Common Core State Standards

Mathematics II

Integrated Pathway

Student Resource Book
Unit 1

WALCH EDUCATION®

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Introduction

Welcome to the *CCSS Integrated Pathway: Mathematics II Student Resource Book.* This book will help you learn how to use algebra, geometry, data analysis, and probability to solve problems. Each lesson builds on what you have already learned. As you participate in classroom activities and use this book, you will master important concepts that will help to prepare you for mathematics assessments and other mathematics courses.

This book is your resource as you work your way through the Math II course. It includes explanations of the concepts you will learn in class; math vocabulary and definitions; formulas and rules; and exercises so you can practice the math you are learning. Most of your assignments will come from your teacher, but this book will allow you to review what was covered in class, including terms, formulas, and procedures.

- In **Unit 1: Extending the Number System**, you will learn about rational exponents and the properties of rational and irrational numbers. This is followed by operating with polynomials. Finally, you will define an imaginary number and learn to operate with complex numbers.

- In **Unit 2: Quadratic Functions and Modeling**, you will begin by exploring and interpreting the the graphs of quadratic functions. Then you will learn how to build quadratic functions from a context and how to carry out operations with functions. This gives way to the exploration of other types of functions, including square root, cube root, absolute value, step, and piecewise functions. The unit progresses to analyzing exponential functions and comparing linear, quadratic, and exponential models given in different forms. The unit ends with transforming functions and finding the inverse of functions.

- In **Unit 3: Expressions and Equations**, you will reexamine the basic structures of expressions, but this time apply these structures to quadratic expressions. Then you will learn to solve quadratic equations using various methods, as well as how to apply structures of quadratic expressions in solving these equations. The structures of expressions theme continues into having you create quadratic equations of various forms; here, you will learn how to rearrange formulas to solve for a quadratic variable of interest. The unit builds on previous units by introducing the Fundamental Theorem of Algebra and showing you how complex numbers are solutions to quadratic equations. Then you will be introduced to rational functions. Again, you will learn to write exponentially structured expressions in equivalent forms. The unit ends with returning to a familiar topic—solving systems of equations—but now complex solutions can be determined.

- In **Unit 4: Applications of Probability**, you will start by defining events, applying the addition rule, and learning about independence. Then you will progress toward conditional probabilities and the multiplication rule. This builds into using combinatorics to count and calculate probabilities. Finally, you will learn to make and analyze decisions based on probability.
- In **Unit 5: Simliarity, Right Triangle Trigonometry, and Proof**, you will begin by learning about midpoints and other points of interest in a line segment. Then you will work with dilations and similarity. This builds into learning about and proving the various similarity statements. Then you will learn about special angles in intersecting lines and about relationships among the angles formed by a set of parallel lines intersected by a transversal. You will then return to working with triangles and proving theorems about them, including the Interior Angle Sum Theorem, theorems about isosceles triangles, midsegments, and centers of triangles. The unit ends with an introduction to trigonometric ratios and problem solving with those ratios and the Pythagorean theorem.
- In **Unit 6: Circles With and Without Coordinates**, you will study the properties of circles, including central and inscribed angles, chords of a circle, and tangents of a circle. Then you build on this to explore polygons circumscribed and inscribed in a circle. You will then learn about the properties and construction of tangent lines. The measurement units of radians are introduced, and you will use radians to measure the area of a sector and the circumference and area of a circle. You build from a 1- and 2-dimensional arena to a 3-dimensional one by exploring more deeply the volume formulas for cylinders, pyramids, cones, and spheres. Then you will study the links between algebra and geometry by deriving the equations for circles and parabolas. Finally, you will use coordinates to prove geometric theorems about circles and parabolas.

Each lesson is made up of short sections that explain important concepts, including some completed examples. Each of these sections is followed by a few problems to help you practice what you have learned. The "Words to Know" section at the beginning of each lesson includes important terms introduced in that lesson.

As you move through your Math II course, you will become a more confident and skilled mathematician. We hope this book will serve as a useful resource as you learn.

Lesson 1: Working with the Number System

Common Core State Standards

N–RN.1 Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. *For example, we define $5^{1/3}$ to be the cube root of 5 because we want $(5^{1/3})^3 = 5^{(1/3)3}$ to hold, so $(5^{1/3})^3$ must equal 5.*

N–RN.2 Rewrite expressions involving radicals and rational exponents using the properties of exponents.

N–RN.3 Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.

Essential Questions

1. What is the relationship between a power and a root?
2. How can the Power of a Power Property be used to rewrite an expression with a rational exponent?
3. How can an equation of the form $x^{\frac{m}{n}} = b$ be solved?
4. How can you determine whether the sum or product of two numbers will be rational or irrational?

WORDS TO KNOW

base — the quantity that is being raised to a power in an exponential expression; in a^x, a is the base

exponent — the quantity that shows the number of times the base is being multiplied by itself in an exponential expression; also known as the power. In a^x, x is the power/exponent.

exponential equation — an equation of the form $y = ab^x$, where x is the independent variable, y is the dependent variable, and a and b are real numbers

exponential expression	an expression containing a base and a power/exponent
integer	a number that is not a fraction or a decimal; a whole number
irrational number	a real number that cannot be expressed as the ratio of two integers
power	the quantity that shows the number of times the base is being multiplied by itself in an exponential expression; also known as the exponent. In a^x, x is the power/exponent.
radical expression	an expression containing a root, such as $\sqrt[5]{9}$
rational number	any number that can be written as $\frac{m}{n}$, where both m and n are integers and $n \neq 0$; any number that can be written as a decimal that ends or repeats
real numbers	the set of all rational and irrational numbers
root	the inverse of a power/exponent; the root of a number x is a number that, when multiplied by itself a given number of times, equals x

Recommended Resources

- Math-Play.com. "Rational and Irrational Numbers Game."

  http://www.walch.com/rr/00080

  Users classify numbers as rational or irrational in this fast-paced game.
- Math Warehouse. "Simplify Fraction Exponents."

  http://www.walch.com/rr/00081

  This site provides a brief explanation of properties of rational exponents as well as a few practice problems.
- Oswego City School District Regents Exam Prep Center. "Rational (Fractional) Exponents."

  http://www.walch.com/rr/00082

  This site provides an overview of the rules for an alternate way to express roots and includes worked examples.

Lesson 1.1.1: Defining, Rewriting, and Evaluating Rational Exponents

Introduction

An **exponent** is a quantity that shows the number of times a given number is being multiplied by itself in an exponential expression. In other words, in an expression written in the form a^x, x is the exponent. So far, the exponents we have worked with have all been **integers**, numbers that are not fractions or decimals (whole numbers). Exponents can also be rational numbers, or numbers that can be expressed as the ratio of two integers. Rational exponents are simply another way to write radical expressions. For example, $\sqrt{x} = x^{\frac{1}{2}}$ and $\sqrt[3]{x} = x^{\frac{1}{3}}$.

As we will see in this lesson, the rules and properties that apply to integer exponents also apply to rational exponents.

Key Concepts

- An **exponential expression** contains a base and a power. A **base** is the quantity that is being raised to a power. A **power**, also known as an exponent, is the quantity that shows the number of times the base is being multiplied by itself in an exponential expression. In the exponential expression a^n, a is the base and n is the power.
- A **radical expression** contains a root, which can be shown using the radical symbol, $\sqrt{}$. The **root** of a number x is a number that, when multiplied by itself a given number of times, equals x.
- The root of a function is also referred to as the inverse of a power, and "undoes" the power. For example, $\sqrt[3]{8} = 2$ and $2^3 = 8$.
- In the radical expression $\sqrt[n]{a^n}$, the nth root of the nth power of a is a.
- Roots can be expressed using a rational exponent instead of the radical symbol. For example, $\sqrt[n]{a} = a^{\frac{1}{n}}$ and $\sqrt[n]{x^m} = x^{\frac{m}{n}}$.
- A **rational exponent** is an exponent that is a rational number.
- A **rational number** is any number that can be written as $\frac{m}{n}$, where both m and n are integers and $n \neq 0$.
- The denominator of the rational exponent is the root, and the numerator is the power. For example, $a^{\frac{m}{n}} = \sqrt[n]{a^m}$.

- An **exponential equation** can be written as $y = ab^x$, where x is the independent variable, y is the dependent variable, and a and b are real numbers.
- To evaluate the equation at non-integer values of x, the equation needs to be evaluated at rational exponents.
- The properties of integer exponents apply to rational exponents.

