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Student's Guide for Exploring Geometry, *Second Edition*

Contents

CHAPTER 1

Geometry and the Axiomatic Method

The development of the axiomatic method of reasoning was one of the most profound events in the history of mathematics. In this chapter we explore axiomatic systems and their properties.

One strand running through the chapter is the search for the "ideal." The golden ratio is the ideal in concrete form, realized through natural and man-made constructions. Deductive reasoning from a base set of axioms is the ideal in abstract form, realized in the crafting of clear, concise, and functional definitions, and in the reasoning employed in well-constructed proofs.

Another strand in the chapter, and which runs through the entire text, is that of the interplay between the concrete and the abstract. As you work through this text, you are encouraged to play with concrete ideas, such as how the Golden Ratio appears in nature, but you are also encouraged to play (experiment) when doing proofs and more abstract thinking. The experimentation in the latter is of the mind, but it can utilize many of the same principles of exploration as you would use in a computer lab. When trying to come up with a proof you should consider lots of examples and ask "What if ...?" questions. Most importantly, you should *interact* with the ideas, just as you interact with the software environment.

Interaction with ideas and discovery of concepts is a primary organizing principle for the text. Interaction is encouraged in three ways. First, topics are introduced and developed in the text. Next, lab projects reinforce concepts, or introduce related ideas. Lastly, project results are discussed, and conclusions drawn, in written lab reports.

You will first read about concepts and hear them discussed in class. Then, you will conduct experiments to make the ideas concrete. Finally, you will conceptualize ideas by re-telling them in project reports.

The work you do in the lab and in group projects is a critical component of the course. The projects that are designed to be done in groups have an additional pedagogical advantage. You will find that by speaking with other students, using mathematical terms and concepts, you will better internalize such concepts and make them less abstract.

NOTES ON LAB PROJECTS

Project explorations are designed to be general in scope. They can be carried out using a variety of geometry software packages such as *Geometry Explorer*, *GeoGebra*, *Geometer's Sketchpad*, etc.

Specific instructions for carrying out each project in the text can be found at the author's website http://www.gac.edu/~hvidsten/geom-text.

In order to help with the formatting of lab reports there is sample lab report for a "fake" lab on the Pythagorean Theorem in appendix A of this guide.

SOLUTIONS TO EXERCISES IN CHAPTER 1

1.3 Project 1 - The Ratio Made of Gold

The main difficulty you will face with the first lab project will be in learning the functionality of the dynamic geometry program that your class is using. One major point to watch out for is the notion of "attaching" objects together when doing a construction. For example, when you create a point on top of a line, the point becomes attached to the line. That is, when the point is moved it is constrained to follow the line.

1.4 The Rise of the Axiomatic Method

In this section we focus on *reasoning* in mathematics. The problems in this section may seem quite distant from the geometry you learned in high school, but the goal is to practice reasoning from the definitions and properties that an axiomatic system posits and then create proofs and explanations using just those basic ideas and relationships. This is good mental training. It is all too easy to argue from diagrams when trying to justify geometric statements.

1.4.1 If dictionaries were not circular, there would need to be an infinite number of different words in the dictionary.

1.4.3 Let a set of two different flavors be called a pairing. Suppose there were m children and $n > m$ pairings. By Axiom 2 every pairing is associated to a unique child. Thus, for some two pairings, P_1,P_2 there is a child C associated to both. But this contradicts Axiom 3. Likewise, if $m > n$, then by Axiom 3 some two children would have the same pairing. This contradicts Axiom 2. So, $m = n$ and, since the number of pairings is $4 + 3 + 2 + 1 = 10$, there are 10 children.

1.4.5 There are exactly four pairings possible of a given flavor with the others. By Axiom 2, there is exactly one child associated to each of these four pairings.

1.4.7 If $xyz = e$, then by Axiom 4, $x^{-1}xyz = x^{-1}$. By Axiom 4 and Axiom 3, $x^{-1}xyz = eyz = yz$ and thus $yz = x^{-1}$. Then, $yzx = x^{-1}x = e$

1.4.9 First we show that $1 \in M$. By Axiom 4 we know 1 is not the successor of any natural number. In particular, it cannot be a successor of itself. Thus, $1' \neq 1$ and $1 \in M$. Now, suppose $x \in M$. That is, $x' \neq x$. By Axiom 3 we have that $(x')' \neq x'$, and so $x' \in M$. Both conditions of Axiom 6 are satisfied and thus $M = N$.

1.4.11 Given x, let $M = \{y|x + y \; is \; defined\}$. Then, by definition $1 \in M$. Suppose $y \in M$. Then, $x + y' = (x + y)'$ is defined and $y' \in M$. So, $M = N$ by Axiom 6. Now, since x was chosen arbitrarily, addition is defined for all x and y.

1.4.13 This is a good discussion question. Think about the role of abstraction versus application in mathematics. Think about how abstraction and application cross-fertilize one another.

1.5 Properties of Axiomatic Systems

This is a "meta" section. By this is meant that we are studying properties of axiomatic systems themselves, considering such systems as mathematical objects in comparison to other systems. This may seem quite foreign territory to you, but have an open mind and think about how one really knows that mathematics is true or logically consistent. We often think of mathematics as an ancient subject, but in this section we bring in the amazing results of the twentieth century mathematician Kurt Godel.

If this topic interests you, you may want to further research the area of information theory and computability in computer science. A good reference here is Gregory Chaitin's book *The Limits of Mathematics* (Springer, 1998.) Additionally, much more could be investigated as to the various philosophies of mathematics, in particular the debates between platonists and constructionists, or between intuitionists and formalists. A good reference here is Edna E. Kramer's *The Nature and Growth of Modern Mathematics* (Princeton, 1981), in particular Chapter 29 on Logic and Foundations.

1.5.1 Let S be the set of all sets which are not elements of themselves. Let P be the proposition that "S is an element of itself". And consider the two propositions P and the negation of P, which we denote as $\neg P$. Assume P is true. Then, S is an element of itself. So, S is a set which by definition is not an element of itself. So, $\neg P$ is true. Likewise, if $\neg P$ is true then P is true. In any event we get P and $\neg P$ both true, and the system cannot be consistent.

1.5.3 Good research books for this question are books on the history of mathematics. This could be a good final project idea.

1.5.5 Let P be a point. Each pairing of a point with P is associated to a unique line. There are exactly three such pairings.

1.5.7 Yes. The lines and points satisfy all of the axioms.

1.5.9 If (x, y) is in P, then $x < y$. Clearly, $y < x$ is impossible and the first axiom is satisfied. Also, inequality is transitive on numbers so the second axiom holds and this is a model.

1.5.11 A quick listing of all points and lines and incidence relations shows that Axioms 1 and 2 are satisfied. For Axiom 3, points A, B, C, and D have the property that no subset of three of the points are collinear. For Axiom 4, line $\overleftrightarrow{AB}$ suffices.

1.5.13 The dual to Axiom 1 is "Given two distinct lines, there is exactly one point incident with them both." Proof: Suppose there were two points A and B incident on both lines. This would contradict Axiom A1.

1.5.15 By Axiom 4 there is a line l with $n+1$ points, say $P_1, \ldots,$ P_{n+1}. By Axiom 3, there must be a point Q that is not on l. Let l_1 be the line incident on P_1 and Q, l_2 be the line incident on P_2 and Q, etc. The $n+1$ lines $l_1, \ldots, l_{n+1}$ through Q satisfy the dual statement of Axiom 4.

1.6 Euclid's Axiomatic Geometry

In this section we take a careful look at Euclid's original axiomatic system. We observe some of its inadequacies in light of our modern "meta" understanding of such systems, and discuss the one axiom that has been the creative source of much of modern geometry – the Parallel Postulate.

1.6.1 Good research books for this question are books on the history of mathematics.

1.6.3 An explanation should be given along with a figure like the following:

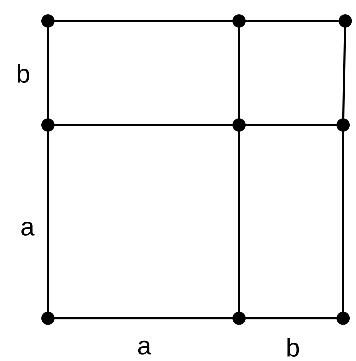


FIGURE 1.1:

1.6.5

$$\begin{aligned} 123 &= 3 \cdot 36 + 15 \\ 36 &= 2 \cdot 15 + 6 \\ 15 &= 2 \cdot 6 + 3 \\ 6 &= 2 \cdot 3 + 0 \end{aligned}$$

Thus, $gcd(123, 36) = 3$.

1.6.7 This exercise is a good starting off point for discussing the importance of definitions in mathematics. One possible definition for a circle is:

Definition 1.1. A circle with center O and radius length r is the set of points P on the sphere such that the distance along the great circle from O to P is r.

Note that this definition is itself not entirely well-defined, as we have not specified what we mean by distance. One workable definition is for distance to be net *cumulative* arc length along a great circle as we move from a point O to a point P.

An angle ABC can be most easily defined as the Euclidean angle made by the tangent lines at B to the circles defining $\overleftrightarrow{AB}$ and $\overleftrightarrow{CB}$.

Then, Postulate 1 is satisfied *most* of the time, as we can construct a unique great circle passing through two points on the sphere, *if* the points are not antipodal. We simply intersect the sphere with the plane through the two points and the center of the sphere.

Postulate 2 is satisfied as we can always extend an arc of a great circle, although we may retrace the existing arc.

Postulate 3 is satisfied if we use the cumulative distance definition as discussed above.

Postulate 4 is automatically satisfied as angles are Euclidean angles.

Postulate 5 is not satisfied, as *every* pair of lines intersects. An easy proof of this is to observe that every line is uniquely defined by a plane through the origin. Two different planes will intersect in a line, and this line must intersect the sphere at two points.

1.6.9 This is true. Given a plane through the origin, we can always find an orthogonal plane. The angle these planes make will equal the angle of the curves they define on the sphere, as the spherical angles are defined by tangent lines to the sphere, and thus lie in the planes.

1.6.11 This is true. An example is the triangle that is defined in the first octant. It has three right angles.

1.7 Project 2 - A Concrete Axiomatic System

After the last few sections dealing with abstract axiomatic systems, this lab is designed so that you can explore another geometric system through concrete manipulation of the points, lines, etc of that system. The idea here is to have you explore the environment first, then make some conjectures about what is similar and what is different in this system as compared to standard Euclidean geometry.

CHAPTER 2

Euclidean Geometry

In this chapter we start off with a very brief overview of basic properties of angles, lines, and parallels. We review, in summary form, some of the most important logical problems of classical Euclidean geometry that axiom writers such as Hilbert attempted to fix, and then to move on to more substantial results in plane geometry. As you work through this material, you may feel unsure of what you can assume and not assume when working on proofs. In each section of Chapter 2 the author tried to carefully describe what results and assumptions were made in that section. For example, in section 2.1, you are asked to use the notion of *betweenness* in the way your tuition would dictate, while at the same time pointing out that this is one of those geometric properties that needs an axiomatic base.

If you desire a more rigorous approach to Euclidean Geometry, the complete foundational development can be found in on-line chapters at the author's website: http://www.gac.edu/~hvidsten/geom-text.

If you want more guidance on the art of writing proofs, consult Appendix A of the text.

SOLUTIONS TO EXERCISES IN CHAPTER 2

2.1 Angles, Lines, and Parallels

This section may be the least satisfying section in the chapter for you, since many theorems are referenced without proof. These results were (hopefully) covered in great detail in your high school geometry course and we will only briefly review them. A full and consistent development of the results in this section would entail a "filling in" of many days foundational work based on Hilberts axioms. This foundational material can be found in on-line chapters at the author's website: http://www.gac.edu/˜hvidsten/geom-text.

A significant number of the exercises deal with parallel lines. This is for two reasons. First of all, historically there was a great effort to prove Euclid's fifth Postulate by converting it into a logically equivalent statement that was hoped to be easier to prove. Thus, many of the exercises nicely echo this history. Secondly, parallels and the parallel postulate are at the heart of one of the greatest revolutions in math —the discovery of non-Euclidean geometry. This section foreshadows that development, which is covered in Chapters 7 and 8.

2.1.1 It has already been shown that $\angle FBG \cong \angle DAB$. Also, by the vertical angle theorem (Theorem 2.3) we have $\angle FBG \cong \angle EBA$ and thus, $\angle DAB \cong \angle EBA$.

Now, $\angle DAB$ and $\angle CAB$ are supplementary, thus add to two right angles. Also, $\angle CAB$ and $\angle ABF$ are congruent by the first part of this exercise, as these angles are alternate interior angles. Thus, $\angle DAB$ and $\angle ABF$ add to two right angles.

2.1.3 a. False, right angles are defined solely in terms of congruent angles.

b. False, an angle is defined as *just* the two rays plus the vertex.

c. True. This is part of the definition.

d. False. The term "line" is undefined.

2.1.5 Proposition I-23 states that angles can be copied. Let A and B be points on l and n respectively and let m be the line through A and B. If $t = m$ we are done. Otherwise, let D be a point on t that is on the same side of n as l. (Assuming the standard properties of betweenness) Then, $\angle BAD$ is smaller than the angle at A formed by m and n. By Theorem 2.9 we know that the interior angles at B and A sum to two right angles, so $\angle CBA$ and $\angle BAD$ sum to less than two right angles. By Euclid's fifth postulate t and l must meet.

2.1.7 First, assume Playfair's Postulate, and let lines l and m be parallel, with line t perpendicular to l at point A. If t does not intersect m then, t and l are both parallel to m, which contradicts Playfair. Thus, t intersects m and by Theorem 2.9 t is perpendicular at this intersection.

Now, assume that whenever a line is perpendicular to one of two parallel lines, it must be perpendicular to the other. Let l be a line and P a point not on l. Suppose that m and n are both parallel to l at P. Let t be a perpendicular from P to l. Then, t is perpendicular to m and n at P. By Theorem 2.4 it must be that m and n are coincident.

2.1.9 Assume Playfair and let lines m and n be parallel to line l. If $m \neq n$ and m and n intersect at P, then we would have two different lines parallel to l through P, contradicting Playfair. Thus, either m and n are parallel, or are the same line.

Conversely, assume that two lines parallel to the same line are equal or themselves parallel. Let l be a line and suppose m and n are parallel to l at a point P not on l. Then, n and m must be equal, as they intersect at P.

2.2 Congruent Triangles and Pasch's Axiom

This section introduces many results concerning triangles and also discusses several axiomatic issues that arose from Euclid's treatment of triangles.

2.2.1 Yes, it could pass through points A and B of ΔABC. It does not contradict Pasch's axiom, as the axiom stipulates that the line cannot pass through A, B, or C.

