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STRUCTURAL TOPOLOGY OPTIMIZATION FOR BLAST MITIGATION USING HYBRID CELLULAR AUTOMATA

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ABSTRACT

Design for structural topology optimization is a method of distributing material within a design domain of prescribed dimensions. This domain is discretized into a large number of elements in which the optimization algorithm removes, adds, or maintains the amount of material. The resulting structure maximizes a prescribed mechanical performance while satisfying functional and geometric constraints. Among different topology optimization algorithms, the hybrid cellular automaton (HCA) method has proven to be efficient and robust in problems involving large, plastic deformations. The HCA method has been used to design energy absorbing structures subject to crash impact. The goal of this investigation is to extend the use of the HCA algorithm to the design of an advanced composite armor (ACA) system subject to a blast load. The ACA model utilized consists of two phases: ceramic and metallic. In this work, the proposed algorithm drives the optimal distribution of a metallic phase within the design domain. When the blast pressure wave hits the targeted structure, the fluids kinetic energy is transformed into strain energy (SE) inside the solid medium. Maximum attenuation is reached when SE is maximized. Along with an optimum use of material, this condition is satisfied when SE is uniformly distributed in the design domain. This work makes use of the CONWEP model developed by the Army Research Laboratory. The resulting structure shows the potential of the HCA method when designing ACAs.

1 Introduction

The design of blast mitigating methods for crew survivability in light ground vehicles is a relatively new field of study. Expected loads and injuries sustained during a blast event are drastically different from those seen in an automotive crash event. The recent interest in vehicle blast mitigation has led to a surge in publications concerning all aspects of ground vehicle blast dynamics, from numerical simulations to experimentation and injury assessment. It is the designer's task and responsibility to account for ongoing research in order to capture all aspects of the design problem prior to generating a solution. In this spirit, the design for blast mitigation must be approached in a multidisciplinary manner.

Similar to automotive crash testing, the key factor in determining blast survivability relies on being able to develop injury criteria by which test data are intended to measure. The conversion of quantitative values such as accelerations, strains, or displacements obtained from a numerical solution or experiment to qualitative measures of occupant injury is crucial for the work involved in assessing survivability. Though the field of injury assessment in vehicle blast events is a relatively unexplored territory, there have been extensive publications concerning injuries associated with high vertical accelerations and loading in the helicopter industry. As encountered in a helicopter crash event, a rapid vertical acceleration has been shown to cause serious injury or death in a blast event occurring beneath a ground vehicle.

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Blast worthiness design using optimization techniques presents a new application for a well understood method. The goal of blast mitigation is to maximize passenger safety during a blast event subject to certain geometric and manufacturing constraints [1]. Physical nonlinearities are abundant in the process of a blast event. As with crash worthiness design problems investigated in the past, passenger safety can be interpreted as the objective of controlling the force transferred to the passenger with the geometric constraint of preserving the integrity of the cockpit. A Dynamic Response Index (DRI) has been developed as a criterion by which to measure the survivability of an occupant under a blast load event [2]. The DRI is based on the acceleration of the occupant, specifically the acceleration of the critical regions of the human body (i.e., spinal cord).

The current state of the art in injury assessments due to mechanical loadings involved in a blast event is focused on characterizing the acceleration profiles of the vehicle, critical components and its occupants. The objective for the design of occupant injury mitigation in the following work is therefore taken to be the gross vehicle and substructure level accelerations. By formulating the design problem as a minimization of the blast reaction impulse, the Hybrid Cellular Automata (HCA) optimization algorithm may be implemented in this design problem with the proper local design update rule. The following report details the development and implementation of the HCA algorithm for the design of injury mitigating structures subjected to blast loading.

The objective of this investigation is to see if the HCA method can be applied to the optimization of structures subject to blast loading. Unlike previous crash worthiness problems, the kinetic energy of the blast event is transferred to the passenger by means of a complex fluid structure interaction between the blast wave and the vehicle. The dynamics of such blast events and their interactions with solid structures is an area that has been extensively researched as per the summary given by [3]. Incorporating multidomain optimization techniques described by [4], the topology optimization problem for blast mitigation can be dissected into multiple design domains with specific design objectives. This investigation will look exclusively at the sub-structure level of the vehicle, thus removing the necessity of accounting for the vehicle-level fluid interaction. At this level, the CONWEP algorithm developed by the Army Research Laboratory can be used to simulate the blast loading on the sub-structural design domain.

