Structural reliability for structural engineers evaluating and strengthening a tall building

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SUMMARY

This paper addresses the topic of evaluating and strengthening a tall building in the Los Angeles region. Failure is characterized by the reliability index in terms that can be readily understood by structural engineers with only a basic knowledge of probability theory. The presented formulation requires the structural engineer to believe that the assumption of a normal or log-normal probability density function for capacity and demand is acceptable for the evaluation and strengthening of a tall building in the Los Angeles region. Copyright © 2012 John Wiley & Sons, Ltd.

1. INTRODUCTION

We are about to start a journey. Like a vacation, we must have a starting 'city' and an ending 'city'. In this paper, our starting city is called 'transparency' and our ending city is called 'limit state design capacity'. The first leg of our journey is along a path we can call the 'normal capacity and demand trail'. It is paved with math and has several valleys and mountains to traverse that require effort. At the end of this first leg, we reach the first city, and it is called 'normal design capacity'. We then start the second leg of the journey, and this path is called 'log-normal capacity and demand trail'. As with the first leg, it has math, valleys and mountains. At the end of this leg, we reach our second city, and it is called 'log-normal design capacity'. Our third leg is perhaps the most difficult, and its trail is called 'professional judgment'. On this trail, we compare the first two cities by using our education and training and make some decisions by using our judgment and typically less information than we would desire. Finally, we reach our destination, and it is a city called 'limit state design capacity'.

The formulation in this paper is viewed as a direct extension of the over half century old tradition of a safety factor used by structural engineers. The references listed at the end of the paper provide examples of published literature for the interested reader. These references are: Ang and Cornell (1974), Ang and Tang (1990), Ellingwood (1994), Ellingwood (2000), Ellingwood *et al.* (1982), Freudenthal (1947), Freudenthal (1956), Freudenthal *et al.* (1957), Hart (1982), Melchers (2002), and Rosenblueth and Esteva (1972). In fact, there are mathematically more sophisticated structural reliability methods available, but what is being proposed is focused on the goal of performance-based design with transparency and professional structural engineering input for a specific building. The proposed approach does not preclude such approaches if so desired, and the mathematics and assumptions are understood and accepted by the structural engineer.

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Structural engineers very often are turned off to structural reliability theory because they either lack the mathematical background, do not want to put in the effort to learn it or leave the definitions of failure and reliability index that have formed the basis of the building code for decades to others because they do not care to learn. The intent of this paper is to present the concept of failure in structural reliability terms that can be easily understood by any college graduate in structural engineering. The reader is also referred to Hart (2012a, 2012b).

Structural reliability analysis requires the definition of the limit states to be addressed and the random variables that are used to formulate the limit states. Very often, it involves assuming probability density functions for all random variables. Selecting and defining the 'best' probability density function (e.g. normal or log-normal distribution, truncated log-normal and beta) are not simple, and selections are often based on mathematical convenience without regard to the need for the end user to really understand all of the mathematics and assumptions involved. This paper presents structural reliability analysis methods for limit states with either normal or log-normal capacity and demand terms. Assuming either normal or log-normal random variables for capacity and demand will satisfy the needs of a structural engineer performing an evaluation and strengthening of an existing tall building in the Los Angeles region. The reader is referred to Hart (2012a, 2012b) for discussion of normal and log-normal random variables.

Failure in this paper is defined in one of the two ways, which are

$$F = C - D < 0 \qquad \text{(safety margin formulation)} \qquad (1)$$

or

$$F = C/D < 1 \qquad \text{(safety factor formulation)} \qquad (2)$$

where C = capacity term for a given limit state D = demand term for a given limit state.

Note that this is a focus on one limit state and the failure of that limit state. Failure for the structure in total must consider the failure of all limit states.

Equation (1) is desirable when C and D are assumed to be jointly normal random variables because then, F is a normal random variable. Equation (2) is desirable when C and D are assumed to be jointly log-normal random variables because then, F is a log-normal random variable or, equivalently, $Z = \ell n(F)$ is a normal random variable.

2. THE STRUCTURAL RELIABILITY INDEX USING $F = C - D$ (CASE IN WHICH C AND D ARE JOINTLY NORMAL)

Capacity is the ability of the structure or structural member considered to resist the demand imposed on the limit state. Demand, e.g., could be wind or earthquake loading. Capacity is often called strength. Capacity could be a strength/force-based or a deformation-based capacity. For example, the strain when one steel bar reaches a strain value equal to its yield strain. With the current view of good structural engineering being displacement-based design, a deformation-based capacity in terms of strain, displacement or rotation is preferred whenever possible.

Define the following variables and parameters:

C = capacity term of limit state = normal
D = demand term of limit state = normal
$\bar{C}$ = expected (or mean) value of C = $E[C]$
$\bar{D}$ = expected (or mean) value of D = $E[D]$
σ_C^2 = variance of capacity (C)
σ_C = standard deviation of capacity (C)
σ_D^2 = variance of demand (D)
σ_D = standard deviation of demand (D)
ρ_{CD} = statistical correlation coefficient between capacity (C) and demand (D)

Failure occurs if the capacity (C) is less than the demand (D).

Define the failure event as

$$C < D \tag{3}$$

and the 'safety margin', F, as

$$F = C - D \tag{4}$$

Therefore, the failure event can also be expressed as $\{F < 0\}$.

Because the capacity (C) and the demand (D) are jointly normal random variables, the safety margin, F, is also a normal random variable.