2.2.3 No. Here is a counter-example.

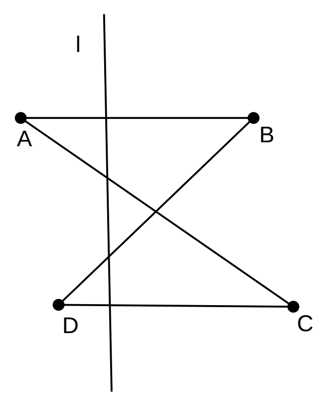


FIGURE 2.1:

2.2.5 If $A = C$ we are done. If A, B, and C are collinear, then B cannot be between A and C, for then we would have two points of intersection for two lines. If A is between B and C, then l cannot intersect $\overline{AC}$. Likewise, C cannot be between A and B.

If the points are not collinear, suppose A and C are on opposite sides. Then l would intersect all three sides of ΔABC, contradicting Pasch's axiom.

2.2.7 Let $\angle ABC \cong \angle ACB$ in ΔABC. Let $\overrightarrow{AD}$ be the angle bisector of $\angle BAC$ meeting side $\overline{BC}$ at D. Then, by AAS, ΔDBA and ΔDCA are congruent and $\overline{AB} \cong \overline{AC}$.

2.2.9 Suppose that two sides of a triangle are not congruent. Then, the angles opposite those sides cannot be congruent, as if they were, then by the previous exercise, the triangle would be isosceles.

Suppose in ΔABC that $\overline{AC}$ is greater than $\overline{AB}$. On $\overline{AC}$ we can find a point D between A and C such that $\overline{AD} \cong \overline{AB}$. Then, $\angle ADB$ is an exterior angle to ΔBDC and is thus greater than $\angle DCB$. But, ΔABD is isosceles and so $\angle ADB \cong \angle ABD$, and $\angle ABD$ is greater than $\angle DCB = \angle ACB$.

2.2.11 Let ΔABC and ΔXYZ be two right triangles with right angles at A and X, and suppose $\overline{BC} \cong \overline{YZ}$ and $\overline{AC} \cong \overline{XZ}$. Suppose $\overline{AB}$ is greater than $\overline{XY}$. Then, we can find a point D between A and B such that $\overline{AD} \cong \overline{XY}$. By SAS $\Delta ADC \cong \Delta XYZ$. Now, $\angle BDC$ is exterior to ΔADC and thus must be greater than 90 degrees. But, ΔCDB is isosceles, and thus $\angle DBC$ must also be greater than 90 degrees. This is impossible, as then ΔCDB would have angle sum greater than 180 degrees.

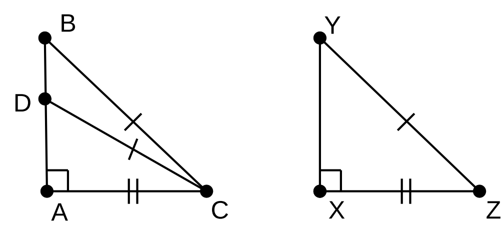


FIGURE 2.2:

2.2.13 We use AAS to show that $\Delta BFH \cong \Delta AFG$ and $\Delta CEI \cong \Delta AEG$. Thus $\overline{BH} \cong \overline{AG} \cong \overline{CI}$ and $BHIC$ is Saccheri. Also, by adding congruent angles in the left case we get that the sum of the angles in the triangle is the same as the sum of the summit angles. In the right case, we need to re-arrange congruent angles.

2.2.15 Given quadrilaterals $ABCD$ and $WXYZ$ we say the two quadrilaterals are congruent if there is some way to match vertices so that corresponding sides are congruent and corresponding angles are congruent.

SASAS Theorem: If $\overline{AB} \cong \overline{WX}$, $\angle ABC \cong \angle WXY$, $\overline{BC} \cong \overline{XY}$,

$\angle BCD \cong \angle XYZ$, and $\overline{CD} \cong \overline{YZ}$, then quadrilateral $ABCD$ is congruent to quadrilateral $WXYZ$.

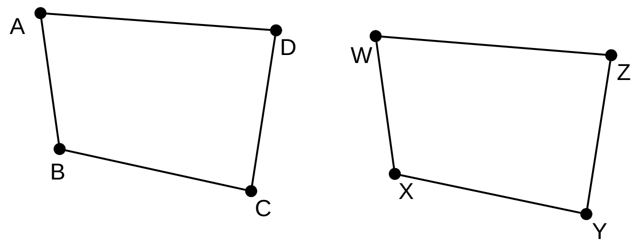


FIGURE 2.3:

Proof: ΔABC and ΔWXY are congruent by SAS. This implies that ΔACD and ΔWYZ are congruent. This shows that sides are correspondingly congruent, and two sets of angles are congruent ($\angle ABC \cong \angle WXY$ and $\angle CDA \cong \angle YZW$). Since $\angle BAC \cong \angle XWY$ and $\angle CAD \cong \angle YWZ$, then by angle addition $\angle BAD \cong \angle XWZ$. Similarly, $\angle BCD \cong \angle XYZ$. □

2.3 Project 3 - Special Points of a Triangle

You are encouraged to explore and experiment in this lab project. Are there any other sets of intersecting lines that one could construct for a given triangle? Are there interesting properties of constructed intersecting lines in other polygons?

You may be taking this class to become a secondary math teacher. This project is one that could be easily transferred to the high school setting.

2.4.1 Mini-Project:Area in Euclidean Geometry

This section includes the first "mini-project" for the course. These projects are designed to be done in the classroom, in groups of three or four students. Each group should elect a Recorder. The Recorder's sole job is to outline the group's solutions to exercises. The summary should not be a formal write-up of the project, but should give enough a brief synopsis of the group's reasoning process.

The main goal for the mini-projects is to have a discussion of geometric ideas. Through the group process, you can clarify your understanding of concepts, and help others better grasp abstract ways of thinking. There is no better way to conceptualize an idea than to have to explain it to another person.

In this mini-project, you are asked to grapple with the notion of "area". The notion of area is not that simple or obvious. For example, what does it mean for two figures to have the same area?

2.4.1 Construct a diagonal and use the fact that alternate interior angles of a line falling on parallel lines are congruent to generate an ASA congruence for the two sub-triangles created in the parallelogram.

2.4.3 We have that $\overline{AE} \cong \overline{DF}$. Theorem 2.9 says that $\angle AEB \cong \angle DFC$. So, by SAS $\Delta AEB \cong \Delta DFC$. Let G be the point where $\overline{CD}$ intersects $\overline{BE}$. (Such a point exists by Pasch's axiom applied to ΔAEB) Now, parallelogram $ABCD$ can be split into ΔAEB plus ΔBGC minus ΔDGE. Also, parallelogram $EBCF$ can be split into ΔDFC plus ΔBGC minus ΔDGE.

2.4.5 By Theorem 2.9 we know that $\angle BAE$ and $\angle FBA$ are right angles, and thus $ABFE$ is a rectangle. By Theorem 2.9 we have that $\angle DAB \cong \angle CBG$, where G is a point on $\overrightarrow{AB}$ to the right of B. Subtracting the right angles, we get $\angle DAE \cong \angle CBF$. By SAS, $\Delta DAE \cong \Delta CBF$. Then rectangle $AEFB$ can be split into $AECB$ and ΔCBF and parallelogram $DABC$ can be split into $AECB$ and ΔDAE and the figures are equivalent.

Hidden Assumptions? One hidden assumption is the notion that areas are additive. That is, if we have two figures that are not overlapping, then the area of the union is the sum of the separate areas.

2.4.2 Cevians and Area

2.4.7 Let the triangle and medians be labeled as in Theorem 2.24. The area of ΔAYB will be equal to AYh, where h is the length of a perpendicular dropped from B to $\overleftrightarrow{AC}$. The area of ΔCYB will be equal to CYh, Since $\overline{AY} \cong \overline{CY}$, these areas will be the same and ΔABC will balance along $\overleftrightarrow{BY}$. A similar argument shows that ΔABC balances along each median, and thus the centroid is a balance point for the triangle.

2.4.9 Consider median $\overline{BD}$ in ΔABC, with E the centroid. Let $a = BE$ and $b = DE$. Then, the area of ΔDBC is $\frac{(a+b)h}{2}$ where h is the height of the triangle. This is 3 times the area of ΔDEC by the previous exercise. Thus, $\frac{(a+b)h}{2} = 3\frac{bh}{2}$, or $a = 2b$.

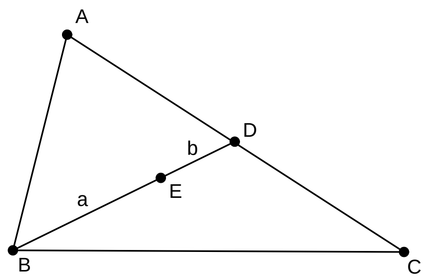


FIGURE 2.4:

2.5 Similar Triangles

As stated in the text, similarity is one of the most useful tools in the geometer's toolkit. It can be used in the definition of the trigonometric functions and in proofs of theorems like the Pythagorean Theorem.

2.5.1 Since $\overleftrightarrow{DE}$ cuts two sides of triangle at the midpoints, then by Theorem 2.27, this line must be parallel to the third side $\overline{BC}$. Thus $\angle ADE \cong \angle ABC$ and $\angle AED \cong \angle ACB$. Since the angle at A is congruent to itself, we have by AAA that ΔABC and ΔADE are similar, with proportionality constant of $\frac{1}{2}$.

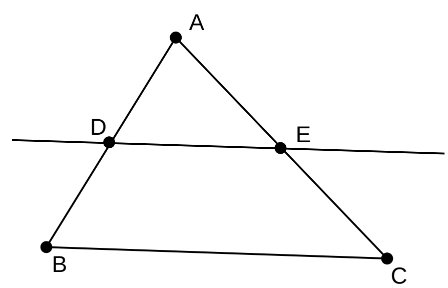


FIGURE 2.5:

2.5.3 Let ΔABC and ΔDEF have the desired SSS similarity property. That is sides $\overline{AB}$ and $\overline{DE}$, sides $\overline{AC}$ and $\overline{DF}$, and sides $\overline{BC}$ and $\overline{EF}$ are proportional. We can assume that $\overline{AB}$ is at least as large as $\overline{DE}$. Let G be a point on $\overline{AB}$ such that $\overline{AG} \cong \overline{DE}$. Let $\overleftrightarrow{GH}$ be the parallel to $\overleftrightarrow{BC}$ through G. Then, $\overleftrightarrow{GH}$ must intersect $\overleftrightarrow{AC}$, as otherwise $\overleftrightarrow{AC}$ and $\overleftrightarrow{BC}$ would be parallel. By the properties of parallels,

$\angle AGH \cong \angle ABC$ and $\angle AHG \cong \angle ACB$. Thus, ΔAGH and ΔABC are similar.

Therefore, $\frac{AB}{AG} = \frac{AC}{AH}$. Equivalently, $\frac{AB}{DE} = \frac{AC}{AH}$. We are given that $\frac{AB}{DE} = \frac{AC}{DF}$. Thus, $\overline{AH} \cong \overline{DF}$.

Also, $\frac{AB}{AG} = \frac{BC}{GH}$ and $\frac{AB}{AG} = \frac{AB}{DE} = \frac{BC}{EF}$. Thus, $GH \cong EF$.

By SSS ΔAGH and ΔDEF are congruent, and thus ΔABC and ΔDEF are similar.

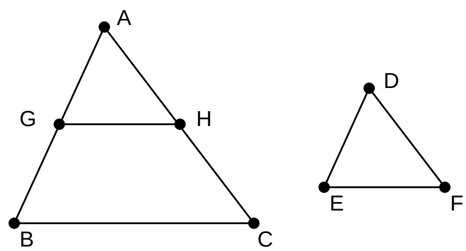


FIGURE 2.6:

2.5.5 Any right triangle constructed so that one angle is congruent to $\angle A$ must have congruent third angles, and thus the constructed triangle must be similar to ΔABC. Since *sin* and *cos* are defined in terms of ratios of sides, then proportional sides will have the same ratio, and thus it does not matter what triangle one uses for the definition.

2.5.7 If the parallel to $\overleftrightarrow{AC}$ does not intersect $\overleftrightarrow{RP}$, then it would be parallel to this line, and since it is already parallel to $\overleftrightarrow{AC}$, then by Exercise 2.1.9 $\overleftrightarrow{RP}$ and $\overleftrightarrow{AC}$ would be parallel, which is impossible.

By the properties of parallels, $\angle RAP \cong \angle RBS$ and $\angle RPA \cong \angle RSB$. Thus, by AAA ΔRBS and ΔRAP are similar. ΔPCQ and ΔSBQ are similar by AAA using an analogous argument for two of the angles and the vertical angles at Q.

Thus, $\frac{CP}{BS} = \frac{CQ}{BQ} = \frac{PQ}{QS}$, and $\frac{AP}{BS} = \frac{AR}{BR} = \frac{PR}{SR}$. So, $\frac{CP}{AP}\frac{BQ}{QC} = \frac{CP}{AP}\frac{BS}{CP} = \frac{BS}{AP}$ And, $\frac{CP}{AP}\frac{BQ}{QC}\frac{AR}{RB} = \frac{BS}{AP}\frac{AR}{RB} = \frac{BS}{AP}\frac{AP}{BS} = 1$.

2.5.1 Mini-Project: Finding Heights

This mini-project is a very practical application of the notion of similarity. The mathematics in the first example for finding height is extremely easy, but the interesting part is the data collection. You will need to determine how to get the most accurate measurements using the materials on hand.

The second method of finding height is a calculation using two similar triangles. The interesting part of this project is to see the connection between the mirror reflection and the calculation you made in part I.

You should work in small groups with a Recorder, but make sure the Recorder position gets shifted around from project to project.

2.6 Circle Geometry

This section is an introduction to the basic geometry of the circle. The properties of inscribed angles and tangents are the most important properties to focus on in this section.

2.6.1 Case 2: A is on the diameter through $\overline{OP}$. Let $\alpha = m\angle PBO$ and $\beta = m\angle POB$. Then, $\beta = 180-2\alpha$. Also, $m\angle AOB = 180-\beta = 2\alpha$.

Case 3: A and B are on the same side of $\overleftrightarrow{PO}$. We can assume that $m\angle OPB > m\angle OPA$. Let $m\angle OPB = \alpha$ and $m\angle OPA = \beta$. Then, we can argue in a similar fashion to the proof of the Theorem using $\alpha - \beta$ instead of $\alpha + \beta$.

2.6.3 Consider $\angle AQP$. This must be a right angle by Corollary 2.33. Similarly, $\angle BQP$ must be a right angle. Thus, A, Q, and B are collinear.

2.6.5 Let $\overline{AB}$ be the chord, O the center, and M the midpoint of $\overline{AB}$. Then $\Delta AOM \cong \Delta BOM$ by SSS and the result follows.

2.6.7 Consider a triangle on the diagonal of the rectangle. This has a right angle, and thus we can construct the circle on this angle. Since the other triangle in the rectangle also has a right angle on the same side (the diameter of the circle) then it is also inscribed in the same circle.

