Stellar Astrophysics Chapter 3: Heat Transfer

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While we have crudely obtained the Fick's law in Chapter 1, we will here derive it more precisely, based on radiative transfer theory, which will also be used to construct a stellar atmosphere model. We discuss various energy transfer mechanisms in stars: radiative, convective, and conductive.

Outline

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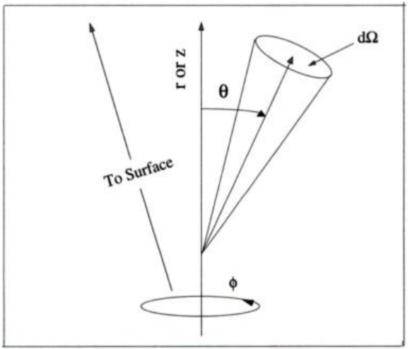
The Diffusion Equation

Central to the discussion of radiative transfer is the specific intensity, $I(\theta)$, which is the energy flux per steradian, in the direction of θ.



Noticing that $dz = \mu ds$ ($\mu = \cos(\theta)$).

The net change in $I(\theta)$ along a path ds with the mass emissivity (j) and opacity (κ) is

$$dI(\theta) = [j - \kappa I(\theta)]\rho ds$$

Defining the optical depth for an outward-directed ray as

$$\tau(z) \equiv -\int_0^z \kappa\rho dz$$

in which $z = 0$ is selected to represent the "true surface" of the the star where density is approximately zero, we have

$$\mu \frac{dI(\tau,\mu)}{d\tau} - I(\tau,\mu) = -S(\tau,\mu),$$

where the *source function* $S(\tau,\mu) = j/\kappa$.

Multiplying both sides of the above equation with $e^{-\tau/\mu}/\mu$ and then integrating lead to

$$\frac{d}{d\tau}\left[e^{-\tau/\mu} I\right] = -e^{-\tau/\mu}\frac{S}{\mu},$$

$$I(\tau,\mu) = e^{-(\tau_0-\tau)/\mu} I(\tau_0,\mu) + \int_{\tau}^{\tau_0} e^{-(t-\tau)/\mu}\frac{S(t,\mu)}{\mu}dt.$$

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For an outward direction ($\mu \geq 0$), choosing $\tau_0 \to \infty$ (i.e., the reference level lies deep within the star) gives

$$I(\tau,\mu \geq 0) = \int_{\tau}^{\infty} e^{-(t-\tau)/\mu}\frac{S(t,\mu)}{\mu}dt, \qquad (1)$$

and for an inward direction, adopting $I(0,\mu < 0) = 0$ at the true surface gives

$$I(\tau,\mu < 0) = \int_{\tau}^{0} e^{-(t-\tau)/\mu}\frac{S(t,\mu)}{\mu}dt.$$

Of course, the above quantities are all energy dependent (i.e., we have left out the subscript, ν).

If the deep interior is nearly in LTE, we expect S to be almost isotropic. Hence its Planck function B can be used, and it is reasonable to expand S in a Taylor series in τ, to first order (*Eddington approximation*):

$$S(t) = B(\tau) + (t - \tau)\left(\frac{\partial B}{\partial \tau}\right)_{\tau}.$$

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Inserting this into the above equation for $I(\tau, \mu \geq 0)$, we have

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where the azimuthal symmetry around the z-direction is assumed. Note that if I is a constant, then F is zero. Therefore, a net flux, or a radiative transfer, requires that $I(\theta)$ must be anisotropic.

Using the above $I(\tau)$ expression and including the subscript ν, we get

$$F_\nu(\tau_\nu) = \frac{4\pi}{3}\frac{\partial B_\nu}{\partial \tau_\nu} = -\frac{4\pi}{3}\frac{1}{\rho\kappa_\nu}\frac{dT}{dr}\frac{\partial B_\nu}{\partial T}$$

The total flux is

$$F(r) = -\frac{4\pi}{3}\frac{1}{\rho\kappa}\frac{dT}{dr}\int_0^\infty \frac{\partial B_\nu}{\partial T} d\nu$$

where the *Rosseland mean opacity*, κ, is defined by

$$\frac{1}{\kappa} \equiv \left[\int_0^\infty \frac{1}{\kappa_\nu}\frac{\partial B_\nu}{\partial T} d\nu\right]\left[\int_0^\infty \frac{\partial B_\nu}{\partial T} d\nu\right]^{-1}$$

Because

$$\int_0^\infty \frac{\partial B_\nu}{\partial T} d\nu = \frac{\partial}{\partial T}\int_0^\infty B_\nu(T) d\nu = \frac{c}{4\pi}\frac{dU}{dT},$$

we then have

$$F(r) = -\frac{c}{3\rho\kappa}\frac{dU}{dr} = -K\frac{dT}{dr}, \quad \text{where } K \equiv \frac{4caT^3}{3\rho\kappa}$$

This version of F is in the Fick's law form introduced in Chapter 1.

