SAMPLE CONTENT

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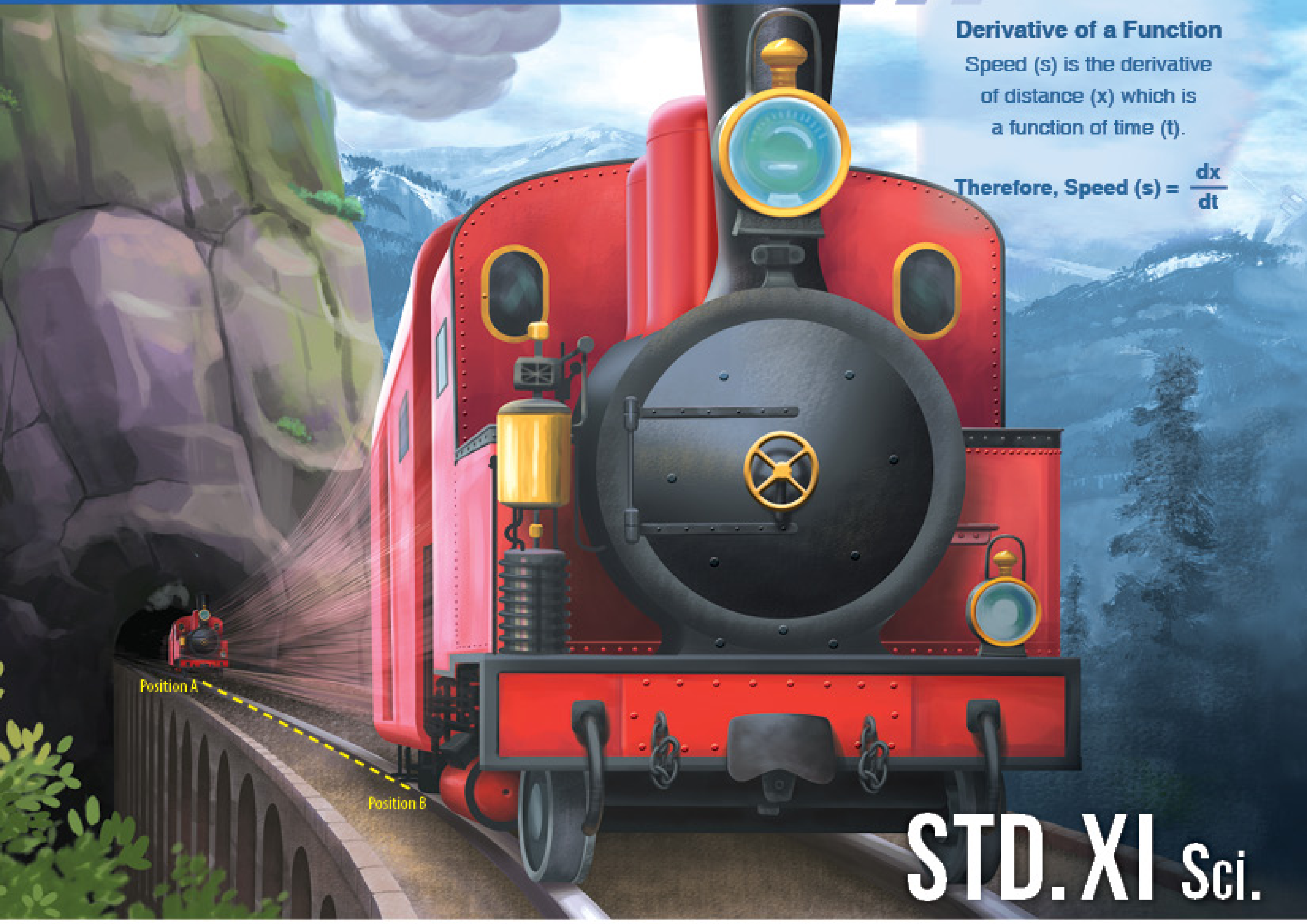
MATHEMATICS - II



STD. XI Sci.

Target Publications® Pvt. Ltd.

Written as per the latest textbook prescribed by the Maharashtra State Bureau of Textbook Production and Curriculum Research, Pune.

PERFECT

MATHEMATICS - II

Std. XI Sci. & Arts

Salient Features

- Written as per the new textbook
- Exhaustive coverage of entire syllabus
- Topic-wise distribution of textual questions and practice problems at the start of every chapter.
- Precise theory for every topic
- Covers answers to all exercises and miscellaneous exercises given in the textbook.
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PREFACE

"The only way to learn Mathematics is to do Mathematics" – Paul Halmos

"Mathematics – II : Std. XI" forms a part of **'Target Perfect Notes'** prepared as per the **New Textbook**. It is a complete and thorough guide critically analysed and extensively drafted to boost the students' confidence.

The book provides **answers to all textbook questions** included in exercises as well as miscellaneous exercises. Apart from these questions, we have provided **ample questions for additional practice** to students based on each every exercise of the textbook. Only the final answer has been provided for such additional practice questions. At the start of the chapter, we have provided a **table to birfucate the textbook questions and additional practice questions** as per the different type of problems/concepts in the chapter. This will help in systematic study of the entire chapter.

Precise theory has been provided at the required places for better understanding of concepts. Further, all **derivations and theorems have been covered** wherever required. A **recap of all important formulae** has been provided at the end of the book for quick revision. We have also included **activity based questions** in every chapter. We have newly introduced **'competitive corner'** in this book wherein we have included questions from prominent competitive exams. It will help students to get an idea about the type of questions that are asked in Competitive Exams. We all know that there are certain sums that can be solved by multiple methods. Besides, there are also other ways to check your answer in Maths. **'Smart Check'** has been included to help you understand how you can check the correctness of your answer.

The journey to create a complete book is strewn with triumphs, failures and near misses. If you think we've nearly missed something or want to applaud us for our triumphs, we'd love to hear from you. Pls write to us on: mail@targetpublications.org

A book affects eternity; one can never tell where its influence stops.

Best of luck to all the aspirants!

From,
Publisher

Edition: First

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8 Continuity

Type of Problems	Exercise	Q. Nos.
Examine the Continuity of a Function at a given point	8.1	Q.1, 2, 4, 5, 10, 12, 13
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Syllabus

- Continuity of a function at a point.
- Continuity of a functiuon over an interval.
- Intermediate value theorem.

Let's Study

Continuous and Discontinuous Functions

Continuity is 'the state of being continuous' and continuous means 'without any interruption or disturbance'.

An activity that takes place gradually, without interruption or abrupt change is called a continuous process

For example, the flow time in human life is continuous, i.e., we are getting older continuously, the flow of water in the river.

Note:

There are no jumps, breaks, gaps or holes in the graph of the function.

Continuity of a function at a point:

Consider the functions indicated by following graphs where $y = f(x)$:

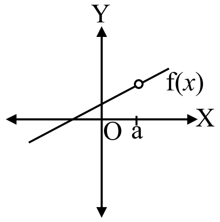


Fig. (i)

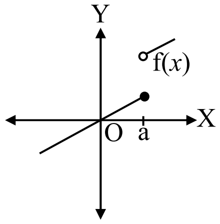


Fig. (ii)

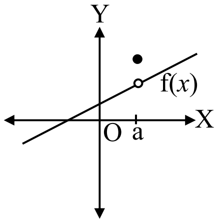


Fig. (iii)

i. The function in figure (i) has a hole at $x = \text{a}$.

$\therefore$ f(x) is not defined at $x = \text{a}$.

ii. The function in figure (ii) has a break at $x = \text{a}$.

iii. For the function in figure (iii), f(a) is not in the continuous line.

Definition of Continuity :

1. A function f(x) is said to be continuous at a point $x = \text{a}$, if the three conditions are satisfied:

i. f is defined at every point on an open interval containing a.

ii. $\lim\limits_{x\to \text{a}} f(x)$ exists

iii. $\lim\limits_{x\to \text{a}} f(x) = f(\text{a})$.

Example:

Consider $f(x) = x^2 - 4$ and let us discuss the continuity of f at $x = 3$

Solution:

i. Here, f(x) is a polynomial function

$\therefore$ It is defined at every point on an open interval containing $x = 3$

ii. $\lim\limits_{x\to 3} f(x) = \lim\limits_{x\to 3} x^2 - 4 = 3^2 - 4 = 5$

$\therefore$ $\lim\limits_{x\to 3} f(x)$ exists.

iii. $f(x) = x^2 - 4$

$\therefore$ $f(3) = 3^2 - 4 = 5$

$\therefore$ $\lim\limits_{x\to 3} f(x) = f(3) = 5$

Here, all the 3 conditions are satisfied.

$\therefore$ f(x) is continuous at a point $x = 3$

2. The condition in above fig. (iii) can be reformulated and the continuity of f(x) at $x = \text{a}$, can be restated as follows:

A function f(x) is said to be continuous at a point $x = \text{a}$ if it is defined in some nighborhood of 'a' and if $\lim\limits_{h\to 0} [f(\text{a}+h) - f(\text{a})] = 0$.

Continuity from the right and from the left:

i. There are some functions, which are defined in two different ways on either side of a point. In such cases we have to consider the limits of function from left as well as right of that point.

ii. A function f(x) is said to be continuous from the right at $x = \text{a}$ if $\lim\limits_{x\to \text{a}^+} f(x) = f(\text{a})$

iii. A function f(x) is said to be continuous from the left at $x = \text{a}$ if $\lim\limits_{x\to \text{a}^-} f(x) = f(\text{a})$

iv. If a function is continuous on the right and also on the left of a then it is continuous at a because

$\lim\limits_{x\to \text{a}^+} f(x) = \lim\limits_{x\to \text{a}^-} f(x) = f(\text{a})$

Example:

Consider the function,

$f(x) = 2x + 7, \quad x < 4$

$= 5x - 5, \quad x \geq 4$

Since, f(x) has different expressions for the value of x

$\therefore$ left hand and right hand limits have to be found out.

$\therefore$ $\lim\limits_{x\to 4^+} f(x) = \lim\limits_{x\to 4} (5x - 5) = 5 \times 4 - 5 = 15$

Also, $f(4) = 5(4) - 5 = 15$

and $\lim\limits_{x\to 4^-} f(x) = \lim\limits_{x\to 4} (2x + 7) = 2 \times 4 + 7 = 15$

$\therefore$ $\lim\limits_{x\to 4^-} f(x) = \lim\limits_{x\to 4^+} f(x) = f(4)$

$\therefore$ f(x) is continuous at $x = 4$.

Examples of continuous functions:

1. Constant function: The constant function $f(x) = k$ (where $k \in R$ is a constant). The function is continuous for all x belonging to its domain.

Example:

$f(x) = 10$, $f(x) = \log_{10} 100$, $f(x) = e^7$

2. Polynomial function: The function $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$, where $n \in N$, $a_0, a_1 \ldots a_n \in R$ is continuous for all x belonging to domain of x.

Example:

$f(x) = x^2 + 5x + 9$, $f(x) = x^3 - 5x + 9$, $f(x) = x^4 - 16, \forall x \in R$.

3. Rational function: If f and g are two polynomial functions having same domain then the rational function $\frac{f}{g}$ is continuous in its domain at points where $g(x) \neq 0$.

Example:

Consider the function, $\frac{x^2+5x+6}{x^2-9}$

Here, $f(x) = x^2 + 5x + 6$ and $g(x) = x^2 - 9$

Given function is continuous on its domain,
where $x^2 - 9 \neq 0$
i.e., $(x + 3)(x - 3) \neq 0$
i.e., $x + 3 \neq 0, x - 3 \neq 0$
i.e., $x \neq -3, x \neq 3$

$\therefore$ The function is continuous on its domain except at $x = 3, -3$.

4. Trigonometric function: sin (ax + b) and cos (ax + b), where $a, b \in R$ are continuous functions for all $x \in R$.

Example:
$\sin(5x + 2), \cos(7x - 11)\ \forall\ x \in R$.

Note: Tangent, cotangent, secant and cosecant functions are continuous on their respective domains.

5. Exponential function: $f(x) = a^x$, $a > 0, a \neq 1$, $x \in R$ is an exponential function, which is continuous for all $x \in R$.

Example:

$f(x) = 3^x$, $f(x) = \left(\frac{1}{2}\right)^x$, $f(x) = e^x\ \forall\ x \in R$,

where $a > 0, a \neq 1$.

6. Logarithmic function: $f(x) = \log_a x$ where $a > 0, a \neq 1$ is a logarithmic function which is continuous for every positive real number i.e. for all $x \in R^+$

Example:
$f(x) = \log_a 7x$, $f(x) = \log_a 9x^2\ \forall\ x \in R$, where $a > 0, a \neq 1$.

Properties of Continuous Functions

If the functions f and g are continuous at $x = a$, then,

1. their sum, that is (f + g) is continuous at $x = a$.
2. their difference, that is (f − g) or (g − f) is continuous at $x = a$
3. the constant multiple of f(x), that is k.f, for any $k \in R$, is continuous at $x = a$.
4. their product, that is (f.g) is continuous at $x = a$.
5. their quotient, that is $\frac{f}{g}$, if $g(a) \neq 0$, is continuous at $x = a$.
6. their composite function, f[g(x)] or g[f(x)], that is fog(x) or gof(x), is continuous at $x = a$.

Types of discontinuities

i. Jump discontinuity:

A function f(x) has a Jump Discontinuity at $x = 0$. If the left hand and right hand limits both exist but are different, that is $\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$

Example:

i. Examine the continuity of the following functions at the given point.
(All functions are defined on $R \to R$)

$f(x) = x^2 - x + 9$, for $x \leq 3$
$= 4x - 3$, for $x > 3$; at $x = 3$.

Solution:

$\lim_{x \to 3^-} f(x) = \lim_{x \to 3} (x^2 - x + 9)$

$= (3)^2 - 3 + 9 = 15$

and $\lim_{x \to 3^+} f(x) = \lim_{x \to 3} (4x + 3)$

$= 4(3) - 3 = 9$

Here, $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ both exist.

But, $\lim_{x \to 3^-} f(x) \neq \lim_{x \to 3^+} f(x)$

$\therefore$ f(x) has jump discontinuity at $x = 3$.

ii. Removable discontinuity:

A function f(x) has a discontinuity at $x = a$, and $\lim_{x \to a} f(x)$ exists, but either f(a) is not defined or $\lim_{x \to a} f(x) \neq f(a)$. In such case we define or redefine f(a) as $\lim_{x \to a} f(x)$. Then with new definition, the function f(x) becomes continuous at $x = a$. Such a discontinuity is called as Removable discontinuity.

Example:

Consider the function,

$f(x) = \frac{x^2 - 5x + 6}{x - 2}$, $x \neq 2$

$= 2$, $x = 2$

Here, $\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2 - 5x + 6}{x - 2}$

$= \lim_{x \to 2} \frac{(x-3)(x-2)}{(x-2)}$

$= \lim_{x \to 2} x - 3$

...[$\because x \to 2, \therefore x \neq 2,$
$\therefore x - 2 \neq 0$]

$= 2 - 3$

$= -1$

$\therefore \lim_{x \to 2} f(x)$ exists

Also, f (2) = 2 …(given)

$\therefore \lim_{x \to 2} f(x) \neq f(2)$

$\therefore$ function f is discontinuous at $x = 2$,
This discontinuity can be removed by redefining f as follows:

$f(x) = \frac{x^2 - 5x + 6}{x - 2}$, $x \neq 2$

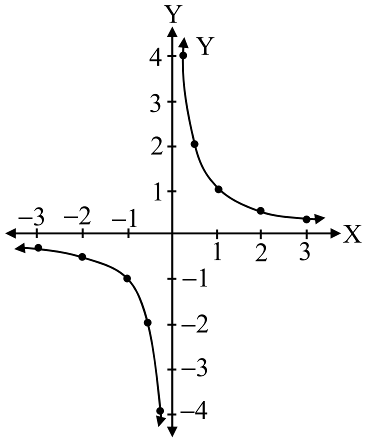
$= -1$, $x = 2$

$\therefore$ $x = 2$ is a point of removable discontinuity.

Extension of the original function:
If the original function is not defined at a and the new definition of f makes it continuous at a, then the new definition is called the extension of the original function.

Infinite discontinuity:
Consider the graph of $xy = 1$

i.e., $y = f(x) = \frac{1}{x}$



Here, $f(x) \to \infty$ as $x \to 0^+$ and
$f(x) \to -\infty$ as $x \to 0^-$.

∴ f(0) is not defined

∴ function is discontinuous at $x = 0$.

A function $f(x)$ is said to have an infinite discontinutiy at $x = a$,
If $\lim_{x\to a^-} f(x) = \pm\infty$ or $\lim_{x\to a^+} f(x) = \pm\infty$

Then, from the figure, $f(x)$ has an infinite discontinutiy.

Continuity over an interval

Let (a, b) be an open interval.
If for every $x \in (a, b)$, f is continuous at x then f is continuous on (a, b).

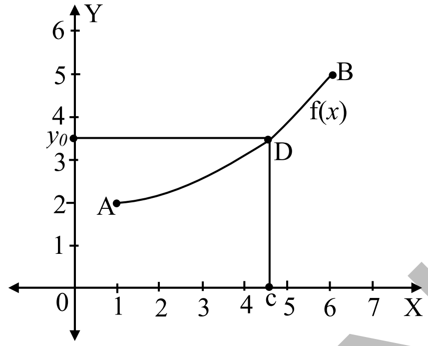
i. Consider f defined on [a, b). If f is continuous on (a, b) and f is continuous to the right of a, $\lim_{x\to a^+} f(x) = f(a)$ then f is continuos on [a, b)

ii. Consider f defined on (a, b]. If f is continuous on (a, b) and f is continuous to the left of b, $\lim_{x\to a^+} f(x) = f(b)$ then f is continuos on (a, b]

iii. Consider a function f continuous on the open interval (a, b). If $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ exists, then we can extend the fuction to [a, b] so that it is continuous on [a, b].

The intermediate value theorem for continuous function

Theorem : If f is a continuous function on a closed interval [a, b], and if y_0 is any value between f(a) and f(b) then $y_0 = f(c)$ for some c in [a, b]



Geometrically, Theorem says that any horizontal line $y = y_0$ crossing the Y-axis between the numbers f(a) and f(b) will cross the curve $y = f(x)$ at least once over the interval [a, b].

Textual Activity

1. Discuss the continuity of $f(x)$ where

$$f(x) = \frac{\log x - \log 5}{x - 5}, \text{ for } x \neq 5$$
$$= \frac{1}{5} \qquad \text{for } x = 5$$

(Textbook page no. 171)

Solution:

Given, $f(5) = \boxed{\frac{1}{5}}$ …(i)

∴ $\lim_{x\to 5} f(x) = \lim_{x\to 5}\left[\frac{\log x - \log 5}{x - 5}\right]$

Put $x - 5 = t$, $x = 5 + t$.
As $x \to 5$, $t \to 0$

$$= \lim_{t\to\boxed{0}}\left[\frac{\log\left(\boxed{5+t}\right) - \log 5}{t}\right]$$

$$= \lim_{t\to\boxed{0}}\left[\frac{\log\left[\frac{\boxed{5+t}}{5}\right]}{t}\right] = \lim_{t\to\boxed{0}}\left[\frac{\log\left(1+\frac{t}{5}\right)}{t}\right]$$

$$= \lim_{t\to\boxed{0}}\left[\frac{\log\left(1+\frac{t}{5}\right)}{\frac{t}{\boxed{5}}}\right] \times \frac{1}{\boxed{5}}$$

$$= 1 \times \frac{1}{\boxed{5}} \qquad \ldots\left[\because \lim_{x\to 0}\left[\frac{\log(1+px)}{px}\right] = 1\right]$$

∴ $\lim_{x\to 0} f(x) = \frac{1}{\boxed{5}}$ …(ii)

∴ From (i) and (ii),
$\lim_{x\to 5} f(x) = f(5)$

∴ The function $f(x)$ is continuous at $x = 5$.

Exercise 8.1

1. Examine the continuity of

i. $f(x) = x^3 + 2x^2 - x - 2$ **at** $x = -2$.

ii. $f(x) = \sin x,\quad$ **for** $x \le \frac{\pi}{4}$

$= \cos x,\quad$ **for** $x > \frac{\pi}{4}$, **at** $x = \frac{\pi}{4}$

iii. $f(x) = \frac{x^2 - 9}{x - 3},\quad$ **for** $x \ne 3$

$= 8\quad$ **for** $x = 3$

Solution :

i. Given, $f(x) = x^3 + 2x^2 - x - 2$

f(x) is a polynomial function and hence it is continuous for all $x \in R$.

