

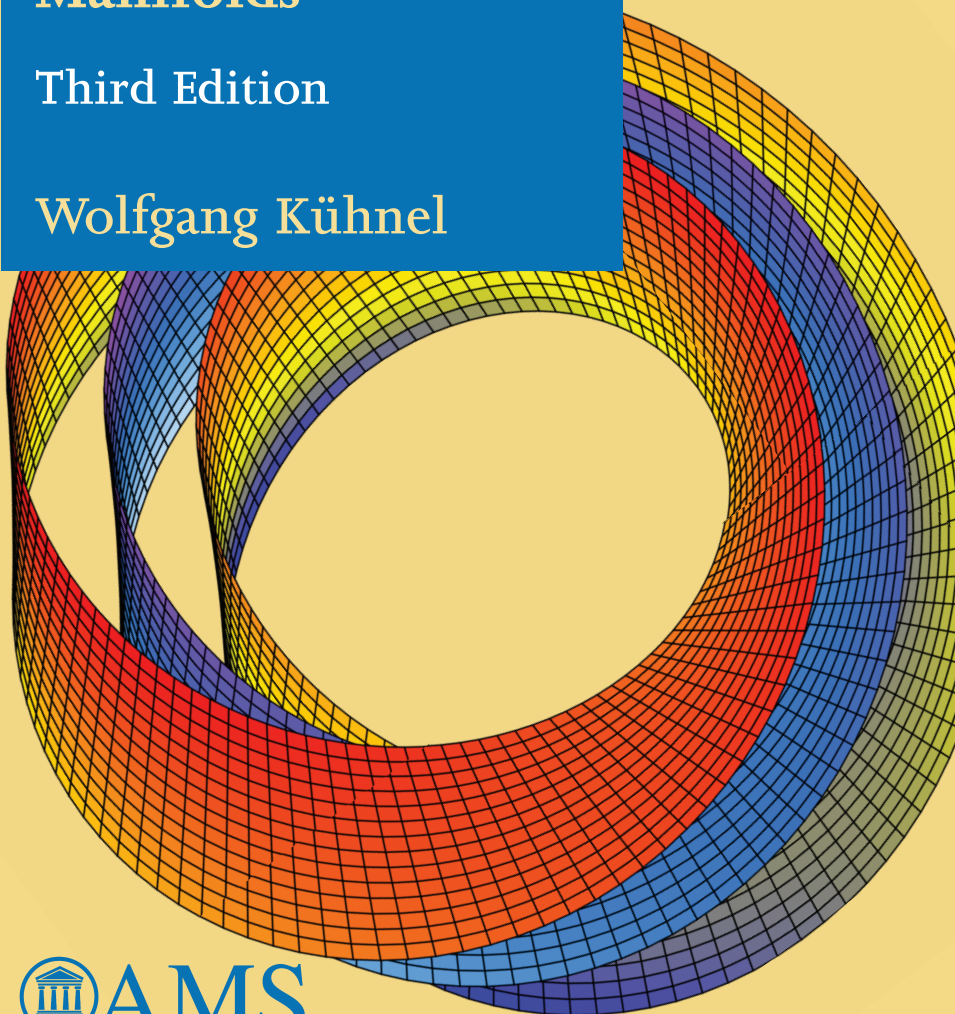
STUDENT MATHEMATICAL LIBRARY
Volume 77

Differential Geometry

Curves – Surfaces –
Manifolds

Third Edition

Wolfgang Kühnel



 **AMS**
AMERICAN MATHEMATICAL SOCIETY

Differential
Geometry
Curves—Surfaces—
Manifolds

STUDENT MATHEMATICAL LIBRARY
Volume 77

Differential
Geometry
Curves—Surfaces—
Manifolds

Third Edition

Wolfgang Kühnel

Translated by
Bruce Hunt



American Mathematical Society
Providence, Rhode Island

Editorial Board

Satyan L. Devadoss John Stillwell (Chair)
Erica Flapan Serge Tabachnikov

Translation from German language edition: *Differentialgeometrie* by Wolfgang Kühnel, ©2013 Springer Vieweg | Springer Fachmedien Wiesbaden GmbH JAHR (formerly Vieweg+Teubner). Springer Fachmedien is part of Springer Science+Business Media. All rights reserved.

Translated by Bruce Hunt, with corrections and additions by the author.

Front and back cover image by Mario B. Schulz.

2010 *Mathematics Subject Classification*. Primary 53-01.

For additional information and updates on this book, visit
www.ams.org/bookpages/stml-77

Page 403 constitutes an extension of this copyright page.

Library of Congress Cataloging-in-Publication Data

Kühnel, Wolfgang, 1950–

[Differentialgeometrie. English]

Differential geometry : curves, surfaces, manifolds / Wolfgang Kühnel ; translated by Bruce Hunt.— Third edition.

pages cm. — (Student mathematical library ; volume 77)

Includes bibliographical references and index.

ISBN 978-1-4704-2320-9 (alk. paper)

1. Geometry, Differential. 2. Curves. 3. Surfaces. 4. Manifolds (Mathematics)

I. Hunt, Bruce, 1958– II. Title.

QA641.K9613 2015

516.3'6—dc23

2015018451

© 2015 by the American Mathematical Society. All rights reserved.

The American Mathematical Society retains all rights
except those granted to the United States Government.

