Stochastic Calculus for Finance II
-
some Solutions to Chapter V

Matthias Thul*

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Exercise 5.1

(i) Let $f(t,x) = S(0)e^x$. We have

$$\frac{\partial f}{\partial t} = 0, \quad \frac{\partial f}{\partial x} = f(t,x) \quad \frac{\partial^2 f}{\partial x^2} = f(t,x)$$

and

$$\begin{aligned} dX(t) &= \left(\alpha(t) - R(t) - \frac{1}{2}\sigma^2(t)\right) dt + \sigma(t) dW(t) \\ (dX(t))^2 &= \sigma^2(t) dt \end{aligned}$$

By the Itô formula, the differential of the discounted stock price $D(t)S(t)$ is given by

$$\begin{aligned} d\left(D(t)S(t)\right) &= df(t, X(t)) \\ &= D(t)S(t)dX(t) + \frac{1}{2}D(t)S(t)\left(dX(t)\right)^2 \\ &= \left(\alpha(t) - R(t)\right) d(t)S(t)dt + \sigma(t)D(t)S(t)dW(t) \qquad \text{(q.e.d.)} \end{aligned}$$

(ii) We first note that the cross variation $dD(t)dS(t) = 0$, since $D(t)$ is a nonrandom function of time and thus has zero quadratic variation. We thus obtain

*The author can be contacted via <<firstname>>.<<lastname>>@gmail.com and http://www.matthiasthul.com.

$$\begin{aligned} d\left(D(t)S(t)\right) &= -R(t)D(t)S(t)dt + \alpha(t)D(t)S(t)dt + \sigma(t)D(t)S(t)dW(t) \\ &= \left(\alpha(t) - R(t)\right) d(t)S(t)dt + \sigma(t)D(t)S(t)dW(t) \qquad \text{(q.e.d.)} \end{aligned}$$

Exercise 5.2 (State Price Density Process)

This assertion follows from Equation (5.2.30) and Lemma 5.2.2.

$$D(t)V(t) = \tilde{\mathbb{E}}\left[D(T)V(T)|\,\mathcal{F}(t)\right] = \frac{1}{Z(t)}\mathbb{E}\left[D(T)Z(T)V(T)|\,\mathcal{F}(t)\right]$$

and thus

$$D(t)Z(t)V(t) = \mathbb{E}\left[D(T)Z(T)V(T)|\,\mathcal{F}(t)\right]. \qquad \text{(q.e.d.)}$$

Exercise 5.3

(i) Differentiating inside the expected value and applying the chain rule yields

$$\begin{aligned} c_x(0,x) &= \tilde{\mathbb{E}}\left[e^{-rT}\mathbb{I}_{\left\{x\exp\left\{\sigma\tilde{W}(T)+\left(r-\frac{1}{2}\sigma^2\right)T\right\}>K\right\}}\exp\left\{\sigma\tilde{W}(T)+\left(r-\frac{1}{2}\sigma^2\right)T\right\}\right] \\ &= \tilde{\mathbb{E}}\left[\mathbb{I}_{\{S(T)>K\}}\exp\left\{\sigma\tilde{W}(T)-\frac{1}{2}\sigma^2 T\right\}\right]. \end{aligned}$$

Here, we have defined $S(t)$ by

$$S(t) = x\exp\left\{\sigma\tilde{W}(t)+\left(r-\frac{1}{2}\sigma^2\right)t\right\}$$

(ii) Let $\hat{\mathbb{P}}$ be a probability measure equivalent to $\tilde{\mathbb{P}}$ and let $Z(t)$ be a Radon-Nikodým. By Lemma 5.2.1, By Lemma 5.2.1, the expected value in of an $\mathcal{F}(t)$-measureable random variable Y satisfies

$$\hat{\mathbb{E}}\left[Y(t)\right] = \tilde{\mathbb{E}}\left[Z(t)Y(t)\right]$$

We note that $\mathbb{I}_{\{S(T)>K\}}$ is $\mathbb{F}(T)$-measurable and can thus write

$$
\begin{aligned}
\hat{\mathbb{P}}\left(S(T) > K\right) &= \hat{\mathbb{E}}\left[\mathbb{I}_{\{S(T)>K\}}\right] \\
&= \tilde{\mathbb{E}}\left[Z(T)\mathbb{I}_{\{S(T)>K\}}\right] \\
&= \tilde{\mathbb{E}}\left[\mathbb{I}_{\{S(T)>K\}} \exp\left\{\sigma\tilde{W}(T) - \frac{1}{2}\sigma^2 T\right\}\right] \\
&= c_x(0,x) \qquad \text{(q.e.d.)}
\end{aligned} \tag{1}
$$

Here, we have defined $Z(t)$ to be

$$
Z(t) = \exp\left\{\sigma\tilde{W}(t) - \frac{1}{2}\sigma^2 t\right\}.
$$

By Girsanov's theorem (Theorem 5.2.3), this is just the Radon-Nikodým derivative process that renders a $\tilde{\mathbb{P}}$-Brownian motion into a $\hat{\mathbb{P}}$-Brownian motion if

$$
\hat{W}(t) = \tilde{W}(t) - \sigma t.
$$

Thus, the $Z(t)$ fulfills all required properties such that the change of measure in Equation (1) is defined.

(iii) Substituing for $S(T)$ and using $\tilde{W}(T) = \hat{W}(T) + \sigma T$ yields

$$
\begin{aligned}
\hat{\mathbb{P}}\left(S(T) > K\right) &= \hat{\mathbb{P}}\left(x \exp\left\{\sigma\left(\hat{W}(T) + \sigma T\right) + \left(r - \frac{1}{2}\sigma^2\right)T\right\} > K\right) \\
&= \hat{\mathbb{P}}\left(\sigma\hat{W}(T) + \left(r + \frac{1}{2}\sigma^2\right)T > \ln\left(\frac{K}{x}\right)\right) \\
&= \hat{\mathbb{P}}\left(\frac{\hat{W}(T)}{\sqrt{T}} > \frac{\ln\left(\frac{K}{x}\right) - \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) \\
&= \hat{\mathbb{P}}\left(-\frac{\hat{W}(T)}{\sqrt{T}} < d_+(T,x)\right)
\end{aligned}
$$

Since $-\frac{\hat{W}(T)}{\sqrt{T}} \sim \mathcal{N}(0,1)$, we finally obtain

$$
\hat{\mathbb{P}}\left(S(T) > K\right) = N\left(d_+(T,x)\right) \qquad \text{(q.e.d.)}
$$

Exercise 5.4

(i) Let $f(t,x) = \ln x$. We have

$$\frac{\partial f}{\partial t} = 0, \quad \frac{\partial f}{\partial x} = \frac{1}{x}, \quad \frac{\partial^2 f}{\partial x^2} = -\frac{1}{x^2}$$

and

$$(dS(t))^2 = \sigma^2(t) S^2(t) dt.$$

Applying Itô's lemma, the differential of the log stock price $d \ln S(t)$ becomes

$$\begin{aligned} d\ln S(t) &= df(t, S(t)) \\ &= \frac{1}{S(t)} dS(t) - \frac{1}{2}\frac{1}{S^2(t)} (dS(t))^2 \\ &= r(t)dt + \sigma(t) d\tilde{W}(t) - \frac{1}{2}\sigma^2(t)dt \\ &= \left(r(t) - \frac{1}{2}\sigma^2(t) \right) dt + \sigma(t) d\tilde{W}(t). \end{aligned}$$