Radiative Opacity Sources

Various processes that are responsible for the dependence of the opacity on temperature and density:

- Electron scattering:

$$\kappa = \frac{\sigma n_e}{\rho}.$$

  In the non-relativistic case ($T \ll 6 \times 10^9$ K), the Thompson cross section can be used. Because $n_e \approx \rho/(\mu_e m_A)$, where $\mu_e = 2/(1+X)$,

$$\kappa_e = 0.2(1+X)\ \mathrm{cm}^2\ \mathrm{g}^{-1}$$

- Free-free absorption (or the inverse of normal bremsstrahlung):

$$\begin{aligned}\kappa_{ff} &= j_{brem}/B(T) \\ &= 40(X+Y)(1+X)\rho T_6^{-3.5}\ \mathrm{cm}^2\ \mathrm{g}^{-1}\end{aligned}$$

  where hydrogen is assumed to be completely ionized.

- Bound-free absorption:

$$\kappa_{bf} = 9 \times 10^{10}(Z/0.02)(1+X)\rho(T/5000K)^{-3.5}\ \mathrm{cm}^2\ \mathrm{g}^{-1}$$

- Bound-bound opacity is more complicated and is typically much less than κ_{ff} and κ_{bf}.

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  The formation of H^- requires the presence of both the neutral hydrogens (i.e., not too high temperature) and electrons (mostly from partially ionized metals).

$$\kappa_{H^-} = 5 \times 10^2 (Z/0.02) \rho^{1/2} (T/5000K)^9 \text{ cm}^2 \text{ g}^{-1},$$

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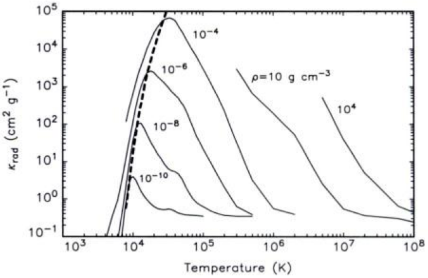
The reduced exponent of the density is presumably due to the inverse-density dependence of the H^{-1} and free electron populations, which is also very sensitive to the temperature (high positive exponent).
At even lower temperatures, opacity due to the presence of molecules or small grains becomes important.

Opacity Summary

This figure is very informative. Please explain the shapes of the curves!



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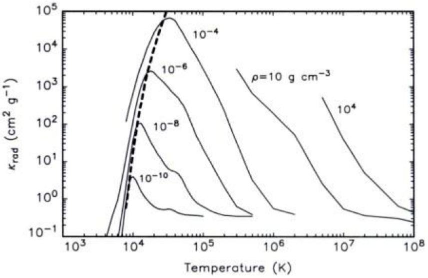

HKT Fig. 4.2: Plots of the LANL radiative opacity for the mixture $X = 0.7$, $Y = 0.28$, and $Z = 0.02$. The dashed line shows the half-ionization curve for pure hydrogen.

Notice the steep half-ionization curve for pure hydrogen, which characterizes the termination of the the H^- opacity region and starts the Kramer's law and then the electron scattering at higher temperatures.

Radiative Opacity and Emissivity

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- Photons can also be scattered into the beam from other direction. If the scattering is isotropic, then $dI/ds = +\rho\kappa^{sca} J$, where J is the mean intensity: $J \equiv \frac{1}{4\pi} \int I d\Omega$.

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- Photons can also be scattered into the beam from other direction. If the scattering is isotropic, then $dI/ds = +\rho\kappa^{sca}J$, where J is the mean intensity: $J \equiv \frac{1}{4\pi}\int I d\Omega$.
- There is an intimate relationship between the specific opacity κ_ν^{abs} and emission j_ν (ν is added back again), which is related to the radiative equilibrium.
  - A *condition of a steady-state* LTE is that the gas does not gain or lose energy to the radiation, which requires the balance the emission, $\int j_\nu d\nu d\Omega$, and absorption, $\int \kappa_\nu^{abs} I_\nu d\nu d\Omega$, or $\int_0^\infty \left(j_\nu - \kappa_\nu^{abs} J_\nu\right) d\nu = 0$, where we have assumed j_ν is isotropic. Under the LTE, we assume that Kirchhoff's law, $j_\nu = \kappa_\nu^{abs} B_\nu(T)$, can be applied.
    If κ_ν^{abs} is independent of ν, then
    $\int_0^\infty J_\nu d\nu = \int_0^\infty B_\nu(T) d\nu = B(T)$ and thus
    $\int_0^\infty j_\nu d\nu = \kappa^{abs} B(T)$, as will be used later.
  - If, in addition, $J_\nu(T) = B_\nu(T)$ (a sufficient, but not necessary, condition for the LTE), we then have a complete thermodynamic equilibrium.

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A model for stellar atmospheres should achieve the following:

- It provides a stellar surface boundary condition for the stellar structure modeling, which is much more realistic than just setting everything to zero at the surface.
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Here we will consider the (so-called grey) radiation transfer model for a simple atmosphere, which characterizes the temperature and density as a function of the optical depth.

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Basic assumptions used to construct our atmosphere model:

- The LTE for the gas is still a reasonably good approximation;
- Plane parallel slab;
- A constant gravity throughout the atmosphere;
- Energy is carried out by radiation, and there are no sources of energy within the atmosphere;
- grey atmosphere: the structure is affected by the continuum opacity only, assumed to be wavelength-independent.