2.6.9 If point P is inside the circle c, then Theorem 2.41 applies. But, this theorem says that $m\angle BPA = \frac{1}{2}(m\angle BOA + m\angle COD)$, where C and D are the other points of intersections of $\overleftrightarrow{PA}$ and $\overleftrightarrow{PB}$ with the circle. If P is inside c, then C and D are different points. The assumption of Theorem 2.42 says that $m\angle BPA = \frac{1}{2}m\angle BOA$. But, $m\angle BPA = \frac{1}{2}(m\angle BOA + m\angle COD)$ would then imply that $m\angle COD = 0$, which is impossible as C and D are not collinear with O.

2.6.11 The angle made by $\overline{BT}$ and l must be a right angle by Theorem 2.36. Likewise, the angle made by $\overline{AT}$ and l is a right angle. Thus, A, T, and B are collinear.

2.6.13 Suppose one of the circles had points A and B on opposite sides of the tangent line l. Then $\overline{AB}$ would intersect l at some point P which is interior to the circle. But, then l would pass through an interior point of the circle and by continuity must intersect the circle in two points which is impossible. Thus, either all points of one circle are on opposite sides of l from the other circle or are on the same side.

2.6.15 By Theorem 2.36, we have that $\angle OAP$ is a right angle, as is $\angle OBP$. Since the hypotenuse ($\overline{OP}$) and leg ($\overline{OA}$) of right triangle ΔOAP are congruent to the hypotenuse ($\overline{OP}$) and leg ($\overline{OB}$) of right triangle ΔOBP, then by Exercise 2.2.10 the two triangles are congruent. Thus $\angle OPA \cong \angle OPB$.

2.6.17 Let A and B be the centers of the two circles. Construct the two perpendiculars at A and B to $\overleftrightarrow{AB}$ and let C and D be the intersections with the circles on one side of $\overleftrightarrow{AB}$.

If $\overleftrightarrow{CD}$ does not intersect $\overleftrightarrow{AB}$, then these lines are parallel, and the angles made by $\overleftrightarrow{CD}$ and the radii of the circles will be right angles. Thus, this line will be a common tangent.

Otherwise, $\overleftrightarrow{CD}$ intersects $\overleftrightarrow{AB}$ at some point P. Let $\overleftrightarrow{PE}$ be a tangent to the circle with center A. Then, since ΔPAC and ΔPBD are similar, we have $\frac{AP}{BP} = \frac{AC}{BD}$. Let $\overleftrightarrow{BF}$ be parallel to $\overleftrightarrow{AE}$ with F the intersection of the parallel with the circle centered at B. Then, $\frac{AC}{BD} = \frac{AE}{BF}$. So, $\frac{AP}{BP} = \frac{AE}{BF}$. By SAS similarity, ΔPAE and ΔPBF are similar, and so F is on $\overleftrightarrow{PE}$ and $\angle PFB$ is a right angle. Thus, $\overleftrightarrow{PE}$ is a tangent to the circle centered at B.

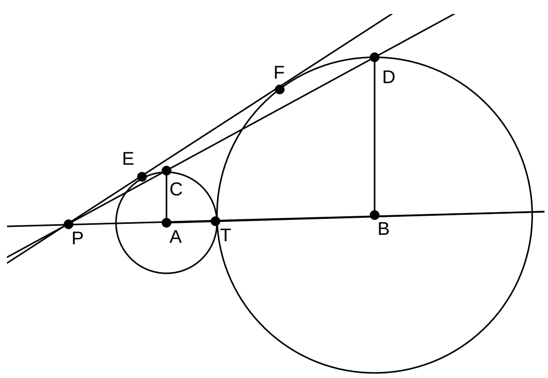


FIGURE 2.7:

2.7 Project 4 - Circle Inversion and Orthogonality

This section is crucial for the later development of the Poincaré model of non-Euclidean (hyperbolic) geometry. It is also has some of the most elegant mathematical results found in the course.

CHAPTER 3

Analytic Geometry

This chapter is a very quick review of analytic geometry. In succeeding chapters, analytic methods will be utilized freely.

SOLUTIONS TO EXERCISES IN CHAPTER 3

3.2 Vector Geometry

3.2.1 If A is on either of the axes, then so is B and the distance result holds by the definition of coordinates. Otherwise, A (and B) are not on either axis. Drop perpendiculars from A and B to the x-axis at P and Q. By SAS similarity, ΔAOP and ΔBOQ are similar, and thus $\angle AOP \cong \angle BOQ$, which means that A and B are on the same line $\overleftrightarrow{AO}$, and the ratio of BO to AO is k.

3.2.3 The vector from P to Q is in the same direction (or opposite direction) as the vector v. Thus, since the vector from P to Q is $\vec{Q}-\vec{P}$, we have $\vec{Q}-\vec{P}=tv$, for some real number t. In coordinates we have $(x,y)-(a,b)=(tv_1,tv_2)$, or $(x,y)=(a,b)+t(v_1,v_2)$.

3.2.5 By Exercise 3.2.3 the line through A and B can be represented by the set of points of the form $\vec{A}+t(\vec{B}-\vec{A})$. Then, $M=\frac{1}{2}(\vec{A}+\vec{B})=\vec{A}+\frac{1}{2}(\vec{B}-\vec{A})$ is on the line through A and B, and is between A and B. Let $A=(x_1,y_1)$ and $B=(x_2,y_2)$, then the distance from A to M is $\sqrt{(\frac{x_1}{2}-\frac{x_2}{2})^2+(\frac{y_1}{2}-\frac{y_2}{2})^2}$, which is equal to the distance from B to M.

3.3 Project 5 - Bézier Curves

3.4 Angles in Coordinate Geometry

3.4.1 Let $\vec{A} = (\cos(\alpha), \sin(\alpha))$ and $\vec{B} = (\cos(\beta), \sin(\beta))$. Then, from Theorem 3.11 we have $\cos(\alpha - \beta) = \vec{A} \circ \vec{B}$, since $\vec{A}$ and $\vec{B}$ are unit length vectors. The result follows immediately.

3.4.3 By Exercise 3.4.1,

$$\begin{aligned} \cos(\frac{\pi}{2} - (\alpha + \beta)) &= \cos(\frac{pi}{2}) \cos(\alpha + \beta) + \sin(\frac{pi}{2} 90) \sin(\alpha + \beta)) \\ &= \sin(\alpha + \beta). \end{aligned}$$

Then, use the formula from Exercise 3.4.2 with the term inside cos being $(\frac{pi}{2} - \alpha) + (-\beta)$.

3.5 The Complex Plane

3.5.1

$$\begin{aligned} e^{i\theta} e^{i\phi} &= (\cos(\theta) + i\ \sin(\theta))(\cos(\phi) + i\ \sin(\phi)) \\ &= (\cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi)) + i(\cos(\theta)\sin(\phi) + \sin(\theta)\cos(\phi)) \\ &= \cos(\theta + \phi) + i\ \sin(\theta + \phi) \\ &= e^{i(\theta + \phi)} \end{aligned}$$

3.5.3 Let $z = e^{i\theta}$ and $w = e^{i\phi}$ and use Exercise 3.4.1.

3.5.5 The rationalized complex numbers have the form $i\frac{-1}{2}$, i, and $\frac{1}{10} - i\frac{1}{5}$.

3.6 Birkhoff's Axiomatic System for Analytic Geometry

3.6.1 First, if A is associated to $x_A = t_A\sqrt{dx^2 + dy^2}$, where $A = (x, y) = (x_0, y_0) + t_A(dx, dy)$, and B is associated to x_B in a similar fashion, then $|x_A - x_B| = |t_A - t_B|\sqrt{dx^2 + dy^2}$. On the other hand,

$$d(A, B) = \sqrt{(t_A dx - t_B dx)^2 + (t_A dy - t_B dy)^2} = \sqrt{dx^2 + dy^2}|t_A - t_B|$$

3.6.3 Given a point O as the vertex of the angle, set O as the origin of the coordinate system. Then, identify a ray $\overrightarrow{OA}$ associated to the angle θ, with $A = (x, y)$. Let $a = ||\vec{A}|| = \sqrt{x^2 + y^2}$. Then, $\sin^2(\theta) + \cos^2(\theta) = (\frac{x}{a})^2 + (\frac{y}{a})^2 = \frac{x^2+y^2}{a^2} = 1$.

3.6.5 Discussion question. One idea is that analytic geometry allows one to study geometric figures by the equations that define them. Thus, geometry can be reduced to the arithmetic (algebra) of equations.

CHAPTER 4

Constructions

In this chapter we cover some of the basic Euclidean constructions and also have a lot of fun with lab projects. The origami project should be especially interesting, as it is an axiomatic system with which you can *physically* interact and explore.

The third section on constructibility may be a bit heavy and abstract, but the relationship between geometric constructibility and algebra is a fascinating one, especially if you have had some exposure to abstract algebra. Also, any mathematically literate person should know what the three classical construction problems are, and how the pursuit of solutions to these problems has had a profound influence on the development of modern mathematics.

SOLUTIONS TO EXERCISES IN CHAPTER 4

4.1 Euclidean Constructions

4.1.1 Use SSS triangle congruence on ΔABF and ΔDGH.

4.1.3 Use the SSS triangle congruence theorem on ΔADE and ΔABE to show that $\angle EAB \cong \angle BAE$.

4.1.5 Use the fact that both circles have the same radius.

4.1.7 Let the given line be l and let P be the point not on l. Construct the perpendicular m to l through P. At a point Q on m, but not on l, construct the perpendicular n to m. Theorem 2.7 implies that l and n are parallel.

4.1.9 On $\overrightarrow{BA}$ construct A' such that $BA' = a$. On $\overrightarrow{BC}$ construct C' such that $BC' = b$. Then, SAS congruence gives $\Delta AB'C'$ congruent to any other triangle with the specified data.

4.2 Project 6 - Euclidean Eggs

Consider the contrast between the method of drawing curves used in this project and the method of Bézier Curves covered in Chapter 3.

4.3 Constructibility

4.3.1 Just compute the formula for the intersection.

4.3.3 Reverse the roles of the product construction.

4.3.5 For $\sqrt{3}$, use a right triangle with hypotenuse 2 and one side 1. For $\sqrt{5}$, use a right triangle with sides of length 1 and 2.

4.3.7 Consider $\frac{a}{\pi}$. This is less than a.

4.3.9 If a circle of radius r and center (x, y) has x not constructible, then $(x, y + r)$ and $(x, y - r)$ are non-constructible on the circle. We can use the same reasoning if y is not constructible. If the center is constructible, then the previous exercise gives at least two non-constructible points for a circle of radius r whose center is at the origin. Add (x, y) to these two points to get two non-constructible points on the original circle.

4.4 Mini-Project: Origami Construction

For this project, you will need a good supply of square paper. Commercial origami paper is quite expensive. Equally as good paper can be made by taking notepads and cutting them into squares using a paper-cutter. (Cutting works best a few sheets at a time)

4.3.11 Given $\overline{AB}$, we can fold A onto B by axiom O2. Let l be the fold line of reflection created, and let l intersect AB at C. Then, since the fold preserves length, we have that $AC = CB$, and $\angle ACE \cong \angle ECB$, as show in Fig. 4.1. The result follows.

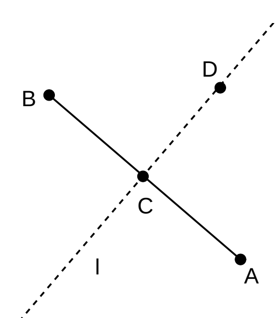


FIGURE 4.1:

4.3.13 Since the reflection fold across t preserves length, we have $PR = P'R$. Also, the distance from a point to a line is measured along the perpendicular from the point to the line. Thus, the distance from R to l is equal to $P'R$. Thus, the distance from R to P equals the distance from R to l and R is on the parabola with focus P and directrix l.

An interesting extra credit problem for this construction would be to show that t is *tangent* to the parabola at R. One proof is as follows:

Suppose t intersected at another point R' on the parabola. Then, by definition, R' must have been constructed in the same way that R was, so there must be a folding (reflection) across t taking P to some point P'' on l such that $\overleftrightarrow{P''R'}$ is perpendicular to l at P'', and intersects t at R'. Then, by a triangle argument, we can show that $\overleftrightarrow{PP'}$ and $\overleftrightarrow{PP''}$ must both be perpendicular to t at R and R'. Since perpendiculars are unique, we must have that $R = R'$.

(To show, for example, that $\overleftrightarrow{PP'}$ is perpendicular to t at R, we can easily show that $\Delta PQR \cong \Delta P'QR$ by using the angle- and distance-preserving properties of reflections, and then use a second congruent triangle argument to show that $\overleftrightarrow{PP'}$ crosses t at right angles.)

CHAPTER 5

Transformational Geometry

In this chapter we make great use of functional notation and somewhat abstract notions such as $1-1$ and onto, inverses, composition, etc. You may wonder how such computations are related to geometry, but that is the very essence of the chapter—that we can understand and investigate geometric ideas with more than one set of foundational lenses.

With that in mind, we will make use of synthetic geometric techniques where they are most elegant and can aid intuition, and at other times we will rely on analytical techniques.

SOLUTIONS TO EXERCISES IN CHAPTER 5

5.1 Euclidean Isometries

5.1.1 Define the function f^{-1} by $f^{-1}(y) = x$ if and only if $f(x) = y$. Then, f^{-1} is well-defined, as suppose $f(x_1) = f(x_2) = y$. Then, since f is $1-1$ we have that $x_1 = x_2$. Since f is onto, we have that for every y in S there is an x such that $f(x) = y$. Thus, f^{-1} is defined on all of S. Finally, $f^{-1}(f(x)) = f^{-1}(y) = x$ and $f(f^{-1}(y)) = f(x) = y$. So, $f \circ f^{-1} = f^{-1} \circ f = id_S$.

Suppose g was another function on S such that $f \circ g = g \circ f = id_S$. Then, $g \circ f \circ f^{-1} = f^{-1}$, or $g = f^{-1}$.

5.1.3 Since $g^{-1} \circ f^{-1} \circ f \circ g = g^{-1} \circ g = id$ and $f \circ g \circ g^{-1} \circ f^{-1} = f \circ f^{-1} = id$, then $g^{-1} \circ f^{-1} = (f \circ g)^{-1}$.

5.1.5 Let f be an isometry and let c be a circle centered at O of radius $r = OA$. Let $O' = f(O)$ and $A' = f(A)$. Let P be any point on c. Then, $O'f(P) = f(O)f(P) = OP = r$. Thus, the image of c under f is contained in the circle centered at O' of radius r. Let P' be any other point on the circle centered at O' of radius r. Then, $Of^{-1}(P') = f^{-1}(O')f^{-1}(P') = O'P' = r$. Thus, $f^{-1}(P')$ is a point on c and every such point P' is the image of a point on c, under the map T.

5.1.7 Label the vertices of the triangle A, B, and C. Then, consider vertex A. Under an isometry, consider the actual position of A in the plane. After applying the isometry, A might remain or be replaced by one of the other two vertices. Thus, there are three possibilities for the position occupied by A. Once that vertex has been identified, consider position B. There are now just two remaining vertices to be placed in this position. Thus, there are a maximum of 6 isometries. We can find 6 by considering the three basic rotations by 0, 120, and 240 degrees, and the three reflections about perpendicular bisectors of the sides.

