Student Solutions Manual

for

Web Sections

Elementary Linear Algebra

4th Edition

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Lines and Planes and the Cross Product in $\mathbb{R}^3$

(1) (a) Let $(x_0, y_0, z_0) = (3, -1, 0)$ and $[a, b, c] = [0, 1, -4]$. Then, from Theorem 1, parametric equations for the line are $x = 3+0t,\ \ y = -1+1t,\ \ z = 0-4t\ \ (t \in \mathbb{R})$; that is, $x = 3,\ \ y = -1+t,\ \ z = -4t$ $(t \in \mathbb{R})$.

(c) Let $(x_0, y_0, z_0) = (6, 2, 1)$ and $(x_1, y_1, z_1) = (4, -3, 7)$. Then a vector in the direction of the line is $[x_1 - x_0,\ y_1 - y_0,\ z_1 - z_0] = [-2, -5, 6]$. Let $[a, b, c] = [-2, -5, 6]$. Then, from Theorem 1, parametric equations for the line are $x = 6 - 2t,\ \ y = 2 - 5t,\ \ z = 1 + 6t\ \ (t \in \mathbb{R})$.
(Reversing the order of the points gives $(x_0, y_0, z_0) = (4, -3, 7)$ and $[a, b, c] = [2, 5, -6]$, and so another valid set of parametric equations for the line is: $x = 4 + 2t,\ \ y = -3 + 5t,\ \ z = 7 - 6t$ $(t \in \mathbb{R})$.)

(e) Let $(x_0, y_0, z_0) = (1, -5, -7)$. A vector in the direction of the given line is $[a, b, c] = [-2, 1, 0]$ (since these values are the respective coefficients of the parameter t), and therefore this is a vector in the direction of the desired line as well. Then, from Theorem 1, parametric equations for the line are: $x = 1 - 2t,\ \ y = -5 + t,\ \ z = -7 + 0t = -7\ \ (t \in \mathbb{R})$.

(2) (a) Setting the expressions for x equal gives $-6 + 6t = 9 + 3s$, which means $s = 2t - 5$. Setting the expressions for y equal gives $3 - 2t = 13 + 4s$, which means $2s + t = -5$. Substituting $2t - 5$ for s into $2s + t = -5$ leads to $t = 1$ and $s = -3$. Plugging these values into both expressions for x, y, and z in turn results in the same values: $x = 0$, $y = 1$, and $z = 7$. Hence, the given lines intersect at a single point: $(0, 1, 7)$.

(c) Setting the expressions for x equal gives $4 - 6t = 8 - 4s$, which means $s = 1 + \frac{3}{2}t$. (The same result is obtained by setting the expressions for y equal.) Setting the expressions for z equal gives $6 - 3t = 9 - 2s$, which means $s = \frac{3}{2}t + \frac{3}{2}$. Since it is impossible for these two expressions for s to be equal $(1 + \frac{3}{2}t \neq \frac{3}{2}t + \frac{3}{2})$ for any value of t, the given lines do not intersect.

(e) Setting the expressions for x equal gives $4t - 7 = 8s - 19$, which means $t = 2s - 3$. Similarly, setting the expressions for y equal and setting the expressions for z equal also gives $t = 2s - 3$. (In other words, substituting $2s - 3$ for t in the expressions for the first given line for x, y, z, respectively, gives the expressions for the second given line for x, y, z, respectively.) Thus, the lines are actually identical. The intersection of the lines is therefore the set of all points on (either) line; that is, the intersection consists of all points on the line $x = 4t - 7,\ y = 3t + 2,\ z = t - 5$ $(t \in \mathbb{R})$.

(3) (b) A vector in the direction of l_1 is $\mathbf{a} = [-3, 0, 4]$. A vector in the direction of l_2 is $\mathbf{b} = [-3, 5, 4]$. If θ is the angle between vectors $\mathbf{a}$ and $\mathbf{b}$, then $\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \frac{9+0+16}{5\sqrt{50}} = \frac{\sqrt{50}}{10} = \frac{\sqrt{2}}{2}$. Then, $\theta = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = 45°$.

(4) (b) Let $(x_0, y_0, z_0) = (6, 2, 1)$. In the solution to Exercise 1(c) above, we found that $[a, b, c] = [-2, -5, 6]$ is a vector in the direction of the line. Substituting these values into the formula for the symmetric equations of the line gives $\frac{x-6}{-2} = \frac{y-2}{-5} = \frac{z-1}{6}$ (If, instead, we had considered the points in reverse order, and chosen $(x_0, y_0, z_0) = (4, -3, 7)$ and calculated $[a, b, c] = [2, 5, -6]$ as a vector in the direction of the line, we would have obtained another valid form for the symmetric equations of the line: $\frac{x-4}{2} = \frac{y+3}{5} = \frac{z-7}{-6}$.)

(5) (a) Let $(x_0, y_0, z_0) = (1, 7, -2)$, and let $[a, b, c] = [6, 1, 6]$. Then by Theorem 2, the equation of the plane is $6x + 1y + 6z = (6)(1) + (1)(7) + (6)(-2)$, which simplifies to $6x + y + 6z = 1$.

(d) This plane goes through the origin $(x_0, y_0, z_0) = (0, 0, 0)$ with normal vector $[a, b, c] = [0, 0, 1]$. Then by Theorem 2, the equation for the plane is $0x + 0y + 1z = (0)(0) + (0)(0) + (1)(0)$, which simplifies to $z = 0$.

(6) (a) A normal vector to the given plane is $\mathbf{a} = [2, -1, 2]$. Since $||\mathbf{a}|| = 3$, a unit normal vector to the plane is given by $\left[\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right]$.

(7) (a) Normal vectors to the given planes are, respectively, $\mathbf{a} = [7, -7, -8]$ and $\mathbf{b} = [4, 5, -11]$. If θ is the angle between vectors $\mathbf{a}$ and $\mathbf{b}$, then $\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{||\mathbf{a}||||\mathbf{b}||} = \frac{28-35+88}{\sqrt{162}\sqrt{162}} = \frac{81}{162} = \frac{1}{2}$. Then, $\theta = \cos^{-1}\left(\frac{1}{2}\right) = 60°$.