A grey transfer model

By integrating over all frequencies the radiative transfer equation

$$\mu \frac{dI_\nu}{d\tau_\nu} = I_\nu - S_\nu, \text{where } S_\nu \equiv \frac{1}{\kappa_\nu} \left(j_\nu + \kappa_\nu^{sca} J_\nu \right)$$

is the source function, and assuming that the atmosphere is "grey" (i.e., κ is independent of ν), we get

$$\mu \frac{dI}{d\tau} = I - S, \text{where } S = (1 - A)B + AJ, \tag{2}$$

in which $A \equiv \kappa^{sca}/\kappa$ is the *albedo* and $\int j_\nu d\nu = \kappa^{abs} B$ (assuming the steady state condition).

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Further averaging over all angles gives

$$\frac{1}{4\pi} \frac{dF}{d\tau} = J - S = (1 - A)(J - B) = 0 \tag{3}$$

which holds because there is no energy sink or source in the atmosphere (hence $F =$ constant) and even if $A \neq 1$.
Note that $J = B$ *does not* necessarily imply that $I_\nu = B_\nu$!

Now we can integrate the Fick's law, $F = \frac{c}{3}\frac{dU}{d\tau}$, to obtain

$$U = \frac{3}{c}F(\tau + \tau_0), \tag{4}$$

where the integration constant τ_0 is to be determined and the drop of U with decreasing τ is due to the net loss of inward moving photons due to absorption.

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$$I(\tau = 0, \mu \geq 0) = \int_0^\infty \frac{3}{4\pi\mu}F(t + \tau_0)e^{-t/\mu}dt = \frac{3}{4\pi}F(\mu + \tau_0). \tag{5}$$

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$$F \equiv 2\pi \int_0^1 I(\tau = 0, \mu \geq 0)\mu d\mu, \tag{6}$$

where we have assumed that the star is not irradiated by another source and hence all of the flux must be outward-directed ($\mu > 0$) at $\tau = 0$ (i.e., $I(\mu < 0, \tau = 0) = 0$).

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Thus the photosphere, where $T(\tau_p) = T_{eff}$, lies at $\tau_p = 2/3$. To find the pressure, one needs to solve the hydrostatic equation

$$\frac{dP}{d\tau} = \frac{g_s}{\kappa}$$

where $g_s = GM/R^2$ is a constant at the surface. Thus

$$P(\tau) = g_s \int_0^\tau \frac{d\tau}{\kappa} + P(\tau = 0)$$

Consider a simple case where κ is constant (as a version of the "grey" atmosphere). The pressure at the photosphere is

$$P(\tau_p) = \frac{2g_s}{3\kappa} + P(\tau = 0).$$

At the true surface $P(\tau = 0)$ should be dominated by the radiation pressure,

$$P_{rad}(\tau = 0) = 2\pi \int_0^1 \frac{I(\mu, \tau = 0)}{c} \mu^2 d\mu = \frac{17}{24} \frac{F}{c},$$

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$$P(\tau_p) = \frac{2g_s}{3\kappa} \left[1 + \frac{17}{16} \frac{L}{L_{edd}} \right].$$

Here $L_{edd} = \frac{4\pi cGM}{\kappa}$ is the Eddington limit, above which the radiative force exceeds the gravitational force, i.e.,

$$\frac{L\kappa}{4\pi R^2 c} > \frac{GM}{R^2}.$$

The opacity in such a case (e.g., the photospheres of very massive stars) is usually due to electron scattering. With a hydrogen mass fraction of $X = 0.7$ and hence $\kappa_e = 0.34 \text{ cm}^2 \text{ g}^{-1}$, the limit is

$$\left(\frac{L_{edd}}{L_\odot} \right) \approx 3.5 \times 10^4 \left(\frac{M}{M_\odot} \right).$$

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$P(\tau_p)$ may be balanced by $\frac{1}{3}aT_{eff}^4 + \frac{\rho N_A}{\mu}kT_{eff}$.

We can then find the density (implicitly) at the photosphere (assuming $\kappa = \kappa_0 \rho^n T_{eff}^{-s}$).

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Realistic atmosphere models ($P(\tau)$ or $\rho(\tau)$ and $T(\tau)$) are constructed for a grid of g_s and T_{eff} values, which give the boundary conditions required at the photosphere.

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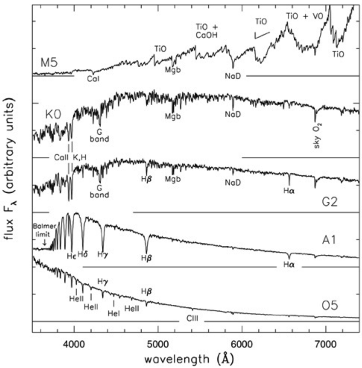
To get the spectral distribution, we need to go back to Eq. 1 and set $\tau = 0$ and $S_\nu = B_\nu(T)$. We then have

$$I_\nu(\mu) = \frac{1}{\mu}\int_0^\infty e^{-t/\mu} B_\nu[T(t)]dt. \qquad (7)$$

Sample Stellar Spectra

Stars are classified according to their luminosities and effective temperatures. Here are some key features:



Sample spectra of main-sequence stars (from Sparke/Gallagher 2007).

- The O- and B-stars have strong lines due to HeI λ4471 and HeII λ4541 (Å).
- The early (e.g., hotter) A-stars show maximum strength in hydrogen Balmer lines. F- and G-stars are week in hydrogen lines, but exhibits metal lines.
- The “cliff” near 4,000Å is due to the H and K lines of Ca-II.
- The depression known as the “G-band” is primarily due to Fe.
- The spectra of the cooler K- and M-stars are dominated by metallic lines and molecular bands.

