} \left((m+1)\lambda_1-\lambda_1^2+\frac{\lambda_1^2}{\lambda_1-\lambda_m}\right) e^{-\lambda_m}\mathcal{I}(\lambda)d\lambda\,. \end{aligned} \tag{8.18}$$

In the last equality we used the fact that, for $1<i<m$,

$$\int_{\mathbb{R}_+^m}\left(\frac{\lambda_1^2}{\lambda_1-\lambda_i}\right) e^{-\lambda_m}\mathcal{I}(\lambda)d\lambda = \int_{\mathbb{R}_+^m}\left(\lambda_1+\frac{\lambda_1\lambda_i}{\lambda_1-\lambda_i}\right) e^{-\lambda_m}\mathcal{I}(\lambda)d\lambda = \int_{\mathbb{R}_+^m}\lambda_1\, e^{-\lambda_m}\mathcal{I}(\lambda)d\lambda\,.$$

Here the second term in the second integral vanishes since the transposition $\lambda_1 \leftrightarrow \lambda_i$ just changes the sign of the integrand. We then apply (8.7) to (8.18) to obtain (8.8).

To obtain (8.9), we see that

$$
\begin{aligned}
0 &= \int_{\mathbb{R}_+^m} \frac{\partial}{\partial \lambda_1} \left( \frac{\lambda_1^2}{\lambda_1 - \lambda_m}\, e^{-\lambda_m} \mathcal{I}(\lambda) \right) d\lambda \\
&= \int_{\mathbb{R}_+^m} \left( \frac{3\lambda_1}{\lambda_1 - \lambda_m} - \frac{\lambda_1^2}{\lambda_1 - \lambda_m} + \sum_{i=2}^{m-1} \frac{\lambda_1^2}{(\lambda_1 - \lambda_i)(\lambda_1 - \lambda_m)} + \frac{2\lambda_1^2\, \delta(\lambda_1 - \lambda_m)}{|\lambda_1 - \lambda_m|} \right) e^{-\lambda_m} \mathcal{I}(\lambda) d\lambda \\
&= \int_{\mathbb{R}_+^m} \left( \frac{3\lambda_1}{\lambda_1 - \lambda_m} - \frac{\lambda_1^2}{\lambda_1 - \lambda_m} + \frac{(m-2)\lambda_1^2}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_m)} + \frac{2\lambda_1^2\, \delta(\lambda_1 - \lambda_m)}{|\lambda_1 - \lambda_m|} \right) e^{-\lambda_m} \mathcal{I}(\lambda) d\lambda\,.
\end{aligned}
$$

Here we used the fact that for $2 < i < m$, the symmetry in $\mathcal{I}(\lambda)$ implies that

$$
\int_{\mathbb{R}_+^m} \frac{\lambda_1^2}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_m)}\, e^{-\lambda_m} \mathcal{I}(\lambda) d\lambda = \int_{\mathbb{R}_+^m} \frac{\lambda_1^2}{(\lambda_1 - \lambda_i)(\lambda_1 - \lambda_m)}\, e^{-\lambda_m} \mathcal{I}(\lambda) d\lambda\,.
$$

For the fourth identity we have

$$
\begin{aligned}
0 &= \int_{\mathbb{R}_+^m} \frac{\partial}{\partial \lambda_1} \left( \lambda_1 \lambda_2\, e^{-\lambda_m} \mathcal{I}(\lambda) \right) d\lambda \\
&= \int_{\mathbb{R}_+^m} \left( 2\lambda_2 - \lambda_1\lambda_2 + \sum_{i=2}^{m} \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_i} \right) e^{-\lambda_m} \mathcal{I}(\lambda) d\lambda \\
&= \int_{\mathbb{R}_+^m} \left( \frac{m+1}{2} \lambda_1 - \lambda_1\lambda_2 + \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_m} \right) e^{-\lambda_m} \mathcal{I}(\lambda) d\lambda\,. \qquad (8.19)
\end{aligned}
$$