In integral form, we get

$$\ln S(t) = \ln S(0) + \int_0^t \left(r(s) - \frac{1}{2}\sigma^2(s) \right) ds + \int_0^t \sigma(s) d\tilde{W}(s). \tag{2}$$

Taking the exponential yields

$$S(t) = S(0) e^{X(t)}$$

where

$$X(t) = \int_0^t \left(r(s) - \frac{1}{2}\sigma^2(s) \right) ds + \int_0^t \sigma(s) d\tilde{W}(s).$$

By Theorem 4.4.9, Itô integrals of a deterministic integrand are normally distributed with zero mean. Thus, $X(t)$ is normally distributed with

$$X_t \sim \mathcal{N}\left(\int_0^t \left(r(s) - \frac{1}{2}\sigma^2(s) \right) ds, \int_0^t \sigma^2(s) ds \right) \tag{3}$$

(ii) Since the payoff of a European call option only depends on the asset price S_T at maturity, two different diffusion processes for the underlying that have imply a common risk-neutral density for S_T at time T will yield the same option prices. By setting $r(t) = R$ and $\sigma(t) = \Sigma$ in Equation (3), we see that the distribution of X_t under constant interest rate and volatility is given by

$$\begin{aligned} X_t &\sim \mathcal{N}\left(\int_0^t \left(R - \frac{1}{2}\Sigma^2\right) ds, \int_0^t \Sigma^2 ds\right) \\ &\sim \mathcal{N}\left(\left(R - \frac{1}{2}\Sigma^2\right) T, \Sigma^2 T\right). \end{aligned} \tag{4}$$

We now equate the mean and variance in Equations (3) and (4) to obtain

$$\Sigma^2 T = \int_0^t \sigma^2(s) ds \quad \Leftrightarrow \quad \Sigma = \frac{1}{T}\sqrt{\int_0^t \sigma^2(s) ds}$$

and

$$\left(R - \frac{1}{2}\Sigma^2\right) T = \int_0^t \left(r(s) - \frac{1}{2}\sigma^2(s)\right) ds \quad \Leftrightarrow \quad R = \int_0^t r(s) ds$$

It follows that we can replace R by $\int_0^t r(s)ds$ and Σ by $\sqrt{\int_0^t \sigma^2(s)ds}$ in the Black-Scholes formula for European call options to obtain the value under deterministic interest rates and volatilities.

Exercise 5.5

(i) Let $f(t,x) = \frac{1}{x}$. We have

$$\frac{\partial f}{\partial t} = 0, \quad \frac{\partial f}{\partial x} = -\frac{1}{x^2}, \quad \frac{\partial^2 f}{\partial x^2} = \frac{2}{x^3}$$

and thus

$$\begin{aligned} d\left(\frac{1}{Z(t)}\right) &= -\frac{1}{Z^2(t)} dZ(t) + \frac{1}{2}\frac{2}{Z^3(t)} (dZ(t))^2 \\ &= \frac{\Theta(t)}{Z(t)} dW(t) + \frac{\Theta^2}{Z(t)} dt. \end{aligned}$$

Here, we have used that

$$dZ(t) = -\Theta(t)Z(t)dW(t),$$

as shown in the proof of Theorem 5.2.3.

(ii) As shown in the proof of Theorem 5.2.3, $Z(t)$ is a Radon-Nikodým derivative process. Thus, Lemma 5.2.2 applies and we get

$$\mathbb{E}\left[\tilde{M}(t)Z(t)\middle|\mathcal{F}(s)\right] = Z(s)\tilde{\mathbb{E}}\left[\tilde{M}(t)\middle|\mathcal{F}(s)\right] = Z(s)M(s) \qquad \text{(q.e.d.)}.$$

Here, the first equality follows by multiplying both sides of the equality in Lemma 5.2.2 by $Z(s)$ and in the second step we use that $\tilde{M}(t)$ is a martingale under $\tilde{\mathbb{P}}$.

(iii) By the Itô product rule (Corollary 4.6.3), we get

$$\begin{aligned}
d\tilde{M}(t) &= d\left(\frac{M(t)}{Z(t)}\right) \\
&= M(t)d\left(\frac{1}{Z(t)}\right) + \frac{1}{Z(t)}dM(t) + dM(t)d\left(\frac{1}{Z(t)}\right) \\
&= \frac{M(t)}{Z(t)}\left[\Theta^2(t)dt + \Theta(t)dW(t)\right] + \frac{\Gamma(t)}{Z(t)}dW(t) + \frac{\Gamma(t)\Theta(t)}{Z(t)}dt \\
&= \frac{1}{Z(t)}\left[\left(M(t)\Theta^2(t) + \Gamma(t)\Theta(t)\right)dt + \left(M(t)\Theta(t) + \Gamma(t)\right)dW(t)\right].
\end{aligned}$$

(iv) Corollary 5.3.2 defines

$$d\tilde{W}(t) = dW(t) + \Theta(t)dt$$

and substituting for $W(t)$ in the differential of $\tilde{M}(t)$ yields

$$\begin{aligned}
d\tilde{M}(t) &= \frac{1}{Z(t)}\left[\left(M(t)\Theta^2(t) + \Gamma(t)\Theta(t)\right)dt + \left(M(t)\Theta(t) + \Gamma(t)\right)\left(d\tilde{W}(t) - \Theta(t)dt\right)\right] \\
&= \frac{1}{Z(t)}\left(M(t)\Theta(t) + \Gamma(t)\right)d\tilde{W}(t) \\
&= \left(\tilde{M}(t)\Theta(t) + \frac{\Gamma(t)}{Z(t)}\right)d\tilde{W}(t).
\end{aligned}$$

We now define

$$\tilde{\Gamma}(t) = \tilde{M}(t)\Theta(t) + \frac{\Gamma(t)}{Z(t)}$$

and integrate to obtain

$$\tilde{M}(t) = \tilde{M}(0) + \int_0^t \tilde{\Gamma}(u) d\tilde{W}(u) \qquad \text{(q.e.d.)}.$$

Note, that $\Gamma(t)$ is adapted to the filtration $\mathcal{F}(t)$ as required by Corollary 5.3.2, since $\tilde{M}(t)$, $\Theta(t)$, $\Gamma(t)$ and $Z(t)$ are adapted processes.

Exercise 5.6

We will proof the more general case of the multidimensional Girsanov theorem, i.e. Theorem 5.4.2 for any $d \in \{1, 2, \ldots\}$. We note, that by the two dimensional Lévy theorem (Theorem 4.6.5) two continuous martingales $M_1(t)$ and $M_2(t)$ with respect to some filtration $\mathcal{F}(t)$ that start at zero and have unit quadratic variation and zero cross variation are independent Brownian motions. If we thus consider the multidimensional martingale

$$M(t) = (M_1(t), M_2(t), \ldots, M_d(t)),$$

then we require that the conditions of the two dimensional Lévy theorem are satisfied for any two $M_i(t)$ and $M_j(t)$ where $i, j \in \{1, 2, \ldots, d\}$. I.e. we want to show that

$$dM_i(t)dM_j(t) = \begin{cases} dt & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

or equivalently in matrix notation

$$dM(t)dM'(t) = I_d dt,$$

where $'$ is used to denote the vector/matrix transpose and I_d is the identity matrix of dimension d.

