PEARSON EDEXCEL INTERNATIONAL A LEVEL

PURE MATHEMATICS 4

Student Book

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ABOUT THIS BOOK

The following three themes have been fully integrated throughout the Pearson Edexcel International Advanced Level in Mathematics series, so they can be applied alongside your learning.

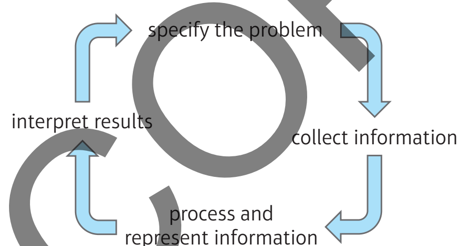
1. Mathematical argument, language and proof

- Rigorous and consistent approach throughout
- Notation boxes explain key mathematical language and symbols

2. Mathematical problem-solving

- Hundreds of problem-solving questions, fully integrated into the main exercises
- Problem-solving boxes provide tips and strategies
- Challenge questions provide extra stretch

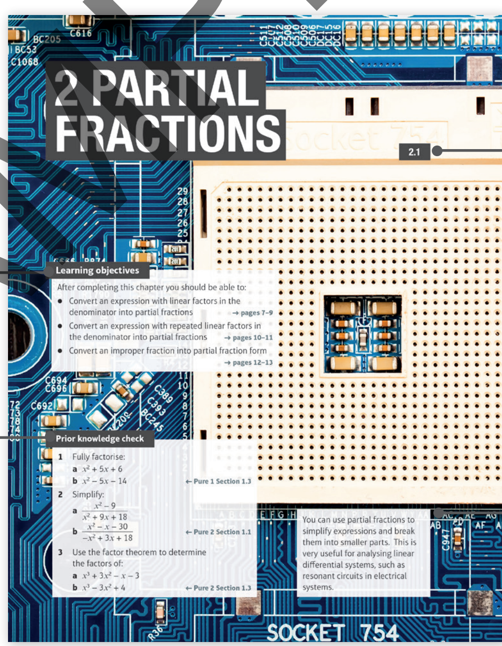
The Mathematical Problem-Solving Cycle



3. Transferable skills

- Transferable skills are embedded throughout this book, in the exercises and in some examples
- These skills are signposted to show students which skills they are using and developing

Finding your way around the book


2 PARTIAL FRACTIONS

2.1

Learning objectives

After completing this chapter you should be able to:
- Convert an expression with linear factors in the denominator into partial fractions → pages 7–9
- Convert an expression with repeated linear factors in the denominator into partial fractions → pages 10–11
- Convert an improper fraction into partial fraction form → pages 12–13

Prior knowledge check

1 Fully factorise:
a $x^2 + 5x + 6$
b $x^2 - 5x - 14$ ← Pure 1 Section 1.3

2 Simplify:
a $\frac{x^2 - 9}{x^2 + 9x + 18}$
b $\frac{x^2 - x - 30}{-x^2 + 3x + 18}$ ← Pure 2 Section 1.1

3 Use the factor theorem to determine the factors of:
a $x^3 + 3x^2 - x - 3$
b $x^3 - 3x^2 + 4$ ← Pure 2 Section 1.3

You can use partial fractions to simplify expressions and break them into smaller parts. This is very useful for analysing linear differential systems, such as resonant circuits in electrical systems.

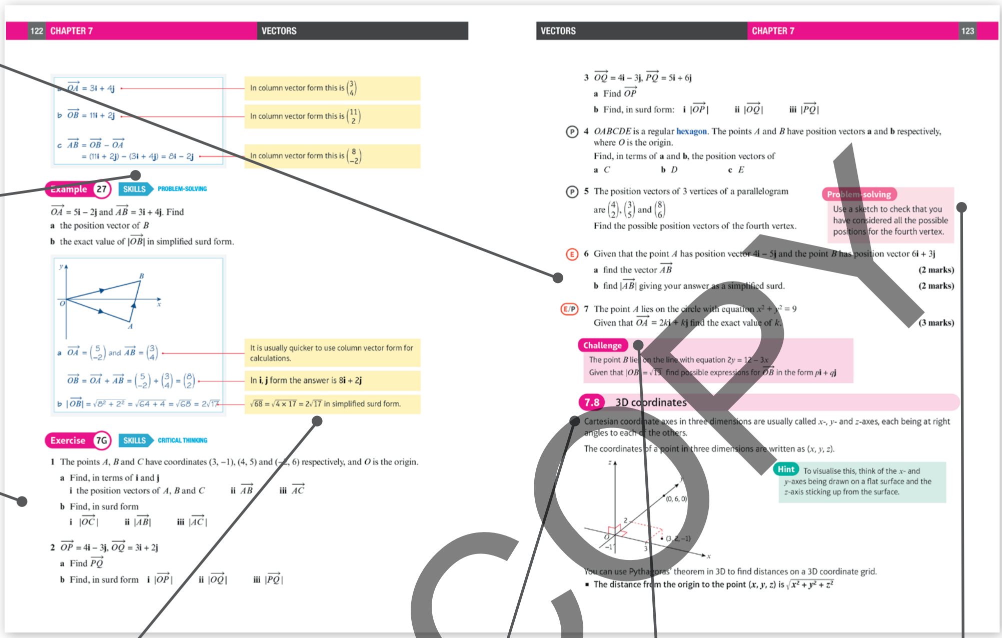
Each chapter is mapped to the specification content for easy reference

Each chapter starts with a list of Learning objectives

The *Prior knowledge check* helps make sure you are ready to start the chapter

The real world applications of the maths you are about to learn are highlighted at the start of the chapter

Glossary terms will be identified by bold blue text on their first appearance



Exercise questions are carefully graded to increase in difficulty and gradually bring you up to exam standard

Transferable skills are signposted where they naturally occur in the exercises and examples

Exercises are packed with exam-style questions to ensure you are ready for the exams

Step-by-step worked examples focus on the key types of questions you'll need to tackle

Each section begins with an explanation and key learning points

Challenge boxes give you a chance to tackle some more difficult questions

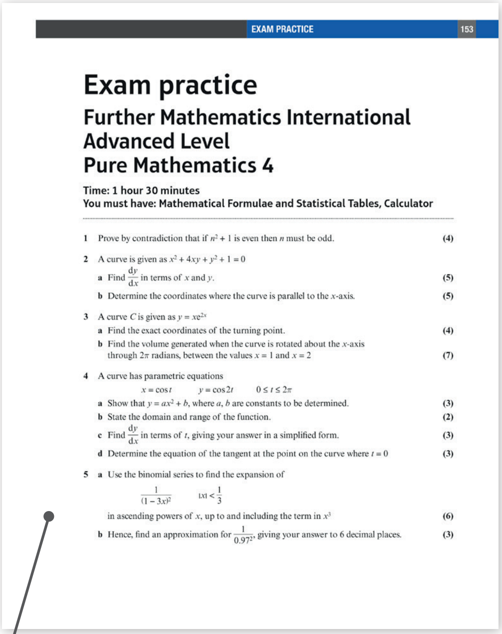
Problem-solving boxes provide hints, tips and strategies, and *Watch out* boxes highlight areas where students often lose marks in their exams