(8) (a) Let $\mathbf{x} = [x_1, x_2, x_3] = [1, 2, -1]$ and $\mathbf{y} = [y_1, y_2, y_3] = [3, 7, 0]$. Then $[x_2y_3 - x_3y_2,\ x_3y_1 - x_1y_3,\ x_1y_2 - x_2y_1] = [(2)(0) - (-1)(7),\ (-1)(3) - (1)(0),\ (1)(7) - (2)(3)] = [7, -3, 1]$.

(g) First note that $[2, 0, -1] \times [-1, 2, 0] = [(0)(0) - (-1)(2),\ (-1)(-1) - (2)(0),\ (2)(2) - (0)(-1)] = [2, 1, 4]$. Thus, $[1, 2, -3] \cdot ([2, 0, -1] \times [-1, 2, 0]) = [1, 2, -3] \cdot [2, 1, 4] = (1)(2) + (2)(1) + (-3)(4) = -8$.

(15) (a) Let $A = (1, 3, 0)$, $B = (-1, 4, 1)$, $C = (3, 2, 2)$. Let $\mathbf{x}$ be the vector from A to B. Then $\mathbf{x} = [-1 - 1, 4 - 3, 1 - 0] = [-2, 1, 1]$. Let $\mathbf{y}$ be the vector from A to C. Then $\mathbf{y} = [3 - 1, 2 - 3, 2 - 0] = [2, -1, 2]$. Hence, a normal vector to the plane is $\mathbf{x} \times \mathbf{y} = [3, 6, 0]$. Therefore, the equation of the plane is $3x + 6y + 0z = (3)(1) + (6)(3) + (0)(0)$, or, $3x + 6y = 21$, which reduces to $x + 2y = 7$.

(17) (b) From the solution to Exercise 3(b) above, we know that a vector in the direction of l_1 is $[-3, 0, 4]$, and a vector in the direction of l_2 is $[-3, 5, 4]$. The cross product of these vectors gives $[-20, 0, -15]$ as a normal vector to the plane. Dividing by -5, we find that $[4, 0, 3]$ is also a normal vector to the plane.

We next find a point of intersection for the lines. Setting the expressions for y equal gives $1 = 5s - 4$,.which means that $s = 1$. Setting the expressions for x equal gives $9 - 3t = 3 - 3s$, which means $s = t - 2$. Substituting 1 for s into $s = t - 2$ leads to $t = 3$. Plugging these values for s and t into both expressions for x, y and z, respectively, yields the same values: $x = 0$, $y = 1$, $z = 13$. Hence, the given lines intersect at a single point: $(0, 1, 13)$.

Finally, the equation for the plane is $4x + 0y + 3z = (4)(0) + (0)(1) + (3)(13)$, which reduces to $4x + 3z = 39$.

(18) (c) The normal vector for the plane $x + z = 6$ is $[1, 0, 1]$, and the normal vector for the plane $y + z = 19$ is $[0, 1, 1]$. The cross product $[-1, -1, 1]$ of these vectors produces a vector in the direction of the line of intersection. To find a point of intersection of the planes, choose $z = 0$ to obtain $x = 6$ and $y = 19$. That is, one point where the planes intersect is $(6, 19, 0)$. Thus, the line of intersection is: $x = 6 - t,\ \ y = 19 - t,\ \ z = t\ \ (t \in \mathbb{R})$.

(19) (a) A vector in the direction of the line l is $[a, b, c] = [-2, 1, 2]$. By setting $t = 0$, we obtain the point $(x_0, y_0, z_0) = (4, -1, 5)$ on l. The vector from this point to the given point $(x_1, y_1, z_1) = (3, -1, 2)$ is $[x_1 - x_0,\ y_1 - y_0,\ z_1 - z_0] = [-1, 0, -3]$. Finally, using Theorem 6, the shortest distance from the given point to l is:

$$\frac{||[-2, 1, 2] \times [-1, 0, -3]||}{\sqrt{(-2)^2 + 1^2 + 2^2}} = \frac{||[-3, -8, 1]||}{3} = \frac{\sqrt{74}}{3} \approx 2.867$$

(21) (a) Since the given point is $(x_1, y_1, z_1) = (5, 2, 0)$, and the given plane is $2x - y - 2z = 12$, we have $a = 2$, $b = -1$, $c = -2$, and $d = 12$. From the formula in Theorem 7, the shortest distance from

the given point to the given plane is:

$$\frac{|ax_1+by_1+cz_1-d|}{\sqrt{a^2+b^2+c^2}} = \frac{|2\cdot 5+(-1)\cdot 2+(-2)\cdot 0-12|}{\sqrt{2^2+(-1)^2+(-2)^2}} = \frac{|-4|}{3} = \frac{4}{3}$$

(c) The given point is $(x_1, y_1, z_1) = (5, 0, -3)$. First, we find the equation of the plane through the points $A = (3, 1, 5)$, $B = (1, -1, 2)$, and $C = (4, 3, 5)$. Note that the vector from A to B is $[-2, -2, -3]$ and the vector from A to C is $[1, 2, 0]$. Thus, a normal vector to the plane is the cross product $[6, -3, -2]$ of these vectors. Hence, the equation of the plane is $6x + (-3)y + (-2)z = (6)(3) + (-3)(1) + (-2)(5)$, or, $6x - 3y - 2z = 5$. That is, $a = 6$, $b = -3$, $c = -2$, and $d = 5$. From the formula in Theorem 7, the shortest distance from the given point to the given plane is:

$$\frac{|ax_1+by_1+cz_1-d|}{\sqrt{a^2+b^2+c^2}} = \frac{|6\cdot 5+(-3)\cdot 0+(-2)\cdot(-3)-5|}{\sqrt{6^2+(-3)^2+(-2)^2}} = \frac{31}{7} \approx 4.429\ .$$