(i) **Continuity:**

Clearly,

$$\tilde{W}(t) = W(t) + \int_0^t \Theta(u)du$$

is continuous since by Definition 3.3.1 the Brownian motion $W(t)$ has continuous sample paths. Similarly, the ordinary Lebesgue integral is a continuous function of the upper limit of integration.

(ii) **Starting at zero:**

It is obvious that $\tilde{W}(0) = W(0) = 0$.

(iii) **Unit quadratic and zero cross variation:**

We have

$$\begin{aligned} d\tilde{W}(t)d\tilde{W}'(t) &= (dW(t) + \Theta(t)dt)\,(dW(t) + \Theta(t)dt)' \\ &= (dW(t) + \Theta(t)dt)\,(dW'(t) + \Theta'(t)dt) \\ &= dW(t)dW'(t) = I_d dt \qquad \text{(q.e.d.)}. \end{aligned}$$

(iv) **Martingale property:**

We first define

$$X(t) = -\int_0^t \Theta(u)\cdot dW(u) - \frac{1}{2}\int_0^t ||\Theta(u)||^2 du$$

such that

$$dX(t) = -\Theta(t)\cdot dW(t) - \frac{1}{2}||\Theta(t)||^2 dt$$

and

$$\begin{aligned}
dX(t)dX(t) &= \left(-\sum_{j=1}^{d} \Theta_j(t)dW_j(t) - \frac{1}{2}\sum_{j=1}^{d} \Theta_j^2(t)dt \right)^2 \\
&= \sum_{j=1}^{d}\sum_{k=1}^{d} \Theta_j(t)\Theta_k(t)dW_j(t)dW_k(t) \\
&= \sum_{j=1}^{d} \Theta_j^2(t)dt \\
&= ||\Theta(t)||^2 dt.
\end{aligned}$$

In analogy to the proof of Theorem 5.2.3 (Girsanov, one dimension), we now define $f(t,x) = e^x$ and apply the Itô formula to obtain the differential of $Z(t)$ as

$$\begin{aligned}
dZ(t) &= Z(t)dX(t) + \frac{1}{2}Z(t)dX(t)dX(t) \\
&= -Z(t)\Theta(t) \cdot dW(t) - \frac{1}{2}Z(t)||\Theta(t)||^2 dt + \frac{1}{2}Z(t)||\Theta(t)||^2 dt \\
&= -Z(t)\Theta(t) \cdot dW(t).
\end{aligned}$$

Since the differential of $Z(t)$ contains no dt term, it follows that $Z(t)$ is a $\mathbb{P}$ martingale with $\mathbb{E}\left[Z(T)\right] = Z(0) = 1$ and thus qualifies as a Radon-Nikoým derivative process. The differential of $\tilde{W}(t)Z(t)$ can be computed using the Itô product rule (Corollary 4.6.3) as

$$\begin{aligned}
d\left(\tilde{W}(t)Z(t)\right) &= \tilde{W}(t)dZ(t) + Z(t)d\tilde{W}(t) + d\tilde{W}(t)dZ(t) \\
&= -\tilde{W}(t)Z(t)\Theta(t) \cdot dW(t) + Z(t)\left(dW(t) + \Theta(t)dt\right) \\
&\quad + \left(dW(t) + \Theta(t)dt\right)\left(-Z(t)\Theta(t) \cdot dW(t)\right) \\
&= -\tilde{W}(t)Z(t)\Theta(t) \cdot dW(t) + Z(t)dW(t) + Z(t)\Theta(t)dt \\
&\quad -Z(t)\Theta(t)dt \\
&= -\tilde{W}(t)Z(t)\Theta(t) \cdot dW(t) + Z(t)dW(t) \\
&= Z(t)\left(-\tilde{W}(t) \cdot \Theta(t) + 1\right) dW(t).
\end{aligned}$$

By the same argument at before, it follows that $\tilde{W}(t)Z(t)$ is a $\mathbb{P}$ martingale. Using Lemma 5.2.2, we obtain

$$\begin{aligned}
\tilde{\mathbb{E}}\left[\tilde{W}(t)\middle|\mathcal{F}(s)\right] &= \frac{1}{Z(s)}\mathbb{E}\left[\tilde{W}(t)Z(t)\middle|\mathcal{F}(s)\right] \\
&= \frac{1}{Z(s)}\tilde{W}(s)Z(s) = \tilde{W}(s) \qquad \text{(q.e.d.)}.
\end{aligned}$$

Thus, $\tilde{W}(t)$ is a d-dimensional $\tilde{\mathbb{P}}$ martingale.

Since all conditions of the multidimensional Lévy theorem are satisfied, we conclude that $\tilde{W}(t)$ is a d-dimensional Brownian motion under $\tilde{\mathbb{P}}$.

Exercise 5.7

(i) We can obtain this portfolio process by investing the amount $X_2(0) > 0$ in the money market such that the value at time T is

$$X_2(0)\exp\left\{\int_0^T R(t)dt\right\} = \frac{X_2(0)}{D(T)}.$$

In addition, we invest in the portfolio process $X_1(t)$. The initial cost of setting up this portfolio are $X_2(0) + X_1(0) = X_2(0)$. At time T, we have

$$\mathbb{P}\left\{X_2(T) \geq \frac{X_2(0)}{D(T)}\right\} = \mathbb{P}\left\{\frac{X_2 0}{D(T)} + X_1(T) \geq \frac{X_2(0)}{D(T)}\right\} = \mathbb{P}\left\{X_1(T) \geq 0\right\} = 1$$

and

$$\mathbb{P}\left\{X_2(T) > \frac{X_2(0)}{D(T)}\right\} = \mathbb{P}\left\{\frac{X_2 0}{D(T)} + X_1(T) > \frac{X_2(0)}{D(T)}\right\} = \mathbb{P}\left\{X_1(T) > 0\right\} > 0.$$

(ii) The same idea as in (i) applies here. We can obtain the portfolio process $X_1(t)$ by borrowing the amount $X_2(0)$ in the money market and investing the proceeds in the portfolio process $X_2(t)$. At time T, we have to repay $X_2(0)/D(T)$ and obtain

$$\mathbb{P}\left\{X_1(T) \geq 0\right\} = \mathbb{P}\left\{X_2(T) - \frac{X_2(0)}{D(T)} \geq 0\right\} = \mathbb{P}\left\{X_2(T) \geq \frac{X_2(0)}{D(T)}\right\} = 1$$

and

$$\mathbb{P}\left\{X_1(T) > 0\right\} = \mathbb{P}\left\{X_2(T) - \frac{X_2(0)}{D(T)} > 0\right\} = \mathbb{P}\left\{X_2(T) > \frac{X_2(0)}{D(T)}\right\} > 0.$$

Exercise 5.8 (Every strictly positive Asset is a generalized geometric Brownian Motion)