Exam-style questions are flagged with (E)

Problem-solving questions are flagged with (P)

Each chapter ends with a *Chapter review* and a *Summary of key points*

After every few chapters, a *Review exercise* helps you consolidate your learning with lots of exam-style questions



A full practice paper at the back of the book helps you prepare for the real thing

QUALIFICATION AND ASSESSMENT OVERVIEW

Qualification and content overview

Pure Mathematics 4 (P4) is a **compulsory** unit in the following qualifications:

International Advanced Level in Mathematics

International Advanced Level in Pure Mathematics

Assessment overview

The following table gives an overview of the assessment for this unit.

We recommend that you study this information closely to help ensure that you are fully prepared for this course and know exactly what to expect in the assessment.

Unit	Percentage	Mark	Time	Availability
P4: Pure Mathematics 4 Paper code WMA14/01	$16\frac{2}{3}$ % of IAL	75	1 hour 30 mins	January, June and October First assessment June 2020

IAL: International Advanced A Level.

Assessment objectives and weightings

		Minimum weighting in IAS and IAL
AO1	Recall, select and use their knowledge of mathematical facts, concepts and techniques in a variety of contexts.	30%
AO2	Construct rigorous mathematical arguments and proofs through use of precise statements, logical deduction and inference and by the manipulation of mathematical expressions, including the construction of extended arguments for handling substantial problems presented in unstructured form.	30%
AO3	Recall, select and use their knowledge of standard mathematical models to represent situations in the real world; recognise and understand given representations involving standard models; present and interpret results from such models in terms of the original situation, including discussion of the assumptions made and refinement of such models.	10%
AO4	Comprehend translations of common realistic contexts into mathematics; use the results of calculations to make predictions, or comment on the context; and, where appropriate, read critically and comprehend longer mathematical arguments or examples of applications.	5%
AO5	Use contemporary calculator technology and other permitted resources (such as formulae booklets or statistical tables) accurately and efficiently; understand when not to use such technology, and its limitations. Give answers to appropriate accuracy.	5%

Relationship of assessment objectives to units

P4	Assessment objective				
	AO1	AO2	AO3	AO4	AO5
Marks out of 75	25–30	25–30	5–10	5–10	5–10
%	$33\frac{1}{3}$–40	$33\frac{1}{3}$–40	$6\frac{2}{3}$–$13\frac{1}{3}$	$6\frac{2}{3}$–$13\frac{1}{3}$	$6\frac{2}{3}$–$13\frac{1}{3}$

Calculators

Students may use a calculator in assessments for these qualifications. Centres are responsible for making sure that calculators used by their students meet the requirements given in the table below.

Students are expected to have available a calculator with at least the following keys: +, −, ×, ÷, π, x^2, $\sqrt{x}$, $\frac{1}{x}$, x^y, ln x, e^x, $x!$, sine, cosine and tangent and their inverses in degrees and decimals of a degree, and in radians; memory.

Prohibitions

Calculators with any of the following facilities are prohibited in all examinations:

- databanks
- retrieval of text or formulae
- built-in symbolic algebra manipulations
- symbolic differentiation and/or integration
- language translators
- communication with other machines or the internet

Extra online content

Whenever you see an *Online* box, it means that there is extra online content available to support you.

SolutionBank

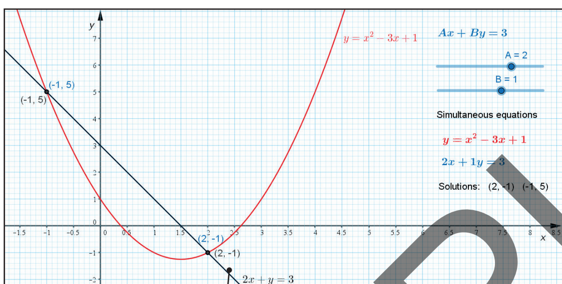
SolutionBank provides worked solutions for questions in the book. Download the solutions as a PDF or quickly find the solution you need online.

Use of technology

Explore topics in more detail, visualise problems and consolidate your understanding. Use pre-made GeoGebra activities or Casio resources for a graphic calculator.

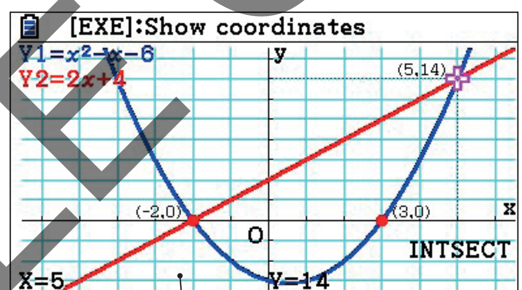
Online Find the point of intersection graphically using technology.

GeoGebra-powered interactives



Interact with the maths you are learning using GeoGebra's easy-to-use tools

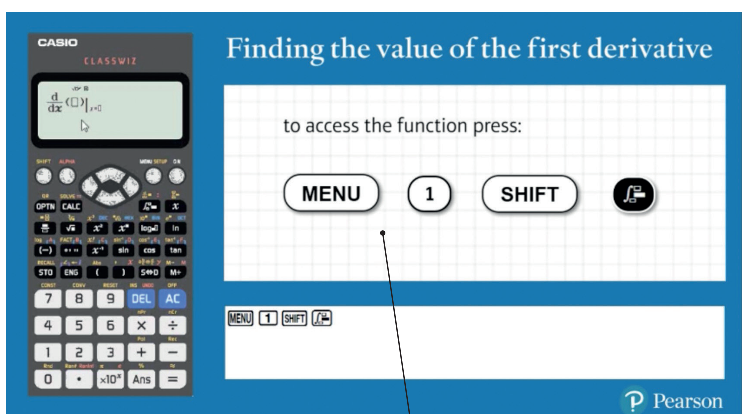
Graphic calculator interactives



Explore the maths you are learning and gain confidence in using a graphic calculator