2 Principles of HCA and crasHCA

The hybrid cellular automaton (HCA) method, and its extension to crashworthiness (crasHCA), combine the basis of the cellular automaton (CA) paradigm, introduced by Stanislaw Ulam and John von Neumann in 1950s [5], and the theory of finite element-based structural optimization, introduced by Lucien Schmit in 1960s [6]. The HCA method presented by Tovar et al. [7, 8] incorporates local updating schemes such as control rules (i.e., on-off, proportional, integral and derivative controllers) [9, 10] and ratio techniques [11]. These local rules drive a defined field variable to an optimum state or set point. The expression for the field variable and the value of the set point are derived from the optimality conditions of the structural design problem [12, 13]. HCA has been applied to a variety of disciplines, including the design of compliant mechanisms [14], non-compliant structures subject to mass, stress, and displacement constraints [15], crash-worthy structures [16, 17], and in bone remodeling [18, 19]. Recently, a proof of the global convergence of the HCA technique, under certain circumstances, to an optimal design has been derived [20].

2.1 Components

An HCA/crasHCA model has three components: (1) a lattice of cells, (2) a set of states for each cell, and (3) a set of rules associated with the set of states. For topology optimization, the lattice of cells is defined in two or three dimensions. For each cell or cellular automaton (CA) in the lattice, there is a set of states which can be described as

$$\underset{\sim}{\alpha}_i = \left[\underset{\sim}{x}_i, \underset{\sim}{y}_i\right], \tag{1}$$

where $\underset{\sim}{x}_i$ are a set of design variables and $\underset{\sim}{y}_i$ are set of state variables for the cell in the i-th discrete location. For each state, there is a corresponding updating rule that defines its evolution over time. In the HCA algorithm, the design variables are modified during the optimization stage (e.g., local control rule) while state variables are updated during the analysis stage (i.e., finite element analysis). The set of updating rules is identical for all the cells and is applied simultaneously in the whole lattice. In other words, any rule is homogeneous, that is, it does not depend on the position of the cell.

The set of local rules operates according to local information collected in the neighborhood of each cell. The neighborhood does not have any restriction on size or location, except that it is the same for all the cells. In practice, the size of the neighborhood is often limited to the adjacent cells but can also be extended. Figures 1 and 2 depict some common neighborhood layouts for two and three dimensional lattices.

To define neighborhoods in the boundary cells, the lattice is virtually extended. The set of states of the virtually added cells determines the boundary conditions of the design domain. Figure 3 depicts some common types of boundary conditions. A fixed boundary is defined so the neighborhood is completed with cells having a pre-assigned fixed state. An adiabatic boundary condition is obtained by duplicating the value of the cell in an extra virtual neighbor. In a reflecting boundary, the state of the opposite neighbor is replicated by the virtual cell. Periodic boundary

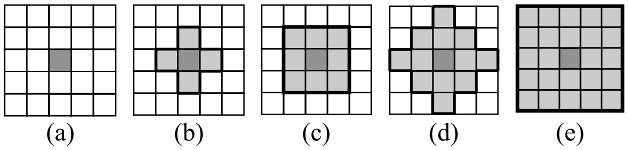

(a) (b) (c) (d) (e)

Figure 1. Neighborhood layouts for a cellular automaton in a 2D lattice. (a) Empty, $\hat{N} = 0$; (b) Von Neumann, $\hat{N} = 4$; (c) Moore, $\hat{N} = 8$; (d) Radial, $\hat{N} = 12$; (e) Extended, $\hat{N} = 24$.

