Stochastic Calculus - An Introduction

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Outline

Random Walk

Consider a very simplistic model of an ant (or a missile) as we observe it over equi-distant time epochs $0 < t_1 < \cdots < t_n = t$. At time zero it is at 0. At times $t_1, t_2, \cdots$, its position moves by the amounts $R_1, R_2, \cdots$, which forms a sequence of random variables

- $\mathbb{E}(R_i) = \mu\,\delta t$,
- $Var(R_i) = \sigma^2\,\delta t$
- $R_1, R_2, \cdots$ are independent.

Next define a new process by taking their partial sums, representing the trajectory (or path) of the ant/missile,

$$T_0 = 0, \quad T_1 = R_1, \quad T_2 = R_1 + R_2, \quad T_3 = R_1 + R_2 + R_3, \quad \cdots, \quad T_k = \sum_{i=1}^{k} R_i,$$

and take $T_0 = 0$ just for convenience. This new process, $\{T_k, k = 0, 1, 2, \cdots\}$, is called a random walk with drift μ and volatility σ. It has two important properties:

Random Walk

Since $T_{k+1} = T_k + R_{k+1}$, it says that the next value, T_{k+1}, is determined by the current value T_k and a totally independent future observation R_{k+1}. The past values of the process, $T_0, T_1, T_2, \cdots, T_{k-1}$ play no role in the evolution of T_{k+1}. Such a property of a process is called a Markovian property. Next consider the conditional expectation,

$$\begin{aligned}
\mathbb{E}(T_{k+1}|T_1, T_2, \cdots, T_k) &= \mathbb{E}(T_k + R_{k+1}|T_1, T_2, \cdots, T_k) \\
&= T_k + \mathbb{E}(R_{k+1}) \\
&= \begin{cases} T_k + \mu\delta t & \text{if } \mu \neq 0, \\ T_k & \text{if } \mu = 0. \end{cases}
\end{aligned}$$

When there is no drift, $\mu = 0$, the expected future value is the same as the current value regardless of the past. This is called the martingale property:

$$\boxed{\mathbb{E}(T_{k+1}|T_1, T_2, \cdots, T_k) = T_k, \qquad k = 1, 2, \cdots}$$

For instance, when $R_i = \pm 1$, this is called Gambler's ruin process. Let us work with this case for simplicity.

Random Walk

Now we are ready to take the limit as $\delta t \to 0$, or equivalently as $n \to \infty$. Since the coin is fair, $R_i \sim (2X_i - 1)\sigma\sqrt{\delta t}$, where $X_1, X_2, \cdots, X_n \overset{iid}{\sim} Binom(1, \frac{1}{2})$. A little bit of algebra reveals that

$$\begin{aligned} T_n &= \sum_{i=1}^{n} R_i \sim \sum_{i=1}^{n}(2X_i - 1)\sigma\sqrt{\delta t} \\ &= \frac{\sigma\sqrt{t}}{\sqrt{n}}\sum_{i=1}^{n}(2X_i - 1) = \sigma\sqrt{t}\left(\frac{\sum_{i=1}^{n} X_i - \frac{n}{2}}{\sqrt{n\frac{1}{2}(1-\frac{1}{2})}}\right). \end{aligned}$$

By the central limit theorem (CLT) the last expression is approximately normally distributed. That is,

$$T_n \overset{approx}{\sim} \sigma\sqrt{t}\,N(0,1) \sim N(0, \sigma^2 t),$$

Usually the limiting random variable is denoted by $W_\sigma(t)$, and we write

$$T_n \overset{dist}{\to} W_\sigma(t) \quad \text{as } n \text{ gets large.}$$

This limiting process $\{W_\sigma(t), t \geq 0\}$ is called a Brownian motion.

Brownian Motion

A process $\{W_\sigma(t), t \geq 0\}$ is called a Brownian motion process with volatility σ if

- $W_\sigma(0) = 0$.
- $W_\sigma(t) \sim N(0, \sigma^2 t)$, for all $t > 0$.
- For any $0 \leq s_1 < u_1 \leq s_2 < u_2 \leq \cdots \leq s_n < u_n$, the increments

$$W_\sigma(u_1) - W_\sigma(s_1), \quad W_\sigma(u_2) - W_\sigma(s_2), \quad \cdots, \quad W_\sigma(u_n) - W_\sigma(s_n)$$

  are mutually independent normal random variables. The first one is $N(0, \sigma^2(u_1 - s_1))$, the second one is $N(0, \sigma^2(u_2 - s_2))$ and so on.
- $W_\sigma(t)$, as a function of t, traces out continuous curves, called sample paths.

In particular, when the volatility is $\sigma = 1$, the resulting Brownian motion is called *standard*, or the Wiener process. We will denote a Wiener process by $\{W(t), t \geq 0\}$.