Properties of Exponents

Words	Symbols	Numbers
Zero Exponent Property A base raised to the power of 0 is equal to 1.	$a^0 = 1$	$12^0 = 1$
Negative Exponent Property A negative exponent of a number is equal to the reciprocal of the positive exponent of the number.	$a^{-\frac{m}{n}} = \frac{1}{a^{\frac{m}{n}}}, a \neq 0, n \neq 0$	$64^{-\frac{2}{3}} = \frac{1}{64^{\frac{2}{3}}} = \frac{1}{16}$
Product of Powers Property To multiply powers with the same base, add the exponents.	$a^m \bullet a^n = a^{m+n}$	$3^{\frac{1}{4}} \bullet 3^{\frac{7}{4}} = 3^{\frac{1}{4}+\frac{7}{4}} = 3^2 = 9$
Quotient of Powers Property To divide powers with the same base, subtract the exponents.	$\frac{a^m}{a^n} = a^{m-n}$	$\frac{8^{\frac{4}{9}}}{8^{\frac{1}{9}}} = 8^{\frac{4}{9}-\frac{1}{9}} = 8^{\frac{1}{3}} = 2$
Power of a Power Property To raise one power to another power, multiply the exponents.	$(a^m)^n = a^{m \bullet n}$	$\left(5^{\frac{2}{3}}\right)^3 = 5^{\frac{2}{3} \bullet 3} = 5^2 = 25$

Power of a Product Property To find the power of a product, distribute the exponent.	$(ab)^m = a^m b^m$	$(25 \bullet 36)^{\frac{1}{2}} = 25^{\frac{1}{2}} \bullet 36^{\frac{1}{2}} = 5 \bullet 6 = 30$
Power of a Quotient Property To find the power of a quotient, distribute the exponent.	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$	$\left(\frac{25}{49}\right)^{\frac{1}{2}} = \frac{25^{\frac{1}{2}}}{49^{\frac{1}{2}}} = \frac{5}{7}$

- Either the power or root can be determined first when evaluating an exponential expression with a rational exponent.
- Rational exponents can be reduced to simplest form before evaluating a radical expression, but use caution when writing equivalent expressions.
- Use absolute value for expressions with an even root or variable roots. For example, the square root of x^2 can be written as $(x^2)^{\frac{1}{2}}$, which is equal to $|x|$.
- An even root is always positive, so even if a rational exponent can be reduced to a simpler form, the solution should match the original exponential expression.
- Sometimes rational exponents appear as decimals. For example, $x^{0.25}$ is equal to $x^{\frac{1}{4}}$ or $\sqrt[4]{x}$.

Guided Practice 1.1.1

Example 1

How can the expression $3^{\frac{6}{5}}$ be rewritten using roots and powers?

1. Identify the power.

   The power is the numerator of the rational exponent: 6.

2. Identify the root. If the root is even, the solution is the absolute value of the expression.

   Since the root is not even, the root is the denominator of the rational exponent: 5.

3. Rewrite the expression in either of the following forms: $\sqrt[\text{root}]{\text{base}^{\text{power}}}$ or $\left(\sqrt[\text{root}]{\text{base}}\right)^{\text{power}}$, where the base is the quantity being raised to the rational exponent.

   $$3^{\frac{6}{5}} = \sqrt[5]{3^6} = \left(\sqrt[5]{3}\right)^6$$

Example 2

How can the expression $\sqrt[8]{a^c}$ be rewritten using a rational exponent?

1. Identify the numerator of the rational exponent.

   The numerator is the power: c.

2. Identify the denominator of the rational exponent.

   The denominator is the root: 8.

3. Rewrite the expression in the form $\text{base}^{\frac{\text{power}}{\text{root}}}$, where the base is the quantity raised to a power and of which the root is being taken.

   $$\sqrt[8]{a^c} = a^{\frac{c}{8}}$$

Example 3

Evaluate the exponential expression $\left(3^{\frac{1}{2}}\right)^{\frac{4}{3}}$. Round your answer to the nearest thousandth.

1. Simplify the expression using properties of exponents.

   An expression with a power of a power can be rewritten using the product of the powers.

   $$\left(3^{\frac{1}{2}}\right)^{\frac{4}{3}} = 3^{\frac{1}{2} \bullet \frac{4}{3}} = 3^{\frac{4}{6}}$$

2. Write the rational exponent in simplest form. Be sure to include absolute value if the original expression involved finding an even root.

   The exponent, $\frac{4}{6}$, can be reduced to $\frac{2}{3}$. The original root is even, so include an absolute value.

   $$3^{\frac{4}{6}} = \left|3^{\frac{4}{6}}\right| = \left|3^{\frac{2}{3}}\right|$$

3. Evaluate the power and root of the function, using a calculator if needed.

   Note that the power of a power exponent property can be used to rewrite the expression $x^{\frac{a}{b}}$ as $\left(x^{\frac{1}{b}}\right)^{a}$ or $\left(x^{a}\right)^{\frac{1}{b}}$, so either the root or power can be evaluated first.

   The third root of 3 is not an integer, so a calculator will be needed to approximate the root.

   The power, 2, can be evaluated first without using a calculator: $3^2 = 9$.

   $$\left|3^{\frac{2}{3}}\right| = \left|\left(3^{2}\right)^{\frac{1}{3}}\right| = \left|9^{\frac{1}{3}}\right| \approx 2.080$$

Example 4

Evaluate the expression $\sqrt[8]{4^{10}}$. Round your answer to the nearest thousandth.

1. Evaluate the power.

   $4^{10} = 1{,}048{,}576$

2. Find an exact root or approximate root using a calculator.

   Use a calculator to approximate the eighth root of 1,048,576, since this is not a common root.

   $\sqrt[8]{1{,}048{,}576} \approx 5.657$

Example 5

A town's population is decreasing. The population in the year 2000 was 4,000, and the population t years after 2000 can be found by using the function $f(t) = 4000(0.96)^t$. What was the town's approximate population 2.5 years after the year 2000?

1. Replace the variable in the equation with the known value.

   The variable, t, is the number of years after 2000. To find the approximate population 2.5 years after 2000, replace t with 2.5.

   $f(2.5) = 4000(0.96)^{2.5}$

2. Evaluate the expression, either with the rational exponent or by first rewriting as a power and a root.

   The base of the exponential expression is a decimal. In this case, use a calculator to approximate the population for $t = 2.5$. Since the evaluated function is a population, round to the nearest whole number.

   $f(2.5) = 4000(0.96)^{2.5} \approx 3612$

   2.5 years after the year 2000, the town's approximate population was 3,612 people.

Practice 1.1.1: Defining, Rewriting, and Evaluating Rational Exponents

Rewrite each expression using powers and roots. Do not evaluate.

1. $(-14)^{\frac{4}{2}}$

2. $8^{\frac{w}{12}}$

Rewrite each expression using a rational exponent. Do not evaluate.

3. $\sqrt[7]{9}$

4. $\sqrt[4]{j^k}$

5. $\sqrt[8]{(-12)^6}$

Evaluate each expression.

6. $\sqrt[4]{(-3)^2}$

7. $-81^{\frac{5}{4}}$

continued

Use the information given in each scenario to solve the problems.

8. A clothing store manager hires an advertising agency to help find new customers. The advertising agency is using social media to contact potential customers. The agency uses the equation $y = 4 \bullet 5^w$ to estimate the number of people who will have been contacted through social media after w weeks. Approximately how many people will have been contacted after $8\frac{1}{7}$ weeks?

9. The balance in a bank account with an annual interest rate of 2%, compounded annually, can be represented using the function $f(t) = 1000(1.02)^t$, where t is the time in years after opening the account. What was the approximate account balance 6 years and 3 months, or $6\frac{1}{4}$ years, after the account was opened?

10. A newspaper is losing subscribers. In 2010, the newspaper had 30,000 subscribers. The newspaper publisher estimates that t years after 2010, the newspaper will have approximately $30{,}000(0.92)^t$ subscribers. Approximately how many subscribers will the newspaper have $3\frac{5}{12}$ years after 2010?

Lesson 1.1.2: Rational and Irrational Numbers and Their Properties

Introduction

The properties of integer exponents also apply to irrational exponents. In this section, we will see how the following properties can be used to write equivalent exponential expressions involving irrational exponents. We will also learn how to tell if a sum or product will be rational or irrational, and we will solve equations that have rational exponents as well as equations that have irrational exponents.

Key Concepts

- An **irrational number** is a number that cannot be expressed as the ratio of two integers. In other words, it cannot be written as a fraction that has whole numbers for both the numerator and denominator.
- The decimal representations of irrational numbers neither end nor repeat.
- The root of a number for which there is no exact root is an irrational number, such as $\sqrt{2}$ or $\sqrt[3]{5}$.
- Addition and multiplication are closed operations within integers, meaning that the sum of any integers is an integer, and the product of any integers is also an integer.
- Equations involving a power can be solved by taking a root, and equations involving a root can be solved by raising to a power.
- In order to solve an equation this way, one side of the equation must be a single variable and the other side must be a quantity.
- If the exponent is rational, raise both sides to the multiplicative inverse (reciprocal) of the fraction.
- If the expression is written using a power and a root, it can be rewritten using a rational exponent and then solved using the multiplicative inverse of the rational fraction. Or, each operation, the root and the power, can be "undone" in separate steps.
- The same operation must be performed on both sides of the equation.

