MA 231 Vector Analysis

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Preface

Health Warning: These notes give the skeleton of the course and are not a substitute for attending lectures. They are meant to make note-taking easier so that you can concentrate on the lectures. An important part in vector analysis are figures and pictures. These will not be contained in these notes. For any figure which appears on the blackboard in my lectures I leave some empty space with a reference number which coincides with the number I am using in the lectures. You can fill the diagrams and figures by your own.

These notes grew out of hand written notes from Jochen Voß who gave this course 2005 and 2006. I thank him very much for letting me using his notes.
Any remarks and suggestions for improvements would help to create better notes for the next year.

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Motivation

What is Vector Analysis?
In analysis differentiation and integration were mostly considered in one dimension. Vector analysis generalises this to curves, surfaces and volumes in $\mathbb{R}^n, n \in \mathbb{N}$. As an example consider the "normal" way to calculate a one dimensional integral: You may find a primitive of a function f and use the fundamental theorem of calculus, i.e. for $f = F'$ we get

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a).$$

The value of the integral can be determined by looking at the boundary points of the interval $[a, b]$. Does this also work in higher dimensions? The answer is given by Gauss's divergence theorem.

Notation

One of the main problems in vector analysis is that there are many books with all possible different notations. During the whole course I outline alternative notations in use. It is one of the objectives to acquaint you with the different notations and symbols. Note that most of the material originated from physics and hence many books are using notations and symbols known by people in physics.

Vectors: $x \in \mathbb{R}^n$ with $x = (x_1, \ldots, x_n)$ and with norm $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$. Alternative notations are: $\vec{x}, \underline{x}, \boldsymbol{x}$ and in dimension $n = 3$ $\vec{r} = (x, y, z)$ for the vector and $|x|$ or r for the norm of $\vec{x}, \underline{x}, \boldsymbol{x}$ or $\vec{r}$.
Properties of the norm are $(i) \|x\| \geq 0$; $(ii) \|\lambda x\| = |\lambda| \|x\|$ for $\lambda \in \mathbb{R}$; $(iii) \|x + y\| \leq \|x\| + \|y\|$ for $x, y \in \mathbb{R}^n$.
Functions: $f \colon \mathbb{R}^m \to \mathbb{R}^n$ with component functions $f_1, \ldots, f_n \colon \mathbb{R}^m \to \mathbb{R}$. Alternative notations are: $\vec{f}$ or $\underline{f}$ or $\boldsymbol{f}$.
Partial derivatives: Let $\mathrm{e}_i, 1 \leq i \leq n$, be the canonical basis vectors in $\mathbb{R}^n$ (i.e. $\langle \mathrm{e}_i, \mathrm{e}_j \rangle$ for $i, j{+}1, \ldots, n$). The partial derivative of a function $f \colon \mathbb{R}^n \to \mathbb{R}$ with respect to the i-th direction at point $x \in \mathbb{R}^n$ is given by

$$\frac{\partial f(x)}{\partial x_i} = \lim_{h \to 0} \frac{f(x + h\mathrm{e}_i) - f(x)}{h}, \qquad x \in \mathbb{R}^n.$$

Alternative notation is: $\partial_i f(x)$.
Scalar product: The scalar product (dot product) is defined as $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$ for $x, y \in \mathbb{R}^n$. Alternative expression is $x \cdot y$. Recall that $\langle x, y \rangle = \|x\| \|y\| \cos\theta$ where θ is the angle between the two vectors x and y. Note that $\|y\| \cos\theta$ is the component of the vector y in direction of the vector x.

Part I: Real Vectoranalysis

1 Gradients and Directional Derivatives

In this section we ask how does a function $f\colon \mathbb{R}^n \to \mathbb{R}$ change when we move from $x \in \mathbb{R}^n$ in direction of a vector $y \in \mathbb{R}^n$.

Figure 1

We can reduce the problem to one dimension. We define for the given $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ the function

$$\begin{aligned}\varphi_{(x)}\colon \mathbb{R} &\to \mathbb{R}\\ t &\mapsto \varphi_{(x)}(t) = f(x+ty).\end{aligned}$$

The change of f in direction y equals the change of $\varphi_{(x)}$ and is thus given by $\varphi'_{(x)}$.

Definition 1.1 *The* ***directional derivative*** $D_y f(x)$ *of a function* $f\colon \mathbb{R}^n \to \mathbb{R}$ *at the point* $x \in \mathbb{R}^n$ *in direction of* $y \in \mathbb{R}^n$ *is given by*

$$\boxed{D_y f(x) = \lim_{t\to 0} \frac{f(x+ty) - f(x)}{t}}$$

if the indicted limit exists. $D_y f(x)$ *is also called the derivative along the vector* y *at the point* x.

Remark 1.2 *(a) With* $\varphi_{(x)}(t) = f(x+ty)$ *the directional derivative of* f *at the point* $x \in \mathbb{R}^n$ *in direction* $y \in \mathbb{R}^n$ *is given as*

$$D_y f(x) = \varphi'_{(x)}(0).$$

(b) We have defined the directional derivative via the limit as $t \to 0$ *for the differential quotient. If one takes the limit* $t \downarrow 0$*, that is* $t > 0$ *and* $t \to 0$*, one gets the directional derivative from the right hand side. The same applies to the limit* $t \uparrow 0$ *for the directional derivative from the left.*

We calculate directional derivatives in the next example.

Example 1.3 Let $f\colon \mathbb{R}^2 \to \mathbb{R}$, $f(x) = x_1^2 + x_2^2$ be given. Then for $x, y \in \mathbb{R}^2$ we have

$$\begin{aligned} f(x+ty) &= (x_1 + ty_1)^2 + (x_2 + ty_2)^2 \\ &= x_1^2 + x_2^2 + 2t(x_1y_1 + x_2y_2) + t^2(y_1^2 + y_2^2) \end{aligned}$$

Hence $D_y f(x) = \varphi'_{(x)}(0) = 2\langle x, y\rangle$.

Figure 2

◇

Recall $\varphi_{(x)}(t) = f(x+ty), t \in \mathbb{R}, x, y \in \mathbb{R}^n$. The function $\varphi_{(x)} \colon \mathbb{R} \to \mathbb{R}$ is the composition of the function $g \colon \mathbb{R} \to \mathbb{R}^n, t \mapsto x + ty$, and the function $f \colon g(\mathbb{R}) \to \mathbb{R}$ (i.e. the restrition of f on $g(\mathbb{R}) \subset \mathbb{R}^n$), that is

$$\varphi_{(x)}(t) = (f \circ g)(t) = f(g(t)).$$

The chain rule gives

$$\begin{aligned}(f \circ g)'(t) &= \sum_{i=1}^{n} \partial_i f(g(t)) g_i'(t) \\ &= \sum_{i=1}^{n} \partial_i f(x+ty) y_i,\end{aligned}$$

where $g_i(t) = x_i + ty_i, t \in \mathbb{R}, 1 \leq i \leq n$, are the component functions of g. Hence we get

$$D_y f(x) = \varphi'_{(x)}(0) = \sum_{i=1}^{n} \partial_i f(x) y_i. \tag{1.1}$$

We can write (1.1) in a shorter way with the following definition.

Definition 1.4 *Let the function $f \colon \mathbb{R}^n \to \mathbb{R}$ be differentiable. The mapping $\nabla f \colon \mathbb{R}^n \to \mathbb{R}^n, x \mapsto \nabla f(x) = (\partial_1 f(x), \ldots, \partial_n f(x))$ is called the gradient mapping and the vector $\nabla f(x) = (\partial_1 f(x), \ldots, \partial_n f(x))$ is called the **gradient** of f at the point x. Alternative notations are* grad f, $\underline{\nabla f}$, $\underline{\nabla} f$ *or* $\nabla f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \mathrm{e}_i$.

Note that $\nabla f \colon \mathbb{R}^n \to \mathbb{R}^n$ is a "vector" of the component functions $\partial_i f \colon \mathbb{R}^n \to \mathbb{R}, x \mapsto \partial_i f(x), 1 \leq i \leq n$.

Definition 1.5 *A scalar or a vector quantity is said to be a **field** if it is a function of the spatial position. Examples: Let $D \subset \mathbb{R}^m$, then $f \colon D \to \mathbb{R}^n$ is called a **vector field** if $n > 1$, and $f \colon D \to \mathbb{R}$ is called a **scalar field**.*

Examples for vector fields are the magnetic, the electric or the velocity (vector) field, whereas temperature and pressure are scalar fields.

Our calculation in (1.1) shows that the directional derivative $D_y f$ at any point $x \in \mathbb{R}^n$ is linear in y and we only need to know the gradient $\nabla f(x)$ in order to calculate $D_y f(x)$

$$\boxed{D_y f(x) = \langle \nabla f(x), y \rangle} \tag{1.2}$$

Remark 1.6 *Recall the Cauchy-Schwarz inequality*

$$|\langle x, y\rangle| \leq \|x\|\|y\| \qquad \textit{for } x, y \in \mathbb{R}^n.$$

We get

$$|D_y f(x)| = |\langle \nabla f(x), y\rangle| \leq \|\nabla f(x)\|\|y\|,$$

and for $y \in \mathbb{R}^n$ with $\|y\| = 1$ we have $|D_y f(x)| \leq \|\nabla f(x)\|$. Assume that $\nabla f(x) \neq 0$ and pick $y = \frac{\nabla f(x)}{\|\nabla f(x)\|}$ with $\|y\| = 1$. This gives

$$D_y f(x) = \Big\langle \nabla f(x), \frac{\nabla f(x)}{\|\nabla f(x)\|}\Big\rangle = \|\nabla f(x)\|.$$

We conclude that $\nabla f(x)$ (under the assumption that $\nabla f(x) \neq 0$) points in the direction along which f is increasing the fastest. Alternatively you may prove this via $\langle \nabla f(x), y\rangle = \|\nabla f(x)\| \cos\theta$ for any unit vector y having an angle θ with $\nabla f(x)$.

2 Visualisation of functions $f\colon \mathbb{R}^m \to \mathbb{R}^n$

Visualisation of functions is hard and needs practice. One of the objectives of the course is to learn basic techniques to create sketches/figures showing basic features of a given function.

2.1 Scalar fields, $n = 1$

We recall the notion of a graph of a function and introduce the notion of a level set of a function. The graph of a function $f\colon \mathbb{R}^m \to \mathbb{R}^n$ is easily pictured for $m = n = 1$ and $m = 2, n = 1$, see Figure 3.

Figure 3:

Definition 2.1 *Let $f \colon D \to \mathbb{R}$ be a function for some domain $D \subset \mathbb{R}^m$.*

*(a) The **graph** of f is the subset of $\mathbb{R}^{m+1}$ consisting of all points $(x, f(x)), x \in D$. In symbols,*

$$\operatorname{graph} f = \{(x, f(x)) \in \mathbb{R}^{m+1} \colon x \in D\}.$$

(b) Let $c \in \mathbb{R}$. The set

$$f^{-1}(c) = \{x \in D \colon f(x) = c\}$$

*is called the **c-level set** of the function f. In dimension $m = 2$ we speak also of a level curve and in dimension $m = 3$ of a level surface.*

The behaviour or structure of a function is determined in part by the shape of its level sets; consequently, understanding these sets is of great help understanding the functions in question. The idea of level sets is also used in drawing contour maps, where one draws lines to represent constant altitude. Walking along such a line would mean walking on a level path. In the case of a hill rising from the $x - y$ plane, a graph of all level curves gives us a good idea of the 'height' function $h(x, y)$, which represents the height of the hill at point (x, y).

Example 2.2 (a) $f\colon \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto f(x, y) = x^2 + y^2$.

Figure 4:

(b) $f\colon \mathbb{R}^3 \to \mathbb{R}, (x, y, z) \mapsto f(x, y, z) = z^2 - x^2 - y^2$.
Let $c = 0$: $f^{-1}(0)$ is a cone,

$$f(x, y, z) = 0 \Leftrightarrow r^2 := x^2 + y^2 = z^2 \Leftrightarrow r = |z|,$$

where $r = \sqrt{x^2 + y^2}$.

Let $c > 0$:

$$f(x, y, z) = c \Leftrightarrow r = \sqrt{z^2 - c} \text{ with } z^2 \geq c \Leftrightarrow z = \pm\sqrt{x^2 + y^2 + c}.$$

Hence the level set is a hyperboloid of two sheets around the z axis, passing through the z axis at the points $(0, 0, \pm\sqrt{c})$.

Figure 5: cone

Figure 6: hyperboloid (two sheets)

Let $c < 0$:

$$f(x, y, z) = c \Leftrightarrow r = \sqrt{z^2 - c} \Leftrightarrow z = \pm\sqrt{x^2 + y^2 - c}.$$

The level set (surface) is the single-sheeted hyperboloid of revolution around the z axis, which intersects the $x-y$ plane in the circle of radius $\sqrt{-c}$.

Figure 7: hyperboloid (single-sheet)

$\diamond$

2.2 Vector fields, $n > 1$

We are going to sketch vector fields. As an example think about the velocity vector field inside a fluid or the electric and magnetic field of power currents.

Example 2.3 (a) $f\colon \mathbb{R}^2 \to \mathbb{R}^2, (x,y) \mapsto f(x,y) = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$, see figure 8 below.

Figure 8:

(b) $f\colon \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto f(x, y) = \begin{pmatrix} x \\ y \end{pmatrix}$, see figure 9 below.

Figure 9:

(c) $f\colon \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto f(x, y) = \begin{pmatrix} y \\ x \end{pmatrix}$, see figure 10 below.

Figure 10:

◇

Vector fields in two dimensions can be visualised by a sketch. In this case the simplest procedure is to evaluate the vector field at a sequence of points and draw vectors indicating the magnitude and direction of the vector field at each point. An example of this procedure is the drawing of wind speeds and directions on weather maps. In the above example 2.2 the third vector field at point is $f(1, 0) = (0, 1)$, so at this point a vector of magnitude 1 in the y-direction is drawn. By considering a few additional points, a sketch of the vector field can be built up (see figure 10).

2.3 Curves and Surfaces

We study now curves and surfaces, i.e. we consider $m < n$. We consider mainly two examples, $m = 1, 2$. For $m = 1$ we have parametric curves in $\mathbb{R}^n$ and for $m = 2$ we have parametric surfaces in $\mathbb{R}^n$. Parametric curves and surfaces are defined via mappings from subsets of $\mathbb{R}$ respectively subsets of $\mathbb{R}^2$ into $\mathbb{R}^n$. In the next example we study the case $n = 3$. That is we have no axis available for the arguments of the mappings, we only sketch the range of these mappings.

Example 2.4 (a)

$$f\colon [0,2\pi] \to \mathbb{R}^3, t \mapsto f(t) = \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix}.$$

This defines the helix seen in the figure 11 below.

Note that for $c \in \mathbb{R}$ the mapping $f_c\colon [0,2\pi] \to \mathbb{R}^3, t \mapsto f_c(t) = \begin{pmatrix} \cos t \\ \sin t \\ c \end{pmatrix}$ defines a circle line in the $x-y$ plane shifted in z direction by c.

(b)

$$f\colon [0,1] \times [0,2\pi] \to \mathbb{R}^3, (s,t) \mapsto f(s,t) = \begin{pmatrix} s\cos t \\ s\sin t \\ t \end{pmatrix}.$$

This mapping defines the helicoid seen in figure 12.

Note that for fixed parameter t we have lines within the surface of the helicoid and through any point of the helicoid there is a helix going through that point. ◇

Curves can be given in two ways. A parametric curve is a map $\varphi\colon [a,b] \to \mathbb{R}^n$, e.g. $\varphi(t) = (\cos t, \sin t) \in \mathbb{R}^2$ for $t \in [0,2\pi]$. Curves in $\mathbb{R}^n$, i.e. subsets $\mathcal{C} \subset \mathbb{R}^n$, can be given as the level set of some real valued function, e.g. consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}, (x,y) \mapsto f(x,y) = x^2+y^2$. The curve $\mathcal{C}$ is then the level set

$$\mathcal{C} = \{(x,y) \in \mathbb{R}^2\colon x^2+y^2 = 1\},$$

which is again the circle line in $x-y$ plane.

For a parametric curve $\varphi\colon \mathbb{R} \to \mathbb{R}^n$ any point $\varphi(t)$ gives the 'position' at 'time' t. The derivative with respect to the parameter t gives the velocity vector $\varphi'(t)$ at time "time" t. Both are vectors in $\mathbb{R}^n$ with the following components

$$\varphi(t) = (\varphi_1(t), \dots, \varphi_n(t)) \in \mathbb{R}^n \text{ and } \varphi'(t) = (\varphi_1'(t), \dots, \varphi\prime_n(t)) \in \mathbb{R}^n.$$

If $\varphi'(t) \neq 0$, then $\varphi'(t)$ is a tangent vector of the curve. The **tangent line** $T_{\varphi(t)}$ at a point $\varphi(t)$ is given by

$$T_{\varphi(t)}\colon \mathbb{R} \to \mathbb{R}^n, \lambda \mapsto T_{\varphi(t)}(\lambda) = \varphi(t) + \lambda\varphi'(t).$$

This is a straight line through the point $\varphi(t)$ in direction of $\varphi'(t)$.

Figure 11: helix

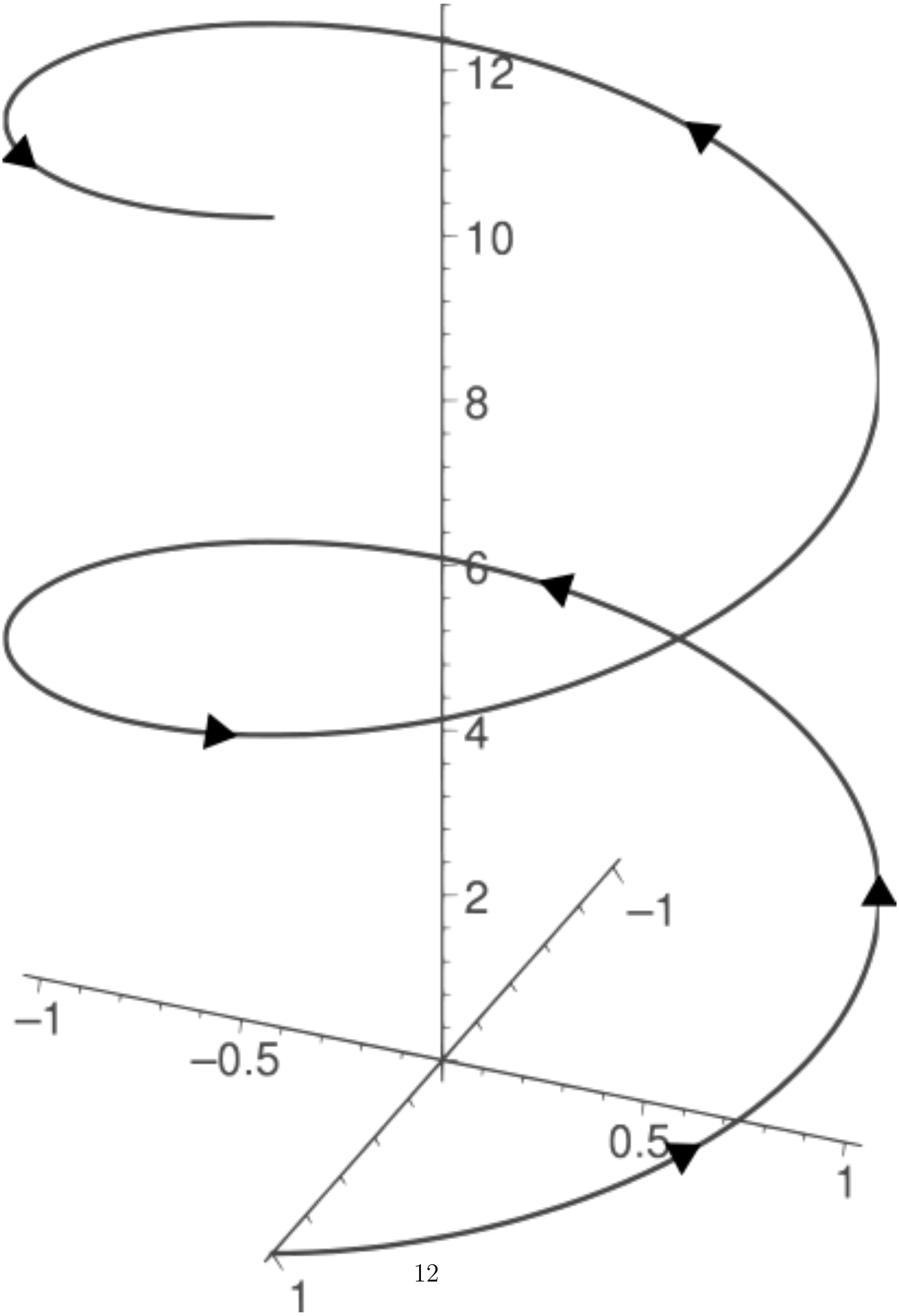
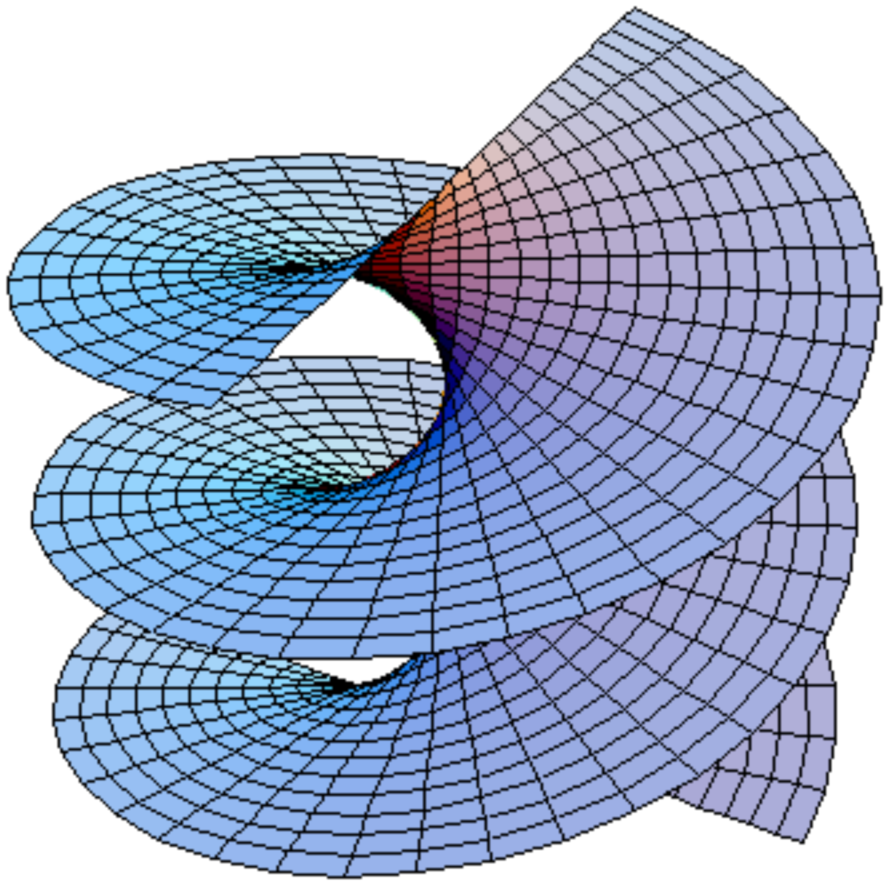


Figure 12: helicoid



Definition 2.5 *A vector $x \in \mathbb{R}^n$ is* ***orthogonal*** *to a parametric curve $\varphi\colon \mathbb{R} \to \mathbb{R}^n$ at the point $\varphi(t)$ if $\langle x, \varphi'(t)\rangle = 0$, i.e. if it is orthogonal to the tangent line.*

Lemma 2.6 *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be differentiable and $a \in \mathbb{R}^n$. Then*

$$\nabla f(a) \perp \{x \in \mathbb{R}^n \colon f(x) = f(a)\} =: \mathcal{L}(f(a)).$$

Proof. Let $\varphi\colon \mathbb{R} \to \mathcal{L}(f(a))$ be a differentiable parametric curve in the surface $\mathcal{L}(f(a))$ with $\varphi(0) = a$. We apply the chain rule (see Proposition 2.8 below) and the notion of differentiability in Definition 2.7. Note that $Df(x_0) \in \mathbf{Lin}(\mathbb{R}^n, \mathbb{R}), x_0 \in \mathbb{R}^n$, is a linear mapping given by the $(1 \times n)$-matrix

$$\begin{pmatrix} \frac{\partial f}{\partial x_1}(x_0) & \frac{\partial f}{\partial x_2}(x_0) & \ldots & \frac{\partial f}{\partial x_n}(x_0) \end{pmatrix},$$

and that $D\varphi(t) \in \mathbf{Lin}(\mathbb{R}, \mathbb{R}^n)$ is a linear mapping given by the $(n \times 1)$-matrix

$$\begin{pmatrix} \varphi_1'(t) \\ \varphi_2'(t) \\ \cdot \\ \cdot \\ \varphi_n'(t) \end{pmatrix}.$$

The chain rule gives for the derivative of the composition $f \circ \varphi$ with respect to t at $t = 0$ as

$$D(f \circ \varphi)(t = 0) = Df(\varphi(0)) \circ D\varphi(0),$$

where the $\circ$-operation on the right hand side is the matrix product which in this case is the corresponding scalar product in $\mathbb{R}^n$. With that we get

$$\begin{aligned} 0 = \frac{\mathrm{d}}{\mathrm{d}t} f(\varphi(t))\big|_{t=0} &= \langle \nabla f(\varphi(0)), \varphi'(0)\rangle \\ &= \langle \nabla f(a), \varphi'(0)\rangle. \end{aligned}$$

This implies $\nabla f(a) \perp \varphi'(0)$ and $\varphi'(0)$ is tangent vector, i.e. it is in $\mathcal{L}(f(a))$. $\square$

Recall the notion of (total) differentiability in the following definition. It generalises the notion of differentiability for real-valued functions defined on the real line. Roughly speaking, the existence of the differential quotient is equivalent to a linear approximation (tangent line) to that function.

Definition 2.7 (Differentiability) *Let $D \subset \mathbb{R}^n$ be a domain and $f\colon D \to \mathbb{R}^m, m \geq 1, f = (f_1, \ldots, f_m)$. We say that f is* ***differentiable*** *at $x_0 \in D$ if the partial derivatives of f exist at x_0 and if there exists a linear mapping $L\colon \mathbb{R}^n \to \mathbb{R}^m$ with*

$$\lim_{x \to x_0} \frac{\|f(x) - f(x_0) - L(x - x_0)\|}{\|x - x_0\|} = 0,$$

where the linear mapping L is given by the so-called $(m \times n)-$ ***Jacobi matrix*** *at the point x_0, i.e.*

$$L = Df(x_0) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x_0) & \frac{\partial f_1}{\partial x_2}(x_0) & \ldots & \frac{\partial f_1}{\partial x_n}(x_0) \\ \frac{\partial f_2}{\partial x_1}(x_0) & \frac{\partial f_2}{\partial x_2}(x_0) & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \frac{\partial f_m}{\partial x_n}(x_0) \end{pmatrix}.$$

*The matrix $Df(x_0)$ is said to be the (****total****)* ***derivative*** *of f at x_0. We say that f is differemtiable if it is differentiable at every point of its domain D. In that case the derivative Df of f is mapping*

$$Df\colon D \to \mathbf{Lin}(\mathbb{R}^n, \mathbb{R}^m),$$

where $\mathbf{Lin}(\mathbb{R}^n, \mathbb{R}^m)$ is the space of linear mappings from $\mathbb{R}^n$ to $\mathbb{R}^m$ isomorphic to the space of real $(n \times m)-$ matrices.

Proposition 2.8 (Chain rule) *Pick $n, p, q \in \mathbb{N}$. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^p$ be open subset sets and consider mappings $f\colon U \to \mathbb{R}^p$ and $g\colon V \to \mathbb{R}^q$. Let $x_0 \in U$ be such that $f(x_0) \in V$. Suppose f is differentiable at x_0 and g at $f(x_0)$. Then $g \circ f\colon U \to \mathbb{R}^q$, the composition of g and f, is differentiable at x_0, and we have*

$$D(g \circ f)(x_0) = Dg(f(x_0)) \circ Df(x_0) \in \mathbf{Lin}(\mathbb{R}^n, \mathbb{R}^q),$$

where the $\circ$-operation on the right hand side means composition of the linear mappings which corresponds to the matrix product of the matrices describing the linear mappings.

Notation 2.9 *Let a curve $\mathcal{C} \subset \mathbb{R}^n$ be given as a level set of some function or via some equation. A parametric curve $\gamma\colon [a, b] \to \mathbb{R}^n$ is called a* ***parametrisation*** *(resp C^1 parametrisation if γ is C^1, that is, γ is differentiable and both, γ and γ' are continuous) of the curve $\mathcal{C}$ if $\gamma([a, b]) = \mathcal{C}$. A given curve can have several parametrisations.*

3 Line integrals

In this section we want to take integrals along a curve $\mathcal{C} \subset \mathbb{R}^n$. A curve is a one-dimensional object in $\mathbb{R}^n$.

3.1 Integrating scalar fields

If we integrate the constant scalar field $u\colon \mathbb{R}^n \to \mathbb{R}, x \mapsto u(x) = 1$ along a curve with given parametrisation γ we shall get the length of the γ. And if we integrate some density along that curve we get the total mass of a thin curve.

Figure 13:

Definition 3.1 (Scalar line integral) *Let $\gamma\colon [a,b] \to \mathbb{R}^n$ be a C^1 parametrisation of the curve $\mathcal{C} \subset \mathbb{R}^n$. The **scalar line integral** of a scalar field $u\colon \mathbb{R}^n \to \mathbb{R}$ along the curve $\mathcal{C}$ is given by*

$$\boxed{\int_{\mathcal{C}} u := \int_a^b u(\gamma(t))\|\gamma'(t)\|\,\mathrm{d}t. \qquad (3.3)}$$

*Alternative notations are: $\int_\gamma u, \int_{\mathcal{C}} u\,\mathrm{d}s$ and $\int_\gamma u\,\mathrm{d}s$ and the integral is sometimes also called the **path integral** along the path $\mathcal{C}$.*

Note that the scalar field u needs only to be defined along the curve, i.e. $u\colon \mathcal{C} \to \mathbb{R}$.

Remark 3.2 *The length of a parametric curve $\gamma\colon [a,b] \to \mathbb{R}^n$ is given by*

$$\int_\gamma 1 = \int_a^b \|\gamma'(t)\| \, \mathrm{d}t.$$

As an example consider $n = 2$ and calculate the length of the circle line

$$\gamma\colon [0, 2\pi] \to \mathbb{R}^2, t \mapsto \gamma(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}.$$

As

$$\gamma'(t) = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \quad \textit{and} \quad \|\gamma'(t)\| = \sqrt{(-\sin t)^2 + \cos^2 t} = 1,$$

we have $\int_\gamma 1 = \int_0^{2\pi} \|\gamma'(t)\| \, \mathrm{d}t = 2\pi$.

3.2 Integrating vector fields

As an introductory example consider a particle moving along a curve (path) $\mathcal{C} \subset \mathbb{R}^3$. The particle is acted on by a force $\vec{F}(\vec{x}), \vec{x} \in \mathbb{R}^3$, which is a vector field $\vec{F}\colon \mathbb{R}^3 \to \mathbb{R}^3, \vec{x} \mapsto \vec{F}(\vec{x})$.
What is the total amount of work done as the particles moves along the curve $\mathcal{C}$? We consider a small displacement $\mathrm{d}\vec{x}$ at position $\vec{x}$ within the curve $\mathcal{C}$. Then the work that is done when the particle moves from position $\vec{x}$ to position $\vec{x} + \mathrm{d}\vec{x}$ along the curve $\mathcal{C}$ is precisely $-\vec{F} \cdot \mathrm{d}\vec{x} = -\langle \vec{F}, \mathrm{d}\vec{x} \rangle$.

Figure 14:

Hence, heuristically the total amount of work shall be an integral (the sum of all these small contributions)

$$-\int_{\mathcal{C}} \vec{F} \cdot \mathrm{d}\vec{x}.$$

We make this notion mathematically precise in the following definition.

Definition 3.3 (Tangent line integral) *Let $\mathcal{C} \subset \mathbb{R}^n$ be a curve with a C^1 parametrisation $\gamma\colon [a,b] \to \mathbb{R}^n$. The* ***tangent line integral*** *of a vector field $f\colon \mathbb{R}^n \to \mathbb{R}^n$ along $\mathcal{C}$ is defined as*

$$\boxed{\int_{\mathcal{C}} f = \int_a^b \langle f(\gamma(t), \gamma'(t) \rangle \,\mathrm{d}t.} \tag{3.4}$$

Alternative notations are $\int_{\mathcal{C}} f \cdot \mathrm{d}\vec{s}$, $\int_\gamma f \cdot \mathrm{d}\vec{s}$ or $\int_\gamma f \cdot \hat{T} \,\mathrm{d}s$.

Remark 3.4 The tangent line integral of a vector field $f\colon \mathbb{R}^n \to \mathbb{R}^n$ along the curve $\mathcal{C}$ with parametrisation $\gamma\colon [a,b] \to \mathbb{R}^n$ can be written as

$$\int_{\mathcal{C}} f = \int_a^b \Big\langle f(\gamma(t), \frac{\gamma'(t)}{\|\gamma'(t)\|} \Big\rangle \|\gamma'(t)\| \,\mathrm{d}t,$$

where $\left\langle f(\gamma(t), \frac{\gamma'(t)}{\|\gamma'(t)\|}\right\rangle$ is the projection of f onto the tangent line, see figure 15 below. Hence, the tangent line intergal $\int_\gamma f$ of a vector field f is the scalar line integral of the component of f along the tangent direction.