$\therefore$ f(x) is continuous at $x = -2$

---

ii. $f(x) = \sin x\ ;\quad x \le \frac{\pi}{4}$

$= \cos x\ ;\quad x > \frac{\pi}{4}$

$\lim_{x \to \frac{\pi}{4}^-} f(x) = \lim_{x \to \frac{\pi}{4}^-} (\sin x)$

$= \sin \frac{\pi}{4}$

$= \frac{1}{\sqrt{2}}$

$\lim_{x \to \frac{\pi}{4}^+} f(x) = \lim_{x \to \frac{\pi}{4}^+} (\cos x)$

$= \cos \frac{\pi}{4}$

$= \frac{1}{\sqrt{2}}$

Also $f\left(\frac{\pi}{4}\right) = \sin \frac{\pi}{4}$

$= \frac{1}{\sqrt{2}}$

$\therefore \lim_{x \to \frac{\pi}{4}^-} f(x) = \lim_{x \to \frac{\pi}{4}^+} f(x) = f\left(\frac{\pi}{4}\right)$

$\therefore$ f(x) is continuous at $x = \frac{\pi}{4}$

---

iii. $f(3) = 8$ …(given)

$\lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{x^2 - 9}{x - 3}$

$= \lim_{x \to 3} \frac{(x+3)(x-3)}{x - 3}$

$= \lim_{x \to 3} (x + 3)$

…[$\because x \to 3, x \ne 3\ \therefore x - 3 \ne 0$]

$= 3 + 3 = 6$

$\therefore \lim_{x \to 3} f(x) \ne f(3)$

$\therefore$ f(x) is discontinuous at $x = 3$

---

2. Examine whether the function is continuous at the points indicated against them.

i. $f(x) = x^3 - 2x + 1,\quad$ **if** $x \le 2$

$= 3x - 2,\quad$ **if** $x > 2$, **at** $x = 2$.

ii. $f(x) = \frac{x^2 + 18x - 19}{x - 1},\quad$ **for** $x \ne 1$

$= 20,\quad$ **for** $x = 1$, **at** $x = 1$

iii. $f(x) = \frac{x}{\tan 3x} + 2,\quad$ **for** $x < 0$

$= \frac{7}{3},\quad$ **for** $x \ge 0$, **at** $x = 0$.

Solution :

i. $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x^3 - 2x + 1)$

$= (2)^3 - 2(2) + 1$

$= 5$

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (3x - 2)$

$= 3(2) - 2$

$= 4$

$\therefore \lim_{x \to 2^-} f(x) \ne \lim_{x \to 2^+} f(x)$

$\therefore$ f(x) is discontinuous at $x = 2$

---

ii. $\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x^2 + 18x - 19}{x - 1}$

$= \lim_{x \to 1} \frac{x^2 + 19x - x - 19}{x - 1}$

$= \lim_{x \to 1} \frac{x(x+19) - 1(x+19)}{(x-1)}$

$= \lim_{x \to 1} \frac{(x-1)(x+19)}{(x-1)}$

$= \lim_{x \to 1} (x + 19)$

…[$\because x \to 1, \therefore x \ne 1, \therefore x - 1 \ne 0$]

$= 1 + 19 = 20$

Also, $f(1) = 20$

$\therefore \lim_{x \to 1} f(x) = f(1)$

$\therefore$ f(x) is continuous at $x = 1$

---

iii. $\lim_{x \to 0^+} f(x) = \frac{7}{3}$ …(given)

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \left(\frac{x}{\tan 3x} + 2\right)$

$= \lim_{x \to 0^-} \frac{x}{\tan 3x} + \lim_{x \to 0^-} 2$

$$= \lim_{x\to0^-}\frac{1}{\frac{\tan 3x}{x}} + \lim_{x\to0^-} 2$$

$$= \lim_{x\to0^-}\left(\frac{1}{\frac{\tan 3x}{3x}\times 3}\right) + \lim_{x\to0^-} 2$$

$$= \frac{\lim_{x\to0^-} 1}{3\lim_{x\to0^-}\left(\frac{\tan 3x}{3x}\right)} + \lim_{x\to0^-} 2$$

$$= \frac{1}{3(1)} + 2$$

$$= \frac{7}{3}$$

$\therefore \quad \lim_{x\to0^+} f(x) = \lim_{x\to0^-} f(x)$

$\therefore \quad$ f(x) is continuous at $x = 0$.

---

3. Find all the points of discontinuities of f(x) = $\lfloor x\rfloor$ on the interval (–3, 2).

Solution :

f(x) = $\lfloor x\rfloor$, $x \in (-3, 2)$

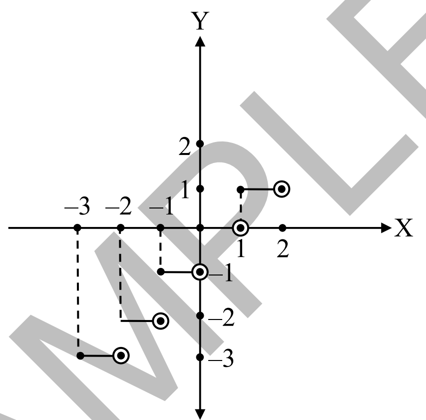
i.e. f(x) = –3, $x \in (-3, -2)$

= –2, $x \in [-2, -1)$

= –1, $x \in [-1, 0)$

= 0, $x \in [0, 1)$

= 1, $x \in [1, 2)$



At $x = -2$,

$\lim_{x\to-2^-} f(x) = \lim_{x\to-2^-} \lfloor x\rfloor$

= –3

$\lim_{x\to-2^+} f(x) = \lim_{x\to-2^+} \lfloor x\rfloor$

= –2

$\therefore \quad \lim_{x\to-2^-} f(x) \neq \lim_{x\to-2^+} f(x)$

$\therefore \quad$ f(x) is discontinuous at $x = -2$

similarly f(x) is discontinuous at the point $x = -1, x = 0, x = 1$.

Thus all the integer values of x in the interval (–3, 2) i.e. the points $x = -2$, $x = -1$, $x = 0$ and $x = 1$ are the required points of discontinuities.

---

4. Discuss the continuity of the function f(x) = $|2x + 3|$, at $x = \frac{-3}{2}$.

Solution :

f(x) = $|2x+3|$, $x = \frac{-3}{2}$

$|2x+3| = 2x + 3 \quad ; \ x \geq \frac{-3}{2}$

$= -(2x+3) \ ; \ x < \frac{-3}{2}$

$\lim_{x\to\frac{-3}{2}^-} f(x) = \lim_{x\to\frac{-3}{2}^-} |2x+3|$

$= \lim_{x\to\frac{-3}{2}^-} [-(2x+3)]$

$= -\left[2\left(\frac{-3}{2}\right)+3\right]$

= 0

$\lim_{x\to\frac{-3}{2}^+} f(x) = \lim_{x\to\frac{-3}{2}^+} |2x+3|$

$= \lim_{x\to\frac{-3}{2}^+} (2x+3)$

$= 2\left(\frac{-3}{2}\right) + 3$

= 0.

$f\left(\frac{-3}{2}\right) = \left|2\left(\frac{-3}{2}\right)+3\right|$

$= |0|$

= 0

$\therefore \quad \lim_{x\to\frac{-3}{2}^-} f(x) = \lim_{x\to\frac{-3}{2}^+} f(x) = f\left(\frac{-3}{2}\right)$

$\therefore \quad$ f(x) is continuous at $x = \frac{-3}{2}$.

---

5. Test the continuity of the following functions at the points or interval indicated against them.

i. $f(x) = \frac{\sqrt{x-1}-(x-1)^{\frac{1}{3}}}{x-2}$, **for $x \neq 2$**

$= \frac{1}{5}$, **for $x = 2$; at $x = 2$**

ii. $f(x) = \frac{x^3-8}{\sqrt{x+2}-\sqrt{3x-2}}$, **for $x \neq 2$**

$= -24$ **for $x = 2$ at $x = 2$**

iii. $f(x) = 4x + 1$, **for $x \leq \frac{8}{3}$**

$= \frac{59-9x}{3}$, **for $x > \frac{8}{3}$, at $x = \frac{8}{3}$.**

iv. $f(x) = \dfrac{(27-2x)^{\frac{1}{3}}-3}{9-3(243+5x)^{\frac{1}{5}}}$, **for $x \neq 0$**

= 2 **for $x = 0$, at $x = 0$**

v. $f(x) = \dfrac{x^2+8x-20}{2x^2-9x+10}$, **for $0 < x < 3; x \neq 2$**

= 12, **for $x = 2$**

$= \dfrac{2-2x-x^2}{x-4}$ **for $3 \leq x < 4$ at $x = 2$**

Solution :

i. $f(2) = \dfrac{1}{5}$ …(given)

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{\sqrt{x-1}-(x-1)^{\frac{1}{3}}}{x-2}$$

Put $x - 1 = y$

$\therefore$ $x = 1 + y$

As $x \to 2$, $y \to 1$

$$\therefore \lim_{x\to 2} f(x) = \lim_{y\to 1} \frac{\sqrt{y}-y^{\frac{1}{3}}}{1+y-2}$$

$$= \lim_{y\to 1} \frac{\left(y^{\frac{1}{2}}-1\right)-\left(y^{\frac{1}{3}}-1\right)}{y-1}$$

$$= \lim_{y\to 1}\left(\frac{y^{\frac{1}{2}}-1}{y-1}-\frac{y^{\frac{1}{3}}-1}{y-1}\right)$$

$$= \lim_{y\to 1}\frac{y^{\frac{1}{2}}-1^{\frac{1}{2}}}{y-1} - \lim_{y\to 1}\frac{y^{\frac{1}{3}}-1^{\frac{1}{3}}}{y-1}$$

$$= \frac{1}{2}(1)^{\frac{-1}{2}} - \frac{1}{3}(1)^{\frac{-2}{3}}$$

$$\ldots\left[\because \lim_{x\to a}\frac{x^n-a^n}{x-a} = n.a^{n-1}\right]$$

$$= \frac{1}{2}-\frac{1}{3}$$

$$= \frac{1}{6}$$

$\therefore$ $\lim_{x\to 2} f(x) \neq f(2)$

$\therefore$ $f(x)$ is discontinuous at $x = 2$

---

ii. $f(2) = -24$ …(given)

$$\lim_{x\to 2} f(x) = \lim_{x\to 2}\frac{x^3-8}{\sqrt{x+2}-\sqrt{3x-2}}$$

$$= \lim_{x\to 2}\frac{x^3-8}{\sqrt{x+2}-\sqrt{3x-2}}\times\frac{\sqrt{x+2}+\sqrt{3x-2}}{\sqrt{x+2}+\sqrt{3x-2}}$$

$$= \lim_{x\to 2}\frac{(x^3-8)\left(\sqrt{x+2}+\sqrt{3x-2}\right)}{(x+2)-(3x-2)}$$

$$= \lim_{x\to 2}\frac{(x^3-2^3)\left(\sqrt{x+2}+\sqrt{3x-2}\right)}{-2x+4}$$

$$= \lim_{x\to 2}\frac{(x-2)(x^2+2x+4)\left(\sqrt{x+2}+\sqrt{3x-2}\right)}{-2(x-2)}$$

$$= \lim_{x\to 2}\frac{(x^2+2x+4)\left(\sqrt{x+2}+\sqrt{3x-2}\right)}{-2}$$

$$\ldots\left[\begin{array}{l}\because x\to 2, x\neq 2\\ \therefore x-2\neq 0\end{array}\right]$$

$$= \frac{-1}{2}\lim_{x\to 2}(x^2+2x+4)\left(\sqrt{x+2}+\sqrt{3x-2}\right)$$

$$= \frac{-1}{2}\lim_{x\to 2}(x^2+2x+4)\lim_{x\to 2}\left(\sqrt{x+2}+\sqrt{3x-2}\right)$$

$$= \frac{-1}{2}\times\left[2^2+2(2)+4\right]\times\left(\sqrt{2+2}+\sqrt{3(2)-2}\right)$$

$$= \frac{-1}{2}\times 12\times(2+2)$$

$$= -24$$

$\therefore$ $\lim_{x\to 2} f(x) = f(2)$

$\therefore$ $f(x)$ is continuous at $x = 2$

---

iii. $$\lim_{x\to \frac{8}{3}^-} f(x) = \lim_{x\to \frac{8}{3}^-}(4x+1)$$

$$= 4\left(\frac{8}{3}\right)+1$$

$$= \frac{35}{3}$$

$$\lim_{x\to \frac{8}{3}^+} f(x) = \lim_{x\to \frac{8}{3}^+}\frac{59-9x}{3}$$

$$= \frac{59-9\left(\frac{8}{3}\right)}{3}$$

$$= \frac{59-24}{3}$$

$$= \frac{35}{3}$$

$f(x) = 4x + 1$

$\therefore$ $f\left(\frac{8}{3}\right) = 4\left(\frac{8}{3}\right)+1 = \frac{35}{3}$

$\therefore$ $\lim_{x\to \frac{8}{3}^-} f(x) = \lim_{x\to \frac{8}{3}^+} f(x) = f\left(\frac{8}{3}\right)$

$f(x)$ is continuous at $x = \frac{8}{3}$.

---

iv. $f(0) = 2$ …(given)

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{(27-2x)^{\frac{1}{3}}-3}{9-3(243+5x)^{\frac{1}{5}}}$$

$$= \lim_{x\to 0}\frac{(27-2x)^{\frac{1}{3}}-3}{-3\left[(243+5x)^{\frac{1}{5}}-3\right]}$$

$$= \frac{-1}{3}\lim_{x\to 0}\frac{(27-2x)^{\frac{1}{3}}-(27)^{\frac{1}{3}}}{(243+5x)^{\frac{1}{5}}-(243)^{\frac{1}{5}}}$$

$$= \frac{-1}{3}\lim_{x\to 0}\frac{\frac{(27-2x)^{\frac{1}{3}}-27^{\frac{1}{3}}}{(27-2x)-27}\times[(27-2x)-27]}{\frac{(243+5x)^{\frac{1}{5}}-(243)^{\frac{1}{5}}}{(243+5x)-243}\times[(243+5x)-243]}$$

$$\ldots\left[\begin{array}{l}\text{As } x\to 0, -2x\to 0 \text{ and } 5x\to 0\\ \therefore (27-2x)-27\to 0 \text{ and } (243+5x)-243\to 0\\ \therefore (27-2x)-27\neq 0 \text{ and } (243+5x)-243\neq 0\end{array}\right]$$

$$= \frac{-1}{3}\,\frac{\lim_{x\to 0}\frac{(27-2x)^{\frac{1}{3}}-27^{\frac{1}{3}}}{(27-2x)-27}\times(-2x)}{\lim_{x\to 0}\frac{(243+5x)^{\frac{1}{5}}-(243)^{\frac{1}{5}}}{(243+5x)-243}\times(5x)}$$

$$= \frac{-1}{3}\times\frac{-2}{5}\times\frac{\lim_{x\to 0}\frac{(27-2x)^{\frac{1}{3}}-27^{\frac{1}{3}}}{(27-2x)-27}}{\lim_{x\to 0}\frac{(243+5x)^{\frac{1}{5}}-(243)^{\frac{1}{5}}}{(243+5x)-243}}$$

$$\ldots[\because x\to 0, x\neq 0]$$

$$= \frac{2}{15}\times\frac{\frac{1}{3}(27)^{\frac{-2}{3}}}{\frac{1}{5}(243)^{\frac{-4}{5}}} \qquad \ldots\left[\because \lim_{x\to a}\frac{x^n-a^n}{x-a}=n\,a^{n-1}\right]$$

$$= \frac{2}{15}\times\frac{5}{3}\times\frac{(3^3)^{\frac{-2}{3}}}{(3^5)^{\frac{-4}{5}}}$$

$$= \frac{2}{9}\times\frac{(3)^{-2}}{(3)^{-4}} = \frac{2}{9}\times(3)^2$$

$= 2$

$\therefore$ $\lim_{x\to 0} f(x) = f(0)$

$\therefore$ $f(x)$ is continuous at $x = 0$

---

v. $f(2) = 12$ …(given)

$$\lim_{x\to 2} f(x) = \lim_{x\to 2}\frac{x^2+8x-20}{2x^2-9x+10}$$

$$= \lim_{x\to 2}\frac{(x+10)(x-2)}{(2x-5)(x-2)}$$

$$= \lim_{x\to 2}\frac{x+10}{2x-5} \qquad \ldots\left[\begin{array}{l}\because x\to 2, x\neq 2\\ \therefore x-2\neq 0\end{array}\right]$$

$$= \frac{\lim_{x\to 2}(x+10)}{\lim_{x\to 2}(2x-5)} = \frac{2+10}{2(2)-5}$$

$$= \frac{12}{-1} = -12$$

$\therefore$ $\lim_{x\to 2} f(x) \neq f(2)$

$\therefore$ $f(x)$ is discontinuous at $x = 2$.

6. Identify discontinuities for the following functions as either a jump or a removable discontinuity.

i. $f(x) = \dfrac{x^2-10x+21}{x-7}$

ii. $f(x) = x^2 + 3x - 2$, for $x \le 4$
$\quad = 5x + 3$, for $x > 4$.

iii. $f(x) = x^2 - 3x - 2$, for $x < -3$
$\quad = 3 + 8x$, for $x > -3$.

iv. $f(x) = 4 + \sin x$, for $x < \pi$
$\quad = 3 - \cos x$ for $x > \pi$

Solution :

i. Given, $f(x) = \dfrac{x^2-10x+21}{x-7}$

It is rational function and is discontinuous if $x - 7 = 0$ i.e., $x = 7$

$\therefore$ $f(x)$ is continuous for all $x \in R$, except at $x = 7$.

$\therefore$ $f(7)$ is not defined.

Now, $\lim_{x\to 7} f(x) = \lim_{x\to 7}\dfrac{x^2-10x+21}{x-7}$

$$= \lim_{x\to 7}\frac{(x-7)(x-3)}{x-7}$$

$$= \lim_{x\to 7}(x-3) \ldots\left[\begin{array}{l}\because x\to 7, \therefore x\neq 7\\ \therefore x-7\neq 0\end{array}\right]$$

$= 7 - 3$

$= 4$

Thus $\lim_{x\to 7} f(x)$ exist but $f(7)$ is not defined

$\therefore$ $f(x)$ has a removable discontinuity.

---

ii. $f(x) = x^2 + 3x - 2, x \le 4$
$\quad = 5x + 3, \quad x > 4$

$f(x)$ is a polynomial function for both the intervals.

$\therefore$ $f(x)$ is continuous for both the open intervals $(-\infty, 4)$ and $(4, \infty)$.

Let us test the continuity at $x = 4$

$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-}(x^2 + 3x - 2)$

$= (4)^2 + 3(4) - 2$

$= 26.$

$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+}(5x + 3)$

$= 5(4) + 3$

$= 23.$

$\therefore$ $\lim_{x\to 4^-} f(x) \neq \lim_{x\to 4^+} f(x)$

$\therefore$ $\lim_{x\to 4} f(x)$ does not exist.

$\therefore$ $f(x)$ is discontinuous at $x = 4$

$\therefore$ $f(x)$ has a jump discontinuity at $x = 4$

iii. $f(x) = x^2 - 3x - 2,\ x < -3$
$= 3 + 8x, \quad x > -3$

$f(x)$ is a polynomial function for both the intervals.

∴ $f(x)$ is continuous for both the given intervals.

Let us test the continuity at $x = -3$

$$\lim_{x\to -3^-} f(x) = \lim_{x\to -3^-} (x^2 - 3x - 2)$$
$$= (-3)^2 - 3(-3) - 2$$
$$= 9 + 9 - 2$$
$$= 16$$
$$\lim_{x\to -3^+} f(x) = \lim_{x\to -3^+} (3 + 8x)$$
$$= 3 + 8(-3)$$
$$= -21$$

∴ $\lim_{x\to -3^-} f(x) \neq \lim_{x\to -3^+} f(x)$

∴ $\lim_{x\to -3} f(x)$ does not exist.

∴ $f(x)$ is discontinuous at $x = -3$

∴ $f(x)$ has a jump discontinuity at $x = -3$.

---

iv. $f(x) = 4 + \sin x,\ x < \pi$
$= 3 - \cos x,\ x > \pi$

$\sin x$ and $\cos x$ are continuous for all $x \in R$.

4 and 3 are constant functions.

∴ $4 + \sin x$ and $3 - \cos x$ are continuous for all $x \in R$.

∴ $f(x)$ is continuous for both the given intervals.

Let us test the continuity at $x = \pi$.

∴ $$\lim_{x\to \pi^-} f(x) = \lim_{x\to \pi^-} (4 + \sin x)$$
$$= 4 + \sin \pi$$
$$= 4 + 0$$
$$= 4$$
$$\lim_{x\to \pi^+} f(x) = \lim_{x\to \pi^+} (3 - \cos x)$$
$$= 3 - \cos \pi$$
$$= 3 - (-1)$$
$$= 4$$

∴ $\lim_{x\to \pi^-} f(x) = \lim_{x\to \pi^+} f(x)$

∴ $\lim_{x\to \pi} f(x) = 4$

But $f(\pi)$ is not defined.

∴ $f(x)$ has a removable discontinuity at $x = \pi$.

---

7. Show that following functions have continuous extension to the point where f(x) is not defined. Also find the extension

i. $f(x) = \dfrac{1-\cos 2x}{\sin x}$, **for** $x \neq 0$.

ii. $f(x) = \dfrac{3\sin^2 x + 2\cos x\,(1-\cos 2x)}{2(1-\cos^2 x)}$, **for** $x \neq 0$.

iii. $f(x) = \dfrac{x^2 - 1}{x^3 + 1}$, **for** $x \neq -1$.

Solution :

i. $f(x) = \dfrac{1-\cos 2x}{\sin x}$, for $x \neq 0$

Here, f(0) is not defined.

Consider,

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1-\cos 2x}{\sin x}$$
$$= \lim_{x\to 0} \frac{2\sin^2 x}{\sin x}$$
$$= 2\lim_{x\to 0} (\sin x) \quad \ldots\begin{bmatrix}\because x\to 0, \therefore x\neq 0\\ \therefore \sin x \neq 0\end{bmatrix}$$
$$= 2(\sin 0) = 2\times 0$$
$$= 0.$$

∴ $\lim_{x\to 0} f(x)$ exists.

But f(0) is not been defined.

∴ $f(x)$ has a removable discontinuity at $x = 0$.