Printed in the United States of America.

⊗ The paper used in this book is acid-free and falls within the guidelines
established to ensure permanence and durability.

Visit the AMS home page at <http://www.ams.org/>

10 9 8 7 6 5 4 3 2 1 20 19 18 17 16 15

Contents

| | |
|---|-----|
| Preface to the English Edition | ix |
| Preface to the German Edition | xi |
| Chapter 1. Notations and Prerequisites from Analysis | 1 |
| Chapter 2. Curves in \mathbb{R}^n | 7 |
| 2A Frenet curves in \mathbb{R}^n | 7 |
| 2B Plane curves and space curves | 14 |
| 2C Relations between the curvature and the torsion | 20 |
| 2D The Frenet equations and the fundamental theorem of the local theory of curves | 27 |
| 2E Curves in Minkowski space \mathbb{R}_1^3 | 33 |
| 2F The global theory of curves | 37 |
| Exercises | 50 |
| Chapter 3. The Local Theory of Surfaces | 55 |
| 3A Surface elements and the first fundamental form | 56 |
| 3B The Gauss map and the curvature of surfaces | 66 |
| 3C Surfaces of rotation and ruled surfaces | 77 |
| 3D Minimal surfaces | 96 |
| 3E Surfaces in Minkowski space \mathbb{R}_1^3 | 113 |

| | |
|--|-----|
| 3F Hypersurfaces in \mathbb{R}^{n+1} | 122 |
| Exercises | 125 |
| Chapter 4. The Intrinsic Geometry of Surfaces | 133 |
| 4A The covariant derivative | 134 |
| 4B Parallel displacement and geodesics | 140 |
| 4C The Gauss equation and the Theorema Egregium | 145 |
| 4D The fundamental theorem of the local theory of surfaces | 152 |
| 4E The Gaussian curvature in special parameters | 157 |
| 4F The Gauss-Bonnet Theorem | 165 |
| 4G Selected topics in the global theory of surfaces | 180 |
| Exercises | 192 |
| Chapter 5. Riemannian Manifolds | 197 |
| 5A The notion of a manifold | 198 |
| 5B The tangent space | 205 |
| 5C Riemannian metrics | 212 |
| 5D The Riemannian connection | 218 |
| Chapter 6. The Curvature Tensor | 233 |
| 6A Tensors | 233 |
| 6B The sectional curvature | 242 |
| 6C The Ricci tensor and the Einstein tensor | 250 |
| Chapter 7. Spaces of Constant Curvature | 265 |
| 7A Hyperbolic space | 266 |
| 7B Geodesics and Jacobi fields | 276 |
| 7C The space form problem | 291 |
| 7D Three-dimensional Euclidean and spherical space forms | 296 |
| Exercises | 306 |
| Chapter 8. Einstein Spaces | 309 |
| 8A The variation of the Hilbert-Einstein functional | 312 |
| 8B The Einstein field equations | 321 |

| | |
|--|-----|
| 8C Homogenous Einstein spaces | 325 |
| 8D The decomposition of the curvature tensor | 331 |
| 8E The Weyl tensor | 341 |
| 8F Duality for four-manifolds and Petrov types | 350 |
| Exercises | 358 |
| Solutions to selected exercises | 361 |
| Bibliography | 391 |
| List of notation | 395 |
| Index | 397 |

Preface to the English Edition

The German original was intended for courses on differential geometry for students in the middle of their academic education, that is, in the second or third year. In the Anglo-American system of university education, the contents of this textbook corresponds to an undergraduate course in elementary differential geometry (Chapters 1 – 4), followed by a beginning course in Riemannian geometry (Chapters 5 – 8). This led to the idea of having a translation of the German original into English.

I am very glad that the American Mathematical Society supported this project and published the present English version. I thank the translator, Bruce Hunt, for the hard work he had spent on the translation. From the beginning he was surprised by the quantity of text, compared to the quantity of formulas. In addition he had to struggle with complicated and long paragraphs in German. One of the major problems was to adapt the terminology of special notions in the theory of curves and surfaces to the English language. Another problem was to replace almost all references to German texts by references to English texts, in particular, all references to elementary textbooks on calculus, linear algebra, geometry, and topology. Ultimately all these problems could be solved, at least to a certain approximation. The

bibliography contains only books in English, with just three exceptions. Therefore, the English version can be used as a textbook for third-year undergraduates and beginning graduate students.

Furthermore, I am grateful to Edward Dunne from the AMS who was extremely helpful at all stages of the project, not only for editorial and technical matters, but also for questions concerning the terminology and the tradition of notations. He pointed out that the ordinary spherical coordinates on the sphere, denoted by φ, ϑ in this book, are denoted ϑ, φ (that is, the other way around) in many English textbooks on calculus. We hope that this does not lead to major confusions.

In the second English edition a number of errors were corrected and a number of additional figures were added, following the second German edition. Most of the additional figures were provided by Gabriele Preissler and Michael Steller. The illustrations play an important rôle in this book. Hopefully they make the book more readable. The concept of having boxes around important statements was kept from the German original, even though now we have a few very large boxes covering major parts of certain pages.