Figure 15:

We close with two examples.

Example 3.5 (Tangent line integral) We calculate the work done when moving a mass $m > 0$ along a line/curve with parametrisation $\gamma\colon [0,\pi] \to \mathbb{R}^2, t \mapsto \begin{pmatrix} t \\ -\cos t \end{pmatrix}$ in the gravity field $f\colon \mathbb{R}^2 \to \mathbb{R}^2, (x,y) \mapsto f(x,y) = \begin{pmatrix} 0 \\ -mg \end{pmatrix}$. Here g is a constant, called the earth acceleration. Then the work to be done moving the mass from $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ to $\begin{pmatrix} \pi \\ 1 \end{pmatrix}$ is given by

$$\begin{aligned} -\int_\gamma f &= -\int_0^\pi \left\langle \begin{pmatrix} 0 \\ -mg \end{pmatrix}, \begin{pmatrix} 1 \\ \sin t \end{pmatrix} \right\rangle \mathrm{d}t \\ &= mg \int_0^\pi \sin t \,\mathrm{d}t = 2mg. \end{aligned}$$

Figure 16:

◇

Example 3.6 (Scalar line integral) Consider the curve $\mathcal{C}$ with parametrisation $\gamma\colon [-\pi, \pi] \to \mathbb{R}^3, t \mapsto \gamma(t) = (\cos t, \sin t, t)$. The scalar line integral of $u\colon \mathbb{R}^3 \to \mathbb{R}, (x, y, z) \mapsto x^2 + y^2$ along the curve $\mathcal{C}$ is

$$\int_{\mathcal{C}} u = \int_{\mathcal{C}} (x^2 + y^2) = \int_{-\pi}^{\pi} (\cos^2(t) + \sin^2(t)) \|\gamma'(t)\| \,\mathrm{d}t,$$

and as $\gamma'(t) = (-\sin t, \cos t, 1)$ with $\|\gamma'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$ we get

$$\int_{\mathcal{C}} (x^2 + y^2) = 2\pi\sqrt{2}.$$

◇

4 Gradient Vector Fields

In this section we will study gradient vector fields. We will prove the Fundamental Theorem of vector calculus (FTC) for vector fields. Moreover, integrals (tangent line integrals) of gradient vector fields along curves can easily be calculated once the potential (primitive) is known.

4.1 FTC for gradient vector fields

Definition 4.1 *A vector field $f\colon D \to \mathbb{R}^n, D \subset \mathbb{R}^n$, is called a* ***gradient vector field****, if there exists a differentiable scalar field $\Phi\colon D \to \mathbb{R}$ with*

$$f(x) = \nabla\Phi(x) \qquad \text{for all } x \in D. \tag{4.5}$$

Φ is called the ***potential*** *of the vector field f if (4.5) is satisfied. Sometimes a gradient vector field is also called a vector field of gradient type.*

Remark 4.2 *(a) If Φ is a potential for the vector field f, then $\Phi + c$ is a potential for f for any $c \in \mathbb{R}$ as well because $\nabla(\Phi + c)(x) = \nabla\Phi(x), x \in D$. That is, the potential is not unique.*

(b) Not every vector field is of gradient type, e.g.

$$f\colon \mathbb{R}^3 \to \mathbb{R}^3, (x, y, z) \mapsto f(x, y, z) = (z, z, y).$$

If $\Phi\colon \mathbb{R}^3 \to \mathbb{R}$ would be a potential it would satisfy

$$\begin{aligned}\partial_x\Phi(x, y, z) = z &\Rightarrow \Phi(x, y, z) = zx + c_1(y, z)\\ \partial_z\Phi(x, y, z) = y &\Rightarrow \Phi(x, y, z) = yz + c_2(x, y),\end{aligned}$$

which has no solution because of xz versus yz.

The most important result about gradient vector fields is the following theorem.

Theorem 4.3 (FTC for gradient vector fields) *Let $\Phi\colon \mathbb{R}^n \to \mathbb{R}$ be a scalar field (C^1- differentiable) and let $\mathcal{C} \subset \mathbb{R}^n$ be a curve with a (piecewise C^1- differentiable) parametrisation $\gamma\colon [a, b] \to \mathbb{R}^n$. Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a vector field with $\nabla\Phi = f$. Then the tangent line integral of the vector field is given by*

$$\boxed{\int_{\mathcal{C}} f = \Phi(\gamma(b)) - \Phi(\gamma(a)).} \tag{4.6}$$

This is the Fundamental Theorem of calculus for gradient vector fields.

Proof. The map $[a, b] \to \mathbb{R}, t \mapsto \Phi(\gamma(t))$ has derivative

$$\begin{aligned}(\Phi(\gamma(t))' &= \sum_{i=1}^{n} \frac{\partial\Phi}{\partial x_i}(\gamma(t))\frac{\mathrm{d}\gamma_i(t)}{\mathrm{d}t}\\ &= \langle\nabla\Phi(\gamma(t)), \gamma'(t)\rangle = \langle f(\gamma(t)), \gamma'(t)\rangle.\end{aligned}$$

Thus we get

$$\int_{\mathcal{C}} f = \int_a^b \langle f(\gamma(t)), \gamma'(t)\rangle \mathrm{d}t = \int_a^b (\Phi(\gamma(t))'\mathrm{d}t$$
$$= \Phi(\gamma(b)) - \Phi(\gamma(a)),$$

where we used the continuity of Φ' and the Fundamental Theorem of Calculus for functions on the real line.

$\square$

Example 4.4 (Revisit of Example 3.5) Moving a mass $m > 0$ in the gravity field $f\colon \mathbb{R}^2 \to \mathbb{R}^2, (x,y) \mapsto f(x,y) = \begin{pmatrix} 0 \\ -mg \end{pmatrix}$ with potential $v\colon \mathbb{R}^2 \to \mathbb{R}, (x,y) \mapsto \Phi(x,y) = -mgy$. Then the tangent line integral along any curve $\mathcal{C} \subset \mathbb{R}^2$ with parametrisation $\gamma\colon [a,b] \to \mathbb{R}^2$ gives the work moving the mass along $\mathcal{C}$, that is

$$-\int_{\mathcal{C}} f = -\int_{\gamma} f = -\Phi(\gamma(b)) + \Phi(\gamma(a)).$$

In Example 3.5 we have a curve $\mathcal{C}$ with parametrisation $\gamma\colon [0,\pi] \to \mathbb{R}^2, t \mapsto \gamma(t) = \begin{pmatrix} t \\ -\cos t \end{pmatrix}$, hence the work (tangent line intergal) can be computed directly with the potential

$$-\int_{\mathcal{C}} f = mg(\cos\pi - (-1)) = 2mg.$$

$\diamond$

Notation 4.5 *(a) A simple curve $\mathcal{C} \subset \mathbb{R}^n$ is a curve which is the image of a piecewise C^1-parametrisation $\gamma\colon [a,b] \to \mathbb{R}$ that is one-to-one on the interval $[a,b]$. The points $\gamma(a)$ and $\gamma(b)$ are the* ***endpoints*** *of the curve.*

Figure 17: Simple and not simple curve

(b) Let $\gamma\colon [a,b] \to \mathbb{R}^n$ *be a (piecewise* C^1*- differentiable) parametrisation. Then the map*

$$\omega\colon [a,b] \to \mathbb{R}^n, t \mapsto \omega(t) = \gamma(a+b-t)$$

is the parametrisation of the reversed (inverse direction) path called $\mathcal{C}^{-1}$*.*

Figure 18:

Remark 4.6 If f is a gradient vector field with potential Φ and γ is a line (path or parametrisation), then the tangent line integral depends only on the endpoints of the line (path or parametrisation) and on the direction of γ.

Figure 19:

Consider $v, w \in \mathbb{R}^n$ and a parametrised path $\gamma_1 \colon [a_1, b_1] \to \mathbb{R}^n$ from v to w

and a parametrised path $\gamma_2 \colon [a_2, b_2] \to \mathbb{R}^n$ from w to v. Taking the direction into account one gets

$$\begin{aligned}\int_{\gamma_1} f &= \Phi(\gamma_1(b_1)) - \Phi(\gamma_1(a_1)) = \Phi(w) - \Phi(v) = -(\Phi(\gamma(a_2)) - \Phi(\gamma_2(b_2)) \\ &= -\int_{\gamma_2} f\end{aligned}$$

because of $\gamma_1(a_1) = v = \gamma_2(b_2)$ and $\gamma_1(b_1) = w = \gamma_2(a_2)$, see figure 19. Moreover it follows with the same arguments that the tangent line integral along a closed curve (path or line) is zero, see figure below.

Figure 20: Integral of a closed curve

The gradient vector fields are important in particular in physics.
Physics notation: A C^1 vector field $f \colon D \to \mathbb{R}^n, D \subset \mathbb{R}^n$, which is defined on D except possibly for a finite number of points, is said to be **conservative** if it has the property that the (tangent) line integral of f along any closed simple curve $\mathcal{C} \subset \mathbb{R}^n$ is zero:

$$\int_{\mathcal{C}_{closed}} f = 0.$$

Equivalent: The vector field f is conservative if the (tangent) line integral of f along a curve only depends on the endpoints of the curve $\mathcal{C}$, not on the particular path taken by the curve. That is

$$\int_{\mathcal{C}_1} f = \int_{\mathcal{C}_2} f$$

for any two curves connecting two points $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^n$ in the same direction, i.e. both curves travel form a to b.
Then one can show that a vector field f is of gradient type if it is conservative.

This provides us with another method to check if a given vector field f is of gradient type or not. We are left to check if an integral (tangent line) along a closed curve is zero. However, here one has to ensure that the closed curve lies in the domain of definition of the vector field.

Example 4.7 Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto f(x, y) = (-y, x)$ be vector field and integrate this along the circle line with parametrisation $\gamma\colon [0, 2\pi] \to \mathbb{R}^2, t \mapsto (\cos t, \sin t)$. The tangent line along this closed curve is

$$\int_\gamma f = \int_0^{2\pi} \left\langle \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}, \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \right\rangle \mathrm{d}t = \int_0^{2\pi} 1 \,\mathrm{d}t = 2\pi \neq 0.$$

Hence the vector field f cannot be of gradient type. $\diamond$

4.2 Finding a potential

Corollary 4.8 *If $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a gradient vector field and $\Phi\colon \mathbb{R}^n \to \mathbb{R}$ a potential, then for any $x \in \mathbb{R}^n \setminus \{0\}$*

$$\Phi(x) = \Phi(0) + \int_{\gamma_x} f,$$

where $\Phi(0)$ is a constant and where γ_x is the straight line from the origin 0 to x, that is

$$\gamma_x\colon [0, 1] \to \mathbb{R}^n, t \mapsto \gamma_x(t) = tx.$$

Figure 21:

We can choose any other reference point $x_0 \in \mathbb{R}^n$ instead the origin. In that case one needs a path connecting x_0 and x. It is important that the point and the path connecting any point with the reference point are lying in the domain of definition of Φ and f. If Φ_1 and Φ_2 are two potentials for the vector field f, then

$$\Phi_1(x) - \Phi_2(x) = \Phi_1(0) - \Phi_2(0) = \text{constant} \qquad x \in \mathbb{R}^n.$$

Definition 4.9 *If $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a vector field, a flow line for f is a parametric curve (path) $\gamma\colon [a,b] \to \mathbb{R}^n$ such that*

$$\boxed{\gamma'(t) = f(\gamma(t)). \qquad (4.7)}$$

If one considers the flow of a liquid in a pipe, then the vector field f yields the velocity vector field of the parametric curve (path) γ. The velocity vector of the fluid is tangent to a flow line, see figure 22 below.

Figure 22:

If one is given a vector field it is easy to draw the flow line. It is the line threading its way through the vector field in the plane as shown in figure 23.

Figure 23:

Example 4.10 (Finding a potential) Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto f(x, y) = \frac{1}{2}(-x, y)$. We sketch the vector field in figure 24 below.

Figure 24:

This is done by considering the mappings

$$\begin{aligned} f(x, 0) &= -\frac{1}{2}\begin{pmatrix} x \\ 0 \end{pmatrix} \\ f(0, y) &= \frac{1}{2}\begin{pmatrix} 0 \\ y \end{pmatrix} \\ f(x, x) &= \frac{1}{2}\begin{pmatrix} -x \\ x \end{pmatrix}. \end{aligned}$$

Assume that $f = \nabla\Phi$ for some $\Phi\colon \mathbb{R}^2 \to \mathbb{R}$. Then

$$\begin{aligned} \partial_x\Phi(x, y) &= -\frac{1}{2}x \Leftrightarrow \Phi(x, y) = -\frac{x^2}{4} + c_1(y) \\ \partial_y\Phi(x, y) &= \frac{1}{2}y \Leftrightarrow \Phi(x, y) = \frac{y^2}{4} + c_2(x), \end{aligned}$$

i.e. $\Phi(x, y) = \frac{y^2}{4} - \frac{x^2}{4}$ is a solution ($C_1 = C_2 = 0$). $\diamond$

Example 4.11 Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto f(x, y) = (2y, x + y)$. Assume that f is a gradient vector field, that is $\nabla\Phi = f$ for some $\Phi\colon \mathbb{R}^2 \to \mathbb{R}$. Then we can conclude

$$\partial_x \Phi(x,y) = 2y \Leftrightarrow \Phi(x,y) = 2xy + C_1(y)$$
$$\partial_y \Phi(x,y) = x + y \Leftrightarrow \Phi(x,y) = xy + \frac{1}{2}y^2 + C_2(x).$$

But it is impossible to find any such functions C_1 and C_2 because one can never match $2xy$ versus xy. Thus f is not a gradient vector field. $\diamond$

4.3 Radial vector fields

The next example is very important. It contains vector fields which are radial symmetric.

Example 4.12 (Radial vector fields) For the scalar valued function $g\colon (0,\infty) \to \mathbb{R}$ define the vector field $f\colon \mathbb{R}^n \to \mathbb{R}^n$ by

$$f(x) = \begin{cases} g(\|x\|)\frac{x}{\|x\|}\,, & \text{for } x \in \mathbb{R}^n \setminus \{0\} \\ 0\,, & \text{for } x = 0 \end{cases}.$$

Assume that we can always find a primitive $G\colon (0,\infty) \to \mathbb{R}$ with $G'(r) = g(r)$ for all $r > 0$.
We say that $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a **radial vector field** if $\|f(x)\|$ is constant within concentric spheres, i.e.

$$\|f(x)\| = \mathit{const} \text{ for all } x \in \mathbb{R}^n \text{ with } \|x\| = c \text{ for any } c > 0.$$

Define

$$\Phi\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}, x \mapsto G(\|x\|) = G\Big(\sqrt{x_1^2 + \cdots + x_n^2}\Big).$$

Then for $x \in \mathbb{R}^n \setminus \{0\}$ and any $i \in \{1, \ldots, n\}$, we compute

$$\begin{aligned}\partial_i \Phi(x) &= G'(\|x\|)\frac{2x_i}{2\sqrt{x_1^2 + \cdots + x_n^2}} = g(\|x\|)\frac{x_i}{\|x\|} \\ &= f_i(x).\end{aligned}$$

Hence, $\nabla\Phi(x) = f(x)$ for all $x \neq 0$.

$\boxed{\text{Result: On } \mathbb{R}^n \setminus \{0\} \text{ radial vector fields are always gradient vector fields.}}$

We outline two important examples of radial vector fields in physics.

(a) **Gravitational force field:** Put the origin at the centre of the earth or some other planet having acceleration g_{planet} and mass $M > 0$. The gravitational force on some testing body with mass $m > 0$ is the vector field

$$F\colon \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^3, x \mapsto F(x) = -\frac{g_{planet} mM}{\|x\|^3} x,$$

which has the potential

$$\Phi\colon \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}, x \mapsto \Phi(x) = \frac{g_{planet} mM}{\|x\|}.$$

The minus sign in the force field ensures that the force is directed to the centre of the planet.

(b) **Coulomb's law:** The force acting on a charge q at position x is due to a charge Q at the origin with electrostatic constant ε

$$F\colon \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^3, x \mapsto F(x) = \frac{\varepsilon q Q}{\|x\|^3} x,$$

which has the potential

$$\Phi\colon \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}, x \mapsto \Phi(x) = \frac{\varepsilon q Q}{\|x\|}.$$

◇

Since the potential Φ is constant on the level surface of Φ they are called **equipotential surfaces**. Note that the force field is orthogonal to the equipotential surfaces, compare the example of radial vector fields above. There, the force field is radial and the equipotential surfaces are concentric spheres.

We finish this section coming back to the notion of a flow line of a vector field. We connect it to first order differential equations. Geometrically, a flow line for a given vector field f is a parametric curve (path) that threads its way through the domain of the vector field in such a way that the tangent vector of the parametric curve (path) coincides with the vector field (see figure 23 above). A flow line may be viewed as a solution of a system of differential equations. Let $\Gamma(x, t)$ for $t \geq 0$ and $x \in \mathbb{R}^n$ be the position at time t of the point on the flow line through x (at time $t = 0$) after time t has elapsed.

Figure 25:

The mapping $\Gamma\colon \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^n$ with

$$\begin{aligned} \frac{\partial \Gamma}{\partial t}(x,t) &= f(\Gamma(x,t)) \\ \Gamma(x,0) &= x \end{aligned}$$

is also called the flow line of the vector field f. If we denote the differentiation of Γ with respect to the spatial variable x by D_x (t is fixed) we can interchange the partial derivative with respect to time t with D_x (under our general assumptions that all maps are continuous differentiable etc). Then one can derive the following **equation of first variation**

$$\boxed{\partial_t D_x \Gamma(x,t) = Df(\Gamma(x,t)) D_x \Gamma(x,t).}$$

Here $Df(x), x \in \mathbb{R}^n$, is the $n \times n$-matrix with the first partial derivatives of the vector field at point x, i.e.

$$Df(x) = \begin{pmatrix} \partial_1 f_1(x) & \cdots & \partial_n f_1(x) \\ \partial_1 f_2(x) & \cdots & \partial_n f_2(x) \\ . & .... & \\ . & \cdots & . \\ . & \cdots & . \\ \partial_1 f_n(x) & \cdots & \partial_n f_n(x) \end{pmatrix}.$$

Similarly, $D_x\Gamma(x,t)$ is a $n \times n$-matrix.

5 Surface Integrals

We want to extend the notion of line integrals to surface integrals in $\mathbb{R}^3$. One can generalises this to arbitrary k-dimensional surfaces in $\mathbb{R}^n, 1 < k \leq n$, but we deal first with the case $k = 2$ and $n = 3$.

5.1 Surfaces

We need methods for describing surfaces in $\mathbb{R}^3$. A surface $\mathcal{S} \subset \mathbb{R}^3$ as a subset of points in $\mathbb{R}^3$ can be described by two methods:
1.) Level sets of real-valued functions $f\colon \mathbb{R}^3 \to \mathbb{R}$. That is,

$$\mathcal{S} = f^{-1}(c) = \{x \in \mathbb{R}^3 \colon f(x) = c\}$$

for some $c \in \mathbb{R}$.
2.) Parametrisation

$$\alpha\colon Q \to \mathbb{R}^3, (s,t) \mapsto \alpha(s,t) = \begin{pmatrix} \alpha_1(s,t) \\ \alpha_2(s,t) \\ \alpha_3(s,t) \end{pmatrix}$$

with some parameter domain $Q \subset \mathbb{R}^2$.

Notation 5.1 *Let $\mathcal{S} \subset \mathbb{R}^3$ be a surface. A (C^1-)* ***parametrisation*** *of the surface $\mathcal{S}$ is a map (respectively a C^1-map) $\alpha\colon Q \to \mathbb{R}^3$ with $Q \subset \mathbb{R}^2$ and $\alpha(Q) = \mathcal{S}$.*

Example 5.2 (a) **Cylinder** (volume). Let $R > 0, H > 0$. The parametrisation

$$\alpha\colon [0,R] \times [0,2\pi] \times [0,H] \to \mathbb{R}^3, (r,\varphi,h) \mapsto \alpha(r,\varphi,h) = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ h \end{pmatrix}$$

describes a cylinder (see figure 26 below). Note that when the angle variable varies only in a subset of $[0,2\pi]$ one gets not the whole cylinder but one where a 'piece of cake' is removed. If one fixes the radius $r \in (0,R]$ one gets the parametrisation of the cylindrical surface of radius r and height H.

Figure 26:

(b) **Graph of a function** Let $h\colon Q \to \mathbb{R}^1$ be a continuous function, $Q \subset \mathbb{R}^2$. The map

$$\alpha\colon Q \to \mathbb{R}^{2+1}, (s,t) \mapsto \alpha(s,t) = \begin{pmatrix} s \\ t \\ h(s,t) \end{pmatrix}$$

is a parametrisation of a surface $\mathcal{S}$, the graph of the function h, i.e. $\mathcal{S} = \alpha(Q) =$ graph h (see figure 27).

Figure 27:

(c) **Polar coordinates** Define $Q_1 = [-\pi, \pi]$ and

$$\alpha^1 \colon Q_1 \to \mathbb{R}^2, \varphi \mapsto \alpha^1(\varphi) = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}.$$

Then define recursively the following parametric k-dimensional surfaces in $\mathbb{R}^n, 1 < k \leq n$. $Q_k = Q_{k-1} \times [-\pi/2, \pi/2]$ and

$$\alpha^k \colon Q_k \to \mathbb{R}^{k+1}, (\varphi_1, \ldots, \varphi_k) \mapsto \begin{pmatrix} \alpha^{k-1}(\varphi_1, \ldots, \varphi_{k-1}) \cos\varphi_k \\ \sin\varphi_k \end{pmatrix}.$$

The set $\mathbb{S}^1 = \alpha^1(Q_1)$ is the circle line in the plane and the set $\mathbb{S}^k = \alpha^k(Q_k)$ is a k-dimensional surface. One can prove that

$$\mathbb{S}^k = \{x \in \mathbb{R}^{k+1} \colon \|x\| = 1\}. \tag{5.8}$$

$\mathbb{S}^k$ is called the k-dimensional sphere in $\mathbb{R}^{k+1}$. For $k = 2$ and $n = 3$ we get the polar coordinates (earth coordinates)

$$\alpha^2(\varphi_1, \varphi_2) = \begin{pmatrix} \cos\varphi_1 \cos\varphi_2 \\ \sin\varphi_1 \cos\varphi_2 \\ \sin\varphi_2 \end{pmatrix},$$

where the angle $\varphi_1 \in [0, 2\pi]$ is the geographical longitude and respectively the angle $\varphi_2 \in [-\pi/2, \pi/2]$ is the geographical latitude. Note that sometimes one takes the latitude angle $\Theta = \frac{\pi}{2} - \varphi_2 \in [0, 2\pi]$ where one takes the angle away from the north pole. In the latter case one take the angle from the equator counting negative towards the north pole and positive towards south pole. Recall that $\sin\varphi_2 = \cos\Theta$ respectively $\cos\varphi_2 = \sin\Theta$ for $\varphi_2 \in [-\pi/2, \pi/2]$ and $\Theta \in [0, \pi]$.

Figure 28: Polar coordinates in $\mathbb{R}^3$

$\diamond$

5.2 Integral

For an integration over a surface in $\mathbb{R}^3$ we need surface unit normal vectors $\widehat{N}$. The unit normal vectors have norm 1 and they are perpendicular to the plane at each single point of the surface (more precisely they are perpendicular to the tangent plane at that point).
If we describe our surface as a level set of a differentiable real-valued function $f\colon \mathbb{R}^3 \to \mathbb{R}$ we know from Lemma 2.6 that the gradient is orthogonal to the level surfaces, i.e.

$$\nabla f \perp \mathcal{S} \Rightarrow \widehat{N} := \frac{\nabla f}{\|\nabla f\|} \in \mathbb{R}^3.$$

What is the tangent plane at a point to a given surface $\mathcal{S} \subset \mathbb{R}^3$? When our surface $\mathcal{S} \subset \mathbb{R}^3$ is given with a parametrisation $\alpha\colon Q \to \mathbb{R}^3$ we see from figure 29 that at each point $\alpha(s,t)$ of the surface $\mathcal{S}$ the vectors $\frac{\partial\alpha}{\partial s}(s,t)$ and $\frac{\partial\alpha}{\partial t}(s,t)$

are both tangent to the surface (that is, they are lying in the tangent plane). Hence, the cross-product

$$\frac{\partial\alpha}{\partial s}(s,t) \times \frac{\partial\alpha}{\partial t}(s,t)$$

is a normal to the surface at the point $\alpha(s,t)$.

Notation 5.3 *Assume that $\alpha\colon Q \to \mathbb{R}^3$ is a C^1-parametrisation of the surface $\mathcal{S} \subset \mathbb{R}^3$. If we fix one of the two arguments of the mapping α we get families of paths/curves lying in the surface $\mathcal{S}$. Fixing t the partial derivative $\frac{\partial\alpha(s,t)}{\partial s} =: T_s(\alpha(s,t))$ is called the* ***tangent vector*** *in s-direction and fixing s the partial derivative $\frac{\partial\alpha(s,t)}{\partial t} =: T_t(\alpha(s,t))$ is called the* ***tangent vector*** *in t-direction. The tangent vectors at a given point span the tangent plane at this point of the given surface.*
We say that a parametrisation (respectively surface) is ***regular*** *or* ***smooth*** *at the point $\alpha(s,t)$ if $T_s \times T_t \neq 0$ at the point $\alpha(s,t)$. Intuitively, a smooth surface has no "corners".*

Figure 29: Tangent vectors of the surface

We normalise this vector to get the unit normal for the surface $\mathcal{S}$

$$\widehat{N}(\alpha(s,t)) = \frac{\partial\alpha}{\partial s}(s,t) \times \frac{\partial\alpha}{\partial t}(s,t) \qquad (s,t) \in Q. \tag{5.9}$$

We will often write $\widehat{N}$ without the argument. Note that $\widehat{N}\colon \mathcal{S} \to \mathbb{R}^3$ is a vector field whose domain is the surface $\mathcal{S}$, but one can conceive it also as the map $\widehat{N}\colon Q \to \mathbb{R}^3$ for a given parametrisation.

Reminder: Cross-product $x, y \in \mathbb{R}^3$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_2 y_3 - x_3 y_2 \\ x_3 y_1 - x_1 y_3 \\ x_1 y_2 - x_2 y_1 \end{pmatrix}$$

Definition 5.4 *Let $\alpha\colon Q \to \mathbb{R}^3, Q \subset \mathbb{R}^2$, be a parametrisation of a surface $\mathcal{S} \subset \mathbb{R}^3$. Then the **scalar surface integral** of a continuous scalar field $u\colon \mathbb{R}^3 \to \mathbb{R}$ over the surface $\mathcal{S}$ is defined as*

$$\boxed{\int_{\mathcal{S}} u = \iint_Q u(\alpha(s,t)) \Big\| \frac{\partial \alpha}{\partial s}(s,t) \times \frac{\partial \alpha}{\partial t}(s,t) \Big\| \,\mathrm{d}s\mathrm{d}t. \qquad (5.10)}$$

Remark 5.5 *(a) If we integrate 1 over a surface we get the surface area of $\mathcal{S}$,*

$$\int_{\mathcal{S}} 1 = \iint_Q \Big\| \frac{\partial \alpha}{\partial s}(s,t) \times \frac{\partial \alpha}{\partial t}(s,t) \Big\| \,\mathrm{d}s\mathrm{d}t = \mathbf{area}\,(\mathcal{S}).$$

(b) If we take a small rectangle $q \subset Q$ in the parameter domain Q whose left bottom corner is $(s,t) \in Q$ and whose side length are Δs and Δt respectively, we can map this rectangle with the parametrisation to $\mathbb{R}^3$. The area of the image rectangle $\alpha(q)$ can be computed as

$$\mathbf{area}\,(\alpha(q)) = \Big\| \frac{\partial \alpha}{\partial s}(s,t) \times \frac{\partial \alpha}{\partial t}(s,t) \Big\| \Delta s \Delta t.$$

Figure 30: Image of the small rectangle q

If $\mathcal{S}$ is a surface in $\mathbb{R}^3$ do we have several parametrisations? The answer is yes, and we will formulate this result for any k-dimensional surface ($1 < k \leq n$) in $\mathbb{R}^n$.

Definition 5.6 *Let $1 < k \leq n$ and Q_1 and Q_2 two bounded and closed sets in $\mathbb{R}^k$ (with a smooth boundary).*

*(a) A bijective map $T\colon Q_1 \to Q_2$ is called a **C^1-parameter transformation** if (i) T and T^{-1} are C^1-mappings, (ii) $\det DT\big|_t > 0$ for all $t \in Q_1 \subset \mathbb{R}^k$.*

*(b) Let $\alpha_1\colon Q_1 \to \mathbb{R}^n$ and $\alpha_2\colon Q_2 \to \mathbb{R}^n$ be two parametrisations (C^1-mappings). α_1 and α_2 are said to be **equivalent** if there exists a C^1 parameter transformation $T\colon Q_1 \to Q_2$ such that $\alpha_1 = \alpha_2 \circ T$.*

The matrix $DT\big|_t$ is the total derivative of the map $T\colon Q_1 \to Q_2$, i.e.

$$DT\Big|_t = \begin{pmatrix} \frac{\partial T_1}{\partial t_1} & \frac{\partial T_2}{\partial t_2} & \dots & \frac{\partial T_1}{\partial t_k} \\ \frac{\partial T_2}{\partial t_1} & \frac{\partial T_2}{\partial t_2} & \dots & \frac{\partial T_2}{\partial t_k} \\ . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \\ \frac{\partial T_k}{\partial t_1} & \frac{\partial T_k}{\partial t_2} & \dots & \frac{\partial T_k}{\partial t_k} \end{pmatrix}.$$

5.3 Kissing problem

Example 5.7 (a) **Spherical cap $\boldsymbol{\mathcal{S}}$:** Fix $\theta \in (0, \pi/2)$ and define

$$\alpha\colon [-\pi, \pi] \times [\theta, \pi/2] \to \mathbb{R}^3, (\varphi_1, \varphi_2) \mapsto \alpha(\varphi_1, \varphi_2) = \begin{pmatrix} \cos\varphi_1 \cos\varphi_2 \\ \sin\varphi_1 \cos\varphi_2 \\ \sin\varphi_2 \end{pmatrix}$$

Figure 31: Spherical cap

We want to compute the surface area of the spherical cap defined by the parametrisation α. We compute the surface normal vectors at (φ_1, φ_2):

$$\frac{\partial\alpha}{\partial\varphi_1} = \begin{pmatrix} -\sin\varphi_1 \cos\varphi_2 \\ \cos\varphi_1 \cos\varphi_2 \\ 0 \end{pmatrix} \qquad \frac{\partial\alpha}{\partial\varphi_2} = \begin{pmatrix} -\cos\varphi_1 \sin\varphi_2 \\ -\sin\varphi_1 \sin\varphi_2 \\ \cos\varphi_2 \end{pmatrix}$$

and their cross-product

$$\frac{\partial\alpha}{\partial\varphi_1} \times \frac{\partial\alpha}{\partial\varphi_2} = \begin{pmatrix} \cos\varphi_1 \cos^2\varphi_2 \\ -\sin\varphi_1 \cos^2\varphi_2 \\ -\sin\varphi_2 \cos\varphi_2 \end{pmatrix}$$

with norm

$$\left\|\frac{\partial\alpha}{\partial\varphi_1} \times \frac{\partial\alpha}{\partial\varphi_2}\right\| = \sqrt{\cos^2\varphi_2} = \cos\varphi_2,$$

where the last equality follows due to $\theta \in (-\pi/2, \pi/2)$. The surface area is then given as the surface integral of 1,

$$\begin{aligned}\mathbf{area}\,(\mathcal{S}) &= \int_{\mathcal{S}} 1 = \int_{\theta}^{\pi/2} \int_{-\pi}^{\pi} 1 \cos\varphi_2 \,\mathrm{d}\varphi_1 \mathrm{d}\varphi_2 = 2\pi \sin\varphi_2 \Big|_{\varphi_2=\theta}^{\varphi_2=\pi/2} \\ &= 2\pi(1-\sin\theta).\end{aligned}$$

Special case: total surface area ($\theta = -\pi/2$) is 4π.

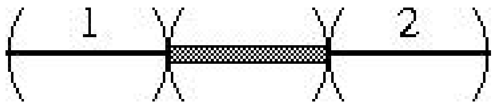
(c) **Newton's kissing problem** How many unit balls can simultaneously touch a given ball in $\mathbb{R}^3$? We calculate the shadowed surface area seen in figure 32.

Figure 32: Kissing problem

We see that $2\sin(\pi/2-\theta) = 1$ which implies $\pi/2 - \theta = \pi/6$ giving $\theta = \pi/3$. Each ball shadows the surface area $2\pi(1-\sin\theta) = 2\pi(1-\sqrt{3/4})$ of the central ball in figure 32. As the full surface area is 4π we can have at most

$$\frac{4\pi}{2\pi(1-\sqrt{3/4})} = \frac{4}{2-\sqrt{3}} \approx 14.928$$

balls. Hence, the number in question is less than 14. Is the maximum number now $12, 13$ or 14 ?
What about the other dimensions?