∴ The extension of the original function is

$$f(x) = \frac{1-\cos 2x}{\sin x}; \text{ for } x\neq 0$$
$$= 0 \quad ; \text{ for } x = 0$$

∴ $f(x)$ is continuous at $x = 0$

---

ii. $f(x) = \dfrac{3\sin^2 x + 2\cos x\,(1-\cos 2x)}{2(1-\cos^2 x)}$; $x \neq 0$

Here f(0) is not defined.

Consider,

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{3\sin^2 x + 2\cos x\,(1-\cos 2x)}{2(1-\cos^2 x)}$$
$$= \lim_{x\to 0} \frac{3\sin^2 x + 2\cos x\cdot(2\sin^2 x)}{2\sin^2 x}$$
$$= \lim_{x\to 0} \frac{\sin^2 x\,(3 + 4\cos x)}{2\sin^2 x}$$
$$= \lim_{x\to 0} \frac{3 + 4\cos x}{2}$$
$$\ldots[\because x\to 0, \therefore x\neq 0, \therefore \sin x\neq 0, \therefore \sin^2 x\neq 0]$$
$$= \frac{1}{2}\lim_{x\to 0}(3 + 4\cos x) = \frac{1}{3}(3 + 4\cos 0)$$
$$= \frac{1}{2}(3+4) = \frac{7}{2}$$

$\lim_{x\to 0} f(x)$ exists but f(0) is not defined.

∴ $f(x)$ has a removable discontinuity at $x = 0$

∴ The extension of the original function is

$$f(x) = \frac{3\sin^2 x + 2\cos x\,(1-\cos 2x)}{2(1-\cos^2 x)} \quad ; x\neq 0$$
$$= \frac{7}{2} \quad ; x = 0$$

∴ $f(x)$ is continuous at $x = 0$

iii. $f(x) = \dfrac{x^2-1}{x^3+1}$; $x \neq -1$.

Here f(– 1) has not been defined.

Consider

$$\lim_{x\to -1} f(x) = \lim_{x\to -1}\left(\frac{x^2-1}{x^3+1}\right)$$

$$= \lim_{x\to -1}\frac{(x+1)(x-1)}{(x+1)(x^2-x+1)}$$

$$= \lim_{x\to -1}\frac{x-1}{x^2-x+1}$$

…[$\because x \to -1, \therefore x \neq -1, \therefore x+1 \neq 0$]

$$= \frac{-1-1}{(-1)^2-(-1)+1} = -\frac{2}{3}$$

Thus $\lim_{x\to -1} f(x)$ exists but f(–1) is not defined.

$\therefore$ f(x) has a removable discontinuity at x = – 1

$\therefore$ The extension of the original function is

$f(x) = \dfrac{x^2-1}{x^3+1}$; $x \neq -1$

$= -\dfrac{2}{3}$; $x = -1$.

f(x) is continuous at $x = -\dfrac{2}{3}$.

8. Discuss the continuity of the following functions at the points indicated against them.

i. $f(x) = \dfrac{\sqrt{3}-\tan x}{\pi - 3x}$, $x \neq \dfrac{\pi}{3}$

$= \dfrac{3}{4}$, **for** $x = \dfrac{\pi}{3}$, **at** $x = \dfrac{\pi}{3}$.

ii. $f(x) = \dfrac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}$, **for** $x \neq 0$

$= 1$, **for** $x = 0$, **at** $x = 0$.

iii. $f(x) = \dfrac{4^x - 2^{x+1}+1}{1-\cos 2x}$, **for** $x \neq 0$

$= \dfrac{(\log 2)^2}{2}$, **for** $x = 0$, **at** $x = 0$.

Solution :

i. $f\left(\dfrac{\pi}{3}\right) = \dfrac{3}{4}$ …(given)

$$\lim_{x\to\frac{\pi}{3}} f(x) = \lim_{x\to\frac{\pi}{3}}\frac{\sqrt{3}-\tan x}{\pi-3x}$$

Put $\dfrac{\pi}{3} - x = h$,

$\therefore$ $x = \dfrac{\pi}{3} - h$

$\therefore$ $$\lim_{x\to\frac{\pi}{3}}\frac{\sqrt{3}-\tan x}{\pi-3x} = \lim_{h\to 0}\frac{\sqrt{3}-\tan\left(\frac{\pi}{3}-h\right)}{\pi-3\left(\frac{\pi}{3}-h\right)}$$

$$= \lim_{h\to 0}\frac{\sqrt{3}-\dfrac{\tan\frac{\pi}{3}-\tan h}{1+\tan\frac{\pi}{3}\tan h}}{3h}$$

$$= \lim_{h\to 0}\frac{\sqrt{3}-\dfrac{\sqrt{3}-\tan h}{1+\sqrt{3}\tan h}}{3h}$$

$$= \lim_{h\to 0}\frac{\sqrt{3}(1+\sqrt{3}\tan h)-(\sqrt{3}-\tan h)}{3h(1+\sqrt{3}\tan h)}$$

$$= \lim_{h\to 0}\frac{\sqrt{3}+3\tan h-\sqrt{3}+\tan h}{3h(1+\sqrt{3}\tan h)}$$

$$= \lim_{h\to 0}\frac{4\tan h}{3h(1+\sqrt{3}\tan h)}$$

$$= \lim_{h\to 0}\frac{4}{3(1+\sqrt{3}\tan h)}\times\frac{\tan h}{h}$$

$$= \frac{4}{3}\left(\lim_{h\to 0}\frac{1}{1+\sqrt{3}\tan h}\right)\left(\lim_{h\to 0}\frac{\tan h}{h}\right)$$

$$= \frac{4}{3}\left[\frac{1}{1+\sqrt{3}(0)}\right](1) = \frac{4}{3}$$

$\therefore$ $\lim_{x\to\frac{\pi}{3}} f(x) \neq f\left(\dfrac{\pi}{3}\right)$

$\therefore$ f(x) is discontinuous at $x = \dfrac{\pi}{3}$.

ii. f(0) = 1 …(given)

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0}\frac{e^{\frac{1}{(0-h)}}-1}{e^{\frac{1}{(0-h)}}+1}$$

$$= \lim_{h\to 0}\frac{e^{\frac{-1}{h}}-1}{e^{\frac{-1}{h}}+1}$$

$$= \lim_{h\to 0}\frac{\dfrac{1}{e^{\frac{1}{h}}}-1}{\dfrac{1}{e^{\frac{1}{h}}}+1}$$

$$= \frac{0-1}{0+1} = -1$$

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0}\frac{e^{\frac{1}{(0+h)}}-1}{e^{\frac{1}{0+h}}+1}$$

$$= \lim_{h\to 0}\frac{e^{\frac{1}{h}}-1}{e^{\frac{1}{h}}+1}$$

$$= \lim_{h \to 0} \frac{e^{\frac{1}{h}}\left(1 - \frac{1}{e^{\frac{1}{h}}}\right)}{e^{\frac{1}{h}}\left(1 + \frac{1}{e^{\frac{1}{h}}}\right)}$$

$$= \frac{1-0}{1+0} = 1$$

$$\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$$

∴ $f(x)$ is discontinuous at $x = 0$

---

iii. $f(0) = \frac{(\log 2)^2}{2}$ …(given)

$$\lim_{x\to0} f(x) = \lim_{x\to0} \frac{4^x - 2^{x+1} + 1}{1 - \cos 2x}$$

$$= \lim_{x\to0} \frac{(2^2)^x - 2^x \cdot 2^1 + 1}{2\sin^2 x}$$

$$= \lim_{x\to0} \frac{(2^x)^2 - 2(2^x) + 1}{2\sin^2 x}$$

$$= \lim_{x\to0} \frac{(2^x - 1)^2}{2\sin^2 x}$$

…[∵ $a^2 - 2ab + b^2 = (a-b)^2$]

$$= \lim_{x\to0} \frac{\frac{(2^x-1)^2}{x^2}}{\frac{2\sin^2 x}{x^2}}$$

…[∵ $x \to 0$, ∴ $x \neq 0$, ∴ $x^2 \neq 0$]

$$= \frac{\left(\lim_{x\to0}\frac{2^x-1}{x}\right)^2}{2\lim_{x\to0}\left(\frac{\sin x}{x}\right)^2}$$

$$= \frac{(\log 2)^2}{2(1)^2} = \frac{(\log 2)^2}{2}$$

∴ $\lim_{x\to0} f(x) = f(0)$

∴ $f(x)$ is continuous at $x = 0$.

---

9. Which of the following functions has a removable discontinuity? If it has a removable discontinuity, redefine the function so that it becomes continuous.

i. $f(x) = \frac{e^{5\sin x} - e^{2x}}{5\tan x - 3x}$, **for** $x \neq 0$

$= \frac{3}{4}$, **for** $x = 0$, **at** $x = 0$.

ii. $f(x) = \log_{(1+3x)}(1+5x)$ **for** $x > 0$

$= \frac{32^x - 1}{8^x - 1}$, **for** $x < 0$, **at** $x = 0$.

iii. $f(x) = \left(\frac{3-8x}{3-2x}\right)^{\frac{1}{x}}$, **for** $x \neq 0$.

iv. $f(x) = 3x + 2$, **for** $-4 \leq x \leq -2$

$= 2x - 3$, **for** $-2 < x \leq 6$.

v. $f(x) = \frac{x^3 - 8}{x^2 - 4}$, **for** $x > 2$

$= 3$, **for** $x = 2$

$= \frac{e^{3(x-2)^2} - 1}{2(x-2)^2}$, **for** $x < 2$

Solution :

i. $f(0) = \frac{3}{4}$ …(given)

$$\lim_{x\to0} f(x) = \lim_{x\to0} \frac{e^{5\sin x} - e^{2x}}{5\tan x - 3x}$$

$$= \lim_{x\to0} \frac{(e^{5\sin x} - 1) - (e^{2x} - 1)}{5\tan x - 3x}$$

$$= \lim_{x\to0} \left[\frac{\frac{(e^{5\sin x} - 1) - (e^{2x} - 1)}{x}}{\frac{5\tan x - 3x}{x}}\right]$$

…$\left[\begin{array}{l}\text{Divide numerator and denominator by } x \\ \because x \to 0, \therefore x \neq 0\end{array}\right]$

$$= \frac{\lim_{x\to0}\left(\frac{e^{5\sin x} - 1}{x} - \frac{e^{2x} - 1}{x}\right)}{\lim_{x\to0}\left(\frac{5\tan x}{x} - 3\right)}$$

$$= \frac{\lim_{x\to0}\left(\frac{e^{5\sin x} - 1}{5\sin x} \cdot \frac{5\sin x}{x}\right) - \lim_{x\to0}\left(\frac{e^{2x} - 1}{2x} \times 2\right)}{\lim_{x\to0}\frac{5\tan x}{x} - \lim_{x\to0} 3}$$

$$= \frac{5\lim_{x\to0}\left(\frac{e^{5\sin x} - 1}{5\sin x}\right) \cdot \lim_{x\to0}\left(\frac{\sin x}{x}\right) - 2\lim_{x\to0}\frac{e^{2x} - 1}{2x}}{5\lim_{x\to0}\frac{\tan x}{x} - \lim_{x\to0}(3)}$$

$$= \frac{5(1)(1) - 2(1)}{5(1) - 3}$$

…$\left[\begin{array}{l}\because x \to 0,\ 2x \to 0,\ \sin x \to 0,\ 5\sin x \to 0 \text{ and} \\ \lim_{x\to0}\left(\frac{e^x - 1}{x}\right) = 1,\ \lim_{x\to0}\frac{\sin x}{x} = 1\end{array}\right]$

$$= \frac{3}{2}$$

∴ $\lim_{x\to0} f(x) \neq f(0)$

∴ $f(x)$ is continuous at $x = 0$.

∴ $f(x)$ has a removable discontinuity at $x = 0$

This discontinuity can be removed by redefining $f(0) = \frac{3}{2}$.

$\therefore$ f(x) can be redefined as

$$f(x) = \frac{e^{5\sin x} - e^{2x}}{5\tan x - 3x}, \quad x \neq 0$$
$$= \frac{3}{2}, \quad x = 0$$

---

ii. $f(x) = \log_{(1+3x)}(1 + 5x)$, $x > 0$

$$= \frac{32^x - 1}{8^x - 1}, \quad x < 0$$

Here, f(0) is not defined.

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \log_{(1+3x)}(1 + 5x)$$
$$= \lim_{x\to 0^+} \frac{\log(1+5x)}{\log(1+3x)}$$
$$= \lim_{x\to 0^+} \left[\frac{\frac{\log(1+5x)}{x}}{\frac{\log(1+3x)}{x}}\right]$$
$$= \frac{\lim_{x\to 0^+} \frac{\log(1+5x)}{5x} \times 5}{\lim_{x\to 0^+} \frac{\log(1+3x)}{3x} \times 3}$$
$$= \frac{1\times 5}{1\times 3} \quad \ldots\left[\begin{array}{l}\because x\to 0, 3x\to 0, 5x\to 0 \\ \lim_{x\to 0} \frac{\log(1+x)}{x} = 1\end{array}\right]$$
$$= \frac{5}{3}$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{32^x - 1}{8^x - 1}$$
$$= \lim_{x\to 0^-} \left[\frac{\frac{32^x - 1}{x}}{\frac{8^x - 1}{x}}\right] \ldots[\because x \to 0, \therefore x \neq 0]$$
$$= \frac{\lim_{x\to 0^-} \frac{32^x - 1}{x}}{\lim_{x\to 0^-} \frac{8^x - 1}{x}}$$
$$= \frac{\log 32}{\log 8} \quad \ldots\left[\because \lim_{x\to 0} \frac{a^x - 1}{x} = \log a\right]$$
$$= \frac{\log(2)^5}{\log(2)^3}$$
$$= \frac{5\log 2}{3\log 2}$$
$$= \frac{5}{3}$$

$\therefore$ $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x)$

$\therefore$ $\lim_{x\to 0} f(x)$ exists

But f(0) is not defined.

$\therefore$ f(x) has a removable discontinuity at x = 0.

This discontinuity can be removed by defining $f(0) = \frac{5}{3}$

$\therefore$ f(x) can be redefined as

$$f(x) = \log_{(1+3x)}(1 + 5x) \quad ; x > 0$$
$$= \frac{5}{3} \quad ; x = 0$$
$$= \frac{32^x - 1}{8^x - 1} \quad ; x < 0$$

---

iii. $f(x) = \left(\frac{3 - 8x}{3 - 2x}\right)^{\frac{1}{x}}$; $x \neq 0$

Here, f(0) is not defined.

Consider, $\lim_{x\to 0} f(x) = \lim_{x\to 0}\left(\frac{3 - 8x}{3 - 2x}\right)^{\frac{1}{x}}$

$$= \lim_{x\to 0}\left[\frac{3\left(1 - \frac{8x}{3}\right)}{3\left(1 - \frac{2x}{3}\right)}\right]^{\frac{1}{x}}$$
$$= \lim_{x\to 0}\frac{\left(1 - \frac{8x}{3}\right)^{\frac{1}{x}}}{\left(1 - \frac{2x}{3}\right)^{\frac{1}{x}}}$$
$$= \frac{\lim_{x\to 0}\left[\left(1 - \frac{8x}{3}\right)^{\frac{-3}{8x}}\right]^{\frac{-8}{3}}}{\lim_{x\to 0}\left[\left(1 - \frac{2x}{3}\right)^{\frac{-3}{2x}}\right]^{\frac{-2}{3}}}$$
$$= \frac{e^{\frac{-8}{3}}}{e^{\frac{-2}{3}}} \quad \ldots\left[\begin{array}{l}\because x\to 0, \frac{-8x}{3} \to 0, \frac{-2x}{3} \to 0 \\ \text{and } \lim_{x\to 0}(1+x)^{\frac{1}{x}} = e\end{array}\right]$$
$$= e^{\frac{-6}{3}} = e^{-2}$$

$\therefore$ $\lim_{x\to 0} f(x)$ exists

But f(0) is not defined.

$\therefore$ f(x) has a removable discontinuity at x = 0.

This discontinuity can be removed by defining $f(0) = e^{-2}$.

$\therefore$ f(x) can be redefined as

$$f(x) = \left(\frac{3 - 8x}{3 - 2x}\right)^{\frac{1}{x}} \quad ; x \neq 0$$
$$= e^{-2} \quad ; x = 0$$

---

iv. $f(x) = 3x + 2$, for $-4 \leq x \leq -2$

$= 2x - 3$, for $-2 < x \leq 6$.

$$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} (3x + 2)$$
$$= 3(-2) + 2 = -4$$

$$\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} (2x-3)$$
$$= 2(-2) - 3$$
$$= -7$$

$\therefore \lim_{x\to -2^-} f(x) \neq \lim_{x\to -2^+} f(x)$

$\therefore \lim_{x\to -2^-} f(x)$ does not exist

$\therefore$ f(x) is discontinuous at $x = -2$.
This discontinuity is irremovable.

---

v. f(2) = 3 …(Given)

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{x^3-8}{x^2-4} = \lim_{x\to 2^-} \frac{x^3-2^3}{x^2-2^2}$$
$$= \lim_{x\to 2^-} \frac{(x-2)(x^2+2x+4)}{(x-2)(x+2)}$$
$$= \lim_{x\to 2^-} \frac{x^2+2x+4}{x+2}$$
$$= \frac{\lim_{x\to 2^-}(x^2+2x+4)}{\lim_{x\to 2^-}(x+2)}$$
$$= \frac{(2)^2+2(2)+4}{2+2} = \frac{12}{4}$$
$$= 3$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{e^{3(x-2)^2}-1}{2(x-2)^2}$$

Put $x - 2 = h$

$\therefore$ $x = 2 + h$

As $x \to 2$, $h \to 0$

$$\therefore \lim_{x\to 2^+} f(x) = \lim_{h\to 0} \frac{e^{3h^2}-1}{2h^2}$$
$$= \frac{1}{2}\lim_{h\to 0} \frac{e^{3h^2}-1}{3h^2} \times 3$$
$$= \frac{1}{2} \times 1 \times 3 \quad \ldots\left[\because h\to 0 \ \therefore h^2\to 0 \text{ and } \lim_{x\to 0}\frac{e^x-1}{x}=1\right]$$
$$= \frac{3}{2}$$

$\therefore \lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$

$\therefore \lim_{x\to 2} f(x)$ does not exist

$\therefore$ f(x) is discontinuous at $x = 2$
This discontinuity is irremovable.

---

10. i. If $f(x) = \dfrac{\sqrt{2+\sin x}-\sqrt{3}}{\cos^2 x}$, for $x \neq \dfrac{\pi}{2}$, Is continuous at $x = \dfrac{\pi}{2}$ then find $f\left(\dfrac{\pi}{2}\right)$.

ii. If $f(x) = \dfrac{\cos^2 x - \sin^2 x - 1}{\sqrt{3x^2+1}-1}$ for $x \neq 0$, is continuous at $x = 0$ then find f(0).

iii. If $f(x) = \dfrac{4^{x-\pi}+4^{\pi-x}-2}{(x-\pi)^2}$ for $x \neq \pi$, is continuous at $x = \pi$, then find f(π).