The expected (or mean) value of the safety margin, F, is

$$\bar{F} = \bar{C} - \bar{D} \tag{5}$$

and the variance of the safety margin, F, is

$$\sigma_F^2 = \sigma_C^2 + \sigma_D^2 - 2\rho_{CD}\sigma_C\sigma_D \tag{6}$$

The standard deviation of F, σ_F, is the square root of the variance of F.

The probability of failure, p_F, for the case in which C and D are jointly normal is given by

$$p_F = P[C < D] = P[F < 0] = \Phi(-\beta) \tag{7}$$

where $P[\ldots]$ is the probability of the event defined inside the square brackets, $\Phi(\ldots)$ denotes the standard normal cumulative distribution function (of a normal random variable with zero mean and unit standard deviation) and

$$\beta = \frac{\bar{F}}{\sigma_F} = (\bar{C} - \bar{D}) / \sqrt{\sigma_C^2 + \sigma_D^2 - 2\rho_{CD}\sigma_C\sigma_D} \tag{8}$$

is referred to in the literature as the reliability (or safety) index.

As shown in Figure 1, as the reliability index increases, there is less probability that a failure occurs.

Look at Equation (8) closely because the reliability index (β) takes a different form when C and D are jointly log-normal random variables.
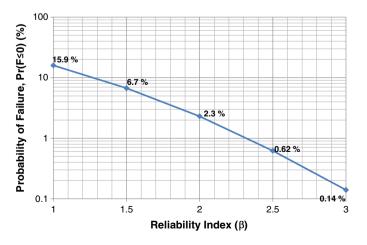

Figure 1. Probability of failure versus reliability index.

Now define the coefficient of variation of the capacity (ρ_C) and coefficient of variation of the demand (ρ_D) to be

$$\rho_C = \sigma_C / \bar{C} \tag{9}$$

from which

$$\sigma_C = \rho_C \bar{C} \tag{10}$$

and

$$\rho_D = \sigma_D / \bar{D} \tag{11}$$

from which

$$\sigma_D = \rho_D \bar{D} \tag{12}$$

Therefore, Equation (8) becomes

$$\beta = \frac{(\bar{C} - \bar{D})}{\sqrt{\rho_C^2 \bar{C}^2 + \rho_D^2 \bar{D}^2 - 2\rho_{CD}\rho_C \bar{C} \rho_D \bar{D}}} \tag{13}$$

Define the 'central safety factor' (β_C) as

$$\beta_C = \bar{C} / \bar{D} \tag{14}$$

from which

$$\bar{D} = \bar{C} / \beta_C \tag{15}$$

Substituting Equation (15) into Equation (13) yields

$$\beta = \frac{[\bar{C} - (\bar{C}/\beta_C)]}{\sqrt{\left[\rho_C^2 \bar{C}^2 + \left(\rho_D^2 \bar{C}^2 / \beta_C^2\right) - 2\rho_{CD}\rho_C \bar{C} \rho_D (\bar{C}/\beta_C)\right]}} \tag{16}$$

and finally,

$$\beta = \frac{[1 - (1/\beta_C)]}{\sqrt{\left[\rho_C^2 + (\rho_D^2/\beta_C^2) - 2(\rho_{CD}\rho_C \rho_D/\beta_C)\right]}} \tag{17}$$

Equation (17) is used to evaluate an existing building limit state when the demand and capacity are jointly normal random variables. If the structural engineer determines or assumes that demand and capacity are uncorrelated (or, equivalently, statistically independent since C and D are jointly normal), then Equation (17) becomes

$$\beta = \frac{1 - (1/\beta_C)}{\sqrt{\rho_C{}^2 + (\rho_D{}^2/\beta_C{}^2)}} = \frac{(\beta_C - 1)}{\sqrt{(\beta_C \rho_C)^2 + \rho_D{}^2}} \tag{18}$$

Using either Equation (17) or (18), the structural engineer can, once ρ_C, ρ_D and ρ_{CD} are determined, calculate the value of the reliability index for a specified value of the central safety factor. Inversely, one can determine the required value of the central safety factor (β_C) for a desired (or target) value of the reliability index. Inverting Equation (18) (for uncorrelated C and D) for β_C gives

$$\beta_C = \frac{1 + \sqrt{\beta^2(\rho_C{}^2 + \rho_D{}^2) - \beta^4 \rho_C{}^2 \rho_D{}^2}}{1 - \beta^2 \rho_C{}^2} \tag{19}$$

Similarly, inverting Equation (17) (for correlated C and D) for β_C yields

$$\beta_C = \frac{1 - \beta^2 \rho_{CD} \rho_C \rho_D + \sqrt{\beta^4 \rho_C{}^2 \rho_D{}^2 (\rho_{CD}^2 - 1) + \beta^2(\rho_C{}^2 + \rho_D{}^2 - 2\rho_{CD}\rho_C\rho_D)}}{1 - \beta^2 \rho_C{}^2} \tag{20}$$

Notice that Equation (20) reduces to Equation (19) when $\rho_{CD} = 0$. Experience has shown that structural engineers relate to and like to adjust their designs by using a target central safety factor. Therefore, to illustrate this, Table 1 reports the required central safety factor β_C for a given value of the reliability index β for the case $\rho_{CD} = 0$. Figure 2 further illustrates this interdependence and show plots of the reliability index (β) versus the central safety factor (β_C). Observe in Figure 2 the significant effect of the statistical correlation ρ_{CD} between the demand and capacity variables for specified values of ρ_D and ρ_C. Recall that a negative value for the correlation coefficient means that values of the demand D larger than the mean of D tend to be correlated with values of the capacity C lower than the mean value of C and vice versa (i.e. the demand tends to increase as the capacity decreases and vice versa). A positive correlation coefficient between C and D is favorable to the safety of the structure, whereas a negative correlation coefficient is unfavorable.