Line Profiles

The properties of a line are intimately related to its profile. For a line transition at a stellar atmosphere, the natural broadening is small compared to other broadening factors: e.g., due to pressure (perturbation due to the encounter with a particle) and the Doppler effect (due to stellar rotation and thermal/turbulent motion).

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The line profile is typically a convolution of the Lorentz profile (mostly due to the pressure broadening) and the Doppler effect.

Line Profiles

The properties of a line are intimately related to its profile. For a line transition at a stellar atmosphere, the natural broadening is small compared to other broadening factors: e.g., due to pressure (perturbation due to the encounter with a particle) and the Doppler effect (due to stellar rotation and thermal/turbulent motion).
The line profile is typically a convolution of the Lorentz profile (mostly due to the pressure broadening) and the Doppler effect.
The cross-section for a line transition $i \to j$ is

$$\sigma_\nu = \frac{e^2}{m_e c} f_{ij} \phi_\nu$$

where f_{ij} is the *oscillator strength* contains the quantum mechanical details, and ϕ_ν is the *Lorentz profile*

$$\phi_\nu = \frac{\gamma/4\pi}{(\triangle\nu)^2 + (\gamma/4\pi)^2},$$

in which γ is the sum of the damping constants for the two levels involved and $\triangle\nu = \nu - \nu_0$. Integrating this profile gives the total cross-section

$$\sigma_{tot} = \frac{\pi e^2}{m_e c} f_{ij}$$

The convolution with an assumed Gaussian velocity distribution results in

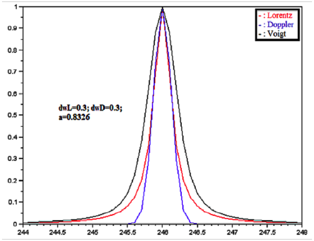
$$\sigma_\nu = \sigma_{tot} \frac{1}{\sqrt{\pi} \triangle\nu_D} H(a, u),$$

and $H(a, u)$ is the well-studied 'Voigt function'

$$H\left(a = \frac{\gamma}{4\pi\triangle\nu_D}, u = \frac{\triangle\nu}{\triangle\nu_D}\right) = \frac{a}{\pi}\int_{-\infty}^{\infty} \frac{e^{-y^2} dy}{a^2 + (u-y)^2},$$

where $\triangle\nu_D = \nu_0 \left(\frac{2kT}{m_e c^2}\right)^{1/2}$. The Voigt function has the following basic properties:

- $H(a \ll 1, u = 0) \approx 1, \int_{-\infty}^{\infty} H(a, u) du = \sqrt{\pi}$
- for small u (line core), $H(a, u) \approx e^{-u^2} \frac{a}{\pi} \int_{-\infty}^{\infty} dy \frac{1}{a^2+(u-y)^2} = e^{-u^2}$
- for $u \gg 1$ (line wind) and a fairly small a, $H(a, u) \approx \frac{a}{\pi} \int_{-\infty}^{\infty} dy \frac{e^{-y^2}}{u^2} = \frac{a}{\sqrt{\pi} u^2}$ close to a Lorentz profile.



To calculate the line absorption, we need to combine the line opacity with the continuum opacity and solve the radiation transfer equation. For simplicity, we are going to assume pure absorption in both the continuum and the line.
The opacity is given by

$$\kappa_\nu = \kappa_\nu^C + \kappa_\nu^L,$$

where κ_ν^C is the continuum opacity and $\kappa_\nu^L = \frac{\kappa_{\nu,0}^L}{\sqrt{\pi}\triangle\nu_D} H(a,\mu)$ is the line opacity, with

$$\kappa_{\nu,0}^L = \frac{n_i}{\rho}\sigma_{tot}(1 - e^{-h\nu_0/kT})$$

being the line opacity at the line center ν_0. The term $e^{-h\nu_0/kT}$ accounts for the stimulated transition from $j \to i$.
We can usually ignore the variation with ν in κ_ν^C over the width of the line. Let us also assume that $\beta_\nu \equiv \kappa_\nu^L/\kappa_\nu^C$ is independent of τ_ν. We can then write $d\tau_\nu = (1 + \beta_\nu)d\tau$, where $d\tau = -\rho\kappa_\nu^C dz$.

Finally, assuming that the temperature does not vary much in the line forming region, we can expand B_ν to first order in τ only,

$$B[T(\tau_\nu)] \approx B_0 + B_1\tau,$$

where B_0 and $B_1 = \left(\frac{\partial B_\nu}{\partial \tau}\right)_0$ are constant.

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Inserting these approximations into Eq. 7, multiplying by μ and integrating over outward bound rays gives the flux as,

$$\begin{aligned} F_\nu =& 2\pi \int_0^1 \int_0^\infty [B_0 + B_1\tau] \exp\left[-\frac{\tau}{\mu}(1+\beta_\nu)\right](1+\beta_\nu) d\tau d\mu \\ =& \pi \left[B_0 + \frac{2B_1}{3(1+\beta_\nu)}\right]. \end{aligned}$$

Far from the line center, $\beta_\nu \to 0$, we get the continuum flux as

$$F_\nu^C = \pi \left[B_0 + \frac{2B_1}{3}\right].$$

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$$F_\nu^C = \pi \left[B_0 + \frac{2B_1}{3}\right].$$

Hence the depth of the line is

$$A_\nu \equiv 1 - \frac{F_\nu}{F_\nu^C} = A_0 \frac{\beta_\nu}{1+\beta_\nu}, \text{where } A_0 \equiv \frac{2B_1}{3B_0 + 2B_1} \quad (8)$$

is the depth of an infinitely opaque ($\beta_\nu \to \infty$) line.