(i) We have that

$$D(t)V(t) = \tilde{\mathbb{E}}\left[D(T)V(T) \middle| \mathcal{F}(t) \right].$$

is a $\tilde{\mathbb{P}}$-martingale since for $s < t$

$$\begin{aligned} \tilde{\mathbb{E}}\left[D(t)V(t) \middle| \mathcal{F}(s) \right] &= \tilde{\mathbb{E}}\left[\tilde{\mathbb{E}}\left[D(T)V(T) \middle| \mathcal{F}(t) \right] \middle| \mathcal{F}(s) \right] \\ &= \tilde{\mathbb{E}}\left[D(T)V(T) \middle| \mathcal{F}(s) \right] \\ &= D(s)V(s). \end{aligned}$$

By Corollary 5.3.2, there exists a an adapted process $\tilde{\Gamma}(t)$ such that

$$D(t)V(t) = V(0) + \int_0^t \tilde{\Gamma}(u) d\tilde{W}(u)$$

or in differential form

$$d(D(t)V(t)) = \tilde{\Gamma}(t) d\tilde{W}(t).$$

By the Itô product rule,

$$\begin{aligned} dV(t) &= D^{-1}(t)d(D(t)V(t)) + D(t)V(t)dD^{-1}(t) + d(D(t)V(t))dD^{-1}(t) \\ &= R(t)V(t)dt + \frac{\tilde{\Gamma}(t)}{D(t)} d\tilde{W}(t). \end{aligned}$$

Here we used that

$$dD^{-1}(t) = R(t)D^{-1}(t)dt$$

is a process of finite variation and thus the cross variation term in the Itô differential drops out.

(ii) We define a new random variable $X = 0$ with $\mathbb{E}[X] = 0$. Note that the constant X satisfies Definition 1.2.1 for random variables. Since both $D(T)$ and $V(T)$ are $\tilde{\mathbb{P}}$-almost surely positive, it follows that $X < D(T)V(T)$ $\tilde{\mathbb{P}}$-almost surely. Consequently, $0 = \mathbb{E}[X] < \mathbb{E}[D(T)V(T)] = D(t)V(t)$. This does not follow directly from Theorem 1.3.1 but is a straight forward modification of it. Now, since $D(t)$ is positive $\tilde{\mathbb{P}}$-almost surely, it follows that $V(t)$ is too.

(iii) Since $V(t)$ is $\tilde{\mathbb{P}}$-almost surely positive, we can write its differential as

$$\begin{aligned} dV(t) &= R(t)V(t)dt + \frac{\tilde{\Gamma}(t)}{D(t)V(t)}V(t)d\tilde{W}(t) \\ &= R(t)V(t)dt + \sigma(t)V(t)d\tilde{W}(t), \end{aligned}$$

where we defined $\sigma(t) = \tilde{\Gamma}(t)/(D(t)V(t))$. $\sigma(t)$ is adapted to the filtration $\mathcal{F}(t)$ since all three processes that define it are.d

Exercise 5.9 (Implying the risk-neutral distribution)

We apply the Leibniz integral rule to obtain

$$\begin{aligned} c_K &= \frac{\partial}{\partial K}\left(e^{-rT}\int_K^\infty (y-K)\tilde{p}(0,T,x,y)dy\right) \\ &= e^{-rT}\int_K^\infty \frac{\partial}{\partial K}(y-K)\tilde{p}(0,T,x,y)dy \\ &= -e^{-rT}\int_K^\infty \tilde{p}(0,T,x,y)dy. \end{aligned}$$

Applying the Leibniz integral rule for a second time yields

$$c_{KK} = e^{-rT}\tilde{p}(0,T,x,K).$$

Thus, the implied risk-neutral density can be recovered from a continuum of call prices by

$$\tilde{p}(0,T,x,K) = e^{rT}c_{KK}.$$

Exercise 5.10 (Chooser Option)

i) At time t_0 (choice date) the holder of a long position in a chooser option has the right to choose whether his option is a call or a put. If he acts rationally, he will always choose the higher priced option. Thus, the price $V(t_0)$ of a chooser option at $t = t_0$ has to be the greater of the call option price $C(t_0)$ and the put option price $P(t_0)$. Using put-call parity we obtain

$$\begin{aligned} V(t_0) &= \max\{C(t_0), P(t_0)\} \\ &= \max\{C(t_0), C(t_0) - F(t_0)\} \\ &= C(t_0) + \max\{0, -F(t_0)\} \\ &= C(t_0) + \left(e^{-r(T-t_0)}K - S(t_0)\right)^+. \quad \text{(q.e.d.)} \end{aligned}$$

ii) By the risk-neutral pricing formula we have

$$\begin{aligned} V(0) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-rt_0}V(t_0)\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[e^{-rt_0}\left(C(t_0) + \left(e^{-r(T-t_0)}K - S(t_0)\right)^+\right)\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[e^{-rt_0}\mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t_0)}C(T)\middle|\,\mathcal{F}(t_0)\right]\right] + \mathbb{E}^{\mathbb{Q}}\left[e^{-rt_0}\left(e^{-r(T-t_0)}K - S(t_0)\right)^+\right] \\ &= C(0) + \mathbb{E}^{\mathbb{Q}}\left[e^{-rt_0}\max\left\{0, e^{-r(T-t_0)}K - S(t_0)\right\}\right] \\ &= C(0,T,K) + P(0,t_0,e^{-r(T-t_0)}K). \quad \text{(q.e.d)} \end{aligned}$$

While the first term is the current value of a call option with strike price K and maturity in T, the second term is the current value of a put option with strike price $e^{-r(T-t_0)}K$ and maturity in t_0. Thus the value of a chooser option is equal to a portfolio consisting of a long call and a long put with different strike prices and expiry dates.

Exercise 5.11 (Hedging a Cash Flow)

Following the hint, we start by defining the market price of risk by

$$\Theta(t) = \frac{\alpha(t) - R(t)}{\sigma(t)}.$$

The process $\Theta(t)$ is adapted to the filtration $\mathcal{F}(t)$ since all three processes that define it are. It is well defined since $\sigma(t)$ is strictly positive. Next, we define

$$Z(t) = \exp\left\{-\int_0^t \Theta(u)dW(u) - \frac{1}{2}\int_0^t \Theta^2(u)du\right\}$$

and

$$\tilde{W}(t) = W(t) + \int_0^t \Theta(u)du.$$

By Girsanov's theorem (Theorem 5.2.3), $\tilde{W}(t)$ is a Brownian motion under the probability measure $\tilde{\mathbb{P}}$ defined by

$$\tilde{\mathbb{P}}(A) = \int_A Z(\omega)dP(\omega)$$

for all $A \in \mathcal{F}$. Further following the hint, we define

$$\tilde{M}(t) = \tilde{\mathbb{E}}\left[\int_0^T D(u)C(u)du \middle| \mathcal{F}(t)\right].$$