5.1.9 First, we show that f is a transformation. To show it is $1-1$, suppose $f(x, y) = f(x', y')$. Then, $kx+a = kx'+a$ and $ky+b = ky'+b$. So, $x = x'$ and $y = y'$.

To show it is onto, let (x', y') be a point. Then, $f(\frac{x'-a}{k}, \frac{y'-b}{k}) = (x', y')$.

f is not, in general, an isometry, since if $A = (x, y)$ and $B = (x', y')$ then $f(A)f(B) = kAB$.

5.1.11 Let ABC be a triangle and let $A'B'C'$ be its image under f. By the previous exercise, these two triangles are similar. Thus, there is a $k > 0$ such that $A'B' = kAB$, $B'C' = kBC$, and $A'C' = kAC$. Let D be any other point not on $\overleftrightarrow{AB}$. Then, using triangles ABD and $A'B'D'$ we get that $A'D' = kAD$.

Now, let $\overline{DE}$ be any segment with D not on $\overleftrightarrow{AB}$. Then, using triangles ADE and $A'D'E'$ we get $D'E' = kDE$, since we know that $A'D' = kAD$.

Finally, let $\overline{EF}$ be a segment entirely on $\overleftrightarrow{AB}$, and let D be a point off $\overleftrightarrow{AB}$. Then, using triangles DEF and $D'E'F'$ we get $E'F' = kEF$, since we know that $D'E' = kDE$.

Thus, in all cases, we get that $f(A)f(B) = kAB$.

5.2.1 Mini-Project:Isometries Through Reflection

In this mini-project, you will be led through a guided discovery of the amazing fact that, given any two congruent triangles, one can find a sequence of at most three reflections taking one triangle to the other.

5.2.1 First of all, suppose that C and R are on the same side of $\overleftrightarrow{AB}$. Then, since there is a unique angle with side $\overline{AB}$ and measure equal to the measure of $\angle BAC$, then R must lie on $\overrightarrow{AC}$. Likewise, R must lie on $\overrightarrow{BC}$. But, the only point common to these two rays is C. Thus, $R = C$.

If C and R are on different sides of $\overleftrightarrow{AB}$, then drop a perpendicular from C to $\overleftrightarrow{AB}$, intersecting at P. By SAS, ΔPAC and ΔPAR are congruent, and thus $\angle APR$ must be a right angle, and R is the reflection of C across $\overleftrightarrow{AB}$.

5.2.3 If two triangles (ΔABC and ΔPQR) share no point in common, then by Theorem 5.6 there is a reflection mapping A to P, and by the previous exercise, we would need at most two more reflections to map $\Delta Pr(B)r(C)$ to ΔPQR.

5.2.2 Reflections

5.2.5

5.2.7 Let G be the midpoint of $\overline{AB}$. Then $\Delta AED \cong \Delta BCD$ by SAS and $\Delta AGD \cong \Delta BGD$ by SSS. Thus, $\overleftrightarrow{DG}$ is the perpendicular bisector of $\overline{AB}$, and reflection across $\overleftrightarrow{DG}$ takes A to B. Also, $\overleftrightarrow{DG}$ must bisect the angle at D and by the previous exercise the bisector is a line of reflection. This proof would be easily extendable to regular n-gons, for n odd, by using repeated triangle congruences to show the perpendicular bisector is the angle bisector of the opposite vertex.

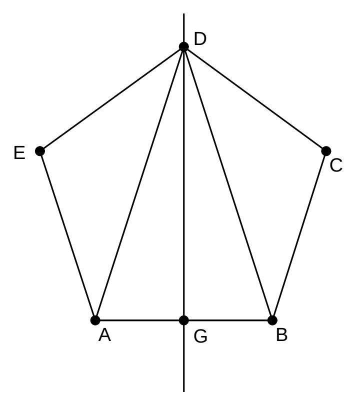


FIGURE 5.1:

5.2.9 Suppose that a line of symmetry l for parallelogram $ABCD$

is parallel to side $\overline{AB}$. Then, clearly reflection across l cannot map A to B, as this would imply that l is the perpendicular bisector of $\overline{AB}$.

If reflection mapped A to C, then l would be the perpendicular bisector of a diagonal of the parallelogram. But, since l is parallel to $\overline{AB}$, this would imply that the diagonal must be perpendicular to $\overline{AB}$ as well. A similar argument can be used to show that the other diagonal ($\overline{BD}$) must also be perpendicular to $\overline{AB}$. If this were the case, one of the triangles formed by the diagonals would have angle sum greater than 180 degrees, which is impossible.

Thus, reflection across l must map A to D, and l must be the perpendicular bisector of $\overline{AD}$. Clearly, using the property of parallels, we get that the angles at A and D in the parallelogram are right angles.

5.2.11 Let r be a reflection across $\overleftrightarrow{AB}$ and let C be a point not on $\overleftrightarrow{AB}$. Then, $r(C)$ is the unique point on the perpendicular dropped to $\overleftrightarrow{AB}$ at a point P on this line such that $CP = r(C)P$, with $r(C) \neq C$. Now, $r(r(C))$ is the unique point on this same perpendicular such that $r(C)P = r(r(C))P$, with $r(r(C)) \neq r(C)$. But since $r(C)P = CP$ and $C \neq r(C)$, then $r(r(C)) = C$. But, then $r \circ r$ fixes three non-collinear points A, B, and C, and so must be the identity.

5.2.13 Let A and B be distinct points on l. Then, $r_m \circ r_l \circ r_m(r_m(A)) = r_m(r_l(A)) = r_m(A)$ and likewise, $r_m \circ r_l \circ r_m(r_m(B)) = r_m(B)$. Thus, the line l' through $r_m(A)$ and $r_m(B)$ is fixed by $r_m \circ r_l \circ r_m$ and this triple composition must be equivalent to reflection across l'.

5.3 Translations

5.3.1 There are few examples in nature that have perfect translational symmetry. One example might be the atoms in a crystal atomic lattice. But there are some partial examples, like the legs on a millipede.

5.3.3 Since $(r_2 \circ r_1) \circ (r_1 \circ r_2) = id$, and $(r_1 \circ r_2) \circ (r_2 \circ r_1) = id$, then $r_2 \circ r_1$ is the inverse of $r_1 \circ r_2$. Also, if T has translation vector v, then $T(x, y) = (x, y) + v$. Let S be the translation defined by $S(x, y) = (x, y) - v$. Then, $S \circ T(x, y) = ((x, y) + v) - v = (x, y)$ and $T \circ S((x, y) - v) + v = (x, y)$. Thus, S is the inverse to T.

5.3.5 Represent T_1 and T_2 in coordinate form.

5.3.7 Let (x, K) be a point on the line $y = K$. If T is a translation with translation vector $v = (0, -K)$, then, by Exercise 5.3.3, T^{-1} has translation vector of $-v = (0, K)$. Thus, $T^{-1} \circ r_x \circ T(x, K) = T^{-1} \circ r_x(x, 0) = T^{-1}(x, 0) = (x, K)$. So, $T^{-1} \circ r_x \circ T$ fixes the line $y = K$ and so must be the reflection across this line. The coordinate equation for r

is given by $T^{-1} \circ r_x \circ T(x, y) = T^{-1} \circ r_x(x, y - K) = T^{-1}(x, -y + K) = (x, -y + 2K)$. So, $r(x, y) = (x, -y + 2K)$.

5.3.9 Let T be a translation with (non-zero) translation vector parallel to a line l. Let m be perpendicular to l at point P. Let n be the perpendicular bisector of $\overline{PT(P)}$, intersecting $\overline{PT(P)}$ at point Q. Then, r_n, reflection about n maps P to $T(P)$. Consider $r_n \circ T$. We have $r_n \circ T(P) = P$. Let $R \neq P$ be another point on m. Then, $PRT(R)T(P)$ is a parallelogram, and thus $\angle PRT(R)$ and $\angle RT(R)T(P)$ are right angles. Let S be the point where n intersects $\overline{RT(R)}$. Then, $\angle RSQ$ is also a right angle. Also, by a congruent triangle argument, we have $\overline{RS} \cong \overline{ST(R)}$, and so n is the perpendicular bisector of $\overline{RT(R)}$ and $r_n \circ T(R) = R$. Since $r_n \circ T$ fixes two points on m we have $r_n \circ T = r_m$, or $T = r_n \circ r_m$.

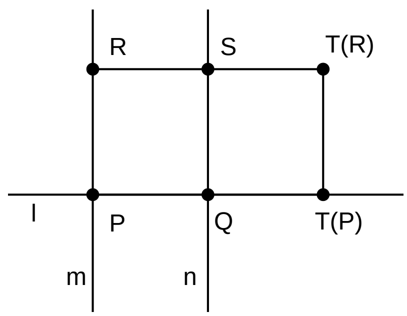


FIGURE 5.2:

5.4 Rotations

5.4.1 First,

$$
\begin{aligned}
T^{-1} \circ Rot_\phi \circ T(C) &= T^{-1} \circ Rot_\phi \circ T(x, y) \\
&= T^{-1} \circ Rot_\phi(0, 0) \\
&= T^{-1}(0, 0) \\
&= (x, y) \\
&= C
\end{aligned}
$$

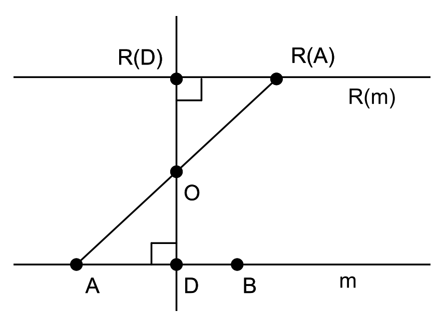
Suppose $T^{-1} \circ Rot_\phi \circ T$ fixed another point P. Then, $Rot_\phi \circ T(P) = T(P)$, which implies that $T(P) = (0, 0)$, or $P = T^{-1}(0, 0) = (x, y) = C$. Thus, $T^{-1} \circ Rot_\phi \circ T$ must be a rotation. What is the angle for this rotation? Consider a line l through C that is parallel to the x-axis. Then, T will map l to the $x - axis$ and Rot_ϕ will map the x-axis to a line m making an angle of ϕ with the x-axis. Then, T^{-1} will preserve this angle, mapping m to a line making an angle of ϕ with l. Thus, the rotation angle for $T^{-1} \circ Rot_\phi \circ T$ is ϕ.

5.4.3 A book on flowers or diatoms (algae) would be a good place to start.

5.4.5 By the preceding exercise, the invariant line must pass through the center of rotation. Let A be a point on the invariant line. Then, $R_O(A)$ lies on $\overleftrightarrow{OA}$ and $\overline{OA} \cong \overline{OR_O(A)}$. Either A and $R_O(A)$ are on the same side of O or are on opposite sides. If they are on the same side, then $A = R_O(A)$, and the rotation is the identity, which is ruled out. If they are on opposite sides, then the rotation is 180 degrees. If the rotation is 180 degrees, then for every point $A \neq O$ we have that A, O, and $R_O(A)$ are collinear, which means that the line $\overleftrightarrow{OA}$ is invariant.

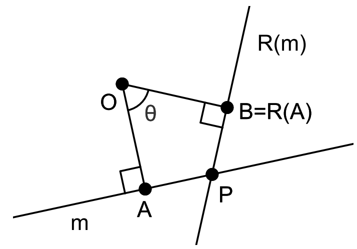
5.4.7 Suppose $R(m)$ is parallel to m. Let O be the center of rotation. If m passes through O, then $R(m)$ also passes through O, which contradicts the lines being parallel. So, we assume m does not pass through O. Let A, B be two points on m and construct $\overline{AR(A)}$. If the angle $\angle BAO$ is a right angle, then $\angle R(B)R(A)R(O)$ is also a right angle, and the rotation angle must be 180 degrees, which is impossible.

So, WLOG we assume that $\angle BAO$ is less than a right angle. Drop a perpendicular from O to m at D. Since R is an isometry , then $\Delta DAO \cong \Delta R(D)R(A)O$ and the vertical angles at O are congruent, which means A, O, and $R(A)$ are collinear. This would mean the rotation angle is 180 degrees, which is impossible.



So, m and $R(m)$ intersect at a point P.

Drop perpendiculars from O to m and $R(m)$ at A and B. Note that $A \neq P$, as if $A = P$ then m and $R(m)$ would coincide. Thus, $OBPA$ is a quadrilateral. Let θ be the rotation angle. Since the sum of the angles in a quadrilateral is 360 degrees, then $\angle BPA$ must have measure $180 - \theta$. But, then the vertical angle at P has measure $180 - m\angle BPA = \theta$.



5.4.9 Let R_1 and R_2 be two rotations about P. Let l and m be the lines of reflection for R_1. By Theorem 5.15 we can choose m as a defining line of reflection for R_2 and there is a unique line n such that $R_2 = r_n \circ r_m$. Then, $R_2 \circ R_1 = r_n \circ r_m \circ r_m \circ r_l = r_n \circ r_l$, which is again a rotation about P.

5.4.11 If $r_1 \circ R = r_2$, then $R = r_1 \circ r_2$. Thus, the lines for r_1 and r_2 must intersect. If they intersected at a point other than the center of rotation for R, then R would fix more than one point, which is impossible.

5.4.13 Let $m = \overleftrightarrow{AB}$. Then, we can choose lines l and n such that $H_A = r_m \circ r_l$ and $H_B = r_n \circ r_m$. Note that l and n are both perpendicular to m and thus parallel. Then, $H_B \circ H_A = r_n \circ r_l$, which is a translation parallel to l.

5.5 Project 7 -Quilts and Transformations

This project is another great opportunity for the future teachers in the class to develop similar projects for use in their own teaching. One idea to incorporate into a high school version of the project is to bring into the class the cultural and historical aspects of quilting.

5.6 Glide Reflections

5.6.1 As with translations, it will be hard to find a perfect example of a glide symmetry in nature. But, the are many plants whose branches alternate in a glide fashion.

5.6.3 Suppose m is invariant. Then, the glide reflection can be written as $G = T_{AB} \circ r_l = r_l \circ T_{AB}$. If $G(G(m)) = m$, then $(T_{AB} \circ r_l) \circ (r_l \circ T_{AB})(m) = T_{2AB}(m) = m$. So, m must be parallel or equal to l, if it is invariant under T_{2AB}. Suppose m is parallel to l. Then, $T_{AB}(m) = m$. So, $G(m) = r_l \circ T_{AB}(m) = r_l(m)$. But, reflection of a line m that is parallel to l cannot be equal to m. Thus, the only line invariant under the glide reflection is l itself.

5.6.5 The glide reflection can be written as $G = T_{AB} \circ r_l = r_l \circ T_{AB}$. So, $G \circ G = (T_{AB} \circ r_l) \circ (r_l \circ T_{AB}) = T_{2AB}$.