- In the following equations, x is the variable and a, b, m, and n represent integers, with $m \neq 0$ and $n \neq 0$.

Power:

$$x^a = b$$
$$\sqrt[a]{x^a} = \sqrt[a]{b}$$
$$x = \sqrt[a]{b}$$

Root:

$$\sqrt[a]{x} = b$$
$$\left(\sqrt[a]{x}\right)^a = b^a$$
$$x = b^a$$

Rational exponent case 1:

$$x^{\frac{m}{n}} = b$$
$$\left(x^{\frac{m}{n}}\right)^{\frac{n}{m}} = b^{\frac{n}{m}}$$
$$x^{\frac{m}{n} \bullet \frac{n}{m}} = b^{\frac{n}{m}}$$
$$x^1 = b^{\frac{n}{m}}$$

Rational exponent case 2:

$$\sqrt[n]{x^m} = b$$
$$\left(\sqrt[n]{x^m}\right)^n = b^n$$
$$x^m = b^n$$
$$\sqrt[m]{x^m} = \sqrt[m]{b^n}$$
$$x = \sqrt[m]{b^n}$$

- The sum of two rational numbers is a rational number.
- The product of two rational numbers is a rational number.
- The sum of a rational number and an irrational number is an irrational number.
- The product of a rational number and an irrational number is an irrational number.

Guided Practice 1.1.2

Example 1

Simplify the expression $a^{\frac{6}{5}} \bullet a^{\frac{3}{2}}$.

1. Identify which property can be used to simplify the expression.

   This is the product of two exponential expressions with the same base. Use the Product of Powers Property to simplify.

2. Apply the property to simplify the expression.

   The Product of Powers Property states that if the bases are the same, the expression can be written as the single base raised to the sum of the powers.

   $$a^{\frac{6}{5}} \bullet a^{\frac{3}{2}} = a^{\frac{6}{5}+\frac{3}{2}} = a^{\frac{27}{10}}$$

Example 2

Simplify the expression $\dfrac{b^{\frac{7}{9}}}{b^{\frac{8}{3}}}$.

1. Identify which property can be used to simplify the expression.

   This is the quotient of two exponential expressions with the same base. Use the Quotient of Powers Property to simplify.

2. Apply the property to simplify the expression.

   The Quotient of Powers Property states that since the bases are the same, the expression can be written as the base raised to the difference between the power of the numerator and the power of the denominator. If the exponent is negative, the Negative Exponent Property can be used to rewrite the expression.

   $$\frac{b^{\frac{7}{9}}}{b^{\frac{8}{3}}} = b^{\frac{7}{9}-\frac{8}{3}} = b^{-\frac{17}{9}} \text{ or } \frac{1}{b^{\frac{17}{9}}}$$

Example 3

Lochlan has a savings account. The total account balance, y, after any number of years t can be found using the equation $y = 5000(x)^t$, where x is equal to 1 plus the annual interest rate. The total balance in the account is currently \$6,203.74, and Lochlan has had the account for $5\frac{1}{2}$ years. What is the annual interest rate?

1. Replace any variables with known quantities.

   The equation showing the total account balance y after t years is $y = 5000(x)^t$. If the total account balance is \$6,203.74, then $y = \$6{,}203.74,\ t = 5\frac{1}{2} = \frac{11}{2}$, and x is unknown.

   $$6203.74 = 5000 \bullet x^{\frac{11}{2}}$$

2. Use inverse operations to find an equivalent equation in the form $x^a = b$, where x is the only unknown variable.

   Divide both sides by 5,000 to isolate the exponential expression, $x^{\frac{11}{2}}$.

   $$\left(\frac{1}{5000}\right) \bullet 6203.74 = \left(\frac{1}{5000}\right) \bullet 5000 \bullet x^{\frac{11}{2}}$$

   $$1.2407 = x^{\frac{11}{2}}$$

3. Solve by raising both sides to the multiplicative inverse of the rational exponent.

   $$(1.2407)^{\frac{2}{11}} = \left(x^{\frac{11}{2}}\right)^{\frac{2}{11}}$$

   $$1.040 = x$$

   Since x is equal to 1 plus the interest rate, the interest rate must be $1.04 - 1 = 0.04$, or 4%.

Example 4

Solve the equation $\sqrt[4]{x^3} = 125$.

1. Rewrite the equation using a rational exponent.

   The variable x is being raised to the third power with a fourth root, so the rational exponent is $\frac{3}{4}$.

   $\sqrt[4]{x^3} = 125$

   $x^{\frac{3}{4}} = 125$

2. Determine the multiplicative inverse of the rational exponent.

   The rational exponent is $\frac{3}{4}$. The multiplicative inverse is $\frac{1}{\frac{3}{4}} = \frac{4}{3}$.

3. Raise both sides to the multiplicative inverse of the rational exponent.

   $\left(x^{\frac{3}{4}}\right)^{\frac{4}{3}} = (125)^{\frac{4}{3}}$

   $x = 125^{\frac{4}{3}}$

4. If possible, find the root of the quantity before raising to the power to find x.

   $125 = 5^3$, so the third root of 125 is 5.

   $x = 125^{\frac{4}{3}} = \left(125^{\frac{1}{3}}\right)^4 = 5^4$

   $x = 625$

Practice 1.1.2: Rational and Irrational Numbers and Their Properties

Use the properties of exponents to simplify the expressions. Do not evaluate.

1. $\dfrac{n^{\frac{10}{11}}}{n^{\frac{1}{2}}}$

2. $\left(15^{\frac{8}{5}}\right)^{\frac{9}{2}}$

3. $20^{\frac{1}{3}} \bullet 20^{\frac{10}{4}}$

Simplify each expression, and then determine whether each answer is rational or irrational.

4. $12+\sqrt[3]{27}$

5. $\sqrt[3]{81}+\dfrac{1}{3}$

6. $7^{\frac{1}{2}} \bullet \sqrt{16}$

continued

Solve each equation for the unknown variable.

7. $w^{\frac{2}{3}} = 49$

8. $\sqrt[5]{m^9} = -21$

Use the information given in each scenario to solve the problems.

9. Rumors in a school can spread exponentially. The equation $y = 3 \bullet n^d$ can estimate the total number of students who have heard a rumor, y, after d days of being spread. n is the number of people who are told the rumor by each person. If after 1.5 days, 24 people have heard the rumor, what is the value of n?

10. A store has seen an increase in sales after advertising on a local billboard. The store's total daily sales, y, can be estimated using the equation $y = 600 \bullet s^w$, where w is the number of weeks after putting up the billboard. The store discovers that 3 weeks and 2 days, or $3\frac{2}{7}$ weeks, after putting up the billboard, daily sales are approximately \$704.33. Find s, and use it to write an equation to estimate the daily sales for any number of weeks, w.

Lesson 2: Operating with Polynomials

Common Core State Standard

A–APR.1 Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

Essential Questions

1. How can a variable and its power be used to determine which terms are like terms?
2. What is the relationship between addition of polynomials and subtraction of polynomials?
3. How can we determine if polynomials are closed under addition, subtraction, or multiplication?

WORDS TO KNOW

closure a system is closed, or shows closure, under an operation if the result of the operation is within the system

like terms terms that contain the same variables raised to the same power

monomial an expression with one term, consisting of a number, a variable, or the product of a number and variable(s)

polynomial a monomial or the sum of monomials

term a number, a variable, or the product of a number and variable(s)

Recommended Resources

- Khan Academy. "Adding Polynomials."

  http://www.walch.com/rr/00083

  This video tutorial explains how to add polynomials. Practice problems are also provided on the content covered in the video.

- MathIsFun.com. "Multiplying Polynomials."

  http://www.walch.com/rr/00084

  This website defines polynomials and provides examples for multiplying them.

- MathIsFun.com. "Polynomials."

  http://www.walch.com/rr/00085

  This website defines polynomials and their components, and provides examples of expressions that are polynomials as well as expressions that are not.

Lesson 1.2.1: Adding and Subtracting Polynomials

Introduction

Polynomials can be added and subtracted like real numbers. Adding and subtracting polynomials is a way to simplify expressions. It can also allow us to find a shorter way to represent a sum or difference.