Stuttgart, June 2005

W. Kühnel

The present third edition is a corrected and updated version that incorporates the development of altogether six editions in German, the last one from 2013. Each of these German editions was corrected, extended and improved in several directions. As an example, a number of proofs were made more precise if they turned out to be too short in the first edition. In comparison to the second English edition, the third edition includes many improvements, there are more figures and more exercises, and - as a new feature - at the end a number of solutions to selected exercises are given.

Stuttgart, July 2014

W. Kühnel

Preface to the German Edition

This book arose from courses given on the topic of “Differential geometry”, which the author has given several times in different places. The amount of material corresponds roughly to a course in classical differential geometry of one semester length (Chapters 1-4 of the book), followed by a second one-semester course on Riemannian geometry (Chapters 5-8). The prerequisites are the standard courses in calculus (including several variables) and linear algebra. Only in section 3D (on minimal surfaces) do we assume some familiarity with complex function theory. For this reason the book is appropriate for a course in the latter part of the undergraduate curriculum, not only for students majoring in mathematics, but also those majoring in physics and other natural sciences. Accordingly, we do not present any material which could in any way be considered original. Instead, our intent is to present the basic notions and results which will enable the interested student to go on and study the masters. Especially in the introductory chapters we will take particular care in presenting the material with emphasis on the geometric intuition which is so characteristic of the topic of differential geometry; this is supported by a large number of figures in this part of the book. The results which the author considers particularly important are placed

in boxes to emphasize them. These results can be thought of as a kind of skeleton of the theory.

This book wouldn't have been possible without the generous help of my students and colleagues, who found numerous mistakes in the distributed notes of the first version of this book. In particular I would like to mention Gunnar Ketelhut, Eric Sparla, Michael Steller and Gabriele Preissler, who spent considerable time and effort in reading the original notes. G. Ketelhut also supplied numerous suggestions for improvements in the text, as well as writing Section 8F himself. Martin Renner provided almost all the figures, which were produced with the computer algebra system MAPLE. Marc-Oliver Otto provided some figures for Chapter 7, and Ilva Maderer typed the original version in \LaTeX . Finally, Michael Grüter accompanied the whole production process with helpful suggestions, as well as giving me personal support in several ways. The work and insistence of Dr. Ulrike Schmickler-Hirzebruch is responsible for the speed with which these lectures were nonetheless accepted for the series "Vieweg-Studium Aufbaukurs Mathematik" and then also appeared almost on time. My thanks goes to all of them.

Stuttgart, June 1999

W. Kühnel

Bibliography

Textbooks on Differential Geometry

- [1] M. do Carmo, “Differential Geometry of Curves and Surfaces”, Prentice Hall, Englewood Cliffs, NJ, 1976.
 - [2] W. Klingenberg, “A Course in Differential Geometry”, Springer, New York, 1978.
 - [3] D. Laugwitz, “Differential and Riemannian Geometry”, Academic Press, New York, 1965.
 - [4] D. J. Struik, “Lectures on Classical Differential Geometry”, Addison-Wesley 1950/1961, reprinted by Dover, New York, 1988.
 - [5] E. Kreyszig, “Differential Geometry”, Dover, New York, 1991.
 - [6] J. J. Stoker, “Differential Geometry”, Wiley-Interscience, New York, 1969, reprint 1989.
 - [7] M. Spivak, “A Comprehensive Introduction to Differential Geometry”, Volumes I – V, Publish or Perish, Wilmington, Del., 1979.
 - [8] T.G.Feeman, *Portraits of the Earth: A Mathematician Looks at Maps* American Mathematical Society, 2002. Paperback, 123 pages Mathematical World, vol 18 MAWRLD/18
- With special emphasis on computer graphics:**
- [9] A. Gray, “Modern Differential Geometry of Curves and Surfaces”, 2nd edition, CRC Press, Boca Raton, 1998.
 - [10] J. Oprea, “The Mathematics of Soap Films: Explorations with Maple”, Student Math. Library Vol. 10, AMS, Providence, 2000.

On minimal surfaces:

- [11] U. Dierkes, S. Hildebrandt, A. Küster, O. Wohlrab, "Minimal Surfaces I", Springer, 1992.
- [12] D. Hoffmann & H. Karcher, *Complete embedded minimal surfaces of finite total curvature*, Geometry V (R. Osserman, ed.), Encycl. Math. Sci. **90**, 5–93, Springer 1997.

Textbooks on Riemannian Geometry

- [13] P. Petersen, "Riemannian Geometry", 2nd edition, Springer, New York, 2006.
- [14] M. do Carmo, "Riemannian Geometry", Birkhäuser, Boston-Basel-Berlin, 1992.
- [15] J. A. Schouten, "Der Ricci-Kalkül", Springer, Heidelberg, 1924, reprint Springer, Berlin-New York, 1978.
- [16] J. A. Schouten, "Ricci-calculus", Springer, Heidelberg, 1954.
- [17] T. J. Willmore, "Total Curvature in Riemannian Geometry", Ellis Horwood, 1982.
- [18] J. Cheeger, D. G. Ebin, "Comparison Theorems in Riemannian Geometry", North Holland, 1975.
- [19] S. Kobayashi, K. Nomizu, "Foundations of Differential Geometry" I, Wiley-Interscience, 1963.
- [20] J. Wolf, "Spaces of Constant Curvature", Publish or Perish, Boston, 1974.