(23) (a) A vector in the direction of the line l_1 is $[-2, 2, 1]$, and a vector in the direction of the line l_2 is $[0, 1, 4]$. (Note that the lines are not parallel.) The cross product of these vectors is $\mathbf{v} = [7, 8, -2]$. By setting $t = 0$ and $s = 0$, respectively, we find that $(x_1, y_1, z_1) = (5, 3, -1)$ is a point on l_1, and $(x_2, y_2, z_2) = (2, 5, 0)$ is a point on l_2. Hence, the vector from (x_1, y_1, z_1) to (x_2, y_2, z_2) is $\mathbf{w} = [x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1] = [-3, 2, 1]$. Finally, from the formula given in this exercise, we find that the shortest distance between the given lines is

$$\frac{|(\mathbf{v}\cdot\mathbf{w})|}{\|\mathbf{v}\|} = \frac{|-21+16-2|}{\sqrt{7^2+8^2+(-2)^2}} = \frac{7}{3\sqrt{13}} = \frac{7\sqrt{13}}{39} \approx 0.647\ .$$

(c) A vector in the direction of the line l_1 is $[-7, 4, 0]$, and a vector in the direction of the line l_2 is $[-1, 3, -3]$. (Note that the lines are not parallel.) The cross product of these vectors is $\mathbf{v} = [-12, -21, -17]$. By setting $t = 0$ and $s = 0$, respectively, we find that $(x_1, y_1, z_1) = (3, 1, -8)$ is a point on l_1, and $(x_2, y_2, z_2) = (-1, -4, 1)$ is a point on l_2. Hence, the vector from (x_1, y_1, z_1) to (x_2, y_2, z_2) is $\mathbf{w} = [x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1] = [-4, -5, 9]$. Finally, from the formula given in this exercise, we find that the shortest distance between the given lines is

$$\frac{|(\mathbf{v}\cdot\mathbf{w})|}{\|\mathbf{v}\|} = \frac{|48+105-153|}{\sqrt{(-12)^2+(-21)^2+(-17)^2}} = \frac{0}{\sqrt{874}} = 0\ .$$

That is, the given lines actually intersect. (In fact, it can easily be shown that the lines intersect at $(-4, 5, -8)$.)

(24) (a) Labeling the first plane as $ax + by + cz = d_1$ and the second plane as $ax + by + cz = d_2$, we have $[a, b, c] = [3, -1, 4]$, and $d_1 = 10$, $d_2 = 7$. From the formula in Exercise 24, the shortest distance between the given planes is

$$\frac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}} = \frac{|10-7|}{\sqrt{3^2+(-1)^2+4^2}} = \frac{3}{\sqrt{26}} = \frac{3\sqrt{26}}{26} \approx 0.588\ .$$

(c) Divide the equation of the first plane by 2, and divide the equation of the second plane by 3 to obtain equations for the planes whose coefficients agree: $2x + 3y - 4z = \frac{9}{2}$ and $2x + 3y - 4z = -\frac{5}{3}$ Labeling the first of these new equations as $ax + by + cz = d_1$ and the second as $ax + by + cz = d_2$, we have $[a, b, c] = [2, 3, -4]$, and $d_1 = \frac{9}{2}$, $d_2 = -\frac{5}{3}$. From the formula in Exercise 24, the shortest distance between the given planes is

$$\frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|\frac{9}{2} - \left(-\frac{5}{3}\right)|}{\sqrt{2^2 + (3)^2 + (-4)^2}} = \frac{\frac{37}{6}}{\sqrt{29}} = \frac{37\sqrt{29}}{174} \approx 1.145 .$$

(25) (a) Let $(x_1, y_1, z_1) = (2, -1, 0)$, $(x_2, y_2, z_2) = (3, 0, 1)$, and $(x_3, y_3, z_3) = (2, 2, 7)$. Then we have $[x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1] = [1, 1, 1]$, and $[x_3 - x_1,\ y_3 - y_1,\ z_3 - z_1] = [0, 3, 7]$. Hence, from the formula in this exercise, the area of the triangle with vertices $(2, -1, 0)$, $(3, 0, 1)$, and $(2, 2, 7)$ is

$$\frac{1}{2}\|\,[x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1] \times [x_3 - x_1,\ y_3 - y_1,\ z_3 - z_1]\,\| = \frac{\|[4, -7, 3]\|}{2} = \frac{\sqrt{74}}{2} \approx 4.301 .$$

(29) (a) The vector $\mathbf{r} = [6, 3, -2]$ ft, and the vector $\boldsymbol{\omega} = [0, 0, \frac{2\pi}{12}] = [0, 0, \frac{\pi}{6}]$ rad/sec. Hence, the velocity vector $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} = [-\frac{\pi}{2}, \pi, 0]$ ft/sec; speed $= ||\mathbf{v}|| = \frac{\pi\sqrt{5}}{2} \approx 3.512$ ft/sec.

(32) (a) Let $\mathbf{r} = [x, y, z]$ represent the direction vector from the origin to a point (x, y, z) on the Earth's surface at latitude θ. Since the rotation is counterclockwise about the z-axis and one rotation of 2π radians occurs every 24 hours = 86400 seconds, we find that $\boldsymbol{\omega} = [0, 0, \frac{2\pi}{86400}] = [0, 0, \frac{\pi}{43200}]$ rad/sec. Then,

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} = \left[\frac{-\pi y}{43200}, \frac{\pi x}{43200}, 0\right].$$

By considering the right triangle with hypotenuse = Earth's radius = 6369 km, and altitude z, we see that $z = 6369 \sin\theta$. But since $||\mathbf{r}|| = \sqrt{x^2 + y^2 + z^2} = 6369$, we must have $x^2 + y^2 + (6369 \sin\theta)^2 = (6369)^2$, which means $x^2 + y^2 = (6369 \cos\theta)^2$. Thus,

$$||\mathbf{v}|| = \sqrt{\left(\frac{-\pi y}{43200}\right)^2 + \left(\frac{\pi x}{43200}\right)^2 + 0^2} = \frac{\pi}{43200}\sqrt{(x^2 + y^2)} = \frac{6369\pi}{43200}\cos\theta$$

$\approx 0.4632 \cos\theta$ km/sec.