Then, for $0 \leq s \leq t$ we get

$$\begin{aligned}
\tilde{\mathbb{E}}\left[\tilde{M}\middle|\mathcal{F}(s)\right] &= \tilde{\mathbb{E}}\left[\tilde{\mathbb{E}}\left[\int_0^T D(u)C(u)du\middle|\mathcal{F}(t)\right]\middle|\mathcal{F}(s)\right] \\
&= \tilde{\mathbb{E}}\left[\int_0^T D(u)C(u)du\middle|\mathcal{F}(s)\right] \\
&= \tilde{M}(s)
\end{aligned}$$

and thus $\tilde{M}(t)$ is a $\tilde{\mathbb{P}}$-martingale. Since the filtration $\mathcal{F}(t)$ is the one generated by the Brownian motion $W(t)$, we can apply Corollary 5.3.2. There exists an adapted process $\tilde{\Gamma}(t)$ such that

$$\tilde{M}(t) = \tilde{M}(0) + \int_0^t \tilde{\Gamma}(u)d\tilde{W}(u).$$

Using Itô's product rule, we now compute the differential of the discounted portfolio value under $\mathbb{P}$ as

$$\begin{aligned}
d(D(t)X(t)) &= D(t)dX(t) + X(t)dD(t) + dD(t)dX(t) \\
&= D(t)\Delta(t)dS(t) + R(t)D(t)(X(t) - Delta(t)S(t))dt - D(t)C(t)dt - R(t)D(t)X(t) \\
&= \Delta(t)D(t)(dS(t) - R(t)S(t)dt) - D(t)C(t)dt \\
&= \Delta(t)D(t)S(t)((\alpha(t) - R(t))dt + \sigma(t)dW(t)) - D(t)C(t)dt \\
&= \Delta(t)D(t)S(t)\left((\alpha(t) - R(t))dt + \sigma(t)\left(\tilde{W}(t) - \Theta(t)dt\right)\right) - D(t)C(t)dt \\
&= \Delta(t)\sigma(t)D(t)S(t)d\tilde{W}(t) - D(t)C(t)dt.
\end{aligned}$$

In integral form, we have

$$D(T)X(T) = X(0) + \int_0^T \Delta(u)\sigma(u)D(u)S(u)d\tilde{W}(u) - \int_0^T D(u)C(u)du.$$

where we used that $D(0) = 1$. Choose $X(0) = \tilde{M}(0)$ and

$$\Delta(t) = \frac{\tilde{\Gamma}(t)}{\sigma(t)D(t)S(t)},$$

then

$$\begin{aligned}
D(T)X(T) &= \tilde{M}(0) + \int_0^T \tilde{\Gamma}(u)d\tilde{W}(0) - \int_0^T D(u)C(u)du \\
&= \tilde{M}(0) + \tilde{M}(T) - \tilde{M}(0) - \int_0^T D(u)C(u)du \\
&= \tilde{\mathbb{E}}\left[\int_0^T D(u)C(u)du \middle| \mathcal{F}(T)\right] - \int_0^T D(u)C(u)du \\
&= 0.
\end{aligned}$$

Here, we used that the integral in the second last equality is $\mathcal{F}(T)$-measurable. Since $D(T)$ is $\tilde{\mathbb{P}}$-almost surely positive, it follows that $X(T) = 0$ $\tilde{\mathbb{P}}$-almost surely. It follows that if we start with an initial wealth of

$$X(0) = \tilde{\mathbb{E}}\left[\int_0^T D(u)C(u)\right]$$

and choose the portfolio process given by $\Delta(t)$ above, then we can hedge the random cash-flow $\tilde{\mathbb{P}}$-almost surely.

Exercise 5.12 (Correlation under Change of Measure)

(i) Remember that $B_i(t)$ is defined by

$$B_i(t) = \sum_{j=1}^{d} \int_0^t \frac{\sigma_{ij}(u)}{\sigma_i(u)} dW_j(u).$$

We have

$$\begin{aligned}
\tilde{B}_i(t) &= \sum_{i=1}^{d} \int_0^t \left(\frac{\sigma_{ij}(u)}{\sigma_i(u)} dW_j(u) + \gamma_i(u) du \right) \\
&= \sum_{i=1}^{d} \int_0^t \frac{\sigma_{ij}(u)}{\sigma_i(u)} \left(dW_j(u) + \Theta(u) du \right) \\
&= \sum_{i=1}^{d} \int_0^t \frac{\sigma_{ij}(u)}{\sigma_i(u)} d\tilde{W}_j(u).
\end{aligned}$$

It has been shown using Lévy's theorem (Theorem 4.6.4) that $B_i(t)$ is a Brownian motion under $\mathbb{P}$. Repeating the same proof but replacing $W_j(t)$ by $\tilde{W}_j(t)$ it can be shown that $\tilde{B}_i(t)$ is a Brownian motion under $\mathbb{P}$. Note that by Theorem 4.3.1, each of the Itô integrals in the definition of $\tilde{B}_i(t)$ is a continuous martingale starting at zero and thus $\tilde{B}_i(t)$ is too. Its quadratic variation is

$$\begin{aligned}
d\tilde{B}_i(t) d\tilde{B}_i(t) &= \sum_{j=1}^{d} \frac{\sigma_{ij}^2(t)}{\sigma_i^2(t)} dt \\
&= dt,
\end{aligned}$$

where we used that by definition

$$\sigma_i(t) = \sqrt{\sum_{j=1}^{d} \sigma_{ij}^2(t)}.$$

As anticipated, it follows by Theorem 4.6.4 that $\tilde{B}_i(t)$ is a $\tilde{\mathbb{P}}$-martingale.

(ii) We have

$$
\begin{aligned}
dS_i(t) &= \alpha_i(t)S_i(t)dt + \sigma_i(t)S_i(t)dB_i(t) \\
&= \alpha_i(t)S_i(t)dt + \sigma_i(t)S_i(t)\sum_{j=1}^{d}\frac{\sigma_{ij}(t)}{\sigma_i(t)}dW_j(t) \\
&= \alpha_i(t)S_i(t)dt + \sigma_i(t)S_i(t)\sum_{j=1}^{d}\frac{\sigma_{ij}(t)}{\sigma_i(t)}\left(d\tilde{W}_j(t) - \Theta_j(t)dt\right) \\
&= \alpha_i(t)S_i(t)dt + \sigma_i(t)S_i(t)\left(d\tilde{B}_i(t) - \sum_{j=1}^{d}\frac{\sigma_{ij}(t)\Theta_j(t)}{\sigma_i(t)}dt\right) \\
&= \left(\alpha_i(t) - \sum_{j=1}^{d}\sigma_{ij}(t)\Theta_j(t)\right)S_i(t)dt + \sigma_i(t)S_i(t)d\tilde{B}_t \\
&= R(t)S_i(t)dt + \sigma_i(t)S_i(t)d\tilde{B}_i(t) \qquad \text{(q.e.d.)}.
\end{aligned}
$$

Here, we used the expression previously obtained for $\tilde{B}_i(t)$ in equality four. The last equality follows from the market price of risk equations

$$
\alpha_i(t) - R(t) = \sum_{j=1}^{d}\sigma_{ij}(t)\Theta_j(t).
$$

(iii) We have for $i \neq k$

$$
\begin{aligned}
d\tilde{B}_i(t)d\tilde{B}_k(t) &= \left(dB_i(t) + \gamma_i(t)dt\right)\left(dB_k(t) + \gamma_k(t)dt\right) \\
&= dB_i(t)dB_k(t) \\
&= \rho_{ik}(t)dt \qquad \text{(q.e.d.)}.
\end{aligned}
$$

Here, we used that the covariation between a process of bounded variation and any other process is zero in the second step.