Including the general theory of relativity:

- [21] T. Levi-Civita, "The Absolute Differential Calculus (Calculus of Tensors)", Dover, New York, 1977.
- [22] B. O'Neill, "Semi-Riemannian Geometry", Academic Press, San Diego, 1983.
- [23] A. Petrov, "Einstein Spaces", Pergamon Press, Oxford, 1969.
- [24] A. Besse, "Einstein Manifolds", Springer, Heidelberg-New York, 1987.
- [25] F. de Felice & C. J. S. Clarke, "Relativity on Curved Manifolds", Cambridge University Press, 1990.
- [26] R. K. Sachs & H. Wu, "General Relativity for Mathematicians", Springer, New York, 1977.

Other textbooks**On analysis:**

- [27] S. Lang, "Undergraduate Analysis", 2nd ed., Springer, New York, 1997.

- [28] S. Lang, "Calculus of Several Variables", Addison-Wesley, 1973.
- [29] M. Spivak, "Calculus on Manifolds", Benjamin, New York, 1965.
- [30] L. C. Evans, "Partial Differential Equations", Graduate Studies in Math. 19, AMS, Providence, 1998.

On algebra:

- [31] S. Lang, "Linear Algebra", 3rd ed., Springer, New York, 1987.
- [32] W. Greub, "Linear Algebra", 3rd ed., Springer, Heidelberg, 1967.
- [33] W. Greub, "Multilinear Algebra", Springer, Heidelberg, 1967.
- [34] H. S. M. Coxeter & W. O. J. Moser, "Generators and Relations for Discrete Groups", 4th ed., Springer, Heidelberg, 1980.

On the theory of functions of a complex variable:

- [35] J. Bak & D. J. Newman, "Complex Analysis", Springer, New York, 1982.
- [36] R. V. Churchill, J. W. Brown, R. F. Verhey, "Complex Variables and Applications", McGraw-Hill, New York 1974.

On topology:

- [37] R. E. Schwartz, "Mostly Surfaces", Student Math. Library Vol. 60, AMS, Providence, 2011.
- [38] M. A. Armstrong, "Basic Topology", Springer, 1983.
- [39] T. Bröcker & K. Jänich, "Introduction to Differential Topology", Cambridge University Press, 1982.
- [40] H. Schubert, "Topology", MacDonald Technical and Scientific, 1968.
- [41] H. Seifert & W. Threlfall, "Lehrbuch der Topologie", Teubner 1934 (reprint Chelsea 1980); Engl. transl.: "Seifert and Threlfall: A Textbook of Topology". Academic Press, New York, 1980.

On Lie groups:

- [42] S. Arvanitoyeorgos, "An Introduction to Lie Groups and the Geometry of Homogeneous Spaces", Student Math. Library Vol. 22, AMS, Providence, 2003.
- [43] H. Pollatsek, "Lie Groups: A problem-oriented introduction via matrix groups", Math. Association of America, 2009
- [44] F. W. Warner, "Foundations of Differentiable Manifolds and Lie Groups", Springer, Heidelberg-New York, 1983.
- [45] J. J. Duistermaat & J. A. C. Kolk, "Lie Groups", Springer, Heidelberg, 2000.
- [46] S. Kobayashi, "Transformation Groups in Differential Geometry", Springer, Heidelberg, 1972.

On geometry:

- [47] J. Stillwell, "Geometry of Surfaces", Springer, New York, 1992.
- [48] H. S. M. Coxeter, "Regular Polytopes", Dover, New York, 1948.
- [49] F. Klein, "Vorlesungen über Nicht-Euklidische Geometrie", Springer 1928 (reprint 1968).
- [50] J. M. Montesinos, "Classical Tessellations and Three-manifolds", Springer, Heidelberg-New York, 1987.
- [51] P. J. Ryan, "Euclidean and Non-Euclidean Geometry", Cambridge Univ. Press, 1986.
- [52] M. A. Armstrong, "Groups and Symmetry", Springer, New York, 1988.

On the history of mathematics:

- [53] F. Klein, "Development of Mathematics in the 19th Century", Lie groups: History, Frontiers and Appl. IX, Math. Sci. Press, Brookline MA, 1979.
- [54] P. Dombrowski, "150 years after Gauss' *Disquisitiones generales circa superficies curvas*", With the original text of Gauss. Astérisque, 62, Société Mathématique de France, Paris, 1979.

List of notation

\mathbb{Z}, \mathbb{R} integers, real numbers

\mathbb{R}^n real vector space, also Euclidean space with fixed origin

E^n Euclidean space without fixed origin

S^n n -dimensional unit sphere in \mathbb{R}^{n+1}

\mathbb{R}_1^n Minkowski space or Lorentzian space

H^n hyperbolic space

\mathbb{C}, \mathbb{H} complex numbers, quaternions

\langle , \rangle Euclidean scalar product, in Chapters 5 to 8 also a Riemannian metric

\langle , \rangle_1 Lorentzian metric in Minkowski space \mathbb{R}_1^3

I, II, III first, second and third fundamental forms

g_{ij}, h_{ij}, e_{ij} first, second and third fundamental forms in local coordinates

g^{ij} inverse matrix to g_{ij}

$h_i^k = \sum_j h_{ij} g^{jk}$ Weingarten mapping in local coordinates

E, F, G Gaussian symbols for the first fundamental form $E = g_{11}, F = g_{12}, G = g_{22}$

g Riemannian metric

κ curvature of a plane or space curve

τ torsion of a space curve

e_1, \dots, e_n Frenet n -frame of a Frenet curve

$\kappa_1, \dots, \kappa_{n-1}$ Frenet curvatures of a Frenet curve in \mathbb{R}^n (in Ch. 2)