It is not uncommon that structural engineers, at least for preliminary design, select a deterministic value for the demand and, therefore to develop a design, assume no uncertainty in the demand. For example, the service level earthquake can be taken as the 2% damped elastic response spectra for a '50% probability of being exceeded in 30 years' earthquake, or the maximum considered earthquake can be selected as the earthquake with a 2% probability of being exceeded in 50 years. In both of these cases, the structural engineer may assume, out of convenience, certainly not reality, that the coefficient of variation of the demand is zero, i.e. $\rho_D = 0$. In the terminology of probabilistic analysis, this operation of assuming (temporarily) a deterministic value of the demand is referred to as 'conditioning with respect to a specified value of the demand'.

Table 1. Target central safety factors for a target reliability index $\beta = 3.5$ ($\rho_{CD} = 0$).

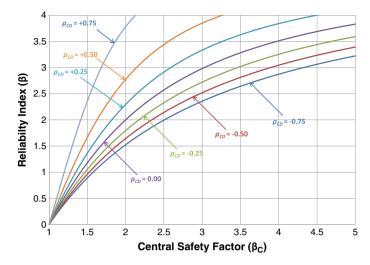| Coefficient of variation of demand ρ_D (%)	Coefficient of variation of capacity ρ_C (%)			
	10	15	20	25
0	1.54	2.11	3.33	8.00
10	1.69	2.21	3.42	8.07
15	1.83	2.33	3.52	8.15
20	1.99	2.48	3.65	8.27
25	2.16	2.64	3.80	8.41
30	2.33	2.81	3.97	8.58
35	2.51	2.99	4.16	8.78
40	2.69	3.18	4.35	8.99

Figure 2. Target reliability index versus target central safety factor for correlated C and D
($\rho_D = 30\,\%$, $\rho_C = 20\,\%$).

If the demand is considered to be known (deterministic) for preliminary design (i.e. not a random variable), then $\bar{D}$ is taken as this deterministic value and $\rho_D = 0$. Therefore, Equation (18) becomes

$$\beta = \frac{[1 - (1/\beta_C)]}{\rho_C} \qquad (21)$$

If we rearrange Equation (21), then the central safety factor can be expressed as

$$\beta_C = (\bar{C}/\bar{D}) = 1/(1 - \beta\rho_C) \qquad (22)$$

Figure 3 shows a plot of the minimum required central safety factor (β_C) as a function of the coefficient of variation of the capacity (ρ_C) and the target reliability index (β) using Equation (22).
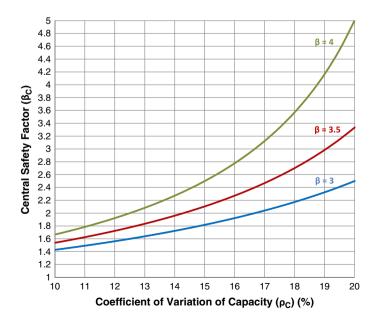

Figure 3. Target central safety factor β_C versus target reliability index β and coefficient of variation of capacity ρ_C.

Now consider the reliability index for uncorrelated C and D ($\rho_{CD} = 0$), i.e. C and D are statistically independent normal random variables. Using Equation (8), we obtain

$$\beta = \text{Reliability Index} = (\bar{F}/\sigma_F) \\ = (\bar{C} - \bar{D})/\sqrt{\sigma_C{}^2 + \sigma_D{}^2} \tag{23}$$

It can be shown, see Hart (2012a, 2012b), that $0.75(\sigma_C + \sigma_D)$ is a good approximation to $\sqrt{\sigma_C{}^2 + \sigma_D{}^2}$ over quite a wide practical range of (σ_C, σ_D), and therefore,

$$\beta = (\bar{C} - \bar{D})/[0.75(\sigma_C + \sigma_D)] \tag{24}$$

from which

$$0.75\beta\sigma_C + 0.75\beta\sigma_D = \bar{C} - \bar{D} \tag{25}$$

and

$$\bar{C}(1 - 0.75\beta\rho_C) = \bar{D}(1 + 0.75\beta\rho_D) \tag{26}$$

Defining the capacity reduction factor as

$$\phi = 1 - 0.75\beta\rho_C \tag{27}$$

and the load amplification factor as

$$\gamma = 1 + 0.75\beta\rho_D \tag{28}$$

it follows that

$$\gamma\bar{D} = \phi\bar{C} \tag{29}$$

Equation (29) is the form of the load and resistance factor design equation expressed in terms of the expected (mean) demand $\bar{D}$ and the expected (mean) resistance $\bar{C}$. Note that it is the expected value of capacity that is multiplied by ϕ and the expected value of demand that is multiplied by γ.

Now define

$$\text{Design Demand} = \gamma\bar{D} \tag{30}$$

$$\text{Design Capacity} = \phi\bar{C} \tag{31}$$

Equation (29) can be rearranged and expressed in terms of the central safety factor as

$$\beta_C = \frac{\bar{C}}{\bar{D}} = \frac{\gamma}{\phi} = \frac{(1 + 0.75\beta\rho_D)}{(1 - 0.75\beta\rho_C)} \tag{32}$$

3. THE PRESCRIBED EARTHQUAKE LOADING APPROACH WITH C AND D NORMAL RANDOM VARIABLES

Now consider the situation where the structural engineer prescribes an earthquake loading to be used for evaluating and strengthening an existing building. Remember that this is performance-based

design, and therefore, there are many limit states, and what follows is to be carried out for each selected limit state.