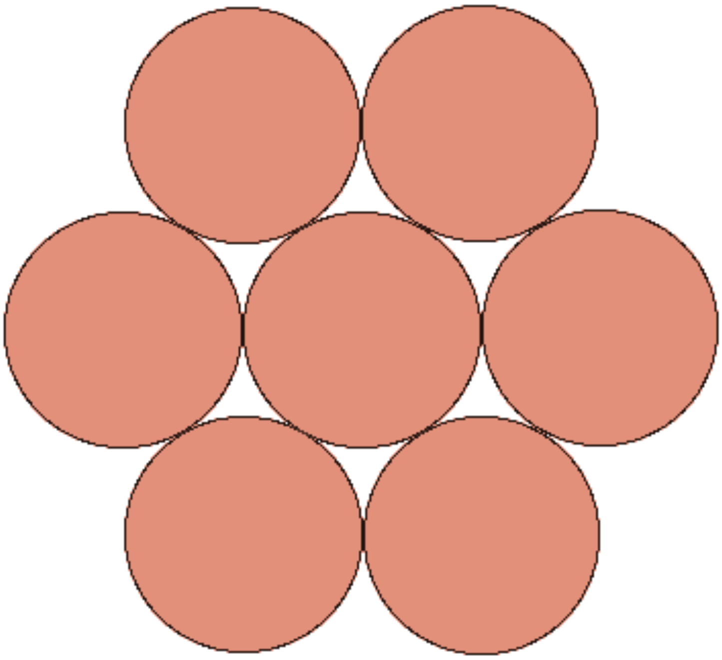
Kissing in $1d$

In one dimension, the kissing number is obviously 2.

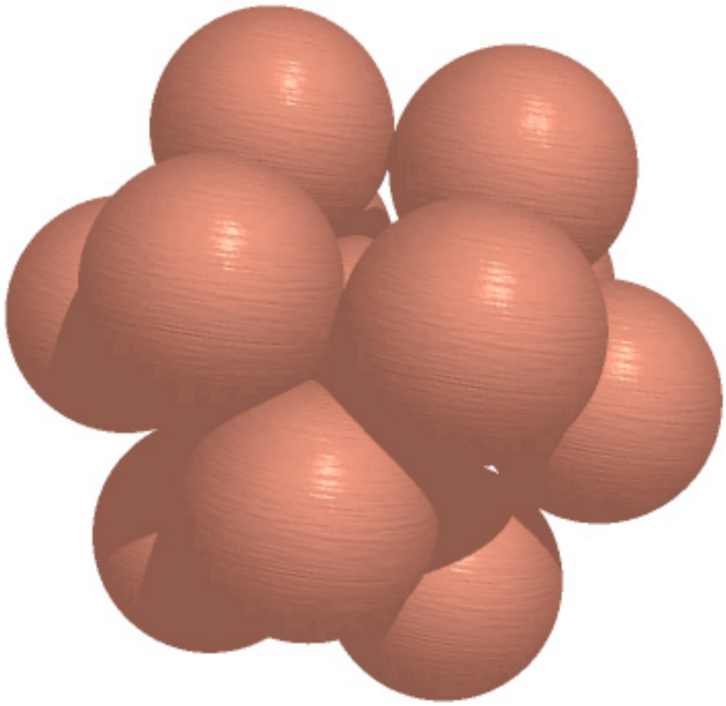
It is easy to see (and to prove) that in two dimensions the kissing number is 6.

In three dimensions the answer is not so clear. It is easy to arrange 12 spheres so that each touches a central sphere, but there is a lot of space left over, and it is not obvious that there is no way to pack in a 13th sphere. (In fact, there is so much extra space that any two of the 12 outer spheres can exchange places without any of the outer spheres losing contact with the centre one.) This was the subject of a famous disagreement between mathematicians Isaac Newton and David Gregory. Newton thought that the limit was 12, and Gregory that a 13th could fit. The question was not resolved until 1874; Newton was correct. In four dimensions, it was known for some time that the answer is either 24 or 25. It is easy to produce a packing of 24 spheres around a central sphere (one can place the spheres at the vertices of a suitably scaled 24-cell centred at the origin). As in the three-dimensional case, there is a lot of space left over - even more, in fact, than for $n = 3$ - so the situation was even less clear. Finally, in 2003, Oleg Musin proved the kissing number for $n = 4$ to be 24, using a subtle trick. The kissing number in n dimensions is unknown for $n > 4$, except for $n = 8(240)$, and $n = 24(196, 560)$. The results in these dimensions stem from the existence of highly symmetrical lattices: the $E8$ lattice and the Leech lattice. In fact, the only way to arrange spheres in these dimensions with the above kissing numbers is to centre them at the minimal vectors in these lattices. There is no space whatsoever for any additional balls. Rough volume estimates show that the kissing number in n dimensions grows exponentially. The base of exponential growth is not known.

The following table lists some known kissing number in various dimensions.



Kissing in $2d$



Kissing in $3d$

dimension	kissing $\sharp$	year
1	2	
2	6	
3	12	(1874)
4	24	(2003)
8	240	(1979)
24	196560	(1979)

$\diamond$

6 Divergence of Vector Fields

6.1 Flux across a surface

We consider the flow through a pipe. Let some fluid flowing with velocity $\vec{v}$ through a pipe. What is the total volume of fluid passing through the pipe per unit time? This volume flow rate is often called the flux of the fluid through the pipe or the flux across the surface $\mathcal{S}$ that forms the end of the pipe.

Figure 33: Flow through a pipe

If the velocity $\vec{v}$ is parallel to the walls and if $\vec{v}$ is constant, i.e. $\|\vec{v}\| = v_0$, the flow rate (flux) is given as $v_0 A$, where A is the area of the surface $\mathcal{S}$.

Figure 34: Velocity not parallel

If the velocity $\vec{v}$ is not perpendicular to the small surface $\mathrm{d}S$ in figure 34 (described by the unit normal vector $\widehat{N}$), only the component of $\vec{v}$ perpendicular to $\mathrm{d}S$ contributes to the flux across the small surface $\mathrm{d}S$.

Definition 6.1 *Let $\mathcal{S} \subset \mathbb{R}^3$ be a surface with unit normal vector field $\widehat{N} \colon \mathcal{S} \to \mathbb{R}^3$ defined with a C^1-parametrisation $\alpha \colon Q \to \mathbb{R}^3, Q \subset \mathbb{R}^2$. Then the* ***flux*** *of the vector field $f \colon \mathbb{R}^3 \to \mathbb{R}^3$ across the surface $\mathcal{S}$ in direction $\widehat{N}$ is defined as the scalar surface integral of $\langle f, \widehat{N} \rangle$, i.e.*

$$\boxed{\int_{\mathcal{S}} \langle f, \widehat{N} \rangle = \iint_Q \big\langle f(\alpha(s,t)), \widehat{N}(\alpha(s,t)) \big\rangle \Big\| \frac{\partial \alpha}{\partial s}(s,t) \times \frac{\partial \alpha}{\partial t}(s,t) \Big\| \, \mathrm{d}s\mathrm{d}t \quad (6.11)}$$

Alternative notations: $\int_{\mathcal{S}} f \cdot \widehat{N} \, \mathrm{d}s$ or $\int_{\mathcal{S}} \vec{f} \cdot \mathrm{d}\vec{s}$.

Example 6.2 (Flux out of a box) $\Omega = [a_x, b_x] \times [a_y, b_y] \times [a_z, b_z]$ with $a_x, a_y, a_z, b_x, b_y, b_z \in \mathbb{R}$ and let $f \colon \mathbb{R}^3 \to \mathbb{R}^3$ be a continuous differentiable vector field with component functions $f = (f_1, f_2, f_3)$. We calculate the flux through any side/surface of the box in figure 35.

Figure 35: box Ω

Calculation of the flux through the six surfaces of the box Ω

1.) Flux through the top of the box. A parametrisation is given by

$$\alpha_{\text{top}}(x,y) = \begin{pmatrix} x \\ y \\ b_z \end{pmatrix} \qquad \text{for } \; x \in [a_x, b_x], y \in [a_y, b_y].$$

The unit normal vector $\widehat{N}_{\text{top}}$ can already be seen in figure 35 but we show the exact calculation.

$$\Big\| \frac{\partial \alpha_{\text{top}}}{\partial x} \times \frac{\partial \alpha_{\text{top}}}{\partial y} \Big\| = \Big\| \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \Big\| = \Big\| \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Big\| = 1.$$

Thus, $\widehat{N}_{\text{top}}(x,y,b_z) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ with $(x,y) \in [a_x, b_x] \times [a_y, b_y]$. Hence, the flux through the top is

$$\int_{\mathcal{S}_{\text{top}}} \langle f, \widehat{N} \rangle = \int_{a_x}^{b_x} \int_{a_y}^{b_y} f_3(x,y,b_z) \, \mathrm{d}y \mathrm{d}x.$$

2.) Flux through the bottom. This is the same calculation but with the opposite direction, hence a minus sign appears for the unit surface normal

$\widehat{N}_{\text{bottom}}(x, y, a_z) = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$ with $(x, y) \in [a_x, b_x] \times [a_y, b_y]$ and the component function has to be evaluated at a_z for the third entry.

$$\int_{\mathcal{S}_{\text{bottom}}} \langle f, \widehat{N} \rangle = -\int_{a_x}^{b_x} \int_{a_y}^{b_y} f_3(x, y, a_z) \, \mathrm{d}y\mathrm{d}x.$$

We sum up both contributions and use the Fundamental Theorem of Calculus (recall that the vector field f is continuously differentiable) to get

$$\begin{aligned} \int_{\mathcal{S}_{\text{top}} \cup \mathcal{S}_{\text{bottom}}} \langle f, \widehat{N} \rangle &= \int_{a_x}^{b_x} \int_{a_y}^{b_y} \big(f_3(x, y, b_z) - f_3(x, y, a_z)\big) \, \mathrm{d}y\mathrm{d}x \\ &= \int_{a_x}^{b_x} \int_{a_y}^{b_y} \int_{a_z}^{b_z} \frac{\partial f_3}{\partial z}(x, y, z) \, \mathrm{d}z\mathrm{d}y\mathrm{d}x. \end{aligned}$$

3.) and 4.) Similarly we derive the contribution from the left and the right surface, $\mathcal{S}_\text{l}$ and $\mathcal{S}_\text{r}$, as

$$\int_{\mathcal{S}_\text{l} \cup \mathcal{S}_\text{r}} \langle f, \widehat{N} \rangle = \int_{a_x}^{b_x} \int_{a_y}^{b_y} \int_{a_z}^{b_z} \frac{\partial f_1}{\partial x}(x, y, z) \, \mathrm{d}z\mathrm{d}y\mathrm{d}x.$$

5.) and 6.) Finally we get the contribution from the back and the front

$$\int_{\mathcal{S}_{\text{back}} \cup \mathcal{S}_{\text{front}}} \langle f, \widehat{N} \rangle = \int_{a_x}^{b_x} \int_{a_y}^{b_y} \int_{a_z}^{b_z} \frac{\partial f_2}{\partial y}(x, y, z) \, \mathrm{d}z\mathrm{d}y\mathrm{d}x.$$

Taking the sum over all contributions of the six surfaces gives the total flux across the surface

$$\partial\Omega := \mathcal{S}_{\text{top}} \cup \mathcal{S}_{\text{bottom}} \cup \mathcal{S}_\text{l} \cup \mathcal{S}_\text{r} \cup \mathcal{S}_{\text{back}} \cup \mathcal{S}_{\text{front}}$$

as

$$\boxed{\int_{\partial\Omega} \langle f, \widehat{N} \rangle = \int_{a_x}^{b_x} \int_{a_y}^{b_y} \int_{a_z}^{b_z} \Big(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \Big) \, \mathrm{d}z\mathrm{d}y\mathrm{d}x. \qquad (6.12)}$$

◇

6.2 Divergence

In the following we will often write $\partial\Omega$ for the surface of some domain $\Omega \subset \mathbb{R}^3$. The integrand in (6.12) motivates the following definition.

Definition 6.3 *Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable vector field. The* ***divergence*** *of the vector field $f = (f_1, \ldots, f_n)$ is the scalar field*

$$\boxed{\begin{aligned} \operatorname{div} f\colon \mathbb{R}^n &\to \mathbb{R} \\ x &\mapsto \operatorname{div}(f)(x) = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}(x). \end{aligned}}$$

Alternative notations are: $\nabla \cdot f$ or $\vec{\nabla} \cdot \vec{f}$ or $\underline{\nabla} \cdot \underline{f}$.

Note the following relations of scalar and vector fields with the corresponding operations.

scalar field	**operation**	**vector field**
$\Phi\colon \mathbb{R}^n \to \mathbb{R}$	$\xrightarrow{\text{grad}}$	$\operatorname{grad}\Phi\colon \mathbb{R}^n \to \mathbb{R}^n$
$\operatorname{div} f\colon \mathbb{R}^n \to \mathbb{R}$	$\xleftarrow{\text{div}}$	$f\colon \mathbb{R}^n \to \mathbb{R}^n$

Definition 6.4 *Given a two times differentiable scalar field $\Phi\colon \mathbb{R}^n \to \mathbb{R}$, the* ***Laplacian*** *of the scalar field Φ is the scalar field*

$$\boxed{\begin{aligned} \Delta\Phi\colon \mathbb{R}^n &\to \mathbb{R} \\ x &\mapsto \Delta\Phi(x) = \operatorname{div}\operatorname{grad}\Phi(x) = \sum_{i=1}^{n} \frac{\partial \Phi^2}{\partial x_i^2}(x). \end{aligned}}$$

Alternative notation is $\nabla^2\Phi$. Note that the Laplacian of scalar field is the divergence of the gradient field of the scalar field.

Note:

$$\Delta\Phi = \operatorname{div}(\operatorname{grad}\Phi) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\frac{\partial \Phi}{\partial x_i} = \sum_{i=1}^{n} \frac{\partial^2 \Phi}{\partial x_i^2}.$$

We have seen in Example 6.2 that

$$\int_{\partial\Omega} \langle f, \widehat{N} \rangle = \int_{\Omega} \operatorname{div} f(x)\, \mathrm{d}x$$

(for boxes only so far). If Ω is a small box centred at $a \in \mathbb{R}^3$, then

$$\int_{\Omega} \operatorname{div} f(x)\, \mathrm{d}x \approx \operatorname{div} f(a)\mathbf{vol}(\Omega),$$

where $\mathbf{vol}(\Omega)$ is the volume of the small box Ω, implies that

$$\operatorname{div} f(a) \approx \frac{\int_{\partial\Omega} \langle f, \widehat{N} \rangle}{\mathbf{vol}(\Omega)}. \tag{6.13}$$

The divergence of a vector field gives the outgoing flux per volume.

Consider the following example.

Example 6.5 (a) $f \colon \mathbb{R}^3 \to \mathbb{R}^3, (x, y, z) \mapsto f(x, y, z) = (x, 0, 0)$.

Figure 36: vector field expanding

As seen $\operatorname{div} f(a)$ corresponds to the amount of flux of the vector field out of the small volume divided by the volume. This is a rate of 'expansion' or 'stretching' of the vector field (see figure 36 above). The divergence is $\operatorname{div} f(x) = 1, x \in \mathbb{R}^3$, and the figure 36 may represent a gas which is expanding.

(b) $f \colon \mathbb{R}^3 \to \mathbb{R}^3, (x, y, z) \mapsto f(x, y, z) = (-x, 0, 0)$.

Figure 37: vector field contracting

This vector field is contracting (see figure 37), and its divergence is $\operatorname{div} f(x) = -1, x \in \mathbb{R}^3$.

(c) $f\colon \mathbb{R}^3 \to \mathbb{R}^3, (x, y, z) \mapsto f(x, y, z) = (0, x, 0)$.

Figure 38: vector field with divergence zero

This vector field is neither expanding nor contracting (see figure 38), and its divergence is $\operatorname{div} f(x) = 0, x \in \mathbb{R}^3$.

$\diamond$

7 Gauss's Divergence Theorem

Theorem 7.1 (Gauss's Divergence Theorem) *Let $\Omega \subset \mathbb{R}^3$ be a bounded region with surface $\partial\Omega$ and outward normal $\widehat{N}$. Let $f\colon \Omega \cup \partial\Omega \to \mathbb{R}^3$ be continuously differentiable. Then*

$$\boxed{\int_\Omega \operatorname{div}(f)(x)\,\mathrm{d}x = \int_{\partial\Omega} \langle f, \widehat{N}\rangle. \qquad (7.14)}$$

Remark 7.2 (a) *The theorem also works for dimension $n \neq 3$ but one has to define the surface integral.*

$n = 2:$ *$\partial\Omega$ is a line and $\int_{\partial\Omega}\langle f, \widehat{N}\rangle$ is a (tangent) line integral.*

(b) *The left hand side of (7.14), $\int_\Omega \operatorname{div}(f)(x)\,\mathrm{d}x$, is just a volume Integral, that is an iterated integral $\iiint_\Omega \operatorname{div} f(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$ (Fubini's theorem, compare Analysis I).*

Alternative notations are: $\int_\Omega \operatorname{div} f\,\mathrm{d}V$ for $n = 3$ or $\int_\Omega \operatorname{div} f\,\mathrm{d}A$ for $n = 2$.

Example 7.3 (a) Let $\Omega = \overline{B}(0,R) \subset \mathbb{R}^n$ a ball of radius $R > 0$, $\overline{B}(0,R) = \{x \in \mathbb{R}^n \colon \|x\| \leq R\}$, and denote by $\partial\Omega$ the surface (sphere) of Ω, i.e. $\partial\Omega = \partial\overline{B}(0,R) = \{x \in \mathbb{R}^n \colon \|x\| = R\}$. The unit outward normal is then

$$\widehat{N}(x) = \frac{x}{R}, \qquad \text{for all } x \in \partial\Omega.$$

Define $f\colon \mathbb{R}^n \to \mathbb{R}^n, x \mapsto f(x) = x$ and check that $\operatorname{div} f(x) = n$ for all $x \in \mathbb{R}^n$. Gauss's Divergence Theorem gives

$$\int_{\overline{B}(0,R)} n\,\mathrm{d}x = \int_{\partial\overline{B}(0,R)} \langle x, \frac{x}{R}\rangle = \int_{\partial\overline{B}(0,R)} R.$$

$$\begin{aligned}
\text{RHS } &= R\,\mathbf{area}(\overline{B}(0,R)) = R \times \text{ surface area of } \overline{B}(0,R)\\
\text{LHS } &= n\,\mathbf{vol}(\overline{B}(0,R)) = n \times \text{ volume of } \overline{B}(0,R).\\
n = 2\colon\ & \mathbf{vol}(\overline{B}(0,R)) = \pi R^2, \mathbf{length}(\partial\overline{B}(0,R)) = 2\pi R\\
n = 3\colon\ & \mathbf{vol}(\overline{B}(0,R)) = \frac{4}{3}\pi R^3, \mathbf{area}(\partial\overline{B}(0,R)) = 4\pi R^2.
\end{aligned}$$

(b) Differentiability in Gauss's Divergence Theorem is needed as the following example shows.

$$f\colon \mathbb{R}^2 \to \mathbb{R}^2, (x,y) \mapsto f(x,y) = \frac{1}{x^2+y^2}\begin{pmatrix} x \\ y \end{pmatrix}.$$

$\operatorname{div} f(x,y) = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) = 0$. Set $\Omega = B(0,1) \subset \mathbb{R}^2$ having unit normal $\widehat{N}\colon \partial\Omega \to \mathbb{R}^2, (x,y) \mapsto \widehat{N}(x,y) = \begin{pmatrix} x \\ y \end{pmatrix}$. We get $\int_\Omega \operatorname{div} f(x,y)\, \mathrm{d}x\mathrm{d}y = 0$ but

$$\int_{\partial\Omega} \langle f, \widehat{N}\rangle = \int_{\partial\Omega} 1 = \mathbf{length}(\partial\Omega) = 2\pi.$$

The theorem does not work due to the singularity at the origin of the vector field f.

$\diamond$

Sketch for the proof of Theorem 7.1. Let $\Omega \subset \mathbb{R}^3$ be a region in $\mathbb{R}^3$ and divide it in finitely many small regions Ω_i centred at $a_i \in \Omega$ with outward surfaces $\partial\Omega_i$ (see figure 39) having unit normal $\widehat{N}_i$.

Figure 39: partition of Ω into small regions

We apply the approximation of the divergence in (6.13) to every small region Ω_i, that is

$$\operatorname{div} f(a_i) \approx \frac{1}{\mathbf{vol}(\Omega_i)} \int_{\partial\Omega_i} \langle f, \widehat{N}_i\rangle. \tag{7.15}$$

The approximation becomes exact in the limit $\mathbf{vol}(\Omega_i) \to 0$. Hence, multiply (7.15) by $\mathbf{vol}(\Omega_i)$ and sum up to get

$$\sum_i \operatorname{div} f(a_i)\mathbf{vol}(\Omega_i) \approx \sum_i \int_{\partial\Omega_i} \langle f, \widehat{N} \rangle. \tag{7.16}$$

Here, the left hand side will become the volume integral in the limit $\mathbf{vol}(\Omega_i) \to 0$ (Riemann sum). But what about the right hand side in (7.16)? Consider in figure 40 two small adjacent regions Ω_1 and Ω_2.

Figure 40: contributions of the common surface

Along the common surface of $\partial\Omega_1$ and $\partial\Omega_2$ we get $\langle f, \widehat{N}_1 \rangle + \langle f, \widehat{N}_2 \rangle = 0$. All contributions from the interior of Ω to the sum of the right hand side in (7.16) cancel out, leaving only the surface integral over the exterior surface $\partial\Omega$. Hence, in the limit the right hand side of (7.16) becomes $\int_{\partial\Omega} \langle f, \widehat{N} \rangle$. $\square$

Example 7.4 Let

$$f\colon \mathbb{R}^3 \to \mathbb{R}^3, (x,y,z) \mapsto f(x,y,z) = \begin{pmatrix} 4x \\ -2y^2 \\ z^2 \end{pmatrix}$$

and $\Omega = \{(x,y,z) \in \mathbb{R}^3 \colon x^2 + y^2 \leq 4, z \in [0,3]\}$ the cylinder in figure 41. We are going to check Gauss's Divergence Theorem. For that we compute both the volume integral of the divergence and the surface (flux) integral.

Figure 41: Cylinder

Volume integral: We compute $\operatorname{div} f(x,y,z) = 4 - 4y + 2z$.

$$\begin{aligned}
\iiint_\Omega (4-4y+2z)\,\mathrm{d}z\mathrm{d}y\mathrm{d}x &= \int_{-2}^{2}\Big[\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\Big(\int_0^3 (4-4y+2z\Big)\,\mathrm{d}z\Big)\,\mathrm{d}y\Big]\,\mathrm{d}x \\
&= \int_{-2}^{2}\Big[\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\big(21-12y\big)\,\mathrm{d}y\Big]\,\mathrm{d}x \\
&= \int_{2-}^{2} 42\sqrt{4-x^2}\,\mathrm{d}x \\
&= 42\int_{-2}^{2}\sqrt{4-x^2}\,\mathrm{d}x \\
&= 42\Big[\frac{x}{2}\sqrt{4-x^2} + 2\arcsin(\frac{x}{2})\Big]_{-2}^{2} = 84\pi.
\end{aligned}$$

Surface integral: The surface $\mathcal{S}$ of the cylinder Ω has the following three single parts $\mathcal{S} = \mathcal{S}_{\text{bottom}} \cup \mathcal{S}_{\text{top}} \cup \mathcal{S}_{\text{cyl.}}$

$\boldsymbol{\mathcal{S}}_{\text{bottom}} = \{(x,y,z)\mathbb{R}^3 \colon x^2+y^2 \le 4, z=0\}$:

$$\widehat{N}(x,y,z) = -\begin{pmatrix}0\\0\\1\end{pmatrix} \quad\text{and}\quad f(x,y,z) = \begin{pmatrix}4x\\-2y^2\\0\end{pmatrix} \quad\text{for}\quad (x,y,z) \in \mathcal{S}_{\text{bottom}}.$$

As $\langle f(x,y,z), \widehat{N}(x,y,z)\rangle = 0$ for $(x,y,z) \in \mathcal{S}_{\text{bottom}}$ we get $\int_{\mathcal{S}_{\text{bottom}}} \langle f, \widehat{N}\rangle = 0$.
$\boldsymbol{\mathcal{S}}_{\text{top}} = \{(x,y,z)\mathbb{R}^3 \colon x^2+y^2 \leq 4, z=3\}$:

$$\widehat{N}(x,y,z) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad f(x,y,z) = \begin{pmatrix} 4x \\ -2y^2 \\ 9 \end{pmatrix} \quad \text{for} \quad (x,y,z) \in \mathcal{S}_{\text{top}}.$$

A parametrisation is given by

$$\alpha_{\text{top}} \colon [0,2] \times [0,2\pi] \to \mathbb{R}^3, (s,t) \mapsto \alpha_{\text{top}}(s,t) = \begin{pmatrix} s\cos(t) \\ s\sin(t) \\ 3 \end{pmatrix}$$

with

$$\frac{\partial \alpha_{\text{top}}}{\partial s}(s,t) = \begin{pmatrix} \cos(t) \\ \sin(t) \\ 0 \end{pmatrix} \quad \text{and} \quad \frac{\partial \alpha_{\text{top}}}{\partial t}(s,t) = \begin{pmatrix} -s\sin(t) \\ s\cos(t) \\ 0 \end{pmatrix}$$

and cross-product

$$\frac{\partial \alpha_{\text{top}}}{\partial s} \times \frac{\partial \alpha_{\text{top}}}{\partial t} = \begin{pmatrix} 0 \\ 0 \\ s \end{pmatrix}.$$

Hence, the top contributes to the flux

$$\int_{\mathcal{S}_{\text{top}}} \langle f, \widehat{N}\rangle = \int_0^{2\pi} \int_0^2 9s \, \mathrm{d}s\mathrm{d}t = 36\pi.$$

$\boldsymbol{\mathcal{S}}_{\text{cyl.}}$: The parametrisation

$$\alpha_{\text{cyl.}} \colon [0,2\pi] \times [0,3] \to \mathbb{R}^3, (t,z) \mapsto \alpha_{\text{cyl.}}(t,z) = \begin{pmatrix} 2\cos(t) \\ 2\sin(t) \\ z \end{pmatrix}$$

gives

$$\frac{\partial \alpha_{\text{cyl.}}}{\partial t}(t,z) = \begin{pmatrix} -2\sin(t) \\ 2\cos(t) \\ 0 \end{pmatrix} \quad \text{and} \quad \frac{\partial \alpha_{\text{cyl.}}}{\partial z}(t,z) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and

$$\frac{\partial \alpha_{\text{cyl.}}}{\partial t} \times \frac{\partial \alpha_{\text{cyl.}}}{\partial z} = \begin{pmatrix} 2\cos(t) \\ 2\sin(t) \\ 0 \end{pmatrix} \Rightarrow \left\| \frac{\partial \alpha_{\text{cyl.}}}{\partial t} \times \frac{\partial \alpha_{\text{cyl.}}}{\partial z} \right\| = 2.$$

Hence, $\widehat{N}_{\text{cyl.}}(t,z) = \frac{1}{2}\begin{pmatrix} 2\cos t \\ 2\cos t \\ 0 \end{pmatrix}$, and the cylinder barrel contributes to the flux

$$\begin{aligned}
\int_{\mathcal{S}_{\text{cyl.}}} \langle f, \widehat{N} \rangle &= \int_0^{2\pi}\int_0^3 \big(16\cos^2(t) - 8\sin^3(t)\big)\,\mathrm{d}z\mathrm{d}t \\
&= \int_0^{2\pi}\int_0^3 \big(16\cos^2(t) - 8\sin^3(t)\big)\,\mathrm{d}z\mathrm{d}t \\
&= \int_0^{2\pi}\int_0^3 \big(16\cos^2(t)) + 8\sin(t)\cos^2(t) - 8\sin(t)\big)\,\mathrm{d}z\mathrm{d}t \\
&= 3\int_0^{2\pi} \big(16\cos^2(t)) + 8\sin(t)\cos^2(t) - 8\sin(t)\big)\,\mathrm{d}t \\
&= 48\int_0^{2\pi} \cos^2(t)\,\mathrm{d}t = 48\Big[\frac{1}{2}\cos(t)\sin(t) + \frac{1}{2}t\Big]_0^{2\pi} = 48\pi,
\end{aligned}$$

where we used that

$$\int_0^{2\pi} \sin(t)\,\mathrm{d}t = \big[-\cos(t)\big]_0^{2\pi} = 0$$
$$\int_0^{2\pi} \big(\sin(t)\cos^2(t)\big)\,\mathrm{d}t = \big[-\cos^3(t)\big]_0^{2\pi} = 0.$$

The total flux is

$$\int_{\mathcal{S}} \langle f, \widehat{N} \rangle = \int_{\mathcal{S}_{\text{top}}} \langle f, \widehat{N} \rangle + \int_{\mathcal{S}_{\text{cyl.}}} \langle f, \widehat{N} \rangle = (36+48)\pi = 84\pi.$$

We have thus proved the Divergence Theorem follows for this example, i.e.

$$\int_{\mathcal{S}} \langle f, \widehat{N} \rangle = \iiint_{\Omega} \operatorname{div} f(x,y,z)\,\mathrm{d}z\mathrm{d}y\mathrm{d}x.$$

$\diamond$

Example 7.5 (Conservation of mass of a fluid) We consider the flow of fluid for a domain $\Omega \subset \mathbb{R}^3$ with surface $\partial\Omega$ and outward normal $\widehat{N}$, see figure 42. The function

$$\varrho\colon \mathbb{R}^3 \times \mathbb{R}_+ \to \mathbb{R}$$

gives the density $\varrho(x,t)$ of the fluid a point $x = (x_1,x_2,x_3) \in \mathbb{R}^3$ and time $t \in \mathbb{R}_+$. The mass of the fluid in Ω at fixed time t is the volume integral

$\iiint_\Omega \varrho(x,t)\,\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3$. The rate of mass flow into the domain Ω is given by the flux integral

$$-\int_{\partial\Omega} \langle \varrho\vec{v}, \widehat{N}\rangle,$$

where the minus sign signals that $\widehat{N}$ points outward and where $\vec{v}\colon \mathbb{R}^3 \times \mathbb{R}_+ \to \mathbb{R}^3$ gives the velocity vector $\vec{v}(x,t)$ of the fluid at point $x \in \mathbb{R}^3$ and time t. **Physical law: mass is conserved, that is, the rate of change of mass in Ω equals the rate at which mass enters Ω.**

Figure 42: Flow of fluid in and out the region Ω

This law can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_\Omega \varrho(x,t)\,\mathrm{d}x = -\int_{\partial\Omega} \langle \varrho\vec{v}, \widehat{N}\rangle.$$

The right hand side can be written as the volume integral for the divergence of $\varrho\vec{v}$ and the time derivative can be interchanged with the volume integration. All this gives

$$\iiint_\Omega \Big(\frac{\partial\varrho}{\partial t}(x,t) + \operatorname{div}(\varrho\vec{v})(x)\Big)\,\mathrm{d}x = 0.$$

As the domain is arbitrary the integrand equals identically zero. Hence, we have derived the conservation law for the mass of a fluid.

Mass conservation law

$$\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \vec{v}) = 0.$$

$\diamond$

8 Integration by Parts

This section introduces integration by parts techniques. We let $\Omega \subset \mathbb{R}^3$ be some region, i.e., a bounded open subset of $\mathbb{R}^3$, and by $\partial\Omega$ we denote the surface of Ω such that $\overline{\Omega} = \Omega \cup \partial\Omega$. The surface normal is the vector field $\widehat{N}\colon \partial\Omega \to \mathbb{R}^3, \widehat{N}(x) = (\widehat{N}_1(x), \widehat{N}_2(x), \widehat{N}_3(x))$.