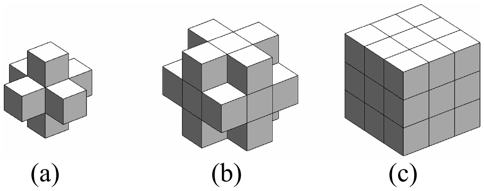
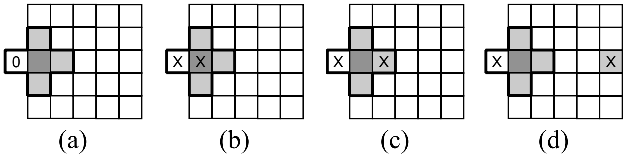

(a) (b) (c)

Figure 2. Neighborhood layouts for a cellular automaton in a 3D lattice. (a) 3D Von Neumann, $\hat{N} = 6$; (b) 3D Radial, $\hat{N} = 18$; (c) 3D Moore, $\hat{N} = 26$.

conditions are used when the design domain is assumed to be wrapped in a torus-like shape. This work makes use of fixed boundary conditions where the extra cells are considered empty spaces without physical or mechanical properties [10].



(a) (b) (c) (d)

Figure 3. Boundary conditions for a cellular automaton. (a) Fixed; (b) Adiabatic; (c) Reflecting; (d) Periodic.

2.2 Global and local formulation

The goal of crashworthiness design is to maximize the energy absorption during a crash event. In addition, strain constraints have to be provided in order to prevent structural failure. For this reason, the maximum effective plastic strain in the structure has to be kept under a limit set by the material's strength.

In a previous investigation, Patel et al. [16] introduced the crasHCA methodology for crashworthiness using the idea of fully stressed design to nonlinear transient events. In their work, the algorithm drives the internal energy density of every cell to a prescribed target or set point. Iteratively, the crasHCA algorithm modifies the stiffness properties of the cells (or field variable, y_i) by varying a defined relative density (or design variable, x_i). The structural problem to be solved can be stated as finding the optimal distribution of material such that local and global constraints are satisfied. The *global formulation*, which refers to the general distribution of material within the design domain, can be stated as

$$\begin{aligned} \text{find} \quad & \underset{\sim}{x} \\ \text{s.t.} \quad & \underset{\sim}{h}(\underset{\sim}{x}) = \underset{\sim}{0} \\ & \underset{\sim}{g}(\underset{\sim}{x}) \leq \underset{\sim}{0} \\ & \underset{\sim}{H}(\underset{\sim}{x}) = \underset{\sim}{0} \\ & \underset{\sim}{G}(\underset{\sim}{x}) \leq \underset{\sim}{0} \\ & x_i \in \{0,1\}, \quad i = 1,\ldots,n, \end{aligned} \tag{2}$$

where $\underset{\sim}{h}$ and $\underset{\sim}{g}$ are sets of local equality and inequality constraints, $\underset{\sim}{H}$ and $\underset{\sim}{G}$ are sets global equality and inequality constraints, and $\underset{\sim}{x}$ is the set of design variables x_i representing presence or absence of material (i.e., binary variables). An example of a local constraint could be given by the strain or nodal displacement. A global constraint could be imposed on the mass of the structure, for example. In general, this problem is over-constrained so it is customary expressed as the minimization of one or more constraint violations. The problem it is usually relaxed so the design variable is allowed to change continuously between 0 (or a small number) and 1.

The *local formulation* refers to the distribution of material within a single cell. In this formulation, only one design variable x_i is modified. Acting in parallel under CA principles, the crasHCA finds the optimal distribution of material across the whole design domain. A simple local problem can be stated as

$$\begin{aligned} \text{find} \quad & x_i \\ \text{s.t.} \quad & y_i(x_i) - y^* = 0 \\ & x_i \in \{0,1\}, \end{aligned} \tag{3}$$

where y_i is the state variable and y^* corresponds its target value. Once the design variable reaches its limits, the equality constraint may be violated. The corresponding relaxed optimization problem can be stated as

$$\begin{aligned} \min_{x_i} \quad & |y_i(x_i) - y^*| \\ \text{s.t.} \quad & 0 \leq x_i \leq 1. \end{aligned} \tag{4}$$