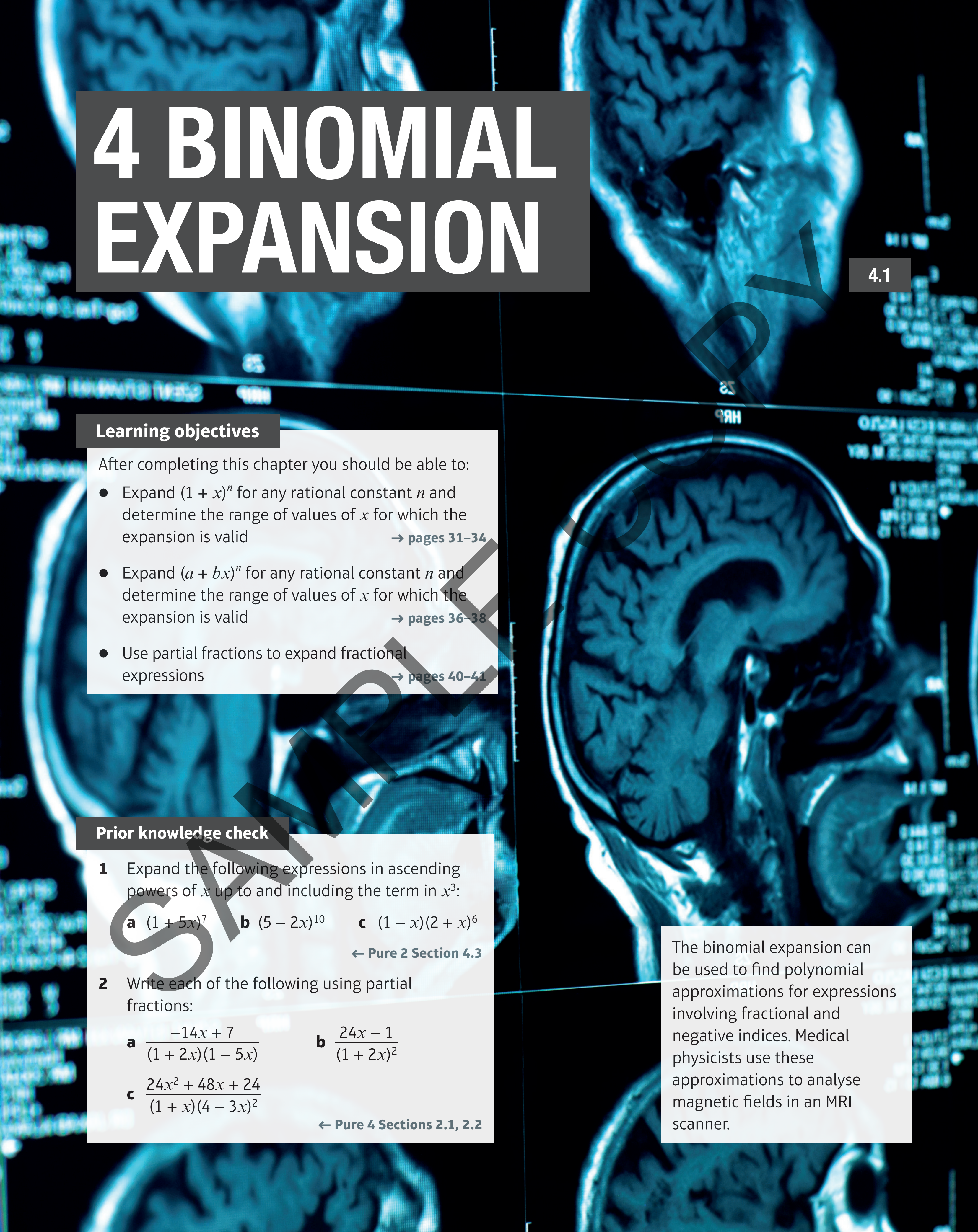
Calculator tutorials

Our helpful video tutorials will guide you through how to use your calculator in the exams. They cover both Casio's scientific and colour graphic calculators.



Step-by-step guide with audio instructions on exactly which buttons to press and what should appear on your calculator's screen

Online Work out each coefficient quickly using the ${}^nC_r$ and power functions on your calculator.



4 BINOMIAL EXPANSION

4.1

Learning objectives

After completing this chapter you should be able to:

- Expand $(1 + x)^n$ for any rational constant n and determine the range of values of x for which the expansion is valid → pages 31–34
- Expand $(a + bx)^n$ for any rational constant n and determine the range of values of x for which the expansion is valid → pages 36–38
- Use partial fractions to expand fractional expressions → pages 40–41

Prior knowledge check

1 Expand the following expressions in ascending powers of x up to and including the term in x^3:

a $(1 + 5x)^7$ **b** $(5 - 2x)^{10}$ **c** $(1 - x)(2 + x)^6$

← Pure 2 Section 4.3

2 Write each of the following using partial fractions:

a $\dfrac{-14x + 7}{(1 + 2x)(1 - 5x)}$ **b** $\dfrac{24x - 1}{(1 + 2x)^2}$

c $\dfrac{24x^2 + 48x + 24}{(1 + x)(4 - 3x)^2}$

← Pure 4 Sections 2.1, 2.2

The binomial expansion can be used to find polynomial approximations for expressions involving fractional and negative indices. Medical physicists use these approximations to analyse magnetic fields in an MRI scanner.

4.1 Expanding $(1 + x)^n$

If n is a natural number you can find the binomial expansion for $(a + bx)^n$ using the formula:

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n, \quad (n \in \mathbb{N})$$

Hint There are $n + 1$ terms, so this formula produces a **finite** number of terms.

If n is a **fraction** or a **negative number** you need to use a different version of the binomial expansion.

- **This form of the binomial expansion can be applied to negative or fractional values of n to obtain an infinite series.**

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + \left(\frac{n(n-1)\ldots(n-r+1)}{r!}\right)x^r + \ldots$$

- **The expansion is valid when $|x| < 1$, $n \in \mathbb{R}$**

When n is not a natural number, none of the factors in the expression $n(n-1)\ldots(n-r+1)$ are equal to zero. This means that this version of the binomial expansion produces an **infinite number** of terms.