Key Concepts

- A **monomial** is an expression with one term, consisting of a number, a variable, or the product of a number and variable(s).
- A **polynomial** is a monomial or the sum of monomials that contains variables, numeric quantities, or both. The variables of polynomials are raised to integer powers ≥ 0. For example, $mn^4 - 6n$ and 12 are both polynomials.
- Each part of a polynomial is called a **term**. A term is a number, a variable, or the product of a number and variable(s). For example, the second term in the polynomial $8p + 4r^3 + 9$ is $4r^3$.
- **Like terms** are terms that contain the same variables raised to the same power. Numeric quantities are like terms; for example, 1 and 3.4 are like terms. The terms $2x^3$ and $-4x^3$ are also like terms.
- Polynomials are typically written in descending order of the exponents; that is, from left to right, the term with the highest exponent is written first, followed by the term with the next highest exponent, and so on down to the term with the lowest exponent or no exponent.
- When there are two or more variables in a polynomial, the terms are written in descending order alphabetically. For example, $x^2 + x + y^2 + y + 12$ is written in descending order alphabetically.
- To add or subtract like terms containing a variable, use the Distributive Property to add or subtract the variable's coefficients. If a and b are real numbers, and n is an integer greater than 0, then adding like terms with the variable x can be represented as $ax^n + bx^n = (a + b)x^n$.
- Subtraction can be represented in a similar way: $ax^n - bx^n = (a - b)x^n$.
- To add polynomials, add any like terms.
- Before subtracting one polynomial from another, rewrite the difference as a sum: $(n + ax) - (m + bx) = (n + ax) + [-(m + bx)]$, then distribute the negative in the second term: $(n + ax) + [-(m + bx)] = (n + ax) + (-m + -bx)$.

- The subtraction of polynomials is then the same as adding two polynomials.
- A system shows **closure** or is closed if the result of the operation is in the system.
- For example, when an integer is added to an integer, the result is an integer, so the integers are closed under addition.
- The result of adding two polynomials is still a polynomial, so the system of polynomials is closed under addition.
- The result of subtracting one polynomial from another polynomial is still a polynomial; therefore, the system of polynomials is closed under subtraction.

Guided Practice 1.2.1

Example 1

Find the sum of $(4 + 3x) + (2 + x)$.

1. Rewrite the sum so that like terms are together.

   There are two numeric quantities, 4 and 2, and two terms that contain a variable, $3x$ and x. All the terms are positive.

   $(4 + 3x) + (2 + x)$

   $= 4 + 2 + 3x + x$

2. Find the sum of any numeric quantities.

   The numeric quantities in this example are 4 and 2.

   $4 + 2 + 3x + x$

   $= 6 + 3x + x$

3. Find the sum of any terms with the same variable raised to the same power.

   The two terms $3x$ and x both contain only the variable x raised to the first power.

   $6 + 3x + x$

   $= 6 + 4x$

   The result of $(4 + 3x) + (2 + x)$ is $6 + 4x$.

Example 2

Find the sum of $(7x^2 - x + 15) + (6x + 12)$.

1. Rewrite the sum so that like terms are together.

   Be sure to keep any negatives with the expression that follows, such as $-x$.

   $(7x^2 - x + 15) + (6x + 12)$

   $= 7x^2 - x + 6x + 15 + 12$

2. Find the sum of any numeric quantities.

   $7x^2 - x + 6x + 15 + 12$

   $= 7x^2 - x + 6x + 27$

3. Find the sum of any terms with the same variable raised to the same power.

   There is only one term with the variable x raised to the second power.

   There are two terms with the variable x raised to the first power, $-x$ and $6x$, so these can be combined.

   Add the coefficients of the variable.

   $7x^2 - x + 6x + 27$

   $= 7x^2 + 5x + 27$

   The result of $(7x^2 - x + 15) + (6x + 12)$ is $7x^2 + 5x + 27$.

Example 3

Find the difference of $(x^5 + 8) - (3x^5 + 5x)$.

1. Rewrite the difference as a sum.

   A difference can be written as the sum of a negative quantity.

   Distribute the negative in the second polynomial.

   $(x^5 + 8) - (3x^5 + 5x)$

   $= (x^5 + 8) + [-(3x^5 + 5x)]$

   $= (x^5 + 8) + (-3x^5 - 5x)$

2. Rewrite the sum so that any like terms are together.

   Be sure to keep any negatives with the expression that follows, such as $-3x^5$.

   $(x^5 + 8) + (-3x^5 - 5x)$

   $= x^5 + (-3x^5) + (-5x) + 8$

3. Find the sum of any terms with the same variable raised to the same power.

   There are two terms with the variable x raised to the fifth power.

   There is only one term with x raised to the first power, and only one numeric quantity.

   The sum of the two terms with x^5 can be combined to simplify the expression.

   $x^5 + (-3x^5) + (-5x) + 8$

   $= -2x^5 - 5x + 8$

   The result of $(x^5 + 8) - (3x^5 + 5x)$ is $-2x^5 - 5x + 8$.

Practice 1.2.1: Adding and Subtracting Polynomials

Find each sum or difference.

1. $(x + 18) + (-x + 4)$

2. $(-7x^3 + 3) - (x^2 + 9)$

3. $(x^2 - 2) + (-x^3 + 2x - 12)$

4. $(x^6 + x^3) - (-3x^6 + x^2)$

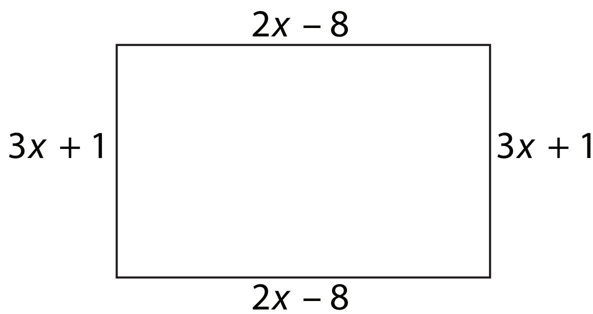
5. $(6x^2 - 6) - (x^3 - x)$

6. $(8x^3 + x^2 - 3) + (x^2 - 4)$

continued

The perimeter of a polygon is the sum of the lengths of the sides of the polygon. For problems 7–10, find the perimeter of each shape. All lengths are given in centimeters.

7.



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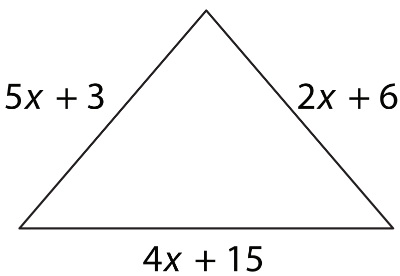

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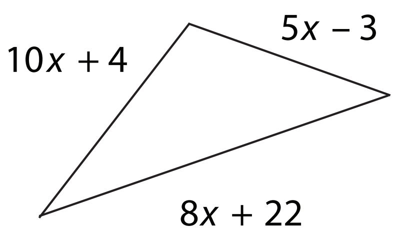

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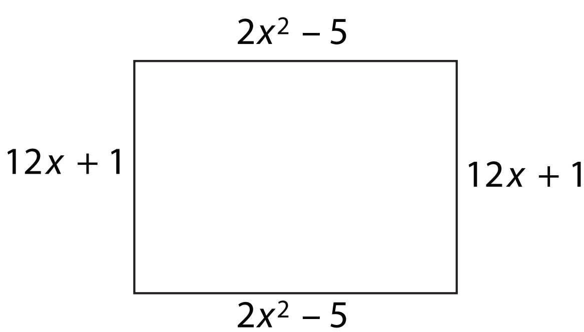


Lesson 1.2.2: Multiplying Polynomials

Introduction

To simplify an expression, such as $(a + bx)(c + dx)$, polynomials can be multiplied. Unlike addition and subtraction of polynomial terms, any two terms can be multiplied, even if the variables or powers are different. Laws of exponents and combining like terms can be used to simplify products of polynomials.

Key Concepts

- To multiply two polynomials, multiply each term in the first polynomial by each term in the second.
- The Distributive Property can be used to simplify the product of two or more polynomials. For example, if each polynomial has two terms, with real numbers a, b, c, and d, then $(a + b)(c + d) = (a + b)c + (a + b)d = ac + bc + ad + bd$.
- Another way to represent the product is to multiply first terms, outside terms, inside terms, and last terms: $(a + b)(c + d) = ac + ad + bc + bd$.
- If each polynomial has three terms, then $(a + b + c)(d + e + f) = (a + b + c)d + (a + b + c)e + (a + b + c)f = ad + bd + cd + ae + be + ce + af + bf + cf$.
- This procedure will work for multiplying any number of polynomials with any number of terms.
- To find the product of two variables raised to a power, use the properties of exponents. If the bases are the same, add the exponents: $x^n \bullet x^m = x^{n+m}$.
- If the bases are not the same, then the exponents cannot be added. Example: $x^n \bullet y^m = x^n y^m$.
- To find the product of a variable with a coefficient and a numeric quantity, multiply the coefficient by the numeric quantity. If a and b are real numbers, then $ax \bullet b = abx$.
- After multiplying all terms, simplify the expression by combining like terms.
- The product of two polynomials is a polynomial, so the system of polynomials is closed under multiplication.

Guided Practice 1.2.2

Example 1

Find the product of $(2x - 1)(x + 18)$.