2.3 Updating rule

A local control rule updates the design variable x_i according the following rule

$$x_i(t+1) = \bar{x}_i(t) + f(\bar{y}_i(t)), \tag{5}$$

where t represents a discrete time step or iteration and $\bar{x}_i(t)$ and $\bar{y}_i(t)$ are effective values for the design and the field variables. These effective values can be expressed as

$$\bar{x}_i(t) = \frac{x_i(t) + \sum_{j=1}^{\hat{N}} x_j(t)}{\hat{N}+1}, \tag{6}$$

$$\bar{y}_i(t) = \frac{y_i(t) + \sum_{j=1}^{\hat{N}} y_j(t)}{\hat{N}+1}, \tag{7}$$

where subindex j refers the neighboring cells. For proportional control, the function $f(\bar{y}_i(t))$ in 5 is defined as

$$f(\bar{y}_i(t)) = c_P \times (\bar{y}_i(t) - y^*(t)), \tag{8}$$

where $y^*(t)$ is the set point defined for the field variable $y_i(t)$ and c_P is a constant referred to as the proportional gain. Other updating rules, such as on-off, derivative, integral, and ratio technique are defined in [10]. This work makes use of the proportional control updating rule defined in 8.

3 Proposed Methodology

3.1 Material Model and Design Variables

In the version of the HCA algorithm developed by Patel [17]for use in crash analysis, a piecewise-linear elastic-plastic material model (MAT 024 in LS-DYNA) was used for the analysis. Since the dynamics of a blast event involves a high loading rate combined with high temperatures, the piecewise-linear model utilized in the original algorithm is not sufficient for blast applications. A model that incorporates strain-rate and temperature effects is the Johnson-Cook material model [21]. Johnson-Cook defines a von-mises flow stress according to the relationship

$$\sigma_y = (A + B(\bar{\varepsilon}^p)^n)(1 + C\log\dot{\varepsilon}^*)(1 - (T^*)^m), \tag{9}$$

where A, B, C, n, and m are user defined constants that are dependent on the material being modeled. In the paper by Patel et al. [16] the SIMP penalization modified the elastic modulus, the yield stress. In the case of the Johnson-Cook material model the von-mises flow stress must be modified in place of the yield stress. This can be done by penalizing the material parameters A, B, and C according to the relationship:

$$A = A_0 x^q, \quad B = B_0 x^q, \quad C = C_0 x^q \tag{10}$$

$$x_{min} \leq x \leq 1 \tag{11}$$

where the subscript, 0, refers to the material parameter at full density, q is the penalization parameter, and x is the relative density. The Elastic modulus and the shear modulus are penalized according to:

$$E = E_0 x^p, \quad G = G_0 x^p. \tag{12}$$

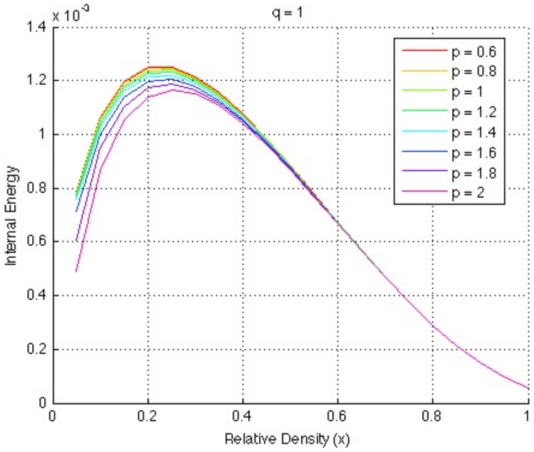
Before taking advantage of this model, it was necessary to perform a study to ensure that it provides the necessary monotonic relationship between SE and element density that allows the optimization algorithm to function. As in the work of Patel et al., a single element was loaded to a fixed value while the density, x, and the penalizations, p and q were varied.

Figure 4 shows that the Johnson-Cook material model does not behave monotonically for the entire range of density. However, as q is decreased, the peak of the curve is pushed to the right. As long as the minimum density allowed in the algorithm is kept above the peak value, then this model is acceptable.