BM Properties

Brownian motion inherits the Markovian and martingale properties from the random walk process. Indeed, for any time epochs $0 = t_0 < t_1 < t_2 < \cdots < t_n = t$, we see that

$$\begin{aligned} & \mathbb{E}(W_\sigma(t) | W_\sigma(t_0), W_\sigma(t_1), \cdots, W_\sigma(t_{n-1})) \\ = \quad & \mathbb{E}((W_\sigma(t) - W_\sigma(t_{n-1})) | W_\sigma(t_0), \cdots, W_\sigma(t_{n-1})) + W_\sigma(t_{n-1}) \\ = \quad & W_\sigma(t_{n-1}), \end{aligned}$$

since $(W_\sigma(t) - W_\sigma(t_{n-1}))$ is independent of $W_\sigma(t_0), W_\sigma(t_1), \cdots, W_\sigma(t_{n-1})$ and its expectation is zero.
We can improve upon the last example. We can construct infinitely many martingales through the Brownian motion. You can easily check that $S(t) = e^{uW_\sigma(t) - \sigma^2 u^2 t/2}$, for any fixed number u, is also a martingale. The key reason is

$$\mathbb{E}(S(t) | W_\sigma(t_0), W_\sigma(t_1), \cdots, W_\sigma(t_{n-1})) = e^{uW_\sigma(t_{n-1}) - \sigma^2 t u^2/2} \mathbb{E}(e^{u(W_\sigma(t) - W_\sigma(t_{n-1}))})$$

In finance uses such martingales to set the prices of various financial assets and their derived contracts. Statistics uses this to derive sequential tests.

Why New Calculus

When f is a deterministic "nice and smooth" function, integration by parts can be used to give a meaning to the integral

$$\int_0^t f(s)\,dW(s) = f(s)W(s)|_0^t - \int_0^t W(s)\,f'(s)\,ds = f(t)W(t) - \int_0^t W(s)\,f'(s)\,ds.$$

For stochastic control theory or financial modeling f are neither differentiable nor deterministic, such as $f(s, W(s)) = W(s)$ itself leading to

$$\int_0^t W(s)\,dW(s).$$

If we treat it as an ordinary Riemann-Stieltjes integral, such as $\int_0^t x\,d\sqrt{x} = \frac{t^{3/2}}{3}$, or $\int_0^t \sqrt{x}\,d\sqrt{x} = \frac{t}{2}$, we run into trouble. These two Riemann-Stieltjes integrals are solved by using the facts that $\frac{d\sqrt{x}}{dx} = \frac{1}{2\sqrt{x}}$, which gives $d\sqrt{x} = \frac{1}{2\sqrt{x}}dx$. Or just pretend $\sqrt{x}$ is a dummy variable to get

$$\int_0^t \sqrt{x}\,d\sqrt{x} = \left.\frac{(\sqrt{x})^2}{2}\right|_0^t = \frac{t}{2}, \quad \int_0^t x d\sqrt{x} = \int_0^t (\sqrt{x})^2 d\sqrt{x} = \left.\frac{(\sqrt{x})^3}{3}\right|_0^t = \frac{t^{3/2}}{3}.$$

This dummy variable idea does not work on our stochastic integral since

$$\text{(pos/neg)} \quad \int_0^t W(s)\,dW(s) = \left.\frac{(W(s))^2}{2}\right|_0^t \neq \frac{W(t)^2}{2}. \quad \text{(pos)}$$

Stoch. Integral

The Ito stochastic integral

$$\int_0^t h(s, X(s))\, dW(s), \qquad \text{abbreviated as} \qquad \int_0^t h(s)\, dW(s),$$

where $h(s, x)$ is some function, which we can approximate by step functions, and $X(t)$ is determined by the path $\{W(s), 0 \le s \le t\}$. This integral is defined as a limit of Riemann-Ito sum

$$\int_0^t h(s)\, dW(s) = \lim_{\delta t \to 0} \sum_{j=0}^{n-1} h(a_j, X(a_j)) \; (W(t_{j+1}) - W(t_j)) \,. \tag{1}$$

Here $0 = t_0 < t_1 < t_2 < \cdots < t_n = t$ and $\delta t = \max_j (t_{j+1} - t_j)$. In Ito calculus three fundamental departures from the ordinary calculus take place:

- Left end evaluation of h
- $h(t)$ should be known at time t,
- Limit is in the sense of convergence in probability.