Curve of Growth

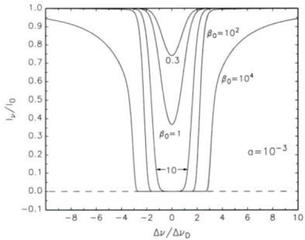
Observing spectral lines with high resolution is not always available. What is often done is to measure the *equivalent width* (EW), defined as

$$W_\nu = \int_0^\infty \left(1 - \frac{I_\nu}{I_0}\right) d\nu. \qquad (9)$$

In a simple case of radiation from a background source incident on a foreground slab of an optical depth of τ_ν, the EW of a spectral line is then

$$W_\nu = \int_0^\infty (1 - e^{-\tau_\nu}) d\nu. \qquad (10)$$

While τ is proportional to the column density of the species that is responsible for the line transition, the plot of W_ν-column density is called a *curve of growth*.



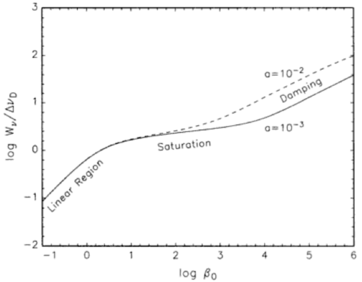
The evolution of a spectral absorption line with increasing opacity of the absorbers for a sample values of β_0 (see HKT).

Now placing Eq. 8 in Eq. 9, we have

$$W_\nu = \int_0^\infty A_0 \frac{\beta_\nu}{1+\beta_\nu} d\nu = 2A_0 \triangle\nu_D \int_0^\infty \frac{\beta_0 H(a,u)}{1+\beta_0 H(a,u)} du,$$

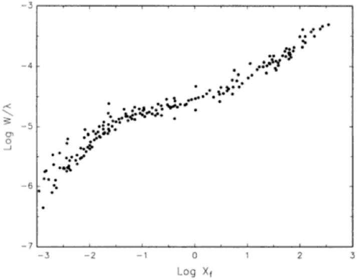
where $\beta_0 = \kappa^L_{\nu,0}/(\kappa^C_\nu \sqrt{\pi}\triangle\nu_D)$ and the factor of 2 comes from the change of the variable from ν to u, since $H(a,u)$ is symmetrical about the line center and the integration now is just over $\triangle\nu > 0$.

- At small (linear) optical depth ($\beta_0 \ll 1$), $W_\nu \approx A_0\sqrt{\pi}\triangle\nu_D\beta_0[1 - \frac{\beta_0}{\sqrt{2}}...]$, where the first term of W_ν is independent of the Doppler broadening, since $\beta_0 \propto \triangle\nu_D^{-1}$.
- In the intermediate (saturation) region, $W_\nu \approx 2A_0\triangle\nu_D\sqrt{\ln\beta_0}$.
- At the (damping) extreme ($\beta_0 \gg 1$), $W_\nu \approx \sqrt{\pi a\beta_0}A_0\triangle\nu_D$, again independent of the Doppler broadening because $a\beta_0 \propto \triangle\nu_D^{-2}$.



HKT Fig. 4.11: The curve of growth corresponding to the line profiles with two different values of a $(=\gamma/(4\pi\triangle\nu_D))$.

- By fitting the curve of growth for a set of lines, one may determine the abundances of individual species and the Doppler broadening.
- The data presented in the right figure was used to determine the temperature (from the Doppler broadening), as well as the metal abundances of the sun.



HKT Fig. 4.12: The composite curve of growth for some 200 lines of ion (Fe I) and titanium (Ti I) in the sun. The quantity X_f is equivalent to our β_0 while $W_\lambda/\lambda(= W_\nu/\nu)$ is the equivalent width divided by the central wavelength of the line.

Outline

Heat Transfer by Conduction

Energy transport by conduction can be important in some cases: mostly in white dwarfs or cores of some red supergiants, where elections are in a degenerate state. Are ions in a degenerate state as well? Why important in such a state?

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The scattering works in the same fashion as radiative transport.
Rewrite Fick's law of particle diffusion (Chapter 1) into

$$F_{cond} = -\frac{v\lambda}{3}\frac{dU}{dr} = -D_e\frac{dT}{dr},$$

in which

$$D_e = \frac{c_V v_e \lambda}{3} = \frac{c_V v_e}{3\sigma_C n_I} \propto \frac{\rho\mu_I T}{\mu_e^2 Z_C^2}.$$

where $c_V = \left(\frac{\partial U}{\partial T}\right)_V \propto (\rho/\mu_e)^{1/3}T$ according to Eq. (3.115) in the HKT text, the ion number density $n_I \sim \rho/\mu_I$, the Coulomb scattering cross section $\sigma_C \propto s^2$, $Z_c e^2/s \sim m_e v_e^2/2$, and $v_e \propto p_F \propto (\rho/\mu_e)^{1/3}$, assuming non-relativity electrons.