(33) (a) False. If $[a, b, c]$ is a direction vector for a given line, then any nonzero scalar multiple of $[a, b, c]$ is another direction vector for that same line. For example, the line $x = t$, $y = 0$, $z = 0$ ($t \in \mathbb{R}$) (that is, the x-axis) has many distinct direction vectors, such as $[1, 0, 0]$ and $[2, 0, 0]$.

(b) False. If the lines are skew lines, they do not intersect each other, and the angle between the lines is not defined. For example, the lines in Exercise 2(c) do not intersect, and hence, there is no angle defined between them.

(c) True. Two distinct nonparallel planes always intersect each other (in a line), and the angle between the planes is the (minimal) angle between a pair of normal vectors to the planes.

(d) True. The vector $[a, b, c]$ is normal to the given plane, and the vector $[-a, -b, -c]$ is a nonzero scalar multiple of $[a, b, c]$, so $[-a, -b, -c]$ is also perpendicular to the given plane.

(e) True. For all vectors $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$, we have: $(\mathbf{x} + \mathbf{y}) \times \mathbf{z} = -(\mathbf{z} \times (\mathbf{x} + \mathbf{y}))$ (by part (1) of Theorem 3) $= -((\mathbf{z} \times \mathbf{x}) + (\mathbf{z} \times \mathbf{y}))$ (by part (5) of Theorem 3) $= -(\mathbf{z} \times \mathbf{x}) - (\mathbf{z} \times \mathbf{y}) = (\mathbf{x} \times \mathbf{z}) + (\mathbf{y} \times \mathbf{z})$ (by part (1) of Theorem 3).

(f) True. For all vectors $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$, we have: $\mathbf{y} \cdot (\mathbf{z} \times \mathbf{x}) = (\mathbf{y} \times \mathbf{z}) \cdot \mathbf{x}$, by part (7) of Theorem 3.

(g) False. From Corollary 5, if $\mathbf{x}$ and $\mathbf{y}$ are parallel, then $\mathbf{x} \times \mathbf{y} = \mathbf{0}$. For example, $[1, 0, 0] \times [2, 0, 0] = [0, 0, 0]$.

(h) True. From Theorem 4, if $\mathbf{x}$ and $\mathbf{y}$ are nonzero, then $\|\mathbf{x} \times \mathbf{y}\| = (||\mathbf{x}||\ ||\mathbf{y}||) \sin\theta$, and dividing both sides by $(||\mathbf{x}||\ ||\mathbf{y}||)$ (which is nonzero) gives the desired result.

(i) False. $\mathbf{k} \times \mathbf{i} = \mathbf{j}$, while $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$. (Since the results of the cross product operations are nonzero here, the anti-commutative property from part (1) of Theorem 3 also shows that the given statement is false.)

(j) True. The vectors $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{x} \times \mathbf{y}$, in that order, form a right-handed system, as in Figure 8, where we can imagine the vectors $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{x} \times \mathbf{y}$ lying along the positive x-, y-, and z-axes, respectively. Then, moving the (right) hand in that figure, so that its fingers are curling instead from the vector $\mathbf{x} \times \mathbf{y}$ (positive z-axis) toward the vector $\mathbf{x}$ (positive x-axis) will cause the thumb of the (right) hand to point in the direction of $\mathbf{y}$ (positive y-axis). Thus, the vectors $\mathbf{x} \times \mathbf{y}$, $\mathbf{x}$, and $\mathbf{y}$, taken in that order, also form a right-handed system.

(k) True. This is precisely the method outlined before Theorem 7 for finding the distance from a point to a plane.

(l) False. If $\boldsymbol{\omega}$ is the angular velocity, $\mathbf{v}$ is the velocity, and $\mathbf{r}$ is the position vector, then $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$, but it is not generally true that $\mathbf{v} \times \mathbf{r} = \boldsymbol{\omega}$. For example, in Example 16, $\mathbf{v} = \left[-\frac{\pi}{2}, -\frac{\pi}{4}, 0\right]$, $\mathbf{r} = [-1, 2, -2]$, and $\boldsymbol{\omega} = \left[0, 0, -\frac{\pi}{4}\right]$, so, $\mathbf{v} \times \mathbf{r} = \left[-\frac{\pi}{2}, -\frac{\pi}{4}, 0\right] \times [-1, 2, -2] = \left[\frac{\pi}{2}, -\pi, -\frac{5\pi}{4}\right] \neq \boldsymbol{\omega}$.

Change of Variables and the Jacobian

(1) (a) The Jacobian matrix $\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Therefore, $dx\,dy = \Big||\mathbf{J}|\Big|\; du\,dv = 2\;du\,dv$.

(c) The Jacobian matrix $\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} 2u & -2v \\ 2v & 2u \end{bmatrix}$. Therefore,

$dx\,dy = \Big||\mathbf{J}|\Big|\; du\,dv = 4(u^2+v^2)\;du\,dv$.

(e) The Jacobian matrix $\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} =$

$\begin{bmatrix} \frac{\left((u+1)^2+v^2\right)(2)-2u(2(u+1))}{((u+1)^2+v^2)^2} & \frac{-4uv}{((u+1)^2+v^2)^2} \\ \frac{\left((u+1)^2+v^2\right)(-2u)-\left(1-(u^2+v^2)\right)(2(u+1))}{((u+1)^2+v^2)^2} & \frac{\left((u+1)^2+v^2\right)(-2v)-\left(1-(u^2+v^2)\right)(2v)}{((u+1)^2+v^2)^2} \end{bmatrix}$. Simplifying yields

$$\mathbf{J} = \frac{2}{((u+1)^2+v^2)^2}\begin{bmatrix} \left((u+1)^2+v^2\right)-2u(u+1) & -2uv \\ -u\left((u+1)^2+v^2\right)-(u+1)\left(1-\left(u^2+v^2\right)\right) & -2uv-2v \end{bmatrix}$$

$$= \frac{2}{((u+1)^2+v^2)^2}\begin{bmatrix} -u^2+v^2+1 & -2uv \\ -u^2-2u-1+v^2 & -2uv-2v \end{bmatrix}.$$