5.6.7 The set does not include the identity element.

5.6.9 The identity (rotation angle of 0) is in the set. The composition of two rotations about the same point is again a rotation by Exercise 5.4.8. The inverse to a rotation is another rotation about the same point by Exercise 5.4.7. Since rotations are functions, associativity is automatic.

5.6.11 A discussion and diagram would suffice for this exercise.

5.6.13 Rotations and translations are defined as the product of two reflections. The identity can also be written as the product of a reflection with itself. These are then both direct and even. Glides and reflections can be written as the product of three or one reflections. These are then odd and opposite.

5.7 Structure and Representation of Isometries

This section is a somewhat abstract digression into ways of representing transformations and of understanding their structure as algebraic elements of a group. An important theme of the section is the usefulness of the matrix form of an isometry, both from a theoretical viewpoint (classification), as well as a practical viewpoint (animation in computer graphics).

Matrix methods (and thus transformations) are used heavily in the field of computer animation. There are many excellent textbooks in computer graphics that one could use as reference for this purpose. For example, the book by F.S. Hill listed in the bibliography of the text is a very accessible introduction to the subject.

5.7.1 Let $G_1 = T_{v_1} \circ r_{l_1}$ and $G_2 = T_{v_2} \circ r_{l_2}$ be two glide reflections. If $G_1 \circ G_2$ is a translation, say T_v, then, since $G_1 \circ G_2 = T_v = (T_{v_1} \circ r_{l_1}) \circ (r_{l_2} \circ T_{v_2})$, then $T_{v-v_1-v_2} = r_{l_1} \circ r_{l_2}$ and thus $l_1 || l_2$.

On the other hand, if the lines are parallel, then $G_1 \circ G_2 = (T_{v_1} \circ r_{l_1}) \circ (r_{l_2} \circ T_{v_2}) = T_{v_1} \circ T_v \circ T_{v_2}$, for some vector v.

If the lines intersect, then the composition of r_{l_1} with r_{l_2} will be a rotation, say R, and $G_1 \circ G_2 = (T_{v_1} \circ r_{l_1}) \circ (r_{l_2} \circ T_{v_2}) = T_{v_1} \circ R \circ T_{v_2}$. This last composition yields a rotation, by Theorem 5.20

5.7.3 First, $f \circ r_m \circ f^{-1}(f(m)) = f(m)$, so $f(m)$ is a fixed line for $f \circ r_m \circ f^{-1}$. Also, $(f \circ r_m \circ f^{-1})^2 = f \circ r_m \circ f^{-1} \circ f \circ r_m \circ f^{-1} = id$. Thus, $f \circ r_m \circ f^{-1}$, which must be a reflection or glide reflection from looking at Table 5.3, is a reflection. Since it fixes $f(m)$ it must be reflection across $f(m)$.

5.7.5 Using the previous exercises we have $f \circ r_m \circ T_{AB} \circ f^{-1} = f \circ r_m \circ f^{-1} \circ f \circ T_{AB} \circ f^{-1} = r_{f(m)} \circ T_{f(A)f(B)}$.

5.7.7 Rotation of (x, y) by an angle ϕ yields $(x\, cos(\phi) - y\, sin(\phi), x\, sin(\phi) + y\, cos(\phi))$. Multiplying $x + iy$ by $cos(\phi) + i\, sin(\phi)$ yields the same point. Translation by $v = (v_1, v_2)$ yields $(x+v_1, y+v_2)$. Adding v_1+iv_2 to $x+iy$ yields the same result. Finally, reflection across x is given by $r_x(x, y) = (x, -y)$. Complex conjugation sends $x + iy$ to $x - iy$. Clearly, this has the same effect.

5.7.9 $T_v \circ R_\beta(z) = (e^{i\beta} z)+v$. To find the fixed point, set $(e^{i\beta} z)+v = z$ and solve for z.

5.8 Project 8 - Constructing Compositions

The purpose of this lab is to make concrete the somewhat abstract notion of composition of isometries. In particular, by carrying out the constructions of the lab, you will see how the conditions on compositions of rotations found in Table 5.3 arise naturally.

If you have difficulty getting started with the first proof, think about how we can write a rotation as the composition of two reflections through the center of rotation. Note that the choice of reflection lines is not important one can choose *any* two lines as long as they make the right angle, namely half the desired rotation angle.

CHAPTER 6

Symmetry

This chapter is quite algebraic in nature—focusing on the different discrete symmetry groups that arise for frieze patterns and wallpaper patterns.

SOLUTIONS TO EXERCISES IN CHAPTER 6

6.1 Finite Plane Symmetry Groups

6.1.1 Flowers and diatoms make good examples.

6.1.3 The dihedral group of order 5. This has order 10, and there can be at most 10 symmetries. (Use an argument like that used in the preceding exercise)

6.1.5 By the previous exercise there are at most $2n$ symmetries. Also, by the work done in Section 5.4 we know there are n rotations, generated by a rotation of $\frac{360}{n}$, that will be symmetries. Let r be a reflection across a perpendicular bisector of a side. This will be a reflection, as will all n compositions of this reflection with the n rotations. This gives $2n$ different symmetries

6.1.7 The number of symmetries is $2n$. The only symmetries that fix a side are the identity and a reflection across the perpendicular bisector of that side. The side can move to n different sides. Thus, the stated product is $2n$ as claimed.

6.2 Frieze Groups

6.2.1 Since $\gamma^2 = \tau$, then $< \tau, \gamma, H >$ is contained in $< \gamma, H >$. Also, it is clear that $< \gamma, H >$ is contained in $< \tau, \gamma, H >$. Thus, $< \tau, \gamma, H >=< \gamma, H >$.

6.2.3 Let r_u and $r_{u'}$ be two reflections across lines perpendicular to m. Then, the composition $r_u \circ r_{u'}$ must be a translation, as these lines will be parallel. Thus, $r_u \circ r_{u'} = T^k$ for some k, and $r_{u'} = r_u \circ T^k$.

6.2.5 Consider g^2. This must be a translation, so $g^2 = T_{kv}$ for some k where T_v is the fundamental translation. Then, $g = T_{\frac{k}{2}v} \circ r_m$, where m is the midline. Suppose $\frac{k}{2}$ is an integer, say $\frac{k}{2} = j$. Then, since $T_{(v-jv)}$ is in the group, we have $T_{(v-jv)} \circ g = T_{(v-jv)} \circ T_{\frac{k}{2}v} \circ r_m = T_v \circ r_m$ is in the group.

Otherwise, $\frac{k}{2} = j + \frac{1}{2}$ for some integer j. We can find T_{-jv} in the group such that $T_{-jv} \circ g = T_{\frac{v}{2}} \circ r_m$ is in the group.

6.2.7 The composition $r_v \circ r_u$ must be a translation. Also, if $r_v \circ r_u(A) = r_v(A) = C$, then the translation vector must be $\vec{AC}$. But, the length of $\overline{AC}$ is twice that of $\overline{AB}$. So, we get that $2\vec{AB} = k'v$ for some k'. Now, either k' is even or it is odd. The result follows.

6.2.9 From Table 5.3 we know that $\tau \circ H$ or $H \circ \tau$ is either a translation or a rotation, so it must be either τ^k for some k or H_A for A on m. Thus, any composition of products of τ and H can be reduced ultimately to a simple translation or half-turn, or to some $\tau^j \circ H_B$ or $H_B \circ \tau_j$, which are both half-turns. Thus, the subgroup generated by τ and H cannot contain r_m or r_u or γ and none of $< \tau, r_m >$ or $< \tau, r_u >$ or $< \tau, r_m >$ can be subgroups of $< \tau, H >$.

6.2.11 The compositions $\tau^k \circ r_m$ or $r_m \circ \tau^k$ generate glide reflections with glide vectors kv. The composition of τ with such glide reflections generates other glide reflections with glide vectors $(k+j)v$. The composition of r_m with a glide in the direction of m will generate a translation. Thus, compositions of the three types of symmetries—glides, r_m, and τ^k—will only generate symmetries within those types. Thus, $< \tau, \gamma >$ cannot be a subgroup of $< \tau, r_m >$, since γ has translation vector of $\frac{v}{2}$ which cannot be generated in $< \tau, r_m >$. Also, neither $< \tau, r_u >$ nor $< \tau, H >$ can be subgroups of $< \tau, r_m >$.

6.2.13 First Row: $< \tau >$, $< \tau, \gamma >$. Second Row: $< \tau, \gamma, H >$, $< \tau, r_u >$. Third Row: $< \tau, r_m, H >$, $< \tau, H >$. Last Row: $< \tau, r_m >$.

6.3 Wallpaper Groups

6.3.1 The first is rectangular, the second rhomboidal, and the third is square.

6.3.3 The translation determined by f^2 will be in the same direction as T, so we do not find two independent directions of translation.

6.3.5 The lattice for G will be invariant under rotations about points of the lattice by a fixed angle. By the previous problem, these rotations must be half-turns. By Theorem 6.18 the lattice must be Rectangular, Centered Rectangular, or Square.

6.3.7 Let C be the midpoint of the vector $v = \vec{AB}$, where v is one of the translation vectors for G. Let m_1 be a line perpendicular to $\overleftrightarrow{AB}$ at A. Then, $T_v = r_{m_1} \circ r_{m_1'}$ where m_1' is a line perpendicular to $\overleftrightarrow{AB}$ at the midpoint of $\overline{AB}$. But since r_{m_1} is in G, then $r_{m_1} \circ T_v = r_{m_1'}$ is in G. Likewise, we could find a line m_2' perpendicular to the other translation vector $w = \vec{AC}$ at its midpoint, yielding another reflection $r_{m_2'}$. The formulas for these two reflections are $r_{m_1'} = r_{m_1} \circ T_v$ and $r_{m_2'} = r_{m_2} \circ T_w$.

6.3.9 In the preceding exercise, we saw that the group of symmetries can be generated from reflections half-way along the translation vectors. Thus, if we reflect the shaded region, we must get another part of the pattern. Thus, three reflections of the shaded area will fill up the rectangle determined by v and w and the rest of the pattern will be generated by translation.

6.3.11 If $A = lv + mw$ and $B = sv + tw$, then $0 \leq s, t \leq 1$. The length between A and B is the length of the vector $B \vec{-} A = (s - l)v + (t - m)w$. This length squared is the dot product of $B \vec{-} A$ with itself, i.e., $(s-l)^2(v \bullet v) + 2(s-l)(t-m)(v \bullet w) + (t-m)^2(w \bullet w)$. If $v \bullet w > 0$, then this will be maximal when both $(s - l)$ and $(t - m)$ are maximal. This occurs when $(s - l) = 1$ and $(t - m) = 1$, which holds only if $s = 1 = t$ and $l = m = 0$. If $v \bullet w < 0$, we need $(s - l)$ to be as negative as possible, and $(t - m)$ to be as positive as possible (or vice-versa). In either case, we get values of 0 or 1 for s, t, l, and m.

6.3.13 A single straight line would have translational symmetries of arbitrarily small size.

6.5 Project 9 - Constructing Tessellations

Tiling is a fascinating subject. If you would like to know more about the mathematics of tiling, a good supplementary source is *Tilings and Patterns*, by Grunbaum and Shephard.

A modern master of the art of tiling is M.C. Escher. A good resource for his work is Doris Schattschnieders book *M. C. Escher, Visions of Symmetry*.

CHAPTER 7

Hyperbolic Geometry

The discovery of non-Euclidean geometry is one of the most important events in the history of mathematics. Much more time could be spent on telling this story, and, in particular, the history of the colorful figures who co-discovered this geometry. The book by Boyer and Merzbach, and the University of St. Andrews web site, both listed in the bibliography of the text, are excellent references for a deeper look at this history. Greenberg's book is also an excellent reference.

SOLUTIONS TO EXERCISES IN CHAPTER 7

In section 7.2 we see for the first time the relevance of our earlier discussion of models in Chapter 1. The change of axioms in Chapter 7 (replacing Euclid's fifth postulate with the hyperbolic parallel postulate) requires a change of models. As you work through this section, it is important to recall that, in an axiomatic system, it not important what the terms actually mean; the only thing that matters is the relationships between the terms.

We introduce two different models at this point to help you recognize the abstraction that lies behind the concrete expression of points and lines in theses models.

7.2.2 Mini-Project: The Klein Model

It may be helpful to do the constructions (lines, etc) of the Klein model on paper as you read through the material.

7.2.1 Use the properties of Euclidean segments.

7.2.3 The special case is where the lines intersect at a boundary point of the Klein disk. Otherwise, use the line connecting the poles of the two parallels to construct a common perpendicular.

7.3 Basic Results in Hyperbolic Geometry

In this section it is important to note the distinction between points at infinity and regular points. Omega triangles share some properties of regular triangles, like congruence theorems and Pasch-like properties, but are not regular trianglesthus necessitating the theorems found in this section.

7.3.1 Use the interpretation of limiting parallels in the Klein model.

7.3.3 First, if m is a limiting parallel to l through a point P, then $r_l(m)$ cannot intersect l, as if it did, then $r_l^2(m) = m$ would also intersect l. Now, drop a perpendicular from $r_l(P)$ to l at Q, and consider the angle made by Q, $r_l(P)$, and the omega point of $r_l(m)$. If there were another limiting parallel (n) to l through $r_l(P)$ that lies within this angle, then by reflecting back by r_l we would get a limiting parallel $r_l(n)$ that lies within the angle made by Q, P and the omega point of l, which is impossible. Thus, $r_l(m)$ must be limiting parallel to l and reflection maps omega points to omega points, as r_l maps limiting parallels to l to other limiting parallels. Also, it must fix the omega point, as it maps limiting parallels on one side of the perpendicular dropped to l to limiting parallels on that same side.

7.3.5 Let P be the center of rotation and let l be a line through P with the given omega point Ω. (Such a line must exist as Ω must correspond to a limiting parallel line m, and there is always a limiting parallel to m through a given point P) Then, we can write $R = r_n \circ r_l$ for another line n passing through P. But, since r_l fixes Ω, and R does as well, then, r_n must fix Ω. But, if n and l are not coincident, then n is not limiting parallel to l and thus cannot have the same omega points as l. By the previous exercise, r_n could not fix Ω. Thus, it must be the case that n and l are coincident and r is the identity.

7.3.7 Let $PQ\Omega$ be an omega triangle and let R be a point interior to the triangle. Drop a perpendicular from Q to $\overleftrightarrow{P\Omega}$ at S. Then, either R is interior to triangle QPS, or it is on $\overleftrightarrow{QS}$, or it is interior to $\angle QS\Omega$. If it is interior to ΔQPS it intersects $\overleftrightarrow{P\Omega}$ by Pasch's axiom for triangles. If it is on $\overleftrightarrow{QS}$ it obviously intersects $\overleftrightarrow{P\Omega}$. If it is interior to $\angle QS\Omega$, it intersects $\overleftrightarrow{P\Omega}$ by the definition of limiting parallels.