To account for the conduction, we can formally define the "conductive opacity" as

$$\kappa_{cond} \equiv \frac{4acT^3}{3D_e\rho} = (4 \times 10^{-8}\ \mathrm{cm}^2\ \mathrm{g}^{-1})\frac{\mu_e^2}{\mu_I}Z_c^2\left(\frac{T}{\rho}\right)^2.$$

We then have

$$F_{cond} = -\frac{4acT^3}{3\kappa_{cond}\rho}\frac{dT}{dr}.$$

The combined heat flux is

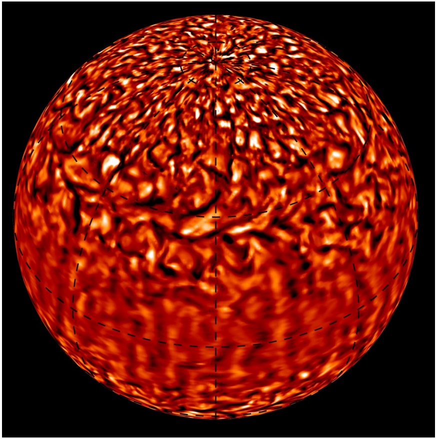
$$F_{tot} = F_{rad} + F_{cond} = -\frac{4acT^3}{3\kappa_{total}\rho}\frac{dT}{dr}$$

where

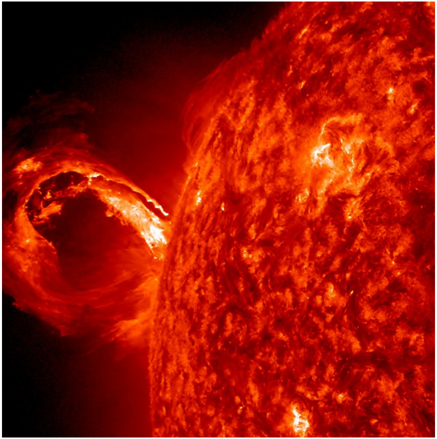
$$\frac{1}{\kappa_{total}} = \frac{1}{\kappa_{rad}} + \frac{1}{\kappa_{cond}}.$$

Whichever opacity is the smaller of κ_{rad} or κ_{cond}, it is also more important in determining the total opacity and hence the heat flow.
For a typical cool white dwarf with $\rho = 10^6\ \mathrm{g\ cm}^{-3}$, $T \sim 10^7$ K, and a composition of carbon ($\mu_e = 2$, $\mu_I = 12$, and $Z_c = 6$),
$\kappa_{cond} = 5 \times 10^{-5}\ \mathrm{cm}^2\ \mathrm{g}^{-1}$, compared to $\kappa_e = 0.2\ \mathrm{cm}^2\ \mathrm{g}^{-1}$, the opacity due to electron scattering.
In normal stellar material, κ_{cond} is too large to be important.

Outline

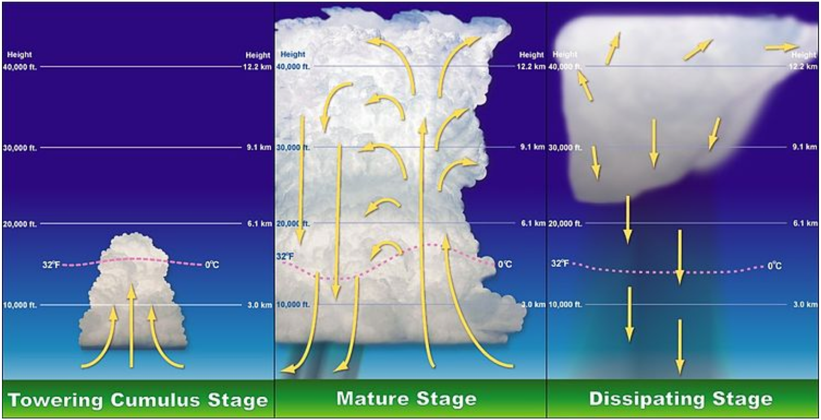


Simulations of global Sun convection. The differential rotation stretches and amplifies the magnetic field, which emerges on the surface, forms sunspots, and produces the eruptions of hot solar plasma known as coronal mass ejections. Gustavo Guerrero



Coronal mass ejection observed by NASA's Solar Dynamics Observatory (SDO), in extreme ultraviolet radiation emitted by ionized helium atoms. The eruption is caused by a magnetic field generated by a dynamo process beneath the surface. NASA/SDO

An Analogy in the Earth's Atmosphere



Stages of cloud development.

Introduction

- When the temperature gradient in a region of a star becomes too large (e.g., due to large opacity), the gas may become convectively unstable.
- But we still don't have an accurate theory to describe stellar convection, which is a *global* and highly *non-linear* phenomenon.

Introduction

- When the temperature gradient in a region of a star becomes too large (e.g., due to large opacity), the gas may become convectively unstable.
- But we still don't have an accurate theory to describe stellar convection, which is a *global* and highly *non-linear* phenomenon.
- The *mixing length theory*, a local phenomenological model, is often used to describe the convection:
  - The heat transportation is via the rise and fall of "eddies" or "parcels or "bubbles".
  - They lose their identify or break up and merge with the surrounding fluid on some characteristic distance, l, the *mixing length*.
  - Large eddies drive progressively small ones until eventually very tiny eddies are excited, with sizes $\lambda \sim \nu / w(\lambda)$, where ν and $w(\lambda)$ are the viscosity and the velocity on scale of λ. This is related to the Navier-Stockes equation

$$(\partial_t + w\partial_x)w = -\partial_x P/\rho + \nu \partial_x^2 w$$

and the Reynolds number $Re(\lambda) \equiv \dfrac{w(l)l}{\nu}$.