Now, for a 2×2 matrix $\mathbf{A}$, $|k\mathbf{A}| = k^2|\mathbf{A}|$, and so,

$|\mathbf{J}| = \frac{4}{((u+1)^2+v^2)^4}\left(\left(-u^2+v^2+1\right)(-2uv-2v)-(-2uv)\left(-u^2-2u-1+v^2\right)\right)$

$= \frac{4}{((u+1)^2+v^2)^4}\left(2u^3v-2uv^3-2uv+2u^2v-2v^3-2v-2u^3v-4u^2v-2uv+2uv^3\right)$

$= \frac{4}{((u+1)^2+v^2)^4}\left(-4uv-2u^2v-2v^3-2v\right) = \frac{-8}{((u+1)^2+v^2)^4}\left(u^2v+2uv+v^3+v\right)$

$= \frac{-8}{((u+1)^2+v^2)^4}\left(\left((u+1)^2+v^2\right)v\right) = \frac{-8v}{((u+1)^2+v^2)^3}$.

Hence, $dx\,dy = \Big||\mathbf{J}|\Big|\; du\,dv = \frac{8|v|}{((u+1)^2+v^2)^3}du\,dv$.

(2) (a) The Jacobian matrix $\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$. Thus, $|\mathbf{J}| = (1)(1)(1)+(1)(1)(1)+$

$(0)(0)(0)-(0)(1)(1)-(1)(1)(0)-(1)(0)(1) = 2$. Therefore, $dx\,dy\,dz = \Big||\mathbf{J}|\Big|\; du\,dv\,dw = 2\;du\,dv\,dw$.

(c) The Jacobian matrix $\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix} =$

$$\frac{1}{(u^2+v^2+w^2)^2}\begin{bmatrix} \left(u^2+v^2+w^2\right)-2u^2 & -2uv & -2uw \\ -2uv & \left(u^2+v^2+w^2\right)-2v^2 & -2vw \\ -2uw & -2vw & \left(u^2+v^2+w^2\right)-2w^2 \end{bmatrix}.$$

To make this expression simpler, let $A = \left(u^2+v^2+w^2\right)$.

Now, for a 3×3 matrix $\mathbf{M}$, $|k\mathbf{M}| = k^3|\mathbf{M}|$, and so by basketweaving,

$|\mathbf{J}| = \frac{1}{A^6}\Big(\left(A-2u^2\right)\left(A-2v^2\right)\left(A-2w^2\right) + (-2uv)(-2vw)(-2uw)$
$+ (-2uw)(-2uv)(-2vw) - (-2uw)(A-2v^2)(-2uw)$
$- (A-2u^2)(-2vw)(-2vw) - (-2uv)(-2uv)(A-2w^2)\Big)$
$= \frac{1}{A^6}\Big(A^3 - 2u^2A^2 - 2v^2A^2 - 2w^2A^2 + 4u^2v^2A + 4u^2w^2A$
$+ 4v^2w^2A - 8u^2v^2w^2 - 8u^2v^2w^2 - 8u^2v^2w^2 - 4u^2w^2A + 8u^2v^2w^2$
$- 4v^2w^2A + 8u^2v^2w^2 - 4u^2v^2A + 8u^2v^2w^2\Big)$
$= \frac{1}{A^6}\left(A^3 - 2(u^2+v^2+w^2)A^2\right) = \frac{1}{A^6}\left(-A^3\right) = -\frac{1}{A^3}.$
Therefore, $dx\,dy\,dz = \Big|\,|\mathbf{J}|\,\Big|\ du\,dv\,dw = \frac{1}{(u^2+v^2+w^2)^3}\,du\,dv\,dw.$

(4) (a) For the given region R, $r^2 = x^2+y^2$ ranges from 1 to 9, and so r ranges from 1 to 3. Since R is restricted to the first quadrant, θ ranges from 0 to $\frac{\pi}{2}$. Therefore, using $x = r\cos\theta$, $y = r\sin\theta$, and $dx\,dy = r\,dr\,d\theta$ yields $\iint\limits_R (x+y)\ dx\,dy = \iint\limits_R (r\cos\theta + r\sin\theta)\ r\,dr\,d\theta$
$= \int_0^{\frac{\pi}{2}}\int_1^3 (r\cos\theta + r\sin\theta)\,r\,dr\,d\theta = \int_0^{\frac{\pi}{2}} \frac{r^3}{3}(\cos\theta+\sin\theta)\Big|_1^3\,d\theta$
$= \int_0^{\frac{\pi}{2}} \frac{26}{3}(\cos\theta+\sin\theta)\ d\theta = \frac{26}{3}\left(\sin\theta - \cos\theta\right)\Big|_0^{\frac{\pi}{2}} = \frac{52}{3}.$

(c) For the given sphere R, ρ ranges from 0 to 1. Since R is restricted to the upper half of the xy-plane, ϕ ranges from 0 to $\frac{\pi}{2}$. However, θ ranges from 0 to 2π, since R is the entire upper half of the sphere; that is, all the way around the z-axis. Therefore, using
$z = \rho\cos\phi$ and $dx\,dy\,dz = (\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta$, we get
$\iiint\limits_R z\ dx\,dy\,dz = \iiint\limits_R \rho\cos\phi(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta$
$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^1 \rho^3\cos\phi\sin\phi\ d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\left(\frac{\rho^4}{4}\cos\phi\sin\phi\right)\Big|_0^1\,d\phi\,d\theta$
$= \frac{1}{4}\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\cos\phi\sin\phi\ d\phi\,d\theta = \frac{1}{4}\int_0^{2\pi}\frac{\sin^2\phi}{2}\Big|_0^{\frac{\pi}{2}}\,d\theta = \frac{1}{4}\int_0^{2\pi}\frac{1}{2}\,d\theta$
$= \frac{1}{8}\theta\Big|_0^{2\pi} = \frac{\pi}{4}.$