An Example

It is not difficult to show that

$$\int_0^t W(s)\, dW(s) = \frac{W(t)^2 - t}{2}. \tag{2}$$

Actually it is equivalent to derive the following quadratic variation

$$\lim_{\delta t \to 0} \sum_{j=0}^{n-1} \left(W(t_{j+1}) - W(t_j)\right)^2 = t. \tag{3}$$

The idea is to use the definition of the stochastic integral

$$S_n(W, W, t) = \sum_{j=0}^{n-1} W(t_j)\left(W(t_{j+1}) - W(t_j)\right) = \sum_{j=0}^{n-1} \left(W(t_j)W(t_{j+1}) - W(t_j)^2\right)$$

and add and subtract $\frac{W(t_{j+1})^2}{2}$ and rewrite this as follows.

$$\begin{aligned} &S_n(W, W, t) \\ &= \sum_{j=0}^{n-1} \left(\underbrace{W(t_j)W(t_{j+1}) - \frac{W(t_{j+1})^2}{2} - \frac{W(t_j)^2}{2}} + \underbrace{\frac{W(t_{j+1})^2}{2} - \frac{W(t_j)^2}{2}} \right) \\ &= \frac{-1}{2} \sum_j \left(W(t_{j+1}) - W(t_j)\right)^2 + \frac{W(t)^2 - W(0)^2}{2}, \quad \text{(telescoping effect)}. \end{aligned}$$

Theorem: Ito Integral

When $\int_0^t \mathbb{E}(h(s)^2)\, ds < \infty$ the Ito integral $\int_0^t hdW$ exists and has the following properties.

- (i) (Martingale) $X(t) := \int_0^t h(s)dW(s)$, $t \geq 0$, is again a stochastic process and forms a martingale, i.e., $\mathbb{E}(X(t) \mid W(u), u \leq s) = X(s)$ for any $0 \leq s < t$. Furthermore, $X(t)$ is continuous with mean and variance

$$\mathbb{E}\left(\int_0^t h(s)\, dW(s)\right) = 0, \qquad Var\left(\int_0^t h(s)\, dW(s)\right) = \int_0^t \mathbb{E}(h(s)^2)\, ds.$$

- (ii) (Linearity) For $h(s)$, $g(s)$ allowing the corresponding Ito integrals to exist, as in (i), and any constants a, b, we have

$$\int_0^t (ah(s) + bg(s))\, dW(s) = a\int_0^t h(s)\, dW(s) + b\int_0^t g(s)\, dW(s).$$

Theorem: Ito Integral

- (iii) (Ito processes & their quadratic variation) When both g, h are non-anticipating, $\int_0^t \mathbb{E}|g(s)|\, ds < \infty$ and $\int_0^t \mathbb{E}(h(s)^2)\, ds < \infty$ then

$$Y(t) - Y(0) \;=\; \int_0^t g(s)ds + \int_0^t h(s)dW(s), \qquad t \geq 0,$$

  is called an *Ito process*, sometimes written in differential form as $dY \;=\; g\,dt \;+\; h\,dW$. Here $Y(0)$ is the initial random variable. Then the quadratic variation of an Ito process, Y, namely $[Y]_t$, written in differential form, is

$$\boxed{d[Y]_t = h(t)^2 d[W]_t = h(t)^2 dt, \qquad [Y]_0 = 0.}$$

Examples

The simplest example of an Ito process is the Brownian motion itself, $B_{\mu,\sigma}(t)$, with drift μ and volatility $\sigma > 0$. In differential form, $dB_{\mu,\sigma} = \mu\, dt + \sigma dW$. In integral form,

$$B_{\mu,\sigma}(t) \;=\; B_{\mu,\sigma}(t) - B_{\mu,\sigma}(0) \;=\; \int_0^t \mu dt + \int_0^t \sigma\, dW \;=\; \mu t + \sigma W(t).$$

Its quadratic variation is $d[B_{\mu,\sigma}]_t = \sigma^2\, dt$.
As another example, recall that $\frac{W(t)^2 - t}{2} = \int_0^t W\, dW$. In other words,

$$Y(t) \;:=\; W(t)^2 \;=\; t + 2\int_0^t W\, dW, \qquad \text{equivalently} \qquad dY = dt + 2W\, dW.$$

Clearly $Y(t)$ is an Ito process, since $dY = d(W^2) = dt + 2WdW$. However, it is not a martingale. Its quadratic variation is $d[Y]_t = 4W^2(t)\, dt$.
This example raises an interesting question: If we take a smooth function, say f, and create a process, $Y(t) \;:=\; Y(0) + f(W(t))$, would this new process be an Ito process? If yes, what should be its g, h, when we write it as $dY = gdt + hdW$? The answer turns out to be in the afirmative, smooth functions of Ito processes are again Ito processes. This is known as Ito's formula.