Solution :

i. f(x) is continuous at $x = \dfrac{\pi}{2}$ …(given)

$$\therefore f\left(\frac{\pi}{2}\right) = \lim_{x\to\frac{\pi}{2}} f(x)$$
$$= \lim_{x\to\frac{\pi}{2}} \frac{\sqrt{2+\sin x}-\sqrt{3}}{\cos^2 x}$$
$$= \lim_{x\to\frac{\pi}{2}} \left[\frac{\sqrt{2+\sin x}-\sqrt{3}}{1-\sin^2 x} \times \frac{\sqrt{2+\sin x}+\sqrt{3}}{\sqrt{2+\sin x}+\sqrt{3}}\right]$$
$$= \lim_{x\to\frac{\pi}{2}} \frac{2+\sin x-3}{(1-\sin x)(1+\sin x)\left(\sqrt{2+\sin x}+\sqrt{3}\right)}$$
$$= \lim_{x\to\frac{\pi}{2}} \frac{\sin x-1}{-(\sin x-1)(1+\sin x)\left(\sqrt{2\sin x}+\sqrt{3}\right)}$$
$$= \lim_{x\to\frac{\pi}{2}} \frac{1}{-(1+\sin x)\left(\sqrt{2+\sin x}+\sqrt{3}\right)}$$
$$\ldots\left[\because x\to\frac{\pi}{2}, \ \therefore \sin x\to 1 \quad \therefore \sin x \neq 1 \ \therefore \sin x - 1 \neq 0\right]$$
$$= \frac{-1}{\lim_{x\to\frac{\pi}{2}}(1+\sin x)\left(\sqrt{2+\sin x}+\sqrt{3}\right)}$$
$$= \frac{-1}{\lim_{x\to\frac{\pi}{2}}(1+\sin x)\cdot\lim_{x\to\frac{\pi}{2}}\left(\sqrt{2+\sin x}+\sqrt{3}\right)}$$
$$= \frac{-1}{(1+1)\left(\sqrt{2+1}+\sqrt{3}\right)} = \frac{-1}{2\times 2\sqrt{3}}$$

$$\therefore f\left(\frac{\pi}{2}\right) = \frac{-1}{4\sqrt{3}}$$

---

ii. f(x) is continuous at $x = 0$ …(given)

$$\therefore f(0) = \lim_{x\to 0} f(x)$$
$$= \lim_{x\to 0} \frac{\cos^2 x - \sin^2 x - 1}{\sqrt{3x^2+1}-1}$$
$$= \lim_{x\to 0} \frac{\cos 2x - 1}{\sqrt{3x^2+1}-1} \times \frac{\sqrt{3x^2+1}+1}{\sqrt{3x^2+1}+1}$$
$$= \lim_{x\to 0} \frac{-(1-\cos 2x)\left(\sqrt{3x^2+1}+1\right)}{(3x^2+1)-1}$$
$$= \lim_{x\to 0} \frac{-2\sin^2 x\cdot\left(\sqrt{3x^2+1}+1\right)}{3x^2}$$
$$= \frac{-2}{3}\lim_{x\to 0} \frac{\sin^2 x}{x^2}\left(\sqrt{3x^2+1}+1\right)$$

$$= \frac{-2}{3} \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^2 \times \lim_{x\to 0} \left(\sqrt{3x^2+1}+1\right)$$

$$= \frac{-2}{3} (1)^2 \times \left(\sqrt{3(0)+1}+1\right)$$

$$= \frac{-2}{3} \times (1+1)$$

$\therefore$ $f(0) = \frac{-4}{3}$

---

iii. $f(x)$ is continuous at $x = \pi$ …(given)

$\therefore$ $f(\pi) = \lim_{x\to\pi} \frac{4^{x-\pi} + 4^{\pi-x} - 2}{(x-\pi)^2}$

Put $x - \pi = h$

As $x \to \pi$, $h \to 0$

$\therefore$ $f(\pi) = \lim_{h\to 0} \frac{4^h + 4^{-h} - 2}{h^2}$

$$= \lim_{h\to 0} \frac{4^h + \frac{1}{4^h} - 2}{h^2}$$

$$= \lim_{h\to 0} \frac{(4^h)^2 + 1 - 2(4^h)}{4^h \cdot (h^2)}$$

$$= \lim_{h\to 0} \frac{(4^h-1)^2}{4^h \cdot h^2}$$

…[$\because a^2 - 2ab + b^2 = (a-b)^2$]

$$= \lim_{h\to 0} \left(\frac{4^h-1}{h}\right)^2 \times \frac{1}{4^h}$$

$$= \lim_{h\to 0} \left(\frac{4^h-1}{h}\right)^2 \times \lim_{h\to 0} \frac{1}{4^h}$$

$$= (\log 4)^2 \times \frac{1}{4^0}$$

$$= (\log 2^2)^2 \times \frac{1}{1}$$

$$= (2 \log 2)^2$$

$\therefore$ $f(\pi) = 4(\log 2)^2$

---

11. i. If $f(x) = \frac{24^x - 8^x - 3^x + 1}{12^x - 4^x - 3^x + 1}$, for $x \neq 0$

$= k$, for $x = 0$

is continuous at $x = 0$, find k.

ii. If $f(x) = \frac{5^x + 5^{-x} - 2}{x^2}$, for $x \neq 0$

$= k$, for $x = 0$

is continuous at $x = 0$, find k.

iii. If $f(x) = \frac{\sin 2x}{5x} - a$, for $x > 0$

$= 4$ for $x = 0$

$= x^2 + b - 3$, for $x < 0$

is continuous at $x = 0$, find a and b.

iv. For what values of a and b is the function

$f(x) = ax + 2b + 18$, for $x \leq 0$

$= x^2 + 3a - b$, for $0 < x \leq 2$

$= 8x - 2$, for $x > 2$,

continuous for every x?

v. For what values of a and b is the function

$f(x) = \frac{x^2-4}{x-2}$, for $x < 2$

$= ax^2 - bx + 3$, for $2 \leq x < 3$

$= 2x - a + b$, for $x \geq 3$

continuous for every x on R ?

Solution :

i. $f(x)$ is continuous at $x = 0$

$\therefore$ $f(0) = \lim_{x\to 0} f(x)$

$\therefore$ $k = \lim_{x\to 0} \frac{24^x - 8^x - 3^x + 1}{12^x - 4^x - 3^x + 1}$

$$= \lim_{x\to 0} \frac{8^x \cdot 3^x - 8^x - 3^x + 1}{4^x \cdot 3^x - 4^x - 3^x + 1}$$

$$= \lim_{x\to 0} \frac{8^x(3^x-1) - 1(3^x-1)}{4^x(3^x-1) - 1(3^x-1)}$$

$$= \lim_{x\to 0} \frac{(3^x-1)(8^x-1)}{(3^x-1)(4^x-1)}$$ …$\left[\begin{array}{l}\because x\to 0,\ 3x \to 3^o \\ \therefore 3^x \to 1 \therefore 3^x \neq 1 \\ \therefore 3^x - 1 \neq 0\end{array}\right]$

$$= \lim_{x\to 0} \frac{8^x-1}{4^x-1}$$

$$= \lim_{x\to 0} \left(\frac{\frac{8^x-1}{x}}{\frac{4^x-1}{x}}\right)$$ …[$\because x \to 0, \therefore x \neq 0$]

$$= \frac{\lim_{x\to 0} \frac{8^x-1}{x}}{\lim_{x\to 0} \frac{4^x-1}{x}}$$

$$= \frac{\log 8}{\log 4}$$ …$\left[\because \lim_{x\to 0}\left(\frac{a^x-1}{x}\right) = \log a\right]$

$$= \frac{\log (2)^3}{\log (2)^2}$$

$$= \frac{3 \log 2}{2 \log 2}$$

$\therefore$ $f(0) = \frac{3}{2}$

---

ii. $f(x)$ is continuous at $x = 0$

$\therefore$ $f(0) = \lim_{x\to 0} f(x)$

$$= \lim_{x\to 0} \frac{5^x + 5^{-x} - 2}{x^2}$$

$= \lim_{x\to 0} \dfrac{5^x + \frac{1}{5^x} - 2}{x^2}$

$= \lim_{x\to 0} \dfrac{\left(5^x\right)^2 + 1 - 2\left(5^x\right)}{5^x \cdot x^2}$

$= \lim_{x\to 0} \dfrac{\left(5^x - 1\right)^2}{5^x \cdot x^2}$

…[$\because$ $a^2 - 2ab + b^2 = (a - b)^2$]

$= \lim_{x\to 0} \left(\dfrac{5^x - 1}{x}\right)^2 \cdot \dfrac{1}{5^x}$

$= \lim_{x\to 0} \left(\dfrac{5^x - 1}{x}\right)^2 \times \lim_{x\to 0} \dfrac{1}{5^x}$

$= (\log 5)^2 \times \dfrac{1}{5^o}$

$= (\log 5)^2$

---

iii. f(x) is continuous at $x = 0$

$\therefore$ $\lim_{x\to 0^+} f(x) = f(0)$

$\therefore$ $\lim_{x\to 0^+} \left(\dfrac{\sin 2x}{5x} - a\right) = 4$

$\therefore$ $\lim_{x\to 0^+} \dfrac{\sin 2x}{5x} - \lim_{x\to 0^+} a = 4$

$\therefore$ $\dfrac{1}{5} \lim_{x\to 0^+} \dfrac{\sin 2x}{2x} \times (2) - \lim_{x\to 0^+} a = 4$

$\therefore$ $\dfrac{1}{5}$ (1) (2) − a = 4 … $\left[\begin{array}{l}\because x \to 0, 2x \to 0 \\ \lim_{x\to 0^+} \dfrac{\sin\theta}{\theta} = 1\end{array}\right]$

$\therefore$ $\dfrac{2}{5} - a = 4$

$\therefore$ $\dfrac{2}{5} - 4 = a$

$\therefore$ $a = -\dfrac{18}{5}$

Also, $\lim_{x\to 0^+} f(x) = f(0)$

$\therefore$ $\lim_{x\to 0^+} (x^2 + b - 3) = 4$

$\therefore$ b − 3 = 4

$\therefore$ b = 7

---

iv. f(x) is continuous for every x.

$\therefore$ f(x) is continuous at $x = 0$ and $x = 2$

As f(x) is continuous at $x = 0$.

$\therefore$ $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$

$\therefore$ $\lim_{x\to 0} (ax + 2b + 18) = \lim_{x\to 0} (x^2 + 3a - b)$

$\therefore$ a (0) + 2b + 18 = $(0)^2$ + 3a − b

$\therefore$ 3a − 3b = 18

$\therefore$ a − b = 6 …(i)

$\because$ f(x) is continous at $x = 2$

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$

$\therefore$ $\lim_{x\to 2} (x^2 + 3a - b) = \lim_{x\to 2} (8x - 2)$

$\therefore$ $(2)^2$ + 3a − b = 8 (2) − 2

$\therefore$ 4 + 3a − b = 14

$\therefore$ 3a − b = 10 …(ii)

Subtracting (i) from (ii), we get

2a = 4

$\therefore$ a = 2

Substituting a = 2 in (i), we get

2 − b = 6

$\therefore$ b = − 4

$\therefore$ a = 2 and b = − 4

---

v. f(x) is continuous for every x on R.

$\therefore$ f(x) is continuous at $x = 2$ and $x = 3$.

f(x) is continuous at $x = 2$.

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$

$\therefore$ $\lim_{x\to 2} \dfrac{x^2 - 4}{x - 2} = \lim_{x\to 2} (ax^2 - bx + 3)$

$\therefore$ $\lim_{x\to 2} \dfrac{(x-2)(x+2)}{x - 2} = \lim_{x\to 2} (ax^2 - bx + 3)$

$\therefore$ $\lim_{x\to 2} (x + 2) = \lim_{x\to 2} (ax^2 - bx + 3)$ … $\left[\begin{array}{l}\because x \to 2 \therefore x \neq 2 \\ \therefore\ x - 2 \neq 0\end{array}\right]$

$\therefore$ 2 + 2 = a$(2)^2$ − b(2) + 3

$\therefore$ 4a − 2b + 3 = 4

$\therefore$ 4a − 2b = 1 …(i)

Also f(x) is continuous at $x = 3$

$\therefore$ $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$

$\therefore$ $\lim_{x\to 3} (ax^2 - bx + 3) = \lim_{x\to 3} (2x - a + b)$

$\therefore$ a$(3)^2$ − b(3) + 3 = 2(3) − a + b

$\therefore$ 9a − 3b + 3 = 6 − a + b

$\therefore$ 10a − 4b = 3 …(ii)

Multiply (i) by 2 and (ii) by 1, we get

8a − 4b = 2 …(iii)

10a − 4b = 3 …(iv)

Subtracting (iv) from (iii)

− 2a = − 1

$\therefore$ $a = \dfrac{1}{2}$

Substituting $a = \dfrac{1}{2}$ in (i), we get

$4\left(\dfrac{1}{2}\right) - 2b = 1$

$\therefore$ 2 − 2b = 1

$\therefore$ 1 = 2b

$\therefore$ $b = \dfrac{1}{2}$

$\therefore$ $a = \dfrac{1}{2}$ and $b = \dfrac{1}{2}$

12. Discuss the continuity of f on its domain, where

$f(x) = |x+1|$, for $-3 \le x \le 2$

$= |x-5|$, for $2 \le x \le 7$

Solution:

$|x+1| = x+1 \quad ; \quad x \ge -1$

$= -(x+1) \quad ; \quad x < -1$

$|x-5| = x-5 \quad ; \quad x \ge 5$

$= -(x-5) \quad ; \quad x < 5$

$\lim\limits_{x\to 2^-} f(x) = \lim\limits_{\substack{x\to 2\\ x<2}} |x+1|$

$= \lim\limits_{x\to 2}(x+1)$

$= 2 + 1$

$= 3$

$\lim\limits_{x\to 2^+} f(x) = \lim\limits_{\substack{x\to 2\\ x>2}} |x-5|$

$= \lim\limits_{x\to 2} -(x-5)$

$= -(2-5)$

$= 3$

$f(2) = |2+1|$

$= 3$

$\therefore \lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^+} f(x) = f(2)$

$\therefore$ $f(x)$ is continuous at $x = 2$

---

13. Discuss the continuity of $f(x)$ at $x = \frac{\pi}{4}$ where,

$$f(x) = \frac{(\sin x + \cos x)^3 - 2\sqrt{2}}{\sin 2x - 1}, \text{ for } x \ne \frac{\pi}{4}$$

$$= \frac{3}{\sqrt{2}}, \qquad \text{for } x = \frac{\pi}{4}$$

Solution:

$f\left(\frac{\pi}{4}\right) = \frac{3}{\sqrt{2}}$

$\lim\limits_{x\to \frac{\pi}{4}} f(x) = \lim\limits_{x\to \frac{\pi}{4}} \frac{(\sin x + \cos x)^3 - 2\sqrt{2}}{\sin 2x - 1}$

$(\sin x + \cos x)^3 = \left[(\sin x + \cos x)^2\right]^{\frac{3}{2}}$

$= (1 + \sin 2x)^{\frac{3}{2}}$

$\therefore \lim\limits_{x\to \frac{\pi}{4}} f(x) = \lim\limits_{x\to \frac{\pi}{4}} \frac{(1+\sin 2x)^{\frac{3}{2}} - 2^{\frac{3}{2}}}{\sin 2x - 1}$

Put $1 + \sin 2x = t$

$\therefore \sin 2x = t - 1$

As $x \to \frac{\pi}{4}$, $t \to 1 + \sin 2\left(\frac{\pi}{4}\right)$

i.e. $t \to 1 + \sin\frac{\pi}{2}$

i.e. $t \to 1 + 1$

i.e. $t \to 2$

$\therefore \lim\limits_{x\to \frac{\pi}{4}} f(x) = \lim\limits_{t\to 2} \frac{t^{\frac{3}{2}} - 2^{\frac{3}{2}}}{t - 1 - 1}$

$= \lim\limits_{t\to 2} \frac{t^{\frac{3}{2}} - 2^{\frac{3}{2}}}{t - 2}$

$= \frac{3}{2}(2)^{\frac{1}{2}} \quad \ldots\left[\lim\limits_{x\to a} \frac{x^n - a^n}{x - a} = na^{n-1}\right]$

$= \frac{3\sqrt{2}}{2} = \frac{3}{\sqrt{2}}$

$\therefore \lim\limits_{x\to \frac{\pi}{4}} f(x) = f\left(\frac{\pi}{4}\right)$

$\therefore$ $f(x)$ is continuous at $x = \frac{\pi}{4}$

---

14. Determine the values of p and q such that the following function is continuous on the entire real number line.

$f(x) = x + 1$, for $1 < x < 3$

$= x^2 + px + q$, for $|x-2| \ge 1$.

Solution:

$|x-2| \ge 1$

$\therefore x - 2 \ge 1$ or $x - 2 \le -1$

$\therefore x \ge 3$ or $x \le 1$

$\therefore f(x) = x^2 + px + q$ for $x \ge 3$ as well as $x \le 1$

Thus $f(x) = x^2 + px + q \quad ; \quad x \le 1$

$= x + 1 \quad ; \quad 1 < x < 3$

$= x^2 + px + q \quad ; \quad x \ge 3$

$f(x)$ is continuous for all $x \in R$

$\therefore$ $f(x)$ is continuous at $x = 1$ and $x = 3$

As $f(x)$ is continuous at $x = 1$

$\therefore \lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^+} f(x)$

$\therefore \lim\limits_{x\to 1^-} (x^2 + px + q) = \lim\limits_{x\to 1^+} (x+1)$

$\therefore (1)^2 + p(1) + q = 1 + 1$

$\therefore 1 + p + q = 2$

$\therefore p + q = 1 \quad \ldots(i)$

Also $f(x)$ is continuous at $x = 3$

$\therefore \lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^+} f(x)$

$\therefore \lim\limits_{x\to 3^-} (x+1) = \lim\limits_{x\to 3^+} (x^2 + px + q)$

$\therefore 3 + 1 = (3)^2 + 3p + q$

$\therefore 3p + q + 9 = 4$

$\therefore 3p + q = -5 \quad \ldots(ii)$

Subtracting (i) from (ii), we get

$2p = -6$

$\therefore p = -3$

Substituting $p = -3$ in (i), we get

$-3 + q = 1$

$\therefore q = 4$

$\therefore p = -3$ and $q = 4$

15. Show that there is a root for the equation $2x^2 - x - 16 = 0$ between 2 and 3.

Solution:

Let $f(x) = 2x^2 - x - 16$

$f(x)$ is a polynomial function and hence it is continuous for all $x \in R$

A root of $f(x)$ exists if $f(x) = 0$ for at least one value of x

$f(2) = 2(2)^3 - 2 - 16$
$= -2 < 0$

$f(3) = 2(3)^3 - 3 - 16$
$= 35 > 0$

$\therefore$ $f(2) < 0$ and $f(3) > 0$

$\therefore$ By intermediate value theorem, there has to be point 'c' between 2 and 3 such that f(c) = 0

$\therefore$ There is a root of the given equation between 2 and 3.

16. Show that there is a root for the equation $x^3 - 3x = 0$ between 1 and 2.

Solution:

Let $f(x) = x^3 - 3x$

$f(x)$ is a polynomial function and hence it is continuous for all $x \in R$

A root of $f(x)$ exists if $f(x) = 0$ for at least one value of x

$f(1) = (1)^3 - 3(1)$
$= -2 < 0$

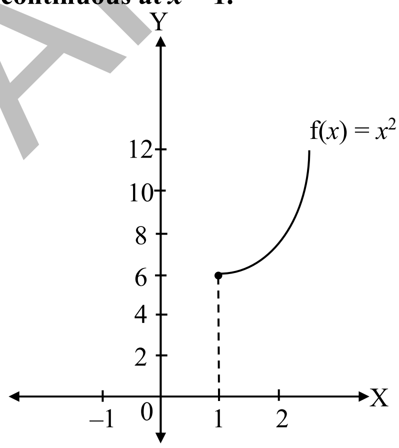
$f(2) = (2)^3 - 3(2)$
$= 2 > 0$

$\therefore$ $f(1) < 0$ and $f(2) > 0$

$\therefore$ By intermediate value theorem, there has to be point 'c' between 1 and 2
Such that f(c) = 0

$\therefore$ There is a root of the given equation between 1 and 2.

17. Activity: Let $f(x) = ax + b$ (where a and b are unknown)
$= x^2 + 5$ for $x \geq 1$
Find the values of a and b, so that $f(x)$ is continuous at $x = 1$.



Solution:

$f(x) = ax + b \qquad x < 1$
$= x^2 + 5 \qquad x \geq 1$

$f(x) = x^2 + 5$

$\therefore$ $f(1) = 1 + 5 = 6$

L.H.L. $= \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (ax + b) = a + b$

R.H.L. $= \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x^2 + 5) = 1 + 5 = 6$

given, $f(x)$ is continuous at n = 1

$\therefore$ L.H.L. = R.H.L.

$\therefore$ a + b = 6 where, a, b $\in$ R

18. Suppose $f(x) = px + 3$ for $a \leq x \leq b$
$= 5x^2 - q$ for $b < x \leq c$
Find the condition on p, q, so that $f(x)$ is continuous on [a, c], by filling in the boxes.

Solution:

$f(b) = \boxed{\mathbf{pb + 3}}$

$\lim_{x \to b^+} f(x) = \boxed{\mathbf{5b^2}} - q$

$\therefore$ $pb + 3 = \boxed{\mathbf{5b^2}} - q$

$\therefore$ $p = \dfrac{\boxed{\mathbf{5b^2 - q - 3}}}{b}$

Miscellaneous Exercise – 8

I. Select the correct answer from the given alternatives.

1. $f(x) = \dfrac{2^{\cot x} - 1}{\pi - 2x}$, for $x \neq \dfrac{\pi}{2}$
$= \log \sqrt{2}$, for $x = \dfrac{\pi}{2}$

(A) f is continuous at $x = \dfrac{\pi}{2}$

(B) f has a jump discontinuity at $x = \dfrac{\pi}{2}$

(C) f has a removable discontinuity

(D) $\lim_{x \to \frac{\pi}{2}} f(x) = 2\log 3$

2. If $f(x) = \dfrac{1 - \sqrt{2}\sin x}{\pi - 4x}$, for $x \neq \dfrac{\pi}{2}$ is continuous at $x = \dfrac{\pi}{4}$, then $\left(\dfrac{\pi}{4}\right) =$

(A) $\dfrac{1}{\sqrt{2}}$ (B) $-\dfrac{1}{\sqrt{2}}$

(C) $-\dfrac{1}{4}$ (D) $\dfrac{1}{4}$

3. If $f(x) = \frac{(\sin 2x)\tan 5x}{(e^{2x}-1)^2}$, for $x \neq 0$ is continuous at $x = 0$, then f(0) is

(A) $\frac{10}{e^2}$ (B) $\frac{10}{e^4}$

(C) $\frac{5}{4}$ (D) $\frac{5}{2}$

4. $f(x) = \frac{x^2-7x+10}{x^2+2x-8}$, for $x \in [-6, -3]$

(A) f is discontinuous at $x = 2$.
(B) f is discontinuous at $x = -4$.
(C) f is discontinuous at $x = 0$.
(D) f is discontinuous at $x = 2$ and $x = -4$.