(e) Since $r^2 = x^2+y^2$, the condition $x^2+y^2 \le 4$ on the region R means that r ranges from 0 to 2 and θ ranges from 0 to 2π. Therefore, using $x^2+y^2 = r^2$ and $dx\,dy\,dz = r\,dr\,d\theta\,dz$ produces $\iiint\limits_R \left(x^2+y^2+z^2\right)\ dx\,dy\,dz = \int_{-3}^5\int_0^{2\pi}\int_0^2\left(r^2+z^2\right) r\ dr\,d\theta\,dz$
$= \int_{-3}^5\int_0^{2\pi}\left(\frac{r^4}{4} + \frac{r^2z^2}{2}\right)\Big|_0^2\,d\theta\,dz = \int_{-3}^5\int_0^{2\pi}\left(4+2z^2\right)\ d\theta\,dz$
$= \int_{-3}^5\left(4+2z^2\right)\theta\Big|_0^{2\pi}\,dz = \int_{-3}^5\left(8\pi + 4\pi z^2\right)\ dz$
$= \left(8\pi z + \frac{4}{3}\pi z^3\right)\Big|_{-3}^5 = \left(40\pi + \frac{500}{3}\pi\right) - (-24\pi - 36\pi) = \frac{800}{3}\pi.$

(5) (a) True. This is because all of the partial derivatives in the Jacobian matrix will be constants, since they are derivatives of linear functions. Hence, the determinant of the Jacobian will be a constant.

(b) True. The determinant of the Jacobian matrix $\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$ is (-1); however,

$dx\,dy = \Big||\mathbf{J}|\Big|\; du\,dv = |-1|\; du\,dv = du\,dv.$

(c) True. This is explained and illustrated in the Jacobian section, just after Example 3, in Figure 3, and again in Example 4.

(d) False. The scaling factor is the absolute value of the determinant of the Jacobian matrix, which will not equal the determinant of the Jacobian matrix when that determinant is negative. For a counterexample, see the solution to part (b) of this Exercise.

Function Spaces

(1) (a) To test for linear independence, we must determine whether $ae^x + be^{2x} + ce^{3x} = 0$ has any nontrivial solutions. Using this equation we substitute the following values for x:

$$\begin{cases} \text{Letting } x = 0 \implies & a + b + c = 0 \\ \text{Letting } x = 1 \implies & a(e) + b(e^2) + c(e^3) = 0 \\ \text{Letting } x = 2 \implies & a(e^2) + b(e^4) + c(e^6) = 0 \end{cases}.$$

But $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ e & e^2 & e^3 & 0 \\ e^2 & e^4 & e^6 & 0 \end{array}\right]$ row reduces to $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$,

showing that the system has only the trivial solution $a = b = c = 0$. Hence, the given set is linearly independent.

(c) To test for linear independence, we must check for nontrivial solutions to

$$a\left(\frac{5x-1}{1+x^2}\right) + b\left(\frac{3x+1}{2+x^2}\right) + c\left(\frac{7x^3-3x^2+17x-5}{x^4+3x^2+2}\right) = 0.$$

Using this equation we substitute the following values for x:

$$\begin{cases} \text{Letting } x = 0 \implies & -a + \frac{1}{2}b - \frac{5}{2}c = 0 \\ \text{Letting } x = 1 \implies & 2a + \frac{4}{3}b + \frac{8}{3}c = 0 \\ \text{Letting } x = -1 \implies & -3a - \frac{2}{3}b - \frac{16}{3}c = 0 \end{cases}.$$

But $\left[\begin{array}{ccc|c} -1 & \frac{1}{2} & -\frac{5}{2} & 0 \\ 2 & \frac{4}{3} & \frac{8}{3} & 0 \\ -3 & -\frac{2}{3} & -\frac{16}{3} & 0 \end{array}\right]$ row reduces to $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$,

which has nontrivial solution $a = -2$, $b = 1$, $c = 1$. Algebraic simplification verifies that

$$(-2)\left(\frac{5x-1}{1+x^2}\right) + (1)\left(\frac{3x+1}{2+x^2}\right) + (1)\left(\frac{7x^3-3x^2+17x-5}{x^4+3x^2+2}\right) = 0$$

is a functional identity. Hence, since a nontrivial linear combination of the elements of S produces $\mathbf{0}$, the set S is linearly dependent.

(4) (a) First, to simplify matters, we eliminate any functions that we can see by inspection are linear combinations of other functions in S. The familiar trigonometric identities $\sin^2 x + \cos^2 x = 1$ and $\sin x \cos x = \frac{1}{2}\sin 2x$ show that the last two functions in S are linear combinations of earlier functions. Thus, the subset $S_1 = \{\sin 2x, \cos 2x, \sin^2 x, \cos^2 x\}$ has the same span as S. However, the identity $\cos^2 x = \cos 2x + \sin^2 x$ (more commonly stated as $\cos 2x = \cos^2 x - \sin^2 x$) shows that $B = \{\sin 2x, \cos 2x, \sin^2 x\}$ also has the same span. We suspect that we have now eliminated all of the functions that we can, and that B is linearly independent, making it a basis for span(S). We verify the linear independence of B by considering the equation $a\sin 2x + b\cos 2x + c\sin^2 x = 0$.

Using this equation we substitute the following values for x:

$$\begin{cases} \text{Letting } x = 0 & \Longrightarrow & & + & b & & & = & 0 \\ \text{Letting } x = \frac{\pi}{6} & \Longrightarrow & \frac{\sqrt{3}}{2}a & + & \frac{1}{2}b & + & \frac{1}{4}c & = & 0 \\ \text{Letting } x = \frac{\pi}{4} & \Longrightarrow & a & & & + & \frac{1}{2}c & = & 0 \end{cases}.$$

Next, row reduce $\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & \frac{1}{4} & 0 \\ 1 & 0 & \frac{1}{2} & 0 \end{array}\right]$ to $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$.