This time we applied (8.17) and used the following facts which follow from the symmetry in $\mathcal{I}(\lambda)$ as was demonstrated above:

$$
\int_{\mathbb{R}_+^m} \lambda_2\, e^{-\lambda_m} \mathcal{I}(\lambda) d\lambda = \int_{\mathbb{R}_+^m} \lambda_1\, e^{-\lambda_m} \mathcal{I}(\lambda) d\lambda
$$

and

$$
\int_{\mathbb{R}_+^m} \frac{\lambda_1 \lambda_2\, e^{-\lambda_m} \mathcal{I}(\lambda) d\lambda}{\lambda_1 - \lambda_2} = 0\,.
$$

Once again we apply (8.7) to (8.19) to obtain (8.10).

To prove (8.11) we see that

$$\begin{aligned}
0 &= \int_{\mathbb{R}_+^m} \frac{\partial}{\partial\lambda_2}\left(\frac{\lambda_1\lambda_2}{\lambda_1-\lambda_m}\, e^{-\lambda_m}\mathcal{I}(\lambda)\right) d\lambda \\
&= \int_{\mathbb{R}_+^m} d\lambda\, \mathcal{I}(\lambda)\, e^{-\lambda_m} \\
&\qquad \times \left(\frac{2\lambda_1}{\lambda_1-\lambda_m} - \frac{\lambda_1\lambda_2}{\lambda_1-\lambda_m} - \frac{\lambda_1\lambda_2}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_m)} + \sum_{i=3}^{m} \frac{\lambda_1\lambda_2}{(\lambda_1-\lambda_m)(\lambda_2-\lambda_i)}\right) \\
&= \int_{\mathbb{R}_+^m} d\lambda\, \mathcal{I}(\lambda)\, e^{-\lambda_m} \\
&\qquad \times \left(\frac{(m+1)\lambda_1}{2(\lambda_1-\lambda_m)} - \frac{\lambda_1\lambda_2}{\lambda_1-\lambda_m} - \frac{\lambda_1\lambda_2}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_m)} + \frac{\lambda_1\lambda_2}{(\lambda_1-\lambda_m)(\lambda_2-\lambda_m)}\right),
\end{aligned}$$

where we applied (8.17) to obtain the last equality.

Finally, for (8.12), we first see that

$$\int_{\mathbb{R}_+^m} \frac{\lambda_1\lambda_2\lambda_m}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_m)(\lambda_2-\lambda_m)}\, e^{-\lambda_m}\mathcal{I}(\lambda)d\lambda = 0\,,$$

since the transposition $\lambda_1 \leftrightarrow \lambda_2$ just changes the sign of the integrand. Then we use the same transposition in the second term to obtain

$$\int_{\mathbb{R}_+^m} \frac{(\lambda_1\lambda_2^2+\lambda_1^2\lambda_2)\, e^{-\lambda_m}\mathcal{I}(\lambda)d\lambda}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_m)(\lambda_2-\lambda_m)} = \int_{\mathbb{R}_+^m} \frac{(\lambda_1\lambda_2^2-\lambda_2^2\lambda_1)\, e^{-\lambda_m}\mathcal{I}(\lambda)d\lambda}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_m)(\lambda_2-\lambda_m)} = 0\,.$$

## 8.3 Proof of Main Theorem 3

The results of the previous two sections show that $\beta_2$ is strictly positive in all dimensions, so now all that is needed to complete the proof of Main Theorem 3 is to define the notion of asymptotic minimization and cite the relevant results from the literature to show that the remainder of the theorem follows from the positivity of $\beta_2$.