Recall that

$\bar{D}$ = expected value of demand using a specified exposure time (e.g. 50 years)

The demand from any prescribed earthquake, wind or any source of loading can be expressed as

D_{PL} = demand from prescribed load

Define

$$\alpha = (D_{PL}/\bar{D}) \tag{33}$$

Substituting Equation (33) into Equation (29), it follows that

$$D_{PL} = (\alpha\phi/\gamma)\bar{C} \tag{34}$$

Define the 'prescribed load capacity reduction factor' to be

$$\phi_{PL} \equiv (\alpha\phi/\gamma) \tag{35}$$

and then Equation (34) can be expressed as

$$D_{PL} = \phi_{PL}\bar{C} \tag{36}$$

Substituting Equations (27) and (28) into Equation (35), we obtain

$$\phi_{PL} = \alpha(1 - 0.75\beta\rho_C)/(1 + 0.75\beta\rho_D) \tag{37}$$

It is noteworthy that the left side of Equation (36) is the prescribed load demand and the right side is the design capacity corresponding to this prescribed load demand. The designer develops a design for each selected limit state such that the prescribed load demand, i.e. D_{PL}, is less than the prescribed load capacity, i.e. $\phi_{PL}\bar{C}$.

The above incorporates the ratio α of the demand from the prescribed load, D_{PL}, (e.g. a service level, 50% in 30 years, or ultimate level, 2% in 50 years, earthquake demand) to the expected value of the demand ($\bar{D}$). It also incorporates the uncertainty in the demand by the inclusion of the coefficient of variation of the demand (ρ_D).

To illustrate the use of Equation (37), consider the case where the prescribed load is equivalent to the expected value of the demand, i.e. when $\alpha = 1$. This corresponds approximately to the service level earthquake. Table 2 gives the prescribed load capacity reduction factors (ϕ_{PL}) for different values of α, reliability index (β), coefficient of variation of the capacity (ρ_C) and coefficient of variation of the demand (ρ_D). As the coefficients of variation of demand (ρ_D) and capacity (ρ_C) increase, the prescribed load capacity reduction factor (ϕ_{PL}) decreases.

It is to be emphasized that the value of α is dependent on the prescribed load selected by the structural engineer. The structural engineer has the freedom to select any time frame of interest and any probability of limit state exceedance. The LATBSDC E&S committee currently has selected the '50% in 30 year' earthquake and the '2% in 50 year' earthquake for their procedure. However, the structural engineer may select a shorter exposure time, e.g. 10 years and a 50% probability. This has been shown to be beneficial for a decision maker such as a company president with a finite span of responsibility and perceived accountability, or he or she can select a longer exposure time, e.g. 100 years, and still the 2% probability. This could be viewed as appropriate for landmark buildings such as a new Los Angeles City Hall. The value of α is the prescribed load divided by the expected load for the exposure time of interest. For example, α could be the 2% in 50-year demand divided by the 50% in 50-year demand.

Table 2. Prescribed load capacity reduction factors.

(a) $\alpha = 1.0$ and $\beta = 0.25$

Coefficient of variation of demand ρ_D (%)	Coefficient of variation of capacity ρ_C (%)					
	10	15	20	25	30	35
10	0.96	0.95	0.94	0.94	0.93	0.92
15	0.95	0.95	0.94	0.93	0.92	0.91
20	0.95	0.94	0.93	0.92	0.91	0.90
25	0.94	0.93	0.92	0.91	0.90	0.89
30	0.93	0.92	0.91	0.90	0.89	0.88
35	0.92	0.91	0.90	0.89	0.89	0.88
40	0.91	0.90	0.90	0.89	0.88	0.87

(b) $\alpha = 1.5$ and $\beta = 3.0$

Coefficient of variation of demand ρ_D (%)	Coefficient of variation of capacity ρ_C (%)					
	10	15	20	25	30	35
10	0.95	0.81	0.67	0.54	0.40	0.26
15	0.87	0.74	0.62	0.49	0.36	0.24
20	0.80	0.69	0.57	0.45	0.34	0.22
25	0.74	0.64	0.53	0.42	0.31	0.20
30	0.69	0.59	0.49	0.39	0.29	0.19
35	0.65	0.56	0.46	0.37	0.27	0.18
40	0.61	0.52	0.43	0.35	0.26	0.17

4. THE STRUCTURAL RELIABILITY INDEX USING $F = C/D$ (CASE IN WHICH C AND D ARE JOINTLY LOG-NORMAL)

Now consider the case in which C and D are assumed to be jointly log-normal random variables. This case is the one most often assumed by building load committees. The 'safety factor' is defined as

$$F = (C/D) \tag{38}$$

from which

$$\ell nF = \ell nC - \ell nD \tag{39}$$

Defining

$$Z = \ell nF \tag{40}$$

$$X = \ell nC \tag{41}$$

$$Y = \ell nD \tag{42}$$

it follows that

$$Z = X - Y \tag{43}$$

Because C and D are assumed to be log-normal random variables, it follows that X and Y are jointly normal random variables. Random variable Z, being a linear combination of jointly random variables X and Y, is also normal.

Note that Z is not the ratio of C over D, which is F, but the natural log of F, i.e. $Z = \ell nF = \ell nC - \ell nD$.