Watch out This expansion is valid for any **real value** of n, but is **only** valid for values of x that satisfy $|x| < 1$, or in other words, when $-1 < x < 1$

Example 1 SKILLS PROBLEM-SOLVING

Find the first four terms in the binomial expansion of $\dfrac{1}{1+x}$

$$\frac{1}{1+x} = (1+x)^{-1}$$

Write in **index** form.

$$= 1 + (-1)x + \frac{(-1)(-2)x^2}{2!} + \frac{(-1)(-2)(-3)x^3}{3!} + \ldots$$

Replace n by -1 in the expansion.

$$= 1 - 1x + 1x^2 - 1x^3 + \ldots$$

As n is not a positive integer, no coefficient will ever be equal to zero. Therefore, the expansion is **infinite**.

$$= 1 - x + x^2 - x^3 + \ldots$$

For the series to be **convergent**, $|x| < 1$

- **The expansion of $(1 + bx)^n$, where n is negative or a fraction, is valid for $|bx| < 1$, or $|x| < \dfrac{1}{|b|}$**

Example 2 SKILLS PROBLEM-SOLVING

Find the binomial expansions of

a $(1 - x)^{\frac{1}{3}}$

b $\dfrac{1}{(1 + 4x)^2}$

up to and including the term in x^3. State the range of values of x for which each expansion is valid.

a $(1 - x)^{\frac{1}{3}}$

$$= 1 + \left(\frac{1}{3}\right)(-x) + \frac{\left(\frac{1}{3}\right)\left(\frac{1}{3} - 1\right)(-x)^2}{2!} + \frac{\left(\frac{1}{3}\right)\left(\frac{1}{3} - 1\right)\left(\frac{1}{3} - 2\right)(-x)^3}{3!} + \ldots$$

$$= 1 + \left(\frac{1}{3}\right)(-x) + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)(-x)^2}{2} + \frac{(1/3)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)(-x)^3}{6} + \ldots$$

$$= 1 - \frac{1}{3}x - \frac{1}{9}x^2 - \frac{5}{81}x^3 + \ldots$$

Expansion is valid as long as $|-x| < 1$

$\Rightarrow |x| < 1$

b $\dfrac{1}{(1 + 4x)^2} = (1 + 4x)^{-2}$

$$= 1 + (-2)(4x) + \frac{(-2)(-2 - 1)(4x)^2}{2!} + \frac{(-2)(-2 - 1)(-2 - 2)(4x)^3}{3!} + \ldots$$

$$= 1 + (-2)(4x) + \frac{(-2)(-3)16x^2}{2} + \frac{(-2)(-3)(-4)64x^3}{6} + \ldots$$

$$= 1 - 8x + 48x^2 - 256x^3 + \ldots$$

Expansion is valid as long as $|4x| < 1$

$\Rightarrow |x| < \frac{1}{4}$

Replace n by $\frac{1}{3}$ and x by $(-x)$.

Simplify brackets.

Watch out Be careful working out whether each term should be positive or negative:
- even number of negative signs means term is positive
- odd number of negative signs means term is negative

The x^3 term here has 5 negative signs in total, so it is negative.

Simplify coefficients.

Terms in expansion are $(-x)$, $(-x)^2$, $(-x)^3$

Write in index form.

Replace n by -2, x by $4x$

Simplify brackets.

Simplify coefficients.

Terms in expansion are $(4x)$, $(4x)^2$, $(4x)^3$

Online Use technology to explore why the expansions are only valid for certain values of x.

Example 3 SKILLS ANALYSIS

a Find the expansion of $\sqrt{1-2x}$ up to and including the term in x^3.

b By substituting in $x = 0.01$, find a decimal approximation to $\sqrt{2}$.

a $\sqrt{1-2x} = (1-2x)^{\frac{1}{2}}$ — Write in index form.

$$= 1 + \left(\frac{1}{2}\right)(-2x) + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)(-2x)^2}{2!} + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)(-2x)^3}{3!} + \ldots$$

Replace n by $\frac{1}{2}$ and x by $(-2x)$

$$= 1 + \left(\frac{1}{2}\right)(-2x) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)(4x^2)}{2!} + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-8x^3)}{6} + \ldots$$

Simplify brackets.

$$= 1 - x - \frac{x^2}{2} - \frac{x^3}{2} + \ldots$$

Simplify coefficients.

Expansion is valid if $|-2x| < 1$

$$\Rightarrow |x| < \frac{1}{2}$$

Terms in expansion are $(-2x)$, $(-2x)^2$, $(-2x)^3$

$x = 0.01$ satisfies the validity **condition** $|x| < \frac{1}{2}$

b $\sqrt{1 - 2 \times 0.01} \approx 1 - 0.01 - \frac{0.01^2}{2} - \frac{0.01^3}{2}$

Substitute $x = 0.01$ into both sides of the expansion.

$$\sqrt{0.98} \approx 1 - 0.01 - 0.00005 - 0.0000005$$

Simplify both sides.
Note that the terms are getting smaller.

$$\sqrt{\frac{98}{100}} \approx 0.9899495$$

Write 0.98 as $\frac{98}{100}$

$$\sqrt{\frac{49 \times 2}{100}} \approx 0.9899495$$

$$\frac{7\sqrt{2}}{10} \approx 0.9899495$$

Use rules of **surds**.

$$\sqrt{2} \approx \frac{0.9899495 \times 10}{7}$$

$$\sqrt{2} \approx 1.414213571$$

This approximation is **accurate** to 7 decimal places.

Example 4 SKILLS CRITICAL THINKING

$$f(x) = \frac{2 + x}{\sqrt{1 + 5x}}$$

a Find the x^2 term in the series expansion of $f(x)$.

b State the range of values of x for which the expansion is valid.

a $f(x) = (2 + x)(1 + 5x)^{-\frac{1}{2}}$

Write in index form.

$$(1 + 5x)^{-\frac{1}{2}} = 1 + \left(-\frac{1}{2}\right)(5x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(5x)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(5x)^3 + \dots$$

Find the binomial expansion of $(1 + 5x)^{-\frac{1}{2}}$

$$= 1 - \frac{5}{2}x + \frac{75}{8}x^2 - \frac{625}{16}x^3 + \dots$$

Simplify coefficients.

Online Use your calculator to calculate the coefficients of the binomial expansion.

$$f(x) = (2 + x)\left(1 - \frac{5}{2}x + \frac{75}{8}x^2 - \frac{625}{16}x^3 + \dots\right)$$

$$2 \times \frac{75}{8} + 1 \times -\frac{5}{2} = \frac{65}{4}$$

x^2 term is $\frac{65}{4}x^2$

Problem-solving

There are two ways to make an x^2 term.