We let $P(M,L)$ denote the class of positively curved metrics, i.e. metrics for which $\frac{i}{2}\Theta_h$ is a positive (1,1)-form. As was noted in [DSZ2], we would not expect $\mathcal{N}_{N,h}^{\mathrm{crit}}$ to have an upper bound as $h$ varies over $P(M,L)$, however it is bounded from below by $|c_m(L \otimes T^{*1,0})|$. It is therefore of interest to determine when a metric which minimizes $\mathcal{N}^{\text{crit}}_{N,h}$, at least in an asymptotic sense, exists. To this end we make the definition:

**Definition 8.7.** *Let $L \to M$ be an ample holomorphic line bundle over a compact Kähler manifold. For $h \in P(M,L)$, we say that $\mathcal{N}^{\text{crit}}_{N,h}$ is asymptotically minimal if for all $h_1 \neq h$ in $P(M,L)$, there exists $N_0 = N_0(h_1)$ such that*

$$\mathcal{N}^{\text{crit}}_{N,h} \leq \mathcal{N}^{\text{crit}}_{N,h_1} \qquad \textit{for} \;\; N \geq N_0 \, . \tag{8.20}$$

As we noted before, the first non-topological term in the asymptotic expansion of $\mathcal{N}^{\text{crit}}_{N,h}$ is $\beta_2$ multiplied by the Calabi functional, $\int_M \rho_h^2 d\text{Vol}_h$. Since $\beta_2$ is strictly positive, in order to find minimizers of $\mathcal{N}^{\text{crit}}_{N,h}$, we need to find minimizers of the Calabi functional.

We note that by results in [Ca1, Ca2, Hw], all critical points of the Calabi functional are local minima. Those that obtain the absolute minimum in a fixed Kähler class are called Calabi extremal metrics. Therefore, the following theorem is an immediate consequence of Lemma 8.2 and Lemma 8.4.

**Main Theorem 3.** *The universal constant $\beta_2(m)$ is strictly positive in all dimensions. Therefore, $\mathcal{N}^{\text{crit}}_{N,h}$ is non-topological, having a metric dependence in the term of order $m-2$ in its asymptotic expansion, in all dimensions. In addition, Calabi extremal metrics asymptotically minimize $\mathcal{N}^{\text{crit}}_{N,h}$, whenever they exist.*

We would like to point out that the positivity of $\beta_{2q}$, for each $q$, was also conjectured in [DSZ2]. Although it seems almost certain that this is true, this remains an open problem.

By a result of Calabi in [Ca2], metrics with constant scalar curvature are extremal, and by a result of Donaldson in [Don], there is at most one Kähler metric of constant scalar curvature in the cohomology class $2\pi\, c_1(L)$. Therefore, we have the following corollary:

**Corollary 8.8.** *Suppose that $L$ possesses a metric $h$ for which the scalar curvature of $\omega_h = \frac{i}{2}\Theta_h$ is constant, then $h$ is the unique metric on $L$ such that $\mathcal{N}^{\mathrm{crit}}_{N,h}$ is asymptotically minimal.*

Thus, for example, the Fubini-Study metric on the hyperplane section bundle $\mathcal{O}(1) \to \mathbb{CP}^m$ is the unique asymptotic minimizer of $\mathcal{N}^{\mathrm{crit}}_{N,h}(\mathbb{CP}^m)$.

## 8.4 Baugher's Conjecture

In this section we will briefly discuss the conjecture that was referred to as Baugher's conjecture in [DSZ2] and provide a proof for the case $q = m$.

The following conjecture was formulated after studying patterns in the computer assisted calculations of the integrals of the individual terms in the formula for $\beta_{2q}$ given in Lemma 3.2, and it was hoped that it would be useful in proving the positivity of $\beta_2$.