(iv) Using the Itô product rule, we get for the first expectation

$$
\begin{aligned}
\mathbb{E}\left[B_i(t)B_k(t)\right] &= \tilde{\mathbb{E}}\left[B_i(0)B_k(0) + \int_0^t d\left(B_i(u)B_k(u)\right)\right] \\
&= \tilde{\mathbb{E}}\left[\int_0^t \left(B_i(u)dB_k(u) + B_k(u)dB_i(u) + dB_i(u)dB_k(u)\right)\right] \\
&= \tilde{\mathbb{E}}\left[\int_0^t dB_i(u)dB_k(u)\right] \\
&= \tilde{\mathbb{E}}\left[\int_0^t \rho_{ik}(u)du\right] \\
&= \int_0^t \rho_{ik}(u)du.
\end{aligned}
$$

Here, we used that $B_i(0) = B_k(0) = 0$ in the second equality. By Theorem 4.3.1, the Itô integrals in the third equality are martingales starting at zero and have thus zero expected value. In the last step, we can drop the expectation operator since the function $\rho_{ik}(t)$ is deterministic. The same steps can be repeated to show that

$$
\tilde{\mathbb{E}}\left[\tilde{B}_i(t)\tilde{B}_k(t)\right] = \int_0^t \rho_{ik}(u)du
$$

and the proof is complete.

(v) Again using the Itô product rule, we get for the first expectation

$$
\begin{aligned}
\mathbb{E}\left[B_1(t)B_2(t)\right] &= \tilde{\mathbb{E}}\left[B_2(0)B_2(0) + \int_0^t d\left(B_1(u)B_2(u)\right)\right] \\
&= \tilde{\mathbb{E}}\left[\int_0^t \left(B_1(u)dB_2(u) + B_2(u)dB_1(u) + dB_1(u)dB_2(u)\right)\right] \\
&= \tilde{\mathbb{E}}\left[\int_0^t dB_1(u)dB_2(u)\right] \\
&= \tilde{\mathbb{E}}\left[\int_0^t \operatorname{sign}\left(W_1(u)\right)du\right] \\
&= \int_0^t \mathbb{E}\left[\operatorname{sign}\left(W_1(u)\right)\right]du \\
&= \int_0^t \left(\mathbb{P}\left\{W_1(u) \geq 0\right\} - \mathbb{P}\left\{W_1(u) < 0\right\}\right)du \\
&= 0.
\end{aligned}
$$

Skipping some intermediate steps, we get for the second expectation

$$
\begin{aligned}
\tilde{\mathbb{E}}\left[\tilde{B}_1(t)\tilde{B}_2(t)\right] &= \int_0^t \tilde{\mathbb{E}}\left[\operatorname{sign}\left(W_1(u)\right)\right] du \\
&= \int_0^t \tilde{\mathbb{E}}\left[\operatorname{sign}\left(\tilde{W}_1(u) - \int_0^u \Theta_1(v)dv\right)\right] du \\
&= \int_0^t \tilde{\mathbb{E}}\left[\operatorname{sign}\left(\tilde{W}_1(u) - u\right)\right] du \\
&= \int_0^t \left(\tilde{\mathbb{P}}\left\{\tilde{W}_1(u) \geq u\right\} - \tilde{\mathbb{P}}\left\{\tilde{W}_1(u) < u\right\}\right) du \\
&\leq 0.
\end{aligned}
$$

Here, strict inequality holds for all $t > 0$. Consequently, for $t > 0$,

$$
\mathbb{E}\left[B_1(t)B_2(t)\right] \neq \tilde{\mathbb{E}}\left[\tilde{B}_1(t)\tilde{B}_2(t)\right] \qquad \text{(q.e.d.).}
$$

Exercise 5.13

(i) We have

$$
\begin{aligned}
\tilde{\mathbb{E}}\left[W_1(t)\right] &= \tilde{\mathbb{E}}\left[\tilde{W}_1(t) - \int_0^t \Theta_1(u)du\right] \\
&= 0,
\end{aligned}
$$

where the last equality follows from $\tilde{W}_1$ being a $\tilde{\mathbb{P}}$-martingale with zero initial value and $\Theta_1(t) = 0$ for all $t \geq 0$. Similarly,

$$
\begin{aligned}
\tilde{\mathbb{E}}\left[W_2(t)\right] &= \tilde{\mathbb{E}}\left[\tilde{W}_2(t) - \int_0^t \Theta_2(u)du\right] \\
&= \tilde{\mathbb{E}}\left[-\int_0^t W_1(u)du\right] \\
&= \tilde{\mathbb{E}}\left[-\int_0^t \left(\tilde{W}_1(u) - \Theta_1(u)\right) du\right) \\
&= \tilde{\mathbb{E}}\left[-\int_0^t \tilde{W}_1(u)du\right].
\end{aligned}
$$

To evaluate the remaining expectation, we define a function $f(t, x) = tx$ where

$$
\frac{\partial f}{\partial t} = x, \quad \frac{\partial f}{\partial x} = t, \quad \frac{\partial^2 f}{\partial x^2} = 0.
$$

Applying Itô's formula yields the differential

$$\begin{aligned} d\left(t\tilde{W}_1(t)\right) &= df\left(t,\tilde{W}_1(t)\right) \\ &= \tilde{W}_1(t)dt + td\tilde{W}_1(t). \end{aligned}$$

Thus, in integral form and after rearranging, we get

$$-\int_0^t \tilde{W}_1(u)du = \int_0^t ud\tilde{W}_1(u) - t\tilde{W}_1(t).$$

Substituting back yields

$$\begin{aligned} \tilde{\mathbb{E}}\left[W_2(t)\right] &= \tilde{\mathbb{E}}\left[\int_0^t ud\tilde{W}_1(u) - t\tilde{W}_1(t)\right] \\ &= 0 \qquad \text{(q.e.d.)}. \end{aligned}$$

Here, we used that by Theorem 4.3.1, both the Itô integral w.r.t. $\tilde{W}(t)$ and $\tilde{W}(t)$ have zero expectation under $\tilde{\mathbb{P}}$.