Lemma 8.1 *Let the function $f\colon \overline{\Omega} \to \mathbb{R}$ be continuously differentiable. Then*

$$\int_\Omega \frac{\partial f}{\partial x_i}(x)\,\mathrm{d}x = \int_{\partial\Omega} f\widehat{N}_i \quad \textit{for } i = 1,2,3. \tag{8.17}$$

Proof. Without loss of generality choose $i = 1$ and define the vector field $v\colon \mathbb{R}^3 \to \mathbb{R}^3, x \mapsto v(x) = (f(x), 0, 0)$. Then $\operatorname{div} v(x) = \frac{\partial f}{\partial x_1}$ and $v \cdot \widehat{N} = \langle v, \widehat{N}\rangle = f\widehat{N}_1$. The claim follows then by the Divergence Theorem, where we put $v = (0, f, 0)$ and $v = (0, 0, f)$ for the other cases. $\square$

Proposition 8.2 *Let the functions $g\colon \overline{\Omega} \to \mathbb{R}$ and $h\colon \overline{\Omega} \to \mathbb{R}$ be continuously differentiable. Then*
(a) For $i = 1,2,3$,

Integration by parts (IBP)

$$\int_\Omega \Big(\frac{\partial g}{\partial x_i}(x)\Big) h(x)\,\mathrm{d}x = \int_{\partial\Omega} gh\widehat{N}_i - \int_\Omega g(x)\Big(\frac{\partial h}{\partial x_i}(x)\Big)\,\mathrm{d}x. \tag{8.18}$$

(b) If $g\colon \overline{\Omega} \to \mathbb{R}$ is two times continuously differentiable,

Green's first identity

$$\int_\Omega \Delta g(x)\,\mathrm{d}x = \int_{\partial\Omega} \langle \nabla g, \widehat{N}\rangle. \tag{8.19}$$

(c) If $f\colon \overline{\Omega} \to \mathbb{R}$ is two times continuously differentiable,

Integration by part (IBP) - vector case
$$\int_\Omega \Delta f(x)h(x)\,\mathrm{d}x = \int_{\partial\Omega} h\langle \nabla f, \widehat{N}\rangle - \int_\Omega \langle \nabla f(x), \nabla h(x)\rangle\,\mathrm{d}x \qquad (8.20)$$

Proof. (a) Apply Lemma 8.1 to the function $f(x) = g(x)h(x), x \in \overline{\Omega}$, and use the chain rule

$$\frac{\partial f}{\partial x_i}(x) = \frac{\partial g}{\partial x_i}(x)h(x) + g(x)\frac{\partial h}{\partial x_i}(x)$$

for $i = 1, 2, 3$.
(b) Use again Lemma 8.1 for $f(x) = \frac{\partial g}{\partial x_i}(x)$ for $i = 1, 2, 3$ to obtain

$$\int_\Omega \frac{\partial^2 g}{\partial x_i^2}(x)\,\mathrm{d}x = \int_{\partial\Omega} \frac{\partial g}{\partial x_i}\widehat{N}_i.$$

Summing up all the terms gives the identity.
(c) Apply part (a) to the function $g = \frac{\partial f}{\partial x_i}$ for $i = 1, 2, 3$ to get

$$\int_\Omega \frac{\partial^2 g}{\partial x_i^2}(x)h(x)\,\mathrm{d}x = \int_{\partial\Omega} h\frac{\partial f}{\partial x_i}\widehat{N}_i - \int_\Omega \frac{\partial f}{\partial x_i}(x)\frac{\partial h}{\partial x_i}(x)\,\mathrm{d}x.$$

Summing up all the terms again gives the identity. □

Application of Proposition 8.2

Let $\Omega \subset \mathbb{R}^3$ be some piece of material and fix the temperature $f(x) \in \mathbb{R}_+$ for every point $x \in \partial\Omega$ on the surface (boundary) of Ω. In a steady state the temperature is a scalar field $T\colon \Omega \cup \partial\Omega \to \mathbb{R}_+$ solving the following boundary value problem

$$\begin{aligned} \Delta T(x) &= 0 \ \text{ for all } \ x \in \Omega, \\ T(x) &= f(x) \ \text{ for all } \ x \in \partial\Omega. \end{aligned} \tag{8.21}$$

The existence of solutions for (8.21) is difficult, but we can use Proposition 8.2 to show the uniqueness of a solution for (8.21). Assume that $T\colon \Omega\cup\partial\Omega \to \mathbb{R}_+$ and $\widetilde{T}\colon \Omega \cup \partial\Omega \to \mathbb{R}_+$ are both a solution to (8.21). Define the scalar field $D\colon \Omega \cup \partial\Omega \to \mathbb{R}_+, x \mapsto D(x) = T(x) - \widetilde{T}(x)$. This solves the following

$$\begin{aligned} \Delta D(x) &= \Delta T(x) - \Delta\widetilde{T}(x) = 0 \ \text{ for all } \ x \in \Omega \\ D(x) &= T(x) - \widetilde{T}(x) = f(x) - f(x) = 0 \text{ for all } \ x \in \partial\Omega. \end{aligned} \tag{8.22}$$

Proposition 8.2 implies that

$$0 = \int_\Omega (\Delta D(x))D(x)\,\mathrm{d}x = -\int_\Omega \langle \nabla D(x), \nabla D(x)\rangle\,\mathrm{d}x + \int_{\partial\Omega} D\langle \nabla D, \widehat{N}\rangle.$$

Henceforth (because the second integral of the right hand side vanishes)

$$\int_\Omega \|\nabla D(x)\|^2\,\mathrm{d}x = 0.$$

Hence, we conclude $\nabla D(x) = 0$ for all $x \in \Omega$ and therefore that D is constant. But as $D(x) = 0$ for all $x \in \partial\Omega$ we get $D(x) = 0$ for all $x \in \Omega \cup \partial\Omega$. Thus $T = \widetilde{T}$, and the uniqueness of the solution of (8.21) follows.

9 Green's theorem and curls in $\mathbb{R}^2$

The Divergence Theorem in Section 7 relates the flux out of region to the volume integral of the divergence of the vector field. In this section we ask about the flow along the boundary line of a region in $\mathbb{R}^2$. The three dimensional case will follow in the next section.

9.1 Green's theorem

Let the vector field $f\colon \mathbb{R}^2 \to \mathbb{R}^2, f = (f_1, f_2)$, and the rectangle $\Omega = [a,b] \times [c,d] \subset \mathbb{R}^2, a \le b, c \le d$, be given. The boundary $\partial\Omega$ of the rectangle is a curve $\mathcal{C}$ in $\mathbb{R}^2$ and we denote the unit tangent vector in counter-clockwise direction (that is, in positive direction of $\mathcal{C}$) by $\widehat{T}\colon \mathcal{C} \to \mathbb{R}^2$.
We are going to calculate the tangent line integral along the curve $\mathcal{C}$, see figure 43.

Figure 43: rectangle Ω

We integrate each single side/edge of the rectangle having parametrisations $\gamma_{\mathrm{R}}, \gamma_{\mathrm{L}}, \gamma_{\mathrm{T}}$ and γ_{B} respectively.

$$\int_{\gamma_{\mathrm{R}}} f = \int_{\gamma_{\mathrm{R}}} \langle f, \widehat{T} \rangle = \int_c^d \Big\langle f(b,y), \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Big\rangle \,\mathrm{d}y = \int_c^d f_2(b,y)\,\mathrm{d}y.$$

The corresponding left hand side is

$$\int_{\gamma_{\mathrm{L}}} f = \int_{\gamma_{\mathrm{L}}} \langle f, \widehat{T} \rangle = \int_0^1 \Big\langle f(a, d + t(c-d)), \begin{pmatrix} 0 \\ c-d \end{pmatrix} \Big\rangle \,\mathrm{d}t = -\int_c^d f_2(a,y)\,\mathrm{d}y.$$

We get similar results for the top and the bottom edge. Summing up these contributions we get the tangent line integral along $\partial\Omega$

$$\begin{aligned}
\int_{\partial\Omega} \langle f, \widehat{T} \rangle &= \int_c^d \big(f_2(b,y) - f_2(a,y)\big)\,\mathrm{d}y + \int_a^b \big(f_1(x,c) - f_1(x,d)\big)\,\mathrm{d}x \\
&= \int_c^d \int_a^b \frac{\partial f_2}{\partial x}(x,y)\,\mathrm{d}x\mathrm{d}y - \int_a^b \int_c^d \frac{\partial f_1}{\partial y}(x,y)\,\mathrm{d}x\mathrm{d}y \\
&= \int_\Omega \Big(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\Big)(x,y)\,\mathrm{d}x\mathrm{d}y,
\end{aligned}$$

where we used the Fundamental Theorem of calculus in the second line. The integrand in the last line motivates the following definition of a scalar field.

Definition 9.1 *Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a differentiable vector field. The* ***curl*** *of the vector field f is the scalar field* $\operatorname{curl} f\colon \mathbb{R}^2 \to \mathbb{R}$ *defined by*

$$\begin{aligned}\operatorname{curl} f\colon \mathbb{R}^2 &\to \mathbb{R}\\ (x,y) &\mapsto \operatorname{curl}(f)(x,y) = \frac{\partial f_2}{\partial x}(x,y) - \frac{\partial f_1}{\partial y}(x,y).\end{aligned}$$

Remark 9.2 $\int_{\partial\Omega}\langle f, \widehat{T}\rangle$ *is the* ***circulation*** *around $\partial\Omega$.*

Example 9.3 (a) The vector field $v\colon \mathbb{R}^2 \to \mathbb{R}^2, (x,y) \mapsto v(x,y) = \begin{pmatrix} -y \\ x \end{pmatrix}$ has positive curl, $\operatorname{curl}(v)(x,y) = 1 - (-1) = 2$ for all $(x,y) \in \mathbb{R}^2$.

Figure 44: positive curl

See the sketch of the vector field in figure 44, where the arrows of the vector field are circulating in positive direction around the origin (that is, anti-clockwise direction).

(b) The vector field $v\colon \mathbb{R}^2 \to \mathbb{R}^2, (x,y) \mapsto v(x,y) = \begin{pmatrix} -x \\ -y \end{pmatrix}$ has vanishing curl, $\operatorname{curl}(v)(x,y) = 0$ for all $(x,y) \in \mathbb{R}^2$. See the sketch of the vector field in figure 45, where the arrows are directed towards the origin.

Figure 45: zero curl

(c) The vector field $v\colon \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto v(x, y) = \begin{pmatrix} 0 \\ x \end{pmatrix}$ has positive curl, $\operatorname{curl}(v)(x, y) = 1$ for all $(x, y) \in \mathbb{R}^2$. In figure 46 the arrows are pointing upwards in the positive half plane and downwards in the negative half plane.

Figure 46: positive curl, a small wheel centred at y-axis would turn

◇

The calculation above for the line integral along the boundary of a rectangle showing that the integral is the surface integral of the curl of the vector field can be proved for any region $\Omega \subset \mathbb{R}^2$. This is the content of the following theorem.

Theorem 9.4 (Green's theorem - Stokes's theorem in $\mathbb{R}^2$) *Let $\Omega \subset \mathbb{R}^2$ be a bounded region and $\widehat{T} \colon \partial\Omega \to \mathbb{R}^2$ be positively oriented unit tangent vectors for the boundary line $\partial\Omega$ of the region Ω. Let $\overline{\Omega} = \Omega \cup \partial\Omega$. If $f \colon \overline{\Omega} \to \mathbb{R}^2, f = (f_1, f_2)$, is continuously differentiable, then*

$$\boxed{\int_{\partial\Omega} \langle f, \widehat{T} \rangle = \int_{\Omega} \operatorname{curl}(f)(x, y)\, \mathrm{d}x\mathrm{d}y. \qquad (9.23)}$$

Proof. We give a sketch of the proof only, because one can use the proof of the Divergence Theorem for it. The contributions of the interior cancel out here as they do for the Divergence Theorem. To apply that theorem directly define the vector field

$$v \colon \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto v(x, y) = \begin{pmatrix} f_2(x, y) \\ -f_1(x, y) \end{pmatrix}.$$

Then,

$$\operatorname{div} v(x,y) = \frac{\partial f_2}{\partial x}(x,y) - \frac{\partial f_1}{\partial y}(x,y) = \operatorname{curl} f(x,y),$$

and

$$\langle v, \widehat{N} \rangle = f_2 \widehat{N}_1 - f_1 \widehat{N}_2 = \left\langle f, \begin{pmatrix} -\widehat{N}_2 \\ \widehat{N}_1 \end{pmatrix} \right\rangle = \langle f, \widehat{T} \rangle.$$

Note that $\widehat{T} \perp \widehat{N}$. Applying Gauss's theorem 7.1 we conclude

$$\int_\Omega \operatorname{curl} f(x,y)\, \mathrm{d}x\mathrm{d}y = \int_\Omega \operatorname{div} v(x,y)\, \mathrm{d}x\mathrm{d}y = \int_{\partial\Omega} \langle v, \widehat{N} \rangle = \int_{\partial\Omega} \langle f, \widehat{T} \rangle.$$

□

In the next proposition we show how one can apply Green's theorem.

Proposition 9.5 *Let $\Omega \subset \mathbb{R}^2$ be a bounded region with closed boundary line $\partial\Omega$, which surrounds Ω. Define the vector field $f\colon \mathbb{R}^2 \to \mathbb{R}^2, (x,y) \mapsto f(x,y) = \begin{pmatrix} -y \\ x \end{pmatrix}$. Then the area of the region is given by the tangent line integral of f, that is,*

$$\boxed{\mathbf{area}\,(\Omega) = \frac{1}{2} \int_{\partial\Omega} \langle f, \widehat{T} \rangle. \qquad (9.24)}$$

Proof.

$$\int_{\partial\Omega} \langle f, \widehat{T} \rangle = \iint_\Omega (1+1)\, \mathrm{d}x\mathrm{d}y = 2 \iint_\Omega \mathrm{d}x\mathrm{d}y = 2\mathbf{area}\,(\Omega).$$

□

9.2 Application

Proposition 9.5 has a direct application for the construction and the use of a planimeter. A planimeter is a measuring instrument used to measure the surface area of an arbitrary two-dimensional shape. The most common use is to measure the area of a plane shape. There are many different kinds of planimeters but all operate in a similar way. A pointer on the planimeter is used to trace around the boundary of the shape. This induces a movement in another part of the instrument and a reading of this is used to establish the area of the shape. The precise way in which they are constructed varies, the main types of mechanical planimeter being polar; linear; and Prytz or "hatchet" planimeters. In the linear and polar planimeter, as one point on a linkage is traced along the shape's perimeter, that linkage rolls a wheel along the drawing. The area of the shape is proportional to the number of turns through which the measuring wheel rotates when the planimeter is traced along the complete perimeter of the shape. The concept having been pioneered by Hermann in 1814, Swiss mathematician Jakob Amsler-Laffon built the first modern planimeter in 1854, the operation of which can be justified by appealing to Green's theorem and in particular to Proposition 9.5. Let us study briefly the so-called linear planimeter, see figure 47 below. The measuring arm (elbow) can move up and down along the y-axis. Note that b is the y-coordinate of that measuring arm. The scalar product of that arm with a vector field $\vec{N} = (N_x, N_y)$ is

$$\langle \vec{EM}, \vec{N} \rangle = xN_x + yN_y.$$

Having $\vec{N}(x, y) = (b - y, x)$ we get $\langle \vec{EM}, \vec{N} \rangle = 0$ and that the length of the measuring arm, $m := \sqrt{(b - y)^2 + x^2}$, is constant. Let Ω be a region in $\mathbb{R}^2$ having boundary $\partial\Omega$. We conclude with Green's theorem that

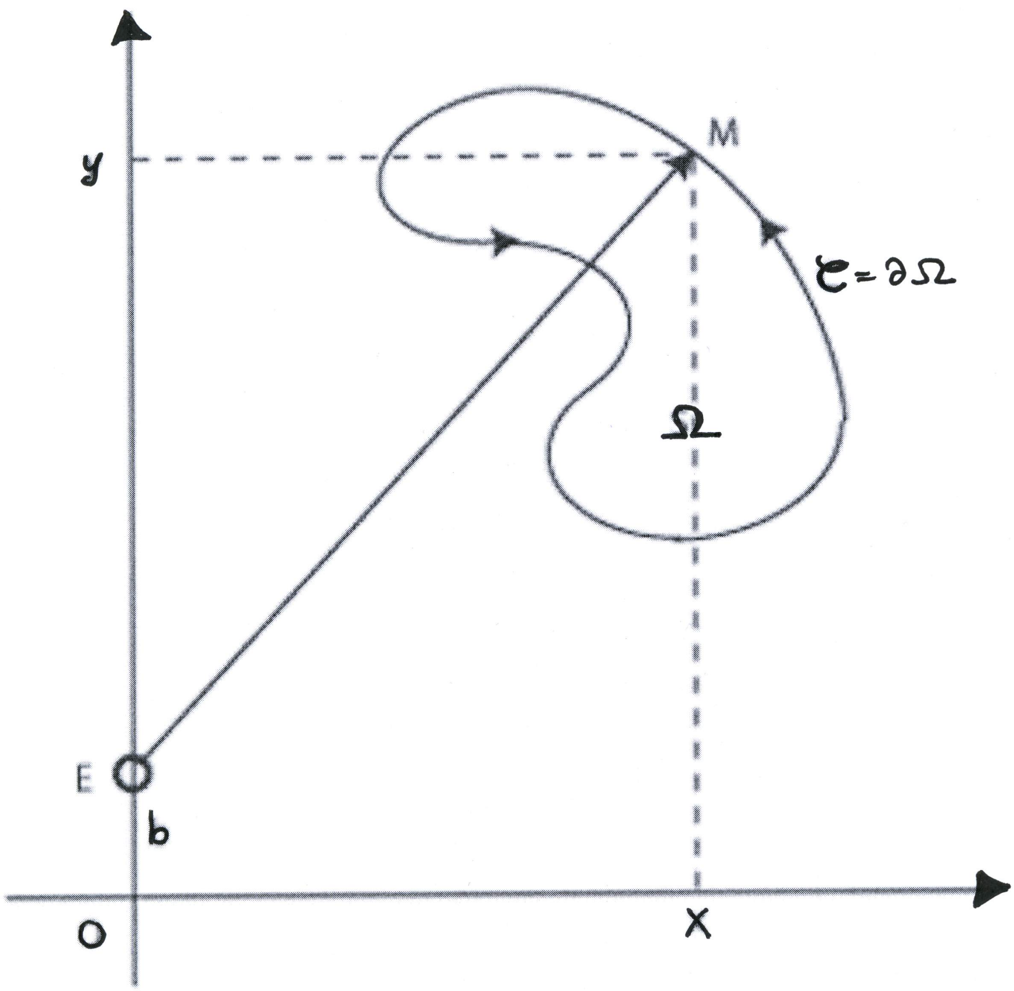
$$\int_{\partial\Omega} \langle \vec{N}, \widehat{T} \rangle = \int_{\Omega} \big(\frac{\partial N_y}{\partial x} - \frac{\partial N_x}{\partial y}\big) \,\mathrm{d}x\mathrm{d}y = \int_{\Omega} (1 + 1) \,\mathrm{d}x\mathrm{d}y = 2\mathbf{area}(\Omega).$$

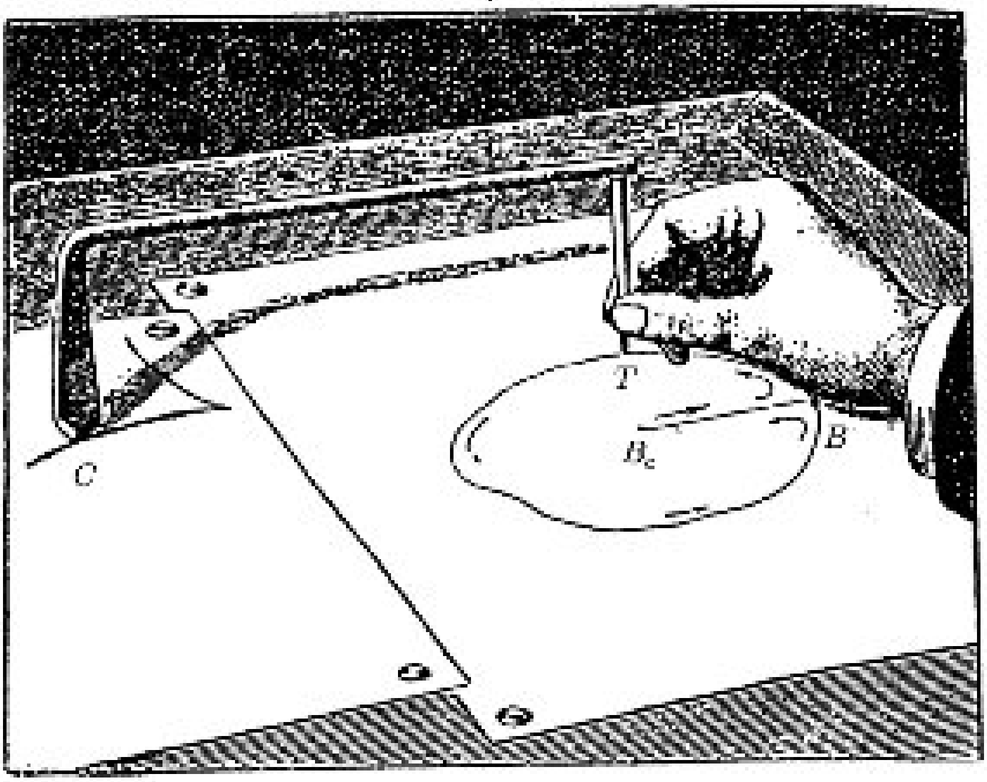
Example 9.6 Verify Green's theorem for the following vector field

$$f \colon \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto f(x, y) = \begin{pmatrix} xy + y^2 \\ x^2 \end{pmatrix},$$

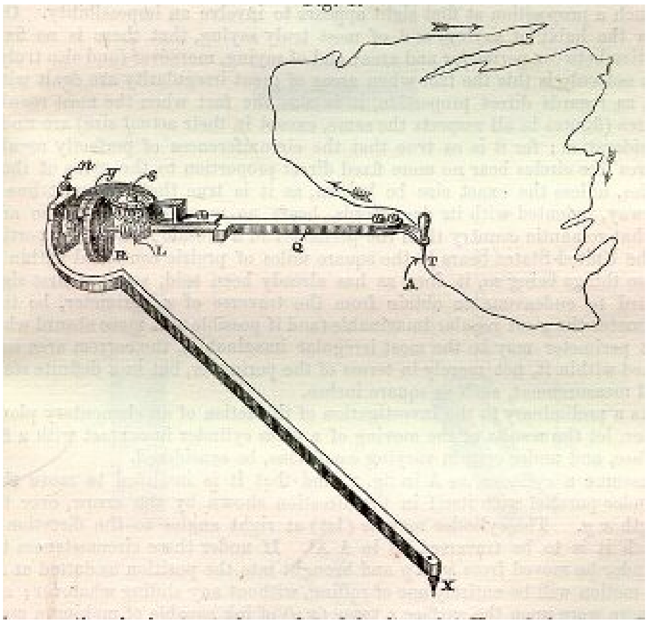
and the region $\Omega = \{(x, y) \in [0, 1] \times [0, 1] \colon x \geq y \geq x^2\}$, see figure 48

Figure 47:

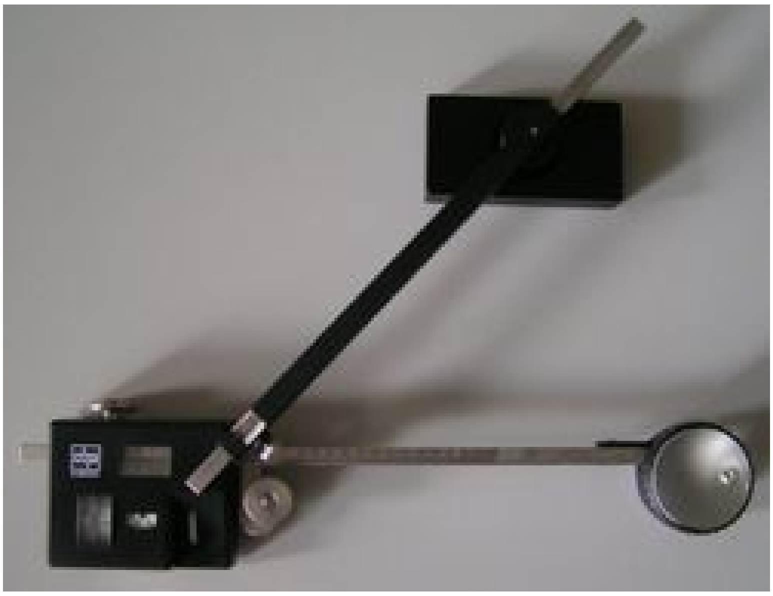




Prytz



Planimeter



Planimeter

Figure 48: region Ω

$\operatorname{curl} f(x,y) = x - 2y$ gives

$$\begin{aligned}\int_\Omega \operatorname{curl} f(x,y)\,\mathrm{d}x\mathrm{d}y &= \int_0^1 \left(\int_{x^2}^{x} (x-2y)\,\mathrm{d}y \right) \mathrm{d}x \\ &= \int_0^1 \Big[xy - y^2 \Big]_{x^2}^{x} \mathrm{d}x \\ &= \int_0^1 (x^4 - x^3)\,\mathrm{d}x = -\frac{1}{20}.\end{aligned}$$

We split the boundary in two pieces $\partial\Omega = \partial\Omega_1 \cup \partial_2$ (see figure 48) with parametrisation

$\partial\Omega_1 : \gamma_1 \colon [0,1] \longrightarrow \mathbb{R}^2, t \mapsto \gamma_1(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}, \gamma_1'(t) = \begin{pmatrix} 1 \\ 2t \end{pmatrix}$

$\partial\Omega_2 : \gamma_2 \colon [0,1] \longrightarrow \mathbb{R}^2, t \mapsto \gamma_2(t) = \begin{pmatrix} 1-t \\ 1-t \end{pmatrix}, \gamma_2'(t) = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$

Note that

$$\widehat{T}_1(\gamma_1(t)) = \frac{\gamma_1'(t)}{\|\gamma_1'(t)\|} = \frac{1}{\sqrt{1+4t^2}} \begin{pmatrix} 1 \\ 2t \end{pmatrix}.$$

Then,

$$\begin{aligned}
\int_{\partial\Omega_1} \langle f, \widehat{T}_1\rangle &= \int_0^1 \Big\langle f(\gamma_1(t)), \widehat{T}_1(\gamma_1(t))\Big\rangle \|\gamma_1'(t)\| \,\mathrm{d}t \\
&= \int_0^1 \left\langle \begin{pmatrix} t^3 + t^4 \\ t^2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2t \end{pmatrix} \right\rangle \mathrm{d}t \\
&= \int_0^1 \left(t^4 + 3t^3\right) \mathrm{d}t = \frac{19}{20}.
\end{aligned}$$

and

$$\begin{aligned}
\int_{\partial\Omega_2} \langle f, \widehat{T}_2\rangle &= \int_0^1 \left\langle \begin{pmatrix} (1-t)^2 + (1-t)^2 \\ (1-t)^2 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right\rangle \sqrt{2}\,\mathrm{d}t \\
&= -3 \int (1-t)^2 \,\mathrm{d}t = -1.
\end{aligned}$$

This gives $\int_{\partial\Omega} \langle f, \widehat{T}\rangle = -\frac{1}{20}$ and thus the proof of Green's theorem for this example. $\diamond$

10 Stokes's theorem

Stokes's theorem gives an alternative expression for the surface integral of the curl of a vector field. This is given by the flow along the boundary.

10.1 Stokes's theorem

Theorem 10.1 (Stokes's theorem) *Let $\mathcal{S} \subset \mathbb{R}^3$ be a bounded surface, $\widehat{N}\colon \mathcal{S} \to \mathbb{R}^3$ be unit normal vectors of the surface $\mathcal{S}$ and $\widehat{T}\colon \partial\mathcal{S} \to \mathbb{R}^3$ unit tangent vectors of the boundary $\partial\mathcal{S}$, and let $f\colon \mathcal{S} \cup \partial\mathcal{S} \to \mathbb{R}^3$ be a continuously differentiable vector field. If $(\widehat{N}, \widehat{T})$ is positively oriented, then*

$$\boxed{\int_{\partial\mathcal{S}} \langle f, \widehat{T}\rangle = \int_{\mathcal{S}} \langle \operatorname{curl} f, \widehat{N}\rangle. \qquad (10.25)}$$

Remark 10.2 *(a) Positively oriented means that the right hand rule for the pair $(\widehat{N}, \widehat{T})$ is satisfied. That is, if you walk around the boundary, the surface should be on your left.*
(b) curl

$f\colon \mathbb{R}^3 \to \mathbb{R}^3, f = (f_1, f_2, f_3)$ differentiable, then the **curl** *in $\mathbb{R}^3$ is the vector field*

$$\operatorname{curl} f\colon \mathbb{R}^3 \to \mathbb{R}^3, (x,y,z) \mapsto \operatorname{curl}(f)(x,y,z) = \begin{pmatrix} \partial_2 f_3(x,y,z) - \partial_3 f_2(x,y,z) \\ \partial_3 f_1(x,y,z) - \partial_1 f_3(x,y,z) \\ \partial_1 f_2(x,y,z) - \partial_2 f_1(x,y,z) \end{pmatrix}.$$

Alternative notation: $\operatorname{curl} f = \nabla \times f$.
If $g\colon \mathbb{R}^2 \to \mathbb{R}^2, g = (g_1, g_2)$ is differentiable we have a scalar field

$$\operatorname{curl} g(x,y) = \partial_x g_2 - \partial_y g_1(x,y),$$

and, if we define the vector field

$$f\colon \mathbb{R}^3 \to \mathbb{R}^3, (x,y,z) \mapsto f(x,y,z) = \begin{pmatrix} g_1(x,y) \\ g_2(x,y) \\ 0 \end{pmatrix},$$

we get $\operatorname{curl} f(x,y,z) = \begin{pmatrix} 0 \\ 0 \\ \operatorname{curl} g \end{pmatrix}$. *Green's theorem 9.4 in $\mathbb{R}^2$ is just a special case of Stokes's theorem in $\mathbb{R}^3$ where $\mathcal{S} \subset \mathbb{R} \times \mathbb{R} \times \{0\}$ (the $x - y$-plane).*

Proof of Theorem 10.1. Following the last remark we prove the theorem by 'lifting' Green's theorem from $\mathbb{R}^2$ to $\mathbb{R}^3$. On the left hand side in figure 49 we have the $2d$-world and on the right hand side of figure 49 we have the $3d$-world.

Figure 49: lifting from $\mathbb{R}^2$ to $\mathbb{R}^3$

$2d$-world: Let $\Omega \subset \mathbb{R}^2$ be a domain (open and bounded subset) with 'smooth' boundary $\partial\Omega$ and denote a parametrisation for the boundary $\partial\Omega$ by $\gamma\colon [a,b] \to \mathbb{R}^2$.

$3d$-world: Let $\alpha\colon \Omega \cup \partial\Omega \to \mathcal{S} \cup \partial\mathcal{S}, (s,t) \mapsto \alpha(s,t)$, be a parametrisation (lifting map from $\mathbb{R}^2$ to $\mathbb{R}^3$) for bounded 'smooth' surface $\mathcal{S}$. The boundary $\partial\mathcal{S}$ can parametrised by

$$\alpha \circ \gamma\colon [a,b] \to \mathbb{R}^3, u \mapsto \alpha(\gamma(u)) = \alpha((\gamma_1(u), \gamma_2(u))$$

with unit tangent vector $\widehat{T}_{\mathcal{S}}$ given by the chain rule

$$\widehat{T}_{\mathcal{S}}(\gamma(u)) = \frac{1}{\|\alpha'(\gamma(u))\|}\alpha'(\gamma(u)) = \frac{1}{\|\frac{\partial\alpha}{\partial s}\frac{\partial\gamma_1(u)}{\partial u} + \frac{\partial\alpha}{\partial t}\frac{\partial\gamma_2(u)}{\partial u}\|}\Big(\frac{\partial\alpha}{\partial s}\frac{\partial\gamma_1(u)}{\partial u} + \frac{\partial\alpha}{\partial t}\frac{\partial\gamma_2(u)}{\partial u}\Big).$$

Note that the norm drops out when we compute the tangent line integral. Henceforth,

$$\begin{aligned}\int_{\partial\mathcal{S}} \langle f, \widehat{T}_{\mathcal{S}}\rangle &= \int_a^b \Big\langle f(\alpha(\gamma(u))), \frac{\partial\alpha}{\partial s}(\gamma(u))\frac{\gamma_1}{\partial u}(u) + \frac{\partial\alpha}{\partial t}(\gamma(u))\frac{\partial\gamma_2}{\partial u}(u)\Big\rangle\,\mathrm{d}u \\ &= \int_a^b \Big\langle \begin{pmatrix}\langle f, \frac{\partial\alpha}{\partial s}\rangle \\ \langle f, \frac{\partial\alpha}{\partial t}\rangle\end{pmatrix}, \gamma'(u)\Big\rangle\,\mathrm{d}u = \int_{\partial\Omega} \Big\langle \begin{pmatrix}\langle f, \frac{\partial\alpha}{\partial s}\rangle \\ \langle f, \frac{\partial\alpha}{\partial t}\rangle\end{pmatrix}, \widehat{T}_{\Omega}\Big\rangle\,\mathrm{d}u.\end{aligned}$$

Similarly we get (after some computation and application of the chain rule)

$$\int_{\mathcal{S}} \langle \operatorname{curl} f, \widehat{N} \rangle = \iint_{\Omega} \left\langle \operatorname{curl} f, \frac{\partial \alpha}{\partial s} \times \frac{\partial \alpha}{\partial t} \right\rangle \mathrm{d}s\mathrm{d}t$$
$$= \iint_{\Omega} \operatorname{curl} \begin{pmatrix} \langle f, \frac{\partial \alpha}{\partial s} \rangle \\ \langle f, \frac{\partial \alpha}{\partial t} \rangle \end{pmatrix} \mathrm{d}s\mathrm{d}t.$$

The result follows now from Green's theorem 9.4.

$\square$

Example 10.3 We prove Stokes's theorem for the following example. Let the surface $\mathcal{S} = \{(x, y, z) \in \mathbb{R}^3 \colon z = x^2 + y^2, z \le 4\}$ with parametrisation

$$\alpha \colon \overline{B(0,2)} \to \mathbb{R}^3, (x, y) \mapsto \alpha(x, y) = \begin{pmatrix} x \\ y \\ x^2 + y^2 \end{pmatrix},$$

where $\overline{B(0,2)} = \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 \le 4\}$ is the ball around the origin having radius 2. We get the surface unit normal $\widehat{N}$ as the gradient of the function $s(x, y, z) = z - x^2 - y^2$ (level set), that is,

$$\widehat{N}(x, y, z) = \frac{\nabla s(x, y, z)}{\|\nabla s(x, y, z)\|} = \frac{1}{3} \begin{pmatrix} -2x \\ -2y \\ 1 \end{pmatrix}, \quad (x, y, z) \in \mathcal{S} = \alpha(\overline{B(0,2)}).$$

The boundary $\partial\mathcal{S}$ is parametrised by

$$\gamma \colon [0, 2\pi] \to \mathbb{R}^3, t \mapsto \gamma(t) = \begin{pmatrix} 2\cos t \\ 2\sin t \\ 4 \end{pmatrix}.$$

Let the vector field $f \colon \mathbb{R}^3 \to \mathbb{R}^3, (x, y, z) \mapsto f(x, y, z) = \begin{pmatrix} y \\ x^2 \\ 3z^2 \end{pmatrix}$ be given.