Either $2 \times \frac{75}{8}x^2$ or $x \times \frac{5}{2}x$

Add these together to find the term in x^2.

b The expansion is valid if $|5x| < 1$

$$\Rightarrow |x| < \frac{1}{5}$$

Example 5 SKILLS PROBLEM-SOLVING

In the expansion of $(1 + kx)^{-4}$ the coefficient of x is 20.

a Find the value of k.

b Find the corresponding coefficient of the x^2 term.

a $(1 + kx)^{-4} = 1 + (-4)(kx) + \frac{(-4)(-5)}{2!}(kx)^2 + \dots$

Find the binomial expansion of $(1 + kx)^{-4}$

$$= 1 - 4kx + 10k^2x^2 + \dots$$

$$-4k = 20$$
$$k = -5$$

Solve to find k.

b Coefficient of $x^2 = 10k^2 = 10(-5)^2 = 250$

Exercise 4A

SKILLS PROBLEM-SOLVING

1 For each of the following:

i find the binomial expansion up to and including the x^3 term

ii state the range of values of x for which the expansion is valid.

a $(1+x)^{-4}$ b $(1+x)^{-6}$ c $(1+x)^{\frac{1}{2}}$

d $(1+x)^{\frac{5}{3}}$ e $(1+x)^{-\frac{1}{4}}$ f $(1+x)^{-\frac{3}{2}}$

2 For each of the following:

i find the binomial expansion up to and including the x^3 term

ii state the range of values of x for which the expansion is valid.

a $(1+3x)^{-3}$ b $\left(1+\frac{1}{2}x\right)^{-5}$ c $(1+2x)^{\frac{3}{4}}$

d $(1-5x)^{\frac{7}{3}}$ e $(1+6x)^{-\frac{2}{3}}$ f $\left(1-\frac{3}{4}x\right)^{-\frac{5}{3}}$

3 For each of the following:

i find the binomial expansion up to and including the x^3 term

ii state the range of values of x for which the expansion is valid.

a $\dfrac{1}{(1+x)^2}$ b $\dfrac{1}{(1+3x)^4}$ c $\sqrt{1-x}$

d $\sqrt[3]{1-3x}$ e $\dfrac{1}{\sqrt{1+\frac{1}{2}x}}$ f $\dfrac{\sqrt[3]{1-2x}}{1-2x}$

Hint In part **f**, write the fraction as a single **power** of $(1-2x)$

E/P 4 $f(x)=\dfrac{1+x}{1-2x}$

a Show that the series expansion of $f(x)$ up to and including the x^3 term is $1+3x+6x^2+12x^3$ **(4 marks)**

b State the range of values of x for which the expansion is valid. **(1 mark)**

Hint First rewrite $f(x)$ as $(1+x)(1-2x)^{-1}$

E 5 $f(x)=\sqrt{1+3x}$, $-\frac{1}{3}<x<\frac{1}{3}$

a Find the series expansion of $f(x)$, in **ascending** powers of x, up to and including the x^3 term. Simplify each term. **(4 marks)**

b Show that, when $x=\frac{1}{100}$, the exact value of $f(x)$ is $\frac{\sqrt{103}}{10}$ **(2 marks)**

c Find the **percentage** error made in using the series expansion in part **a** to **estimate** the value of $f(0.01)$. Give your answer to 2 **significant figures**. **(3 marks)**

P 6 In the expansion of $(1+ax)^{-\frac{1}{2}}$ the coefficient of x^2 is 24.

a Find the possible values of a.

b Find the corresponding coefficient of the x^3 term.

P 7 Show that if x is small, the expression $\sqrt{\frac{1+x}{1-x}}$ is approximated by $1 + x + \frac{1}{2}x^2$

Notation 'x is small' means we can assume the expansion is valid for the x values being considered because high powers become **insignificant** compared to the first few terms.

E/P 8 $h(x) = \frac{6}{1+5x} - \frac{4}{1-3x}$

a Find the series expansion of $h(x)$, in ascending powers of x, up to and including the x^2 term. Simplify each term. **(6 marks)**

b Find the percentage error made in using the series expansion in part **a** to estimate the value of $h(0.01)$. Give your answer to 2 significant figures. **(3 marks)**

c Explain why it is not valid to use the expansion to find $h(0.5)$. **(1 mark)**

E/P 9 **a** Find the binomial expansion of $(1-3x)^{\frac{3}{2}}$ in ascending powers of x up to and including the x^3 term, simplifying each term. **(4 marks)**

b Show that, when $x = \frac{1}{100}$, the exact value of $(1-3x)^{\frac{3}{2}}$ is $\frac{97\sqrt{97}}{1000}$ **(2 marks)**

c Substitute $x = \frac{1}{100}$ into the binomial expansion in part **a** and hence obtain an approximation to $\sqrt{97}$. Give your answer to 5 decimal places. **(3 marks)**

Challenge

$h(x) = \left(1 + \frac{1}{x}\right)^{-\frac{1}{2}}, |x| > 1$

a Find the binomial expansion of $h(x)$ in ascending powers of x up to and including the x^2 term, simplifying each term.

b Show that, when $x = 9$, the exact value of $h(x)$ is $\frac{3\sqrt{10}}{10}$

c Use the expansion in part **a** to find an approximate value of $\sqrt{10}$. Write your answer to 2 decimal places.

Hint Replace x with $\frac{1}{x}$

4.2 Expanding $(a + bx)^n$

The binomial expansion of $(1 + x)^n$ can be used to expand $(a + bx)^n$ for any constants a and b.

You need to take a factor of a^n out of the expression:

$$(a+bx)^n = \left(a\left(1+\frac{b}{a}x\right)\right)^n = a^n\left(1+\frac{b}{a}x\right)^n$$

Watch out Make sure you multiply a^n by **every term** in the expansion of $\left(1+\frac{b}{a}x\right)^n$

- The expansion of $(a + bx)^n$, where n is negative or a fraction, is valid for $\left|\frac{b}{a}x\right| < 1$ or $|x| < \left|\frac{a}{b}\right|$

Example 6 SKILLS ADAPTIVE LEARNING

Find the first four terms in the binomial expansion of **a** $\sqrt{4 + x}$ **b** $\frac{1}{(2 + 3x)^2}$

State the range of values of x for which each of these expansions is valid.

a $\sqrt{4 + x} = (4 + x)^{\frac{1}{2}}$ — Write in index form.

$= \left(4\left(1 + \frac{x}{4}\right)\right)^{\frac{1}{2}}$

$= 4^{\frac{1}{2}}\left(1 + \frac{x}{4}\right)^{\frac{1}{2}}$ — Take out a factor of $4^{\frac{1}{2}}$

$= 2\left(1 + \frac{x}{4}\right)^{\frac{1}{2}}$ — Write $4^{\frac{1}{2}}$ as 2.

$$= 2\left(1 + \left(\frac{1}{2}\right)\left(\frac{x}{4}\right) + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2} - 1\right)\left(\frac{x}{4}\right)^2}{2!} + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2} - 1\right)\left(\frac{1}{2} - 2\right)\left(\frac{x}{4}\right)^3}{3!} + \ldots\right)$$

Expand $\left(1 + \frac{x}{4}\right)^{\frac{1}{2}}$ using the binomial expansion with $n = \frac{1}{2}$ and $x = \frac{x}{4}$

$$= 2\left(1 + \left(\frac{1}{2}\right)\left(\frac{x}{4}\right) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{x^2}{16}\right)}{2} + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{x^3}{64}\right)}{6} + \ldots\right)$$

$$= 2\left(1 + \frac{x}{8} - \frac{x^2}{128} + \frac{x^3}{1024} + \ldots\right)$$

Simplify coefficients.

$$= 2 + \frac{x}{4} - \frac{x^2}{64} + \frac{x^3}{512} + \ldots$$

Multiply every term in the expansion by 2.