1. Distribute the first polynomial over the second.

   Ensure that any negatives are included in the products where appropriate.

   $(2x - 1)(x + 18)$

   $= 2x \bullet x + 2x \bullet 18 + (-1) \bullet x + (-1) \bullet 18$

2. Use properties of exponents to simplify any expressions.

   x is x to the first power, or x^1.

   $2x \bullet x$

   $= 2x^1 \bullet x^1$

   $= 2x^{1+1}$

   $= 2x^2$

   Rewrite the expression with simplified expressions.

   $2x \bullet x + 2x \bullet 18 + (-1) \bullet x + (-1) \bullet 18$

   $= 2x^2 + 2x \bullet 18 + (-1) \bullet x + (-1) \bullet 18$

3. Simplify any remaining products.

   The coefficient of a term can be multiplied by a numeric quantity: $ax \bullet b = abx$.

   $2x^2 + 2x \bullet 18 + (-1) \bullet x + (-1) \bullet 18$

   $= 2x^2 + 36x - x - 18$

4. Combine any like terms using sums.

   $2x^2 + 36x - x - 18$

   $= 2x^2 + 35x - 18$

   The result of $(2x - 1)(x + 18)$ is $2x^2 + 35x - 18$.

Example 2

Find the product of $(x^3 + 9x)(-x^2 + 11)$.

1. Distribute the first polynomial over the second.

   Ensure that any negatives are included in the products where appropriate.

   $(x^3 + 9x)(-x^2 + 11)$

   $= x^3 \bullet (-x^2) + x^3 \bullet 11 + 9x \bullet (-x^2) + 9x \bullet 11$

2. Use properties of exponents to simplify any expressions.

   To multiply terms that have the same base (in this case, *x*), keep this base and add the exponents.

   A negative can be written at the beginning of the term.

   $= x^3 \bullet (-x^2) + x^3 \bullet 11 + 9x \bullet (-x^2) + 9x \bullet 11$

   $= -x^{3+2} + x^3 \bullet 11 - 9x^{1+2} + 9x \bullet 11$

   $= -x^5 + x^3 \bullet 11 - 9x^3 + 9x \bullet 11$

3. Simplify any remaining products.

   The coefficient of a term can be multiplied by a numeric quantity: $ax \bullet b = abx$.

   $-x^5 + 11 \bullet x^3 - 9x^3 + 9x \bullet 11$

   $= -x^5 + 11x^3 - 9x^3 + 99x$

4. Combine any like terms.

   $-x^5 + 11x^3 - 9x^3 + 99x$

   $= -x^5 + 2x^3 + 99x$

   The result of $(x^3 + 9x)(-x^2 + 11)$ is $-x^5 + 2x^3 + 99x$.

Example 3

Find the product of $(3x + 4)(x^2 + 6x + 10)$.

1. Distribute the first polynomial over the second.

   Multiply each term in the first polynomial by each term in the second polynomial.

   $(3x + 4)(x^2 + 6x + 10)$

   $= 3x \bullet x^2 + 3x \bullet 6x + 3x \bullet 10 + 4 \bullet x^2 + 4 \bullet 6x + 4 \bullet 10$

2. Use properties of exponents to simplify any expressions.

   $3x \bullet x^2 + 3x \bullet 6x + 3x \bullet 10 + 4 \bullet x^2 + 4 \bullet 6x + 4 \bullet 10$

   $= 3x^3 + 18x^2 + 3x \bullet 10 + 4 \bullet x^2 + 4 \bullet 6x + 4 \bullet 10$

3. Simplify any remaining products.

   $3x^3 + 18x^2 + 3x \bullet 10 + 4 \bullet x^2 + 4 \bullet 6x + 4 \bullet 10$

   $= 3x^3 + 18x^2 + 30x + 4x^2 + 24x + 40$

4. Combine any like terms.

   Only terms with the same variable raised to the same power can be combined.

   The sum can first be rewritten with the exponents in descending order.

   $3x^3 + 18x^2 + 30x + 4x^2 + 24x + 40$

   $= 3x^3 + 18x^2 + 4x^2 + 30x + 24x + 40$

   $= 3x^3 + 22x^2 + 54x + 40$

   The result of $(3x + 4)(x^2 + 6x + 10)$ is $3x^3 + 22x^2 + 54x + 40$.

Example 4

Find the product of $(x + y + 1)(x^2 + 4y - 5)$.

1. Distribute the first polynomial over the second.

   Multiply each term in the first polynomial by each term in the second polynomial.

   $(x + y + 1)(x^2 + 4y - 5)$

   $= x \bullet x^2 + x \bullet 4y + x \bullet (-5) + y \bullet x^2 + y \bullet 4y + y \bullet (-5) + 1 \bullet x^2 + 1 \bullet 4y + 1 \bullet (-5)$

2. Use properties of exponents to simplify any expressions.

   $x \bullet x^2 + x \bullet 4y + x \bullet (-5) + y \bullet x^2 + y \bullet 4y + y \bullet (-5) + 1 \bullet x^2 + 1 \bullet 4y + 1 \bullet (-5)$

   $= x^3 + x \bullet 4y + x \bullet (-5) + y \bullet x^2 + 4y^2 + y \bullet (-5) + 1 \bullet x^2 + 1 \bullet 4y + 1 \bullet (-5)$

3. Simplify any remaining products.

   $x^3 + x \bullet 4y + x \bullet (-5) + y \bullet x^2 + 4y^2 + y \bullet (-5) + 1 \bullet x^2 + 1 \bullet 4y + 1 \bullet (-5)$

   $= x^3 + 4xy - 5x + x^2y + 4y^2 - 5y + x^2 + 4y - 5$

4. Combine any like terms.

   Only terms with the same variable raised to the same power can be combined.

   The sum can first be rewritten with the exponents in descending order.

   When two variables are in a term, such as $x^n y^m$, both n and m, the powers of the two variables, must be the same to combine the terms.

   $x^3 + 4xy - 5x + x^2y + 4y^2 - 5y + x^2 + 4y - 5$

   $= x^3 + x^2 + x^2y - 5x + 4xy + 4y^2 - 5y + 4y - 5$

   $= x^3 + x^2 + x^2y - 5x + 4xy + 4y^2 - y - 5$

   The result of $(x + y + 1)(x^2 + 4y - 5)$ is $x^3 + x^2 + x^2y - 5x + 4xy + 4y^2 - y - 5$.

Practice 1.2.2: Multiplying Polynomials

Find each product.

1. $(x + 3)(x + 8)$

2. $(x^2 - 9)(x^3 + 3)$

3. $(x + 10)(2x^2 + x - 6)$

4. $(-3x^4 + 1)(-x^2 - 8x + 5)$

5. $(x^3 + x^2 + 2)(x^2 + x - 3)$

6. $(4x^2 + x)(3x^2 - x + 4)$

The area of a rectangle is found using the formula area = lw, where l is the length of the rectangle and w is the width. Find the area of each rectangle with the given lengths and widths.

7. $l = 2x - 15$; $w = x - 4$

8. $l = -x^3 + 2$; $w = x^2 + x$

9. $l = 5x + 2$; $w = x^2 + 1$

10. $l = 8x - 7$; $w = 3x - 3$

Lesson 3: Operating with Complex Numbers

Common Core State Standards

N–CN.1 Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real.

N–CN.2 Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

Essential Questions

1. How are complex numbers and real numbers related?
2. What types of numbers can be the sum or difference of complex numbers?
3. What types of numbers are the result of the product of a complex number and its conjugate?

WORDS TO KNOW

complex conjugate — the complex number that when multiplied by another complex number produces a value that is wholly real; the complex conjugate of $a + bi$ is $a - bi$

complex number — a number in the form $a + bi$, where a and b are real numbers, and i is the imaginary unit

complex number system — all numbers of the form $a + bi$, where a and b are real numbers, including complex numbers (neither a nor b equal 0), real numbers ($b = 0$), and imaginary numbers ($a = 0$)

imaginary number — any number of the form bi, where b is a real number, $i = \sqrt{-1}$, and $b \neq 0$

imaginary unit — the letter i, used to represent the non-real value, $i = \sqrt{-1}$

real numbers — the set of all rational and irrational numbers

wholly imaginary a complex number that has a real part equal to 0; written in the form $a + bi$, where a and b are real numbers, i is the imaginary unit, $a = 0$, and $b \neq 0$: $0 + bi$

wholly real a complex number that has an imaginary part equal to 0; written in the form $a + bi$, where a and b are real numbers, i is the imaginary unit, $b = 0$, and $a \neq 0$: $a + 0i$

Recommended Resources

- The Article 19 Group. "The Ohm Zone."

  http://www.walch.com/rr/00086

  This fun, interactive manipulative lets users construct and test any type of electrical circuit imaginable. Pop-ups provide clear definitions and examples of the different types of circuits. Clicking "visualize" shows which direction current would flow in the circuit.