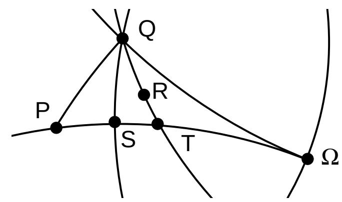


FIGURE 7.1:

7.3.9 Let l be the line passing through R. Then, either l passes within Omega triangle $PR\Omega$ or it passes within $QR\Omega$. In either case, we know by Theorem 7.5 that l must intersect the opposite side, i.e. it must intersect $\overline{P\Omega}$ or $\overline{Q\Omega}$.

7.3.11 Suppose we had another segment $\overline{P'Q'}$ with $\overline{P'Q'} \cong \overline{PQ}$ and let l' be a perpendicular to $\overline{P'Q'}$ at Q'. Let $\overleftrightarrow{P'R'}$ be a limiting parallel to l' at P'. Then, by Theorem 7.8, we know that $\angle QPR \cong \angle Q'P'R'$ and thus, the definition of this angle only depends on h, the length of $\overline{PQ}$.

7.3.13 Suppose $a(h) = a(h')$ with $h \neq h'$. We can assume that $h < h'$. But, then the previous exercise would imply that $a(h) > a(h')$. Thus, if $a(h) = a(h')$ then $h = h'$.

7.4 Project 10 - The Saccheri Quadrilateral

As you do the computer construction of the Saccheri Quadrilateral, you may experience a flip of orientation for your construction when moving the quad about the screen. The construction depends on the orientation of the intersections of circles and these may switch as the quad is moved. A construction of the Saccheri quad that does not have this unfortunate behavior was searched for unsuccessfully by the author. A nice challenge problem would be to see if you can come up with a better construction. If you can, the author would love to hear about it!

7.5 Lambert Quadrilaterals and Triangles

7.5.1 Referring to Figure 7.6, we know ΔACB and ΔACE are congruent by SAS. Thus, $\angle ACB \cong \angle ECA$. Since $\angle ACD \cong \angle FCA$, and both are right angles, then $\angle BCD \cong \angle FCE$. Then, ΔBCD and ΔFCE are congruent by SAS. We conclude that $\overline{BD} \cong \overline{FE}$ and the angle at E is a right angle.

7.5.3 Create two Lambert quadrilaterals from the Saccheri quadrilateral, and then use Theorem 7.13.

7.5.5 Since the angle at O is acute, then OAA' and OBB' are triangles. Also, since $OA < OB$, then A is between O and B, and likewise A' is between O and B'. Thus, the perpendicular n at A to $\overleftrightarrow{AA'}$ will enter $\Delta OBB'$. By Pasch's axiom it must intersect $\overline{OB'}$ or $\overline{BB'}$. It cannot intersect $\overline{OB'}$ as n and $\overleftrightarrow{OB'}$ must be parallel. Thus, n intersects $\overline{BB'}$ at C. Then, $A'ACB'$ is a Lambert Quadrilateral and $B'C > A'A$. Since C is between B and B' we have $B'B > A'A$.

7.5.7 Let m be right limiting parallel to l at P and let P' be a point on m to the right of P (i.e. in the direction of the omega point). Let Q and Q' be the points on l where the perpendiculars from P and P' to l intersect l.

We claim that $m\angle QPP' < m\angle Q'P'R$ where R is a point on m to the right of P'. If these angles were equal we would have $\overline{PQ} \cong \overline{P'Q'}$ by Exercise 7.3.11, and thus $QPP'Q'$ would be a Saccheri quadrilateral, which would imply that $\angle Q'P'R$ is a right angle, which is impossible. If $m\angle QPP' > m\angle Q'P'R$, then $PQ < P'Q'$ by Exercise 7.3.12, which would imply that we could find a point S on $\overline{P'Q'}$ with $PQ = Q'S$, yielding Saccheri quadrilateral $PQQ'S$. Then, $\angle PSQ'$ must be acute, which contradicts the Exterior angle theorem for $\Delta PSP'$.

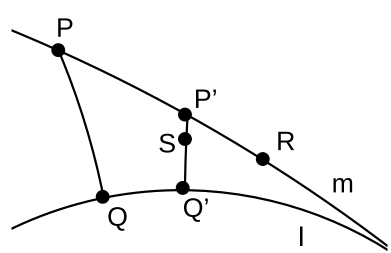


FIGURE 7.2:

Thus, $m\angle QPP' < m\angle Q'P'R$, and the result follows from Exercise 7.3.12.

7.5.9 If they had more than one common perpendicular, then we would have a rectangle.

7.5.11 Suppose Saccheri Quadrilaterals $ABCD$ and $EFGH$ have $\overline{AB} \cong \overline{EF}$ and $\angle ADC \cong \angle EHG$. If $EH > AD$ then we can find I on $\overline{EH}$ and J on $\overline{FG}$ such that $\overline{EI} \cong \overline{FJ} \cong \overline{AD}$. Then, by repeated application of SAS on sub-triangles of $ABCD$ and $EIJF$ we can show that these two Saccheri Quadrilaterals are congruent. But, this implies that the angles at H and I in quadrilateral $IHGJ$ are supplementary, as are the angles at G and J, which means that we can construct a quadrilateral with angles sum of 360. This contradicts Theorem 7.15, by considering triangles created by a diagonal of $IHGJ$.

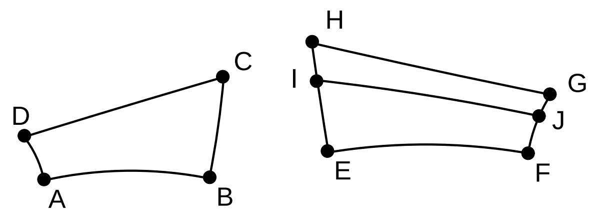


FIGURE 7.3:

7.5.13 No. To construct a scale model, we are really constructing a figure similar to the original. That is, a figure with corresponding angles congruent, and length measurements proportional by a non-unit scale factor. But, AAA congruence implies that any such scale model must have lengths preserved.

7.6 Area in Hyperbolic Geometry

In this section we can refer back to the mini-project we did on area in Chapter 2. That discussion depended on rectangles as the basis for a definition of area. In hyperbolic geometry, no rectangles exist, so the next best shape to base area on is the triangle. This explains the nature of the theorems in this section.

7.6.1 Let J be the midpoint of $\overline{A''B}$ and suppose that $\overleftrightarrow{EF}$ cuts $\overline{A''B}$ at some point $K \neq J$. Then, on $\overleftrightarrow{E''K}$ we can construct a second Saccheri Quadrilateral by the method of dropping perpendiculars from B and C to $\overleftrightarrow{E''K}$. Now, $\overline{BC}$ is the summit of the original Saccheri Quadrilateral $BCIH$ and the new Saccheri Quadrilateral. Thus, if n is the perpendicular bisector of $\overline{BC}$, then n meets $\overleftrightarrow{E''F}$ and $\overleftrightarrow{E''K}$ at right angles. Since E'' is common to both curves, we get a triangle having two right angles, which is impossible.

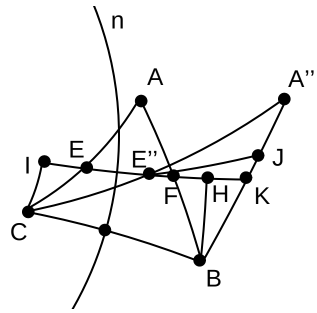


FIGURE 7.4:

7.6.3 This question can be argued both ways. If we could make incredibly precise measurements of a triangle, then we could possibly measure the angle sum to be less than 180. However, since the universe is so vast, we would have to have an incredibly large triangle to measure, or incredibly good instruments. Also, we could never be sure of errors in the measurement overwhelming the actual differential between the angle sum and 180.

7.7 Project 11 - Tiling the Hyperbolic Plane

A nice artisitic example of hyperbolic tilings can be found in M. C. Eschers Circle Limit figures. Consult Doris Schattschnieders book *M. C. Escher, Visions of Symmetry* for more information about these tilings.

CHAPTER 8

Elliptic Geometry

Hyperbolic and Elliptic geometry are the fundamental examples of non-Euclidean geometry. The connection between the three geometries —Euclidean, Hyperbolic, and Elliptic —and the three possible parallel properties —1, > 1, or 0 parallels through a point to a given line —is one of the most interesting and thought-provoking ideas in geometry.

SOLUTIONS TO EXERCISES IN CHAPTER 8

8.2 Perpendiculars and Poles in Elliptic Geometry

8.2.1 Referring to Figure 8.1, we know ΔACB and ΔACE are congruent by SAS. Thus, $\angle ACB \cong \angle ECA$. Since $\angle ACD \cong \angle FCA$, and both are right angles, then $\angle BCD \cong \angle FCE$. Then, ΔBCD and ΔFCE are congruent by SAS. We conclude that $\overline{BD} \cong \overline{FE}$ and the angle at E is a right angle.

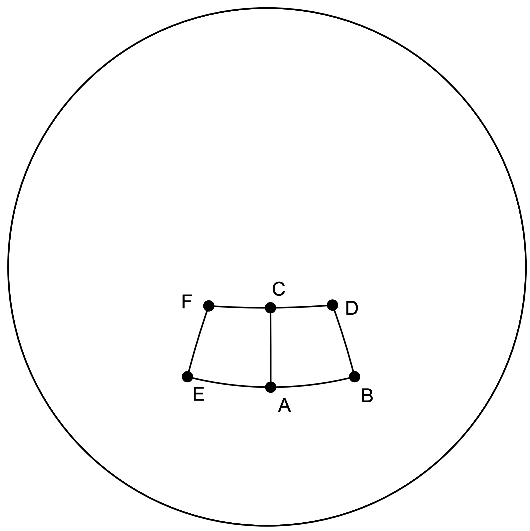


FIGURE 8.1:

8.2.3 By Exercise 8.2.2 we can create a Saccheri quadrilateral from the Lambert quadrilateral. Since the summit angles in the Saccheri quadrilateral will equal the fourth (non-right) angle in the Lambert quadrilateral, the result follows.

8.2.5 Since there are two perpendiculars from n that meet at O, then by the work in this section, all perpendiculars to n meet at O and O is the polar point.

8.3 Project 12 - Models of Elliptic Geometry

All of the projects in Chapter 8 use the software program Geometry Explorer. If you are not already using this program, go to http://www.gac.edu/ hvidsten/geom-text for instructions on how to download and use this software.

8.4 Basic Results in Elliptic Geometry

8.4.1 Consider the line $\overline{OR}$. This line must intersect l at some point Q. Also, it is perpendicular to l at Q, for if it were not perpendicular, then at Q we could construct the perpendicular which must pass through the polar point O. But, then we would have two lines through O and

Q. Since $\overline{OR}$ is perpendicular to l at Q, then the distance from Q to O is the polar distance q. There is a unique point along $\overline{OR}$ in the direction of R that is at this distance. Thus, R must be equal to Q and $R$is on l.

8.4.3 Suppose the polar points were the same. Then, by Exercise 8.4.1 points that are a distance of q away from this common polar would have to be on l and also on m. Thus, the two lines must be identical. The line through the two polar points will be perpendicular to both lines by Exercise 8.4.2.

8.4.5 The *exterior* of a circle is the set of points whose distance to the center is larger than the radius. If the radius is equal to the polar distance then the circle is a line and the center of the circle is the polar point of the circle/line. The interior is defined, but would be all of the elliptic plane, as the distance from a point to another point is at most the polar distance (the length of an entire line is $2q$, so the length of any subset (segment) is at most q).

8.4.7 The rotation R is made of two reflections r_m and r_m where m and n are lines passing through O. If P is a point on l, then for $P' = r_m(P)$, we have that the length of $\overline{OP}(= q)$ is equal to the length of $\overline{OP'}$, as r_m fixes O and preserves length. Thus, by Exercise 8.4.1 we have that P' is on l. A similar argument shows that $r_n(P')$ is also on l. Thus, $R(P) = r_n(r_m(P))$ is on l.

8.4.9 We can assume that $R_1 = r_n \circ r_p$ and $R_2 = r_p \circ r_m$ are the two rotations. (This is similar to the work done in Chapter 5 on compositions of transformations). Then, $R_1 \circ R_2 = r_n \circ r_m$. If $n = m$ then the composition is the identity, which is a rotation of 0 degrees. If $n \neq m$, then we know that m and m must intersect at a point O. Then, $R_1 \circ R_2 = r_n \circ r_m$ is a composition of reflections with a fixed point and thus is a rotation.

8.5 Triangles and Area in Elliptic Geometry

8.5.1 Divide the quadrilateral into two triangles and use Theorem 8.9.

8.5.3 If they had more than one common perpendicular, then we would have a rectangle.

8.5.5 Suppose that the intersection of $\overleftrightarrow{EF}$ with $\overline{A\hat{\ }B}$ is not the midpoint. Connect G to the midpoint K of $\overline{A\hat{\ }B}$ and construct a second Saccheri quadrilateral B. Let M be the midpoint of $\overline{BC}$. We know that the line joining the midpoints of a Saccheri quadrilateral is perpendicular to the base and summit. (Exercise 8.2.1) Thus, the perpendicular

n of $\overline{BC}$ at M will meet both bases of the quadrilaterals at right angles – it will be a common perpendicular. Thus, point G must be the polar point of n (Exercise 8.2.5). But, this is impossible, as the intersection of n with the bases of the Saccheri quadrilaterals happens at the midpoints of the bases and these points cannot be a distance q from G (Q being the polar distance).

8.6 Project 13 - Elliptic Tiling

This project uses the software program Geometry Explorer. If you are not already using this program, go to http://www.gac.edu/ hvidsten/geomtext for instructions on how to download and use this software.

CHAPTER 9

Projective Geometry

Projective geometry is the natural culmination of a study of geometry that proceeds from Euclidean to non-Euclidean geometries. While much of the material in this chapter is much more opaque than in previous chapters, the beauty and elegance realized from abstraction can (hopefully) shine through.

SOLUTIONS TO EXERCISES IN CHAPTER 9

9.2 Project 14 - Perspective and Projection

9.3 Foundations of Projective Geometry

9.3.1 Suppose the lines are l and m. If they both pass through points P and Q, then by axiom A1 the two lines must be the same line.

9.3.3 By the previous exercise, we know that an Affine geometry must have at least four distinct points, say P, Q, R, and S. For each pair of points, we have a unique line, by axiom A1. A quick check of every pairing of one of the six lines with a point not on that line shows that axiom A2 is satisifed. Thus, for these four points and six lines, all three axioms are satisfied.

9.3.5 Let P be a point. By axiom P3 there must be two other points Q and R such that P, Q, and R are non-collinear. Then, $l = \overleftrightarrow{PQ}$ and $m = \overleftrightarrow{PR}$ are two distinct lines. Now, Q and R define a line n by axiom P1, and this line is different than l or m. By axiom P4, n must have a third point, say S. Then, $\overleftrightarrow{PS}$ exists and is distinct from both $\overleftrightarrow{PQ}$ and $\overleftrightarrow{PR}$.