**Conjecture 8.9.**

$$\beta_{2q}(m) = \frac{(-i)^{m(m-1)/2}}{4\,\pi^{2m}\prod_{j=1}^{m-1} j!}\int_{Y_{2m-q}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}} \Delta(\lambda)\,\Delta(\xi)\prod_{j=1}^{m}|\lambda_j|\; e^{i\langle\lambda,\xi\rangle}\,\mathcal{J}(\xi)\; d\xi_1\cdots d\xi_m\, d\lambda\;,$$

*where $Y_{2m-q}$ is as in Theorem 3.1 and*

$$\mathcal{J}(\xi) = \frac{4}{(m+1)(m+2)(m+3)\left(1-\frac{i}{2}\sum_j \xi_j\right)^2 \prod_{j\le k}\left[1+\frac{i}{2}(\xi_j+\xi_k)\right]}\;.$$

*Equivalently,*

$$\int_{\mathbf{S}_{m,q-m}}\left(1-2\,|H_{11}|^2+\frac{1}{2}\,|H_{11}|^4\right)\,\mathcal{G}(H,x)dH\,dx = \frac{4}{\prod_{i=1}^{3}(m+i)}\int_{\mathbf{S}_{m,q-m}}|x|^2\,\mathcal{G}(H,x)dH\,dx\;, \quad (8.21)$$

*where* $\mathbf{S}_{m,q-m}$ *is as in Theorem 2.4 and*

$$\mathcal{G}(H,x) = \frac{1}{4\,\pi^{\binom{m+2}{2}}} \left|\det(2HH^* - |x|^2 I)\right| \, e^{-\langle (H,x),(H,x)\rangle} \,. \tag{8.22}$$

The equivalence of the two formulations of the conjecture follows from Lemma 3.2 and the lemma given below. The proof of this lemma is almost identical to that of Lemma 3.3, and we will not repeat the entire argument here.

**Lemma 8.10.**

$$\int_{\mathbf{S}_{m,q-m}} |x|^2 \, \mathcal{G}(H,x) dH \, dx \; = \int_{Y_{2m-q}} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \mathcal{H}(\lambda,\xi) d\xi_1 \cdots d\xi_m \, d\lambda \,,$$

*where* $\mathcal{G}(H,x)$ *is given by* (8.22) *and*

$$\mathcal{H}(\lambda,\xi) = \frac{(-i)^{m(m-1)/2} \Delta(\lambda)\, \Delta(\xi) \, \prod_{j=1}^m |\lambda_j| \; e^{i\langle \lambda,\xi\rangle}}{4\,\pi^{2m} \prod_{j=1}^{m-1} j! \left(1 - \frac{i}{2}\sum_j \xi_j\right)^2 \prod_{j\le k} \left[1 + \frac{i}{2}(\xi_j + \xi_k)\right]}.$$

*Proof.* Follow the proof of Lemma 3.3 all the way through to (3.6), inserting the factor $|x|^2$ into the integrand at each step. Then, for this lemma we have

$$\begin{aligned}
\mathcal{I}(\lambda,\xi) \;&=\; \frac{1}{\pi^{d_m}} \int_{\mathbb{C}} \int_{\mathrm{Sym}(m,\mathbb{C})} |x|^2 \, e^{\Phi(H,x;\xi)} \, dH \, dx \\
&=\; \frac{1}{\prod_{j\le k}\left(1 + \frac{i}{2}(\xi_j+\xi_k)\right)} \int_{\mathbb{C}} |x|^2 \, e^{-\left(1 - \frac{i}{2}\sum_j \xi_j\right)|x|^2} \, dx \\
&=\; \frac{\pi}{\left(1 - \frac{i}{2}\sum_j \xi_j\right)^2 \prod_{j\le k}\left(1 + \frac{i}{2}(\xi_j+\xi_k)\right)}.
\end{aligned}$$

Observing, as in Chapter 3, that the map

$$(\varepsilon_1, \ldots, \varepsilon_m) \mapsto \int_{\mathbb{R}^m} \Delta(\xi) \;\, e^{i\langle\lambda,\xi\rangle} \, \mathcal{I}(\lambda,\xi) \, e^{-\sum \varepsilon_j \xi_j^2} \, d\xi$$