Expansion is valid if $\left|\frac{x}{4}\right| < 1$

$\Rightarrow |x| < 4$

The expansion is infinite, and **converges** when $\left|\frac{x}{4}\right| < 1$, or $|x| < 4$

b $\dfrac{1}{(2+3x)^2} = (2+3x)^{-2}$

$$= \left(2\left(1+\frac{3x}{2}\right)\right)^{-2}$$

$$= 2^{-2}\left(1+\frac{3x}{2}\right)^{-2}$$

$$= \frac{1}{4}\left(1+\frac{3x}{2}\right)^{-2}$$

$$= \frac{1}{4}\left(1 + (-2)\left(\frac{3x}{2}\right) + \frac{(-2)(-2-1)\left(\frac{3x}{2}\right)^2}{2!} + \frac{(-2)(-2-1)(-2-2)\left(\frac{3x}{2}\right)^3}{3!} + \ldots\right)$$

$$= \frac{1}{4}\left((1 + (-2)\left(\frac{3x}{2}\right) + \frac{(-2)(-3)\left(\frac{9x^2}{4}\right)}{2} + \frac{(-2)(-3)(-4)\left(\frac{27x^3}{8}\right)}{6} + \ldots\right)$$

$$= \frac{1}{4}\left(1 - 3x + \frac{27x^2}{4} - \frac{27x^3}{2} + \ldots\right)$$

$$= \frac{1}{4} - \frac{3}{4}x + \frac{27x^2}{16} - \frac{27x^3}{8} + \ldots$$

Expansion is valid if $\left|\frac{3x}{2}\right| < 1$

$$\Rightarrow |x| < \frac{2}{3}$$

Write in index form.

Take out a factor of 2^{-2}

Write $2^{-2} = \frac{1}{2^2} = \frac{1}{4}$

Expand $\left(1+\frac{3x}{2}\right)^{-2}$ using the binomial expansion with $n = -2$ and $x = \frac{3x}{2}$

Simplify coefficients.

Multiply every term by $\frac{1}{4}$

The expansion is infinite, and converges when $\left|\frac{3x}{2}\right| < 1$, $|x| < \frac{2}{3}$

Exercise 4B

SKILLS ANALYSIS

(P) **1** For each of the following:

i find the binomial expansion up to and including the x^3 term

ii state the range of values of x for which the expansion is valid.

a $\sqrt{4+2x}$ **b** $\dfrac{1}{2+x}$ **c** $\dfrac{1}{(4-x)^2}$ **d** $\sqrt{9+x}$

e $\dfrac{1}{\sqrt{2+x}}$ **f** $\dfrac{5}{3+2x}$ **g** $\dfrac{1+x}{2+x}$ **h** $\sqrt{\dfrac{2+x}{1-x}}$

Hint Write part **g** as $1 - \dfrac{1}{x+2}$

E **2** $f(x) = (5 + 4x)^{-2}$, $|x| < \frac{5}{4}$

Find the binomial expansion of $f(x)$ in ascending powers of x, up to and including the term in x^3. Give each coefficient as a simplified fraction. **(5 marks)**

E **3** $m(x) = \sqrt{4 - x}$, $|x| < 4$

a Find the series expansion of $m(x)$, in ascending powers of x, up to and including the x^2 term. Simplify each term. **(4 marks)**

b Show that, when $x = \frac{1}{9}$, the exact value of $m(x)$ is $\frac{\sqrt{35}}{3}$ **(2 marks)**

c Use your answer to part **a** to find an approximate value for $\sqrt{35}$, and calculate the percentage error in your approximation. **(4 marks)**

P **4** The first three terms in the binomial expansion of $\frac{1}{\sqrt{a + bx}}$ are $3 + \frac{1}{3}x + \frac{1}{18}x^2 + \ldots$

a Find the values of the constants a and b.

b Find the coefficient of the x^3 term in the expansion.

P **5** $f(x) = \frac{3 + 2x - x^2}{4 - x}$

Prove that if x is **sufficiently small**, $f(x)$ may be approximated by $\frac{3}{4} + \frac{11}{16}x - \frac{5}{64}x^2$

E/P **6** **a** Expand $\frac{1}{\sqrt{5 + 2x}}$, where $|x| < \frac{5}{2}$, in ascending powers of x up to and including the term in x^2, giving each coefficient in simplified surd form. **(5 marks)**

b Hence or otherwise, find the first 3 terms in the expansion of $\frac{2x - 1}{\sqrt{5 + 2x}}$ as a series in ascending powers of x. **(4 marks**

E/P **7** **a** Use the binomial theorem to expand $(16 - 3x)^{\frac{1}{4}}$, $|x| < \frac{16}{3}$ in ascending powers of x, up to and including the term in x^2, giving each term as a simplified fraction. **(4 marks)**

b Use your expansion, with a suitable value of x, to obtain an approximation to $\sqrt[4]{15.7}$ Give your answer to 3 decimal places. **(2 marks)**

8 $g(x) = \frac{3}{4 - 2x} - \frac{2}{3 + 5x}$, $|x| < \frac{1}{2}$

a Show that the first three terms in the series expansion of $g(x)$ can be written as $\frac{1}{12} + \frac{107}{72}x - \frac{719}{432}x^2$ **(5 marks)**

b Find the exact value of $g(0.01)$. Round your answer to 7 decimal places. **(2 marks)**

c Find the percentage error made in using the series expansion in part **a** to estimate the value of $g(0.01)$. Give your answer to 2 significant figures. **(3 marks)**

4.3 Using partial fractions

Partial fractions can be used to simplify the expansions of more difficult expressions.

Links You need to be confident expressing **algebraic** fractions as sums of partial fractions.