- Khan Academy. "Adding and Subtracting Complex Numbers."

  http://www.walch.com/rr/00087

  This interactive site provides practice with adding and subtracting complex numbers. The site also includes a video tutorial breaking down sums and differences if additional support is needed.

- Math Worksheets Go! "Multiplying Complex Numbers."

  http://www.walch.com/rr/00088

  Download free PDF worksheets (with answer keys) to practice multiplying complex numbers, including complex conjugates.

- The Physics Classroom. "Circuit Connections."

  http://www.walch.com/rr/00089

  This site provides essential background knowledge on series and parallel electrical circuits, which are cited in several examples in this lesson.

- Purplemath.com. "Complex Numbers: Introduction."

  http://www.walch.com/rr/00090

  This site offers a supplemental explanation of complex numbers, including defining complex numbers and introducing operations with complex numbers.

Lesson 1.3.1: Defining Complex Numbers, *i*, and i^2

Introduction

Until now, you have been told that you cannot take the square of –1 because there is no number that when squared will result in a negative number. In other words, the square root of –1, or $\sqrt{-1}$, is not a real number. French mathematician René Descartes suggested the imaginary unit i be defined so that $i^2 = -1$. The imaginary unit enables us to solve problems that we would not otherwise be able to solve. Problems involving electricity often use the imaginary unit.

Key Concepts

- All rational and irrational numbers are real numbers.
- The **imaginary unit** i is used to represent the non-real value, $\sqrt{-1}$.
- An **imaginary number** is any number of the form bi, where b is a real number, $i=\sqrt{-1}$, and $b \neq 0$.
- Real numbers and imaginary numbers can be combined to create a **complex number system**.
- A complex number contains two parts: a real part and an imaginary part.
- All **complex numbers** are of the form $a + bi$, where a and b are real numbers and i is the imaginary unit.
- In the general form of a complex number, a is the real part of the complex number, and bi is the imaginary part of the complex number. Note that if $a = 0$, the complex number $a + bi$ is **wholly imaginary** and contains no real part: $0 + bi = bi$.
- If $b = 0$, the complex number $a + bi$ is **wholly real** and contains no imaginary part: $a + (0)i = a$.
- Expressions containing imaginary numbers can also be simplified.
- To simplify i^n for $n > 4$, divide n by 4, and use the properties of exponents to rewrite the exponent.
- The exponent can be rewritten using the quotient: $n \div 4 = m$ remainder r, or $\frac{n}{4}=m+\frac{r}{4}$, where r is the remainder when dividing n by 4, and m is a whole number.

- Then $n = 4m + r$, and r will be 0, 1, 2, or 3. Use the properties of exponents to rewrite i^n.

  $i^n = i^{4m+r} = i^{4m} \bullet i^r$

- $i^4 = 1$, so i to a multiple of 4 will also be 1: $i^{4m} = (i^4)^m = (1)^m = 1$.
- The expression i^r will determine the value of i^n.
- Use i^0, i^1, i^2, and i^3 to find i^n.

If $r = 0$, then:

$i^n = i^{4m+r}$

$i^n = i^{4m} \bullet i^r$

$i^n = 1 \bullet i^0$

$i^n = 1 \bullet 1 = 1$

If $r = 1$, then:

$i^n = i^{4m+r}$

$i^n = i^{4m} \bullet i^r$

$i^n = 1 \bullet i^1$

$i^n = i$, or $\sqrt{-1}$

If $r = 2$, then:

$i^n = i^{4m+r}$

$i^n = i^{4m} \bullet i^r$

$i^n = 1 \bullet i^2$

$i^n = 1 \bullet -1 = -1$

If $r = 3$, then:

$i^n = i^{4m+r}$

$i^n = i^{4m} \bullet i^r$

$i^n = 1 \bullet i^3$

$i^n = 1 \bullet -i = -i$, or $-\sqrt{-1}$

- Only the value of the remainder when n is divided by 4 is needed to simplify i^n.

  $i^0 = 1$, $i^1 = i$, $i^2 = -1$, and $i^3 = -i$

- Properties of exponents, along with replacing i with its equivalent value of $\sqrt{-1}$, can be used to simplify the expression i^n. Start with $n = 0$.

  $i^0 = 1$

  $i^1 = i = \sqrt{-1}$

  $i^2 = \left(\sqrt{-1}\right)^2 = -1$

- When simplifying i^3, use the property $i^3 = i^2 \bullet i^1$, and the determined values of i^2 and i.

  $i^3 = i^2 \bullet i = (-1) \bullet \sqrt{-1} = -\sqrt{-1}$, or $-i$

- When simplifying i^4, use the property $i^4 = i^2 \bullet i^2$, and the determined value of i^2.

  $i^4 = i^2 \bullet i^2 = (-1) \bullet (-1) = 1$

Guided Practice 1.3.1

Example 1

Identify the real and imaginary parts of the complex number $8+\frac{1}{3}i$.

1. Identify the real part of the complex number.

   Identify the part that is not a multiple of *i*.

   8 is not a multiple of *i*.

   The real part of $8+\frac{1}{3}i$ is 8.

2. Identify the imaginary part of the complex number.

   Identify the part that is a multiple of *i*.

   $\frac{1}{3}$ is a multiple of *i*.

   The imaginary part of $8+\frac{1}{3}i$ is the term $\frac{1}{3}i$.

Example 2

Rewrite the complex number 2*i* using a radical.

1. Replace *i* with $\sqrt{-1}$.

   $2i=2\bullet\sqrt{-1}$

2. Place the squared value of any whole multiples of *i* under the radical.

   $2\bullet\sqrt{-1}=\sqrt{2^2\bullet(-1)}$

3. Simplify the expression under the radical.

   $\sqrt{2^2\bullet(-1)}=\sqrt{4\bullet(-1)}=\sqrt{-4}$

Example 3

Rewrite the radical $\sqrt{-32}$ using the imaginary unit i.

1. Rewrite the value under the radical as the product of –1 and a positive value.

   $\sqrt{-32} = \sqrt{(-1) \bullet 32}$

2. Rewrite the radical $\sqrt{-1}$ as i.

   $\sqrt{(-1) \bullet 32} = \sqrt{-1} \bullet \sqrt{32} = i \bullet \sqrt{32}$

3. If possible, rewrite the positive value as the product of a square number and another whole number.

   32 = 16 • 2, and 16 is a square number.

   $i \bullet \sqrt{32} = i \bullet \sqrt{16 \bullet 2}$

4. Simplify the radical by finding the square root of the square number. The simplified expression will be in the form *whole number • i • radical*.

   $i \bullet \sqrt{16 \bullet 2} = i \bullet \sqrt{4^2 \bullet 2} = 4i\sqrt{2}$

Example 4

Simplify i^{57}.

1. Find the remainder of the power of i when divided by 4.

   14 • 4 = 56; therefore, 57 ÷ 4 = 14 remainder 1.

   The remainder is 1.

2. Use the remainder to simplify the power of i.

   $i^{57} = 1 \bullet i^1 = i$

Example 5

Impedance is the measure of an object's resistance to an electric current, or its opposition to the flow of a current. Complex numbers are used to represent the impedance of an element in a circuit. The voltage, V, is the real part of the complex number, and the current, I, is the coefficient of the imaginary unit i. So, impedance is equal to $V + Ii$, where I is in milliamperes. A certain element has a voltage of 18 volts and a current of 2 milliamperes. Use a complex number to represent the element's impedance.

1. Use the voltage to write the real part of the complex number.

   The voltage is 18 volts; therefore, the real part of the number is 18.

2. Use the current to write the imaginary part of the complex number.

   The current is the coefficient of i.

   The current is 2 milliamperes; therefore, the coefficient of i is 2.

   The imaginary part of the number is $2i$.

3. The complex representation of impedance is the sum of the real and imaginary parts.

   The element's impedance is $18 + 2i$.

Practice 1.3.1: Defining Complex Numbers, *i*, and i^2

Identify the real and imaginary parts of each complex number.

1. -51

2. $48 + 32i$

Simplify the radical and use the imaginary unit i.

3. $\sqrt{-27}$

4. $\sqrt{-144}$

Rewrite each imaginary number using a radical instead of the imaginary unit i.

5. $-10i$

6. $i\sqrt{115}$

Simplify each imaginary number using the properties of exponents.

7. i^{71}

8. i^{121}

Write a complex number to represent the impedance of each element. The voltage, V, is the real part, and the current, I, is the multiple of the imaginary unit i.

9. $V = 50$ volts; $I = 6$ milliamperes

10. $V = 29$ volts; $I = 3.1$ milliamperes

Lesson 1.3.2: Adding and Subtracting Complex Numbers

Introduction

Finding sums and differences of complex numbers is similar to finding sums and differences of expressions involving numeric and algebraic quantities. The real parts of the complex number are similar to the numeric quantities, and the imaginary parts of the complex number are similar to the algebraic quantities. Before finding sums or differences, each complex number should be in the form $a + bi$. If i is raised to a power n, use the remainder of $n \div 4$ to simplify i^n.