Key assumptions (part of the so-called Boussinesq approximation) are made in the Mixing Length theory:

- The characteristic dimension of a parcel is the same order as l.
- l is much shorter than the stellar scale height (e.g., λ_P).
- The pressure is balanced between the parcel and its surrounding.
- The temperature difference between the parcel interior and exterior is small.

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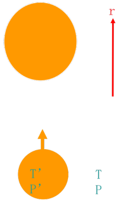
Considering a typical parcel, which has the interior temperature T', pressure P', and density ρ'. Outside the parcel, the corresponding quantities are denoted by T, P, and ρ.

Suppose that $T' > T$, then Archimedes' principle states that the parcel will rise because of a net upward buoyancy force of $(\rho - \rho')gV$ where g is the local gravity and V is the volume of the parcel.



Criteria for Convection

Assuming that there is no heat exchange with the surrounding, T' will vary as the parcel rises:

$$\beta_S \equiv -\left(\frac{dT'}{dz}\right)_S = -T'\left(\frac{d\ln T'}{d\ln P'}\right)_S \frac{d\ln P'}{dz}.$$

The variation of the temperature in the surrounding gas is

$$\beta \equiv -\frac{dT}{dz} = -T\frac{d\ln T}{d\ln P}\frac{d\ln P}{dz}$$

Replacing P' with P (pressure balance) and T' with T and defining $d\ln P/dz \equiv -\lambda_P^{-1}$, as well as $\nabla \equiv \frac{d\ln T}{d\ln P}$ and $\nabla_S \equiv \left(\frac{d\ln T'}{d\ln P'}\right)_S$, we have

$$\beta - \beta_S = \frac{T}{\lambda_P}(\nabla - \nabla_S). \tag{11}$$

When $\nabla > \nabla_S$, the fluid is convectively unstable, or the interior temperature of the parcel drops at a rate slower than that of the exterior temperature (i.e., $\beta > \beta_S$). This is the *Schwarzschild criteria*, which holds even when heat exchange and/or viscosity are considered.

Radiative Leakage

To follow what happens in the parcel as it moves, we consider

$$\frac{dT'}{dt} = \left(\frac{\partial T'}{\partial t}\right)_P + w \cdot \nabla T',$$

where w is the parcel velocity.

- The r.h.s. first term takes care of the instantaneous heat change, $\rho c_P \left(\frac{\partial T'}{\partial t}\right)_P = -\nabla \cdot \mathbf{F}_{\mathbf{rad}} = K\nabla^2 T$. $K = \frac{4acT^3}{3\kappa\rho}$. Approximating $\nabla^2 T = -\Delta T/l^2$, where $\Delta T = T' - T$, we have $\frac{\partial T'}{\partial t} = -\frac{\nu_T}{l^2}\Delta T$, where $\nu_T = K/\rho c_P$.
- The (adiabatic) advective term, $w \cdot \nabla T' = w\frac{\partial T'}{\partial z} = -w\beta_S$, describes how T' behaves without the radiative loss.

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Putting all the above together, we have

$$\frac{d\Delta T}{dt} = -\frac{\nu_T}{l^2}\Delta T + (\beta - \beta_S)w. \tag{12}$$

Equation of Motion

While the above equation describe the time-dependent temperature contrast between the parcel and its immediate surroundings as the parcel moves, the motion caused by the buoyancy force is approximately

$$\frac{dw}{dt} = \frac{(\rho - \rho')}{\rho} g$$

where we have ignored any viscous effects.

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For a small relative density contrast, $\frac{(\rho-\rho')}{\rho} = -Q\frac{(T-T')}{T} = \frac{Q}{T}\Delta T$, where $Q \equiv -\left(\frac{d\ln\rho}{d\ln T}\right)_P$ is the coefficient of thermal expansion.

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$$\frac{dw}{dt} = \frac{Qg}{T}\Delta T. \quad (13)$$

Assuming that all the coefficients are constant, Eqs. 12 and 13 can be solved for ΔT and w as functions of time.

Convective Efficiencies and Fluxes

The solutions have the form $\propto e^{\sigma t}$, where σ is a complex angular frequency and must satisfy the characteristic equation

$$\sigma^2 + \sigma \frac{\nu_T}{l^2} + N^2 = 0, \qquad (14)$$

where the *Brunt-Väisälä frequency*, N, is defined by

$$N^2 = -\frac{Qg}{\lambda_P}(\nabla - \nabla_S), \qquad (15)$$

where the last step used Eq. 11. The solutions of Eq. 14 are

$$\sigma_\pm = \frac{1}{2}\left[-\frac{\nu_T}{l^2} \pm \sqrt{\left(\frac{\nu_T}{l^2}\right)^2 - 4N^2}\right] \qquad (16)$$