Example 7 SKILLS INNOVATION

a Express $\dfrac{4-5x}{(1+x)(2-x)}$ as partial fractions.

b Hence show that the **cubic** approximation of $\dfrac{4-5x}{(1+x)(2-x)}$ is $2-\dfrac{7x}{2}+\dfrac{11}{4}x^2-\dfrac{25}{8}x^3$

c State the range of values of x for which the expansion is valid.

a $\dfrac{4-5x}{(1+x)(2-x)} \equiv \dfrac{A}{1+x}+\dfrac{B}{2-x}$

The denominators must be $(1+x)$ and $(2-x)$

$\equiv \dfrac{A(2-x)+B(1+x)}{(1+x)(2-x)}$

Add the fractions.

$4-5x \equiv A(2-x)+B(1+x)$

Set the numerators equal.

Substitute $x = 2$:

$4-10 = A \times 0 + B \times 3$

Set $x = 2$ to find B.

$-6 = 3B$

$B = -2$

Substitute $x = -1$:

$4+5 = A \times 3 + B \times 0$

Set $x = -1$ to find A.

$9 = 3A$

$A = 3$

so $\dfrac{4-5x}{(1+x)(2-x)} = \dfrac{3}{1+x}-\dfrac{2}{2-x}$

b $\dfrac{4-5x}{(1+x)(2-x)} = \dfrac{3}{1+x}-\dfrac{2}{2-x}$

$= 3(1+x)^{-1}-2(2-x)^{-1}$

Write in index form.

Problem-solving

Use headings to keep track of your working. This will help you stay organised and check your answers.

The expansion of $3(1+x)^{-1}$

$= 3\left(1+(-1)x+(-1)(-2)\dfrac{x^2}{2!} + (-1)(-2)(-3)\dfrac{x^3}{3!}+\ldots\right)$

Expand $3(1+x)^{-1}$ using the binomial expansion with $n = -1$

$= 3(1-x+x^2-x^3+\ldots)$

$= 3-3x+3x^2-3x^3+\ldots$

The expansion of $2(2 - x)^{-1}$

$= 2\left(2\left(1 - \frac{x}{2}\right)\right)^{-1}$

$= 2 \times 2^{-1}\left(1 - \frac{x}{2}\right)^{-1}$ — Take out a factor of 2^{-1}

$= 1 \times \left(1 + (-1)\left(-\frac{x}{2}\right) + \frac{(-1)(-2)\left(-\frac{x}{2}\right)^2}{2!} + \frac{(-1)(-2)(-3)\left(-\frac{x}{2}\right)^3}{3!} + \ldots\right)$ — Expand $\left(1 - \frac{x}{2}\right)^{-1}$ using the binomial expansion with $n = -1$ and $x = \frac{x}{2}$

$= 1 \times \left(1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8} + \ldots\right)$

$= 1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8}$

Hence $\frac{4 - 5x}{(1 + x)(2 - x)}$

$= 3(1 + x)^{-1} - 2(2 - x)^{-1}$ — 'Add' both expressions.

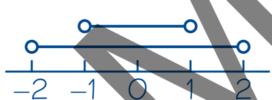
$= (3 - 3x + 3x^2 - 3x^3)$

$-\left(1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8}\right)$

$= 2 - \frac{7}{2}x + \frac{11}{4}x^2 - \frac{25}{8}x^3$

c $\frac{3}{1 + x}$ is valid if $|x| < 1$ — The expansion is infinite, and converges when $|x| < 1$

$\frac{2}{2 - x}$ is valid if $\left|\frac{x}{2}\right| < 1 \Rightarrow |x| < 2$ — The expansion is infinite, and converges when $\left|\frac{x}{2}\right| < 1$ or $|x| < 2$



The expansion is valid when $|x| < 1$

Watch out You need to find the range of values of x that satisfy **both** inequalities.

Exercise 4C

SKILLS INNOVATION

(P) **1** **a** Express $\frac{8x + 4}{(1 - x)(2 + x)}$ as partial fractions.

b Hence or otherwise expand $\frac{8x + 4}{(1 - x)(2 + x)}$ in ascending powers of x as far as the term in x^2.

c State the set of values of x for which the expansion is valid.

P **2** **a** Express $-\frac{2x}{(2+x)^2}$ as partial fractions.

b Hence prove that $-\frac{2x}{(2+x)^2}$ can be expressed in the form $-\frac{1}{2}x + Bx^2 + Cx^3$ where constants B and C are to be determined.

c State the set of values of x for which the expansion is valid.

P **3** **a** Express $\frac{6+7x+5x^2}{(1+x)(1-x)(2+x)}$ as partial fractions.

b Hence or otherwise expand $\frac{6+7x+5x^2}{(1+x)(1-x)(2+x)}$ in ascending powers of x as far as the term in x^3.

c State the set of values of x for which the expansion is valid.

E/P **4** $g(x) = \frac{12x-1}{(1+2x)(1-3x)}, |x| < \frac{1}{3}$

Given that g(x) can be expressed in the form $g(x) = \frac{A}{1+2x} + \frac{B}{1-3x}$

a Find the values of A and B. **(3 marks)**

b Hence, or otherwise, find the series expansion of g(x), in ascending powers of x, up to and including the x^2 term. Simplify each term. **(6 marks)**

P **5** **a** Express $\frac{2x^2+7x-6}{(x+5)(x-4)}$ in partial fractions.

Hint First divide the numerator by the denominator.

b Hence, or otherwise, expand $\frac{2x^2+7x-6}{(x+5)(x-4)}$ in ascending powers of x as far as the term in x^2.

c State the set of values of x for which the expansion is valid.

E/P **6** $\frac{3x^2+4x-5}{(x+3)(x-2)} = A + \frac{B}{x+3} + \frac{C}{x-2}$

a Find the values of the constants A, B and C. **(4 marks)**

b Hence, or otherwise, expand $\frac{3x^2+4x-5}{(x+3)(x-2)}$ in ascending powers of x, as far as the term in x^2. Give each coefficient as a simplified fraction. **(7 marks)**

E/P **7** $f(x) = \frac{2x^2+5x+11}{(2x-1)^2(x+1)}, |x| < \frac{1}{2}$

f(x) can be expressed in the form $f(x) = \frac{A}{2x-1} + \frac{B}{(2x-1)^2} + \frac{C}{x+1}$

a Find the values of A, B and C. **(4 marks)**

b Hence or otherwise, find the series expansion of f(x), in ascending powers of x, up to and including the term in x^2. Simplify each term. **(6 marks)**

c Find the percentage error made in using the series expansion in part **b** to estimate the value of f(0.05). Give your answer to 2 significant figures. **(4 marks)**

Chapter review 4

SKILLS PROBLEM-SOLVING

P **1** For each of the following

i find the binomial expansion up to and including the x^3 term

ii state the range of values of x for which the expansion is valid.

a $(1 - 4x)^3$ **b** $\sqrt{16 + x}$ **c** $\dfrac{1}{1 - 2x}$ **d** $\dfrac{4}{2 + 3x}$

e $\dfrac{4}{\sqrt{4 - x}}$ **f** $\dfrac{1 + x}{1 + 3x}$ **g** $\left(\dfrac{1 + x}{1 - x}\right)^2$ **h** $\dfrac{x - 3}{(1 - x)(1 - 2x)}$

E **2** Use the binomial expansion to expand $\left(1 - \frac{1}{2}x\right)^{\frac{1}{2}}$, $|x| < 2$ in ascending powers of x, up to and including the term in x^3, simplifying each term. **(5 marks)**

3 **a** Give the binomial expansion of $(1 + x)^{\frac{1}{2}}$ up to and including the term in x^3.

b By substituting $x = \frac{1}{4}$, find an **approximation** to $\sqrt{5}$ as a fraction.