The expected (mean) value of Z is

$$\bar{Z} = \bar{X} - \bar{Y} \tag{44}$$

By using the fact that C is log-normal and consequently $X = \ell nC$ is normal, it can be shown (Ang and Tang, 2007) that

$$\bar{X} = \ell n\bar{C} - \left(\sigma_X^2/2\right) \tag{45}$$

and

$$\sigma_X^2 = \ell n\left(1 + \rho_C^2\right) \tag{46}$$

Therefore, substituting Equation (46) into Equation (45), it follows that

$$\bar{X} = \ell n\bar{C} - (1/2)\ell n\left(1 + \rho_C^2\right) \tag{47}$$

Note that the expected value ($\bar{X}$) of the normal random variable $X = \ell nC$ is now expressed in terms of the expected value ($\bar{C}$) and coefficient of variation (ρ_C) of the log-normal random variable C. Similarly to Equation (47), but for the random variable $Y = \ell nD$,

$$\bar{Y} = \ell n\bar{D} - (1/2)\ell n\left(1 + \rho_D^2\right) \tag{48}$$

Also note that the expected value ($\bar{Y}$) of the normal random variable $Y = \ell nD$ is now expressed in terms of the expected value ($\bar{D}$) and coefficient of variation (ρ_D) of the log-normal random variable D. Now, we can write

$$\begin{aligned}\bar{Z} &= \bar{X} - \bar{Y} \\ &= \left[\ell n\bar{C} - \ell n\bar{D}\right] - \left[(1/2)\ell n\left(1 + \rho_C^2\right) - (1/2)\ell n\left(1 + \rho_D^2\right)\right]\end{aligned} \tag{49}$$

and

$$\bar{Z} = \ell n\left[(\bar{C}/\bar{D})\sqrt{\frac{1 + \rho_D^2}{1 + \rho_C^2}}\right] \tag{50}$$

From Equation (14), the central safety factor (β_C) is defined as

$$\beta_c = \bar{C}/\bar{D} \tag{51}$$

so it follows that

$$\bar{Z} = \ell n\left[\beta_c\sqrt{\frac{1 + \rho_D^2}{1 + \rho_C^2}}\right] \tag{52}$$

The above equation expresses the expected value of the normal random variable Z in terms of the expected values, $\bar{C}$ and $\bar{D}$, and the coefficients of variation, ρ_C and ρ_D, of jointly log-normal random variables C and D.

The variance of Z is

$$\sigma_Z^2 = \sigma_X^2 + \sigma_Y^2 - 2\rho_{XY}\sigma_X\sigma_Y$$
$$= \ell n\left(1 + \rho_C^2\right) + \ell n\left(1 + \rho_D^2\right) - 2\rho_{XY}\sqrt{\ell n\left(1 + \rho_C^2\right)}\sqrt{\ell n\left(1 + \rho_D^2\right)} \tag{53}$$

The reliability index is now

$$\beta = \frac{\bar{Z}}{\sigma_Z} = \frac{\ell n\left[\beta_c\sqrt{\frac{1+\rho_D^2}{1+\rho_C^2}}\right]}{\sqrt{\ell n\left(1 + \rho_C^2\right) + \ell n\left(1 + \rho_D^2\right) - 2\rho_{\ell nC,\ell nD}\sqrt{\ell n\left(1 + \rho_C^2\right)}\sqrt{\ell n\left(1 + \rho_D^2\right)}}} \tag{54}$$

Inversely, one can determine the required value of the central safety factor (β_C) for a desired value of the reliability index (β). Inverting Equation (54) (for correlated *C* and *D*) for β_C gives

$$\beta_C = \frac{\exp\left(\beta\sqrt{\ell n\left(1 + \rho_C^2\right) + \ell n\left(1 + \rho_D^2\right) - 2\rho_{\ell nC,\ell nD}\sqrt{\ell n\left(1 + \rho_C^2\right)}\sqrt{\ell n\left(1 + \rho_D^2\right)}}\right)}{\sqrt{\left(1 + \rho_D^2\right)/\left(1 + \rho_C^2\right)}} \tag{55}$$

Now, assuming that *C* and *D* are statistically uncorrelated random variables (or, equivalently, statistically independent since *C* and *D* are jointly log-normal), it follows that the variance of *Z* is

$$\sigma_Z^2 = \sigma_X^2 + \sigma_Y^2$$
$$= \ell n\left(1 + \rho_C^2\right) + \ell n\left(1 + \rho_D^2\right) = \ell n\left[\left(1 + \rho_C^2\right)\left(1 + \rho_D^2\right)\right] \tag{56}$$

So the reliability index in Equation (54) reduces to

$$\beta = \frac{\bar{Z}}{\sigma_Z} = \frac{\ell n\left[\beta_C\sqrt{\frac{1+\rho_D^2}{1+\rho_C^2}}\right]}{\sqrt{\ell n\left[\left(1 + \rho_C^2\right)\left(1 + \rho_D^2\right)\right]}} \tag{57}$$

The probability of failure, p_F, for the case in which *C* and *D* are jointly log-normal, is given by

$$p_F = P\left[\frac{C}{D} < 1\right] = P\left[\ell n\left(\frac{C}{D}\right) < 0\right] = P[Z < 0] = \Phi(-\beta) \tag{58}$$

When we were considering the case in which *C* and *D* are jointly normal random variables, the reliability index was $\beta = \bar{F}/\sigma_F$ where $F = C - D$, but here, the reliability index is $\bar{Z}/\sigma_Z$, where $Z = \ell nC - \ell nD$. Also, note that the reliability index is a different expression of $\bar{C}, \bar{D}, \rho_C$ and ρ_D than in the case in which *C* and *D* are jointly normal, compare Equations (16) and (54) for *C* and *D* correlated and Equations (18 and 57) for *C* and *D* uncorrelated.