9.3.7 It is clear that the example satisfies P3 and P4. Checking all pairs of points and lines will suffice for P1 and P2.

9.3.9 Since Axioms P(n)1 and P(n)2 exactly match P1 and P2, then P1 and P2 are satisfied. P(n)3 guarantees the existence of three non-collinear points, so Axiom P3 is satisfied.

Let P, Q, R, and S be the four points guaranteed by Axiom P(n)3. Then, $\overleftrightarrow{PQ}$, $\overleftrightarrow{PR}$,$\overleftrightarrow{PS}$, $\overleftrightarrow{QR}$, $\overleftrightarrow{QS}$ and $\overleftrightarrow{RS}$ must be distinct lines. Suppose there were a line m with only two points A and B. Since m intersects $\overleftrightarrow{PQ}$ and $\overleftrightarrow{PR}$, then either one of A or B is P, or A is on one of the lines and B is on the other.

Suppose $A = P$. Now m intersects $\overleftrightarrow{QR}$ and $\overleftrightarrow{QS}$ at a point other than P, so $B = Q$. Then, $m = \overleftrightarrow{PQ}$. But, we know that $\overleftrightarrow{PQ}$ must have a third point as it must intersect $\overleftrightarrow{RS}$ at a point other than P or Q. So, $A \neq P$.

Suppose A is on $\overleftrightarrow{PQ}$ and B is on $\overleftrightarrow{PR}$. Since m must intersect $\overleftrightarrow{QR}$ at some point, then A is on $\overleftrightarrow{QR}$ or B is on $\overleftrightarrow{QR}$. Then, $A = Q$ and $B = R$. Then, $m = \overleftrightarrow{QR}$. But, we know that $\overleftrightarrow{QR}$ must have a third point as it must intersect $\overleftrightarrow{PS}$ at a point other than Q or R. So, it cannot be the case that A is on $\overleftrightarrow{PQ}$ and B is on $\overleftrightarrow{PR}$.

We conclude that there cannot be a line with only two points. A similar proof (simpler) shows that there cannot be a line with only one point.

9.3.11 Since Moulton lines are built from pieces of Euclidean lines, then P3 and P4 follow immediately.

For P2, any pair of Moulton lines that are standard Euclidean lines will either intersect in the regular Euclidean plane or will intersect at a point at infinity. For other Moulton lines, if the two lines are defined by different slopes m_1 and m_2, then consider the regular Euclidean lines of those slopes. Either those lines will intersect in the region $x \leq 0$ or $x > 0$. Since the Moulton "pieces" defining the line have the same slope, then the Moulton lines with slopes m_1 and m_2 will also intersect where the Euclidean lines did. The case left is the case where the Moulton lines have the same slope. In this case, they intersect at the point at infinity. So, P2 holds.

For P1, let A and B be two point in the plane. If A and B are both in the left half-plane ($x \leq 0$) we just find the line $y = m'x + b$ joining them and define $y = 2m'x + b$ in the right half-plane ($x > 0$) to get the Moulton line incident on A and B. A similar construction works if A and B are both in the right half-plane ($x > 0$).

If A is in the left half-plane and B in the right, but A is above or at the same level as B, then the slope of the line joining the two points is negative or zero, and the Moulton line is just the Euclidean line incident on the points.

The only case left is where A is in the left half-plane and B in the right, and A is below B. It is clear from the hint that the fraction $\frac{slope(\overleftrightarrow{BY})}{slope(\overleftrightarrow{AY})}$ range from 0 to infinity as Y ranges from Y_1 to Y_0. Thus, by continuity there is a Y value where this fraction is equal to 2. Let $m = slope(\overleftrightarrow{BY})$. This will generate the unique Moulton line from A to B.

9.4 Transformations in Projective Geometry and Pappus's Theorem

9.4.1 Closure: Let π_1 and π_2 be projectivities. Then, each is constructed from a composition of perspectivities. The composition of π_1 with π_2 will again be a composition of perspectivities and thus is a projectivity.

Associativity: This is automatic (inherited) from function composition.

Identity: A perspectivity from a pencil of points back to itself (or pencil of lines back to itself) is allowed, and thus the identity is a projectivity.

Inverses: Given any basic perspectivity from, say a pencil of points on l to a pencil of points on l', will have its inverse be the revers mapping from the pencil at l' to the pencil at l. Since a projectivity is the composition of perspectivities, then the inverse will be the composition of inverse perspectivities (in reverse order).

9.4.3 Dual to Theorem 9.6: Let a,b,c be three distinct lines incident on point P and a', b', and c' be three distinct lines incident on point P', with $P \neq P'$. Then, there is a projectivity taking a,b,c to a',b',c'.

Dual to Corollary 9.7: Let a,b,c and a', b', and c' be two sets of distinct lines incident on point P. Then, there is a projectivity taking a,b,c to a',b',c'.

9.4.5 Let π_1 and π_2 be projectivities having the same value on points A, B, and C on l. Then $\pi_1 \circ \pi_2^{-1}$ will leave the points A, B, and C on l invariant. By P7, $\pi_1 \circ \pi_2^{-1} = id$, or $\pi_1 = \pi_2$.

9.4.7 Consider a projectivity of pencils of points from line l to line l'. Theorem 9.6 gives us a construction for a projectivity as the composition of two perspectivities if $l \neq l'$. If $l = l'$, then let A, B, and C be projectively related to A', B', and C' on l'. Let m be a line not identical to l and P a point not on m. Let π be the perspectivity defined by center P and line m. Let X, Y, and Z be the points on m that are perspective from P to A', B', and C'. Then, by Theorem 9.6 we can find a sequence of two perspectivities taking A, B, and C to X, Y, and Z. Then, π maps X, Y, Z to A', B', C'.

9.4.9 WLOG assume $A = A'$. Then, $P = AB \cdot A'B' = A$ and $R = AC \cdot A'C' = A$. Clearly, there is a line defined on P, Q, and R as $P = R$.

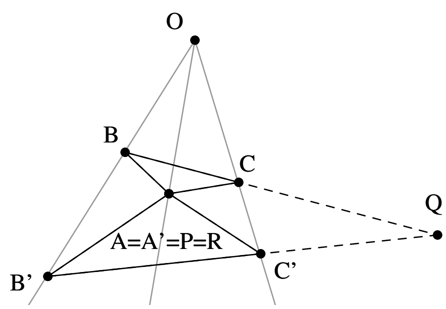


FIGURE 9.1:

9.5 Models of Projective Geometry

9.5.1 P1: Given two "regular" points $(x_1, y_1, 1)$ and $(x_2, y_2, 1)$, the Euclidean line defined by these points will be the only line so defined in the plane $z = 1$, and thus will be the unique projective line on these points. Given $(x, y, 1)$ and a point at infinity $(x', y', 0)$. there is a unique line through $(x, y, 1)$ with slope $\frac{y'}{x'}$. This is the projective line through the points. Given two points at infinity, the line at infinity is the only line through these points.

P2: For two regular lines in the plane $z = 1$, there is only one point of intersection. For a regular line l and the line at infinity, let $\frac{y}{x}$ be the slope of l, Then, the point $(x, y, 0)$ is the point at infinity that is on l.

P3: $(1, 0, 1)$, $(0, 1, 1)$ and $(1, 1, 1)$ are not collinear.

P4: Every regular line clearly has at least three points. The line at infinity contains an infinite number of points of the form $(x, y, 0)$.

9.5.3 All points at infinity must lie on the line at infinity. Points at infinity are of the form $(x, y, 0)$ where both x and y cannot be simultaneously zero. Let $[u, v, w]$ be the coordinate for the line at infinity. Then $ux + vy + 0w = 0$ for all choices of x and y. Clearly, $u = v = 0$. Since the coordinate vector for a line cannot be the zero vector, then

$w \neq 0$. Thus, $[0, 0, 1]$ must be the homogeneous coordinates for the line at infinity.

9.5.5 Let $P = (x_1, y_1, 1)$ and $Q = (x_2, y_2, 1)$. Let $x = x_2 - x_1$ and $y = y_2 - y_1$. Then, the point $(x, y, 0)$ will be the point at infinity, as it points (as a vector) in the direction of the slope of the line.

9.5.7 Let P, Q, R, and S be the points of the first complete quadrangle. Let $X_1 = (1, 0, 0)$, $X_2 = (0, 1, 0)$, $X_3 = (0, 0, 1)$ and $X_4 = (1, 1, 1)$. By Theorem 9.24 there is a unique collineation π_1 taking P to X_1, Q to X_2, R to X_3 and S to X_4. Let P', Q', R', and S' be the points of the second complete quadrangle. Then, there is a unique collineation π_2 taking P' to X_1, Q' to X_2, R' to X_3 and S' to X_4. Then, $\pi_2^{-1} \circ \pi_1$ will take P, Q, R, and S to P', Q', R', and S'.

Suppose there was another collineation π' taking P, Q, R, and S to P', Q', R', and S'. Then, $\pi_2 \circ \pi'$ takes P to X_1, Q to X_2, R to X_3 and S to X_4. Uniqueness implies that $\pi_1 = \pi_2 \circ \pi'$ or $\pi' = \pi_2^{-1} \circ \pi_1$.

9.5.9 Let A be the collineation and let P and Q be distinct points on l. Let l' be the line that is the image of l under A. Let P' and Q' be distinct points of l'. Given X on l, we have $X = \alpha P + \beta Q$. Also, $AX = \alpha' P' + \beta' Q'$. Now,

$$\begin{aligned} AX &= A(\alpha P + \beta Q) \\ &= \alpha AP + \beta AQ \\ &= \alpha(\lambda_1 P' + \lambda_2 Q') + \beta(\mu_1 P' + \mu_2 Q') \\ &= (\alpha\lambda_1 + \beta\mu_1)P' + (\alpha\lambda_2 + \beta\mu_2)Q' \end{aligned}$$

Thus, $\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = \begin{bmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \end{bmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

9.5.11 Let $X = (x, y, 0)$ be the homogeneous coordinates for an ideal point. A collineation can be represented by

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Then, $AX = (\alpha, \beta, gx + hy + 0)$. This must again be an ideal point by Exercise 9.5.11. So, $gx + hy = 0$ for all choices of x and y. Then, $x = 0 = y$. For A to be non-singular $i \neq 0$. Then, $\frac{1}{i}A$ will be equivalent to A as a collineation and will have third row equivalent to $[0\ 0\ 1]$.

9.6 Project 15 - Ratios and Harmonics

9.7 Harmonic Sets

9.7.1 For the quadrangle, $ABCD$ let $E = AB \cdot CD$, $F = AC \cdot BD$, and $G = AD \cdot BC$.

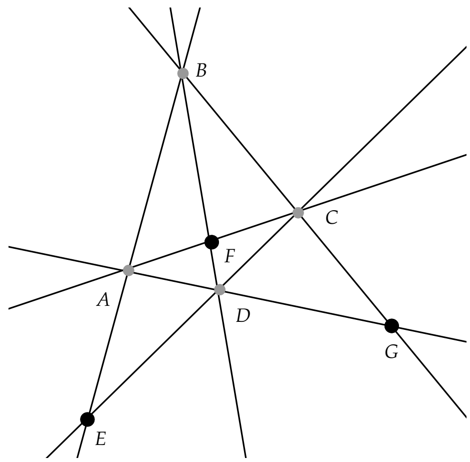


FIGURE 9.2:

Consider a line through two of the diagonal points, say the line EF. Suppose A was on EF. Then, F is on $AE = AB$. But, this implies that D is on AB, which contradicts the fact that A, B, and D must be non-collinear. A similar argument will show that EF cannot intersect any of the other three points B, C, or D. Similarly, the result holds for EG and FG.

9.7.3 Let O be the intersection of FF' and HH'. Then, the perspectivity from O maps EF to $E'F'$, thus maps l to l'. (See Fig. 9.3) The perspectivity maps E,F, and H to E', F', and H'. Let I'' be th image of I under this perspectivity. Then, by Corollary 9.31 we have that $H(E', F'; H', I'')$. But, by Theorem 9.29 we know that the fourth point in a harmonic set is uniquely defined, based on the first three points. Thus, $I' = I''$.

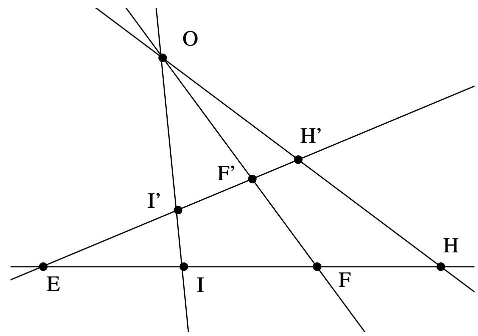


FIGURE 9.3:

9.7.5 Let A,B,C, and D have parametric homogeneous coordinates (α_1, α_2), (β_1, β_2), (γ_1, γ_2) and (δ_1, δ_2). We can assume that the homogeneous parameters for the four points are equivalently $(\alpha, 1)$, $(\beta, 1)$, $(\gamma, 1)$ and $(\delta, 1)$, where $\alpha = \frac{\alpha_1}{\alpha_2}$, $\beta = \frac{\beta_1}{\beta_2}$, $\gamma = \frac{\gamma_1}{\gamma_2}$, and $\delta = \frac{\delta_1}{\delta_2}$. From Theorem 9.33 we have

$$\begin{aligned} R(A, B; C, D) &= \frac{dd(A, C)\, dd(B, D)}{dd(B, C)\, dd(A, D)} \\ &= \frac{(\gamma - \alpha)(\delta - \beta)}{(\gamma - \beta)(\delta - \alpha)} \end{aligned}$$

If we do a replacement using $\alpha = \frac{\alpha_1}{\alpha_2}$, $\beta = \frac{\beta_1}{\beta_2}$, $\gamma = \frac{\gamma_1}{\gamma_2}$, and $\delta = \frac{\delta_1}{\delta_2}$. we get

$$\begin{aligned} R(A, B; C, D) &= \frac{(\frac{\gamma_1}{\gamma_2} - \frac{\alpha_1}{\alpha_2})(\frac{\delta_1}{\delta_2} - \frac{\beta_1}{\beta_2})}{(\frac{\gamma_1}{\gamma_2} - \frac{\beta_1}{\beta_2})(\frac{\delta_1}{\delta_2} - \frac{\alpha_1}{\alpha_2})} \\ &= \frac{\frac{1}{\gamma_2\alpha_2}(\gamma_1\alpha_2 - \gamma_2\alpha_1)\frac{1}{\delta_2\beta_2}(\delta_1\beta_2 - \delta_2\beta_1)}{\frac{1}{\gamma_2\beta_2}(\gamma_1\beta_2 - \gamma_2\beta_1)\frac{1}{\delta_2\alpha_2}(\delta_1\alpha_2 - \delta_2\alpha_1)} \\ &= \frac{(\gamma_1\alpha_2 - \gamma_2\alpha_1)(\delta_1\beta_2 - \delta_2\beta_1)}{(\gamma_1\beta_2 - \gamma_2\beta_1)(\delta_1\alpha_2 - \delta_2\alpha_1)} \end{aligned}$$

9.7.7 Use the coordinates for A, B, C, and D as set up in Theorem 9.33. Let D' have coordinates $(\delta', 1)$. Then,

$$\begin{aligned} R(A, B; C, D) &= \frac{(\gamma - \alpha)(\delta - \beta)}{(\gamma - \beta)(\delta - \alpha)} \\ &= R(A, B; C, D') \\ &= \frac{(\gamma - \alpha)(\delta' - \beta)}{(\gamma - \beta)(\delta' - \alpha)} \end{aligned}$$

Thus, $\frac{\delta' - \beta}{\delta' - \alpha} = \frac{\delta - \beta}{\delta - \alpha}$. Or, $(\delta' - \beta)(\delta - \alpha) = (\delta - \beta)(\delta' - \alpha)$. So, $-\delta'\alpha - \delta\beta = -\delta'\beta - \delta\alpha$. Thus, $\delta'(\alpha - \beta) = \delta(\alpha - \beta)$ and $\delta' = \delta$.