Since the system has only the trivial solution, B is linearly independent, and is thus the desired basis for span(S).

(c) First, to simplify matters, we eliminate any functions that we can see by inspection are linear combinations of other functions in S. We start with the familiar trigonometric identities $\sin(a + b) = \sin a \cos b + \sin b \cos a$ and $\cos(a + b) = \cos a \cos b - \sin a \sin b$. Hence,

$\sin(x + 1) = (\sin x)(\cos 1) + (\sin 1)(\cos x)$,
$\cos(x + 1) = (\cos x)(\cos 1) - (\sin x)(\sin 1)$,
$\sin(x + 2) = (\sin x)(\cos 2) + (\sin 2)(\cos x)$, and
$\cos(x + 2) = (\cos x)(\cos 2) - (\sin x)(\sin 2)$.

Therefore, each of the elements of S is a linear combination of $\sin x$ and $\cos x$. That is, $\text{span}(S) \subseteq \text{span}(\{\sin x, \cos x\})$. Thus, $\dim(\text{span}(S)) \leq 2$. Now, if we can find two linearly independent vectors in S, they form a basis for span(S). We claim that the subset $B = \{\sin(x+1), \cos(x+1)\}$ of S is linearly independent, and hence is a basis contained in S.

To show that B is linearly independent, we plug two values for x into $a\sin(x+1)+b\cos(x+1) = 0$:

$$\begin{cases} \text{Letting } x = 0 & \Longrightarrow & (\sin 1)a & + & (\cos 1)b & = & 0 \\ \text{Letting } x = -1 & \Longrightarrow & & & b & = & 0 \end{cases}.$$

Using back substitution and the fact that $\sin 1 \neq 0$ shows that $a = b = 0$, proving that B is linearly independent.

(5) (a) Since $\mathbf{v} = 5e^x + 0e^{2x} + (-7)e^{3x}$, $[\mathbf{v}]_B = [5, 0, -7]$ by the definition of $[\mathbf{v}]_B$.

(c) We start with the familiar trigonometric identity $\sin(a + b) = \sin a \cos b + \sin b \cos a$, which yields

$\sin(x + 1) = (\sin x)(\cos 1) + (\sin 1)(\cos x)$, and
$\sin(x + 2) = (\sin x)(\cos 2) + (\sin 2)(\cos x)$.

Dividing the first equation by $\cos 1$ and the second by $\cos 2$, and then subtracting and rearranging terms yields $\frac{1}{\cos 1}\sin(x + 1) - \frac{1}{\cos 2}\sin(x + 2) = \left(\frac{\sin 1}{\cos 1} - \frac{\sin 2}{\cos 2}\right)\cos x$. Now, $\frac{\sin 1}{\cos 1} - \frac{\sin 2}{\cos 2} = \frac{\sin 1 \cos 2 - \sin 2 \cos 1}{\cos 1 \cos 2} = \frac{\sin(-1)}{\cos 1 \cos 2} = -\frac{\sin 1}{\cos 1 \cos 2}$. Hence, $\cos x = -\frac{\cos 2}{\sin 1}\sin(x + 1) + \frac{\cos 1}{\sin 1}\sin(x + 2)$. Therefore, the definition of $[\mathbf{v}]_B$ yields $[\mathbf{v}]_B = [-\frac{\cos 2}{\sin 1}, \frac{\cos 1}{\sin 1}]$, or approximately $[0.4945, 0.6421]$.

An alternate approach is to solve for a and b in the equation $\cos x = a\sin(x + 1) + b\sin(x + 2)$. Plug in the following values for x:

$$\begin{cases} \text{Letting } x = -1 & \Longrightarrow & \cos 1 & = & & (\sin 1)b \\ \text{Letting } x = -2 & \Longrightarrow & \cos 2 & = & -(\sin 1)a & \end{cases},$$

producing the same results for a and b as above.

(6) (a) True. In *any* vector space, $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent if and only if either of the vectors in S can be expressed as a linear combination of the other. Since there are only two vectors in the set, neither of which is zero, this happens precisely when $\mathbf{v}_1$ is a nonzero constant multiple of $\mathbf{v}_2$.

(b) True. Since the polynomials all have different degrees, none of them is a linear combination of the others.

(c) False. By the definition of linear independence, the set of vectors would be linearly independent, *not* linearly dependent.

(d) False. It is possible that the three values chosen for x do not produce equations proving linear independence, but choosing a different set of three values might. This principle is illustrated directly after Example 1 in the Functions Spaces section. We are only assured of linear dependence when we have found values of a, b, and c, not all zero, such that $a\mathbf{f}_1(x) + b\mathbf{f}_2(x) + c\mathbf{f}_3(x) = 0$ is a functional identity.

Max-Min Problems in $\mathbb{R}^n$ and the Hessian Matrix

(1) (a) First, $\nabla f = [3x^2 + 2x + 2y - 3, 2x + 2y]$. We find critical points by setting $\nabla f = \mathbf{0}$. Setting the second coordinate of ∇f equal to zero yields $y = -x$. Plugging this into $3x^2 + 2x + 2y - 3 = 0$ produces $3x^2 - 3 = 0$, which has solutions $x = 1$ and $x = -1$. This gives the two critical points $(1, -1)$ and $(-1, 1)$. Next, $\mathbf{H} = \begin{bmatrix} 6x+2 & 2 \\ 2 & 2 \end{bmatrix}$. At the first critical point, $\mathbf{H}\Big|_{(1,-1)} = \begin{bmatrix} 8 & 2 \\ 2 & 2 \end{bmatrix}$, which is positive definite because its determinant is 12, which is positive, and its $(1, 1)$ entry is 8, which is also positive. (See the comment just before Example 3 in the Hessian section for easily verified necessary and sufficient conditions for a 2×2 matrix to be either positive definite or negative definite.) So, f has a local minimum at $(1, -1)$. At the second critical point, $\mathbf{H}\Big|_{(-1,1)} = \begin{bmatrix} -4 & 2 \\ 2 & 2 \end{bmatrix}$. We refer to this matrix as $\mathbf{A}$. Now $\mathbf{A}$ is neither positive definite nor negative definite, since its determinant is negative. Also note that $p_{\mathbf{A}}(x) = x^2 + 2x - 12$, which, by the Quadratic Formula, has roots $-1 \pm \sqrt{13}$. Since one of these is positive and the other is negative, we see that $(-1, 1)$ is not a local extreme value.