Using the first-order approximation $\ln(1 + V) \cong V$, we have

$$\ln\left(1 + \rho_C^2\right) \cong \rho_C^2$$
$$\ln\left(1 + \rho_D^2\right) \cong \rho_D^2 \tag{59}$$

First, consider the numerator of the right-hand side of Equation (57), which is

$$\bar{Z} = \ln\left\{ [\bar{C}/\bar{D}]\sqrt{\frac{1 + \rho_D^2}{1 + \rho_C^2}} \right\} \tag{60}$$

Therefore,

$$\bar{Z} \cong \ln(\bar{C}/\bar{D}) - \left(\rho_C^2/2\right) + \left(\rho_D^2/2\right)$$
$$= \ln(\bar{C}/\bar{D}) + \left(^1/_2\right)\left[\rho_D^2 - \rho_C^2\right] \tag{61}$$

Using a first-order approximation, it follows that

$$(1/2)\left[\rho_D^2 - \rho_C^2\right] \approx 0 \tag{62}$$

and therefore,

$$\bar{Z} \cong \ln(\bar{C}/\bar{D}) \tag{63}$$

Using Equations (53) and (59) and assuming $\rho_{XY} = 0$, we obtain

$$\sigma_Z^2 = \sigma_X^2 + \sigma_Y^2 = \ln\left(1 + \rho_C^2\right) + \ln\left(1 + \rho_D^2\right) \cong \rho_C^2 + \rho_D^2 \tag{64}$$

The standard deviation of Z is the square root of the sum of squares of the coefficients of variation of C and D, and it has no units.

The reliability index in Equation (57) then becomes

$$\beta = (\bar{Z}/\sigma_Z) \cong [\ln(\bar{C}/\bar{D})]/\sqrt{\rho_C^2 + \rho_D^2} \tag{65}$$

It can be shown that $0.75(\rho_D + \rho_C)$ is a good approximation to $\sqrt{\rho_D^2 + \rho_C^2}$ over quite a wide range of (ρ_C, ρ_D), and therefore,

$$\beta = \ln(\bar{C}/\bar{D})/[0.75(\rho_C + \rho_D)] = \ln(\beta_C)/[0.75(\rho_C + \rho_D)] \tag{66}$$

and then,

$$0.75\beta\rho_C + 0.75\beta\rho_D = \ln(\bar{C}/\bar{D}) \tag{67}$$

from which

$$\bar{D}\exp[0.75\beta\rho_D] = \bar{C}\exp[-0.75\beta\rho_C] \tag{68}$$

Defining the capacity reduction factor as

$$\phi = \exp[-0.75\beta\rho_C] \tag{69}$$

and the load amplification factor as

$$\gamma = \exp[0.75\beta\rho_D] \tag{70}$$

it follows that

$$\gamma\bar{D} = \phi\bar{C} \tag{71}$$

Equation (71) is the form of the load and resistance factor design equation expressed in terms of the expected (mean) demand $\bar{D}$ and expected (mean) resistance $\bar{C}$. Note that it is the expected value of capacity that is multiplied by ϕ and the expected value of demand that is multiplied by γ.

It is very important to note that Equation (71) is the same as Equation (29), but Equations (69) and (70) are not the same as Equations (27) and (28).

Now, as before, define

$$\text{Design Demand} = \gamma\bar{D} \tag{72}$$

$$\text{Design Capacity} = \phi\bar{C} \tag{73}$$

Therefore, Equation (71) can be rearranged and expressed in terms of the central safety factor as

$$\beta_C = \frac{\bar{C}}{\bar{D}} = \frac{\gamma}{\phi} = \exp[0.75\beta(\rho_C + \rho_D)] \tag{74}$$

Table 3 provides values for the target central safety factor using Equation (74). This table can now be compared with Table 1 to show the impacts of the decision where C and D are normal (Table 1) or log-normal (Table 3).

5. THE PRESCRIBED EARTHQUAKE LOADING APPROACH WITH C AND D LOG-NORMAL RANDOM VARIABLES

As before, from our discussion of the normal C and D, $\bar{D}$ = expected value of demand using a specified exposure time (e.g. 50 years) D_{PL} = demand from prescribed load and repeating Equation (33),

$$\alpha = (D_{PL}/\bar{D})$$

Also, the 'prescribed load capacity reduction factor' from Equations (71) and (35) is

$$\phi_{PL} \equiv (\alpha\phi/\gamma)$$

Equation (71) can be rewritten as was carried out to obtain Equation (36) to obtain

$$D_{PL} = \phi_{PL}\bar{C}$$

By using ϕ and γ from Equations (69) and (70), respectively, it follows that

Table 3. Target central safety factor for target reliability index of 3.5.

Coefficient of variation of demand ρ_D (%)	Coefficient of variation of capacity ρ_C (%)					
	10	15	20	25	30	35
10	1.69	1.93	2.20	2.51	2.86	3.26
15	1.93	2.20	2.51	2.86	3.26	3.72
20	2.20	2.51	2.86	3.26	3.72	4.24
25	2.51	2.86	3.26	3.72	4.24	4.83
30	2.86	3.26	3.72	4.24	4.83	5.51
35	3.26	3.72	4.24	4.83	5.51	6.28
40	3.72	4.24	4.83	5.51	6.28	7.16

$$\phi_{PL} = \alpha[\exp(-0.75\beta\rho_C)] / \exp[0.75\beta\rho_D] \\ = \alpha\exp[-0.75\beta(\rho_C + \rho_D)] \tag{75}$$

Equation (75) gives the prescribed load capacity reduction factor for log-normal *C* and *D*. It is important to note that Equation (75) is different from Equation (37), which applies for normal *C* and *D*.

6. COMPARISON OF NORMAL AND LOG-NORMAL CASES

With all of the math that preceded this part of the paper, it seems appropriate to take a step back and to summarize the key points. It is especially important to reflect on the impact of the decision whether the capacity and demand are either normal or log-normal random variables.