5. If $f(x) = ax^2 + bx + 1$, for $|x-1| \geq 3$ and
$= 4x + 5$, for $-2 < x < 4$ is
continuous everywhere then,

(A) $a = -\frac{1}{2}, b = 5$ (B) $a = -\frac{1}{2}, b = -5$

(C) $a = \frac{1}{2}, b = -5$ (D) $a = \frac{1}{2}, b = 3$

[Note: *The option has been modified.***]**

6. $f(x) = \frac{(16^x-1)(9^x-1)}{(27^x-1)(32^x-1)}$, for $x \neq 0$
$= k$, for $x = 0$
is continuous at $x = 0$, then 'k' =

(A) $\frac{8}{3}$ (B) $\frac{8}{15}$

(C) $-\frac{8}{15}$ (D) $\frac{20}{3}$

7. $f(x) = \frac{32^x-8^x-4^x+1}{4^x-2^{x+1}+1}$, for $x \neq 0$ is continuous at $x = 0$, then value of 'k' is

(A) 6 (B) 4
(C) (log 2)(log 4) (D) 3 log 4

8. If $f(x) = \frac{12^x-4^x-3^x+1}{1-\cos 2x}$, for $x \neq 0$ is continuous at $x = 0$ then the value of f(0) is

(A) $\frac{\log 12}{2}$ (B) log2. log3

(C) $\frac{\log 2.\log 3}{2}$ (D) None of these

9. If $f(x) = \left(\frac{4+5x}{4-7x}\right)^{\frac{4}{x}}$, for $x \neq 0$ and $f(0) = k$, is continuos at $x = 0$, then k is

(A) e^7 (B) e^3

(C) e^{12} (D) $e^{\frac{3}{4}}$

10. If $f(x) = \lfloor x \rfloor$ for $x \in (-1, 2)$ then f is discontinuous at

(A) $x = -1, 0, 1, 2$ (B) $x = -1, 0, 1$
(C) $x = 0, 1$ (D) $x = 2$

Answers:

1.	(A)	2.	(D)	3.	(D)	4.	(B)
5.	(D)	6.	(B)	7.	(A)	8.	(B)
9.	(C)	10.	(C)				

Hints:

1. $f\left(\frac{\pi}{2}\right) = \log\sqrt{2}$

$$\lim_{x\to\frac{\pi}{2}} f(x) = \lim_{x\to\frac{\pi}{2}} \frac{2^{\cot x}-1}{\pi-2x}$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{2^{\tan\left(\frac{\pi}{2}-x\right)}-1}{2\left(\frac{\pi}{2}-x\right)}$$

Put $\frac{\pi}{2} - x = h$

As $x \to \frac{\pi}{2}$, $h \to 0$

$\therefore$ $$\lim_{x\to\frac{\pi}{2}} f(x) = \lim_{h\to 0} \frac{2^{\tan h}-1}{2h}$$

$$= \frac{1}{2}\lim_{h\to 0}\left(\frac{2^{\tan h}-1}{\tan h} \times \frac{\tan h}{h}\right)$$

...($\because$ h→0, $\therefore$ tan h→0 $\therefore$ tan h ≠ 0)

$$= \frac{1}{2}\lim_{h\to 0}\frac{2^{\tan h}-1}{\tan h} \times \lim_{h\to 0}\frac{\tan h}{h}$$

$$= \frac{1}{2}\cdot\log 2\cdot(1)$$

$$= \log\sqrt{2} = f\left(\frac{\pi}{2}\right)$$

$\therefore$ f(x) is continuous at $x = \frac{\pi}{2}$

2. f(x) is continuous at $x = \frac{\pi}{4}$

$\therefore$ $$f\left(\frac{\pi}{4}\right) = \lim_{x\to\frac{\pi}{4}} f(x)$$

$$= \lim_{x\to\frac{\pi}{4}} \frac{1-\sqrt{2}\sin x}{\pi-4x}$$

$$= \lim_{x\to\frac{\pi}{4}} \frac{\sqrt{2}\left(\sin x - \frac{1}{\sqrt{2}}\right)}{4\left(x-\frac{\pi}{4}\right)}$$

$$= \frac{\sqrt{2}}{4}\lim_{x\to\frac{\pi}{4}} \frac{\sin x - \sin\frac{\pi}{4}}{x-\frac{\pi}{4}}$$

$$= \frac{\sqrt{2}}{4} \lim_{x \to \frac{\pi}{4}} \frac{2\cos\left(\frac{x+\frac{\pi}{4}}{2}\right) \cdot \sin\left(\frac{x-\frac{\pi}{4}}{2}\right)}{x-\frac{\pi}{4}}$$

$$= \frac{\sqrt{2}}{4} \cdot \lim_{x \to \frac{\pi}{4}} \cos\left(\frac{x}{2}+\frac{\pi}{8}\right) \cdot \lim_{x \to \frac{\pi}{4}} \frac{\sin\left(\frac{x-\frac{\pi}{4}}{2}\right)}{\frac{x-\frac{\pi}{4}}{2}}$$

$$= \frac{\sqrt{2}}{4} \cdot \cos\left(\frac{x}{8}+\frac{\pi}{8}\right) \times 1$$

$$\ldots\left[\begin{array}{l} \because x \to \frac{\pi}{4}, x-\frac{\pi}{4} \to 0 \\ \therefore \frac{x-\frac{\pi}{4}}{2} \to 0 \text{ and } \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1 \end{array}\right]$$

$$= \frac{\sqrt{2}}{4} \times \cos\frac{\pi}{4}$$

$$= \frac{1}{4}$$

3. f(x) is continuous at $x = 0$

f(0) = $\lim_{x \to 0} f(x)$

$\therefore$ $= \lim_{x \to 0} \frac{(\sin 2x)(\tan 5x)}{\left(e^{2x}-1\right)^2}$

$$= \frac{\lim_{x \to 0} \frac{\sin 2x}{2x} \times \lim_{x \to 0} \frac{\tan 5x}{5x} \times 2 \times 5}{\left(\lim_{x \to 0} \frac{e^{2x}-1}{2x}\right) \times (2)^2}$$

$$= \frac{1 \times 1 \times 2 \times 5}{(1)^2 \times 4}$$

$$\ldots\left[\begin{array}{l} \because x \to 0,\ 2x \to 0,\ 5x \to 0 \\ \text{and } \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1,\ \lim_{\theta \to 0} \frac{\tan\theta}{\theta} = 1 \end{array}\right]$$

$$= \frac{5}{2}$$

4. f(x) = $\frac{x^2-7x+10}{x^2+2x-8}$; $x \in [-6, -3]$

$= \frac{x^2-7x+10}{(x+4)(x-2)}$

Here f(x) is a rational function and is continuous everywhere except at the points Where denominator becomes zero.

Here, denominator becomes zero when

$x = -4$ OR $x = 2$

But $x = 2$ does not lie in the given interval

$\therefore$ $x = -4$ is the point of discontinuity

5. f(x) = $ax^2 + bx + 1$, $|x-1| \ge 3$

$= 4x + 5$; $-2 < x < 4$

The first interval is

$|x-1| \ge 3$

$\therefore$ $x - 1 \ge 3$ OR $x - 1 \le -3$

$\therefore$ $x \ge 4$ OR $x \le -2$

$\therefore$ f(x) is same for $x \le -2$ as well as $x \ge 4$

$\therefore$ f(x) is defined as:

f(x) = $ax^2 + bx + 1$; $x \le -2$

$= 4x + 5$; $-2 < x < 4$

$= ax^2 + bx + 1$; $x \ge 4$

f(x) is continuous everywhere

$\therefore$ f(x) is continuous at $x = -2$ and $x = 4$

As f(x) is continuous at $x = -2$

$\lim_{x \to -2^-} f(x) = \lim_{x \to -2^+} f(x)$

$\therefore$ $\lim_{x \to -2} \left(ax^2 + bx + 1\right) = \lim_{x \to -2} (4x + 5)$

$\therefore$ $a(-2)^2 + b(-2) + 1 = 4(-2) + 5$

$\therefore$ $4a - 2b + 1 = -3$

$\therefore$ $4a - 2b = -4$

$\therefore$ $2a - b = -2$ …(i)

Also f(x) is continuous at $x = 4$

$\therefore$ $\lim_{x \to 4^-} f(x) = \lim_{x \to 4^+} f(x)$

$\therefore$ $\lim_{x \to 4} (4x + 5) = \lim_{x \to 4} \left(ax^2 + bx + 1\right)$

$\therefore$ $4(4) + 5 = (4)^2 + b(4) + 1$

$\therefore$ $16a + 4b + 1 = 21$

$\therefore$ $16a + 4b = 20$

$\therefore$ $4a + b = 5$ …(ii)

Adding (i) and (ii)

$6a = 3$

$\therefore$ $a = \frac{1}{2}$

Substitute $a = \frac{1}{2}$ in (ii)

$4\left(\frac{1}{2}\right) + b = 5$

$\therefore$ $2 + b = 5$

$\therefore$ $b = 3$

$\therefore$ $a = \frac{1}{2}$, $b = 3$

6. f(x) is continuous at $x = 0$

$\therefore$ f(0) = $\lim_{x \to 0} f(x)$

$\therefore$ $k = \lim_{x \to 0} \frac{\left(16^x-1\right)\left(9^x-1\right)}{\left(27^x-1\right)\left(32^x-1\right)}$

$$= \frac{\lim_{x \to 0}\left(\frac{16^x-1}{x}\right) \times \lim_{x \to 0}\left(\frac{9^x-1}{x}\right)}{\lim_{x \to 0}\left(\frac{27^x-1}{x}\right) \times \lim_{x \to 0}\left(\frac{32^x-1}{x}\right)}$$

$$= \frac{\log 16 \times \log 9}{\log 27 \times \log 32} \quad \ldots\left[\because \lim_{x \to 0} \frac{a^x - 1}{x} = \log a\right]$$

$$= \frac{4\log 2 \times 2\log 3}{3\log 3 \times 5\log 2}$$

$$= \frac{8}{15}$$

7. f(x) is continuous at $x = 0$

$\therefore$ $f(0) = \lim_{x \to 0} f(x)$

$\therefore$ $k = \lim_{x \to 0} \frac{32^x - 8^x - 4^x + 1}{4^x - 2^{x+1} + 1}$

$$= \lim_{x \to 0} \frac{(4^x - 1)(8^x - 1)}{(2^x - 1)^2}$$

$$= \frac{\lim_{x \to 0} \left(\frac{4^x - 1}{x}\right)\left(\frac{8^x - 1}{x}\right)}{\lim_{x \to 0} \left(\frac{2^x - 1}{x}\right)^2}$$

$$= \frac{\lim_{x \to 0} \left(\frac{4^x - 1}{x}\right) \cdot \lim_{x \to 0} \left(\frac{8^x - 1}{x}\right)}{\left(\lim_{x \to 0} \frac{2^x - 1}{x}\right)^2}$$

$$= \frac{\log 4 \times \log 8}{(\log 2)^2}$$

$$= \frac{2\log 2 \times 3\log 2}{(\log 2)^2} = 6$$

8. If f(x) is continuous at $x = 0$ (given)

$\therefore$ $f(0) = \lim_{x \to 0} f(x)$

$$= \lim_{x \to 0} \frac{12^x - 4^x - 3^x + 1}{1 - \cos 2x}$$

$$= \frac{1}{2} \lim_{x \to 0} \frac{4^x (3^x - 1)(3^x - 1)}{\sin^2 x}$$

$$= \frac{1}{2} \lim_{x \to 0} \frac{(3^x - 1)(4^x - 1)}{\sin^2 x}$$

$$= \frac{1}{2} \frac{\lim_{x \to 0} \left(\frac{3^x - 1}{x}\right) \cdot \lim_{x \to 0} \left(\frac{4^x - 1}{x}\right)}{\left(\lim_{x \to 0} \frac{\sin x}{x}\right)^2}$$

$$= \frac{1}{2} \times \frac{(\log 3) \times (\log 4)}{(1)^2}$$

$$= \frac{1}{2} \times \log 3 \times \log (2)^2$$

$= \log 3 . \log 2$

9. f(x) is continuous at $x = 0$

$\therefore$ $f(0) = \lim_{x \to 0} f(x)$

$$= \lim_{x \to 0} \left(\frac{4 + 5x}{4 - 7x}\right)^{\frac{4}{x}}$$

$$= \lim_{x \to 0} \left[\frac{4\left(1 + \frac{5x}{4}\right)}{4\left(1 - \frac{7x}{4}\right)}\right]^{\frac{4}{x}}$$

$$= \frac{\lim_{x \to 0} \left[\left(1 + \frac{5x}{4}\right)^{\frac{4}{5x}}\right]^5}{\lim_{x \to 0} \left[\left(1 - \frac{7x}{4}\right)^{\frac{-4}{7x}}\right]^{-7}}$$

$$= \frac{e^5}{e^{-7}} \quad \ldots \left[\begin{array}{l}\because x \to 0, \frac{5x}{4} \to 0, \frac{-7x}{4} \to 0 \\ \text{and } \lim_{x \to 0} (1 + x)^{\frac{1}{x}} = e\end{array}\right]$$

$= e^{12}$

10. $f(x) = \lfloor x \rfloor$, $x \in (-1, 2)$

This function is discontinuous at all integer values of x between –1 and 2.

$\therefore$ f(x) is discontinuous at $x = 0$ and $x = 1$.

II. Discuss the continuity of the following functions at the point (s) or no the interval indicated against them.

1. $f(x) = \frac{x^2 - 3x - 10}{x - 5}$, **for** $3 \le x \le 6$,

$= 10$, **for** $x = 5$

$= \frac{x^2 - 3x - 10}{x - 5}$, **for** $6 < x \le 9$

Solution:

$\frac{x^2 - 3x - 10}{x - 5}$ is not defined at $x = 5$

$\therefore$ $f(x) = \frac{x^2 - 3x - 10}{x - 5}$ where $x \in [3, 5) \cup (5, 6]$

We can write f(x) explicitly, as follows:

$f(x) = \frac{x^2 - 3x - 10}{x - 5}$, $3 \le x < 5$

$= 10$, $x = 5$

$= \frac{x^2 - 3x - 10}{x - 5}$, $5 < x \le 6$

$= \frac{x^2 - 3x - 10}{x - 5}$, $6 < x \le 9$

$\because$ $x^2 - 3x - 10 = (x - 5)(x + 2)$

$\therefore$ $f(x) = x + 2$, $3 < x < 5$

$= 10$, $x = 5$

$= x + 2$, $5 < x$

$f(5) = 10$

$\lim_{x \to 5^-} f(x) = \lim_{x \to 5^-} (x + 2) = 5 + 2 = 7$

$\lim_{x \to 5^+} f(x) = \lim_{x \to 5^+} (x + 2) = 5 + 2 = 7$

$\therefore$ $f(5) \ne \lim_{x \to 5} f(x)$

$\therefore$ f(x) is continuous on its domain except at $x = 5$

2. $f(x) = 2x^2 - 2x + 5$, **for** $0 \le x \le 2$

$= \dfrac{1-3x-x^2}{1-x}$, **for** $2 < x < 4$

$= \dfrac{x^2-25}{x-5}$, **for** $4 \le x$ **7 and** $x \ne 5$

$= 7$ **for** $x = 5$

Solution:

2. The domain of f(x) is [0, 7]

i. For $0 \le x < 2$

$f(x) = 2x^2 - 2x + 5$

It is a polynomial function and is Continuous at all point in [0, 2)

ii. For $2 < x < 4$

$f(x) = \dfrac{1-3x-x^2}{1-x}$

It is a rational function and is continuous everwhere except at points where its denominator becomes zero.

Denominator becomes zero at $x = 1$

But $x = 1$ does not lie in the interval.

f(x) is continuous at all points in (2, 4)

iii. For $4 < x \le 7, x \le 5$

$f(x) = \dfrac{x^2-25}{x-5}$

It is a rational function and is continuous everwhere except at points where its denominator becomes zero.

Denominator becomes zero at $x = 5$

But $x = 5$ does not lie in the interval.

$\therefore$ f(x) is continuous at all points in (4, 7] – {5}.

iv. For continuity at $x = 2$:

$$\lim_{x\to2^-} f(x) = \lim_{x\to2^-} f(x)\ (2x^2 - 2x + 5)$$
$$= 2(2)^2 - 2(2) + 5$$
$$= 8 - 4 + 5$$
$$= 9$$

$$\lim_{x\to2^+} f(x) = \lim_{x\to2^+} \frac{1-3x-x^2}{1-x}$$
$$= \frac{\lim_{x\to2^+}(1-3x-x^2)}{\lim_{x\to2^+}(1-x)}$$
$$= \frac{1-3(2)-(2)^2}{1-2}$$
$$= \frac{1-6-4}{-1}$$
$$= \frac{-9}{-1}$$
$$= 9$$

Also $f(2) = 2(2)^2 - 2(2) + 5$
$$= 8 - 4 + 5$$
$$= 9$$

$\therefore$ $\lim_{x\to2^-} f(x) = \lim_{x\to2^+} f(x) = f(2)$

$\therefore$ f(x) is continuous at $x = 2$

v. For continuity at $x = 4$:

$$\lim_{x\to4^-} f(x) = \lim_{x\to4^-} \frac{1-3x-x^2}{1-x}$$
$$= \frac{\lim_{x\to4^-}(1-3x-x^2)}{\lim_{x\to4^-}(1-x)}$$
$$= \frac{1-3(4)-(4)^2}{1-4}$$
$$= \frac{1-12-16}{1-4} = \frac{-27}{-3}$$
$$= 9$$

$$\lim_{x\to4^+} f(x) = \lim_{x\to4^+} \frac{x^2-25}{x-5}$$
$$= \frac{\lim_{x\to4^+}(x^2-25)}{\lim_{x\to4^+}(x-5)}$$
$$= \frac{(4)^2-25}{4-5}$$
$$= \frac{16-25}{-1}$$
$$= 9$$

Also $f(4) = \dfrac{(4)^2-25}{4-5}$
$$= \frac{16-25}{-1}$$
$$= 9$$

$\therefore$ $\lim_{x\to4^-} f(x) = \lim_{x\to4^+} f(x) = f(4)$

$\therefore$ f(x) is continuous at $x = 4$

vi. For continuity at $x = 5$

f(5) = 7

$$\lim_{x\to5} f(x) = \lim_{x\to5} \frac{x^2-25}{x-5}$$
$$= \lim_{x\to5} \frac{(x-5)(x+5)}{x-5}$$
$$= \lim_{x\to5}(x+5) \quad \ldots \begin{pmatrix} \text{As } x \to 5, x \ne 5 \\ \therefore\ x-5 \ne 0 \end{pmatrix}$$
$$= 5 + 5$$
$$= 10$$

$\therefore$ $\lim_{x\to5} f(x) \ne f(5)$

$\therefore$ f(x) is discontinuous at $x = 5$

Thus f(x) is continuous at all points on its domain except at $x = 5$

3. $\mathbf{f(x) = \dfrac{\cos 4x - \cos 9x}{1 - \cos x}}$ **, for** $\mathbf{x \neq 0}$

$\mathbf{f(0) = \dfrac{68}{15}}$ **, at** $\mathbf{x = 0}$ **on** $\mathbf{-\dfrac{\pi}{2} \leq x \leq \dfrac{\pi}{2}}$

Solution:

The domain of f(x) is $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

i. For $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right] - \{0\}$

$f(x) = \dfrac{\cos 4x - \cos 9x}{1 - \cos x}$

It is a rational function and is continuous everwhere except at points where its denominator becomes zero.