is a continuous map from $[0,+\infty)^m$ to the tempered distributions, we have

$$\begin{aligned}
\int_{\mathbf{S}_{m,q-m}} |x|^2 \, \mathcal{G}(H,x) dH \, dx = \frac{(-i)^{m(m-1)/2}}{4\,\pi^{2m} \prod_{j=1}^m j!} \; \lim_{\varepsilon' \to 0^+} \; \lim_{\varepsilon_1,\ldots,\varepsilon_m \to 0^+} m! \int_{Y_{2m-q}} d\lambda \\
\times \int_{\mathbb{R}^m} \Delta(\lambda)\Delta(\xi) \prod_{j=1}^m |\lambda_j| \; e^{i\langle\lambda,\xi\rangle} \, \mathcal{I}(\lambda,\xi) \, e^{-\sum \varepsilon_j \xi_j^2 - \varepsilon' \sum \lambda_j^2} \, d\xi \;,
\end{aligned}$$

and then we obtain the desired result by letting $\varepsilon_1 \to 0, \ldots, \varepsilon_m \to 0, \varepsilon' \to 0$ sequentially.

$\square$

It is easy to see from (8.21) that Conjecture 8.9 implies the positivity of $\beta_{2q}$ for each $q$. Unfortunately, we were unable to prove the conjecture for all $q$, and instead used the method detailed in previous chapters to prove the positivity of $\beta_2$, which was the most important thing to show. Although our intial interest in this conjecture was as a means of proving our positivity result, it seems to us that the matrix integral identity may be of interest independent of this result. Of course the conjecture may still prove useful in establishing the positivity of $\beta_{2q}$ for $q > m$.

We will now utilize our work in the above sections to prove the $q = m$ case of the conjecture.

**Theorem 8.11.**

$$\beta_{2m}(m) = \frac{(-i)^{m(m-1)/2}}{4\,\pi^{2m}\prod_{j=1}^{m-1} j!} \int_{Y_m}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}} \Delta(\lambda)\,\Delta(\xi)\prod_{j=1}^{m}|\lambda_j|\; e^{i\langle\lambda,\xi\rangle}\,\mathcal{J}(\xi)\; d\xi_1\cdots d\xi_m\, d\lambda\,, \tag{8.23}$$

*where*

$$\mathcal{J}(\xi) = \frac{4}{(m+1)(m+2)(m+3)\left(1-\frac{i}{2}\sum_j \xi_j\right)^2 \prod_{j\le k}\left[1+\frac{i}{2}(\xi_j+\xi_k)\right]}\,.$$

*Equivalently,*

$$\int_{\mathbf{S}_{m,0}} \left(1 - 2\,|H_{11}|^2 + \frac{1}{2}\,|H_{11}|^4\right)\,\mathcal{G}(H,x)dH\,dx$$
$$= \frac{4}{\prod_{i=1}^{3}(m+i)}\int_{\mathbf{S}_{m,0}} |x|^2\,\mathcal{G}(H,x)dH\,dx\,,$$

*where*

$$\mathcal{G}(H,x) = \frac{1}{4\,\pi^{\binom{m+2}{2}}}\left|\det(2HH^* - |x|^2 I)\right|\, e^{-\langle (H,x),(H,x)\rangle}\,.$$

*Proof.* Let $\Gamma_{2m}$ denote the integral on the RHS of (8.23). We first apply the change of variable argument as in the proof of Lemma 8.3, followed by Lemma 4.1 with $s = 1$ and $c = m + 2$ to obtain

$$\Gamma_{2m}(m) = \frac{2^{\frac{m^2+m}{2}}}{4\,\pi^m \prod_{j=1}^{m-1} j!} \int_{Y_m} d\lambda \prod_{j=1}^{m} |\lambda_j|\, \Delta(\lambda)\, e^{-\sum \lambda_j} \left( \frac{16}{(m+1)(m+2)^3(m+3)} \right).$$

Then, we apply Lemma 6.3 to the above equation and simplify to obtain

$$\Gamma_{2m}(m) = \frac{4m!}{\pi^m (m+2)^3 (m+3)},$$

which agrees with the formula for $\beta_{2m}$ given in Lemma 8.2.

□

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# Vita

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