Ito's Formula

Ito's formula tells us how an Ito process, $dY = g\,dt + h\,dW$, produces a new Ito process for $S = f(t, Y(t))$ when a smooth functions $f(t, x)$ is used as a transform. For instance, we already know that the Brownian motion, with drift μ could be written as $dY = \mu dt + \sigma dW$. If we transform Y by a function, say $S(t) = f(t, Y(t))$ for some smooth function f, will $S(t)$ be an Ito process? The answer is yes, and is a famous result of Ito.

Theorem: Ito's Formula

Let $f(t, x)$ be a real valued function with continuous partial derivative in t and continuous second order partial derivative in x, and let $dY = g\,dt + h\,dW$ be an Ito process. Then $S(t) = f(t, Y(t))$ is again an Ito process. Moreover,

$$dS = f_t(t, Y)\,dt + f_x(t, Y)\,dY + \frac{f_{xx}(t, Y)}{2}\,d[Y]_t = \left(f_t + gf_x + \frac{h^2}{2}f_{xx}\right)dt + f_x h\,dW.$$

[The middle form uses Y and its quadratic variation $[Y]$ as its "drivers". The last form uses the Brownian motion W as its driver to reveal that it is an Ito process.]

Example: Ito's Formula

Computing stochastic integrals. Recall our old friend

$$\int_0^t W(s)\,dW(s) \;=\; \frac{W(t)^2 - t}{2}.$$

Ito's formula gives another proof. Take $Y(t) = W(t)$, in other words, $dY = 0dt + dW$, making $g(t) = 0$ and $h(t) = 1$ in Ito's setup. Now take $f(t, x) = x^2$ for which $f_t = 0$, $f_x = 2x$ and $f_{xx} = 2$. For $S(t) = f(t, Y(t)) = W^2(t)$, Ito's formula gives

$$dS \;=\; \left(f_t + gf_x + \frac{h^2}{2} f_{xx}\right) dt + f_x h\, dW \;=\; \left(\frac{2}{2}\right) dt + 2W\,dW.$$

Restating this in integral form, we have shown that

$$W^2(t) - W^2(0) \;=\; S(t) - S(0) \;\;=\;\; t + 2\int_0^t W(s)dW(s).$$

Rearranging the terms gives the result $\int_0^t W(s)\,dW(s) = \frac{W(t)^2 - t}{2}$. The reader can try $f(t, x) = x^n$ for higher n to get other varieties of Ito integrals,

Example: Ito's Formula

Martingales. When $S(t) = e^{\theta W(t) - (\theta^2 t/2)}$, for a constant θ, find dS.
In this case we may just take $f(t,x) = e^{\theta x - (\theta^2 t/2)}$ and $Y(t) = W(t)$, giving $g = 0$, $h = 1$ and $f_t = -\frac{\theta^2}{2} f$, $f_x = \theta f$ and $f_{xx} = \theta^2 f$. Then Ito's formula gives

$$dS = \left(\frac{-\theta^2 f}{2} + \frac{\theta^2 f}{2} \right) dt + \theta f dW = \theta f dW.$$

So, $dS = \theta S dW = \theta e^{\theta W(t) - (\theta^2 t/2)} dW$, or $S(t) - 1 = \theta \int_0^t e^{\theta W(s) - (\theta^2 s/2)} dW(s)$, in constrast with $e^x - 1 = \int_0^x e^w \, dw$. Also since Ito integral is a martingale, $\{S(t), t \geq 0\}$ is a martingale for each choice of real number θ.

Example: Ito's Formula

Moments. Consider the last example one more time. Suppose we wanted to find $\mathbb{E}(S(t))$ when we only knew that $dS = \theta S\, dW$, for a constant θ. In integral form it says that $S(t) - S(0) = \theta \int_0^t S\, dW$. But since the expectation of any Ito integral is zero, we immediately get $\mathbb{E}(S(t)) = S(0)$ for all $t \geq 0$. Of course the differential form does not explicitly tell us what $S(0)$ is equal to. Since $S(t) = e^{\theta W(t) - (\theta^2 t/2)}$, from the last example, we happen to know $S(0) = 1$. This gives

$$\mathbb{E}(e^{\theta W(t)}) = e^{\theta^2 t/2},$$

and, in particular, we have found the moment generating function $N(0, t)$.

What did we see?

- We started with random walk over discrete time steps.
- Taking smaller step sizes random walk led to Brownian motion.
- Using Brownian motion we constructed a Ito stochastic integral.
- Using Ito stochastic integral we introduced .
- Finally we learnt that smooth transform of an Ito processes is again an Ito processes. This result was called Ito's formula
- Among many applications of Ito's formula we saw that it can be used to compute
    - Ito integrals,
    - Create new Ito processes,
    - Even compute moments.

What we did not see are the concepts of stochastic differential equations and how Ito's formula may help solve them.
We also did not see how stochastic calculus is used in Finance to price various financial contracts. Just Google and see a lot more.

Questions?