Key Concepts

- First, find the sum or difference of the real parts of the complex number.
- Then, to find the sum or difference of the imaginary numbers, add or subtract the coefficients of i.
- The resulting sum of the real parts and the imaginary parts is the solution.
- In the following equation, let a, b, c, and d be real numbers.

  $$(a + bi) + (c + di) = a + c + bi + di = (a + c) + (b + d)i$$

- $a + c$ is the real part of the sum, and $(b + d)i$ is the imaginary part of the sum.
- When finding the difference, distribute the negative throughout both parts of the second complex number.

  $$(a + bi) - (c + di) = a + bi - c - di = (a - c) + (b - d)i$$

- $a - c$ is the real part of the difference, and $(b - d)i$ is the imaginary part of the difference.
- The sum or difference of two complex numbers can be wholly real (having only real parts), wholly imaginary (having only imaginary parts), or complex (having both real and imaginary parts).

Guided Practice 1.3.2

Example 1

Is $(6 + 5i) + (8 - 3i)$ wholly real or wholly imaginary, or does it have both a real and an imaginary part?

1. Find the sum of the real parts.

   The real parts are 6 and 8.

   $6 + 8 = 14$

2. Find the sum of the imaginary parts by summing the multiples of i.

   The imaginary parts are $5i$ and $-3i$.

   $5i + (-3i) = [5 + (-3)]i = 2i$

3. Write the solution as the sum of the real and imaginary parts.

   $14 + 2i$

4. Use the form of the sum to determine if it is wholly real or wholly imaginary, or if it has both a real and an imaginary part.

   A wholly real number has no imaginary part, a wholly imaginary number has no real part, and a complex number includes both parts.

   $14 + 2i$ has a real part, 14, and an imaginary part, $2i$.

Example 2

Is $(5 + 6i^9) - (5 + 3i^{15})$ wholly real or wholly imaginary, or does it have both a real and an imaginary part?

1. Simplify any expressions containing i^n.

   Two expressions, $6i^9$ and $3i^{15}$, contain i^n.

   Divide each power of i by 4 and use the remainder to simplify i^n.

   $9 \div 4 = 2$ remainder 1, so $9 = 2 \bullet 4 + 1$.

   $i^9 = i^{2 \bullet 4} \bullet i^1 = i$

   $15 \div 4 = 3$ remainder 3, so $15 = 3 \bullet 4 + 3$.

   $i^{15} = i^{3 \bullet 4} \bullet i^3 = -i$

   Replace each occurrence of i^n in the expressions with the simplified versions, and replace the original expressions in the difference with the simplified expressions.

   $6i^9 = 6 \bullet (i) = 6i$

   $3i^{15} = 3 \bullet (-i) = -3i$

   $(5 + 6i^9) - (5 + 3i^{15}) = (5 + 6i) - [5 + (-3i)]$

2. Distribute the difference through both parts of the complex number.

   $(5 + 6i) - [5 + (-3i)]$

   $= (5 + 6i) - 5 - (-3i)$

   $= 5 + 6i - 5 + 3i$

3. Find the sum or difference of the real parts.

   $5 - 5 = 0$

4. Find the sum or difference of the imaginary parts.

   $6i + 3i = 9i$

5. Find the sum of the real and imaginary parts.

   $0 + 9i = 9i$

6. Use the form of the sum to determine if it is wholly real or wholly imaginary, or if it has both a real and an imaginary part.

   $9i$ has only an imaginary part, $9i$, so the difference is wholly imaginary.

Example 3

Is $(12 - i^{20}) + (-18 - 4i^{18})$ wholly real or wholly imaginary, or does it have both a real and an imaginary part?

1. Simplify any expressions containing i^n.

   Two expressions, i^{20} and $4i^{18}$, contain i^n.

   Divide each power of i by 4 and use the remainder to simplify i^n.

   $20 \div 4 = 5$ remainder 0, so $20 = 5 \bullet 4 + 0$.

   $i^{20} = i^{5 \bullet 4} = 1$

   $18 \div 4 = 4$ remainder 2, so $18 = 4 \bullet 4 + 2$.

   $i^{18} = i^{4 \bullet 4} \bullet i^2 = -1$

   Replace each occurrence of i^n in the expressions with the simplified versions, and replace the original expressions in the sum with the simplified expressions.

   $i^{20} = 1$

   $4i^{18} = 4 \bullet (-1) = -4$

   $(12 - i^{20}) + (-18 - 4i^{18}) = (12 - 1) + [-18 - (-4)]$

2. Find the sum or difference of the real parts.

$(12 - 1) + [-18 - (-4)]$

$= 12 - 1 - 18 + 4$

$= -3$

3. Find the sum or difference of the imaginary parts.

The expression contains no multiples of i, so there are no imaginary parts, and the multiple of i is 0: $0i$.

4. Find the sum of the real and imaginary parts.

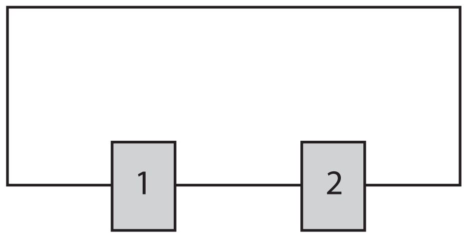
$-3 + 0i = -3$

5. Use the form of the sum to determine if it is wholly real or wholly imaginary, or if it has both a real and an imaginary part.

–3 has only a real part, –3, so the sum is wholly real.

Example 4

A circuit in series is a circuit where the power flows in only one direction and goes through each part of the circuit. A flashlight with two batteries is a series circuit, because the power goes through the batteries to the lightbulb. The impedance (resistance to current) of an element can be represented using the complex number, $V + Ii$, where V is the element's voltage and I is the element's current. If two elements are used in a circuit in series, the total impedance is the sum of the impedance of each element. The following diagram of a circuit contains two elements, 1 and 2, in series.



The total impedance of the circuit is the sum of the impedance of elements 1 and 2. Element 1 has a voltage of 25 volts and a current of 1 milliampere. Element 2 has a voltage of 20 volts and a current of 1.5 milliamperes. What is the total impedance of the circuit?

1. Write the impedance of each element as a complex number.

   Impedance = $V + Ii$

   Element 1: $25 + i$

   Element 2: $20 + 1.5i$

2. The two elements are in series in a circuit. The total impedance of the circuit is the sum of the impedance of each element. Write the sum.

   Total impedance: $(25 + i) + (20 + 1.5i)$

3. Evaluate the sum.

   $(25 + i) + (20 + 1.5i)$

   $= 25 + 20 + i + 1.5i$

   $= 45 + 2.5i$

   The total impedance is $45 + 2.5i$.

Practice 1.3.2: Adding and Subtracting Complex Numbers

Find each sum or difference. Identify whether each sum or difference is wholly real or wholly imaginary, or if it has both a real and an imaginary part.

1. $(12 + i) + (-4 - i)$

2. $(8 + 2i) - (-8 + 8i)$

3. $(2 + 25i) - (17 + 24i)$

4. $(-18 + 3i) + (18 - 9i)$

5. $(10 + i^{53}) + (-23 + i)$

6. $(30 - 24i) - (-6 + 3i^{34})$

7. $(-14 + 12i) - (-14 + 5i^{23})$

8. $(-44 + 10i) + (39 - 11i)$

Use the following information to solve problems 9 and 10.

> The impedance of an element can be written in the form $V + Ii$, where V is the voltage and I is the current in milliamperes. For two elements in series in a circuit, the total impedance is the sum of each element's impedance. Find the total impedance of two given elements if the elements are in series in a circuit.

9. Element 1: $V = 29$ volts, $I = 1.1$ milliamperes

   Element 2: $V = 27$ volts, $I = 0.9$ milliamperes

10. Element 1: $V = 15.5$ volts, $I = 2$ milliamperes

    Element 2: $V = 23$ volts, $I = 2.3$ milliamperes

Lesson 1.3.3: Multiplying Complex Numbers

Introduction

The product of two complex numbers is found using the same method for multiplying two binomials. As when multiplying binomials, both terms in the first complex number need to be multiplied by both terms in the second complex number. The product of the two binomials $x + y$ and $x - y$ is the difference of squares: $x^2 - y^2$. If y is an imaginary number, this difference of squares will be a real number since $i \bullet i = -1$: $(a + bi)(a - bi) = a^2 - (bi)^2 = a^2 - b^2(-1) = a^2 + b^2$.

Key Concepts

- Simplify any powers of i before evaluating products of complex numbers.
- In the following equations, let a, b, c, and d be real numbers.
- Find the product of the first terms, outside terms, inside terms, and last terms. *Note*: The imaginary unit i follows the product of real numbers.