Some calculation gives then

$$\int_{\partial\mathcal{S}} \langle f, \widehat{T} \rangle = \int_0^{2\pi} \left\langle \begin{pmatrix} 2\sin t \\ 4\cos^2 t \\ 48 \end{pmatrix}, \begin{pmatrix} -2\sin t \\ 2\cos t \\ 0 \end{pmatrix} \right\rangle \mathrm{d}t = -4\pi.$$

$$\operatorname{curl}(f)(x, y, z) = \begin{pmatrix} 0 \\ 0 \\ 2x - 1 \end{pmatrix},$$

and therefore

$$\int_{\mathcal{S}} \langle \operatorname{curl} f, \widehat{N} \rangle = \int_{\overline{B(0,2)}} \Big\langle \begin{pmatrix} 0 \\ 0 \\ 2x-1 \end{pmatrix}, \frac{1}{3} \begin{pmatrix} -2x \\ -2y \\ 1 \end{pmatrix} \Big\rangle 3 \,\mathrm{d}x\mathrm{d}y$$
$$= \int_{\overline{B(0,2)}} 2x \,\mathrm{d}x\mathrm{d}y - \int_{\overline{B(0,2)}} \mathrm{d}x\mathrm{d}y = 0 - \mathbf{vol}(\overline{B(0,2)}) = -4\pi.$$

Thus we have shown Stokes's theorem for this example. $\diamond$

Remark 10.4 *(a) If $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a differentiable gradient vector field (that is, there is a potential $V\colon \mathbb{R}^3 \to \mathbb{R}$ with $\nabla V(x) = f(x)$), we get*

$$\int_{\mathcal{S}} \langle \operatorname{curl} f, \widehat{N} \rangle = \int_{\partial\mathcal{S}} \langle f, \widehat{T} \rangle = 0$$

either by FTC for line integrals or by observing

$$\operatorname{curl} \operatorname{grad} V(x,y,z) = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \times \begin{pmatrix} \frac{\partial V}{\partial x} \\ \frac{\partial V}{\partial y} \\ \frac{\partial V}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 V}{\partial y \partial z} - \frac{\partial^2 V}{\partial z \partial y} \\ \frac{\partial^2 V}{\partial z \partial x} - \frac{\partial^2 V}{\partial x \partial z} \\ \frac{\partial^2 V}{\partial x \partial y} - \frac{\partial^2 V}{\partial y \partial x} \end{pmatrix} = 0,$$

where we assume that $V\colon \mathbb{R}^3 \to \mathbb{R}$ is two times continuously differentiable.

(b) If the vector field $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ can be represented as the curl of some vector field $v\colon \mathbb{R}^3 \to \mathbb{R}^3$, that is, $f = \operatorname{curl} v$, we get

$$\int_{\partial\Omega} \langle f, \widehat{N} \rangle = \int_{\Omega} \operatorname{div} f(x,y,z) \,\mathrm{d}x\mathrm{d}y\mathrm{d}z = 0$$

by observing $\operatorname{div} \operatorname{curl} v(x,y,z) = 0$ *for all $(x,y,z) \in \mathbb{R}^3$.*

Example 10.5 (Faraday's law) Let the time-dependent electric (E) and magnetic (H) field be given,

$$E\colon \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3,$$
$$H\colon \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3.$$

For a bounded surface $\mathcal{S} \subset \mathbb{R}^3$ with closed boundary $\partial\mathcal{S} = \mathcal{C}$ the voltage around $\mathcal{C}$ is given by the tangent-line integral

$$\int_{\mathcal{C}} \langle E, \widehat{T} \rangle,$$

which is time-dependent, and the time-dependent magnetic flux is given by the flux integral

$$\int_{\mathcal{S}} \langle H, \widehat{N} \rangle.$$

Fraday's law: voltage around $\mathcal{C}$ equals the negative rate of change of magnetic flux through $\mathcal{S}$.

We show that Farday's law follows from $\nabla \times E = -\frac{\partial H}{\partial t}$. By Stokes's theorem

$$\int_{\mathcal{C}} \langle E, \widehat{T} \rangle = \int_{\mathcal{S}} \langle \nabla \times E, \widehat{N} \rangle,$$

which implies that

$$-\frac{\partial}{\partial t} \int_{\mathcal{S}} \langle H, \widehat{N} \rangle = \int_{\mathcal{S}} \langle \nabla \times E, \widehat{N} \rangle = \int_{\mathcal{C}} \langle E, \widehat{T} \rangle,$$

which gives finally the mathematical form of Faraday's law

$$\boxed{\int_{\mathcal{C}} \langle E, \widehat{T} \rangle = -\frac{\partial}{\partial t} \int_{\mathcal{S}} \langle H, \widehat{N} \rangle.}$$

◇

We finish this section with a summary on conservative vector fields.

Theorem 10.6 (Conservative vector fields) *Let $f \colon \mathbb{R}^3 \to \mathbb{R}^3$ be a C^1-vector field. Then the following conditions are equivalent.*

(i) For any oriented closed simple curve $\mathcal{C} \subset \mathbb{R}^3$: $\int_{\mathcal{C}} \langle f, \widehat{T} \rangle = 0$.

(ii) For any two oriented simple closed curves $\mathcal{C}_1 \subset \mathbb{R}^3$ and $\mathcal{C}_2 \subset \mathbb{R}^3$ having the same starting and terminal point: $\int_{\mathcal{C}_1} \langle f, \widehat{T}_1 \rangle = \int_{\mathcal{C}_2} \langle f, \widehat{T}_2 \rangle$.

(iii) f is the gradient of some C^2 scalar field $V \colon \mathbb{R}^3 \to \mathbb{R}$.

(iv) $\nabla \times f(x) = \mathrm{curl}\,(f)(x) = 0$ for all $x \in \mathbb{R}^3$.

The vector field f is called **conservative** *is one of the conditions (i)-(iv) is satisfied.*

10.2 Polar coordinates

As a reminder we summarise briefly important facts about spherical coordinates.

Polar coordinates in $\mathbb{R}^3$

$$\boldsymbol{x}\colon [0,\infty)\times[-\pi,\pi]\times[0,2\pi]\to\mathbb{R}^3, (r,\theta,\varphi)\mapsto \boldsymbol{x}(r,\theta,\varphi)=\begin{pmatrix} r\cos\varphi\cos\theta\\ r\sin\varphi\cos\theta\\ r\sin\theta\end{pmatrix}.$$

Here the single coordinates have the following names.
$r\in[0,\infty)$ is called the radius,
$\theta\in[-\pi,\pi]$ is called the (geographical) latitude,
$\varphi\in[0,2\pi]$ is called the (geographical) longitude.
Compare with Example 5.2 part (c) where we defined iteratively polar coordinates for any dimension. A function $f\colon\mathbb{R}^3\to\mathbb{R}$ can be given in polar coordinates by writing

$$\overline{f}\colon [0,\infty)\times[-\pi,\pi]\times[0,2\pi]\to\mathbb{R}, (r,\theta,\varphi)\mapsto\overline{f}(r,\theta,\varphi)=f(\boldsymbol{x}(r,\theta,\varphi)).$$

Basis vectors

$$\mathrm{e}_r=\frac{\partial\boldsymbol{x}}{\partial r}=\begin{pmatrix}\cos(\varphi)\cos(\theta)\\ \sin(\varphi)\cos(\theta)\\ \sin(\theta)\end{pmatrix}, \|\mathrm{e}_r\|=1,\quad \hat{\mathrm{e}}_r=\frac{\mathrm{e}_r}{\|\mathrm{e}_r\|}=\mathrm{e}_r,$$

$$\mathrm{e}_\theta=\frac{\partial\boldsymbol{x}}{\partial\theta}=\begin{pmatrix}-r\cos(\varphi)\sin(\theta)\\ -r\sin(\varphi)\sin(\theta)\\ r\cos(\theta)\end{pmatrix}, \|\mathrm{e}_\theta\|=r,\quad \hat{\mathrm{e}}_\theta=\begin{pmatrix}-\cos(\varphi)\sin(\theta)\\ -\sin(\varphi)\sin(\theta)\\ \cos(\theta)\end{pmatrix},$$

$$\mathrm{e}_\varphi=\frac{\partial\boldsymbol{x}}{\partial\varphi}=\begin{pmatrix}-r\sin(\varphi)\cos(\theta)\\ r\cos(\varphi)\cos(\theta)\\ 0\end{pmatrix}, \|\mathrm{e}_\varphi\|=r\cos(\theta),\quad \hat{\mathrm{e}}_\varphi=\begin{pmatrix}-\sin(\varphi)\\ \cos(\varphi)\\ 0\end{pmatrix}.$$

The system $(\hat{\mathrm{e}}_r,\hat{\mathrm{e}}_\varphi,\hat{\mathrm{e}}_\theta)$ is a right hand orthonormal system.

Warning: In some textbooks a slightly modified version is used. Instead of θ (geographical latitude) one uses $\Theta\in[0,\pi]$. Then $\pi/2-\Theta$ is the latitude, i.e. $\theta=\pi/2-\Theta$.

Any vector field can be expressed in this basis,

$$V\colon [0,\infty)\times[-\pi,\pi]\times[0,2\pi] \to \mathbb{R}^3, (r,\theta,\varphi) \mapsto V_r(r,\theta,\varphi)\hat{\mathrm{e}}_r+V_\theta(r,\theta,\varphi)\hat{\mathrm{e}}_\theta+V_\varphi(r,\theta,\varphi)\hat{\mathrm{e}}_\varphi.$$

Example 10.7 The vector field $V\colon \mathbb{R}^3 \to \mathbb{R}^3, x \mapsto x$ can be written as

$$V\colon [0,\infty)\times[-\pi,\pi]\times[0,2\pi] \to \mathbb{R}^3, (r,\theta,\varphi) \mapsto V(r,\theta,\varphi) = r\hat{\mathrm{e}}_r.$$

$\diamond$

Warning:

$$\operatorname{div} V \neq \frac{\partial V_r}{\partial r} + \frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_\varphi}{\partial \varphi}.$$

Applying the chain rule one obtains after some calculation the differential operators.

Differential operators:

$$\begin{aligned}
\operatorname{grad}(f) &= \frac{\partial f}{\partial r}\hat{\mathrm{e}}_r + \frac{1}{r\cos\theta}\frac{\partial f}{\partial \phi}\hat{\mathrm{e}}_\phi + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\mathrm{e}}_\theta \\
\operatorname{div}(v) &= \frac{1}{r^2}\frac{\partial}{\partial r}\big(r^2 v_r\big) + \frac{1}{r\cos\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{1}{r\cos\theta}\frac{\partial}{\partial \theta}\big(\cos(\theta)v_\theta\big) \\
\operatorname{curl}(v) &= \frac{1}{r\cos\theta}\Big(\frac{\partial v_\theta}{\partial \phi} - \frac{\partial}{\partial \theta}\big(\cos(\theta)v_\phi\big)\Big)\hat{\mathrm{e}}_r + \frac{1}{r}\Big(\frac{\partial v_r}{\partial \theta} - \frac{\partial}{\partial r}\big(rv_\theta\big)\Big)\hat{\mathrm{e}}_\phi \\
&\quad + \frac{1}{r}\Big(\frac{\partial}{\partial r}\big(rv_\phi\big) - \frac{1}{\cos\theta}\frac{\partial v_r}{\partial \phi}\Big)\hat{\mathrm{e}}_\theta
\end{aligned}$$

Part II

Introduction to Complex Analysis

11 Complex Derivatives and Möbius transformations

The second part of the lecture gives an introduction to complex analysis. In the first section we study complex derivatives. The field of complex numbers is the set

$$\mathbb{C} = \{z = x + iy \colon x, y \in \mathbb{R}\} \quad \text{with} \quad i^2 = -1.$$

Our aim is to understand the calculus for complex-valued functions $f \colon \mathbb{C} \to \mathbb{C}$ on the complex numbers. The link to our vector analysis part is given by the representation of any complex number in real and imaginary part,

$$f(z) = f(x + iy) = \mathbf{Re}(f)(x, y) + i\mathbf{Im}(f)(x, y) = u(x, y) + iv(x, y),$$

where

$$\begin{aligned} &\mathbf{Re}(f) = u \colon \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto u(x, y) \quad \text{is the real part,} \\ &\mathbf{Im}(f) = v \colon \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto v(x, y) \quad \text{is the imaginary part.} \end{aligned}$$

There is a bijection between any function $f \colon \mathbb{C} \to \mathbb{C}$ and two two-dimensional vector fields $u, v \colon \mathbb{R}^2 \to \mathbb{R}^2$. This is the connection with the first part of the course. We will prove some results via the corresponding results for real vector fields. **Reminder:**

Figure 50: Graphical representation of a complex number

From figure 50 we get

$$\begin{aligned} z &= x + iy = \mathbf{Re}(z) + i\mathbf{Im}(z) \\ &= r\cos(\varphi) + i\sin(\varphi) \quad \text{with} \quad r = \sqrt{x^2+y^2} = |z| \end{aligned}$$

and the argument $\varphi = \arctan(\frac{y}{x}) = \arg(z)$. We call $x = \mathbf{Re}(z)$ the real part and $y = \mathbf{Im}(z)$ the imaginary part of $z = x+iy \in \mathbb{C}$ and $|z| = \sqrt{x^2+y^2}$ the modulus of $z = x+iy \in \mathbb{C}$.

Addition: $(x_1+iy_1)+(x_2+iy_2) = (x_1+x_2)+i(y_1+y_2), x_1, x_2, y_1, y_2 \in \mathbb{R}$.

Multiplication: Let $z = r\cos(\varphi) + ir\sin(\varphi), w = s\cos(\psi) + is\sin(\psi) \in \mathbb{C}$, then $z \cdot w = rs\cos(\varphi+\psi) + irs\sin(\varphi+\psi) \in \mathbb{C}$.

Complex conjugate: The complex conjugate of $z = x+iy \in \mathbb{C}$ is $\overline{z} = x - iy \in \mathbb{C}$ with the following rules for $z, w \in \mathbb{C}$

(i) $\overline{\overline{z}} = z$.

(ii) $\overline{z+w} = \overline{z} + \overline{w}$.

(iii) $\overline{zw} = \overline{z}\overline{w}$.

(iv) $|\overline{z}| = |z|$.

(v) $|z|^2 = z\overline{z}$.

Inverse: $z \in \mathbb{C}, z \neq 0$, the inverse is defined as

$$\frac{1}{z} = \frac{x}{(x^2+y^2)} - i\frac{y}{(x^2+y^2)} = \frac{1}{|z|^2}\overline{z}.$$

Moivre-Laplace:

$$\boxed{\mathrm{e}^{i\varphi} = \cos(\varphi) + i\sin(\varphi) \quad \text{ for } \varphi \in \mathbb{R}.}$$

Any complex number $z \in \mathbb{C}$ can be written in the *polar form*, i.e. as $z = |z|\mathrm{e}^{i\varphi}$ where $\varphi = \arg(z)$ and $|z| = \sqrt{\mathbf{Re}(z)^2 + \mathbf{Im}(z)^2}$. Let $n \in \mathbb{N}$ and $z = |z|\mathrm{e}^{i\varphi} \in \mathbb{C}$

$$\begin{aligned} z^n = 1 &\Leftrightarrow |z|^n \mathrm{e}^{in\varphi} = 1 \\ &\Leftrightarrow |z| = 1 \;\text{ and }\; \cos(n\varphi) + i\sin(n\varphi) = 1 \\ &\Leftrightarrow |z| = 1 \;\text{ and }\; \varphi = \frac{2k\pi}{n} \quad k = 0, 1, \ldots, n-1. \end{aligned}$$

Hence, the distant roots of $z^n = 1$ are given by

$$z_k = \mathrm{e}^{\frac{2k\pi i}{n}} \quad , k = 0, 1, \ldots, n-1,$$

and they are called the n-th roots of unity.

Definition 11.1 *Let $(z_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{C}$, that is $z_n \in \mathbb{C}$ for all $n \in \mathbb{N}$, and $z \in \mathbb{C}$. The sequence $(z_n)_{n\in\mathbb{N}}$ converges to z as $n \to \infty$ if and only if the real sequence $(|z_n - z|)_{n\in\mathbb{N}}$ converges to 0 as $n \to \infty$. Equivalent formulation with $z_n = x_n + iy_n, z = x + iy$:*

$$\begin{aligned} z_n \to z \;\textit{as}\; n \to \infty &\Leftrightarrow |z_n - z| \to 0 \;\textit{as}\; n \to \infty \\ &\Leftrightarrow |x_n - x| \to 0 \;\textit{as}\; n \to \infty \;\textit{and}\; |y_n - y| \to 0 \;\textit{as}\; n \to \infty. \end{aligned}$$

Note that $|\cdot|$ is used in the first line as the modulus for complex numbers and in the second line as the modulus for real numbers.

Lemma 11.2 (Rules for converging sequences) *Let $(z_n)_{n\in\mathbb{N}}, (w_n)_{n\in\mathbb{N}}$ be complex sequences with $z_n \to z \in \mathbb{C}$ and $w_n \to w \in \mathbb{C}$ as $n \to \infty$. then* *(a)* $z_n + w_n \to z + w$ *as* $n \to \infty$.

(b) $z_n w_n \to zw$ *as* $n \to \infty$.

(c) $\frac{z_n}{w_n} \to \frac{z}{w}$ *as* $n \to \infty$ *if* $w \neq 0$.

Proof. Exercise, use the following lemma. $\square$

Lemma 11.3 (Inequalities) *Let* $z, w \in \mathbb{C}$.

(a) $|\mathbf{Re}(z)| \leq |z|$, $|\mathbf{Im}(z)| \leq |z|$.

(b) $|z + w| \leq |z| + |y|$ ***Triangle Inequality***.

(c) $|z + w| \geq \big||z| - |w|\big|$.

Proof. Exercise. $\square$

Notation 11.4 *Let* $\varepsilon > 0$. *The set*

$$B(z, \varepsilon) = \{z' \in \mathbb{C} \colon |z' - z| < \varepsilon\}$$

is the open ball (circle) in $\mathbb{C}$ *around* $z \in \mathbb{C}$ *with radius* ε, *see figure 51. The circle line is* $\partial B(z, \varepsilon) = \{z' \in \mathbb{C} \colon |z' - z| = \varepsilon\}$ *and the closed ball is*

$$\overline{B(z, \varepsilon)} = B(z, \varepsilon) \cup \partial B(z, \varepsilon) = \{z' \in \mathbb{C} \colon |z' - z| \leq \varepsilon\}.$$

Figure 51: ball of radius ε

The **circle** centre $a \in \mathbb{C}$ and radius $r > 0$ is the locus of points at distance r from a so has equation $|z - a| = r$. There is another equation: let $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq \beta$ and let $\lambda \in \mathbb{R}$, with $\lambda > 0$ and $\lambda \neq 1$. The equation

$$\boxed{\left|\frac{z-\alpha}{z-\beta}\right| = \lambda, \quad \lambda > 0, \lambda \neq 0, \qquad (11.26)}$$

represents a circle. This can be seen using $\alpha = \alpha_1 + i\alpha_2, \beta = \beta_1 + i\beta_2$, and $z = x + iy$. The equation (11.26) can be rewritten as $|z - \alpha|^2 = \lambda^2|z - \beta|^2$, and simplifications lead indeed to the equation

$$\left(x - \frac{\alpha_1 - \lambda^2\beta_1}{1 - \lambda^2}\right)^2 + \left(y - \frac{\alpha_2 - \lambda^2\beta_2}{1 - \lambda^2}\right)^2 = c, \qquad \text{for some } c > 0.$$

We shall investigate the geometric significance of the points α and β in (11.26), which are known as **inverse points** with respect to the circline. First note that for $\lambda = 1$ we have $|z - \alpha| = |z - \beta|$. That is, we have a line ℓ. The points α and β are reflections of each other in ℓ. Let $\lambda \neq 1$. There are exactly two points on the circle which are collinear with α and β. We call them z_1 and z_2 respectively and they satisfy

$$z_1 - \alpha = \lambda(z_1 - \beta) \quad \text{and} \quad z_2 - \alpha = -\lambda(z_2 - \beta).$$

Figure 52: circles from inverse-point representation

These points lie at opposite ends of a diameter and writing the equation of the circle in the form $|z-a|=r$ we get

$$a=\frac{1}{2}(z_1+z_2) \quad \text{and} \quad r=\frac{1}{2}|z_1-z_2|.$$

Hence

$$\alpha-a=\frac{1}{2}\lambda(z_2-z_1) \quad \text{and} \quad \lambda(\beta-a)=\frac{1}{2}(z_2-z_1).$$

Thus

$$(\alpha-a)\overline{(\beta-a)}=\frac{1}{4}(z_2-z_1)\overline{(z_2-z_1)}=r^2.$$

The points α and β in $\mathbb{C}$ are **inverse points** with respect to the circle $|z-a|=r$ if and only if they satisfy $(\alpha-a)\overline{(\beta-a)}=r^2$. Note that we must always have one of α and β inside the circle and the other outside see figure 52.

The extended complex plane and the Riemann sphere

We next introduce an ingenious device, which will allow us to treat lines and circles, and half-lines and circular arcs, in a unified way. Consider $\mathbb{C}$ as embedded in Euclidean space $\mathbb{R}^3$ by identifying $z=x+iy$ with the point $(x,y,0)\in\mathbb{R}^3$. Furthermore, define

$$\Sigma=\{(x,y,w)\in\mathbb{R}^3\colon x^2+y^2+(w-\frac{1}{2})^2=\frac{1}{4}\}.$$

This is a sphere centered at $(0,0,\frac{1}{2})$ having radius $\frac{1}{2}$, called the **Riemann sphere**. It touches the complex plane $\mathbb{C}$ at the point $S=(0,0,0)$ (the 'south pole'). Stereographic projection allows us to set up a one-to-one correspondence between the points of $\mathbb{C}$ and the points of Σ, excluding $N=(0,0,1)$, the 'north pole' of Σ. Geometrically, the line from any point z of $\mathbb{C}$ to the north pole N cuts $\Sigma\setminus\{N\}$ in precisely one point z', and, for every point z' of $\Sigma\setminus\{N\}$, the line through N and z' meets the plane $\mathbb{C}$ in a unique point z. The irritation of the north pole being 'left out in the cold' can be removed: just add to $\mathbb{C}$ an extra point $\infty\notin\mathbb{C}$ and define the **extended complex plane** $\widetilde{\mathbb{C}}$ to be $\mathbb{C}\cup\{\infty\}$.

Figure 53: Riemann sphere - stereographic projection

Correspondence $\widetilde{\mathbb{C}}$ and Σ:

$$\begin{aligned} \mathbb{C} \ni z = x + iy = re^{i\theta} &\longleftrightarrow z' = (x(1+r^2)^{-1}, y(1+r^2)^{-1}, r^2(1+r^2)^{-1}), \\ \infty &\longleftrightarrow (0,0,1). \end{aligned}$$

We shall add some arithmetic rules for the extended complex plane $\widetilde{\mathbb{C}}$:

$$\begin{aligned} a \pm \infty = \pm + a = \infty, a/\infty = 0, &\quad \text{for all } a \in \mathbb{C}, \\ a\infty = \infty a = \infty, a/0 = \infty, &\quad \text{for all } a \in \mathbb{C} \setminus \{0\}, \\ \infty + \infty = \infty\infty = \infty. & \end{aligned}$$

We shall consider lines rather in $\widetilde{\mathbb{C}}$ than in $\mathbb{C}$, by regarding ∞ as being adjoined to any line in $\mathbb{C}$. Having the above stereographic correspondence, circles on Σ which pass through the north pole N project down onto lines in $\widetilde{\mathbb{C}}$. Any circle drawn on Σ parallel to the horizontal plane $w = 0$, necessarily with centre on the vertical axis $x = y = 0$, projects down to a circle in $\mathbb{C}$ (with centre 0). Clearly, any circle on Σ which does not pass through N

projects onto to a circle in $\mathbb{C}$, and that every circle in $\mathbb{C}$ arises in this manner. It is thus natural to regard lines as 'circles through infinity', and to adopt the collective name **circline** for a circle or straight line in $\widetilde{\mathbb{C}}$. Henceforth, (11.26) with $\lambda > 0$ represents a **circline** which is a line if $\lambda = 1$ and a circle (in $\mathbb{C}$) otherwise.

Möbius transformations

We introduce a family of mappings of the extended complex plane onto itself which map circlines to circlines. First some particular mappings.

$$
\begin{aligned}
&z \mapsto z\mathrm{e}^{i\varphi} \quad (\varphi \in \mathbb{R}) \quad \text{anticlockwise rotation through } \varphi,\\
&z \mapsto Rz \quad (R > 0) \quad \text{stretching by a factor } R,\\
&z \mapsto z + a \quad (a \in \mathbb{C}) \quad \text{translation by } a,\\
&z \mapsto 1/z \quad \text{inversion.}
\end{aligned}
$$

It is easy to see that mappings of the first three types take straight lines to straight lines and circles to circles. Consider the inversion $z \mapsto w = 1/z$ as a map from $\widetilde{\mathbb{C}}$ to $\widetilde{\mathbb{C}}$. The image of the line $\mathbf{Re}\, z = 1$ is described in inverse-point form as $|z| = |z - 2|$ (points equidistant from 0 and 2). We get that $|1/w| = |1/w - 2|$, that is, $|2w - 1| = 1$. As inversion is self-inverse, we see also that the circle $|2z - 1| = 1$ maps to the line $\mathbf{Re}\, w = 1$ under $z \mapsto w = 1/z$. Henceforth under inversion a line may map to a circle and a circle may map to a line.

Definition 11.5 *A **Möbius transformation** is a mapping of the form*

$$z \mapsto w = f(z) = \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{C}, ad - bc \neq 0.$$

This mapping is viewed as a mapping f from $\widetilde{\mathbb{C}}$ to $\widetilde{\mathbb{C}}$, when putting $f(-d/c) = \infty$ and $f(\infty) = a/c$, according to the above mentioned rules in $\widetilde{\mathbb{C}}$. Then $f\colon \widetilde{\mathbb{C}} \to \widetilde{\mathbb{C}}$ is one-to-one and onto, with a well-defined inverse given by

$$f^{-1}\colon \widetilde{C} \to \widetilde{\mathbb{C}}, f^{-1}(w) = \frac{dw - b}{a - cw}.$$

Proposition 11.6 (Circlines under Möbius transformations) *Let $\mathcal{C}$ be a circline with inverse points α and β in $\mathbb{C}$ and let f be a Möbius transformation. Then f maps $\mathcal{C}$ to a circline, with inverse points $f(\alpha)$ and $f(\beta)$.*

Remark 11.7 *In real analysis convergence of a sequence is defined with the so-called $\varepsilon - n_0$-criterion. For complex valued sequences $(z_n)_{n\in\mathbb{N}}$ it reads as follows:*

$$z_n \to z \in \mathbb{C} \text{ as } n \to \infty \Leftrightarrow \forall \varepsilon > 0 \, \exists \, n_0(\varepsilon) \in \mathbb{N} \, \forall n \geq n_0(\varepsilon) \colon z_n \in B(z, \varepsilon).$$

Definition 11.8 *A function $f \colon D \to \mathbb{C}, D \subset \mathbb{C}$, is continuous at $z \in D$ if for any sequence $(z_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ with $z_n \to z$ as $n \to \infty$ one gets that $f(z_n) \to f(z)$ as $n \to \infty$. The function f is continuous on D if it is continuous at each $z \in D$.*

Remark 11.9 *The $\varepsilon - \delta$-criterion reads as*

$$\begin{aligned}
&f \colon D \to \mathbb{C} \text{ is continuous at } z \in D \\
&\quad \Leftrightarrow \forall \varepsilon > 0 \, \exists \, \delta > 0 \colon w \in D \text{ and } |z - w| < \delta \Rightarrow |f(z) - f(w)| < \varepsilon \\
&\quad \Leftrightarrow \forall \varepsilon > 0 \, \exists \, \delta > 0 \colon w \in B(z, \delta) \Rightarrow f(w) \in B(f(z), \varepsilon).
\end{aligned}$$

Note that $\delta = \delta(\varepsilon, z)$.

Definition 11.10 *The function $f \colon D \to \mathbb{C}, D \subset \mathbb{C}$, is said to be **differentiable at** $z \in D$ if the limit (differential quotient)*

$$f'(z) := \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists, that is, if the limit of the sequence $\frac{f(z+h_n)-f(z)}{h_n}$ exists for every sequence $(h_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ with $h_n \to 0$ as $n \to \infty$, and this limit is (when it exists) denoted by $f'(z)$.

Lemma 11.11 (Rules) *Let $f, g \colon D \to \mathbb{C}, D \subset C$, be continuous (differentiable) at $z \in D$. Then $f+g, fg, f/g$ (for $g(z) \neq 0$ for $z \in D$) are continuous (differentiable) at $z \in D$.*

So far there seems to be no much difference with real analysis. But the following example sheds some light onto the new features of complex analysis.

Example 11.12 (a) $f \colon \mathbb{C} \to \mathbb{C}, z \mapsto f(z) = z$ is continuous on $\mathbb{C}$ and, because of

$$\frac{f(z+h) - f(z)}{h} = 1, \quad h \in \mathbb{C}, h \neq 0,$$

differentiable at any $z \in \mathbb{C}$ with $f'(z) = 1$.

(b) $f\colon \mathbb{C} \to \mathbb{C}, z \mapsto f(z) = \overline{z} = x - iy$ for $z = x + iy$. As

$$\begin{aligned} z_n = x_n + iy_n \to z \text{ as } n \to \infty \Rightarrow x_n \to x \text{ and } y_n \to y \text{ as } n \to \infty \\ \Rightarrow \overline{z}_n \to \overline{z} \text{ as } n \to \infty, \end{aligned}$$

we get that f is continuous on $\mathbb{C}$. Pick $z \in \mathbb{C}$, for any $h \in \mathbb{C}, h \neq 0$, we have

$$\frac{f(z+h) - f(z)}{h} = \frac{\overline{z+h} - \overline{z}}{h} = \frac{\overline{h}}{h}.$$

If we choose $h_n = \frac{1}{n}, n \in \mathbb{N}$, we get $\frac{\overline{h_n}}{h_n} = 1$ for any $n \in \mathbb{N}$. But if we choose $h_n = i\frac{1}{n}, n \in \mathbb{N}$, we get $\frac{\overline{h_n}}{h_n} = -1$. Hence, the limit of the differential quotient does not exist and therefore f is not differentiable at $z \in \mathbb{C}$. $\diamond$

Example 11.12(b) shows that in complex analysis differentiability of complex-valued function on $\mathbb{C}$ is not equivalent to the differentiability of the real and imaginary part alone.

Warning: Let $f = u+iv$ be a complex valued function on $\mathbb{C}$, $z = x+iy \in \mathbb{C}$,

$$f \text{ continuous on } \mathbb{C} \Leftrightarrow u, v \text{ continuous on } \mathbb{R}^2 \quad \textbf{is true!}$$

$$f \text{ differentiable at } z \in \mathbb{C} \Leftrightarrow u, v \text{ differentiable at } (x, y) \in \mathbb{R}^2 \quad \textbf{is false!}$$

12 Complex Power Series

In the last section we learned that differentiability for complex valued functions on $\mathbb{C}$ is not equivalent to the differentiability of the real and imaginary parts alone. In this section we study another route to complex calculus, the theory for complex power series. In real analysis you heard about real power series and that some real functions can be expressed as a power series (Taylor series).

Definition 12.1 *(a) Let $c_n \in \mathbb{C}, n \in \mathbb{N}_0$ and $c \in \mathbb{C}$. The series $\sum_{n=0}^{\infty} c_n$ converges if and only if the sequence $(s_N)_{N \in \mathbb{N}}, s_N = \sum_{n=0}^{N} c_n$, converges. If $\sum_{n=0}^{\infty}$ converges (in symbols $\sum_{n=0}^{\infty} c_n < \infty$) we call $c = \sum_{n=0}^{\infty} c_n = \lim_{N \to \infty} s_N$ the sum.*

(b) Cauchy-criterion: Let $(z_n)_{n \in \mathbb{N}}, z_n = x_n + iy_n$, be a complex sequence.

$$\begin{aligned} &(z_n)_{n \in \mathbb{N}} \textit{ is a Cauchy sequence} \\ &\quad \Leftrightarrow \forall \varepsilon > 0 \, \exists \, n_0(\varepsilon) \in \mathbb{N}\colon \ \forall m, n \geq n_0(\varepsilon)\colon |z_m - z_n| < \varepsilon \\ &\quad \Leftrightarrow (x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \textit{ are Cauchy sequences in } \mathbb{R}. \end{aligned}$$

The Cauchy criterion is used in the proof of the following lemma.