9.7.9 Let the parametric coordinates of A, B, C, D, and E be $(\alpha, 1)$, $(\beta, 1)$, $(\gamma, 1)$, $(\delta, 1)$, and $(\epsilon, 1)$. Then,

$$\begin{aligned} R(A,B;C,D)R(A,B;D,E) &= \frac{(\gamma-\alpha)(\delta-\beta)}{(\gamma-\beta)(\delta-\alpha)}\frac{(\delta-\alpha)(\epsilon-\beta)}{(\delta-\beta)(\epsilon-\alpha)} \\ &= \frac{(\gamma-\alpha)(\epsilon-\beta)}{(\gamma-\beta)(\epsilon-\alpha)} \\ &= R(A,B;C,E) \end{aligned}$$

9.7.11 If the coordinates for A and C are α and γ then the coordinates for B would be $\beta = \frac{\alpha + \gamma}{2}$. Then, for a fourth point D with coordinate δ we have

$$\begin{aligned} R(A,C;B,D) &= \frac{(\beta-\alpha)(\delta-\gamma)}{(\beta-\gamma)(\delta-\alpha)} \\ &= \frac{(\frac{\gamma-\alpha}{2})(\delta-\gamma)}{(\frac{\alpha-\gamma}{2})(\delta-\alpha)} \\ &= -\frac{\delta-\gamma}{\delta-\alpha} \end{aligned}$$

If D has coordinates of the point at infinity, then the cross-ratio will be equal to -1 and the four points will form a harmonic set.

9.8 Conics and Coordinates

9.8.1 According to Theorem 9.12 the projectivity defining the conic is equivalent to the composition of two perspectivities. The proof is by contradiction. Suppose that the line AB corresponded to itself under the projectivity. Interpret Lemma 9.10 as a statement about the pencils of points on l, m, and n. The dual to this lemma would be:

"Given three pencils of lines at P, R, and Q with $P \neq Q$, suppose there is a projectivity taking the pencil at P to the pencil at Q. If $l = PQ$ is invariant under the projectivity, then, then the pencil at P is perspective to the pencil at Q."

In the case of the exercise, the line l would be AB. If AB it is invariant, then by the lemma the projectivity is a perspectivity. This implies that the point conic is singular.

9.8.3 Suppose the conic is defined by pencils of points at A and

B. Let C, D, E, and F be four other distinct points on the conic. By Theorem 9.41, we know that no subset of three of A, B, C, D, E, and F are collinear. Thus, by Theorem 9.42, the points P, Q, and R defined in the theorem are collinear. If we switch A with C, B with D, and E with F, then the points P, Q, and R are unchanged as points of intersection. Thus, they remain collinear. So, by Theorem 9.42, the point conic is defined by pencils at C and D.

9.8.5 If they intersected in five or more points, they would have to be the same conic, by Theorem 9.45.

9.8.7 Re-write the equations as: $2(x^2 - 2xz + z^2) + 4y^2 + 10yz + 10z^2 = 0$. Let $x' = x - z$. Then, we get $2x'^2 + 4y^2 + 10yz + 10z^2 = 0$.

CHAPTER 10

Fractal Geometry

Much of the material in this chapter is at an advanced level, especially the sections on contraction mappings and fractal dimension—Sections 10.5 and 10.6. But this abstraction can be made quite concrete by the computer explorations developed in the chapter. In fact, the computer projects are the *only* way to really understand these geometric objects on an intuitive level.

SOLUTIONS TO EXERCISES IN CHAPTER 10

10.3 Similarity Dimension

The notion of dimension of a fractal is very hard to make precise. In this section we present one simple way to define dimension, but there are also other ways to define dimension as well, each useful for a particular purpose and all agreeing with integer dimension, but not necessarily with each other.

10.3.1 Theorem 2.27 guarantees that the sides of the new triangles are parallel to the original sides. Then, we can use SAS congruence to achieve the result.

10.3.3 At each successive stage of the construction, 8 new squares are created, each of area $\frac{1}{9}$ the area of the squares at the previous stage. Thus, the pattern for the total area of each successive stage of the construction is

$$\begin{aligned} l &= 1 - \frac{1}{9} - \frac{8}{81} - \frac{64}{9^3} - \dots \\ &= 1 - \frac{1}{9}\sum_{k=0}^{\infty}\left(\frac{8}{9}\right)^k \\ &= 1 - \frac{1}{9}\frac{1}{1-\frac{8}{9}} \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

Thus, the area of the final figure is 0.

10.3.5 The similarity dimension would be $\frac{\log(4)}{\log(3)}$.

10.3.7 Split a cube into 27 sub-cubes, as in the Menger sponge construction, and then remove all cubes except the eight corner cubes and the central cube. Do this recursively. The resulting fractal will have similarity dimension $\frac{\log(9)}{\log(3)}$, which is exactly 2.

10.4 Project 16 - An Endlessly Beautiful Snowflake

If students want a challenge, you could think of other templates based on a simple segment, generalizing the Koch template and the Hat template from exercise 10.4.4.

10.6 Fractal Dimension

Sections 10.5 and 10.6 are quite "thick" mathematically. To get some sense of the Hausdorff metric, you can compute it for some simple pairs of compact sets. For example, two triangles in different positions. Ample practice with examples will help you get a feel for the mini-max approach to the metric and this will also help you be successful with the homework exercises.

10.6.1 A function f is continuous if for each $\epsilon > 0$ we can find $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ when $0 < |x - y| < \delta$. Let S be a contraction mapping with contraction factor $0 \leq c < 1$. Then, given ϵ, let $\delta = \epsilon$ (if $c = 0$) and $\delta = \frac{\epsilon}{c}$ (if $c > 0$).

If $c = 0$ we have $0 = |S(x) - S(y)| \leq |x - y| < \delta = \epsilon$.

If $c > 0$, we have $|S(x) - S(y)| \leq c|x - y| < c\frac{\epsilon}{c} = \epsilon$.

10.6.3 Property (2): Since $d_{\mathcal{H}}(A, A) = d(A, A)$, and since $d(A, A) = max\{d(x, A)|x \in A\}$, then we need to show $d(x, A) = 0$.

But, $d(x, A) = min\{d(x, y)|y \in A\}$, and this minimum clearly occurs when $x = y$; that is, when the distance is 0.

Property (3): If $A \neq B$ then we can always find a point x in A that is not in B. Then, $d(x, B) = min\{d(x, y)|y \in B\}$ must be greater than 0. This implies that $d(A, B) = max\{d(x, B)|x \in A\}$ is also greater than 0.

10.6.5 We know that

$$\begin{aligned} d(A, C \cup D) &= max\{d(x, C \cup D)|x \in A\} \\ &= max\{min\{d(x, y)|x \in A \; and \; y \in C \; or \; D\}\} \\ &= max\{min\{min\{d(x, y)|x \in A \, y \in C\}, min\{d(x, y)|x \in A, y \in D\}\}\} \\ &= max\{min\{d(x, C), d(x, D)\}|x \in A\} \end{aligned}$$

The last expression is clearly less than or equal to $max\{d(x, C)|x \in A\} = d(A, C)$ and also less than or equal to $max\{d(x, D)|x \in A\} = d(A, D)$.

10.6.7 There are three contraction mappings which are used to construct Sierpinski's triangle. Each of them has contraction scale factor of $\frac{1}{2}$. Thus, we want $(\frac{1}{2})^D + (\frac{1}{2})^D + (\frac{1}{2})^D = 1$, or $3(\frac{1}{2})^D = 1$, Solving for D we get $D = \frac{log(3)}{log(2)}$.

10.7 Project 17 - IFS Ferns

Do not worry too much about getting exactly the same numbers for the scaling factor and the rotations that define the fern. The important idea is that you get the right *types* of transformations (in the correct order of evaluation) needed to build the fern image. For exercise 10.7.5 it may be helpful to copy out one piece of the image and then rotate and move it so it covers the other pieces, thus generating the transformations needed.

10.9 Grammars and Productions

This section will be very different from anything you have done before, except for those who have had some computer science courses. The connection between re-writing and axiomatic systems is a deep one. One could view a theorem as essentially a re-writing of various symbols and terms used to initialize a set of axioms. Also, turtle geometry is a very concrete way to view re-writing and so we have a nice concrete realization of an abstract idea.

10.9.1 Repeated use of production rule 1 will result in an expression of the form a^nSb^n. Then, using production rule 2, we get a^nb^n.

10.9.3 The level 1 rewrite is $+RF-LFL-FR+$. This is shown in Fig. 10.1. The level 2 rewrite is $+-LF+RFR+FL-F-+RF-LFL-FR+F+RF-LFL-FR+-F-LF+RFR+FL-+$. This is shown in Fig. 10.2. For the last part of the exercise, the students should recognize that all interior "lattice" points (defined by the length of one segment) are actually visited by the curve. Thus, as the level increases (and we scale the curve back to some standard size) the interior points will cover space, just as the example in section 9.9 did.

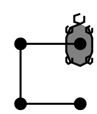


FIGURE 10.1:

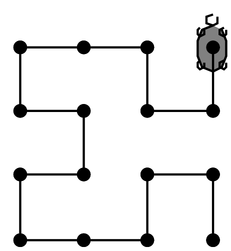


FIGURE 10.2:

10.10 Project 18 - Words Into Plants

Grammars as representations of growth is an idea that can be tied in nicely with the notion of genetics from biology. A grammar is like a blueprint governing the evolution of the form of an object such as a bush, in much the same way that DNA in its expression as proteins governs the biological functioning of an organism.

Sample Lab Report

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MCS 303 Project 0
September 12, 2003
The Amazing Pythagorean Theorem

Introduction

The Pythagorean Theorem is perhaps the most famous theorem in geometry, if not in all of mathematics. In this lab, we look at one method of proving the Pythagorean Theorem by constructing a special square. Part I of this report describes the construction used in the proof and Part II gives a detailed explanation of why this construction works, that is why the construction generates a proof of the Pythagorean theorem. Finally, we conclude with some comments on the many proofs of the Pythagorean Theorem.

Part I:

To start out our investigation of the Pythagorean Theorem, we assume that we have a right triangle with legs b and a and hypotenuse c. Our first task construction is that of a segment sub-divided into two parts of lengths a and b. Since a and b are arbitrary, we just create a segment, attach a point, hide the original segment, and draw two new segments as shown.

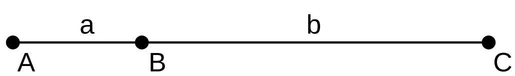

FIGURE A.1:

Then, we construct a square on side a and a square on side b. The purpose of doing this is to create two regions whose total area is a^2+b^2. Clever huh? Constructing the squares involved several rotations, but was otherwise straightforward.

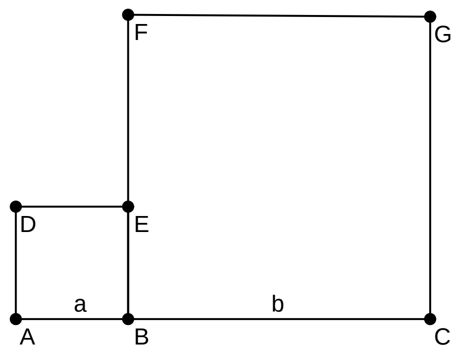

FIGURE A.2:

The next construction was a bit tricky. We define a translation from B to A and translate point C to get point H. Then, we connect H to D and H to G, resulting in two right triangles. In part II, we will prove that both of these right triangles are congruent to the original right triangle.

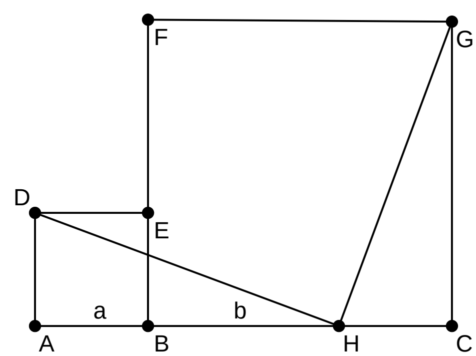

FIGURE A.3:

Next, we hide segment BC and create segments BH and HC. This is so that we have well-defined triangle sides for the next step - rotating right triangle ADH 90 degrees about its top vertex, and right triangle HGC -90 degrees about its top vertex.

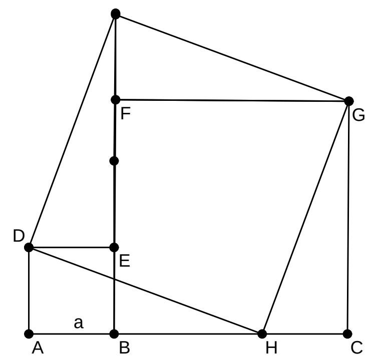

FIGURE A.4:

Part II:

We will now prove that this construction yields a square (on DH) of side length c, and thus, since the area of this square is clearly equal to the sum of the areas of the original two squares, we have $a^2 + b^2 = c^2$, and our proof would be complete. By SAS, triangle HCB must be congruent to the original right triangle, and thus its hypotenuse must be c. Also, by SAS, triangle DAH is also congruent to the original triangle, and so its hypotenuse is also c. Then, angles AHD and CHG(= ADH) must sum to 90 degrees, and the angle DHG is a right angle. Thus, we have shown that the construction yields a square on DH of side length c, and our proof is complete.

Conclusion:

This was a very elegant proof of the Pythagorean Theorem. In researching the topic of proofs of the Pythagorean Theorem, we discovered that over 300 proofs of this theorem have been discovered. Elisha Scott Loomis, a mathematics teacher from Ohio, compiled many of these proofs into a book titled *The Pythagorean Proposition*, published in 1928. This tidbit of historical lore was gleaned from the Ask Dr. Math website (http://mathforum.org/library/drmath/view/62539.html). It seems that people cannot get enough of proofs of the Pythagorean Theorem.