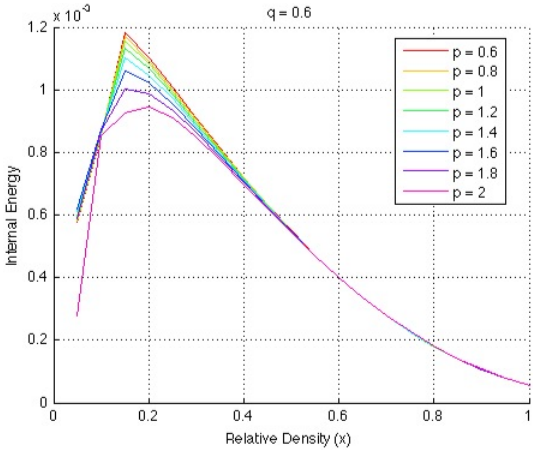
3.2 Blast model and field variables

In recent years the Army Research Laboratory and Raytheon have published works concerning blast and projectile penetration models assuming simple blast loading conditions. The CONWEP model developed by the Army Research Laboratory and currently used in commercial FEA blast simulations is considered adequate for use in engineering studies of vehicle response to the blast from land mines [3, 22]. The CONWEP algorithm does not explicitly simulate the effects of the detonation reaction with air nor does it calculate the shock wave propagation and its reactions with the target structure. For the studies conducted by the Army Research Laboratory concerning blast interactions with an isotropic thick plate (assumed mechanical properties of steel), simplified interaction assumptions adopted by the CONWEP algorithm have proved acceptable. Given the design criteria of the current research in ceramic composites (anisotropic / hexagonal properties) these same assumptions may not be relevant and may require greater consideration. For the purpose of modeling the blast wave effects on a full structure, the balance equations of a compressible fluid will invariably be necessary as well as heat, conservation of energy, and linear momentum. Angular momentum may be considered negligible given this assumption.

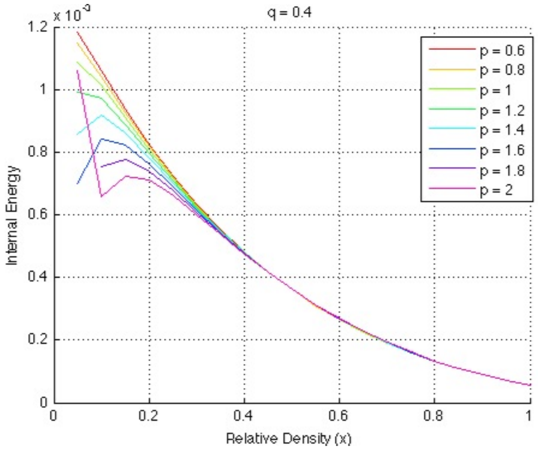
Since the scale of the current application is localized to the substructural level of the vehicle, the need to account for the fluid interaction with the deforming structure is eliminated. For now, the application of blast loading within the HCA framework will be limited to small domains that will comprise a repeating substructure within the overall vehicle system.



(a) q = 1.0

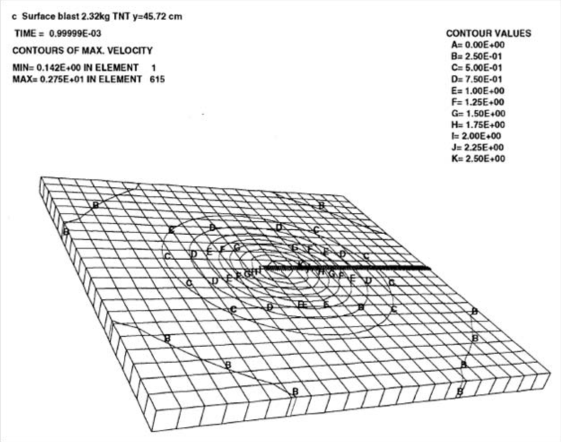


(b) q = 0.8



(c) q = 0.4

Figure 4. Internal energy vs. relative density for varying values of p and q



```
c  Surface blast 2.32kg TNT y=45.72 cm
 TIME =  0.99999E-03
CONTOURS OF MAX. VELOCITY
MIN= 0.142E+00 IN ELEMENT     1
MAX= 0.275E+01 IN ELEMENT   615

CONTOUR VALUES
  A= 0.00E+00
  B= 2.50E-01
  C= 5.00E-01
  D= 7.50E-01
  E= 1.00E+00
  F= 1.25E+00
  G= 1.50E+00
  H= 1.75E+00
  I= 2.00E+00
  J= 2.25E+00
  K= 2.50E+00
```