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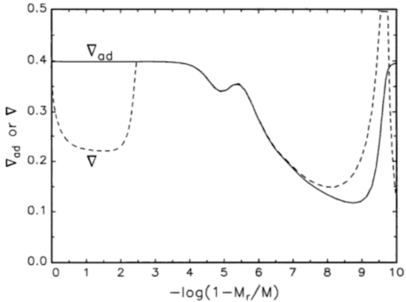
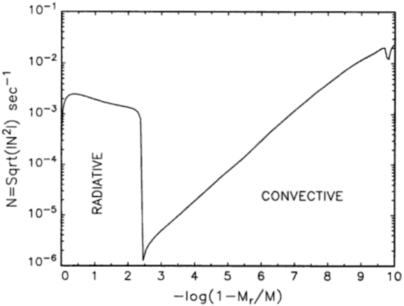
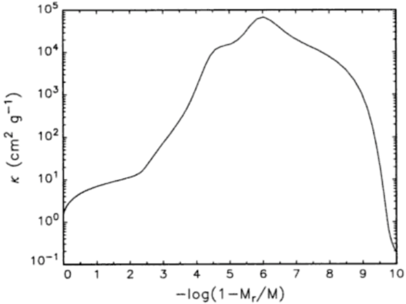
- If $N^2 > 0$, with or without an imaginary part of σ, the parcel's motion will be damped out due to radiative losses (no convection).
- If $N^2 < 0$ [equivalent to $(\beta - \beta_S) > 0$], σ is real with one positive root, both ΔT and w may increase exponentially (convectively unstable).
- If $\nu_T/l^2 \ll |N|$, then $\sigma = |N|$. This is the case of **efficient convection** because the parcel loses essentially no heat during its travel until it breaks up.
- Conversely, if $\sigma = -N^2 l^2/\nu_T \ll 1$, ΔT and w increase, but slowly.





Parameters for a model ZAMS sun.



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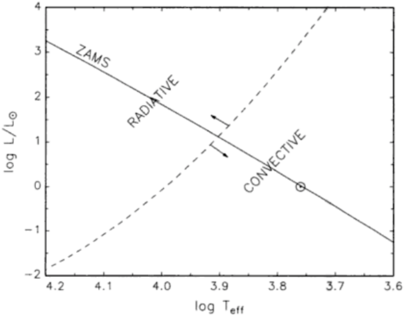

The dashed line in this schematic HR diagram divides those stars with active and efficient outer convection zones from those that have feeble and inefficient convection.

To estimate the convection flux, we first estimate ΔT from Eq. 13, assuming $w \sim \sigma l$

$$\frac{dw}{dt} \sim \sigma^2 l = \frac{Qg}{T}\Delta T \rightarrow \Delta T \sim \frac{\sigma^2 l}{Qg} T.$$

The convection flux is then

$$F_{conv} = w\rho c_P \Delta T \sim \frac{\sigma^3 l^2 \rho c_P T}{Qg}.$$

For adiabatic convection, $\sigma = \sqrt{-N^2}$, using Eq. 15, one gets

$$F_{conv} = \frac{\rho c_P l^2 T (gQ)^{1/2} (\nabla - \nabla_S)^{3/2}}{\lambda_P^{3/2}}. \quad (17)$$

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There is no exact way to deal with l. It may be chosen to be a fraction (α) of the λ_P (e.g., $\alpha \sim 0.5$, which is called the *mixing length parameter* and adjusted from comparison of obtained evolutionary models with observations). The treatment, though rough and not self-consistent, often does not make much a difference if the convection is efficient.

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How is this situation handled?

For simplicity, we consider here only adiabatic convection. To see whether or not the convection may play a role in the heat transfer, we first make believe that all the flux is carried by radiation and compute

$$\nabla_{rad} = \frac{3}{4ac}\frac{r^2 P\kappa}{T^4}\frac{F_{tot}}{GM_r}.$$

This ∇_{rad}, derived from the global quantity (F_{tot}) is more reliable than the local ∇ (from the assumed heat transfer in the present modeling). So we need to check the consistency.

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- If $\nabla = \nabla_{rad}$, then the flux is carried solely by the radiation and you are done.
- However, if $\nabla > \nabla_{rad}$, the convection must then play a role with

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These two equations, together with F_{conv} (Eq. 17), allow us to solve for ∇ and compare it with the present model value. We may need to iterate to make them converge. When the convection is efficient, it dominates. It is often reasonable just to assume $\nabla = \nabla_S$.

Review

Key concepts: radiation transfer, Rosseland mean opacity, grey atmosphere, curve of growth, heat conduction, efficient convection, adiabatic convection.

1. What are the various heat transfer mechanisms? What are the physical conditions for each of the mechanisms to dominate?
2. What the various physical processes that contribute to the stellar opacity (as a function of temperature in a plot)?
3. What are the power law exponents of the processes of the opacity when it is expressed in a power law? Why do they have these particular exponents?
4. What are the relative importance of the opacity processes in various parts from the stellar center to surface?
5. How may the opacity processes depend on metallicity or He abundance?
6. What is the Eddington limit?
7. Please derive the Schwarzschild criteria for the convective instability by yourself. What are the underlying assumptions?
8. What is the mixing length theory?
9. Why do main-sequence massive stars tend to have convective cores and radiative envelops, whereas lower mass stars tend to have the opposite behaviors?
10. What is the practical way in the stellar modeling to decide on whether or not the convection is important? How may it be implemented?