Denominator becomes zero when $\cos x = 1$

i.e. $x = 0$

But $x = 0$ does not lie in the interval

∴ f(x) is continuous at all points in $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right] - \{0\}$

---

ii. For continuity at $x = 0$

$f(0) = \dfrac{68}{15}$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\cos 4x - \cos 9x}{1 - \cos x}$$

$$= \lim_{x\to 0} \frac{2\sin\left(\frac{4x+9x}{2}\right)\cdot \sin\left(\frac{9x-4x}{2}\right)}{2\sin^2 \frac{x}{2}}$$

$$= \lim_{x\to 0} \frac{\sin\left(\frac{13x}{2}\right)\cdot \sin\left(\frac{5x}{2}\right)}{\left(\sin\frac{x}{2}\right)^2}$$

$$= \lim_{x\to 0}\left[\frac{\dfrac{\sin\left(\frac{13x}{2}\right)\cdot \sin\left(\frac{5x}{2}\right)}{x^2}}{\dfrac{\left(\sin\frac{x}{2}\right)^2}{x^2}}\right] \quad \ldots\left[\begin{array}{l}\text{Divide numerator and denominator by } x^2\\ \text{As } x \to 0, x \neq 0 \quad \therefore \quad x^2 \neq 0\end{array}\right]$$

$$= \frac{\lim\limits_{x\to 0}\dfrac{\sin\left(\frac{13x}{2}\right)}{x}\cdot\dfrac{\sin\left(\frac{5x}{2}\right)}{x}}{\lim\limits_{x\to 0}\left(\dfrac{\sin\frac{x}{2}}{x}\right)^2}$$

$$= \frac{\lim\limits_{x\to 0}\dfrac{\sin\left(\frac{13x}{2}\right)}{\frac{13x}{2}} \times \dfrac{13}{2}\cdot \lim\limits_{x\to 0}\dfrac{\sin\left(\frac{5x}{2}\right)}{\frac{5x}{2}} \times \dfrac{5}{2}}{\lim\limits_{x\to 0}\left(\dfrac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2 \times \dfrac{1}{4}}$$

$$= \frac{1 \times \frac{13}{2} \times 1 \times \frac{5}{2}}{(1)^2 \times \frac{1}{4}} \quad \ldots\left[\begin{array}{l}\because x \to 0, \frac{13x}{2} \to 0,\\ \frac{5x}{2} \to 0 \text{ and } \lim\limits_{\theta\to 0}\frac{\sin\theta}{\theta} = 1\end{array}\right]$$

$= 65$

∴ $\lim\limits_{x\to 0} f(x) \neq f(0)$

∴ f(x) is discontinuous at $x = 0$

---

4. $\mathbf{f(x) = \dfrac{\sin^2 \pi x}{3(1-x)^2}}$ **,** **for** $\mathbf{x \neq 1}$

$\mathbf{= \dfrac{\pi^2 \sin^2\left(\frac{\pi x}{2}\right)}{3 + 4\cos^2\left(\frac{\pi x}{2}\right)}}$**,** **for** $\mathbf{x = 1}$**, at** $\mathbf{x = 1}$

Solution:

$$f(1) = \frac{\pi^2 \sin\left(\frac{\pi}{2}\right)}{3 + 4\cos^2\left(\frac{\pi}{2}\right)}$$

$$= \frac{\pi^2 \times 1}{3 + 4(0)^2}$$

$$= \frac{\pi^2}{3}$$

$$\lim_{x\to 1} f(x) = \lim_{x\to 1}\frac{\sin^2 \pi x}{3(1-x)^2}$$

Put $1 - x = h$ ∴ $x = 1 - h$

As $x \to 1$, $h \to 0$

∴ $$\lim_{x\to 1} f(x) = \lim_{h\to 0}\frac{\sin^2 \pi(1-h)}{3[1-(1-h)]^2}$$

$$= \lim_{h\to 0}\frac{[\sin(\pi - \pi h)]^2}{3h^2}$$

$$= \frac{1}{3}\lim_{h\to 0}\frac{(\sin \pi h)^2}{h^2} = \frac{1}{3}\lim_{h\to 0}\frac{(\sin \pi h)^2}{h}$$

$$= \frac{1}{3}\lim_{h\to 0}\frac{(\sin \pi h)^2}{\pi h} \times \pi^2$$

$$= \frac{1}{3} \times (1)^2 \times \pi^2 \ldots\left[\begin{array}{l}\because h \to 0, \therefore \pi h \to 0\\ \text{and } \lim\limits_{\theta\to 0}\frac{\sin\theta}{\theta} = 1\end{array}\right]$$

$$= \frac{\pi^2}{3}$$

∴ $\lim\limits_{x\to 1} f(x) = f(1)$

∴ f(x) is continuous at $x = 1$

5. $\mathbf{f(x) = \dfrac{|x+1|}{2x^2+x-1}}$ **, for $x \neq -1$**

= 0 for $x = -1$ at $x = -1$

Solution:

$|x+1| = x+1 \quad ; x \geq -1$

$= -(x+1) \quad ; x < -1$

$\therefore \quad f(x) = \dfrac{-(x+1)}{2x^2+x-1} \quad ; x < -1$

$= 0 \quad ; x = -1$

$= \dfrac{x+1}{2x^2+x-1} \quad ; x > -1$

$f(-1) = 0$

$\lim\limits_{x\to-1^-} f(x) = \lim\limits_{x\to-1^-} \dfrac{-(x+1)}{2x^2+x-1}$

$= \lim\limits_{x\to-1^-} \dfrac{-(x+1)}{(x+1)(2x-1)}$

$= \lim\limits_{x\to-1^-} \dfrac{-1}{2x-1} \quad \ldots\left[\begin{array}{l}\because x\to-1, \therefore x\neq-1\\ \therefore x+1\neq 0\end{array}\right]$

$= \dfrac{-1}{2(-1)-1}$

$= \dfrac{1}{3}$

$\lim\limits_{x\to-1^+} f(x) = \lim\limits_{x\to-1^+} \dfrac{x+1}{2x^2+x-1}$

$= \lim\limits_{x\to-1^+} \dfrac{x+1}{(x+1)(2x-1)}$

$= \lim\limits_{x\to-1^+} \dfrac{-1}{2x-1} \quad \ldots\left(\begin{array}{l}\text{As } x\to-1, x\neq-1\\ \therefore \ x+1\neq 0\end{array}\right)$

$= \dfrac{1}{2(-1)-1}$

$= \dfrac{-1}{3}$

$\therefore \quad \lim\limits_{x\to-1^-} f(x) \neq \lim\limits_{x\to-1^+} f(x)$

$\therefore \quad f(x)$ is discontinuous at $x = -1$

6. $\mathbf{f(x) = [x+1]}$ **for $x \in [-2, 2)$**

Where [ ] is greatest integer function.

Solution:

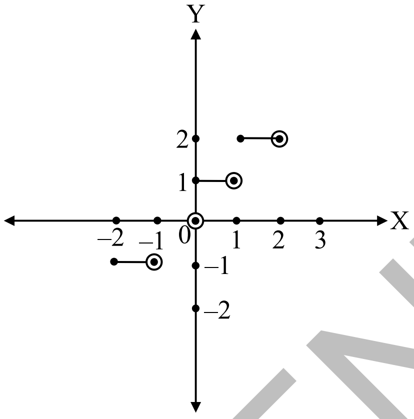
$f(x) = [x+1] \quad ; \quad x \in [-2, 2)$

$\therefore \quad f(x) = -1 \quad ; \quad x \in [-2, -1)$

$= 0 \quad ; \quad x \in [-1, 0)$

$= 1 \quad ; \quad x \in [0, 1)$

$= 2 \quad ; \quad x \in [1, 2)$



For continuity at $x = -1$

$\lim\limits_{x\to-1^-} f(x) = \lim\limits_{x\to-1^-} [x+1]$

$= -1$

$\lim\limits_{x\to-1^+} f(x) = \lim\limits_{x\to-1^+} [x+1]$

$= 0$

$\therefore \quad \lim\limits_{x\to-1^-} f(x) = \lim\limits_{x\to-1^+} f(x)$

$\therefore \quad f(x)$ is discontinuous at $x = -1$

Similarly $f(x)$ is discontinuous at

The points $x = 0$ and $x = 1$

7. $\mathbf{f(x) = 2x^2 + x + 1}$**, for $|x-3| \geq 2$**

$= x^2 + 3$, for $1 < x < 5$

Solution:

$|x-3| \geq 2$

$\therefore \quad x - 3 \geq 2 \quad$ or $\quad x - 3 \leq -2$

$\therefore \quad x \geq 5 \quad$ or $\quad x \leq 1$

$\therefore \quad f(x) = 2x^2 + x + 1 \quad ; \quad x \leq 1$

$= x^2 + 3 \quad ; \quad 1 < x < 5$

$= 2x^2 + x + 1 \quad ; \quad x \geq 5$

Consider the intervals

$x < 1$ i.e. $(-\infty, 1)$

$1 < x < 5$ i.e. $(1, 5)$

$x > 5$ i.e. $(5, \infty)$

In all these intervals $f(x)$ is a polynomial function and hence is continuous at all points.

For continuity at $x = 1$:

$\lim\limits_{x\to1^-} f(x) = \lim\limits_{x\to1^-} (2x^2 + x + 1)$

$= 2(1)^2 + 1 + 1$

$= 4$

$\lim\limits_{x\to1^+} f(x) = \lim\limits_{x\to1^+} (x^2 + 3)$

$= (1)^2 + 3$

$= 4$

Also $f(1) = 2(1)^2 + 1 + 1$

$= 4$

$\therefore \quad \lim\limits_{x\to1^-} f(x) = \lim\limits_{x\to1^+} f(x) = f(1)$

$\therefore \quad f(x)$ is continuous at $x = 1$

For continuity at $x = 5$:

$\lim_{x\to 5^-} f(x) = \lim_{x\to 5^-} (x^2+3)$
$= (5)^2 + 3$
$= 28$

$\lim_{x\to 5^+} f(x) = \lim_{x\to 5^+} (2x^2+x+1)$
$= 2(5)^2 + 5 + 1$
$= 56$

∴ $\lim_{x\to 5^-} f(x) \neq \lim_{x\to 5^+} f(x)$

∴ $f(x)$ is discontinuous at $x = 5$

∴ $f(x)$ is continuous for all $x \in R$, except at $x = 5$

III. Identify discontinuous if any for the following functions as either a jump of a removable discontinuity in their respective domains.

1. $f(x) = x^2 + x - 3$, **for** $x \in [-5, -2)$
$= x^2 - 5$, **for** $x \in (-2, 5]$

Solution:

$f(-2)$ has not been defined

$\lim_{x\to -2} f(x) = \lim_{x\to -2} (x^2 + x - 3)$
$= (-2)^2 + (-2) - 3$
$= 4 - 2 - 3$
$= -1$

$\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} (x^2 - 5)$
$= (-2)^2 - 5$
$= 4 - 5$
$= -1$

$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^+} f(x)$

∴ $\lim_{x\to -2} f(x) = -1$

But $f(-2)$ has not been defined

∴ $f(x)$ has a removable discontinuity at $x = -2$

2. $f(x) = x^2 + 5x + 1$, **for** $0 \le x \le 3$
$= x^3 + x + 5$, **for** $3 < x \le 6$

Solution:

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (x^2 + 5x + 1)$
$= \lim_{x\to -3^-} (3)^2 + 5(3) + 1$
$= 9 + 15 + 1$
$= 25$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (x^3 + x + 5)$
$= (3)^3 + 3 + 5$
$= 35$

∴ $\lim_{x\to 3^-} f(x) \neq \lim_{x\to 3^+} f(x)$

∴ $\lim_{x\to 3} f(x)$ does not exist

∴ $f(x)$ is discontinuous at $x = 3$

∴ $f(x)$ has a jump discontinuity at $x = 3$

3. $f(x) = \dfrac{x^2+x+1}{x+1}$, **for** $x \in [0, 3)$
$= \dfrac{3x+4}{x^2-5}$, **for** $x \in [3, 6]$.

Solution:

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \dfrac{x^2+x+1}{x+1}$

$= \dfrac{\lim_{x\to 3^-}(x^2+x+1)}{\lim_{x\to 3^-}(x+1)}$

$= \dfrac{(3)^2+3+1}{3+1}$

$= \dfrac{13}{4}$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \dfrac{3x+4}{x^2-5}$

$= \dfrac{\lim_{x\to 3^+}(3x+4)}{\lim_{x\to 3^+}(x^2-5)} = \dfrac{3(3)+4}{(3)^2-5}$

$= \dfrac{13}{4}$

Also $f(3) = \dfrac{3(3)+4}{(3)^2-5}$

$= \dfrac{13}{4}$

∴ $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3)$

∴ $f(x)$ is continuous at $x = 3$.

IV. Discuss the continuity of the following functions at the point or on the interval indicated against them. If the function is discontinuous, identify the type of discontinuity and state whether the discontinuity is removable. If it has a removable discontinuity, redefine the function so that it becomes continuous.

1. $f(x) = \dfrac{(x+3)(x^2-6x+8)}{x^2-x-12}$

Solution:

$f(x) = \dfrac{(x+3)(x^2-6x+8)}{x^2-x-12}$

$f(x) = \dfrac{(x+3)(x^2-6x+8)}{(x-4)(x+3)}$

∴ $f(x)$ is not defined at for $x = 4$ and $x = -3$

∴ The domain of function $f = R - \{-3, 4\}$

For $x \neq -3, 4$

$f(x) = \dfrac{(x+3)(x-2)(x-4)}{(x-4)(x+3)}$

∴ $f(x) = x - 2 \quad x \neq -3, 4$

$\therefore$ f(–3) = –5 and f(4) = 2

f(x) is discontinuous $x = 4$ and $x = -3$

This discontinuity is removable.

$\therefore$ f(x) can be redefined as

$$f(x) = \frac{(x+3)(x^2-6x+8)}{x^2-x-12}$$

$= -5$ for $x = -3$

$= 2$ for $x = 4$

---

2. $\mathbf{f(x) = x^2 + 2x + 5}$, **for $x \le 3$**

$\mathbf{= x^2 - 2x^2 - 5}$, **for $x > 3$**

Solution:

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-}\left(x^2 + 2x + 5\right)$$

$= (3)^2 + 2(3) + 5$

$= 9 + 6 + 5$

$= 20$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+}\left(x^3 - 2x^2 - 5\right)$$

$= (3)^3 + 2(3)^2 - 5$

$= 27 - 18 - 5$

$= 4$

$\therefore$ $\lim_{x\to 3^-} f(x) \ne \lim_{x\to 3^+} f(x)$

$\therefore$ $\lim_{x\to 3} f(x)$ does not exist

$\therefore$ f(x) is discontinuous at $x = 3$

This continuity is irremovable.

V. Find k if following functions are continuous at the points indicated against them.

1. $\mathbf{f(x) = \left(\frac{5x-8}{8-3x}\right)^{\frac{3}{2x-4}}}$, **for $x \ne 2$**

$\mathbf{= k,}$ **for $x = 2$ at $x = 2$.**

Solution:

1. f(x) is continuous at $x = 2$

$\therefore$ $f(2) = \lim_{x\to 2} f(x)$

$$\therefore\ k = \lim_{x\to 2}\left(\frac{5x-8}{8-3x}\right)^{\frac{3}{2x-4}}$$

Put $x - 2 = h$

$\therefore$ $x = 2 + h$

As $x \to 2$, $h \to 0$

$$\therefore\ k = \lim_{h\to 0}\left[\frac{5(2+h)-8}{8-3(2+h)}\right]^{\frac{3}{2h}}$$

$$= \lim_{h\to 0}\left(\frac{10+5h-8}{8-6-3h}\right)^{\frac{3}{2h}}$$

$$= \lim_{h\to 0}\left(\frac{2+5h}{2-3h}\right)^{\frac{3}{2h}}$$

$$= \lim_{h\to 0}\left[\frac{2\left(1+\frac{5h}{2}\right)}{2\left(1-\frac{3h}{2}\right)}\right]^{\frac{3}{2h}}$$

$$= \lim_{h\to 0}\frac{\left(1+\frac{5h}{2}\right)^{\frac{3}{2h}}}{\left(1-\frac{3h}{2}\right)^{\frac{3}{2h}}}$$

$$= \frac{\lim_{h\to 0}\left[\left(1+\frac{5h}{2}\right)^{\frac{2}{5h}}\right]^{\frac{5}{2}\times\frac{3}{2}}}{\lim_{h\to 0}\left[\left(1-\frac{3h}{2}\right)^{\frac{-2}{3h}}\right]^{\frac{-3}{2}\times\frac{3}{2}}}$$

$$= \frac{e^{\frac{15}{4}}}{e^{\frac{-9}{4}}} \quad \ldots\left[\begin{array}{l}\because h\to 0, \frac{5h}{2}\to 0, \frac{-3h}{2}\to 0 \\ \text{and } \lim_{x\to 0}(1+x)^{\frac{1}{x}} = e\end{array}\right]$$

$= e^{\frac{24}{4}}$

$= e^6$

---

2. $\mathbf{f(x) = \frac{45^x - 9^x - 5^x + 1}{(k^x - 1)(3^x - 1)}}$, **for $x \ne 0$**

$\mathbf{= \frac{2}{3}}$, **for $x = 0$, at $x = 0$**

Solution:

f(x) is continuous at $x = 0$

$\therefore$ $\lim_{x\to 0} f(x) = f(0)$

$$\therefore\ \lim_{x\to 0}\frac{(45)^x - 9^x - 5^x + 1}{(k^x-1)(3^x-1)} = \frac{2}{3}$$

$$\therefore\ \lim_{x\to 0}\frac{9^x\cdot 5^x - 9^x - 5^x + 1}{(k^x-1)(3^x-1)} = \frac{2}{3}$$

$$\therefore\ \lim_{x\to 0}\frac{9^x(5^x-1) - 1(5^x-1)}{(k^x-1)(3^x-1)} = \frac{2}{3}$$

$$\therefore\ \lim_{x\to 0}\frac{(5^x-1)(9^x-1)}{(k^x-1)(3^x-1)} = \frac{2}{3}$$

$$\therefore\ \lim_{x\to 0}\frac{\frac{(5^x-1)(9^x-1)}{x^2}}{\frac{(k^x-1)(3^x-1)}{x^2}} = \frac{2}{3}$$

$$\ldots\left[\begin{array}{l}\text{Divide Numerator and Denominator} \\ \text{by } x^2 \\ \because x\to 0, \therefore x\ne 0 \quad \therefore x^2 \ne 0\end{array}\right]$$

$\therefore \quad \dfrac{\lim\limits_{x\to 0}\left(\dfrac{5^x-1}{x}\right)\left(\dfrac{9^x-1}{x}\right)}{\lim\limits_{x\to 0}\left(\dfrac{k^x-1}{x}\right)\left(\dfrac{3^x-1}{x}\right)} = \dfrac{2}{3}$

$\therefore \quad \dfrac{\left(\lim\limits_{x\to 0}\dfrac{5^x-1}{x}\right)\cdot\left(\lim\limits_{x\to 0}\dfrac{9^x-1}{x}\right)}{\left(\lim\limits_{x\to 0}\dfrac{k^x-1}{x}\right)\cdot\left(\lim\limits_{x\to 0}\dfrac{3^x-1}{x}\right)} = \dfrac{2}{3}$

$\therefore \quad \dfrac{\log 5\cdot\log 9}{\log k\cdot\log 3} = \dfrac{2}{3}$ $\quad \ldots\left[\because \lim\limits_{x\to 0}\dfrac{a^x-1}{x}=\log a\right]$

$\therefore \quad \dfrac{\log 5\cdot\log(3)^2}{\log k\cdot\log 3} = \dfrac{2}{3}$

$\therefore \quad \dfrac{\log 5\times\log 3}{\log k\times\log 3} = \dfrac{1}{3}$

$\therefore \quad 3\log 5 = \log k$

$\therefore \quad \log(5)^3 = \log k$

$\therefore \quad (5)^3 = k$

$\therefore \quad k = 125$

VI. Find a and b if following functions are continuous at the points or on the interval indicated against them.

1. $f(x) = \dfrac{4\tan x+5\sin x}{a^x-1}$, **for** $x<0$

$\quad = \dfrac{9}{\log 2}$, $\quad x = 0$

$\quad = \dfrac{11x+7x\cdot\cos x}{b^x-1}$, **for** $x<0$

Solution:

1. $f(x)$ is continuous at $x = 0$

$\therefore \quad \lim\limits_{x\to 0^-} f(x) = f(0)$

$\therefore \quad \lim\limits_{x\to 0}\left(\dfrac{4\tan x+5\sin x}{a^x-1}\right) = \dfrac{9}{\log 2}$

$\therefore \quad \lim\limits_{x\to 0}\left(\dfrac{\dfrac{4\tan x+5\sin x}{x}}{\dfrac{a^x-1}{x}}\right)$ $\quad \ldots[\because x\to 0, x\neq 0]$

$= \dfrac{9}{\log 2}$

$\therefore \quad \dfrac{\lim\limits_{x\to 0}\left(\dfrac{4\tan x}{x}+\dfrac{5\sin x}{x}\right)}{\lim\limits_{x\to 0}\dfrac{a^x-1}{x}} = \dfrac{9}{\log 2}$

$\therefore \quad \dfrac{4\lim\limits_{x\to 0}\dfrac{\tan x}{x}+5\lim\limits_{x\to 0}\dfrac{\sin x}{x}}{\lim\limits_{x\to 0}\dfrac{a^x-1}{x}} = \dfrac{9}{\log 2}$

$\therefore \quad \dfrac{4(1)+5(1)}{\log a} = \dfrac{9}{\log 2}$ $\quad \ldots\left[\because \lim\limits_{x\to 0}\dfrac{a^x-1}{x}=\log a\right]$

$\therefore \quad \dfrac{9}{\log a} = \dfrac{9}{\log 2}$

$\therefore \quad \log a = \log 2$

$\therefore \quad a = 2$

Also $\lim\limits_{x\to 0^+} f(x) = f(0)$

$\therefore \quad \lim\limits_{x\to 0}\dfrac{11x+7x\cdot\cos x}{b^x-1} = \dfrac{9}{\log 2}$

$\therefore \quad \lim\limits_{x\to 0}\dfrac{\dfrac{11x+7x\cos x}{x}}{\dfrac{b^x-1}{x}} = \dfrac{9}{\log 2}$ $\quad \ldots[\because x\to 0, x\neq 0]$

$\therefore \quad \dfrac{\lim\limits_{x\to 0}(11+7\cos x)}{\lim\limits_{x\to 0}\left(\dfrac{b^x-1}{x}\right)} = \dfrac{9}{\log 2}$

$\therefore \quad \dfrac{11+7\cos 0}{\log b} = \dfrac{9}{\log 2}$ $\quad \ldots\left[\because \lim\limits_{x\to 0}\dfrac{a^x-1}{x}=\log a\right]$

$\therefore \quad \dfrac{11+7(1)}{\log b} = \dfrac{9}{\log 2}$

$\therefore \quad 9\log b = 18\log 2$

$\therefore \quad \log b = 2\log 2$

$\quad\quad = \log(2)^2$

$\therefore \quad \log b = \log 4$

$\therefore \quad b = 4$

$\therefore \quad a = 2$ and $b = 4$

---

2. $f(x) = ax^2 + bx + 1$, **for** $|2x-3| \geq 2$

$\quad = 3x + 2$, **for** $\dfrac{1}{2} < x < \dfrac{5}{2}$

Solution:

$|2x-3| \geq 2$

$\therefore \quad 2x-3 \geq 2 \quad$ or $\quad 2x-3 \leq -2$

$\therefore \quad 2x \geq 5 \quad$ or $\quad 2x \leq 1$

$\therefore \quad x \geq \dfrac{5}{2} \quad$ or $\quad x \leq \dfrac{1}{2}$

$\therefore \quad f(x)$ is redefined as

$f(x) = ax^2 + bx + 1 \;; \quad x \leq \dfrac{1}{2}$

$\quad = 3x + 2 \;; \quad \dfrac{1}{2} < x < \dfrac{5}{2}$

$\quad = ax^2 + bx + 1 \;; \quad x \geq \dfrac{5}{2}$

$f(x)$ is continuous everywhere on its domain

$\therefore \quad f(x)$ is continuous at $x = \dfrac{1}{2}$ and $x = \dfrac{5}{2}$

As $f(x)$ is continuous at $x = \dfrac{1}{2}$

$\therefore \quad \lim\limits_{x\to \frac{1}{2}^-} f(x) = \lim\limits_{x\to \frac{1}{2}^+} f(x)$

$\therefore \quad \lim\limits_{x\to \frac{1}{2}^-}\left(ax^2+bx+1\right) = \lim\limits_{x\to \frac{1}{2}^+}(3x+2)$

$\therefore \quad a\left(\frac{1}{2}\right)^2 + b\left(\frac{1}{2}\right) + 1 = 3\left(\frac{1}{2}\right) + 2$

$\therefore \quad \frac{a}{4} + \frac{b}{4} + 1 = \frac{7}{2}$

$\therefore \quad a + 2b + 4 = 14$ …[Multiplying by 4]

$\therefore \quad a + 2b = 10$ …(i)

Also f(x) is continuous at $x = \frac{5}{2}$

$\therefore \quad \lim_{x\to\frac{5}{2}^-} f(x) = \lim_{x\to\frac{5}{2}^+} f(x)$

$\therefore \quad \lim_{x\to\frac{5}{2}^-} (3x + 2) = \lim_{x\to\frac{5}{2}^+} (ax^2 + bx + 1)$

$\therefore \quad 3\left(\frac{5}{2}\right) + 2 = a\left(\frac{5}{2}\right)^2 + b\left(\frac{5}{2}\right) + 1$

$\therefore \quad \frac{15}{2} + 2 = \frac{25a}{4} + \frac{5b}{2} + 1$

$\therefore \quad 30 + 8 = 25a + 10b + 4$

…[Multiplying both sides by 4]

$\therefore \quad 25a + 10b = 34$ …(ii)

Multiplyinig (i) by 5, we get

$5a + 10b = 50$ …(iii)

Subtract (iii) from (ii),

$20a = -16$

$\therefore \quad a = \frac{-16}{20} = \frac{-4}{5}$

Substituting $a = \frac{-4}{5}$ in (iii), we get

$5\left(\frac{-4}{5}\right) + 10b = 50$

$\therefore \quad -4 + 10b = 50$

$\therefore \quad 10b = 54$

$\therefore \quad b = \frac{54}{10} = \frac{27}{5}$

$\therefore \quad a = \frac{-4}{5}, b = \frac{27}{5}$

VII. Find f(a), if f is continuous at x = a where,

1. $f(x) = \frac{1+\cos(\pi x)}{\pi(1-x)^2}$, for $x \neq 1$ and at a = 1.