Lemma 12.2 *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions $f_n\colon D \to \mathbb{C}, D \subset \mathbb{C}$, and $M_n \geq 0$ for all $n \in \mathbb{N}$. If $|f_n(z)| \leq M_n$ for all $z \in D, n \in \mathbb{N}$, and $\sum_{n=0}^\infty M_n < \infty$, then the series*

$$f(z) := \sum_{n=0}^\infty f_n(z)$$

converges for all $z \in D$. If all f_n are continuous functions then so is the function f defined via the sum of the converging series for any $z \in D$.

Proof. Pick $z \in D$ and let $s_N(z) := \sum_{n=0}^N f_n(z)$. Then (w.l.o.g. $M > N$)

$$|s_M(z) - s_N(z)| = \Big|\sum_{k=N+1}^M f_k(z)\Big| \leq \sum_{k=N+1}^M M_k.$$

The converges of $\sum_{n=0}^\infty M_n$ implies that $M_n \to 0$ as $n \to \infty$. Hence, $(|s_M(z) - s_N(z)| \to 0$ as $N, M \to \infty$, and thus $f(z)$ exists (i.e., the sum converges). For the proof of the continuity part see real analysis. □

Definition 12.3 (Power series) *A power series is defined to be a series of the form $\sum_{n=0}^\infty c_n(z-a)^n, a \in \mathbb{C}$, and $c_n \in \mathbb{C}$ for all $n \in \mathbb{N}_0$. W.l.o.g. we often assume $a = 0$.*

(a) The ***radius of convergence*** *of the power series $\sum_{n=0}^\infty c_n(z-a)^n$ is defined to be*

$$R = \sup\big\{|z-a|\colon z \in \mathbb{C} \text{ with } \sum_{n=0}^\infty c_n(z-a)^n \text{ converges }\big\}. \tag{12.27}$$

We write $R = \infty$ if $\sum_{n=0}^\infty c_n(z-a)^n$ converges for arbitrarily large $|z|$, i.e., for all $z \in \mathbb{C}$.

(b) The series $\sum_{n=0}^\infty c_n(z-a)^n$ is said to be ***absolutely convergent*** *if the real series $\sum_{n=0}^\infty |c_n(z-a)^n|$ is convergent.*

Theorem 12.4 (Power series Theorem) *Let $(c_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{C}$. Then there is a $R \in [0, \infty]$ such that*

(a) $\sum_{n=0}^\infty c_n z^n$ converges for $z \in C$ with $|z| < R$,

(b) $\sum_{n=0}^{\infty} c_n z^n$ *does not converge for* $z \in C$ *with* $|z| > R$.

(c) If $\sum_{n=0}^{\infty} c_n(z-a)^n$ *converges for* $z_1 \in \mathbb{C}$ *then* $\sum_{n=0}^{\infty} |c_n||(z-a)|^n$ *converges for all* $z \in \mathbb{C}$ *with* $|z-a| < |z_1 - a|$.

(d) If $\sum_{n=0}^{\infty} c_n(z-a)^n$ *diverges for* $z_1 \in \mathbb{C}$ *then* $\sum_{n=0}^{\infty} c_n(z-a)^n$ *diverges for all* $z \in \mathbb{C}$ *with* $|z-a| > |z_1 - a|$.

(e) On the open ball $B(0, R)$ *a function is defined as*

$$f\colon B(0,R) \to \mathbb{C}, z \mapsto f(z) = \sum_{n=0}^{\infty} c_n z^n.$$

The function f *is continuous and differentiable on* $B(0, R)$ *with derivative given by term-by-term differentiation:*

$$\frac{\mathrm{d}f}{\mathrm{d}z}(z) = \sum_{n=1}^{\infty} c_n n z^{n-1}.$$

Proof. The proof for (a)-(d) is easy. We sketch it for (c): If $\sum_{n=0}^{\infty} c_n(z_1 - a)^n$ converges we know that $c_n(z_1 - a)^n \to 0$ as $n \to \infty$. Hence, there is $M > 0$ such that $|c_n(z_1 - a)|^n \le M$. We get

$$|c_n(z-a)| = |c_n(z_1-a)^n| \left|\frac{z-a}{z_1-a}\right|^n \le M \left|\frac{z-a}{z_1-a}\right|^n,$$

and if the right hand side gives a convergent series (geometric series) we get the statement for $|z-a| < |z_1 - a|$. The remaining part will be proved later. The proof is similar to the corresponding one for real analysis. $\square$

Remark 12.5 *(a) The series may or may not converge for* $|z| = R$. *This has to be analysed in detail for any example. Recall that* $\partial B(0, R)$ *is the circle line of radius* R *around the origin and* $\overline{B(0,R)}$ *is the closed ball. The set of complex numbers for which the power series converges includes the open ball* $B(0, R)$ *and possibly some points from the boundary line* $\partial B(0, R)$.

(b) A similar theorem holds in $\mathbb{R}$.

(c) One can apply the theorem repeatedly:

$$f \text{ is } C^\infty \text{ on } B(0,R) \text{ with}$$
$$\frac{\mathrm{d}^k f}{\mathrm{d}z^k}(z) = \sum_{n=k} c_n n(n-1)\cdots(n-k+1) z^{n-k}.$$

(d) We have put $a = 0$ in the theorem because, by shifting,

$$g(z) := \sum_{n=0}^{\infty} c_n (z-a)^n$$

converges for all $z \in B(a, R)$ and $f(z) = g(z + a), z \in B(0, R)$.

Two methods for the calculation of the radius of convergence

A d'Alembert's Ratio test:

If

$$\frac{|c_{n+1}|}{|c_n|} \to \alpha \in [0, \infty] \text{ as } n \to \infty,$$

then $\sum_{n=0}^{\infty} |c_n|$ converges if $\alpha < 1$ (i.e. $\sum_{n=0}^{\infty} c_n$ converges absolutely), and $\sum_{n=0}^{\infty} |c_n|$ diverges if $\alpha > 1$. If $\alpha = 1$ then the test gives no information.

B Cauchy's $\boldsymbol{n}$-th root test:

If

$$\lim_{n\to\infty} \sqrt[n]{|c_n|} = \alpha \in [0, \infty),$$

then $\sum_{n=0}^{\infty} |c_n|$ converges if $\alpha < 1$ (i.e. $\sum_{n=0}^{\infty} c_n$ converges absolutely), and $\sum_{n=0}^{\infty} |c_n|$ diverges if $\alpha > 1$. If $\alpha = 1$ then the test gives no information.

Proof.

A: Let $\alpha < 1$. Then there a $q \in (\alpha, 1)$. Hence, there is $n_0 \in \mathbb{N}$ such that $\left|\frac{c_{n+1}}{c_n}\right| < q$ for all $n \geq n_0$. W.l.o.g. we can assume that the last statement holds for all $n \in \mathbb{N}$ as any addition of finitely many terms does not change the convergence properties of the series. We get

$$\left|\frac{c_{n+1}}{c_n}\right|\left|\frac{c_n}{c_{n-1}}\right| \cdots \left|\frac{c_2}{c_1}\right| = \left|\frac{c_{n+1}}{c_1}\right|, \quad \forall n \in \mathbb{N}.$$

Hence, $|c_{n+1}| < |c_1| q^n$, and $\sum_{n=1}^{\infty} |c_1| q^n < \infty$ gives the convergence of $\sum_{n=0}^{\infty} |c_n|$. If $\alpha > 1$ we get $\left|\frac{c_{n+1}}{c_n}\right| > 1$, and hence $|c_n| < |c_{n+1}|$ for all $n \in \mathbb{N}$. But this implies that the necessary condition for convergence, $c_n \to 0$ as $n \to \infty$, does not hold.

B: Let $\alpha < 1$. From the convergence of the sequence $(\sqrt[n]{|c_n|})_{n\in\mathbb{N}}$ we get for $q \in (\alpha, 1)$ a $n_0(q)$ with $\sqrt[n]{|c_n|} < q$ for all $n \geq n_0(q)$. Hence $|c_n| < q^n$ or all $n \geq n_0(q)$. This implies that $\sum_{n=n_0(q)}^{\infty} q^n$ converges. Therefore $\sum_{n=n_0(q)}^{\infty} c_n$ converges absolutely as well as $\sum_{n=0}^{\infty} c_n$ (addition of only finitely many terms). This proves the first (convergence) part. If $\alpha > 1$ we have a subsequence $(c_{n_k})_{k\in\mathbb{N}}$ with $\lim_{k\to\infty} \sqrt[n_k]{|c_{n_k}|} = \alpha$ such that there is $k_0 \in \mathbb{N}$ with $|c_{n_k}| > 1$ for all $k \geq k_0$. But this contradicts the necessary condition $c_n \to 0$ as $n \to \infty$ for the convergence of the series. Therefore the series diverges for $\alpha > 1$. $\square$

The **convergence set** of a power series $\sum_{n=0}^{\infty} c_n(z-a)^n$ is the set of complex numbers z for which the series converges. This set includes the open disc $|z-a| < R$ if R is the radius of convergence. It may also contain points of the boundary, however, this has to check for each example separately.

Example 12.6 (a) For which complex numbers does the series

$$\sum_{n=0}^{\infty} c_n(z-a)^n = \sum_{n=0}^{\infty} (3+i)(2i)^n(z+i)^n$$

converge? Pick $z \in \mathbb{C}$ and $c_n = (2i)^n(z+i)^n$. Then

$$\frac{|c_{n+1}|}{|c_n|} = 2|z+i|.$$

If $|z+i| < \frac{1}{2}$ the series converges, and if $|z+i| > \frac{1}{2}$ the series diverges. Hence the radius of convergence is $R = \frac{1}{2}$ and the series converges for all points in the open ball $B(-i, \frac{1}{2})$ of radius $\frac{1}{2}$ around $-i$, see figure 54.

Figure 54: ball of convergence (a)

What happens on the boundary of that open ball? For complex numbers $z \in \mathbb{C}$ with $|z+i| = \frac{1}{2}$ we get $\frac{|c_{n+1}|}{|c_n|} = 1$, and hence $(|c_n|)_{n\in\mathbb{N}}$ and therefore $(c_n)_{n\in\mathbb{N}}$ does not converge to zero. Thus the series is divergent for any point on the boundary $\partial B(-i, \frac{1}{2})$. The series converges only in the open ball $B(-i, \frac{1}{2})$.

(b) For which complex numbers does the series $\sum_{n=1}^{\infty} \frac{z^n}{n}$ converge?

$$\frac{|c_{n+1}|}{|c_n|} = |z|\frac{n}{n+1} \to |z| \text{ as } n \to \infty,$$

gives $R = 1$. We know that on the boundary line $\partial B(0,1)$ the series converges for $z = -1$ but it diverges for $z = 1$ (harmonic series). The convergence set is shown in figure 55.

Figure 55: convergence set (b)

What about other boundary points $z \in \partial B(0,1) \setminus \{-1, 1\}$? This is a hard problem. ◇

Definition 12.7 *For $z \in \mathbb{C}$ the following power series are defined*

$$
\begin{aligned}
\mathrm{e}^z &:= \sum_{n=0}^{\infty} \frac{z^n}{n!}, \\
\cos(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}, \\
\cosh(z) &:= \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n}, \\
\sin(z) &:= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}, \\
\sinh(z) &:= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} z^{(2n+1)}.
\end{aligned}
\tag{12.28}
$$

Proposition 12.8 *The power series on the right hand side of* (12.28) *have radius of convergence $R = \infty$.*

Proof. Exercise. The ratio test shows, easily, that all series have infinite radius of convergence. □

How do the functions defined in (12.28) behave for $z \in \mathbb{R}$? From the Power Series Theorem 12.4 we know for example that

$$\frac{\mathrm{d}}{\mathrm{d}z}\mathrm{e}^z = \sum_{n=1}^{\infty} n \frac{z^{n-1}}{n!} = \sum_{n=0}^{\infty} \frac{z^n}{n!} = \mathrm{e}^z.$$

The following lemma gives a representation of some functions with the exponential function.

Lemma 12.9 *The following holds for all $z \in \mathbb{C}$.*

$$
\begin{aligned}
\sin(z) &= \frac{\mathrm{e}^{iz} - \mathrm{e}^{-iz}}{2i}, \\
\cos(z) &= \frac{\mathrm{e}^{iz} + \mathrm{e}^{-iz}}{2}, \\
\sinh(z) &= \frac{\mathrm{e}^{z} - \mathrm{e}^{-z}}{2}, \\
\cosh(z) &= \frac{\mathrm{e}^{z} + \mathrm{e}^{-z}}{2}.
\end{aligned}
$$

Proof. First note that

$$i^n + (-i)^n = \begin{cases} 2i^n \ ; & \text{if } n = 2k \text{ is even} \\ 0 \ ; & \text{if } n \text{ is odd} \end{cases}$$

Using the power series for the exponential function we get

$$\begin{aligned} \frac{1}{2}\big(\mathrm{e}^{iz} + \mathrm{e}^{-iz}\big) &= \frac{1}{2}\sum_{n=0}^{\infty}\Big(\frac{(iz)^n}{n!} + \frac{(-iz)^n}{n!}\Big) \\ &= \frac{1}{2}\sum_{n=0}^{\infty}\frac{z^n}{n!}\big(i^n + (-i)^n\big) \\ &= \sum_{k=0}^{\infty}\frac{z^{2k}(-1)^k}{(2k)!} = \cos(z). \end{aligned}$$

The remaining relations are proved similar. □

The following example shows that the complex sine function is unbounded in contrast to the real sine function.

Example 12.10 The sine function at $z \in \mathbb{C} \setminus \mathbb{R}$ is

$$\sin(iy) = \frac{\mathrm{e}^{-y} - \mathrm{e}^{y}}{2i} = i\frac{\mathrm{e}^{y} - \mathrm{e}^{-y}}{2} = i\sinh(y), \quad y \in \mathbb{R}.$$

Choosing $y = 10$ we get $|\sin(10i)| = |i\sinh(10)| \approx 22000$ in contrast to $|\sin(x)| \leq 1$ for $x \in \mathbb{R}$. ◇

Theorem 12.11 (Properties of the exponential function) *The exponential function*

$$\exp\colon \mathbb{C} \to \mathbb{C}, z \mapsto \exp(z) = \mathrm{e}^z = \sum_{n=0}^{\infty}\frac{z^n}{n!}$$

has the properties

(a) $\mathrm{e}^0 = 1$,

(b) $\mathrm{e}^{z+w} = \mathrm{e}^z\mathrm{e}^w$ *for all* $z, w \in \mathbb{C}$,

(c) $\mathrm{e}^z \neq 0$ *for all* $z \in \mathbb{C}$.

Proof. (a) Follows from the power series. (b) We first sketch the long method (Cauchy product rule for series). Put

$$p_l = \sum_{n,m\in\mathbb{N}_0\colon n+m=l}\frac{z^n}{n!}\frac{w^m}{m!} \qquad \text{for all } l \in \mathbb{N}_0.$$

Then

$$\Big(\sum_{n=0}^{\infty}\frac{z^n}{n!}\Big)\Big(\sum_{m=0}^{\infty}\frac{w^m}{m!}\Big)=\sum_{l=0}^{\infty}p_l=\sum_{l=0}^{\infty}\frac{1}{l!}\sum_{n=0}^{l}\binom{l}{n}z^nw^{l-n}=\sum_{l=0}^{\infty}\frac{(z+w)^l}{l!}.$$

Smart method: Fix $c\in\mathbb{C}$ and consider the function $f(z):=\mathrm{e}^z\mathrm{e}^{c-z}, z\in\mathbb{C}$. The function is differentiable at any $z\in\mathbb{C}$ with (product rule) $f'(z)=\mathrm{e}^z\mathrm{e}^{c-z}-\mathrm{e}^z\mathrm{e}^{c-z}=0, z\in\mathbb{C}$. Hence (see below Proposition 13.7) there is a constant $K\in\mathbb{C}$ such that $f(z)=K$ for all $z\in\mathbb{C}$. To find K we put $z=c$ and obtain $K=\mathrm{e}^c\mathrm{e}^{c-c}=\mathrm{e}^c$ because of (a). Thus $\mathrm{e}^c=\mathrm{e}^z\mathrm{e}^{c-z}$ for all $z\in\mathbb{C}$. Putting $c=w+z$ we conclude with (b). (c) Now $\mathrm{e}^z\mathrm{e}^{-z}=1, z\in\mathbb{C}$, gives (c).

□

Periodicity

Lemma 12.12 *Let $z\in\mathbb{C}$.*

(a) $\mathrm{e}^z=1\Leftrightarrow z=2k\pi i$ *for some* $k\in\mathbb{Z}$.

(b) $\mathrm{e}^z=-1\Leftrightarrow z=(2k+1)\pi i$ *for some* $k\in\mathbb{Z}$.

(c) $\mathrm{e}^{z+\alpha}=\mathrm{e}^z$ *for all* $z\in\mathbb{C}\Leftrightarrow\alpha=2k\pi i$ *for some* $k\in\mathbb{Z}$.

Proof. We prove only (a) as (b) and (c) follow immediately analogously. Put $z=x+iy$. Then

$$\begin{aligned}\mathrm{e}^{x+iy}=\mathrm{e}^x\mathrm{e}^{iy}=\mathrm{e}^x(\cos(y)+i\sin(y))=1&\Leftrightarrow|\mathrm{e}^x|=1\ \text{ and }\ \cos(x)+i\sin(y)=1\\&\Leftrightarrow x=0;\cos(y)=1,\text{and }\sin(y)=0\\&\Leftrightarrow x=0\text{ and }y=2k\pi,k\in\mathbb{Z}.\end{aligned}$$

□

Argument Any complex number $z\in\mathbb{C}$ can be written in the polar form $z=|z|\mathrm{e}^{i\theta}$. Let $z\in\mathbb{C}\setminus\{0\}$, the argument of z is the set

$$\arg(z):=\{\theta\in\mathbb{R}\colon z=|z|\mathrm{e}^{i\theta}\}.$$

Note that $\arg(z)$ is a countably infinite set consisting of all numbers of the form $\theta+2k\pi,k\in\mathbb{Z}$.

Example 12.13 *Let $z,w\in\mathbb{C}\setminus\{0\}$.*

(a) $\arg(i) = \{(4k+1)\frac{\pi}{2} \colon k \in \mathbb{Z}\}$.

(b) $\arg(zw) = \{\theta + \varphi \colon \theta \in \arg(z), \varphi \in \arg(w)\}$.

(c) $\arg(\frac{1}{z}) = \{-\theta \colon \theta \in \arg(z)\}$.

(d) *Let* $w \in \mathbb{C}, z \in \mathbb{C} \setminus \{0\}$. *Put* $z = \mathrm{e}^w = \mathrm{e}^{u+iv}, u, v \in \mathbb{R}$. *Then* $|z| = |\mathrm{e}^u \mathrm{e}^{iv}| = \mathrm{e}^u$ *and* $\arg(z) = \{v + 2k\pi \colon k \in \mathbb{Z}\}$. *Thus*

$$\mathrm{e}^w = z \Leftrightarrow w = \log|z| + i\theta, \quad \theta \in \arg(z).$$

$\diamond$

Definition 12.14 *Let* $z \in \mathbb{C} \setminus \{0\}$ *and* $\alpha \in \mathbb{C}$. *We use* $[\cdot]$*-brackets for a set.*

(a) $[\log z] := \{\log|z| + i\theta \colon \theta \in \arg(z)\}$. *For* $w = u + iv$ *one gets*

$$[\log \mathrm{e}^w] = \{\log \mathrm{e}^u + i(v + 2k\pi) \colon k \in \mathbb{Z}\} = \{w + 2k\pi \colon k \in \mathbb{Z}\}.$$

(b) $[z^\alpha] := \{\mathrm{e}^{\alpha(\log|z| + i\theta)} \colon \theta \in \arg(z)\}$.

Example 12.15 (Graphical representation of the n-th roots) *Solve* $z^6 = -1$, *that is give the six roots* $z_k, k = 0, 1, \ldots, 5$, *with* $z_k^6 = -1$. *Note that* $-1 = \mathrm{e}^{i\theta} = \mathrm{e}^{i\pi}$, *i.e.* $\theta = \pi$. *Hence,*

$$z_k = \Big(\cos\big(\frac{\theta + 2k\pi}{6}\big) + i\sin\big(\frac{\theta + 2k\pi}{6}\big)\Big), k = 0, 1, \ldots, 5.$$

In figure 56 we get the six roots as the corners of the regular six polygon.

Figure 56: the roots of $z^6 = -1$

$\diamond$

13 Holomorphic functions

In this section holomorphic functions will be defined. The starting point is Example 11.12(b). The complex conjugation is not differentiable at any point $z \in \mathbb{C}$. We want to understand this. Therefore we assume that the function $f\colon \mathbb{C} \to \mathbb{C}, z \mapsto f(z) = u(x,y) + iv(x,y)$ is differentiable at $z = x + iy \in \mathbb{C}$.

We study the following two cases for the limit of the differential quotient.

1.) Let $\mathbb{R}_+ \ni \varepsilon > 0$ and put $h = \varepsilon \in \mathbb{C}$, that is $h \to 0$ means $\varepsilon \to 0$. Then

$$\lim_{\varepsilon\to 0} \frac{u(x+\varepsilon,y)-u(x,y)}{\varepsilon} + i\lim_{\varepsilon\to 0}\frac{v(x+\varepsilon,y)-v(x,y)}{\varepsilon} = \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y).$$

2.) Let $\mathbb{R}_+ \ni \varepsilon > 0$ and put $h = \varepsilon i \in \mathbb{C}$, that is $h \to 0$ means $\varepsilon \to 0$. Then

$$\lim_{\varepsilon\to 0} \frac{u(x,y+\varepsilon)-u(x,y)}{i\varepsilon} + i\lim_{\varepsilon\to 0}\frac{v(x,y+\varepsilon)-v(x,y)}{i\varepsilon} = -i\frac{\partial u}{\partial y}(x,y) + \frac{\partial v}{\partial y}(x,y).$$

The function f is differentiable at $z = x + iy$ and therefore all the limits of the differential quotients exist and are equal. Thus we get the following equations

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \qquad (13.29)}$$

The two equations (13.29) are called the **Cauchy-Riemann equations**. We have thus shown that a function which is differentiable at a point $z = x+iy$ satisfies the Cauchy-Riemann equations (13.29) at the point $(x, y) \in \mathbb{R}^2$. The converse is true only if the partial derivatives of the real and imaginary part of the function are continuous in a small neighbourhood of that point.

Theorem 13.1 *Let $f\colon D \to \mathbb{C}, D \subset \mathbb{C}$, with $f(x + iy) = u(x, y) + iv(x, y)$ for $x + iy \in D$.*

(a) If the function f is differentiable at $z_0 = x_0 + iy_0 \in D$, then the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ exist at (x_0, y_0) and satisfy the Cauchy-Riemann equations (13.29) *at the point (x_0, y_0).*

(b) If the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ exists and are continuous in an open ball $B((x_0, y_0), \varepsilon), \varepsilon > 0$, around (x_0, y_0) such that the corresponding open ball in the complex plane is contained in D ($B((x_0 + iy_0), \varepsilon) \subset D$) and the Cauchy-Riemann equations hold at (x_0, y_0), then the function f is differentiable at $z_0 = x_0 + iy_0 \in D$ with

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) - i\frac{\partial u}{\partial y}(x_0, y_0). \quad (13.30)$$

Proof. (a) This follows as indicated above. (b) Apply the mean value theorem of real analysis (exercise). □

Example 13.2 (a) $f\colon \mathbb{C} \to \mathbb{C}, z \mapsto f(z) = z^3$. The real and imaginary parts are

$$\begin{aligned} f(x + iy) &= (x + iy)^3 = x^3 + 3x^2 iy + 3x(iy)^2 + (iy)^3 \\ &= (x^3 - 3xy^2) + i(3x^2 y - y^3), \end{aligned}$$

with partial derivatives

$$\begin{aligned} \frac{\partial u}{\partial x} &= 3x^2 - 3y^2; & \frac{\partial v}{\partial y} &= 3x^2 - 3y^2 \\ \frac{\partial u}{\partial y} &= -6xy; & \frac{\partial v}{\partial x} &= 6xy. \end{aligned}$$

The Cauchy-Riemann equations are satisfied for all $(x,y) \in \mathbb{R}^2$ and the partial derivatives are continuous on $\mathbb{R}^2$, hence the function f is differentiable on $\mathbb{C}$.

(b) $f\colon \mathbb{C} \to \mathbb{C}, z \mapsto f(z) = f(x+iy) = x^2 + iy^2$. The partial derivatives of the real and imaginary part are

$$\frac{\partial u}{\partial x} = 2x; \quad \frac{\partial v}{\partial y} = 2y$$
$$\frac{\partial u}{\partial y} = 0; \quad \frac{\partial v}{\partial x} = 0,$$

they are continuous on $\mathbb{R}^2$ but the Cauchy-Riemann equations are only satisfied on $\{(x,y) \in \mathbb{R}^2 \colon x = y\}$. The function f is differentiable on $\{z \in \mathbb{C} \colon z = x + ix, x \in \mathbb{R}\}$. $\diamond$

The condition in Theorem 13.1 that there is a small ball in the domain of definition is crucial. In the following we will assume that the domain for any function has the property that one can find around any point of that domain a small ball contained in the domain. Such sets have a special name.

Definition 13.3 (Holomorphic function) *(a) Let $D \subset \mathbb{C}$. The set D is called* ***open*** *if for any $z \in D$ there is an $\varepsilon > 0$ such that $B(z,\varepsilon) \subset D$.*

(b) A function $f\colon D \to \mathbb{C}, D \subset \mathbb{C}$ open, is called ***holomorphic*** *at $z_0 \in D$ if f is differentiable at all $z \in B(z_0,\varepsilon) \subset D$ for some $\varepsilon > 0$. The function f is called holomorphic in D if f is holomorphic at all $z \in D$.*

Remark 13.4 *Differentiability in real analysis is defined at a single point. Being holomorphic is not a property of a single point, it depends on a whole neighbourhood of a single point in the complex plane. This is the main difference, and the notion of holomorphic functions will prove to be the most useful one.*

Lemma 13.5 *Let $f,g\colon D \to \mathbb{C}, D \subset \mathbb{C}$ open, be holomorphic in D. Then the following holds.*

(a) $f+g, \lambda f$ and fg are holomorphic in D, where $\lambda \in \mathbb{C}$.

(b) Suppose $f(z) \neq 0$ for all $z \in D$. Then $\frac{1}{f}$ is holomorphic in D with

$$\Big(\frac{1}{f}\Big)'(z) = \frac{-f'(z)}{f^2(z)}, \quad z \in D.$$

(c) Suppose $g(z) \neq 0$ for all $z \in D$. Then $\frac{f}{g}$ is holomorphic in D with

$$\Big(\frac{f}{g}\Big)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{g^2(z)}, \quad z \in D.$$

Proof. Follows analogous to the proofs for real analysis. Exercise. □

Example 13.6 (a) Let $f\colon \mathbb{C} \to \mathbb{C}$ be non-constant and holomorphic in $\mathbb{C}$. Then $u = \mathbf{Re}(f)$ is not holomorphic in $\mathbb{C}$. Assume that u is holomorphic. We conclude from the Cauchy-Riemann equations that $\frac{\partial u}{\partial x}(x,y) = 0$ and $\frac{\partial u}{\partial y}(x,y) = 0$ holds for all $(x,y) \in \mathbb{R}^2$. This implies that u is constant in contrast to f non-constant. Hence, the assumption was wrong and u is not holomorphic.

(b) Let $f\colon \mathbb{C} \to \mathbb{C}$ be non-constant and holomorphic in $\mathbb{C}$. Then $|f|$ is nowhere holomorphic. Denote the real part by $\widetilde{u} = \mathbf{Re}(|f|) = |f|$ and the imaginary part $\widetilde{v} = 0$. Assume that $|f|$ is holomorphic. Then the Cauchy-Riemann equations imply that (u, v are the real and imaginary part of f respectively)

$$\begin{aligned} \partial_x \widetilde{u} = 0 &= \frac{1}{\sqrt{u^2+v^2}}\big(u\partial_x u + v\partial_x v\big) \\ \partial_y \widetilde{u} = 0 &= \frac{1}{\sqrt{u^2+v^2}}\big(u\partial_y u + v\partial_y v\big). \end{aligned}$$

This gives

$$u\partial_x u + v\partial_x v = 0 = u\partial_y u + v\partial_y v,$$

and with the Cauchy-Riemann equations for the function f itself we get, summing the left and right hand side,

$$0 = u(\partial_x u + \partial_y u) + v(\partial_x u - \partial_y u).$$

This equation can be satisfied only by u constant which contradicts our assumption that f is non-constant or by the condition that the terms in the brackets vanish (other choices like $u = -v$ are not possible). The terms in the brackets vanish if $\partial_x u + \partial_y u = 0$ and $\partial_x u = \partial_y u$. This implies $2\partial_y u = 0$ and $\partial_x u = 0$, and thus that u is constant again in contradiction to our assumption. Thus $|f|$ is nowhere holomorphic.

◇

Proposition 13.7 *Let $f\colon D \to \mathbb{C}, D \subset \mathbb{C}$ open, be holomorphic in D. Then any of the following conditions forces the function f to be constant in D.*

(a) $f'(z) = 0$ *for all* $z \in D$.

(b) $|f|$ *is constant in* D.

(c) $f(z) \in \mathbb{R}$ *for all* $z \in D$.

Our aim is to apply the theorems of vector analysis for the study of complex valued functions on $\mathbb{C}$. Let $f\colon \mathbb{C} \to \mathbb{C}$ be a holomorphic function on $\mathbb{C}$ with real and imaginary part u respectively v. Define the two-dimensional vector field

$$F\colon \mathbb{R}^2 \to \mathbb{R}^2, (x,y) \mapsto F(x,y) = \begin{pmatrix} F_1(x,y) \\ F_2(x,y) \end{pmatrix} = \begin{pmatrix} u(x,y) \\ -v(x,y) \end{pmatrix}.$$

Then

$$\begin{aligned} \operatorname{div} F &= \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0 \\ \operatorname{curl} F &= \frac{F_2}{\partial x} - \frac{\partial F_1}{\partial y} \\ &= -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0. \end{aligned}$$

Hence, the theorems of Gauss and Green are useful here. The missing piece is the notion of integration for complex valued functions defined on the complex plane. This is the content of the next section.

14 Complex integration

The complex line integral will be defined, the Fundamental Theorem of Calculus will be proved and examples of non-vanishing integrals along closed paths are studied. A curve (path) $\mathcal{C} \subset \mathbb{C}$ is a set points in the complex plane for which one can find a one-dimensional parametrisation (i.e. a parametrisation that depends only on one real variable). Let a parametrisation for a curve $\mathcal{C}$ in the complex plane $\mathbb{C}$ be given, i.e.

$$\gamma\colon [a,b] \to \mathbb{C}, t \mapsto \gamma(t) = x(t) + iy(t),$$

where $a \leq b$ and where $x(t)$ is the real part $\mathbf{Re}(\gamma(t))$ and $y(t)$ is the imaginary part $\mathbf{Im}(\gamma(t))$ of the curve. Recall that γ is a parametrisation of the curve $\mathcal{C}$ if γ is piecewise C^1 and $\gamma([a,b]) = \mathcal{C}$. The derivative with respect to the real parameter t is $\gamma'(t) = x'(t) + iy'(t), t \in [a,b]$.

Definition 14.1 *The line (path) integral of a function $f\colon D \to \mathbb{C}, D \subset \mathbb{C}$, along the curve (path) $\mathcal{C} \subset D \subset \mathbb{C}$ with parametrisation $\gamma\colon [a,b] \to \mathbb{C}$ is defined as*

$$\boxed{\begin{aligned}\int_\gamma f &= \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t \\ &= \int_a^b \mathbf{Re}(f(\gamma(t))\gamma'(t))\,\mathrm{d}t + i\int_a^b \mathbf{Im}(f(\gamma(t))\gamma'(t))\,\mathrm{d}t\end{aligned} \qquad (14.31)}$$

Alternative notation: $\int_{\mathcal{C}} f$ *and* $\int_\gamma f(z)\,\mathrm{d}z$.

Example 14.2 Let the function $f\colon \mathbb{C} \to \mathbb{C}, z \mapsto f(z) = f(x+iy) = x^2 + iy^2$ and the parametrised path $\gamma\colon [0,1] \to \mathbb{C}, t \mapsto \gamma(t) = t(1+i)$ be given. γ parametrises the straight line from the origin to $1+i$, see figure 57.

Figure 57: path from the origin to $\boldsymbol{1+i}$

As $\gamma'(t) = 1+i$ for all $t \in [0,1]$ we get the path integral

$$\int_\gamma f = \int_0^1 (t^2 + it^2)(1+i)\,\mathrm{d}t = 2i\int_0^1 t^2\,\mathrm{d}t = \frac{2}{3}i.$$

$\diamond$

The following properties are easily proved (compare part I of the lecture).

Lemma 14.3 *Let $\mathcal{C} \subset \mathbb{C}$ be a curve in the complex plane with parametrisation $\gamma \colon [a, b] \to \mathbb{C}$ (i.e., $\gamma([a, b]) = \mathcal{C}$) and let $f \colon \mathcal{C} \to \mathbb{C}$ be a continuous function.*

(a) If $\gamma^- \colon [a, b] \to \mathbb{C}, t \mapsto \gamma^-(t) = \gamma(a + b - t)$ is the reversed parametrisation, then
$$\int_{\gamma^-} f = -\int_\gamma f.$$

(b) Let $\widetilde{\gamma} \colon [\widetilde{a}, \widetilde{b}] \to \mathbb{C}$ and suppose that $\widetilde{\gamma} = \gamma \circ \psi$ with $\widetilde{\gamma}([\widetilde{a}, \widetilde{b}]) = \mathcal{C}$ and that the map $\psi \colon [\widetilde{a}, \widetilde{b}] \to [a, b]$ is bijective and has a positive continuous derivative. Then
$$\int_{\widetilde{\gamma}} f = \int_\gamma f.$$

In the next example we integrate the complex conjugate along a closed path. Recall from Example 11.12 that the complex conjugate is not differentiable but continuous.