E/P **4** The binomial expansion of $(1 + 9x)^{\frac{2}{3}}$ in ascending powers of x up to and including the term in x^3 is $1 + 6x + cx^2 + dx^3$, $|x| < \frac{1}{9}$

a Find the value of c and the value of d. **(4 marks)**

b Use this expansion with your values of c and d together with an appropriate value of x to obtain an estimate of $(1.45)^{\frac{2}{3}}$ **(2 marks)**

c Obtain $(1.45)^{\frac{2}{3}}$ from your calculator and hence make a comment on the accuracy of the estimate you obtained in part **b**. **(1 mark)**

P **5** In the expansion of $(1 + ax)^{\frac{1}{2}}$ the coefficient of x^2 is -2.

a Find the possible values of a.

b Find the corresponding coefficients of the x^3 term.

E **6** $f(x) = (1 + 3x)^{-1}$, $|x| < \frac{1}{3}$

a Expand $f(x)$ in ascending powers of x up to and including the term in x^3. **(5 marks)**

b Hence show that, for small x:

$$\frac{1 + x}{1 + 3x} \approx 1 - 2x + 6x^2 - 18x^3$$ **(4 marks)**

c Taking a suitable value for x, which should be stated, use the series expansion in part **b** to find an approximate value for $\frac{101}{103}$, giving your answer to 5 decimal places. **(3 marks)**

E/P **7** When $(1 + ax)^n$ is expanded as a series in ascending powers of x, the coefficients of x and x^2 are -6 and 27 respectively.

a Find the values of a and n. **(4 marks)**

b Find the coefficient of x^3. **(3 marks)**

c State the values of x for which the expansion is valid. **(1 mark)**

8 Show that if x is sufficiently small then $\frac{3}{\sqrt{4+x}}$ can be approximated by $\frac{3}{2} - \frac{3}{16}x + \frac{9}{256}x^2$

E 9 a Expand $\frac{1}{\sqrt{4-x}}$, where $|x| < 4$, in ascending powers of x up to and including the term in x^2. Simplify each term. **(5 marks)**

b Hence, or otherwise, find the first 3 terms in the expansion of $\frac{1+2x}{\sqrt{4-x}}$ as a series in ascending powers of x. **(4 marks)**

E 10 a Find the first four terms of the expansion, in ascending powers of x, of $(2+3x)^{-1}$, $|x| < \frac{2}{3}$ **(4 marks)**

b Hence or otherwise, find the first four **non-zero** terms of the expansion, in ascending powers of x, of:

$\frac{1+x}{2+3x}$, $|x| < \frac{2}{3}$ **(3 marks)**

E/P 11 a Use the binomial theorem to expand $(4+x)^{-\frac{1}{2}}$, $|x| < 4$, in ascending powers of x, up to and including the x^3 term, giving each answer as a simplified fraction. **(5 marks)**

b Use your expansion, together with a suitable value of x, to obtain an approximation to $\frac{\sqrt{2}}{2}$. Give your answer to 4 decimal places. **(3 marks)**

E 12 $q(x) = (3+4x)^{-3}$, $|x| < \frac{3}{4}$

Find the binomial expansion of $q(x)$ in ascending powers of x, up to and including the term in the x^2. Give each coefficient as a simplified fraction. **(5 marks)**

E/P 13 $g(x) = \frac{39x+12}{(x+1)(x+4)(x-8)}$, $|x| < 1$

$g(x)$ can be expressed in the form $g(x) = \frac{A}{x+1} + \frac{B}{x+4} + \frac{C}{x-8}$

a Find the values of A, B and C. **(4 marks)**

b Hence, or otherwise, find the series expansion of $g(x)$, in ascending powers of x, up to and including the x^2 term. Simplify each term. **(7 marks)**

E/P 14 $f(x) = \frac{12x+5}{(1+4x)^2}$, $|x| < \frac{1}{4}$

For $x \neq -\frac{1}{4}$, $\frac{12x+5}{(1+4x)^2} = \frac{A}{1+4x} + \frac{B}{(1+4x)^2}$, where A and B are constants.

a Find the values of A and B. **(3 marks)**

b Hence, or otherwise, find the series expansion of $f(x)$, in ascending powers of x, up to and including the term x^2, simplifying each term. **(6 marks)**

E/P **15** $q(x) = \dfrac{9x^2 + 26x + 20}{(1 + x)(2 + x)}$, $|x| < 1$

a Show that the expansion of $q(x)$ in ascending powers of x can be approximated to $10 - 2x + Bx^2 + Cx^3$ where B and C are constants to be found. **(7 marks)**

b Find the percentage error made in using the series expansion in part **a** to estimate the value of $q(0.1)$. Give your answer to 2 significant figures. **(4 marks)**

Challenge

Obtain the first four non-zero terms in the expansion, in ascending powers of x, of the function $f(x)$ where $f(x) = \dfrac{1}{\sqrt{1 + 3x^2}}$, $3x^2 < 1$

Summary of key points

1 This form of the binomial expansion can be applied to negative or fractional values of n to obtain an infinite series:

$$(1 + x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \ldots + \frac{n(n-1)\ldots(n-r+1)x^r}{r!} + \ldots$$

The expansion is valid when $|x| < 1$, $n \in \mathbb{R}$.

2 The expansion of $(1 + bx)^n$, where n is negative or a fraction, is valid for $|bx| < 1$, or $|x| < \dfrac{1}{|b|}$

3 The expansion of $(a + bx)^n$, where n is negative or a fraction, is valid for $\left|\dfrac{b}{a}x\right| < 1$ or $|x| < \left|\dfrac{a}{b}\right|$

4 If an expression is of the form $\dfrac{f(x)}{g(x)}$ where $g(x)$ can be split into linear factors, then split $\dfrac{f(x)}{g(x)}$ into partial fractions before expanding each part of the new expression.