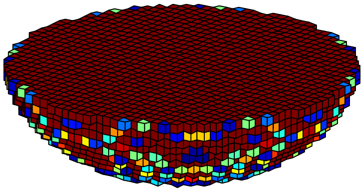
Figure 5. Example output of a surface blast event in air simulated using CONWEP algorithm

The original crasHCA algorithm was developed to design energy absorbing structures. As a field variable, crasHCA utilized the Strain Energy (SE) as the field variable of each element in the FE lattice and attempted to drive the SE to a constant value. This is the concept of a uniformly stressed design and is equivalent to minimizing the maximum strain. In crasHCA, only the value of the SE at the final time step, which was picked sufficiently long to allow for any elastic relaxation, was used, thus accounting for only the plastic deformation in the structure. The time step required to resolve a blast event is on the order of micro-seconds, making it computationally infeasible to run a blast simulation until elastic relaxation occurs. Furthermore, the majority of the damage is done within the first several hundred micro-seconds, and it is this initial period of damage and acceleration that needs to be mitigated, therefore, setting the final time to be on the order of seconds is unnecessary.

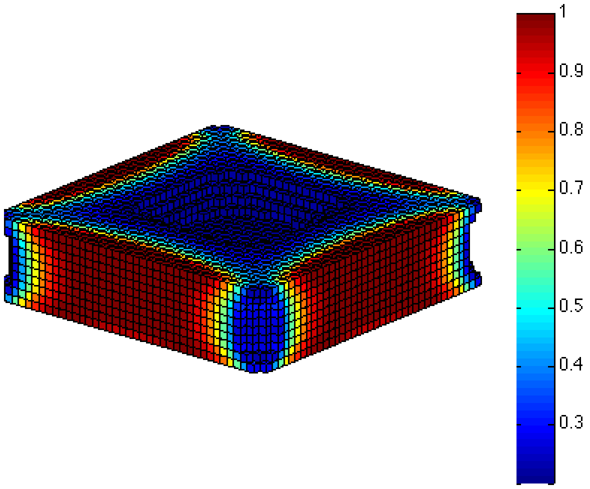
Initial investigations with the CONWEP algorithm have shown that the topology produced by HCA is different depending on the selected simulation end time. Figure 6 illustrates the effect of simulation end time on the resulting topology under the original crasHCA methodology. In order to prevent the end-time dependence of the final topology, a new method is proposed that will take the integral of the SE over the duration of the blast event. This is equivalent to saying

$$S_i = \int_{t=0}^{t=t_f} U_i(t)\,dt, \tag{13}$$

where the subscript i denotes the i^{th} CA and t_f is the final simulation time. Using this methodology, the HCA algorithm should drive the topology towards a design that is approximately fully stressed over the duration of the simulation.



(a) Simulation end time immediately after peak pressure



(b) Simulation end time after 800μs

Figure 6. Effect of simulation end time on final topology of a plate subjected to blast loading (relative element density shown)

4 Implementation: Single Material Optimization

Using the updated version of the HCA algorithm a test scenario was run to study the effect of the integrated SE method for blast mitigation. A 40cm x 40cm x 10cm 7039 aluminum plate was chosen for the design domain to represent the substructure and a 40cm x 40cm x 1cm aluminum plate was placed on the blast-ward face to absorb the CONWEP blast loading (See Figure 7). The plate was fixed in translational degrees of freedom on all sides. The actual model used in LS-DYNA was one quarter of the entire design domain with reflected boundary conditions along the interior faces.

The loading on the model is a CONWEP model given by the *LOAD_BLAST keyword. The center of the blast source is set to be 100 cm from the origin, which corresponds to 99 cm from the surface of the plate (the top of the design domain is set to be at 0 cm). Hourglass control has been added to the model to help alleviate complex sound speed errors.