Solution :

f(x) is continuous at $x = 1$

$\therefore \quad f(1) = \lim_{x\to1} f(x)$

$\therefore \quad f(1) = \lim_{x\to1} \frac{1+\cos\pi x}{\pi(1-x)^2}$

Put $1 - x = h$

$\therefore \quad x = 1 - h$

As $x \to 1$, $h \to 0$

$\therefore \quad f(1) = \lim_{h\to0} \frac{1+\cos[\pi(1-h)]}{\pi h^2}$

$= \lim_{h\to0} \frac{1+\cos(\pi - \pi h)}{\pi h^2}$

$= \lim_{h\to0} \frac{1-\cos\pi h}{\pi h^2}$

$= \lim_{h\to0} \frac{1-\cos\pi h}{\pi h^2} \times \frac{1+\cos\pi h}{1+\cos\pi h}$

$= \lim_{h\to0} \frac{1-\cos^2\pi h}{\pi h^2(1+\cos\pi h)}$

$= \frac{1}{\pi}\lim_{h\to0} \frac{\sin^2\pi h}{h^2(1+\cos\pi h)}$

$= \frac{1}{\pi}\lim_{h\to0}\left(\frac{\sin\pi h}{h}\right)^2 \times \frac{1}{1+\cos\pi h}$

$= \frac{1}{\pi}\lim_{h\to0}\left(\frac{\sin\pi h}{\pi h}\right)^2 \times \pi^2 \times \frac{1}{\lim_{h\to0}(1+\cos\pi h)}$

$= \frac{1}{\pi} \times (1)^2 \times \pi^2 \times \frac{1}{1+1}$ …$\left[\begin{array}{l}\text{As } h\to0, \pi h\to0 \\ \text{and } \lim_{\theta\to0}\frac{\sin\theta}{\theta} = 1\end{array}\right]$

$= \frac{\pi}{2}$

2. $f(x) = \frac{1-\cos[7(x-\pi)]}{5(x-\pi)^2}$, for $x \neq \pi$ and at a = π.

Solution:

f is continuous at $x = \pi$.

$\therefore \quad f(\pi) = \lim_{x\to\pi} f(x) = \lim_{x\to\pi} \frac{1-\cos[7(x-\pi)]}{5(x-\pi)^2}$

Put $x - \pi = h$, as $x \to \pi$, $h \to 0$

$\therefore \quad f(\pi) = \lim_{h\to0} \frac{1-\cos 7h}{5h^2}$

$= \lim_{h\to0} \frac{2\sin^2\left(\frac{7h}{2}\right)}{5h^2}$

$= \frac{2}{5}\lim_{h\to0} \frac{\sin^2\left(\frac{7h}{2}\right)}{\left(\frac{7h}{2}\right)^2} \times \left(\frac{7}{2}\right)^2$

$= \frac{2}{5}\left[\lim_{h\to0} \frac{\sin\left(\frac{7h}{2}\right)}{\left(\frac{7h}{2}\right)}\right]^2 \times \frac{49}{4}$

$= \frac{2}{5} \times (1)^2 \times \frac{49}{4}$ …$\left[\because \lim_{\theta\to0}\frac{\sin\theta}{\theta} = 1\right]$

$\therefore \quad f(\pi) = \frac{49}{10}$

VIII. Solve using intermediate value theorem.

1. Show that $5^x - 6x = 0$ has a root in [1, 2]

Solution :

1. Let $f(x) = 5^x - 6x$.

5^x and $6x$ are continuous functins for all $x \in R$.

$\therefore \quad 5^x - 6x$ is also continuous for all $x \in R$.

i.e. f(x) is continuous for all $x \in R$.
A root of f(x) exists if f(x) = 0 for at least one value of x.
$f(1) = 5^1 - 6(1)$
$= -1 < 0$
$f(2) = (5)^2 - 6(2)$
$= 13 > 0$
$\therefore$ f(1) < 0 and f(2) > 0
By intermediate value theorem, there has to be a point 'c' between 1 and 2 such that f(c) = 0.
$\therefore$ There is a root of the given equation in [1, 2].

---

2. Show that $x^3 - 5x^2 + 3x + 6 = 0$ has at least two real roots between $x = 1$ and $x = 5$.

Solution:

Let $f(x) = x^3 - 5x^2 + 3x + 6$.
f(x) is a polynomial function and hence it is continuous for all $x \in R$.
A root of f(x) exists if f(x) = 0 for at least one value of x.
Here we have been asked to show that f(x) has at least two roots between $x = 1$ and $x = 5$.
$f(1) = (1)^3 - 5(1)^2 + 3(1) + 6$
$= 5 > 0$
$f(2) = (2)^3 - 5(2)^2 + 3(2) + 6$
$= 8 - 20 + 6 + 6$
$= 0$
$\therefore$ $x = 0$ is a root of f(x).
Also $f(3) = (3)^3 - 5(3)^2 + 3(3) + 6$
$= 27 - 45 + 9 + 6$
$= -3 < 0$
$F(4) = (4)^3 - 5(4)^2 + 3(4) + 6$
$= 64 - 80 + 12 + 6$
$= 2 > 0$
$\therefore$ f(3) < 0 and f(4) > 0
$\therefore$ By intermediate value theorem, there has to be point 'c' between 3 and 4 such that f(c) = 0.
$\therefore$ There are two roots, $x = 2$ and a root between $x = 3$ and $x = 4$.
Thus there are at least two roots of the given equation between $x = 1$ and $x = 5$.

Activities for Practice

1. Let $f(x) = \dfrac{1 - \sqrt{2}\sin x}{\pi - 4x}$, $x \neq \dfrac{\pi}{4}$

$= a$, $\quad x = \dfrac{\pi}{4}$

If function f is continuous at $x = \dfrac{\pi}{4}$ then evaluate a by completing the activity.

Solution:

$\because$ function f is continuous at $x = \dfrac{\pi}{4}$

$\therefore$ $f\left(\dfrac{\pi}{4}\right) = \lim\limits_{x \to \frac{\pi}{4}} f(x)$

Let $x = \dfrac{\pi}{4} - t$

$\therefore$ $a = \lim\limits_{t \to 0} \dfrac{1 - \sqrt{2}\sin\left(\frac{\pi}{4} - t\right)}{\square}$

$= \square \lim\limits_{t \to 0} \dfrac{1 - \cos t + \sin t}{t}$ …(i)

Consider, $\lim\limits_{t \to 0} \dfrac{1 - \cos t}{t} = \lim\limits_{t \to 0} \dfrac{2\sin^2\left(\frac{t}{2}\right)}{2\left(\frac{t}{2}\right)}$

$= \lim\limits_{t \to 0} \left(\dfrac{\sin\left(\frac{t}{2}\right)}{\frac{t}{2}} \cdot \sin\left(\frac{t}{2}\right)\right)$ …(ii)

By substituting (ii) in (i), we get

$a = \square \left[\lim\limits_{t \to 0}\left(\left(\dfrac{\sin\frac{t}{2}}{\frac{t}{2}}\right) \cdot \sin\left(\frac{t}{2}\right)\right) + \lim\limits_{t \to 0}\left(\dfrac{\sin t}{t}\right)\right]$

$\therefore$ $a = \square$

---

2. Let $f(x) = \cos \pi (|x| + [x])$, $-1 \le x \le 1$, where [ ] represents greatest integer function. Show that f is a discontinuous function, by completing the activity.

Solution:

$f(x) = \cos \pi (\square)$, $\quad -1 \le x < 0$

$= \cos \pi (\square)$, $\quad 0 \le x < 1$

$\therefore$ $f(x) = -\cos \pi x$, $\quad -1 \le x < 0$
$= \cos \pi x$, $0 \le x < 0$
In $(-1, 0) \cup (0, 1)$, f is continuous as cosine function is continuous.

Continuity at $x = 0$:

L. h. lim. = $\lim\limits_{x \to 0^-} (-\cos \pi x) = \square$

R. h. lim. = $\lim\limits_{x \to 0^+} (\cos \pi x) = \square$

$\because$ L. h. lim $\neq$ R. h. lim
$\therefore$ f is not continuous at $x = 0$
$\therefore$ f is a discontinuous function in $(-1, 1)$.

3. Let a function f be defined as

$$f(x) = \frac{\sqrt{1+px}-\sqrt{1-px}}{x}, \quad -1 \le x < 0$$
$$= \frac{2x+1}{x-2}, \quad 0 \le x \le 1$$

If the function f is continuous in the interval [– 1, 1] then to evaluate p, complete the activity.

Solution:

$\because$ f is continuous in [– 1, 1]

$\therefore$ f is continuous at $x = 0$.

$\therefore \lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x) = f(0)$

$\therefore \lim\limits_{x\to 0^-} \dfrac{\sqrt{1+px}-\sqrt{1-px}}{x} = \lim\limits_{x\to 0^+} \dfrac{2x+1}{x-2}$

On Rationalizing, we get

$\therefore \lim\limits_{x\to 0^-} \dfrac{(1+px)-(1-px)}{x\left(\sqrt{1+px}+\sqrt{1-px}\right)} = \square$

$\lim\limits_{x\to 0} \dfrac{\square}{\sqrt{1+px}+\sqrt{1-px}} = \square$

$\therefore$ p = $\square$

---

4. Let a function f be defined as

$f(x) = [\tan x] + \{\tan x\}, \ x \in \left(\frac{\pi}{4} - \delta, \frac{\pi}{4} + \delta\right)$

where δ is a small positive number. Show that function f is a continuous at $x = 0$, by completing the activity.

Solution:

When $x < \frac{\pi}{4}$, then $0 < \tan x < 1$

$\therefore \{\tan x\} = \tan x$

When $x = \frac{\pi}{4}$, then $\tan x$ is an integer

$\therefore \{\tan x\} = 0$

When $x > \frac{\pi}{4}$, then $\tan x > 1$ but less than 2

$\therefore \{\tan x\} = \tan x - [\tan x]$
$= \tan x - 1$

$\therefore$ Let us write the explicit definition of function f, for $x \in \left(\frac{\pi}{4} - \delta, \frac{\pi}{4} + \delta\right)$

$f(x) = 0 + \sqrt{\square} \qquad x < \frac{\pi}{4}$

$= 1 \qquad x = \frac{\pi}{4}$

$= 1 + \sqrt{\square} \qquad x > c$

$\therefore$ L. h. lim = $\lim\limits_{x\to \frac{\pi}{4}^-} f(x) = \square$ and

R. h. lim = $\lim\limits_{x\to \frac{\pi}{4}^+} f(x) = \square$

$\because$ L. h. lim = R. h. lim = $f\left(\frac{\pi}{4}\right)$

$\therefore$ f is continuus at $x = \frac{\pi}{4}$

---

5. If the following function is continuous at $x = 0$, find a and b.

$f(x) = x^2 + a \qquad$, for $x > 0$
$= 2\sqrt{x^2+1} + b \qquad$, for $x < 0$
$= 2 \qquad$, for $x = 0$

Solution:

Given

$f(x) = x^2 + a \qquad$, for $x > 0$
$= 2\sqrt{x^2+1} + b \qquad$, for $x < 0$

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} (x^2 + a)$

$\therefore \lim\limits_{x\to 0^+} f(x) = \square$

Since, f(x) is continuous at $x = 0$

$\therefore \lim\limits_{x\to 0^+} f(x) = f(0)$

$\therefore$ a = $\square$

$\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} (2\sqrt{x^2+1} + b)$

$\therefore \lim\limits_{x\to 0^-} f(x) = \square + b$

Since, f(x) is continuous at $x = 0$.

$\therefore \lim\limits_{x\to 0^-} f(x) = f(0)$

$\therefore$ b = $\square$

---

6. If $f(x) = \dfrac{x^2-4}{x-2}$, for $x \ne 2$ is continuous at $x = 2$, then find f(2).

Solution:

$f(x) = \dfrac{x^2-4}{x-2}$

$\lim\limits_{x\to 2} f(x) = \lim\limits_{x\to 2} \dfrac{x^2-4}{x-2}$

$= \lim\limits_{x\to 2} \dfrac{(x+\square)(x-\square)}{x-2}$

$\therefore \lim\limits_{x\to 2} f(x) = \square$

Since, f(x) is continuous at $x = 2$.

$\therefore \lim\limits_{x\to 2} f(x) = f(2)$

$\therefore$ f(2) = $\square$

7. Determine whether the function 'f' is continuous on its domain

$f(x) = 3x + 1, \quad x < 2$
$= 7, \quad 2 \le x < 4$
$= x^2 - 8, \quad x \ge 4$

Solution:

$\lim_{x\to2^-} f(x) = \lim_{x\to2^-} (3x + 1)$
$= 7$

Now $\lim_{x\to2^+} f(x) = \square$

But $f(x) = 7$ at $x = 2$

∴ The given function is $\square$ at $x = 2$.

Also $\lim_{x\to4^-} f(x) = \square$

$\lim_{x\to4^+} f(x) = \lim_{x\to4^+} (x^2 - 8) = 8$

∴ the given function is $\square$ at $x = 4$.

Answers

1. i. 4t ii. $\frac{1}{4}$
   iii. $\frac{1}{4}$ iv. $\frac{1}{4}$
2. i. $x + 1$ ii. x
   iii. -1 iv. 1
3. i. $-\frac{1}{2}$ ii. 2p
   iii. $-\frac{1}{2}$ iv. $-\frac{1}{2}$
4. i. $\tan x$ ii. $\tan x - 1$
   iii. 1 iv. 1
5. i. a ii. 2
   iii. 2 iv. 0
6. i. 2 ii. 2
   iii. 4 iv. 4
7. i. 7
   ii. continuous
   iii. 7
   iv. discontinuous

Additional Problems for Practice

Based on Exercise 8.1

1. Examine the continuity of the following funtions at the given point:

i. $f(x) = \frac{\sin x}{x} + \cos x$, for $x \ne 0$
$= 2$, for $x = 0$; at $x = 0$

ii. $f(x) = \frac{1}{2} \sin x^2$, for $x \ne 0$
$= 0$, for $x = 0$; at $x = 0$

iii. $f(x) = (1 + 2x)^{1/x}$, for $x \ne 0$
$= e^2$, for $x = 0$; at $x = 0$

iv. $f(x) = \frac{x^2 - x - 6}{x - 3}$, for $x \ne 3$
$= 7$, for $x = 3$; at $x = 3$

v. $f(x) = x^2 + 6x + 10$, for $x \le 4$
$= x^2 - x + 38$, for $x > 4$; at $x = 4$

vi. $f(y) = \frac{(e^{2y} - 1).\sin y}{y^2}$, for $y \ne 0$
$= 4$, for $y = 0$; at $y = 0$

vii. $f(x) = \left(1 + \frac{4x}{5}\right)^{\frac{1}{x}}$, for $x \ne 0$
$= e^{\frac{4}{5}}$, for $x \ne 0$ at $x = 0$

viii. $f(x) = \frac{1}{2} \sin \frac{\pi}{2}(x + 1)$, for $x \le 0$
$= \frac{\tan x - \sin x}{x^3}$, for $x > 0$; at $x = 0$

ix. $f(x) = \frac{x^3 + 2x^2 + 2x - 5}{x^3 + 3x^2 - 3x - 1}$, for $x < 1$
$= \left[\frac{1}{x-1} - \frac{1}{x^4 - x^3}\right]$, for $x \ge 1$; at $x = 1$

x. $f(x) = \frac{\sqrt{x+3} - 2}{x^3 - 1}$, for $x \ne 1$
$= \frac{1}{12}$, for $x = 1$; at $x = 1$

2. Discuss the continuity of the following functions:

i. $f(x) = \frac{a^{3x} - a^{5x}}{x}$, for $x \ne 0$
$= \log a$, for $x = 0$; at $x = 0$

ii. $f(x) = \left(1 + \frac{x}{a}\right)^{\frac{1}{x}}$, for $x \ne 0$
$= e^{\frac{1}{a}}$, for $x = 0$; at $x = 0$

iii. $g(x) = \frac{\log\left(1 + \frac{5}{2}x\right)}{x}$, for $x \ne 0$
$= \frac{5}{2}$, for $x = 0$; at $x = 0$

iv. $f(x) = \frac{5^x - e^x}{\sin 2x}$, for $x \ne 0$
$= \frac{1}{2}(\log 5 + 1)$, for $x = 0$; at $x = 0$

v. $f(x) = \frac{\sin^2 ax}{x^2}$, for $x \ne 0$
$= 1$, for $x = 0$; at $x = 0$

vi. $f(x) = x^2 \sin \frac{1}{x}$, for $x \ne 0$
$= 0$, for $x = 0$; at $x = 0$

vii. $f(x) = \frac{1-\cos x}{x}$, for $x \neq 0$

$= 0$, for $x = 0$; at $x = 0$

viii. $f(x) = \frac{\sqrt{4+x}-2}{3x}$, for $x \neq 0$

$= \frac{1}{12}$, for $x = 0$; at $x = 0$

ix. If $f(x) = \frac{2^{3x}-1}{\tan x}$, for $x \neq 0$

$= 1$, for $x = 0$

+3. Discuss the continuity of the function $f(x) = |x - 3|$ at $x = 3$.

+4. Determine whether the function f is continuous on the set of the real numbers

Where $f(x) = 3x + 1$, for $x < 2$

$= 7$ for $2 \leq x < 4$

$= x^2 - 8$ for $x \geq 4$

+5. Test whether the function f(x) is continuous at $x = -4$, where

$f(x) = \frac{x^2+16x+48}{x+4}$, for $x \neq 4$

$= 8$, for $x = -4$

+6. Discuss the continuity of $f(x) = \sqrt{9-a^2}$, on the interval $[-3, 3]$.

+7. Show that the function $f(x) = \lfloor x \rfloor$ is not continuous at $x = 0$, in the interval $[-1, 2)$

8. Discuss the continuity of the following functions at the points given against them. If the function is discontinuous, determine whether the discontinuity is removable. In that case, redefine the function, so that it becomes continuous.

i. $f(x) = \frac{1-\cos 3x}{x\tan x}$, for $x \neq 0$

$= 9$, for $x = 0$; at $x = 0$

ii. $f(x) = \frac{\sin \pi x}{5x}$, for $x \neq 0$

$= \frac{5}{\pi}$, for $x = 0$; at $x = 0$

iii. $f(x) = \frac{2-\sqrt{x+4}}{\sin 2x}$, for $x \neq 0$

$= 8$, for $x = 0$; at $x = 0$

iv. $f(x) = \frac{\sin(x^2-x)}{x}$, for $x \neq 0$

$= 2$, for $x = 0$; at $x = 0$

+9. Identify discontinuities for the following functions as either a jump or a removable discontinuity on R.

i. $f(x) = \frac{x^2-3x-18}{x-6}$,

ii. $g(x) = 3x + 1$, for $x < 3$

$= 2 - 3x$, for $x \geq 3$

iii. $h(x) = 13 - x^2$, for $x < 5$

$= 13 - 5x$, for $x > 5$

+10. Show that the function

$f(x) = \frac{5^{\cos x} - e^{\left(\frac{\pi}{2}-x\right)}}{\cot x}$, for $x \neq \frac{\pi}{2}$

$= \log 5 - e$, for $x = \frac{\pi}{2}$

has a removable discontinuity at $x = \frac{\pi}{2}$.

