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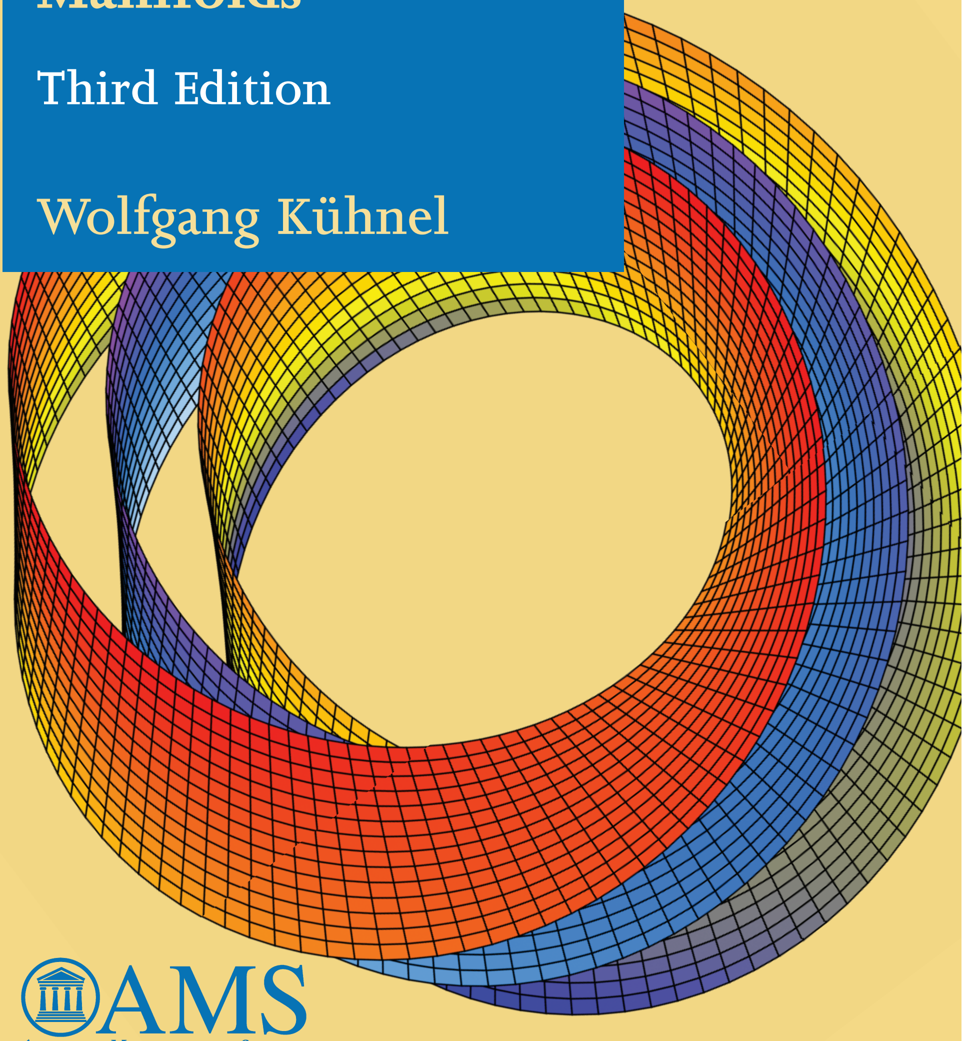
Differential Geometry

Curves – Surfaces – Manifolds

Third Edition

Wolfgang Kühnel



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Translated by
Bruce Hunt

AMS
American Mathematical Society
Providence, Rhode Island

Translated by Bruce Hunt, with corrections and additions by the author.

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Contents

Preface to the English Edition

The German original was intended for courses on differential geometry for students in the middle of their academic education, that is, in the second or third year. In the Anglo-American system of university education, the contents of this textbook corresponds to an undergraduate course in elementary differential geometry (Chapters 1 – 4), followed by a beginning course in Riemannian geometry (Chapters 5 – 8). This led to the idea of having a translation of the German original into English.

I am very glad that the American Mathematical Society supported this project and published the present English version. I thank the translator, Bruce Hunt, for the hard work he had spent on the translation. From the beginning he was surprised by the quantity of text, compared to the quantity of formulas. In addition he had to struggle with complicated and long paragraphs in German. One of the major problems was to adapt the terminology of special notions in the theory of curves and surfaces to the English language. Another problem was to replace almost all references to German texts by references to English texts, in particular, all references to elementary textbooks on calculus, linear algebra, geometry, and topology. Ultimately all these problems could be solved, at least to a certain approximation. The bibliography contains only books in English, with just three exceptions. Therefore, the English version can be used as a textbook for third-year undergraduates and beginning graduate students.

Furthermore, I am grateful to Edward Dunne from the AMS who was extremely helpful at all stages of the project, not only for editorial and technical matters, but also for questions concerning the terminology and the tradition of notations. He pointed out that the ordinary spherical coordinates on the sphere, denoted by φ, ϑ in this book, are denoted ϑ, φ (that is, the other way around) in many English textbooks on calculus. We hope that this does not lead to major confusions.

In the second English edition a number of errors were corrected and a number of additional figures were added, following the second German edition. Most of the additional figures were provided by Gabriele Preissler and Michael Steller. The illustrations play an important rôle in this book. Hopefully they make the book more readable. The concept of having boxes around important statements was kept from the German original, even though now we have a few very large boxes covering major parts of certain pages.