(ii) We have

$$\begin{aligned} \widetilde{\mathrm{Cov}}\left[W_1(T),W_2(T)\right] &= \tilde{\mathbb{E}}\left[W_1(T)W_2(T)\right] - \tilde{\mathbb{E}}\left[W_1(T)\right]\tilde{\mathbb{E}}\left[W_2(T)\right] \\ &= \tilde{\mathbb{E}}\left[W_1(T)W_2(T)\right]. \end{aligned}$$

Here, the second equality follows from the result in (i). Now using the hint, we have in integral form

$$\begin{aligned} W_1(T)W_2(T) &= \int_0^T W_1(u)dW_2(u) + \int_0^T W_2(u)dW_1(u) \\ &= \int_0^T W_1(u)\left(d\tilde{W}_2(u) - W_1(u)du\right) + \int_0^T W_2(u)d\tilde{W}_1(u) \\ &= \int_0^T W_1(u)d\tilde{W}_2(u) - \int_0^T \tilde{W}_1^2(u)du + \int_0^T W_2(u)d\tilde{W}_1(u). \end{aligned}$$

Thus,

$$
\begin{aligned}
\widetilde{\mathrm{Cov}}\left[W_1(T), W_2(T)\right] &= \tilde{\mathbb{E}}\left[\int_0^T W_1(u)d\tilde{W}_2(u) - \int_0^T \tilde{W}_1^2(u)du + \int_0^T W_2(u)d\tilde{W}_1(u)\right] \\
&= \tilde{\mathbb{E}}\left[-\int_0^T \tilde{W}_1^2(u)du\right] \\
&= -\int_0^T \tilde{\mathbb{E}}\left[\tilde{W}_1^2(u)\right]du \\
&= -\int_0^T udu \\
&= -\frac{1}{2}T^2 \qquad \text{q.e.d..}
\end{aligned}
$$

Here, we again used the martingale property of the two Itô integrals w.r.t. $\tilde{W}_1(t)$ and $\tilde{W}_2(t)$ under $\tilde{\mathbb{P}}$ in the second equality.

Exercise 5.14 (Cost of Carry)

(i) The differential of the discounted portfolio value is given by

$$
\begin{aligned}
d(D(t)X(t)) &= D(t)dX(t) + X(t)dD(t) + dD(t)dX(t) \\
&= \Delta(t)D(t)dS(t) - a\Delta(t)D(t)dt + rD(t)(X(t) - \Delta(t)S(t))dt - rD(t)X(t) \\
&= \Delta(t)D(t)\left(dS(t) - rS(t)dt\right) - a\Delta(t)D(t)dt \\
&= \Delta(t)\sigma D(t)S(t)d\tilde{W}(t).
\end{aligned}
$$

Since $\tilde{W}(t)$ is a $\tilde{\mathbb{P}}$-martingale, the claim follows.

(ii) Let $f(t,x) = e^x$ where

$$
\frac{\partial f}{\partial t} = 0, \quad \frac{\partial f}{\partial x} = e^x, \quad \frac{\partial^2 f}{\partial x^2} = e^x
$$

and define

$$
X(t) = \left(r - \frac{1}{2}\sigma^2\right)t + \sigma\tilde{W}(t).
$$

Then,

$$
\begin{aligned}
dY(t) &= df(t, X(t)) \\
&= Y(t)dX(t) + \frac{1}{2}Y(t)dX(t)dX(t) \\
&= \left(r - \frac{1}{2}\sigma^2\right)Y(t)dt + \sigma Y(t)d\tilde{W}(t) + \frac{1}{2}\sigma^2 dt \\
&= rY(t)dt + \sigma Y(t)d\tilde{W}(t) \qquad \text{(q.e.d.)}.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
d(D(t)Y(t)) &= D(t)dY(t) + Y(t)dD(t) + dD(t)dY(t) \\
&= rD(t)Y(t)dt + \sigma D(t)Y(t)d\tilde{W}(t) - rD(t)Y(t)dt \\
&= \sigma D(t)Y(t)d\tilde{W}(t) \qquad \text{(q.e.d.)}.
\end{aligned}
$$

Since the differential contains no dt-term, it follows that $D(t)Y(t)$ is a $\tilde{\mathbb{P}}$-martingale. Finally, from the second definition of $S(t)$, we obtain the differential form by applying the Itô product rule

$$
\begin{aligned}
dS(t) &= S(0)dY(t) + Y(t)d\left(\int_0^t \frac{a}{Y(s)}ds\right) + \left(\int_0^t \frac{a}{Y(s)}ds\right)dY(t) + dY(t)d\left(\int_0^t \frac{a}{Y(s)}ds\right) \\
&= \left(S(0) + \int_0^t \frac{a}{Y(s)}ds\right)dY(t) + Y(t)\frac{a}{Y(t)}dt \\
&= r\left(S(0) + \int_0^t \frac{a}{Y(s)}ds\right)Y(t)dt + \sigma\left(S(0) + \int_0^t \frac{a}{Y(s)}ds\right)Y(t)d\tilde{W}(t) + adt \\
&= rS(t)dt + \sigma S(t)d\tilde{W}(t) + adt \qquad \text{(q.e.d.)}.
\end{aligned}
$$

We could have directly obtained the solution to the linear stochastic differential equation for $S(t)$ using the (yet to be obtained) result from Exercise 6.1. Using the notation of that exercise, we have

$$
dS(t) = (a(t) + b(t)X(t))du + (\gamma(t) + \sigma(t)S(t))d\tilde{W}(t)
$$

where

$$
a(t) = a, \quad b(t) = r, \quad \gamma(t) = 0, \quad \sigma(t) = \sigma.
$$

Let

$$
\begin{aligned}
A(t) &= \exp\left\{\int_0^t \sigma(v)d\tilde{W}(v) + \int_0^t \left(b(v) - \frac{1}{2}\sigma^2(v)dv\right)\right\} \\
&= \exp\left\{\sigma\tilde{W}(t) + \left(r - \frac{1}{2}\sigma^2\right)t\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
B(t) &= S(0) + \int_0^t \frac{a(v) - \sigma(v)\gamma(v)}{A(v)}dv + \int_0^t \frac{\gamma(v)}{Z(v)}d\tilde{W}(v) \\
&= S(0) + \int_0^t \frac{a}{A(v)}dv.
\end{aligned}
$$

Then the solution to the stochastic differential equation is

$$
\begin{aligned}
S(t) &= A(t)B(t) \\
&= S(0)A(t) + A(t)\int_0^t \frac{a}{A(v)}dv.
\end{aligned}
$$

This coincides with the result that we just proved.

(iii) Following the hint, we get

$$
\begin{aligned}
\tilde{\mathbb{E}}\left[S(T)|\,\mathcal{F}(t)\right] &= \tilde{\mathbb{E}}\left[S(0)Y(T) + Y(T)\int_0^T \frac{a}{Y(s)}ds\,\middle|\,\mathcal{F}(t)\right] \\
&= S(0)\tilde{\mathbb{E}}\left[Y(T)|\,\mathcal{F}(t)\right] + \tilde{\mathbb{E}}\left[Y(T)|\,\mathcal{F}(t)\right]\int_0^t \frac{a}{Y(s)}ds \\
&\quad + \tilde{\mathbb{E}}\left[Y(T)\int_t^T \frac{a}{Y(s)}ds\,\middle|\,\mathcal{F}(t)\right] \\
&= S(0)\tilde{\mathbb{E}}\left[Y(T)|\,\mathcal{F}(t)\right] + \tilde{\mathbb{E}}\left[Y(T)|\,\mathcal{F}(t)\right]\int_0^t \frac{a}{Y(s)}ds \\
&\quad + a\int_t^T \tilde{\mathbb{E}}\left[\frac{Y(T)}{Y(s)}\,\middle|\,\mathcal{F}(t)\right]ds.
\end{aligned}
$$