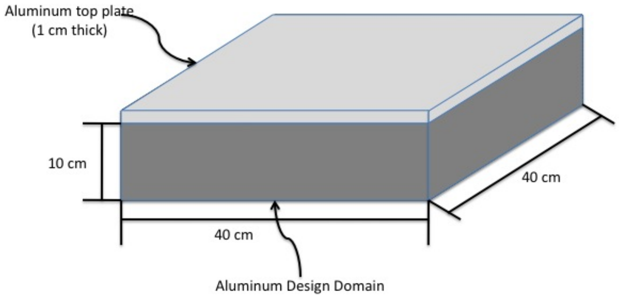


Figure 7. Design domain for HCA CONWEP blast analysis

A baseline model has also been created for comparison. This model is half as thick as the model above (40 cm x 40 cm x 5 cm) and also has the 1 cm thick aluminum layer on the top. The bottom of the comparison model is at the same distance from the blast source as the original model to compare the acceleration of the bottom of the baseline model to the acceleration of the bottom of the HCA model. This allows the HCA topology to be evaluated in terms of the standard injury criterion.

For this experiment a mass target of 50% was set for the aluminum design domain. A mesh size of 0.5 cm^3 elements was selected. Since reducing density in the elements often leads to instability under blast loading, an extra check was added to the algorithm to correct for the appearance of complex sound speeds. When such an error occurs, the algorithm adds a fixed amount of mass to every element in the design domain until the error stops appearing. The algorithm spends more time in the inner mass constraint loop in order to restore the total domain mass to the target value, but it allows the program to continue where it would otherwise fail.

The current implementation ran to 50 iterations without converging. However, Figure 8 shows a promising topology for blast mitigation. As expected, a majority of the mass is concentrated in the center where the peak force of the blast wave impacts the plate. Moving away from the center, the mass is reduced to allow for some crushing of the design domain for the purpose of energy absorption. Figure 9 shows that thicker struts form on the outside of the domain to provide some structural connection to the boundaries. It can also be seen that the regions that look to be of lower density in Figure 8 only penetrate a short way into the design domain. Also, Figure 10 shows that the two low density regions seen in the quarter-domain cut actually represent a larger low-density region surrounding a central column. These low density regions may correspond to regions of crush (and thus energy absorption).

The concept of maximizing the SE absorbed by the domain is to reduce the total energy transferred from the blast to the interior of the vehicle where the occupant is located. If this objective holds true, then there should be a reduction in the acceleration

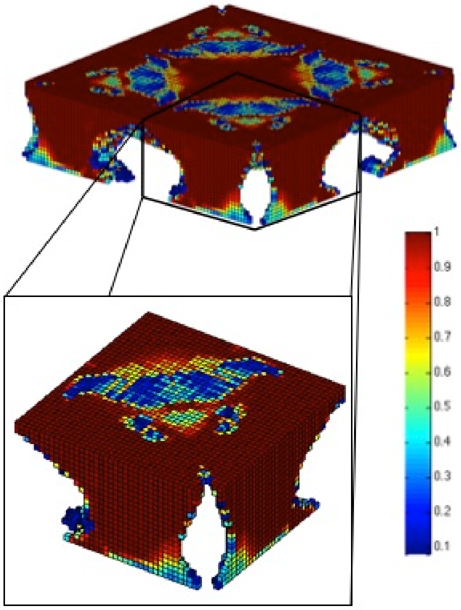


Figure 8. HCA topology after 50 iterations subject to CONWEP blast loading

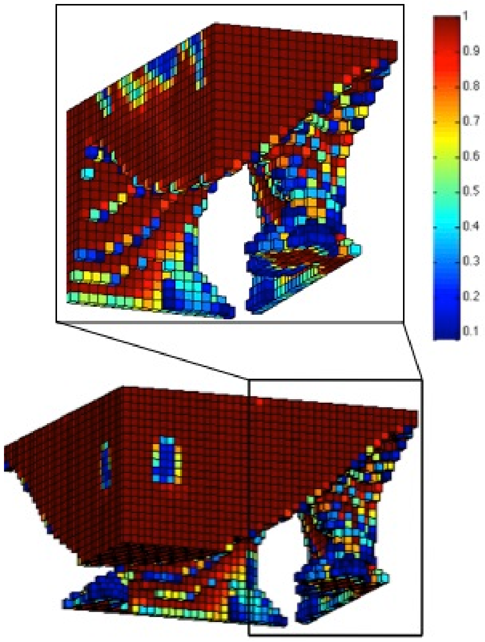


Figure 9. Zoomed in view of HCA topology showing details of the outer portion of the domain