Example 14.4 Let $\mathcal{C} = \partial B(i, 2)$ be the circle line around i with radius 2, see figure 58.

Figure 58: circle line $\boldsymbol{\partial B(i, 2)}$

The line integral for the complex conjugate, i.e., for the function $f \colon \mathbb{C} \to \mathbb{C}, z \mapsto f(z) = \overline{z}$, along $\mathcal{C}$ is computed as follows. A parametrisation for the circle line is given by $\gamma(t) = i + 2e^{it}$ for $t \in [0, 2\pi]$ with derivative $\gamma'(t) = 2ie^{it}, t \in [0, 2\pi]$. Hence,

$$\begin{aligned}\int_\gamma f &= \int_0^{2\pi} \overline{\left(i + 2e^{it}\right)} 2ie^{it}\,\mathrm{d}t = \int_0^{2\pi} \left(-i + 2e^{-it}\right) 2ie^{it}\,\mathrm{d}t \\ &= \int_0^{2\pi} \left(2e^{it} + 4i\right)\mathrm{d}t = 2\frac{e^{it}}{i}\Big|_{t=0}^{2\pi} + 8\pi i = 8\pi i.\end{aligned}$$

It remains to check that the first term vanishes, that is

$$\begin{aligned}\int_0^{2\pi} e^{it}\,\mathrm{d}t &= \int_0^{2\pi} \cos(t)\,\mathrm{d}t + i\int_0^{2\pi} \sin(t)\,\mathrm{d}t \\ &= \sin(t)\Big|_0^{2\pi} - i\cos(t)\Big|_0^{2\pi} = \left[\frac{i\sin(t) + \cos(t)}{i}\right]_0^{2\pi} \\ &= \frac{e^{it}}{i}\Big|_0^{2\pi} = 0.\end{aligned}$$

$\diamond$

Why does the integral in Example 14.4 does not vanish, although we have a closed path? We will come back to this later. We now study the most important integral in complex analysis, the so-called *Fundamental Integral*.

Example 14.5 (Fundamental Integral) Let $a \in \mathbb{C}$ and $r > 0$ and consider the circle line $\partial B(a, r)$ around a with radius r and parametrisation $\gamma(t) = a + re^{it}, t \in [0, 2\pi]$. Then

$$\boxed{\int_{\partial B(a,r)} (z-a)^n = \begin{cases} 0\,;\ n \neq -1 \\ 2\pi i\,;\ n = -1 \end{cases} \qquad (14.32)}$$

This can be seen as follows

$$\begin{aligned}\int_0^{2\pi} \left(re^{it}\right)^n rie^{it}\,\mathrm{d}t &= ir^{n+1}\int_0^{2\pi} e^{i(n+1)t}\,\mathrm{d}t \\ &= \begin{cases} ir^{n+1}\left[\frac{\sin((n+1)t)}{n+1} - i\frac{\cos((n+1)t)}{n+1}\right]_0^{2\pi} & ;\ n \neq -1 \\ i\Big[1\Big]_0^{2\pi} & ;\ n = -1 \end{cases}\end{aligned}$$

$\diamond$

We do have examples of line integrals along closed paths which do not vanish. The integrand functions of these examples are not holomorphic at points on the curve and/or at points surrounded by the curve. However, example 14.5 shows that there are cases when the line integral along the closed circline vanishes although the function is not holomorphic at points inside the circle line. We therefore expect to have vanishing line integrals along closed paths as long the integrand functions are holomorphic. The next theorem is the first step towards the so-called Cauchy-Theorem which proves this conjecture.

Theorem 14.6 (Fundamental Theorem of calculus in $\mathbb{C}$) *Let the function $f\colon D \to \mathbb{C}, D \subset \mathbb{C}$, be holomorphic in D with derivative f' and $\mathcal{C} \subset D$ be a curve (path) with parametrisation $\gamma\colon [a, b] \to D$. Then*

$$\boxed{\int_\gamma f' = f(\gamma(b)) - f(\gamma(a)) \qquad (14.33)}$$

Proof.

$$\begin{aligned}\int_\gamma \frac{\mathrm{d}f}{\mathrm{d}z} &= \int_a^b \frac{\mathrm{d}f}{\mathrm{d}z}(\gamma(t))\gamma'(t)\,\mathrm{d}t = \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}(f(\gamma(t)))\,\mathrm{d}t \\ &= \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{Re}(f(\gamma(t)))\,\mathrm{d}t + i\int_a^b \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{Im}(f(\gamma(t)))\,\mathrm{d}t \\ &= f(\gamma(b)) - f(\gamma(a)),\end{aligned}$$

where we used the chain rule and the FTC for the real functions $\mathbf{Re}(f(\gamma(t)))$ and $\mathbf{Im}(f(\gamma(t)))$. □

15 Cauchy's theorem

In this section the Cauchy Theorem is proved and analysed. The proof involves to specify an admissible class of paths/curves. Before that we shall discuss the connection to Green's and Gauss's theorem. Let $\gamma\colon [a, b] \to \mathbb{C}, t \mapsto \gamma(t) = x(t) + iy(t)$ be a parametrisation of a curve (path) $\mathcal{C} \subset \mathbb{C}$ and let a continuous function $f\colon \mathcal{C} \to \mathbb{C}, f = u + iv$, be given. We compute

$$\begin{aligned}
\int_\gamma f &= \int_a^b (u(\gamma(t)) + iv(\gamma(t))(x'(t) + iy'(t))\,\mathrm{d}t \\
&= \int_a^b \big(u(\gamma(t))x'(t) - v(\gamma(t))y'(t)\big)\,\mathrm{d}t + i\int_a^b \big(u(\gamma(t))y'(t) + v(\gamma(t))x'(t)\big)\,\mathrm{d}t \\
&= \int_a^b \Big\langle \begin{pmatrix} u(\gamma(t)) \\ -v(\gamma(t)) \end{pmatrix}, \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} \Big\rangle\,\mathrm{d}t + \int_a^b \Big\langle \begin{pmatrix} u(\gamma(t)) \\ -v(\gamma(t)) \end{pmatrix}, \begin{pmatrix} y'(t) \\ -x'(t) \end{pmatrix} \Big\rangle\,\mathrm{d}t \\
&= \int_\gamma \langle F, \widehat{T}\rangle + i\int_\gamma \langle F, \widehat{N}\rangle,
\end{aligned}$$

where $\widehat{T} = \gamma'(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$ and $\widehat{N} = \begin{pmatrix} y'(t) \\ -x'(t) \end{pmatrix}$ and $F\colon \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto F(x, y) = \begin{pmatrix} u(x, y) \\ -v(x, y) \end{pmatrix}$. Note that $\widehat{N}$ is orthogonal to $\widehat{T}$. At the end of Section 13 we showed that $\operatorname{div} F = \operatorname{curl} F = 0$. Hence, we can easily derive Cauchy's theorem from Green's theorem and Gauss' s theorem. However, it will turn out that the topological properties of the curves (paths) are linked to the calculus of complex valued function defined on domains in $\mathbb{C}$. We only indicate this issue briefly and defer a detailed study of this relationship to later courses. Hence, we will not define properly what deformation in the following definition means. A deformation of a curve (path) is a continuous transformation in the complex plane.

Definition 15.1 *(a) A subset $D \subset \mathbb{C}$ is* ***connected*** *if it cannot be expressed as the union of non-empty open sets $D_1 \subset \mathbb{C}$ and $D_2 \subset \mathbb{C}$ with $D_1 \cap D_2 = \emptyset$.*

(b) A ***region*** *is a non-empty open connected subset of the complex plane $\mathbb{C}$.*

(c) A region $D \subset \mathbb{C}$ is called ***simply connected*** *if every closed curve (path) in D can be deformed to a single point.*

Theorem 15.2 (Cauchy's theorem) *Let the function $f\colon D \to \mathbb{C}, D \subset \mathbb{C}$ be a simply connected region, be holomorphic on D and $\mathcal{C} \subset D$ a closed curve (path) in D with parametrisation γ. Then*

$$\boxed{\int_\gamma f = 0. \qquad (15.34)}$$

Proof. We sketch the proof ignoring the dependence on the topological details of the curve $\mathcal{C} \subset D$. By Ω we denote the region surrounded by the curve (path), that is the boundary $\partial\Omega = \mathcal{C}$. See figure 59.

Figure 59: region $\boldsymbol{D}$ with $\boldsymbol{\Omega \cup \partial\Omega}$

From the previous calculation we get

$$\begin{aligned}\int_\gamma f &= \int_\gamma \langle F, \widehat{T}\rangle + i\int_\gamma \langle F, \widehat{N}\rangle \\ &= \int_{\partial\Omega} \operatorname{curl}(F) + i\int_\Omega \operatorname{div}(F) = 0 + 0,\end{aligned}$$

where the last equality follows from Green's and Gauss's theorem respectively using the fact that γ is a closed path. □

Remark 15.3 The Theorem also holds for more general curves, see figure 60,

Figure 60: γ union of $\gamma_1 \cup \gamma_2$

where the integral $\int_\gamma f = \int_{\gamma_1} f + \int_{\gamma_2} f = 0$ vanishes because Green's and Gauss's theorem can be applied to Ω_1 and Ω_2 respectively. But the region the curve (path) is encircling has to be simply connected, i.e., it must be possible to contract it to a single point. This contraction is not possible in the example of an annulus in figure 61,

Figure 61: annulus with $\partial\Omega \subset D$ and $\Omega \not\subset D$

where $\gamma = \partial\Omega$ but $\Omega \not\subset D$.

For later purposes we consider the following classes of curves (paths). Recall the definition of a simple path: $a \leq s < t \leq b$ implies that $\gamma(s) \neq \gamma(t)$ unless $s = t$ or $s = a$ and $t = b$ if γ is closed.

Notation 15.4 *(a) Let $u, v \in \mathbb{C}$. The image $\mathcal{C}$ of the parametrisation γ given by $\gamma\colon [0,1] \to \mathbb{C}, t \mapsto \gamma(t) = (1-t)u + tv$ is the **line segment** $[u, v]$ traced from u to v.*

*(b) A **circular arc traced** (anti-clockwise/clockwise) is the image of the parametrisation $\gamma\colon [\Theta_1, \Theta_2] \to \; t \mapsto \gamma(t) = a + r\mathrm{e}^{it}, a \in \mathbb{C}$ and $r > 0$ with $\Theta_1 \leq \Theta_2$. The circline $\partial B(a, r)$ around $a \in \mathbb{C}$ with radius $r > 0$ is parametrised by $\gamma\colon [0, 2\pi] \to \mathbb{C}, t \mapsto \gamma(t) = a + r\mathrm{e}^{it}$.*

*(c) A **circline path** is a path which is the join of finitely many paths of type (a) and (b), and a **contour** is a simple closed circline path.*

(d) Given a closed curve (path) $\mathcal{C} \subset \mathbb{C}$ with parametrisation $\gamma\colon [a, b] \to \mathbb{C}$ and the image by $\gamma^* = \gamma([a, b]) = \mathcal{C}$, the inside $\mathbf{I}(\gamma)$ is the set of points surrounded by the curve (path) and $\mathbf{O}(\gamma)$ is the complement of $\mathbf{I}(\gamma) \cup \gamma^*$, see figure 62.

(e) *The curve (path) parametrised by $\gamma\colon [0,2\pi] \to \mathbb{C}, t \mapsto \gamma(t) = a + r\mathrm{e}^{it}$ is called the* ***positively oriented circle*** *with centre a and radius r, and the curve (path) with parametrisation $\gamma\colon [0,2\pi] \to \mathbb{C}, t \mapsto \gamma(t) = a + r\mathrm{e}^{-it}$ is called the* ***negatively oriented circle****. Positively oriented means anti-clockwise in the following.*

Figure 62: I(γ) and O(γ)

Example 15.5 For any $0 < R < \pi$

$$\int_{\partial B(0,R)} \frac{1}{\mathrm{e}^z + 1}\,\mathrm{d}z = 0.$$

To see this note that for $z = x + iy \in \mathbb{C}$

$$\mathrm{e}^z + 1 = 0 \Leftrightarrow \mathrm{e}^z = -1 \Leftrightarrow x = 0, y = (2n+1)\pi \quad \text{for } n \in \mathbb{Z}.$$

Hence the function $f\colon \mathbb{C} \to \mathbb{C}, z \mapsto f(z) = \frac{1}{\mathrm{e}^z+1}$ is holomorphic on and inside any circle with radius $R < \pi$, see figure 63.

Figure 63: $\partial B(0, R)$

◇

In the next example we analyse again the complex conjugation as in Example 14.4.

Example 15.6 In Example 14.4 it is shown that

$$\int_\gamma \overline{z} = 8\pi i,$$

where γ is the parametrisation for $\partial B(i, 2)$, a circle line of radius 2 around i. As 4π is the area of that circle one can guess that the following holds. Let γ be the parametrisation of any closed curve (path) $\mathcal{C}$. Then

$$\boxed{\int_\gamma \overline{z} = 2i(\mathbf{area}(\mathbf{I}(\gamma))), \qquad (15.35)}$$

where $\mathbf{I}(\gamma)$ is the region enclosed by the curve (path). This can be seen as follows. Recall that for any $a, b \in \mathbb{C}$ one can prove that $\mathbf{Im}(a\overline{b}) = 2$(**area** of the triangle spanned by a and b).

Figure 64: $z, z + \Delta$

Consider in figure 64 the triangle spanned by z (vector from the origin to a point z on γ^*) and the vector $z + \Delta$, where Δ is a small tangential line segment attached to the point z on γ^*. Then the area of the triangle spanned by z and $z + \Delta$ is given by $\frac{1}{2}\mathbf{Im}((z + \Delta)\overline{z}) = \frac{1}{2}\mathbf{Im}(\Delta\overline{z})$. The line integral is approximated by the sum of all these small contributions, that is, the integral is obtained as a limit $\Delta \to 0$. One might be inclined to think that the integral of the complex conjugate $\overline{z}$ never vanishes for a non-trivial closed curve (path). But this is not true as we see for the curve in figure 65.

Figure 65: $\gamma = \gamma_1 \cup \gamma_2$

Here γ is the union of γ_1 and γ_2. Now γ_2 is negatively oriented, that is the integral along γ_2 gets a minus sign. As the two areas of the surrounded regions of γ_1 and γ_2 are equal, the two contributions cancel out and the integral along γ vanishes. $\diamond$

One can generalise our fundamental integral in Example 14.5 to arbitrary positively oriented contours. Let γ be a parametrisation of positively oriented contour with $z_0 \notin \gamma^*$. Then

$$\int_\gamma (z - z_0)^n = \begin{cases} 0\,; \text{if } z_0 \in \mathbf{O}(\gamma) \\ 0\,; \text{if } z_0 \in \mathbf{I}(\gamma) \text{ and } n \neq -1 \\ 2\pi i\,; \text{if } z_0 \in \mathbf{I}(\gamma) \text{ and } n = -1 \end{cases} \tag{15.36}$$

The proof follows with the following theorem which we cite here without proof.

Theorem 15.7 (Deformation) *(a) Let γ be a positively oriented contour and suppose that $\overline{B(a,r)} \subset \mathbf{I}(\gamma), a \in \mathbb{C}, r > 0$. Let f be holomorphic inside (i.e. on $\mathbf{I}(\gamma)$) and on γ except possibly at $a \in \mathbb{C}$. Then*

$$\boxed{\int_\gamma f = \int_{\partial B(a,r)} f.}$$

(b) Suppose that $\gamma, \widetilde{\gamma}$ are positively oriented contours such that $\widetilde{\gamma}$ lies inside γ, i.e., $\widetilde{\gamma}^* \subset \mathbf{I}(\gamma)$. If f is holomorphic inside and on γ

$$\boxed{\int_\gamma f = \int_{\widetilde{\gamma}} f.}$$

Example 15.8 Integrate $\int_{\partial B(0,2)} f$, where $f\colon \mathbb{C} \to \mathbb{C}, z \mapsto f(z) = 2(4z^2 - 1)^{-1}$. The function has two singularities inside the circline $\partial B(0,2)$. We write

$$f(z) = \frac{1}{2z-1} - \frac{1}{2z+1}, \quad z \in \mathbb{C}.$$

The Deformation Theorem 15.7 implies that

$$\begin{aligned}
\int_{\partial B(0,2)} f &= \int_{\partial B(0,2)} \frac{1}{2z-1} - \int_{\partial B(0,2)} \frac{1}{2z+1} \\
&= \int_{\partial B(\frac{1}{2},\frac{1}{4})} \frac{1}{2z-1} - \int_{\partial B(-\frac{1}{2},\frac{1}{4})} \frac{1}{2z+1} \\
&= \frac{1}{2} 2\pi i - \frac{1}{2} 2\pi i = 0,
\end{aligned}$$

where we used the circlines $\partial B(\frac{1}{2}, \frac{1}{4})$ and $\partial B(-\frac{1}{2}, \frac{1}{4})$ such that we avoid one of the singularities respectively, see figure 66 (other circline (paths) would do the same).

Figure 66: $B(\frac{1}{2}, \frac{1}{4})$ and $B(-\frac{1}{2}, \frac{1}{4})$

◇

16 Cauchy's formulae

Armed with Cauchy's theorem we can prove a host of striking results about holomorphic functions. In Subsection 16.2 we prove Liouville's theorem, differentiability of any order, and Taylor's theorem.

16.1 Cauchy's formulae

Cauchy's integral formula expresses the value of a holomorphic function at a point in terms of a 'boundary value integral' taken over a contour encircling the point. The ingredients at hand are

- The Deformation Theorem 15.7
- $$\int_{\partial B(a,r)} \frac{1}{w-a}\,\mathrm{d}w = 2\pi i$$
- $f(w) - f(a) \to 0$ as $w \to a$

Theorem 16.1 (Cauchy's integral formula) *Let the function f be holomorphic inside and on a positively oriented contour with parametrisation $\gamma\colon [a,b] \to \mathbb{C}$. Then*

$$\boxed{f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,\mathrm{d}w \qquad \text{for } z \in \mathbf{I}(\gamma). \qquad (16.37)}$$

Proof. Pick $z \in \mathbf{I}(\gamma)$. Then there is an $\varepsilon > 0$ with $\overline{B(z,\varepsilon)} \subset \mathbf{I}(\gamma)$. Use the Deformation Theorem 15.7, add and subtract an integrand with numerator $f(z)$ to obtain

$$\begin{aligned}
\int_\gamma \frac{f(w)}{w-z}\,\mathrm{d}w &= \int_{\partial B(z,\varepsilon)} \frac{f(w)}{w-z}\,\mathrm{d}w \\
&= \int_{\partial B(z,\varepsilon)} \frac{f(z)}{w-z}\,\mathrm{d}w + \int_{\partial B(z,\varepsilon)} \frac{f(w)-f(z)}{w-z}\,\mathrm{d}w \\
&=: I + II.
\end{aligned}$$

The first term I is the fundamental integral

$$I = f(z)\int_{\partial B(z,\varepsilon)} \frac{1}{w-z}\,\mathrm{d}w = 2\pi i f(z).$$

To conclude we need to show that the second term II vanishes for $\varepsilon \downarrow 0$.

$$\begin{aligned}
|II| &= \Big|\int_0^{2\pi} \frac{f(z+\varepsilon \mathrm{e}^{it}) - f(z)}{\varepsilon \mathrm{e}^{it}} \varepsilon i \mathrm{e}^{it}\,\mathrm{d}t\Big| \\
&\le 2\pi \sup_{t\in[0,2\pi]} \Big| f(z+\varepsilon\mathrm{e}^{it}) - f(z)\Big| \to 0 \text{ as } \varepsilon \downarrow 0,
\end{aligned}$$

because the function f is continuous at z. Now the expression on the left hand side is independent of the radius (Deformation Theorem, the first step above) and so it must be zero. □

Remark 16.2 *Holomorphic function are special: the values of the function at the boundary of a closed curve γ determine the value at $z \in \mathbf{I}(\gamma)$.*

Can we also get integral formulae for the derivatives? Let the function f be holomorphic inside and on a positively oriented contour parametrised by γ. Cauchy's formula (16.37) implies

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,\mathrm{d}w \qquad \text{for } z \in \mathbf{I}(\gamma),$$

so we may differentiate the expression on the right hand side with respect to z neglecting the integral. We obtain

$$f'(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-z)^2}\,\mathrm{d}w \qquad \text{for } z \in \mathbf{I}(\gamma),$$

if we can prove that integration and differentiation can be interchanged. We may proceed further for the second derivative (and any higher one as well) but there is one difficulty. The function f is holomorphic which implies so far only that f' exists. But it is not clear if higher derivatives exist as well. A way out of this is to apply Cauchy's integral formula (16.37) to the differential quotient

$$\frac{f(z+h)-f(z)}{h}.$$

Iteration of that process gives the existence of the derivatives $f^{(n)}(z)$ for $n = 2, 3, \ldots$, and $z \in \mathbf{I}(\gamma)$; all given by differentiation under the integral sign. This is the content of the following theorem.

Theorem 16.3 (Cauchy's formulae for derivatives) *Let the function f be holomorphic inside and on a positively oriented contour parametrised by γ and let $z \in \mathbf{I}(\gamma)$. Then $f^{(n)}(z)$ exists for $n = 1, 2, 3, \ldots$, and the derivative is given as*

$$\boxed{f^{(n)}(z) = \frac{n!}{2\pi i}\int_\gamma \frac{f(w)}{(w-z)^{n+1}}\,\mathrm{d}w \qquad \textit{for } z \in \mathbf{I}(\gamma). \qquad (16.38)}$$

Proof. Pick $\in \mathbf{I}(\gamma)$. Then there is an $\varepsilon > 0$ such that $B(z, 2\varepsilon) \subset \mathbf{I}(\gamma)$. Using formula (16.37) for the differential quotient (for h small enough)

$$\begin{aligned}
&\frac{f(z+h)-f(z)}{h} = \frac{1}{2h\pi i}\int_{\partial B(z,2\varepsilon)} f(w)\Big(\frac{1}{w-z-h} - \frac{1}{w-z}\Big)\,\mathrm{d}w \\
&= \frac{1}{2\pi i}\int_{\partial B(z,2\varepsilon)} \frac{f(w)}{(w-z-h)(w-z)}\,\mathrm{d}w \\
&= \frac{1}{2\pi i}\int_{\partial B(z,2\varepsilon)} \frac{f(w)}{(w-z)^2}\,\mathrm{d}w \\
&\quad + \frac{1}{2\pi i}\int_{\partial B(z,2\varepsilon)} f(w)\Big(\frac{1}{(w-z-h)(w-z)} - \frac{1}{(w-z)^2}\Big)\,\mathrm{d}w \\
&= \frac{1}{2\pi i}\int_{\partial B(z,2\varepsilon)} \frac{f(w)}{(w-z)^2}\,\mathrm{d}w + \frac{1}{2\pi i}\int_{\partial B(z,2\varepsilon)} \Big(\frac{hf(w)}{(w-z-h)(w-z)^2}\Big)\,\mathrm{d}w.
\end{aligned}$$

We have to show that the second term on the right hand side vanishes as $h \to 0$. For that choose h so small that $|h| < \varepsilon$ and conclude for all $w \in \partial B(z, 2\varepsilon)$ that $|w - z - h| \geq |w - z| - |h| > \varepsilon$. As f is continuous there is a $M > 0$ with $|f(w)| \leq M$ for $w \in \partial B(z, 2\varepsilon)$. Hence the second term on the right hand side is bounded by $\frac{|h| M 2\varepsilon}{\varepsilon(2\varepsilon)^2}$ which converges to zero as $h \to 0$. The higher derivatives are proved via induction using formula (16.37) for the difference $f^{(k+1)}(z+h) - f^{(k)}(z)$. □

The last theorem guarantees that a holomorphic function is infinitely often differentiable. As in real analysis one is interested to represent such a function as a power series. If we can interchange differentiation and integration we get an integral formula for the coefficients of that power series. This will be the content of the next theorem in Subsection 16.2.

16.2 Taylor's and Liouville's theorem

Theorem 16.4 (Taylor's theorem) *Let the function f be holomorphic on $B(a, R), a \in \mathbb{C}, R > 0$. Then there exist unique constants $c_n \in \mathbb{C}, n \in \mathbb{N}_0$, such that*

$$f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n \quad \text{for } z \in B(a, R), \tag{16.39}$$

and the coefficients are given by

$$c_n = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w - a)^{n+1}} \, \mathrm{d}w = \frac{f^{(n)}(a)}{n!}, \tag{16.40}$$

where γ is the parametrisation of a circle line $\partial B(a, r), 0 < r < R$, or of any positively oriented contour in $B(a, R)$ enclosing the point a.

Proof. Pick $z \in B(a, R)$ and choose $r > 0$ such that $|z - a| < r < R$ and let γ be the parametrisation of the circline $\partial B(a, r)$. The formula (16.37) gives

$$f(z) = \frac{1}{2\pi i} \int_{\partial B(a,r)} \frac{f(w)}{(w - z)} \, \mathrm{d}w. \tag{16.41}$$

Because of $|z-a| < |w-a| = r$ for all $w \in \partial B(a, r)$ we have $|z-a|/|w-a| < 1$ and can expand (in a geometric series)

$$\frac{1}{w - z} = \frac{1}{w - a} \frac{1}{\left(1 - \frac{z-a}{w-a}\right)}.$$

Inserting this into the integrand in (16.41) and interchanging summation and integration (to be justified later in Theorem 18.7) we conclude with the existence of the coefficients and their representation by (16.40). The uniqueness follows from the following. Assume that $f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$ for all $z \in B(a, r)$ for some $0 < r < R$. Provided summation and integration can be interchanged we have for any $n \in \mathbb{N}_0$

$$\begin{aligned}\int_{\partial B(a,r)} \frac{f(z)}{(z-a)^{n+1}} \, \mathrm{d}z &= \int_{\partial B(a,r)} \Big(\sum_{k=0}^{\infty} c_k (z-a)^k \Big) (z-a)^{-n-1} \, \mathrm{d}z \\ &= \sum_{k=0}^{\infty} c_k \int_{\partial B(a,r)} (z-a)^{k-n-1} \, \mathrm{d}z = 2\pi i c_n\end{aligned}$$

because of the fundamental integral in example 14.5. □

The following theorem is a striking result from Cauchy's formula.

Theorem 16.5 (Liouville's theorem) *Let the function f be holomorphic and bounded on $\mathbb{C}$. Then f is a constant function.*

Proof. Suppose $|f(w)| \leq M$ for all $w \in \mathbb{C}$. Pick $a, b \in \mathbb{C}$ and choose $R > 2 \max\{|a|, |b|\}$. Then $|w-a| \geq \frac{1}{2}R$ and $|w-b| \geq \frac{1}{2}R$ whenever $|w| = R$, i.e., whenever $w \in \partial B(0, R)$. Cauchy's formula (16.37) gives

$$\begin{aligned}f(a) - f(b) &= \frac{1}{2\pi i} \int_{\partial B(0,R)} f(w) \Big(\frac{1}{w-a} - \frac{1}{w-b} \Big) \, \mathrm{d}w \\ &= \Big(\frac{a-b}{2\pi i} \Big) \int_{\partial B(0,R)} \frac{f(w)}{(w-a)(w-b)} \, \mathrm{d}w.\end{aligned}$$

Hence

$$|f(a) - f(b)| \leq \frac{1}{2\pi} \frac{|a-b| 2\pi R M}{(\frac{1}{2}R)^2} \to 0 \text{ as } R \to \infty,$$

implying $f(a) = f(b)$. □

The following example shows how Cauchy's formulae can be used to evaluate integrals.

Example 16.6 (a)

$$\int_{\partial B(0,1)} \Big(\frac{\mathrm{e}^w}{w^3} \Big) \, \mathrm{d}w = \frac{2\pi i}{2!} \Big(\frac{\mathrm{d}^2}{\mathrm{d}z^2} \mathrm{e}^z \Big) \Big|_{z=0} = \pi i.$$

(b)

$$\int_{\partial B(-1,3)} \frac{1}{(w-4)(w+1)^4}\,\mathrm{d}w = \frac{2\pi i}{3!}\Big(\frac{\mathrm{d}^3}{\mathrm{d}z^3}\Big(\frac{1}{z-4}\Big)\Big)\Big|_{z=-1} = -\frac{2\pi i}{5^4}.$$

(c) Let γ be a parametrisation of a positively oriented contour.

$$\int_\gamma \Big(\frac{\sin(w)}{w-i}\Big)\,\mathrm{d}w = \begin{cases} 0\ , \text{ if } i \in \mathbf{O}(\gamma) \\ 2\pi i \sin(i)\ , \text{ if } i \in \mathbf{I}(\gamma) \\ ?\ , \text{ if } i \in \gamma^* \end{cases} ,$$

The problem when singularities are on the curve is studied in the next section. $\diamond$

17 Real Integrals

This section will demonstrate the usefulness of Cauchy's Theorem for evaluating real integrals. We focus on the following example. What is the integral

$$\int_{-\infty}^{\infty} \Big(\frac{\sin(x)}{x}\Big)\,\mathrm{d}x\ ?$$

Real analysis tells us that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$, i.e., the integrand is well-defined for all $x \in \mathbb{R}$. In the following we prove that

$$\boxed{\int_{-\infty}^{\infty} \Big(\frac{\sin(x)}{x}\Big)\,\mathrm{d}x = \pi. \qquad (17.42)}$$

Proof of (17.42)**.** The idea is to consider the complex function

$$f\colon \mathbb{C} \to \mathbb{C}, z \mapsto f(z) = \frac{\mathrm{e}^{iz}}{z},$$

which is holomorphic in the punctuated complex plane $\mathbb{C} \setminus \{0\}$.
Pick some $R > 0$, and choose $\varepsilon > 0$ with $\varepsilon < R$. Consider the closed curve (path) γ in figure 67. It is a union of the curves (paths) $\gamma_1, \gamma_2, \gamma_3$ and γ_4.

Figure 67: path γ

γ_1 is the half-circle of radius $R > 0$ from $(R, 0)$ to $(-R, 0)$, γ_2 is the straight line from $(-R, 0)$ to $(-\varepsilon, 0)$ and γ_4 is the straight line from $(\varepsilon, 0)$ to $(R, 0)$. To avoid the singularity at the origin the path γ_3 is the small half-circle of radius ε from $(-\varepsilon, 0)$ to $(\varepsilon, 0)$. This path is negatively oriented (clockwise). The function $f(z) = \frac{\mathrm{e}^{iz}}{z}$ is inside γ and on γ^* holomorphic. Hence, Cauchy's theorem gives

$$\int_\gamma f = \int_{\cup_{i=1}^4 \gamma_i} f = 0. \tag{17.43}$$

The contributions from the segments on the real axis are evaluated in the limit $\varepsilon \to 0$.

$$\begin{aligned}
\int_{\gamma_2} f + \int_{\gamma_4} f &= \int_{-R}^{-\varepsilon} \frac{\mathrm{e}^{ix}}{x}\,\mathrm{d}x + \int_{\varepsilon}^{R} \frac{\mathrm{e}^{ix}}{x}\,\mathrm{d}x \\
&= \int_{-R}^{-\varepsilon} \frac{\cos(x) + i\sin(x)}{x}\,\mathrm{d}x + \int_{\varepsilon}^{R} \frac{\cos(x) + i\sin(x)}{x}\,\mathrm{d}x \\
&= \int_{-R}^{-\varepsilon} \frac{i\sin(x)}{x}\,\mathrm{d}x + \int_{\varepsilon}^{R} \frac{i\sin(x)}{x}\,\mathrm{d}x \\
&\longrightarrow i\int_{-R}^{R} \frac{\sin(x)}{x}\,\mathrm{d}x \text{ as } \varepsilon \to 0.
\end{aligned}$$

From (17.43) we will conclude

$$\lim_{\varepsilon\to 0}\left(\int_{\gamma_2} f + \int_{\gamma_4} f\right) = i\int_{-R}^{R} \frac{\sin(x)}{x}\,\mathrm{d}x = -\lim_{\varepsilon\to 0}\Big(\int_{\gamma_3} f + \int_{\gamma_1} f\Big). \tag{17.44}$$

To prove (17.42) we divide (17.44) by i and take the limit $R\to\infty$. For that we compute the integrals on the right hand side of (17.44).

(1) Adding and subtracting an integrand we get for the first integral on the right hand side of (17.44)

$$\begin{aligned}\int_{\gamma_3} f = \int_{\gamma_3} \frac{1}{w}\,\mathrm{d}w + \int_{\gamma_3} \frac{\mathrm{e}^{iw}-1}{w}\,\mathrm{d}w &= -\frac{1}{2}\int_{\partial B(0,\varepsilon)} \frac{1}{w}\,\mathrm{d}w + \int_{\gamma_3} \frac{\mathrm{e}^{iw}-1}{w}\,\mathrm{d}w \\ &= -i\pi + \int_{\gamma_3} \frac{\mathrm{e}^{iw}-1}{w}\,\mathrm{d}w =: -i\pi + II,\end{aligned}$$

where the minus sign is due to the negative orientation of γ_3 and the $\frac{1}{2}$ is there because we only have the half-circle. It remains to show that the second term II on the right hand side of the last equation vanishes as $\varepsilon\to 0$.

$$|II| \le \langle\mathbf{length}(\gamma_3)\rangle \max_{w\in\partial B(0,\varepsilon)}\left|\frac{\mathrm{e}^{iw}-1}{w}\right| \le M\pi\varepsilon \to 0 \text{ as } \varepsilon\to 0,$$

because $\frac{\mathrm{e}^{iw}-1}{w}$ is continuous on $\partial B(0,\varepsilon)$ (attains its maximum value because the circline is closed and bounded). Hence,

$$\lim_{\varepsilon\to 0}\int_{\gamma_3} f = -i\pi. \tag{17.45}$$

Thus we need to show in (2) that the second integral on the right hand side of (17.44) vanishes in the limit $R\to\infty$. The other term does not depend on R.