Redefine the function so that it becomes continuous at $x = \frac{\pi}{2}$.

+11. If f(x) is defined on R, discuss the continuity of f at $x = \frac{\pi}{2}$, where

$f(x) = \frac{5^{\cos x} + 5^{-\cos x} - 2}{(3\cot x).\log\left(\frac{2+\pi-2x}{2}\right)}$, for $x \neq \frac{\pi}{2}$

$= \frac{2\log 5}{3}$, for $x = \frac{\pi}{2}$.

+12. Discuss the continuity of the following function at $x = 0$, where

$f(x) = x^2 \sin\left(\frac{1}{x}\right)$, for $x \neq 0$

$= 0$, for $x = 0$

13. If f is continuous at $x = 0$, then find f(0).

i. $f(x) = \frac{\left(4^{\sin x}-1\right)^2}{x\log(1+2x)}$, $x \neq 0$

ii. $f(x) = \frac{\log(1+ax)-\log(1-bx)}{x}$

iii. $f(x) = \frac{\log(2+x)-\log(2-x)}{\tan x}$

iv. $f(x) = \frac{\cos^2 x - \sin^2 x - 1}{\sqrt{x^2+1}-1}$

+v. $f(x) = \left(\frac{3x+2}{2-5x}\right)^{\frac{1}{x}}$, for $x \neq 0$

14. Find f(3) if $f(x) = \frac{x^2-9}{x-3}$, $x \neq 3$ is contiuous at $x = 3$.

15. Find the value of k, if the function

i. $f(x) = \frac{8^x - 2^x}{k^x - 1}$, for $x \neq 0$
$= 2$, for $x = 0$
is continuous at $x = 0$

ii. $f(x) = \frac{\log(1+kx)}{\sin x}$, for $x \neq 0$
$= 5$, for $x = 0$
is continuous $x = 0$

iii. $f(x) = x^2 + k$, for $x \geq 0$
$= -x^2 - k$, for $x < 0$
is continuous at $x = 0$

iv. $f(x) = \frac{x^2+3x+k}{2(x^2-1)}$, for $x \neq 1$
$= \frac{5}{4}$, for $x = 1$
is continuous at $x = 1$

+v. $f(x) = \frac{xe^x + \tan x}{\sin 3x}$, for $x \neq 0$
$= k$ for $x = 0$

+16. If f is continuous at $x = 1$, where
$f(x) = \frac{\sin(\pi x)}{x-1} + a$, for $x < 1$
$= 2\pi$, for $x = 1$
$= \frac{1+\cos(\pi x)}{\pi(1-x)^2} + b$, for > 1,
then find the values of a and b.

17. If f is continuous at $x = 0$, where
$f(x) = x^2 + a$, for $x \geq 0$
$= 2\sqrt{x^2+1} + b$, for $x < 0$
Find a, b given that $f(1) = 2$.

18. If $f(x) = \frac{\tan 2x}{3x} + a$, for $x < 0$
$= 1$, for $x = 0$
$= x + 4 - b$, for $x > 0$
is continuous at $x = 0$, then find the values of a and b.

19. If the function $f(x) = \frac{k\cos x}{\pi - 2x}$, for $x \neq \frac{\pi}{2}$
$= 3$, for $x = \frac{\pi}{2}$
be continuous at $x = \frac{\pi}{2}$, then find k

20. Is the function
$f(x) = 2x^3 + 3x^2 + 3x - \cos x + \sin 5x + 3$
continuous at $x = \frac{\pi}{4}$? Justify

Based on Miscellaneous Exercise – 8

1. Examine the continuity of the following funtions at the given point:

i. $f(x) = \frac{10^x + 7^x - 14^x - 5^x}{1 - \cos x}$, for $x \neq 0$
$= \frac{10}{7}$, for $x = 0$; at $x = 0$

ii. $f(x) = \frac{\sin 3x}{\tan 2x}$, for $x < 0$
$= \frac{3}{2}$, for $x = 0$
$= \frac{\log(1+3x)}{e^{2x}-1}$, for $x > 0$

iii. $f(x) = \frac{\sqrt{23} - \sqrt{4x-1}}{(x-6)}$, for $x \neq 6$
$= \frac{1}{5}$, for $x = 6$; at $x = 6$

iv. $f(x) = \frac{\sqrt{1-2x} - \sqrt{1+2x}}{x}$, for $x < 0$
$= 2x^2 + 3x - 2$, for $x \geq 0$; at $x = 0$

v. $f(x) = \frac{x^3 + x^2 - 16x + 20}{(x-2)^2}$, for $x \neq 2$
$= 7$, for $x = 2$; at $x = 2$

2. Discuss the continuity of the following functions:

i. $f(x) = \frac{2^x - 5^x}{4^x - 3^x}$, for $x \neq 0$
$= \log \frac{3}{10}$, for $x = 0$; at $x = 0$

ii. $f(x) = \frac{(2^x-1)^2}{\tan x . \log(1+x)}$, for $x \neq 0$
$= \log 4$, for $x = 0$

3. Discuss the continuity of the following functions at the points given against them. If the function is discontinuous, determine whether the discontinuity is removable. In that case, redefine the function, so that it becomes continuous:

i. $f(x) = \frac{4^x - e^x}{6^x - 1}$, for $x \neq 0$
$= \log\left(\frac{2}{3}\right)$, for $x = 0$; at $x = 0$

ii. $f(x) = \frac{3^x + 3^{-x} - 2}{x^2}$, for $x \neq 0$
$= 2\log 3$, for $x = 0$; at $x = 0$

iii. $f(x) = \dfrac{x^6 - \frac{1}{64}}{x^3 - \frac{1}{8}}$, for $x \neq \frac{1}{2}$

$= \frac{1}{3}$, for $x = \frac{1}{2}$; at $x = \frac{1}{2}$

iv. $f(x) = \dfrac{(8^x - 1)^2}{\sin x \log\left(1 + \frac{x}{4}\right)}$, for $x \neq 0$

$= 8 \log 8$, for $x = 0$; at $x = 0$

4. If f is continuous at $x = 0$, then find f(0).

i. $f(x) = \dfrac{4^x - 2^{x+1} + 1}{1 - \cos x}$, $x \neq 0$

ii. $f(x) = \dfrac{e^{5x} - e^{2x}}{\sin 3x}$

5. Find the value of k, if the function

$f(x) = \dfrac{\sin^2 \frac{3x}{2}}{x^2}$, for $x \neq 0$

$= k$, for $x = 0$

is continuous at $x = 0$

6. If $f(x) = \dfrac{\sin 4x}{5x} + a$, for $x > 0$

$= x + 4 - b$, for $x < 0$

$= 1$, for $x = 0$

is continuous at $x = 0$, find a and b.

7. If $f(x) = \dfrac{1 - \cos 4x}{x^2}$, for $x < 0$

$= a$, for $x = 0$

$= \dfrac{\sqrt{x}}{\sqrt{\left(16 + \sqrt{x}\right)} - 4}$, for $x > 0$

is continuous at $x = 0$, then find the value of 'a'.

8. Discuss the continuity of the function f at $x = 0$, where $f(x) = \dfrac{5^x + 5^{-x} - 2}{\cos 2x - \cos 6x}$, for $x \neq 0$

$= \frac{1}{8}(\log 5)^2$, for $x = 0$

Multiple Choice Questions

1. If $f(x) = \begin{cases} 2 & , 0 \le x < 1 \\ c - 2x & , 1 \le x \le 2 \end{cases}$ is continuous at $x = 1$, then c =

(A) 2 (B) 4
(C) 0 (D) 1

2. If $f(x) = \begin{cases} 1 & , \text{ if } x \le 3 \\ ax + b & , \text{ if } 3 < x < 5 \\ 7 & , \text{ if } 5 \le x \end{cases}$ is continuous, then the value of a and b is

(A) 3, 8 (B) –3, 8
(C) 3, –8 (D) –3, –8

3. The sum of two discontinuous functions
(A) is always discontinuous.
(B) may be continuous.
(C) is always continuous.
(D) may be discontinuous.

4. For what value of k the function $f(x) = \begin{cases} \dfrac{\sqrt{5x+2} - \sqrt{4x+4}}{x-2}, & \text{if } x \neq 2 \\ k & , \text{if } x = 2 \end{cases}$ is continuous at $x = 2$?

(A) $\dfrac{-1}{4\sqrt{3}}$ (B) $\dfrac{1}{2\sqrt{3}}$
(C) $\dfrac{1}{4\sqrt{3}}$ (D) $\dfrac{-1}{2\sqrt{3}}$

5. The function $f(x) = \dfrac{\log(1 + ax) - \log(1 - bx)}{x}$ is not defined at $x = 0$. The value which should be assigned to f at $x = 0$ so that it is continuous at $x = 0$, is

(A) a – b (B) a + b
(C) log a + log b (D) log a – log b

6. In order that the function $f(x) = (x + 1) \cot x$ is continuos at $x = 0$, f(0) must be defined as

(A) $f(0) = \frac{1}{e}$ (B) $f(0) = 0$
(C) $f(0) = e$ (D) None of these

7. If $f(x) = \begin{cases} \dfrac{\sin 3x}{\sin x}, & x \neq 0 \\ k, & x = 0 \end{cases}$ is a continuous function, then k =

(A) 1 (B) 3 (C) $\frac{1}{3}$ (D) 0

8. A function f is continuous at a point $x = a$ in the domain of 'f' if

(A) $\lim_{x \to a} f(x)$ exists (B) $\lim_{x \to a} f(x) = f(a)$
(C) $\lim_{x \to a} f(x) \neq f(a)$ (D) both (A) and (B)

9. Which of the following function is discontinuous?

(A) $f(x) = x^2$ (B) $g(x) = \tan x$
(C) $h(x) = \dfrac{3x}{x^2 + 1}$ (D) none of these

10. If the function $f(x) = \begin{cases} \dfrac{k \cos x}{\pi - 2x}, & \text{when } x \neq \frac{\pi}{2} \\ 3, & \text{when } x = \frac{\pi}{2} \end{cases}$ is continuous at $x = \frac{\pi}{2}$, then k =

(A) 3 (B) 6
(C) 12 (D) None of these

11. The points at which the function $f(x) = \frac{x+1}{x^2+x-12}$ is discontinuous, are
(A) –3,4 (B) 3,–4
(C) –1,–3,4 (D) –1,3,4

12. Which of the following statement is true for graph $f(x) = \log x$
(A) Graph shows that function is continuous
(B) Graph shows that function is discontinuous
(C) Graph finds for negative and positive values of x
(D) Graph is symmetric along x-axis

13. If $f(x) = \begin{cases} \frac{x^2-1}{x+1}, & \text{when } x \neq -1 \\ -2, & \text{when } x = -1 \end{cases}$, then
(A) $\lim_{x\to(-1)^-} f(x) = -2$
(B) $\lim_{x\to(-1)^+} f(x) = -2$
(C) $f(x)$ is continuous at $x = -1$
(D) All the above are correct

14. If $f(x) = \begin{cases} \frac{|x-a|}{x-a}, & \text{when } x \neq a \\ 1, & \text{when } x = a \end{cases}$, then
(A) $f(x)$ is continuous at $x = a$
(B) $f(x)$ is discontinuous at $x = a$
(C) $\lim_{x\to a} f(x) = 1$
(D) None of these

15. If $f(x) = \begin{cases} \frac{1-\cos 4x}{x^2}, & \text{when } x < 0 \\ a & \text{when } x = 0, \\ \frac{\sqrt{x}}{\sqrt{(16+\sqrt{x})}-4}, & \text{when } x = 0 \end{cases}$
is continuous at $x = 0$, then the value of 'a' will be
(A) 8 (B) –8
(C) 4 (D) None of these

16. If $f(x) = \begin{cases} \frac{x^4-16}{x-2}, & \text{when } x \neq 2 \\ 16, & \text{when } x = 2 \end{cases}$, then
(A) $f(x)$ is continuous at $x = 2$
(B) $f(x)$ is discountinuous at $x = 2$
(C) $\lim_{x\to 2} f(x) = 16$
(D) None of these

17. The values of A and B such that the function $f(x) = \begin{cases} -2\sin x, & x \leq -\frac{\pi}{2} \\ A\sin x + B, & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ \cos x, & x \geq \frac{\pi}{2} \end{cases}$, is continuous everywhere are
(A) A = 0, B = 1 (B) A = 1, B = 1
(C) A = –1, B = 1 (D) A = –1, B = 0

18. If $f(x) = \begin{cases} \frac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, & \text{for } -1 \leq x < 0 \\ 2x^2+3x-2, & \text{for } 0 \leq x \leq 1 \end{cases}$, is continuous at $x = 0$, then k =
(A) –4 (B) –3
(C) –2 (D) –1

19. The function $f(x) = \sin |x|$ is
(A) Continuous for all x
(B) Continuous only at certain points
(C) Differentiable at all points
(D) None of these

20. The function $f(x) = \frac{1-\sin x+\cos x}{1+\sin x+\cos x}$ is not defined at $x = \pi$. The value of $f(\pi)$, so that $f(x)$ is continuous at $x = \pi$, is
(A) $-\frac{1}{2}$ (B) $\frac{1}{2}$
(C) –1 (D) 1

21. The function $f(x) = \frac{2x^2+7}{x^3+3x^2-x-3}$ is discontinuous for
(A) $x = 1$ only
(B) $x = 1$ and $x = -1$ only
(C) $x = 1, x = -1, x = -3$ only
(D) $x = 1, x = -1, x = -3$ and other values of x

22. The function f is defined by $f(x) = 2x - 1$, if $x > 2$, $f(x) = k$ if $x = 2$ and $x^2 - 1$, if $x < 2$ is continuous, then the value of k is equal to
(A) 2 (B) 3
(C) 4 (D) –3

23. Function $f(x) = \frac{1-\cos 4x}{8x^2}$, where $x \neq 0$ and $f(x) = k$, where $x = 0$ is a continous function at $x = 0$ then the value of k will be?
(A) k = 0 (B) k = 1
(C) k = –1 (D) None of these

24. If $f(x) = \begin{cases} x, & \text{when } 0 < x < 1/2 \\ 1, & \text{when } x = 1/2 \\ 1-x, & \text{when } 1/2 < x < 1 \end{cases}$, then

(A) $\lim\limits_{x \to 1/2+} f(x) = 2$

(B) $\lim\limits_{x \to 1/2-} f(x) = 2$

(C) $f(x)$ is continuous at $x = \frac{1}{2}$

(D) $f(x)$ is discontinuous at $x = \frac{1}{2}$

25. If $f(x) = \frac{x^2 - 10x + 25}{x^2 - 7x + 10}$ for $x \neq 5$ and f is continuous at $x = 5$, then $f(5) =$

(A) 0 (B) 5
(C) 10 (D) 25

Answers to Additional Practice Problems

Based on Exercise 8.1

1. i. Continuous ii. Continuous
iii. Continuous iv. Discontinuous
v. Continuous vi. Discontinuous
vii. Continuous viii. Continuous
ix. Discontinuous x. Continuous

2. i. Discontinuous ii. Continuous
iii. Continuous iv. Discontinuous
v. Discontinuous vi. Continuous
vii. Discontinuous viii. Discontinuous
ix. Discontinuous

3. Continuous

4. Discontinuous

5. Continuous

6. Continuous

7. Discontinuous

8. i. Discontinuous, removable
ii. Discontinuous, removable
iii. Discontinuous, removable
iv. Discontinuous, removable

9. i. Discontinuous
ii. Discontinuous
iii. Discontinuous

11. $x = \left(\frac{\pi}{2}\right)$

12. Continuous

13. i. $\frac{(\log 4)^2}{2}$ ii. a + b
iii. 1 iv. –4
v. e^4

14. 6

15. i. 2 ii. 5
iii. 0 iv. –4
v. $\frac{2}{3}$

16. $a = 3\pi$, $b = \frac{3\pi}{2}$

17. a = 1, b = –1

18. $a = \frac{1}{3}$, b = 3

19. 6

20. Addition of continuous functions.
∴ $f(x)$ is continuous.

Based on Miscellaneous Exercise – 8

1. i. Discontinuous ii. Continuous
iii. Discontinuous iv. Continuous
v. Continuous

2. i. Discontinuous ii. Discontinuous

3. i. Discontinuous, removable
ii. Discontinuous, removable
iii. Discontinuous, removable
iv. Discontinuous, removable

4. i. $2(\log 2)^2$ ii. 1

5. $\frac{9}{4}$

6. $a = \frac{1}{5}$, b = 3

7. 8

8. Discontinuous

Answers to Multiple Choice Questions

1.	(B)	2.	(C)	3.	(B)	4.	(C)
5.	(B)	6.	(C)	7.	(B)	8.	(D)
9.	(B)	10.	(B)	11.	(B)	12.	(A)
13.	(D)	14.	(B)	15.	(A)	16.	(B)
17.	(C)	18.	(C)	19.	(A)	20.	(C)
21.	(C)	22.	(B)	23.	(B)	24.	(D)
25.	(A)						

Competitive Corner

1. For what value of k, the function defined by

$$f(x) = \frac{\log(1+2x)\sin x^\circ}{x^2}, \quad \text{for } x \neq 0$$
$$= k \quad , \quad \text{for } x = 0$$

is continuous at $x = 0$?

[MHT CET 2016]

(A) 2 (B) $\frac{1}{2}$

(C) $\frac{\pi}{90}$ (D) $\frac{90}{\pi}$

2. If the function f(x) defined by

$$f(x) = x\sin\frac{1}{x}, \quad \text{for } x \neq 0$$
$$= k \quad , \quad \text{for } x = 0$$

is continuous at $x = 0$, then k =

[MHT CET 2016]

(A) 0 (B) 1

(C) – 1 (D) $\frac{1}{2}$

3. If the function $f(x) = (x + 1)^{\cot x}$ is continuous at $x = 0$, then f(0) = **[MHT CET 2019]**

(A) $\frac{1}{e}$ (B) $\frac{1}{e^2}$

(C) e (D) $\frac{1}{e^3}$

4. If f(x) is continuous at x = a, where $f(x) = \frac{\sqrt{x}-\sqrt{a}+\sqrt{x-a}}{\sqrt{x^2-a^2}}$, for $x \neq$ a, then f(a) =

[MHT CET 2019]

(A) $\frac{1}{\sqrt{2a}}$ (B) $\frac{1}{2\sqrt{a}}$

(C) $\frac{1}{2a}$ (D) $2\sqrt{a}$

Answers:

1. (C) 2. (A) 3. (C) 4. (A)

Hints:

1. For f(x) to be continuous at x = 0,

$f(0) = \lim_{x\to 0} f(x)$

$\therefore$ $k = \lim_{x\to 0} \frac{\log(1+2x)\sin x^\circ}{x^2}$

$\therefore$ $k = \lim_{x\to 0} \frac{\log(1+2x)}{2x} \times 2 \times \frac{\sin\frac{\pi x}{180}}{\frac{\pi x}{180}} \times \frac{\pi}{180}$

$\therefore$ $k = 1 \times 2 \times 1 \times \frac{\pi}{180} = \frac{\pi}{90}$

2. $\lim_{x\to 0} f(x) = \lim_{x\to 0} x\sin\frac{1}{x}$, but $-1 \leq \sin\frac{1}{x} \leq 1$ and $x \to 0$

$\therefore$ $\lim_{x\to 0} f(x) = 0$

Since, f(x) is continuous at x = 0.

$\therefore$ $f(0) = \lim_{x\to 0} f(x)$

$\therefore$ k = 0

3. Since, f(x) is continuous at x = 0.

$\therefore$ $f(0) = \lim_{x\to 0} f(x)$

$= \lim_{x\to 0} (x+1)^{\cot x}$

$= \lim_{x\to 0} \left[(1+x)^{\frac{1}{x}}\right]^{\frac{x}{\tan x}}$

$= e^1 = e$

4. Since, f(x) is continuous at x = a.

$\therefore$ $f(a) = \lim_{x\to a} f(x)$

$= \lim_{x\to a} \frac{\sqrt{x}-\sqrt{a}+\sqrt{x-a}}{\sqrt{x^2-a^2}}$

$= \lim_{x\to a} \frac{\sqrt{x}-\sqrt{a}+\sqrt{x-a}}{\sqrt{x+a}\,.\sqrt{x-a}}$

$= \lim_{x\to a} \frac{1}{\sqrt{x+a}}\left(\frac{\sqrt{x}-\sqrt{a}+\sqrt{x-a}}{\sqrt{x-a}}\right)$

$= \frac{1}{\sqrt{2a}} \lim_{x\to a}\left(\frac{\sqrt{x}-\sqrt{a}}{\sqrt{x-a}}+1\right)$

$= \frac{1}{\sqrt{2a}} \lim_{x\to a}\left[\frac{\left(\sqrt{x}\right)^2-\left(\sqrt{a}\right)^2}{\sqrt{x-a}\left(\sqrt{x}+\sqrt{a}\right)}+1\right]$

$= \frac{1}{\sqrt{2a}} \lim_{x\to a}\left(\frac{x-a}{\sqrt{x-a}\left(\sqrt{x}+\sqrt{a}\right)}+1\right)$

$= \frac{1}{\sqrt{2a}} \lim_{x\to a}\left[\frac{\sqrt{x-a}}{\sqrt{x}+\sqrt{a}}+1\right]$

$= \frac{1}{\sqrt{2a}}(0+1)$

$= \frac{1}{\sqrt{2a}}$