-
- $\dot{c} = \frac{dc}{dt}$ tangent vectors to a curve with parameter t
 $c' = \frac{dc}{ds}$ tangent vectors to a curve with arc length parameter s
 U_c index of a closed plane curve c
 κ_N normal curvature of a curve on a surface
 κ_g geodesic curvature of a curve on a surface
 ν Gaussian normal mapping, Gauss map
 L Weingarten mapping
 κ_1, κ_2 principal curvatures of a surface element in \mathbb{R}^3
 $\kappa_1, \dots, \kappa_n$ principal curvatures of a hypersurface in \mathbb{R}^{n+1} (in Ch. 3)
 λ parameter of distribution of a ruled surface
 dA area element of a two-dimensional surface element
 dV volume element in higher dimensions
 H mean curvature
 K Gaussian curvature
 K_i i th mean curvature (on hypersurface elements)
 D directional derivative in \mathbb{R}^n
 ∇ covariant derivative or Riemannian connection
 $[X, Y]$ Lie bracket of two vector fields X, Y
 $\Gamma_{ij}^k, \Gamma_{ij,m}$ Christoffel symbols
 $R(X, Y)Z$ curvature tensor
 R_{ijk}^s, R_{ijkl} curvature tensor in local coordinates
 $\text{Ric}(X, Y)$ Ricci tensor
 $\text{ric}(X)$ Ricci curvature in the direction X
 R_{ij} Ricci tensor in local coordinates
 S scalar curvature
 W, C Weyl and Schouten tensors
 \exp_p exponential mapping at a point p

Index

- acceleration vector, 141
- angle, 2
- angle preserving, 99, 127
- apex, 12
- arc element, 60
- arc length, 9
- Archimedean spiral, 51
- asymptotic curve, 83, 126
- atlas, 200, 202

- Banchoff, T., 185
- Beltrami, E., 84, 93
- Bertrand curve, 54
- Bianchi identity, 243, 335
- binary dihedral group, 303
- binary icosahedral group, 303
- binary octahedral group, 303
- binary tetrahedral group, 303
- binormal, 17
- biquadratic form, 246
- bivector, 332
- Bonnet, O., 153
- boost, 272

- canal surface, 77
- Cardan angles, 202
- Cartan, É., 165
- Catalan, E. C., 111, 128
- catenary, 11, 109, 193
- catenoid, 109, 128, 156, 193, 262

- Cauchy-Riemann equations, 101
- Cayley map, 201
- Cayley plane, 330
- Cayley ruled surface, 132
- chain rule, 210
- chart, 5, 198, 199, 202
- Christoffel symbols, 139, 166, 223
- circle, 9
- Clifford torus, 32
- Codazzi-Mainardi equation, 147, 154, 168, 196, 242
- Cohn-Vossen, S., 188
- complex manifold, 203
- complex projective space, 329
- complex structure, 204
- cone, 88, 90, 92
- conformal, 99, 103, 216, 274
- conformal curvature, 341
- conformally flat, 124, 346, 347
- conic type, 81
- conjugate point, 307
- conjugate surface, 110
- connection, 221
- connection form, 166
- constant curvature, 25, 36, 80, 161, 189, 249, 269, 289, 310
- constant mean curvature, 191
- contact of k th order, 12
- contraction, 250
- contravariant tensor, 236

- convergence, 204
 convex, 43, 182
 convex hull, 45, 182, 304
 coordinate transformation, 200
 Cornu spiral, 16
 cosmological constant, 325
 countability axiom, 218
 covariant derivative, 136, 138, 166, 221, 239, 268
 covariant tensor, 236
 covector field, 237
 covering, 293, 302, 310
 Coxeter, H.S.M., 303
 CR equations, 101
 critical point, 187
 cubical parabola, 20
 curvature, 14, 17, 20, 36, 70
 curvature tensor, 150, 169, 243, 247, 315
 curve, 7, 8
 curve, closed, 37
 curve, length of, 8
 curve, simply closed, 37, 46
 cyclic group, 302, 303
 cycloid, 50
 cylinder, 88, 90

 Darboux equations, 26, 53
 Darboux vector, 26, 53
 derivative, 3, 6, 210
 developable surface, 88, 119
 dicyclic group, 303
 diffeomorphic, 204
 differentiable, 3
 differentiable manifold, 199
 differentiable structure, 200
 differential, 6, 210, 240
 differential form, 166
 dihedral group, 302
 Dini, U., 93
 directional derivative, 98, 135, 206, 207, 221
 directional vector, 62
 directrix, 84
 distance, 2
 divergence, 251, 257, 319, 320
 double point, 37, 49
 double tangent, 49

 dual basis, 212
 duality, 350
 Dupin indicatrix, 75

 eigenvalue, 71
 eigenvector, 71
 Einstein field equations, 324
 Einstein space, 254, 263, 328, 351
 Einstein tensor, 257, 324
 Einstein, A., 312, 317, 351
 ellipse, 75
 ellipsoid, 130
 elliptic point, 72, 193
 elongated sphere, 81
 energy functional, 280
 Enneper, A., 84, 111
 equations of Gauss and Weingarten, 140, 146
 Euclidean motion, 269
 Euler angles, 202
 Euler characteristic, 177, 179, 180, 188, 218
 evolute, 15
 exponential mapping, 226, 280
 exterior derivative, 167