6.1. Reliability index

Figure 4 shows the plot of the reliability index with varying β_C and ρ_{CD} using the following equations for the normal and log-normal cases.

6.1.1. Normal random variables

$$\beta = \frac{[1 - (1/\beta_C)]}{\sqrt{[\rho_C^2 + (\rho_D^2/\beta_C^2) - 2(\rho_{CD}\rho_C\rho_D/\beta_C)]}} \tag{76}$$

6.1.2. Log-normal random variables

$$\beta = \frac{\bar{Z}}{\sigma_Z} = \frac{\ell n\left[\beta_c\left[\sqrt{(1 + \rho_D^2)/(1 + \rho_C^2)}\right]\right]}{\sqrt{\ell n(1 + \rho_C^2) + \ell n(1 + \rho_D^2) - 2\rho_{\ell nC,\ell nD}\sqrt{\ell n(1 + \rho_C^2)}\sqrt{\ell n(1 + \rho_D^2)}}} \tag{77}$$

6.2. Central safety factor

Figure 5 shows the plot of the central safety factor with varying β and ρ_C with assumed $\rho_D = 0\%, \rho_{CD} = \rho_{\ell nC,\ell nD} = 0$ using the following equations for the normal and log-normal cases.
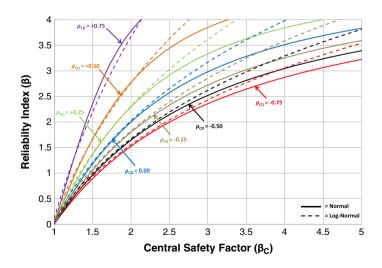

Figure 4. Target reliability index versus target central safety factor for various levels of correlation of
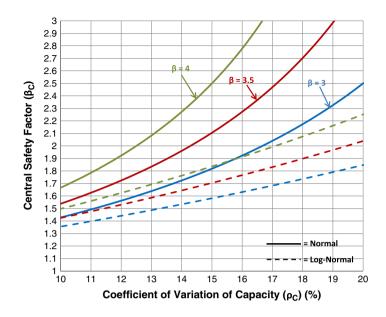*C* and *D* ($\rho_D = 30\%, \rho_C = 20\%$).

Figure 5. Target central safety factor versus coefficient of variation of capacity and target reliability index values $\left(\rho_D = 0\%, \ \rho_{CD} = \rho_{\ell nC, \ell nD} = 0\right)$.

6.2.1. Normal random variables

$$\beta_C = \frac{1 - \beta^2 \rho_{CD} \rho_C \rho_D + \sqrt{\beta^4 \rho_C{}^2 \rho_D{}^2 \left(\rho_{CD}^2 - 1\right) + \beta^2 \left(\rho_C{}^2 + \rho_D{}^2 - 2\rho_{CD} \rho_C \rho_D\right)}}{1 - \beta^2 \rho_C{}^2} \tag{78}$$

6.2.2. Log-normal random variables

$$\beta_C = \frac{\exp\left(\beta \sqrt{\ell n\left(1 + \rho_C^2\right) + \ell n\left(1 + \rho_D^2\right) - 2\rho_{\ell nC, \ell nD} \sqrt{\ell n\left(1 + \rho_C^2\right)} \sqrt{\ell n\left(1 + \rho_D^2\right)}}\right)}{\sqrt{\left(1 + \rho_D^2\right) / \left(1 + \rho_C^2\right)}} \tag{79}$$

It is very important to note that Equations (76)–(79) allow the structural engineer to assign a value for the correlation between capacity and demand and thus account for any correlation.

6.3. Capacity reduction factor and load amplification factor

Following are the equations for the capacity reduction factor and load amplification factor for the normal and log-normal cases. Note that for the derivation of Equations (80)–(85), it was necessary to assume that the capacity and demand are not correlated ($\rho_{CD} = 0$).

6.3.1. Normal random variables

$$\gamma = 1 + 0.75\beta \rho_D \tag{80}$$

$$\phi = 1 - 0.75\beta \rho_C \tag{81}$$

6.3.2. Log-normal random variables

$$\gamma = \exp[0.75\beta \rho_D] \tag{82}$$

$$\phi = \exp[-0.75\beta \rho_C] \tag{83}$$

6.4. Prescribed load capacity reduction factor

Figure 6 shows the plot of the prescribed load capacity reduction factor with varying β and ρ_C with assumed $\rho_D = 30\%$ and $\alpha = 1$ using the following equations for the normal and log-normal cases.
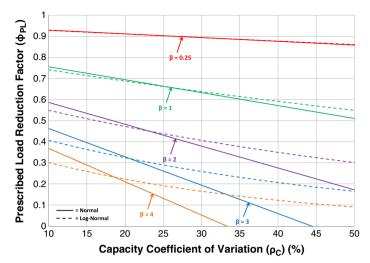
Figure 6. Prescribed load reduction factor as a function of target reliability index and capacity coefficient of variation ($\rho_D = 30\%$, $\alpha = 1$).

6.4.1. Normal random variables

$$\phi_{PL} = \alpha(1 - 0.75\beta\rho_C)/(1 + 0.75\beta\rho_D) \tag{84}$$

6.4.2. Log-normal random variables

$$\begin{aligned} \phi_{PL} &= \alpha[\exp(-0.75\beta\rho_C)]/\exp[0.75\beta\rho_D] \\ &= \alpha\exp[-0.75\beta(\rho_C + \rho_D)] \end{aligned} \tag{85}$$

7. SUMMARY

The step by step procedure to evaluate the prescribed load capacity reduction factor, ϕ_{PL}, is shown in Figure 7 and explained in this section.