(2) Let $\gamma_1\colon [0,\pi] \to \mathbb{C}, t \mapsto \gamma_1(t) = R\mathrm{e}^{it}$ be the parametrisation for the half-circle. We estimate

$$\begin{aligned}\Big|\int_{\gamma_1} \frac{\mathrm{e}^{iz}}{z}\,\mathrm{d}z\Big| &= \Big|\int_0^{\pi} \frac{\mathrm{e}^{iR(\cos(t)+i\sin(t))}}{R\mathrm{e}^{it}} iR\mathrm{e}^{it}\,\mathrm{d}t\Big| \le \int_0^{\pi} \left|\mathrm{e}^{iR(\cos(t)+i\sin(t))}\right|\mathrm{d}t \\ &= \int_0^{\pi} \mathrm{e}^{-R\sin(t)}\,\mathrm{d}t = 2\int_0^{\pi/2} \mathrm{e}^{-R\sin(t)}\,\mathrm{d}t \le 2\int_0^{\pi/2} \mathrm{e}^{-\frac{2tR}{\pi}}\,\mathrm{d}t \\ &= 2\Big[-\frac{\pi}{2R}\mathrm{e}^{-\frac{2Rt}{\pi}}\Big]_{t=0}^{t=\pi/2} = \frac{\pi}{R}\Big(1-\mathrm{e}^{-R}\Big) \le \pi/R \to 0 \text{ as } R\to\infty.\end{aligned}$$

The inequality follows from the estimation $\sin(t) \geq \frac{t}{\pi/2} = \frac{2t}{\pi}$ for all $t \in [0, \pi/2]$, see figure 68. A proof can be found in Lemma 17.1.

Figure 68: estimation $\boldsymbol{\sin(t)}$ on $\boldsymbol{[0, \pi/2]}$

This implies $\mathrm{e}^{-R\sin(t)} \leq \mathrm{e}^{-R\frac{2t}{\pi}}$ for all $t \in [0, \pi/2]$. We conclude from (17.44) and (17.45)

$$\lim_{R\to\infty} i \int_{-R}^{R} \frac{\sin(t)}{x}\,\mathrm{d}x = -\lim_{R\to\infty}\lim_{\varepsilon\to 0}\Big(\int_{\gamma_3} f + \int_{\gamma_1} f\Big) = i\pi$$

to get (17.42). $\square$

Lemma 17.1 (Jordan's inequality)

$$\frac{2}{\pi} \leq \frac{\sin(x)}{x} \leq 1 \qquad \textit{for } 0 < x \leq \frac{\pi}{2}.$$

Proof. It suffices to show that the function $\frac{\sin(x)}{x}$ decreases for $x \in (0, \frac{\pi}{2}]$, i.e., to show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(\frac{\sin(x)}{x}\Big) = \frac{x\cos(x) - \sin(x)}{x^2} \leq 0, \quad x \in (0, \frac{\pi}{2}].$$

This follows easily from $[x\cos(x) - \sin(x)]|_{x=0} = 0$ and from

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(x\cos(x) - \sin(x)\big) = -x\sin(x) \leq 0.$$

$\square$

18 Power series for Holomorphic functions

This section completes our study of power series and its proofs are based on Cauchy's theorem and Cauchy's formulae.

18.1 Power series representation

Theorem 18.1 *Let the power series $\sum_{n=0}^{\infty} c_n z^n$ have radius of convergence $R > 0$ and define on the open ball $B(0, R)$ the function $f\colon B(0, R) \to \mathbb{C}$ by*

$$f(z) := \sum_{n=0}^{\infty} c_n z^n, \quad z \in B(0, R).$$

Then the following statements are true:

(a) $\sum_{n=1}^{\infty} nc_n z^{n-1}$ has radius of convergence R.

(b) f is holomorphic on $B(0, R)$, and $f'(z) = \sum_{n=1}^{\infty} nc_n z^{n-1}$ for all $z \in B(0, R)$.

(c) f has derivatives of all orders $n \in \mathbb{N}$ in the open ball $B(0, R)$ and

$$f^{(n)}(0) = n! c_n, n \in \mathbb{N}_0.$$

Proof. We only sketch the ideas, parts of the proof follow analogously to the real case. Assume that $\sum_{n=0}^{\infty} c_n z^n$ converges for $|z| < R$. Pick $0 < \rho < R$ and let $|z| < \rho$ and assume $z \neq 0$. Now $\frac{|z|}{\rho} < 1$ implies that $\sum_{n=0}^{\infty} n\big(\frac{|z|}{\rho}\big)^n$ converges (Ratio test for example). Hence there is a $M > 0$ such that

$$n\Big(\frac{|z|}{\rho}\Big)^n \leq M \quad \text{for all } n \in \mathbb{N}_0.$$

Thus

$$|nc_n z^{n-1}| \leq \frac{M}{|z|} |c_n \rho^n| \quad \text{for all } n \in \mathbb{N}_0,$$

gives (a) because the series $\sum_{n=0}^{\infty} |c_n \rho^n|$ converges. To conclude with (b) we have to show that

$$(*) := \frac{f(z+h) - f(z)}{h} - g(z) = \sum_{n=1}^{\infty} \Big(\frac{(z+h)^n - z^n}{h} - nz^{n-1} \Big) \to 0 \text{ as } h \to 0,$$

where $g(z) := \sum_{n=1}^{\infty} nc_n z^{n-1}$. We shall use the binomial expansion

$$(z+h)^n = \sum_{k=0}^{n} \binom{n}{k} z^{n-k} k^k, \quad n \in \mathbb{N}, z, h \in \mathbb{C}.$$

For $z, z+h \in B(0,R)$ we must prove that

$$\frac{(z+h)^n - z^n}{h} - nz^{n-1} = h\sum_{k=2}^{n}\binom{n}{k}h^{k-2}z^{n-k}$$
$$= h\sum_{r=0}^{n-2}\frac{n!}{(n-(r+2))!(r+2)!}h^r z^{n-2-r}$$

(with $r = k-2$) vanishes in the limit $h \to 0$. With the infinite version of the triangle inequality the estimation and again with the binomial expansion the following estimate

$$|(*)| = \Big|\sum_{n=1}^{\infty} c_n\Big(h\sum_{r=0}^{n-2}\frac{n!}{(n-r-2)!(r+2)!}h^r z^{n-2-r}\Big)\Big|$$
$$\leq |h|\sum_{n=1}^{\infty} n(n-1)|c_n|\Big(\sum_{r=0}^{n-2}\frac{(n-2)!}{(n-2-r)!r!}|h|^r|z|^{n-2-r}\Big)$$
$$= |h|\sum_{n=1}^{\infty} n(n-1)|c_n|(|z|+|h|)^{n-2}.$$

Fix z and pick ρ with $|z| < \rho < R$, so that $|z| + |h| < \rho$ whenever $|h| < \rho - |z|$. Using part (a) twice, $\sum_{n=2}^{\infty} n(n-1)|c_n|\rho^{n-2}$ converges to a constant independent of h. We conclude that $f'(z)$ does exist and equals $g(z)$. The proof of (c) follows wit (16.38). $\square$

Example 18.2 We know that the geometric series $\sum_{n=0}^{\infty} z^n$ has radius of convergence $R = 1$. Theorem 18.1 implies that

$$(1-z)^{-2} = \frac{\mathrm{d}}{\mathrm{d}z}(1-z)^{-1} = \sum_{n=1}^{\infty} nz^{n-1}, \quad |z| < 1.$$

Via induction one can easily see that

$$\frac{1}{(1-z)^{k+1}} = \sum_{n\geq k}\binom{n}{k}z^{n-k}, \quad |z| < 1, k \in \mathbb{N}.$$

$\diamond$

Definition 18.3 *A function f is called **analytic** if it can be expanded into a power series around every point.*

Theorem 18.1 gives

$$\boxed{f \text{ holomorphic } \Leftrightarrow f \text{ analytic.}}$$

This does not hold in $\mathbb{R}$:

f once differentiable on $\mathbb{R} \not\Rightarrow f$ twice differentiable on $\mathbb{R}$
f infinitely many times differentiable on $\mathbb{R} \not\Rightarrow f$ has a power series expansion

This can be seen in the following example.

Example 18.4 The real-valued function

$$f\colon \mathbb{R} \to \mathbb{R}, x \mapsto f(x) = \begin{cases} 0\,; \text{if } x \leq 0 \\ \mathrm{e}^{-\frac{1}{x}}\,; \text{if } x > 0 \end{cases}$$

is infinitely many times differentiable, $f \in C^\infty$, and $\frac{\mathrm{d}^k f}{\mathrm{d}x^k}(0) = 0$ for all $k \in \mathbb{N}_0$. But the corresponding power series

$$\sum_{k=0}^{\infty} \Big(\frac{\mathrm{d}^k f}{x^k}(0)\Big)\frac{x^k}{k!} = 0$$

in contrast to $f(x) \neq 0$ for $x > 0$. $\diamond$

18.2 Power series representation - further results

Theorem 18.5 (Uniqueness Theorem) *Let the functions f, g be holomorphic on $B(a,R), a \in \mathbb{C}, R > 0$, and $(z_n)_{n\in\mathbb{N}}$ be a sequence in $B(a,R) \setminus \{a\}$ with $z_n \to a$ as $n \to \infty$ and $f(z_n) = g(z_n)$ for all $n \in \mathbb{N}$. Then $f = g$ on $B(a,R)$.*

Proof. The function $h = f - g$ is holomorphic on $B(a,R)$. Thus

$$h(z) = c_0 + c_1(z-a) + c_2(z-a)^2 + \cdots$$

for $z \in B(a,R)$. From continuity and the assumptions we get $h(a) = \lim_{n\to\infty} h(z_n) = 0$ which implies that the coefficient $c_0 = 0$. The limit of

$$\frac{h(z_n)}{(z_n - a)} = c_1 + c_2(z_n - a) + \cdots$$

exists (h is holomorphic) and equals $c_1 = \lim_{n\to\infty} \frac{h(z_n)}{z_n - a} = 0$. Induction gives $c_n = 0$ for all $n \in \mathbb{N}_0$, and from the power series representation of h, Theorem 18.1, we conclude $h = 0$ on $B(a,R)$, i.e., $f = g$ on $B(a,R)$. $\square$

The next examples show the power of this theorem.

Example 18.6 (a) Let $f(z) = \sin^2(z) + \cos^2(z)$ and $g(z) = 1$ defined for all $z \in \mathbb{C}$, both functions are holomorphic on $B(0, R)$ for all $R > 0$. As $f(x) = g(x)$ for all $x \in \mathbb{R}$ we conclude with Theorem 18.5 that $f(z) = g(z)$ for all $z \in \mathbb{C}$.

(b) Define $f(\frac{1}{n}) = \sin(\frac{1}{n})$ for any $n \in \mathbb{N}$, that is the function f is defined only on the countable set $\{\frac{1}{n} : n \in \mathbb{N}\}$. By Thoerem 18.5 we conclude that $f(z) = \sin(z)$ for all $z \in \mathbb{C}$. $\diamond$

We have already used earlier the possibility to interchange summation and integration. The next theorem provide a proof for this.

Theorem 18.7 (Interchange theorem) *Let γ be a parametrisation for a curve (path) in $\mathbb{C}$ and let the functions $U, u_k, k \in \mathbb{N}_0$, be continuous on γ^*. Assume that $\sum_{k=0}^{\infty} u_k(z)$ converges to $U(z)$ for all $z \in \gamma^*$ and that there are constants $M_k, k \in \mathbb{N}_0$, such that $|u_k(z)| \leq M_k$ for all $z \in \gamma^*$ and $k \in \mathbb{N}_0$ and that $\sum_{k=0}^{\infty} M_k$ converges. Then*

$$\sum_{k=0}^{\infty} \int_\gamma u_k(z)\,\mathrm{d}z = \int_\gamma \sum_{k=0}^{\infty} u_k(z)\,\mathrm{d}z = \int_\gamma U(z)\,\mathrm{d}z.$$

Proof. The partial sum $U_N(z) := \sum_{k=0}^{N} u_k(z)$ and U are continuous on γ^*. Finite sums can be interchanged with integration

$$\begin{aligned}
\Big|\int_\gamma U(z)\,\mathrm{d}z - \sum_{k=0}^{N} \int_\gamma u_k(z)\,\mathrm{d}z\Big| &= \Big|\int_\gamma \Big(U(z) - U_N(z)\Big)\,\mathrm{d}z\Big| \\
&\leq \sup_{z\in\gamma^*}\Big\{\Big|U(z) - U_N(z)\Big|\Big\}\langle\mathbf{length}(\gamma)\rangle \\
&\leq \sup_{z\in\gamma^*}\Big\{\sum_{k=N+1}^{\infty} |u_k(z)|\Big\}\langle\mathbf{length}(\gamma)\rangle \\
&\leq \Big(\sum_{k=N+1}^{\infty} M_k\Big)\langle\mathbf{length}(\gamma)\rangle \to 0 \text{ as } N \to \infty,
\end{aligned}$$

because $\sum_{k=0}^{\infty} M_k$ converges. $\square$

The next theorem is the Fundamental Theorem of Algebra which is derived from Liouville's theorem 16.5.

Theorem 18.8 (Fundamental Theorem of Algebra) *Every non-constant polynomial p on $\mathbb{C}$ has a root, i.e., $p(z) = 0$ for some $z \in \mathbb{C}$.*

Proof. Let $p(z) = c_n z^n + c_{n-1} z^{n-1} + \cdots + c_1 z + c_0$ with $c_1, \ldots, c_n \in \mathbb{C}$ and $c_n \neq 0$. Assume that p has no root. Then $f\colon \mathbb{C} \to \mathbb{C}, z \mapsto f(z) = \frac{1}{p(z)}$ is holomorphic on $\mathbb{C}$. As $|p(z)| \to \infty$ as $|z| \to \infty$, there exists $R > 0$ such that $|f(z)| = |\frac{1}{p(z)}| < 1$ for $|z| > R$. Hence, f is bounded and therefore constant due to Liouville's theorem 16.5. This gives the required contradiction. □

Remark 18.9 *Let p a polynomial of order n. If $p(a) = 0$ for $a \in \mathbb{C}$ then $p(z) = (z - a)q(z)$. Iterating this (induction) gives: there are $z_1, \ldots, z_n \in \mathbb{C}$ and a constant $c \in \mathbb{C}$ such that*

$$p(z) = c(z - z_1)(z - z_2) \cdots (z - z_n), \quad z \in \mathbb{C}.$$

Every n-th order polynominal has n roots.

19 Laurent series and Cauchy's Residue Formula

In the first part the we will extend Cauchy's theorem to a larger class of curves (paths). Then we study power expansions of functions around singularities. These power expansions include also negative powers, the so-called Laurent series.

19.1 Index

Definition 19.1 (Index) *Let γ be a parametrisation of a closed path, and consider the complement $\Omega := \mathbb{C} \setminus \gamma^*$ (recall that $\gamma^* = \gamma([a, b])$). The **index** of the curve (path) is the function $\mathbf{Ind}_\gamma\colon \Omega \to \mathbb{C}$ defined by*

$$\mathbf{Ind}_\gamma(z) = \frac{1}{2\pi i} \int_\gamma \frac{\mathrm{d}w}{w - z}, \quad z \in \Omega. \tag{19.46}$$

Without proof we cite the following theorem about the values of the index function.

Theorem 19.2 *The index function $\mathbf{Ind}_\gamma$ is an integer-valued function on Ω and is constant in each component of Ω and is zero in the unbounded component of Ω, see an example in figure 69.*

Figure 69: components of Ω of a path γ

Example 19.3 (Some index functions) The examples in figure 70 show an example with index $\mathbf{Ind}_\gamma(z_0) = 1, \mathbf{Ind}_\gamma(z_0) = -1$ and $\mathbf{Ind}_\gamma(z_0) = 2$ respectively.

Figure 70: three index functions

◇

Example 19.4 Consider the parametrisation $\gamma\colon [0, 4\pi] \to \mathbb{C}, \gamma(t) = (1+i)+ e^{it}$ of a circline around $(1+i)$ with radius 1, see figure 71 around $z_0 = 1+i$. The index is

$$\mathbf{Ind}_\gamma((1+i)) = \frac{1}{2\pi i}\int_\gamma \frac{\mathrm{d}w}{w-1-i} = \frac{1}{2\pi i}\int_0^{4\pi} e^{-it} i e^{it}\,\mathrm{d}t = 2.$$

Hence, we do know that

$$\int_\gamma \left(\frac{2i}{z-1-i}\right)\mathrm{d}z = -8\pi.$$

Is there a connection with the index? This is indeed the case, and we shall show it later. ◇

Figure 71: example with $\mathbf{Ind}_{\gamma}((\mathbf{1}+\boldsymbol{i}))=\mathbf{2}$

We want to enlarge the class of curves (paths) for Cauchy's theorem. We will do this step by step. The following is the first version which extends to closed curves (paths) without specifying the orientation.

Cauchy Theorem (preliminary version)
Let $f\colon D\to\mathbb{C}, D\subset\mathbb{C}$ a region, be holomorphic and let γ be a parametrisation of a closed curve (path) in D. If all points of the complement $\mathbb{C}\setminus D$ are lying in the outside $\mathbf{O}(\gamma)$ of the curve (path) Cauchy's theorem holds, i.e.,

$$\int_{\gamma} f(z)\,\mathrm{d}z=0.$$

We shall explore the meaning of the condition that the complement of the domain of definition lies outside of the curve (path). We do this with a couple of examples. In figure 72 we have two circle lines around the origin, $\partial B(0,R)$ and $\partial B(0,r), r<R$, one circle line is lying inside the other one. Consider the path γ as the union of γ_1, the outer circle line in positive direction, the straight connection of the outer circle with the inner circle γ_2, the inner circle line γ_3 with negative orientation and the straight line back (on the same line as before but in different direction) γ_4.

Figure 72: paths $\gamma_1, \gamma_2, \gamma_3, \gamma_4$

Let f be a holomorphic function on the annulus of the two circle lines in figure 72. Then from Theorem 15.7 we get that

$$\int_{\gamma_1} f(z)\,\mathrm{d}z = \int_{\gamma_3} f(z)\,\mathrm{d}z,$$

but inside of the inner circle $\partial B(0, r)$ we have points which do not belong to the domain of definition and do not belong to the outside of the whole path γ. A way out of this is the following. Pick $\varepsilon > 0$ and open both circle lines by a small piece of length ε to get the key-hole path in figure 73.

Figure 73: key-hole path

The curves (paths) in figure 74 and 75 are encircling both two singularity points. We color the two components around the two singularities by red (the upper one) and by blue (the lower one). Let γ be the closed paths enclosing (surrounding) the two points respectively- one (figure 74) puts the points on the outside and one (figure 75) encircles the upper point for example in one movement, see the figure 74 and 75 respectively. Is there a difference in the methods of encircling? In which components is the index $\mathbf{Ind}_\gamma \neq 0$? If we consider the curve (path) as a fence keeping some animals we would see no difference at all. But there is a difference.

Figure 74: enclosing two singularities - method (a)

In figure 74 both the red and blue component have zero index but in figure 75 the blue component has a non-zero index. This shows that for the curve (path) γ in figure 74 the two singularities are lying outside γ.

Figure 75: enclosing two singularities - method (b)

This discussion shall motivate the following final version of Cauchy's theorem.

Theorem 19.5 (Cauchy's theorem - final version) *Let $D \subset \mathbb{C}$ be a region and the function $f\colon D \to \mathbb{C}, D \subset \mathbb{C}$, be holomorphic and γ be a closed curve (path) in D which does not surround a point in the complement $\mathbb{C} \setminus D$, that is, $\mathbf{Ind}_\gamma(z) = 0$ for all $z \in \mathbb{C} \setminus D$. Then*

$$\boxed{\int_\gamma f = 0.}$$

Theorem 19.6 (Cauchy's formulae - index version) *Let $D \subset \mathbb{C}$ be a region and the function $f\colon D \to \mathbb{C}, D \subset \mathbb{C}$, be holomorphic and γ be a closed curve (path) in D which does not surround a point in the complement $\mathbb{C} \setminus D$, that is, $\mathbf{Ind}_\gamma(z) = 0$ for all $z \in \mathbb{C} \setminus D$. Then*

$$\boxed{\mathbf{Ind}_\gamma(z) f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(w)}{(w-z)^{n+1}}\,\mathrm{d}w, \qquad \textit{for all } z \in D \setminus \gamma^*.}$$

19.2 Laurent series

We study now power expansions with negative powers. We start with the following Binomial expansions derived from the geometric series.

Binomial expansions
For $|z| < 1$ the geometric series gives the expansion of $(1-z)^{-1}$ with positive powers

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n.$$

What about $|z| > 1$? Because of

$$|z| > 1 \Leftrightarrow \left|\frac{1}{z}\right| < 1$$

an expansion of $(1 - \frac{1}{z})^{-1}$ is possible.

$$\frac{1}{1-z} = -\frac{1}{z}\frac{1}{\left(1-\frac{1}{z}\right)} = -\sum_{n=0}^{\infty} z^{-n-1} = -\sum_{m=-\infty}^{-1} z^m \qquad \text{for } |z| > 1.$$

In the same way we can expand $(a-z)^{-1}$ as a series in positive powers of z if $|z| < |a|$ and as a series in negative powers if $|z| > |a|$.

Definition 19.7 *A series $\sum_{n=-\infty}^{\infty} a_n, a_n \in \mathbb{C}$, converges (to $s = s_1 + s_2$) if $\sum_{n=0}^{\infty} a_n$ converges (to s_1) and $\sum_{n=1}^{\infty} a_{-n}$ converges (to s_2). A power series with negative and positive powers is called a* ***Laurent series****.*

Suppose the function f is holomorphic in the punctuated ball $B(a,R) \setminus \{a\}$, and at the point a something nasty happens (singularity). If the function is not holomorphic in a we cannot hope for a power series expansion. But what about to expand the function f as a Laurent series $\sum_{n=-\infty}^{\infty} c_n(z-a)^n$ for $0 < |z-a| < R$?

Theorem 19.8 (Laurent's theorem) *Let an annulus A,*

$$A = \{z \in \mathbb{C} \colon R < |z-a| < S; 0 < R < S \leq \infty, a \in \mathbb{C}\},$$

be given and let the function f be holomorphic on A. Then

$$\boxed{f(z) = \sum_{n=-\infty}^{\infty} c_n(z-a)^n \quad \textit{for } z \in A, \qquad (19.47)}$$

where the coefficients are given by

$$c_n = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w-a)^{n+1}} \, \mathrm{d}w, \quad n \in \mathbb{Z}, \tag{19.48}$$

with $\gamma^* = \partial B(a, r)$ *for any* r *with* $R < r < S$.

Proof. We sketch only the main ideas. W.l.o.g. we set $a = 0$.

Figure 76: $\widetilde{\gamma}$ and $\widehat{\gamma}$ in A

Pick $z \in A$ and choose $R < P < |z| < Q < S$ and consider the two closed paths $\widetilde{\gamma}$ and $\widehat{\gamma}$ in figure 76, where $\widetilde{\gamma}$ is the closed path enclosing $z \in A$ and $z \in \mathbf{O}(\widehat{\gamma})$. From (16.37) we get

$$f(z) = \frac{1}{2\pi i} \int_{\widetilde{\gamma}} \frac{f(w)}{w-z} \, \mathrm{d}w,$$

and from Cauchy's theorem 19.5 we have

$$0 = \frac{1}{2\pi i} \int_{\widehat{\gamma}} \frac{f(w)}{w-z} \, \mathrm{d}w.$$

The integrals along the line segments connecting the two circlines $\partial B(0, Q)$ and $\partial B(0, P)$ in figure 75 cancel out each other and the remaining gives

$$f(z) = \frac{1}{2\pi i} \int_{\partial B(0,Q)} \frac{f(w)}{w-z} \, \mathrm{d}w - \frac{1}{2\pi i} \int_{\partial B(0,P)} \frac{f(w)}{w-z} \, \mathrm{d}w. \tag{19.49}$$

We can expand the integrands $(w-z)^{-1}$ in (19.49) in positive powers for the first integral and in negative powers for the second integral in (19.49) respectively (see Binomial expansions).

$$\frac{1}{w-z} = \frac{1}{w}\frac{1}{\left(1-\frac{z}{w}\right)} = \frac{1}{w}\sum_{n=0}^{\infty}\left(\frac{z}{w}\right)^n \qquad \text{for all } w \in \partial B(0,Q)$$

because $|\frac{z}{w}| < 1$ for $w \in \partial B(0,Q)$.

$$\frac{1}{w-z} = -\frac{1}{z}\frac{1}{\left(1-\frac{w}{z}\right)} = -\frac{1}{z}\sum_{m=0}^{\infty}\left(\frac{w}{z}\right)^m \qquad \text{for all } w \in \partial B(0,P)$$

because $|\frac{w}{z}| < 1$ for $w \in \partial B(0,P)$. Inserting theses expansions into the integrands in (19.49) and using Theorem 18.7 we get

$$f(z) = \sum_{n=0}^{\infty}\Big(\frac{1}{2\pi i}\int_{\partial B(0,Q)}\frac{f(w)}{w^{n+1}}\Big)z^n + \sum_{m=0}^{\infty}\Big(\frac{1}{2\pi i}\int_{\partial B(0,P)} f(w)w^m\,\mathrm{d}w\Big)z^{-m-1}.$$

Putting $n=-m-1$ and an application of the Deformation Theorem 15.7 gives the proof. $\square$

It is often very difficult to calculate the coefficients of a Laurent expansion. An easier way is to find Binomial expansions directly as the following example shows.

Example 19.9 (Laurent expansions of a function) The function

$$f(z) = \frac{1}{z(1-z)^2}$$

is holomorphic in $A_1 = \{z \in \mathbb{C} \colon 0 < |z-1| < 1\}$ and $A_2 = \{z \in \mathbb{C} \colon |z-1| > 1\}$. Hence we have two Laurent expansions. As

$$f(z) = \frac{1}{(z-1)^2}\frac{1}{\Big(1+(z-1)\Big)},$$

we conclude that the Laurent expansion for $0 < |z-1| < 1$ is

$$f(z) = \sum_{n=-2}^{\infty}(-1)^n(z-1)^n.$$

For the annulus A_2 we write

$$f(z) = \frac{1}{(z-1)^3} \frac{1}{\left(1 + \frac{1}{z-1}\right)},$$

and hence the Laurent expansion for $|z-1| > 1$ is

$$f(z) = \sum_{n=-\infty}^{-3} (-1)^{n+1} (z-1)^n.$$

$\diamond$

The notion singularity becomes clear in the following.

Notation 19.10 *Let $f\colon D \to \mathbb{C}, D \subset \mathbb{C}$, be a function.*

*(a) A point $a \in D$ is a **regular** point of the function f if f is holomorphic at a.*

*(b) A point $a \in D$ is a **singularity** of f if a is not a regular point but a limit point of regular points, e.g., a is singularity of*

$$\frac{1}{z-a}$$

because a is limit of the regular points $z_n = a + \frac{1}{n}$.

*(c) A singularity of f is said to be **isolated** if f is holomorphic in the open punctuated ball $B(a,R) \setminus \{a\}$ for some $R > 0$.*

Theorem 19.8 indicates that the isolated singularities are the candidates for Laurent expansions. We need the following classification of isolated singularities.

Classification of isolated singularities Suppose that the function f has an isolated singularity at a, hence f is holomorphic in some annulus

$$A = \{z \in \mathbb{C}\colon 0 < |z-a| < r, r > 0\}.$$

Theorem 19.8 gives

$$f(z) = \sum_{n=-\infty}^{-1} c_n (z-a)^n + \sum_{n=0}^{\infty} c_n (z-a)^n, \quad z \in A. \tag{19.50}$$

The first term on the right hand side of (19.50) is called the **principal part** of the Laurent expansion. The isolated singularity a is said to be

- a **removable** singularity if $c_n = 0$ for all $n < 0$.
- a **pole** of order m ($m \geq 1$) if $c_{-m} \neq 0$ and $c_n = 0$ for all $n < -m$.
- an **essential singularity** if there does not exist m such that $c_n = 0$ for $n < -m$.

The example $1/(z-1)^2$ has at $z = 1$ a pole of order 2, also called double pole.

Definition 19.11 (Residue) *Let $f\colon D \to \mathbb{C}, D \subset \mathbb{C}$, be a function which has a pole at $a \in D$. The* ***residue*** *of the function f at a is the unique coefficient c_{-1} of the term $(z-a)^{-1}$ of the Laurent expansion of f about a and it is denoted by $\mathbf{res}\{f; a\}$.*

We are now able to answer our questions about when integrals of functions along closed paths vanish or not. If we surround poles of the integrand the integral does not vanish, because there is a reminder which is given by the residue of that pole. This is the content of the last theorem.

Theorem 19.12 (Cauchy's residue theorem) *Let the function f be holomorphic inside and on a positively oriented contour γ except at a finite number of poles $a_1, \ldots, a_N \in \mathbf{I}(\gamma)$. Then*

$$\boxed{\int_\gamma f(z)\,\mathrm{d}z = 2\pi i \sum_{k=1}^{N} \mathbf{res}\{f; a_k\}.} \tag{19.51}$$

Proof. The N poles give exactly N principal parts

$$f_k(z) = \sum_{n=-\infty}^{-1} c_n^{(k)} (z - a_k)^n \quad k = 1, \ldots, N.$$

For any $k = 1, \ldots, N$ there is $m_k \in \mathbb{N}$ with $c_{-m_k}^{(k)} \neq 0$ and $c_n^{(k)} = 0$ for all $n < -m_k$. Integration of each principal part gives (see fundamental integral)

$$\int_\gamma f_k(z)\,\mathrm{d}z = \sum_{n=1}^{m_k} \frac{c_{-n}^{(k)}}{(z-a_k)^n} = 2\pi i c_{-1}^{(k)} = 2\pi i\,\mathbf{res}\{f; a_k\}.$$

The function $g\colon \mathbf{I}(\gamma) \to \mathbb{C}$, given by $g(z) = f(z) - \sum_{k=1}^{N} f_k(z)$, is holomorphic and Theorem 19.5, i.e.,

$$0 = \int_\gamma g(z)\,\mathrm{d}z = \int_\gamma f(z)\,\mathrm{d}z - \sum_{k=1}^{N} \int_\gamma f_k(z)\,\mathrm{d}z,$$

and we conclude with the proof. $\square$

In order to take advantage of the last theorem we provide some simple rules for calculating the residue of a function.

Lemma 19.13 *Let $B(a, R)$ be the open ball around $a \in \mathbb{C}$ with radius $R > 0$, and assume that the function $f\colon B(a, R) \to \mathbb{C}$ is holomorphic on $B(a, R) \setminus \{a\}$.*

(a) If a is simple pole (i.e. a pole of order 1), then

$$\mathbf{res}\{f; a\} = \lim_{z\to a} (z - a) f(z).$$

Furthermore, if the function $g\colon B(a, R) \to \mathbb{C}$ is holomorphic and $g(a) \neq 0$ such that $f(z) = \frac{g(z)}{z-a}$ for all $z \in B(a, R) \setminus \{a\}$, then

$$\mathbf{res}\{f; a\} = g(a).$$

(b) Let $h, k\colon B(a, R) \to \mathbb{C}$ be holomorphic on the ball $B(a, R)$ such that $f(z) = \frac{h(z)}{k(z)}$ or all $z \in B(a, R) \setminus \{a\}$. Furthermore, assume that $h(a) \neq 0, k(a) = 0$, and $k'(a) \neq 0$, then

$$\mathbf{res}\{f; a\} = \frac{h(a)}{k'(a)}.$$

(c) Let f have a pole at a of order $m > 1$ and let $f(z) = \frac{g(z)}{(z-a)^m}, z \in B(a, R) \setminus \{a\}$, with g holomorphic on $B(a, R)$ and $g(a) \neq 0$. Then

$$\mathbf{res}\{f; a\} = \frac{1}{(m-1)!} g^{(m-1)}(a).$$

Proof. (a) This follows immediately from the the Laurent series expansion of the function f.
(b)

$$\lim_{z\to a} (z - a) \frac{h(z)}{k(z)} = h(a) \lim_{z\to a} \frac{z - a}{k(z) - k(a)} = \frac{h(a)}{k'(a)}.$$

(c) Cauchy's formulae give

$$g^{(m-1)}(a) = \frac{(m-1)!}{2\pi i} \int_{\partial B(a,R/2)} \frac{g(z)}{(z-a)^m} \, \mathrm{d}z = \frac{(m-1)!}{2\pi i} \int_{\partial B(a,R/2)} f(z) \, \mathrm{d}z$$
$$= (m-1)! \, \mathbf{res}\{f; a\}.$$

$\square$