 Fabricius-Bjerre, Fr., 49
 Fenchel, W., 47
 Fermi coordinates, 160
 first fundamental form, 59
 Flamm's paraboloid, 95
 flow, 230
 focal curve, 15
 four vertex theorem, 46
 free motion, 270
 Frenet curvature, 27
 Frenet curve, 13
 Frenet equations, 14, 17, 27, 36
 Frenet matrix, 25, 27, 37
 Frenet n -frame, 13
 Frobenius, G., 155

 Gauss equation, 147, 150, 154, 168, 196, 261, 268, 359
 Gauss formula, 140
 Gauss lemma, 283
 Gauss map, 63, 66, 100, 116, 122
 Gauss, C. F., 148, 320
 Gauss-Bonnet formula, 172, 321

- Gauss-Kronecker curvature, 123
Gaussian curvature, 72, 117, 148,
193, 242
geodesic, 71, 121, 141, 216, 225,
280, 307
geodesic curvature, 23, 71, 125, 171
geodesic parallel coordinates, 160
geodesic polar coordinates, 272,
283, 290
geodesic torsion, 125
geodesic triangle, 176, 230
geometric linearization, 8, 55
golden ratio, 303
gradient, 4, 98, 240, 313
Gram determinant, 181
Gram-Schmidt orthogonalization,
13
graph, 58, 73
- harmonic function, 100
Hausdorff separation axiom, 203
helicoid, 86, 109, 132, 156
helicoidal motion, 10, 131
helicoidal ruled surface, 86, 93
helix, 10, 20
Henneberg, L., 111
Hesse tensor, 240
Hessian, 73, 240, 264, 345
Hessian matrix, 73
hexagonal torus, 294
Hilbert, D., 162, 190, 312, 317
Hilbert-Einstein functional, 312
Hodge operator, 350, 353
holomorphic, 101, 105
holonomy group, 228
homogenous space, 326
Hopf, H., 41, 180
Hurwitz quaternions, 304
hyperbola, 35, 75
hyperbolic plane, 119, 130, 196
hyperbolic point, 72
hyperbolic space, 267, 269, 292
hyperboloid, 73, 83, 85, 113, 127,
267
hyperboloid type, 81
hyperplane, 124
hypersphere, 124
hypersurface element, 122
- icosahedral group, 302
icosahedron, 303
immersion, 3, 5, 8, 55
implicit function, 3
index, 124
index form, 277
inflection point, 14, 49
inner product, 2, 98, 214
integrability conditions, 146, 147,
149, 155, 168, 264, 348
inverse mapping, 4
irreducible, 327
isometric, 59, 110, 161, 216
isometry, 216
isometry group, 326
isothermal, 99
isotropic, 34, 35, 114, 131
isotropy group, 326
- Jacobi determinant, 64
Jacobi equation, 284
Jacobi field, 284, 286, 307
Jacobi identity, 220, 243
Jacobian, 3
- Killing field, 219
Klein bottle, 201, 218
Koszul formula, 343
Kuiper, N. H., 185
- Lagrange multiplier, 72
Laplace-Beltrami operator, 252
Laplacian, 252, 360
length preserving, 161
lens space, 304, 308
level point, 72, 104, 105
Levi-Civita connection, 221
Lie algebra, 227
Lie bracket, 137, 219, 231
Lie derivative, 219
Lie group, 227
Lie, S., 219
Liebmann, H., 47, 189, 191
light-cone, 34, 113
light-like, 34
light-like line, 35
line, 9, 84, 121
line of curvature, 76

- lines of curvature parameters, 76, 103
 locally compact, 204
 locally isometric, 249
 logarithmic spiral, 51
 Lorentz group, 270
 Lorentz rotation, 117
 Lorentz space, 33, 270
 Lorentz transformation, 271
 Lorentzian metric, 214, 218

 manifold, 199
 Maurer-Cartan equations, 168
 mean curvature, 72, 98, 123, 129
 mean curvature vector, 100
 measure tensor, 234
 Mercator projection, 127
 meridian curve, 77
 meromorphic, 105, 204
 metric tensor, 234
 Meusnier, M., 71
 minimal surface, 98, 131
 Minkowski space, 33, 113, 214, 270
 Möbius strip, 65
 Möbius, A., 65
 Monge coordinates, 73, 75, 124
 Monge surface, 132
 monkey saddle, 73
 multilinear, 236
 multiplicity, 307
 mylar balloon, 95