Stuttgart, June 2005 *W. Kühnel*

The present third edition is a corrected and updated version that incorporates the development of altogether six editions in German, the last one from 2013. Each of these German editions was corrected, extended and improved in several directions. As an example, a number of proofs were made more precise if they turned out to be too short in the first edition. In comparison to the second English edition, the third edition includes many improvements, there are more figures and more exercises, and - as a new feature - at the end a number of solutions to selected exercises are given.

Stuttgart, July 2014 *W. Kühnel*

Preface to the German Edition

This book arose from courses given on the topic of "Differential geometry", which the author has given several times in different places. The amount of material corresponds roughly to a course in classical differential geometry of one semester length (Chapters 1-4 of the book), followed by a second one-semester course on Riemannian geometry (Chapters 5-8). The prerequisites are the standard courses in calculus (including several variables) and linear algebra. Only in section 3D (on minimal surfaces) do we assume some familiarity with complex function theory. For this reason the book is appropriate for a course in the latter part of the undergraduate curriculum, not only for students majoring in mathematics, but also those majoring in physics and other natural sciences. Accordingly, we do not present any material which could in any way be considered original. Instead, our intent is to present the basic notions and results which will enable the interested student to go on and study the masters. Especially in the introductory chapters we will take particular care in presenting the material with emphasis on the geometric intuition which is so characteristic of the topic of differential geometry; this is supported by a large number of figures in this part of the book. The results which the author considers particularly important are placed in boxes to emphasize them. These results can be thought of as a kind of skeleton of the theory.

This book wouldn't have been possible without the generous help of my students and colleagues, who found numerous mistakes in the distributed notes of the first version of this book. In particular I would like to mention Gunnar Ketelhut, Eric Sparla, Michael Steller and Gabriele Preissler, who spent considerable time and effort in reading the original notes. G. Ketelhut also supplied numerous suggestions for improvements in the text, as well as writing Section 8F himself. Martin Renner provided almost all the figures, which were produced with the computer algebra system MAPLE. Marc-Oliver Otto provided some figures for Chapter 7, and Ilva Maderer typed the original version in LaTeX. Finally, Michael Grüter accompanied the whole production process with helpful suggestions, as well as giving me personal support in several ways. The work and insistence of Dr. Ulrike Schmickler-Hirzebruch is responsible for the speed with which these lectures were nonetheless accepted for the series "Vieweg-Studium Aufbaukurs Mathematik" and then also appeared almost on time. My thanks goes to all of them.

Stuttgart, June 1999 *W. Kühnel*

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List of notation

$\mathbb{Z}, \mathbb{R}$ integers, real numbers

$\mathbb{R}^n$ real vector space, also Euclidean space with fixed origin

E^n Euclidean space without fixed origin

S^n n-dimensional unit sphere in $\mathbb{R}^{n+1}$

$\mathbb{R}^n_1$ Minkowski space or Lorentzian space

H^n hyperbolic space

$\mathbb{C}, \mathbb{H}$ complex numbers, quaternions

$\langle\ ,\ \rangle$ Euclidean scalar product, in Chapters 5 to 8 also a Riemannian metric

$\langle\ ,\ \rangle_1$ Lorentzian metric in Minkowski space $\mathbb{R}^3_1$

I, II, III first, second and third fundamental forms

g_{ij}, h_{ij}, e_{ij} first, second and third fundamental forms in local coordinates

g^{ij} inverse matrix to g_{ij}

$h^k_i = \sum_j h_{ij} g^{jk}$ Weingarten mapping in local coordinates

E, F, G Gaussian symbols for the first fundamental form $E = g_{11}, F = g_{12}, G = g_{22}$

g Riemannian metric

κ curvature of a plane or space curve

τ torsion of a space curve

$e_1, \ldots, e_n$ Frenet n-frame of a Frenet curve

$\kappa_1, \ldots, \kappa_{n-1}$ Frenet curvatures of a Frenet curve in $\mathbb{R}^n$ (in Ch. 2)

$\dot{c} = \frac{dc}{dt}$ tangent vectors to a curve with parameter t

$c' = \frac{dc}{ds}$ tangent vectors to a curve with arc length parameter s

U_c index of a closed plane curve c

κ_N normal curvature of a curve on a surface

κ_g geodesic curvature of a curve on a surface

ν Gaussian normal mapping, Gauss map

L Weingarten mapping

κ_1, κ_2 principal curvatures of a surface element in $\mathbb{R}^3$

$\kappa_1, \ldots, \kappa_n$ principal curvatures of a hypersurface in $\mathbb{R}^{n+1}$ (in Ch. 3)

λ parameter of distribution of a ruled surface

dA area element of a two-dimensional surface element

dV volume element in higher dimensions

H mean curvature

K Gaussian curvature

K_i ith mean curvature (on hypersurface elements)

D directional derivative in $\mathbb{R}^n$

∇ covariant derivative or Riemannian connection

$[X, Y]$ Lie bracket of two vector fields X, Y

$\Gamma_{ij}^k, \Gamma_{ij,m}$ Christoffel symbols

$R(X, Y)Z$ curvature tensor

R_{ijk}^s, R_{ijkl} curvature tensor in local coordinates

$\mathrm{Ric}(X, Y)$ Ricci tensor

$\mathrm{ric}(X)$ Ricci curvature in the direction X

R_{ij} Ricci tensor in local coordinates

S scalar curvature

W, C Weyl and Schouten tensors

$\exp_p$ exponential mapping at a point p

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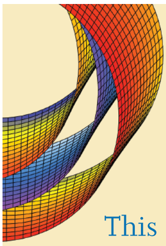

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