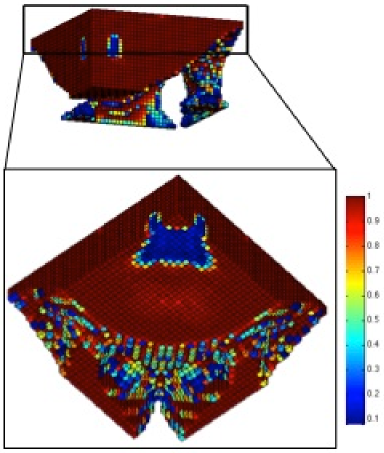


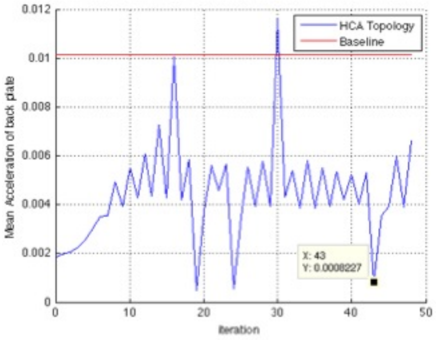
Figure 10. Zoomed in view of the HCA topology showing details of the center of the domain

of the back of the HCA design domain relative to the baseline design. Figure 11 shows the mean and peak accelerations of the two domains per iteration. First, it is clear that the average acceleration of the HCA topology is lower than that of the baseline. It should be noted that every element in the domain starts with half of true density, meaning that the first few iterations are not physical representations of any structure. It is also clear from these plots that the algorithm may never reach a converged design. The addition of mass every time a complex sound speed error arises is the most likely cause for the continuing fluctuation in the performance of the HCA design.

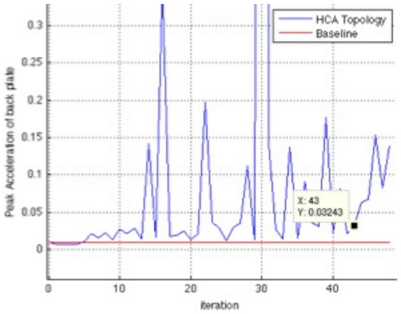
The peak acceleration of the HCA topology is higher than that of the baseline model. This, however, is most likely an artifact of individual nodal accelerations. Figure 12(a) shows that there are a couple of nodes that contribute to the extremely high peak value of iteration 30. These nodes are likely located on or next to elements of extremely low density. Although hourglass control is in effect, these nodes still accelerate beyond what is physical. Iteration 43, which has the lowest acceleration profile, still exhibits the spike nodal acceleration phenomenon (Figure 12(b)).

5 Final Remarks

This investigation found that the HCA algorithm could be modified to produce topologies that helped to mitigate the acceleration transferred to the occupant from a blast loading simulated using the CONWEP model. Although the topology did not converge to a final solution, the initial results showed that the intermediate, unconverged, design showed improvement over the mean acceleration of the baseline model. Future investigations will be conducted to study the effects of mesh refinement and



(a) Mean acceleration



(b) Peak acceleration

Figure 11. Comparison of acceleration of the bottom of the baseline design and HCA topology

multi-material design domains on the topology evolution.

6 Acknowledgments

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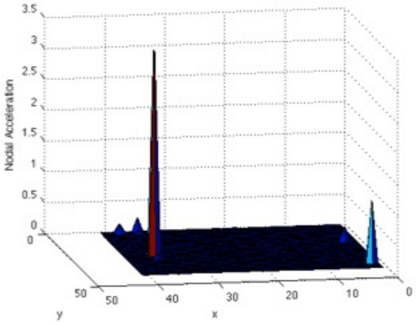
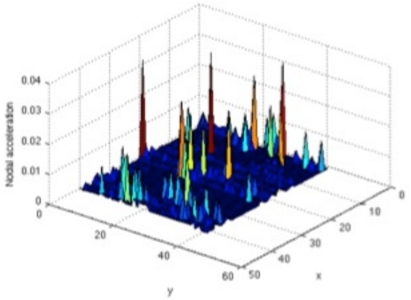


(a) Iteration 30



(b) Iteration 43

Figure 12. Nodal Acceleration profile for iterations 30 and 43

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