 Neil parabola, 19
 non-Euclidean geometry, 120
 norm, 2
 normal coordinates, 281
 normal curvature, 71, 171
 normal plane, 19
 normal section, 71
 normal space, 6, 56
 normal variation, 96
 normal vector, 14, 57, 66, 115, 122
 null cubic, 371
 null vector, 34, 113, 214, 266
 null-cone, 113

 oblate sphere, 81
 octahedral group, 302
 octahedral space, 304

 octahedron, 303
 orientability, 63
 orthogonal group, 201, 270
 osculating plane, 19, 71
 osculating sphere, 20
 ovaloid, 182

 parabola, 19
 parabola of contact, 50
 parabolic point, 72
 paraboloid, 73, 127
 parallel, 141, 225, 263
 parallel displacement, 142, 225
 parallel surface, 65, 129
 parameter, 56
 parameter of distribution, 86
 parameter transformation, 64
 parametrization, 5, 56, 198
 parametrized curve, 8
 partition of unity, 217
 Petrov type, 358
 Pfaffian form, 166
 Poincaré upper half-plane, 194, 216, 267, 306
 polar angle function, 39, 42
 polar coordinates, 38, 272, 273
 polarization, 246
 position vector, 62
 potential equation, 346
 primitive, 103
 principal curvature, 71, 123, 253, 258
 principal normal, 17, 193
 prism space, 304
 product rule, 137, 206, 221, 239
 profile curve, 77
 projective plane, 201, 216
 pseudo-Euclidean space, 37, 266
 pseudo-hyperbolic space, 269
 pseudo-Riemannian metric, 214
 pseudo-sphere, 82, 269

 quadratic integral, 60
 quaternion algebra, 300
 quaternion group, 305
 quaternion space, 304, 305
 quaternionic projective space, 330
 quaternions, 300

- rank, 3
- rank theorem, 4
- rectifying developable, 128
- rectifying plane, 20
- relativity theory, 214
- relativity, special, 33
- Ricci calculus, 207
- Ricci curvature, 255, 260, 286
- Ricci flow, 253
- Ricci tensor, 252, 331
- Ricci, G., 207, 241
- Riemann sphere, 204
- Riemann, B., 197, 263
- Riemannian connection, 221
- Riemannian manifold, 213
- Riemannian metric, 213
- Rodrigues, O., 72
- rotation group, 301
- rotation index, 39, 40
- rotation matrix, 202
- rotational torus, 58, 129
- ruled surface, 77, 84, 92, 119
- ruling, 84, 132, 259, 264, 368

- saddle point, 75
- scalar curvature, 152, 193, 252, 260, 317, 331
- scaling, 249
- Scherk, H. F., 111
- Schmidt orthogonalization, 13
- Schouten tensor, 347
- Schouten, J. A., 347
- Schur, F., 248
- Schwarz, H. A., 112
- Schwarzschild metric, 231, 359
- screw-motion, 10, 87, 131, 372
- scroll, 77
- second fundamental form, 68, 116, 237, 261
- sectional curvature, 151, 242, 245–247, 285, 351
- self-adjoint, 67, 264, 333
- semi-Riemannian metric, 214
- shape operator, 67
- shortest path, 144, 279
- singularity, 56, 80, 82, 90, 95, 102
- slope line, 23, 52
- space curve, 17
- space form, 291
- space-like, 34, 114, 266
- space-time, 309, 311, 324, 353
- sphere, 57, 66, 73, 75, 80, 81, 95, 123, 127, 195, 266
- spherical coordinates, 60, 123
- spherical curve, 20
- spherical dodecahedral space, 304
- spiral, 51
- square torus, 294
- standard parameters, 84
- Stiefel manifold, 14
- Stokes, G., 170, 320
- striction line, 85, 86
- structural equations, 168
- structure, 202
- submanifold, 5, 57
- submersion, 3, 5
- surface, 55, 58, 179, 180
- surface area, 61, 98
- surface classification, 179
- surface element, 56
- surface integral, 61
- surface of revolution, 77
- surface of rotation, 77, 117
- symmetries, 269

- tangent, 8
- tangent bundle, 5, 229, 308
- tangent developable, 88
- tangent hyperplane, 122
- tangent plane, 56
- tangent space, 5, 6, 57, 205
- tangent surface, 90
- tangent vector, 8, 17, 57, 206
- Taylor expansion, 12, 18
- Tchebychev grid, 126
- tensor, 235, 236
- tensor field, 235
- tensor product, 236
- tetrahedral group, 302
- tetrahedron, 303
- theorem on turning tangents, 41, 173
- Theorema Egregium, 148, 151, 157, 233
- Theorema Elegantissimum, 176, 378

-
- theory of relativity, 317
 - third fundamental form, 68
 - tightness, 184, 186, 187
 - time-like, 34, 114, 266
 - topological manifold, 203
 - topology, 2, 203
 - torse, 88
 - torsion, 17, 20, 27, 36
 - torsion tensor, 221
 - torus, 58, 129, 201, 215, 218
 - torus knot, 32
 - torus of revolution, 58, 129
 - total absolute curvature, 43, 45, 47, 182
 - total curvature, 38, 40, 185, 321
 - total mean curvature, 129
 - totally umbilical, 75, 124
 - trace, 250
 - tractrix, 11, 82
 - transition function, 200
 - truncated cube space, 304
 - tubular surface, 132

 - umbilic, 72, 75, 124

 - variation, 98
 - variation of a metric, 313
 - variation of arc length, 277
 - vector field, 62, 212, 237
 - vector space, 2
 - vertex, 46

 - warped product, 195, 230, 261
 - wedge product, 167
 - Weierstrass representation, 106
 - Weingarten equation, 140
 - Weingarten map, 67, 116, 122, 237, 242, 268
 - Weingarten surface, 93, 130, 132
 - Wente-torus, 191
 - Weyl tensor, 341, 342, 347
 - Weyl, H., 347
 - Willmore conjecture, 129
 - winding number, 39
 - Wolf, J., 328

Page iv constitutes the beginning of this copyright page.