We further have

$$
\begin{aligned}
\tilde{\mathbb{E}}\left[Y(T)\middle|\mathcal{F}(t)\right] &= \tilde{\mathbb{E}}\left[\exp\left\{\left(r-\frac{1}{2}\sigma^2\right)T+\sigma\tilde{W}(t)\right\}\right] \\
&= \tilde{\mathbb{E}}\left[Y(t)\exp\left\{\left(r-\frac{1}{2}\sigma^2\right)(T-t)+\sigma\left(\tilde{W}(T)-\tilde{W}(t)\right)\right\}\middle|\mathcal{F}(t)\right] \\
&= Y(t)e^{r(T-t)}\tilde{\mathbb{E}}\left[\exp\left\{-\frac{1}{2}\sigma^2(T-t)+\sigma\left(\tilde{W}(T)-\tilde{W}(t)\right)\right\}\right] \\
&= Y(t)e^{r(T-t)}\tilde{\mathbb{E}}\left[\exp\left\{-\frac{1}{2}\sigma^2(T-t)+\sigma\tilde{W}(T-t)\right\}\right] \\
&= Y(t)e^{r(T-t)}.
\end{aligned}
$$

The third equality uses that $Y(t)$ is $\mathcal{F}(t)$-measurable and that the increment $\tilde{W}(T)-\tilde{W}(t)$ is independent of the σ-algebra $\mathcal{F}(t)$ to drop the conditioning. In the fourth equality we exploit the time homogeniety of the Brownian motion, i.e. that the law of $\tilde{W}(T)-\tilde{W}(t)$ under $\tilde{\mathbb{P}}$ is the same as that of $\tilde{W}(T-t)$. Finally, we use that

$$
e^{-rt}Y(t) = \exp\left\{-\frac{1}{2}\sigma^2 t-\sigma\tilde{W}(t)\right\}
$$

is a $\tilde{\mathbb{P}}$-martingale with an initial value of one as shown in (ii). Next, for $t \le s \le T$ and using similar steps we have

$$
\begin{aligned}
\tilde{\mathbb{E}}\left[\frac{Y(T)}{Y(s)}\middle|\mathcal{F}(t)\right] &= \tilde{\mathbb{E}}\left[\exp\left\{\left(r-\frac{1}{2}\sigma^2\right)(T-s)+\sigma\left(\tilde{W}(T)-\tilde{W}(s)\right)\right\}\right] \\
&= e^{r(T-s)}\tilde{\mathbb{E}}\left[\exp\left\{-\frac{1}{2}\sigma^2(T-s)+\sigma\tilde{W}(T-s)\right\}\right] \\
&= e^{r(T-s)}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\tilde{\mathbb{E}}\left[S(T)\middle|\mathcal{F}(t)\right] &= S(0)Y(t)e^{r(T-t)}+Y(t)e^{r(T-t)}\int_0^t\frac{a}{Y(s)}ds+a\int_t^T e^{r(T-s)}ds \\
&= S(t)e^{r(T-t)}-\frac{a}{r}e^{r(T-s)}\Big|_{s=t}^{s=T} \\
&= S(t)e^{r(T-t)}-\frac{a}{r}\left(1-e^{r(T-t)}\right).
\end{aligned}
$$

(iv) Let $f(t,x)=xe^{r(T-t)}-\frac{a}{r}\left(1-e^{r(T-t)}\right)$ where

$$
\frac{\partial f}{\partial t} = -rxe^{r(T-t)}-ae^{r(T-t)}, \quad \frac{\partial f}{\partial x} = e^{r(T-t)}, \quad \frac{\partial^2 f}{\partial x^2} = 0.
$$

An application of Itô's lemma yields the differential

$$
\begin{aligned}
& d\left(S(t)e^{r(T-t)} - \frac{a}{r}\left(1-e^{r(T-t)}\right)\right) \\
= \; & \left(-rS(t)e^{r(T-t)} - ae^{r(T-t}\right)dt + rS(t)e^{r(T-t)}dt + \sigma S(t)e^{r(T-t)}d\tilde{W}(t) + ae^{r(T-t)}dt \\
= \; & \sigma S(t)e^{r(T-t)}d\tilde{W}(t).
\end{aligned}
$$

Since the differential contains no dt-term it follows that $\tilde{\mathbb{E}}\left[S(T)|\,\mathcal{F}(t)\right]$ is a martingale under $\tilde{\mathbb{P}}$. This is also obvious by letting $s \le t \le T$ and directly computing

$$
\begin{aligned}
\tilde{\mathbb{E}}\left[S(t)e^{r(T-t)} - \frac{a}{r}\left(1-e^{r(T-t)}\right)\middle|\,\mathcal{F}(s)\right] &= \tilde{\mathbb{E}}\left[\tilde{\mathbb{E}}\left[S(T)|\,\mathcal{F}(t)\right]\middle|\,\mathcal{F}(s)\right] \\
&= \tilde{\mathbb{E}}\left[S(T)|\,\mathcal{F}(s)\right] \\
&= S(s)e^{r(T-s)} - \frac{a}{r}\left(1-e^{r(T-s)}\right) \qquad \text{(q.e.d.)}.
\end{aligned}
$$

(v) This result immediately follows from the definition of the value of the forward contract. We have

$$
\tilde{\mathbb{E}}\left[e^{-r(T-t)}\left(S(T) - \mathrm{For}_S(t,T)\right)\middle|\,\mathcal{F}(t)\right] \quad \Leftrightarrow \quad \mathrm{For}_S(t,T) = \tilde{\mathbb{E}}\left[S(T)|\,\mathcal{F}(t)\right] \qquad \text{(q.e.d.)}.
$$

Here, we used that $\mathrm{For}_S(t,T)$ is $\mathcal{F}(t)$-measurable to take it outside the conditional expectation.

(vi) Following the hint, we first compute the differential of the discounted portfolio value

$$
\begin{aligned}
d(D(t)X(t)) &= D(t)dX(t) + X(t)dD(t) + dD(t)dX(t) \\
&= \Delta(t)D(t)dS(t) - a\Delta(t)D(t)dt + rD(t)(X(t) - \Delta(t)S(t))dt - rD(t)X(t)dt \\
&= \Delta(t)D(t)(dS(t) - rS(t)dt) - a\Delta(t)D(t)dt \\
&= \Delta(t)\sigma d(D(t)S(t)) - a\Delta(t)D(t)dt.
\end{aligned}
$$

Integrating and using that $\Delta(t) = 1$ for all $0 \le t \le T$ yields

$$
\begin{aligned}
D(T)X(T) &= \sigma \int_0^T d(D(t)S(t)) - a \int_0^T D(t)dt \\
&= \sigma D(t)S(t)|_{t=0}^{t=T} - a \int_0^T e^{-rt}dt \\
&= D(T)S(T) - S(0) + \frac{a}{r}\left(e^{-rT} - 1\right).
\end{aligned}
$$

Solving or $X(T)$ yields

$$
\begin{aligned}
X(T) &= S(T) - e^{rT}\left(S(0) - \frac{a}{r}\left(e^{-rT} - 1\right)\right) \\
&= S(T) - \left(S(0) - \frac{a}{r}\left(1 - e^{rT}\right)\right) \\
&= S(T) - \mathrm{For}_S(0,T) \qquad \text{(q.e.d.)}.
\end{aligned}
$$