  $(a + bi) \bullet (c + di) = ac$ (product of the first terms) $+ adi$ (product of the outside terms) $+ bci$ (product of the inside terms) $+ bidi$ (product of the last terms)

  $= ac + adi + bci + bdi^2$

  $= ac + bd(-1) + adi + bci$

  $= (ac - bd) + (ad + bc)i$

- $ac - bd$ is the real part of the product, and $ad + bc$ is the multiple of the imaginary unit i in the imaginary part of the product.
- A **complex conjugate** is a complex number that when multiplied by another complex number produces a value that is wholly real.
- The product of a complex number and its conjugate is a real number.
- The complex conjugate of $a + bi$ is $a - bi$, and the complex conjugate of $a - bi$ is $a + bi$.
- The product of a complex number and its conjugate is the difference of squares, $a^2 - (bi)^2$, which can be simplified.

  $a^2 - b^2i^2 = a^2 - b^2 \bullet (-1) = a^2 + b^2$

Guided Practice 1.3.3

Example 1

Find the result of $i \bullet 5i$.

1. Multiply the two terms.

   $i \bullet 5i = 5i^2$

2. Simplify any powers of i.

   $i^2 = -1$

   Substitute –1 in for i^2.

   $5i^2 = 5(-1) = -5$

   The product of $i \bullet 5i$ is –5.

Example 2

Find the result of $(7 + 2i)(4 + 3i)$.

1. Multiply both terms in the first polynomial by both terms in the second polynomial. Find the product of the first terms, outside terms, inside terms, and last terms.

   $(7 + 2i)(4 + 3i) = 7 \bullet 4 + 7 \bullet 3i + 2i \bullet 4 + 2i \bullet 3i$

2. Evaluate or simplify each expression.

   $7 \bullet 4 + 7 \bullet 3i + 2i \bullet 4 + 2i \bullet 3i$

   $= 28 + 21i + 8i + 6i^2$

   $= 28 + 21i + 8i + 6(-1)$

   $= 28 + 21i + 8i - 6$

3. Combine any real parts and any imaginary parts.

$28 + 21i + 8i - 6$

$= 28 - 6 + 21i + 8i$

$= 22 + 29i$

Example 3

Find the complex conjugate of $5 - i$. Use multiplication to verify your answer.

1. Find the complex conjugate.

   The complex conjugate of a number of the form $a - bi$ is $a + bi$; therefore, the complex conjugate of $5 - i$ is $5 + i$.

2. Multiply both terms in the first polynomial by both terms in the second polynomial. Find the product of the first terms, outside terms, inside terms, and last terms.

   $(5 - i)(5 + i)$

   $= 25 + 5i - 5i - i^2$

   $= 25 - i^2$

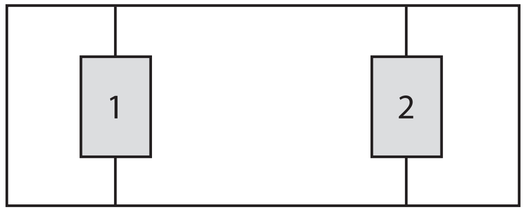
   $= 25 - (-1) = 26$

3. Verify that the product of the complex number and its conjugate contains no imaginary units, i.

   The product of $(5 - i)$ and $(5 + i)$ is 26, which is a real number. $(5 + i)$ is the complex conjugate of $(5 - i)$.

Example 4

A parallel circuit has multiple pathways through which current can flow. The following diagram of a circuit contains two elements, 1 and 2, in parallel.



The impedance of an element can be represented using the complex number $V + Ii$, where V is the element's voltage and I is the element's current in milliamperes. If two elements are in a circuit in parallel, the total impedance is the sum of the reciprocals of each impedance. If the impedance of element 1 is Z_1, and the impedance of element 2 is Z_2, the total impedance of the two elements in parallel is $\frac{1}{Z_1}+\frac{1}{Z_2}$.

Element 1 has a voltage of 10 volts and a current of 3 milliamperes. Element 2 has a voltage of 15 volts and a current of 2 milliamperes. What is the total impedance of the circuit? Leave your result as a fraction.

1. Write each impedance as a complex number.

   Impedance = $V + Ii$

   Element 1: $10 + 3i$

   Element 2: $15 + 2i$

2. Find the reciprocal of each impedance.

   Element 1: $\frac{1}{10+3i}$

   Element 2: $\frac{1}{15+2i}$

3. Find a common denominator of the two fractions.

   Find the product of the two denominators. This will be the common denominator.

   $(10 + 3i)(15 + 2i) = 150 + 20i + 45i + 6i^2 = 150 + 65i - 6 = 144 + 65i$

4. Find an equivalent fraction for each element with a common denominator.

   Multiply the numerator and denominator of the fraction for element 1 by the denominator of element 2. Use the product from step 3 to simplify $(10 + 3i)(15 + 2i)$.

   $$\frac{1}{10+3i} \bullet \frac{15+2i}{15+2i} = \frac{15+2i}{144+65i}$$

   Multiply the numerator and denominator of the fraction for element 2 by the denominator of element 1. Use the product from step 3 to simplify $(10 + 3i)(15 + 2i)$.

   $$\frac{1}{15+2i} \bullet \frac{10+3i}{10+3i} = \frac{10+3i}{144+65i}$$

5. Find the total impedance by summing the reciprocals of the impedance of each element.

   $$\frac{1}{10+3i} + \frac{1}{15+2i}$$

   $$= \frac{15+2i}{144+65i} + \frac{10+3i}{144+65i}$$

   $$= \frac{15+10+2i+3i}{144+65i}$$

   $$= \frac{25+5i}{144+65i}$$

Practice 1.3.3: Multiplying Complex Numbers

Find each product.

1. $(1 + i)(13 + 9i)$

2. $(-10 - 6i)(2 - i)$

3. $(4 + 8i)(3 - 5i)$

4. $(-12 + i)(7 + 7i)$

5. $(-11 - 18i)(5 + 3i)$

Find the complex conjugate of each number. Find the product of the complex number and its conjugate to verify your answer.

6. $42 + i$

7. $-24 + 10i$

8. $-13 - 4i$

Use the following information to solve problems 9 and 10.

> The impedance of an element can be represented using the complex number $V + Ii$, where V is the element's voltage and I is the element's current in milliamperes. If two elements are in a circuit in parallel, the total impedance of the two elements in parallel is $\frac{1}{Z_1} + \frac{1}{Z_2}$. Calculate the total impedance for each pair of elements. Leave your response as a fraction.

9. Element 1: $28 + 2i$

   Element 2: $29 + 2i$

10. Element 1: $19 + 3i$

    Element 2: $21 + 4i$

Answer Key

Lesson 1: Working with the Number System

Practice 1.1.1: Defining, Rewriting, and Evaluating Rational Exponents, pp. 9–10

1. $\sqrt[2]{(-14)^4}$ or $(-14)^2$
3. $9^{\frac{1}{7}}$
5. $(-12)^{\frac{6}{8}}$ or $(-12)^{\frac{3}{4}}$
7. –243
9. $1,131.75

Practice 1.1.2: Rational and Irrational Numbers and Their Properties, pp. 16–17

1. $n^{\frac{9}{22}}$
3. $20^{\frac{17}{6}}$
5. irrational
7. $w = 343$
9. $n = 4$

Lesson 2: Operating with Polynomials

Practice 1.2.1: Adding and Subtracting Polynomials, pp. 25–26

1. 22
3. $-x^3 + x^2 + 2x - 14$
5. $-x^3 + 6x^2 + x - 6$
7. $(10x - 14)$ cm
9. $(23x + 23)$ cm

Practice 1.2.2: Multiplying Polynomials, p. 33

1. $x^2 + 11x + 24$
3. $2x^3 + 21x^2 + 4x - 60$
5. $x^5 + 2x^4 - 2x^3 - x^2 + 2x - 6$
7. $(2x^2 - 23x + 60)$ units^2
9. $(5x^3 + 2x^2 + 5x + 2)$ units^2

Lesson 3: Operating with Complex Numbers

Practice 1.3.1: Defining Complex Numbers, *i*, and *i*², p. 41

1. real: –51; imaginary: $0i$ or none
3. $3i\sqrt{3}$
5. $-\sqrt{-100}$
7. $-i$
9. $50 + 6i$

Practice 1.3.2: Adding and Subtracting Complex Numbers, p. 48

1. 8; wholly real
3. $-15 + i$; both real and imaginary parts
5. $-13 + 2i$; both real and imaginary parts
7. $17i$; wholly imaginary
9. $56 + 2i$

Practice 1.3.3: Multiplying Complex Numbers, p. 54

1. $4 + 22i$
3. $52 + 4i$
5. $-1 - 123i$
7. $-24 - 10i$
9. $\frac{57+4i}{808+114i}$