Copying and reprinting. Individual readers of this publication, and nonprofit libraries acting for them, are permitted to make fair use of the material, such as to copy select pages for use in teaching or research. Permission is granted to quote brief passages from this publication in reviews, provided the customary acknowledgment of the source is given.

Republication, systematic copying, or multiple reproduction of any material in this publication is permitted only under license from the American Mathematical Society. Permissions to reuse portions of AMS publication content are handled by Copyright Clearance Center's RightsLink[®] service. For more information, please visit: <http://www.ams.org/rightslink>.

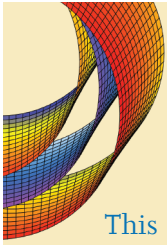
Send requests for translation rights and licensed reprints to reprint-permission@ams.org.

Excluded from these provisions is material for which the author holds copyright. In such cases, requests for permission to reuse or reprint material should be addressed directly to the author(s). Copyright ownership is indicated on the copyright page, or on the lower right-hand corner of the first page of each article within proceedings volumes.

SELECTED PUBLISHED TITLES IN THIS SERIES

- 77 **Wolfgang Kühnel**, *Differential Geometry: Curves — Surfaces — Manifolds*, Third Edition, 2015
- 76 **John Roe**, *Winding Around*, 2015
- 75 **Ida Kantor, Jiří Matoušek, and Robert Šámal**, *Mathematics++*, 2015
- 74 **Mohamed Elhamdadi and Sam Nelson**, *Quandles*, 2015
- 73 **Bruce M. Landman and Aaron Robertson**, *Ramsey Theory on the Integers*, Second Edition, 2014
- 72 **Mark Kot**, *A First Course in the Calculus of Variations*, 2014
- 71 **Joel Spencer**, *Asymptopia*, 2014
- 70 **Lasse Rempe-Gillen and Rebecca Waldecker**, *Primality Testing for Beginners*, 2014
- 69 **Mark Levi**, *Classical Mechanics with Calculus of Variations and Optimal Control*, 2014
- 68 **Samuel S. Wagstaff, Jr.**, *The Joy of Factoring*, 2013
- 67 **Emily H. Moore and Harriet S. Pollatsek**, *Difference Sets*, 2013
- 66 **Thomas Garrity, Richard Belshoff, Lynette Boos, Ryan Brown, Carl Lienert, David Murphy, Junalyn Navarra-Madsen, Pedro Poitevin, Shawn Robinson, Brian Snyder, and Caryn Werner**, *Algebraic Geometry*, 2013
- 65 **Victor H. Moll**, *Numbers and Functions*, 2012
- 64 **A. B. Sossinsky**, *Geometries*, 2012
- 63 **María Cristina Pereyra and Lesley A. Ward**, *Harmonic Analysis*, 2012
- 62 **Rebecca Weber**, *Computability Theory*, 2012
- 61 **Anthony Bonato and Richard J. Nowakowski**, *The Game of Cops and Robbers on Graphs*, 2011
- 60 **Richard Evan Schwartz**, *Mostly Surfaces*, 2011
- 59 **Pavel Etingof, Oleg Golberg, Sebastian Hensel, Tiankai Liu, Alex Schwendner, Dmitry Vaintrob, and Elena Yudovina**, *Introduction to Representation Theory*, 2011
- 58 **Álvaro Lozano-Robledo**, *Elliptic Curves, Modular Forms, and Their L-functions*, 2011
- 57 **Charles M. Grinstead, William P. Peterson, and J. Laurie Snell**, *Probability Tales*, 2011
- 56 **Julia Garibaldi, Alex Iosevich, and Steven Senger**, *The Erdős Distance Problem*, 2011

For a complete list of titles in this series, visit the
AMS Bookstore at www.ams.org/bookstore/stmlseries/.



This carefully written book is an introduction to the beautiful ideas and results of differential geometry. The first half covers the geometry of curves and surfaces, which provide much of the motivation and intuition for the general theory. The second part studies the geometry of general manifolds, with particular emphasis on connections and curvature. The text is illustrated with many figures and examples. The prerequisites are undergraduate analysis and linear algebra. This new edition provides many advancements, including more figures and exercises, and—as a new feature—a good number of solutions to selected exercises.

“This new edition is an improved version of what was already an excellent and carefully written introduction to both differential geometry and Riemannian geometry. In addition to a variety of improvements, the author has included solutions to many of the problems, making the book even more appropriate for use in the classroom.”

— Colin Adams, Williams College

“This book on differential geometry by Kühnel is an excellent and useful introduction to the subject. . . . There are many points of view in differential geometry and many paths to its concepts. This book provides a good, often exciting and beautiful basis from which to make explorations into this deep and fundamental mathematical subject.”

— Louis Kauffman, University of Illinois at Chicago

ISBN: 978-1-4704-2320-9



9 781470 423209

STML/77



For additional information
and updates on this book, visit

www.ams.org/bookpages/stml-77

AMS on the Web
www.ams.org