Step 1: Select limit state for design such as cracking of concrete, first yield of reinforcing steel or shear failure.

Step 2: Calculate the means of capacity and demand ($\bar{C}$ and $\bar{D}$) of selected limit state and then estimate the coefficients of variation of capacity and demand (ρ_C and ρ_D).

Step 3: Determine the consequences of failure: whether the failure is ductile with average consequences or sudden failures with serious consequences.

Step 4: Select target reliability index (β). For example, β is taken between 3 and 3.5 for ductile failures with average consequences of failure and between 3.5 and 4 for sudden failures with serious consequences. The exposure time can have an impact on the target value selected for β.

Step 5: Determine whether the capacity and demand have a normal or log-normal probability distribution pattern. If the capacity and the demand are determined to have a 'normal' probability distribution pattern from Step 5, then

Step 6: Safety margin, F, is $C - D$, and the failure event can be expressed as $F < 0$.

Step 7: Calculate the expected value of the safety margin, F, which is $\bar{F} = \bar{C} - \bar{D}$.

Step 8: Calculate the standard deviation of the safety margin (σ_F). The standard deviation of the safety margin is $\sigma_F = \sqrt{\sigma_C{}^2 + \sigma_D{}^2} = \sqrt{(\bar{C}\rho_C)^2 + (\bar{D}\rho_D)^2}$ if the capacity and the demand are uncorrelated.
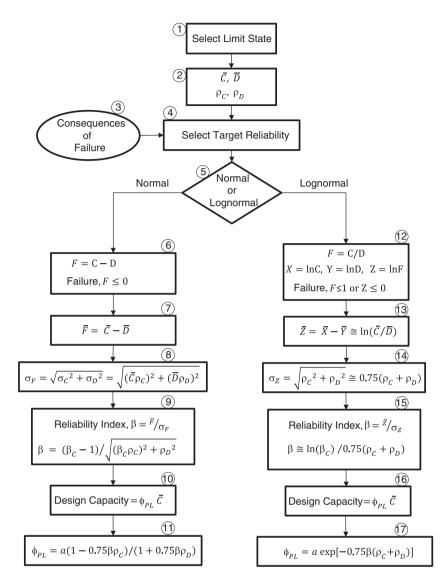
Figure 7. Flow chart for prescribed load capacity reduction factor.

Step 9:   Calculate the reliability index, β, which is

$$\beta = \bar{F}/\sigma_F = (\bar{C} - \bar{D})/\sqrt{(\bar{C}\rho_C)^2 + (\bar{D}\rho_D)^2} = (\beta_C - 1)/\sqrt{(\beta_C\rho_C)^2 + \rho_D{}^2}$$

Steps 10 and 11: Calculate the design capacity for the considered limit state, which is $\phi_{PL}\bar{C}$, where $\phi_{PL} = \alpha(1 - 0.75\beta\rho_C)/(1 + 0.75\beta\rho_D)$.

Step 12:   If the capacity and the demand are determined to have a 'log-normal' probability distribution pattern from Step 5, then safety factor, F, is C/D, and the failure event can be expressed as $F < 1$ From $F = (C/D)$, $\ell n F = \ell n C - \ell n D$ Defining $Z = \ell n F$, $X = \ell n C$ and $Y = \ell n D$, it follows that $Z = X - Y$ and the failure event can be expressed as $Z < 0$ or $\ell n F < 0$.

Step 13:   Calculate the expected value of Z, which is $\bar{Z} = \bar{X} - \bar{Y} \cong \ell n(\bar{C}/\bar{D})$.

Step 14:   Calculate the standard deviation of Z. The standard deviation of Z is $\sigma_Z = \sqrt{\rho_C{}^2 + \rho_D{}^2} \cong 0.75(\rho_C + \rho_D)$ if the capacity and the demand are uncorrelated.

Step 15:   Calculate the reliability index, β, which is

$$\beta = \bar{Z}/\sigma_Z \cong \ell n(\bar{C}/\bar{D})/0.75(\rho_C + \rho_D) = \ell n(\beta_C)/0.75(\rho_C + \rho_D)$$

Steps 16 and 17: Calculate the design capacity for the considered limit state, which is $\phi_{PL}\bar{C}$, where $\phi_{PL} = \alpha \exp[-0.75\beta(\rho_C + \rho_D)]$.

8. LOOKING FORWARD

The capacity reduction factor is one of the three options for evaluation and strengthening but the current preferred approach of the LATBSDC E&S committee for both the service level (50% probability of exceedance in 30 years) and the ultimate level (2% probability of exceedance in 50 years) earthquakes. This approach probably has a long life for the service level earthquake because the structural analysis model is typically a linear analysis model using effective stiffness properties for structural members. However, it is clear that for both nonlinear and linear structural analyses, the demand and capacity cannot by some structural engineers be considered to be independent because the demands (e.g. strains and displacements) are dependent on the limit state capacities. Currently, the minimum computer run time for a well-developed nonlinear time history analysis can be many hours. As this run time decreases, the simulation approach in structural reliability theory (i.e. also called the Monte Carlo approach) will probably become the method of choice for the ultimate level earthquake. This approach is currently used in a low level way since the LATBSDC criterion [Los Angeles Tall Buildings Structural Design Council (2011)] requires multiple nonlinear time history computer analyses using the expected value of the structural parameters in the nonlinear model.

8. CONCLUSION

Transparency has been achieved with the presented development of the capacity reduction factor for demands and capacities that are normal or log-normal random variables. The references that follow are all excellent and present material that formed the foundation of this paper.

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