(c) First, $\nabla f = [4x+2y+2, 2x+2y-2]$. We find critical points by setting $\nabla f = \mathbf{0}$. This corresponds to a linear system with two equations and two variables which has the unique solution $(-2, 3)$. Next, $\mathbf{H} = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}$. $\mathbf{H}$ is positive definite since its determinant is 4, which is positive, and its $(1, 1)$ entry is 4, which is positive. Hence, the critical point $(-2, 3)$ is a local minimum.

(e) First, $\nabla f = [4x + 2y + 2z,\ 2x + 4y^3 + 12y^2z + 12yz^2 - 2y + 4z^3 - 4z,$ $2x+4y^3+12y^2z+12yz^2-4y+4z^3-2z]$. We find critical points by setting $\nabla f = \mathbf{0}$. To make things easier, notice that at the critical point, $\frac{\partial f}{\partial y} = \frac{\partial f}{\partial z}$. This yields $y = z$. With this, and $\frac{\partial f}{\partial x} = 0$, we obtain $x = -y$. Substituting for y and z into $\frac{\partial f}{\partial y} = 0$ and solving, produces $8x - 32x^3 = 0$. Hence, $8x(1 - 4x^2) = 8x(1 - 2x)(1 + 2x) = 0$. This yields the three critical points $(0, 0, 0)$, $\left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, and $\left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$. Next, $\mathbf{H} = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 12y^2 + 24yz + 12z^2 - 2 & 12y^2 + 24yz + 12z^2 - 4 \\ 2 & 12y^2 + 24yz + 12z^2 - 4 & 12y^2 + 24yz + 12z^2 - 2 \end{bmatrix}$.

At the first critical point, $\mathbf{H}\Big|_{(0,0,0)} = \begin{bmatrix} 4 & 2 & 2 \\ 2 & -2 & -4 \\ 2 & -4 & -2 \end{bmatrix}$. We refer to this matrix as $\mathbf{A}$. Then $p_{\mathbf{A}}(x) = x^3 - 36x + 64 = (x - 2)(x^2 + 2x - 32)$. Hence, the eigenvalues of $\mathbf{A}$ are 2, and $-1 \pm \sqrt{33}$. Since $\mathbf{A}$ has both positive and negative eigenvalues, $(0, 0, 0)$ is not an extreme point.

At the second critical point, $\mathbf{H}\Big|_{\left(-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)} = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 10 & 8 \\ 2 & 8 & 10 \end{bmatrix}$. We refer to this matrix as $\mathbf{B}$. Then $p_{\mathbf{B}}(x) = x^3 - 24x^2 + 108x - 128 = (x - 2)(x^2 - 22x + 64)$. Hence, the eigenvalues of $\mathbf{B}$ are all positive (2 and $11 \pm \sqrt{57}$). So f has a local minimum at $\left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ by Theorem 10.4.

At the third critical point, $\mathbf{H}\Big|_{\left(\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)} = \mathbf{H}\Big|_{\left(-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)}$, and so f also has a local minimum at $\left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$.

(4) (a) True. We know from calculus that if a real-valued function f on $\mathbb{R}^n$ has continuous second partial derivatives then $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$, for all i, j.

(b) False. For example, the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ represents neither a positive definite nor negative definite quadratic form since it has both a positive and a negative eigenvalue.

(c) True. In part (a) we noted that the Hessian matrix is symmetric. Theorems 6.18 and 6.20 then establish that the matrix is orthogonally diagonalizable, hence diagonalizable.

(d) True. We established that a symmetric 2×2 matrix $\mathbf{A}$ with $|\mathbf{A}| > 0$ and $a_{11} > 0$ represents a positive definite quadratic form (see the comment just before Example 3 in the Hessian section). In this problem, $|\mathbf{A}| = 1 > 0$ and $a_{11} = 5 > 0$.

(e) False. The eigenvalues of the given matrix are clearly 3, -9, and 4. Hence, the given matrix has both a positive and a negative eigenvalue, and so can not represent a positive definite quadratic form.

Jordan Canonical Form

(1) (c) By parts (a) and (b), $(\mathbf{A}-\lambda\mathbf{I}_k)\,\mathbf{e}_1=\mathbf{0}_k$, and $(\mathbf{A}-\lambda\mathbf{I}_k)\,\mathbf{e}_i=\mathbf{e}_{i-1}$ for $2\le i\le k$. We use induction to prove that $(\mathbf{A}-\lambda\mathbf{I}_k)^i\,\mathbf{e}_i=\mathbf{0}_k$ for all i, $1\le i\le k$. The Base Step holds since we already noted that $(\mathbf{A}-\lambda\mathbf{I}_k)\,\mathbf{e}_1=\mathbf{0}_k$. For the Inductive Step, we assume that $(\mathbf{A}-\lambda\mathbf{I}_k)^{i-1}\,\mathbf{e}_{i-1}=\mathbf{0}_k$ and show that $(\mathbf{A}-\lambda\mathbf{I}_k)^i\,\mathbf{e}_i=\mathbf{0}_k$. But $(\mathbf{A}-\lambda\mathbf{I}_k)^i\,\mathbf{e}_i=(\mathbf{A}-\lambda\mathbf{I}_k)^{i-1}(\mathbf{A}-\lambda\mathbf{I}_k)\,\mathbf{e}_i=(\mathbf{A}-\lambda\mathbf{I}_k)^{i-1}\,\mathbf{e}_{i-1}=\mathbf{0}_k$, completing the induction proof.

Using $(\mathbf{A}-\lambda\mathbf{I}_k)^i\,\mathbf{e}_i=\mathbf{0}_k$, we see that $(\mathbf{A}-\lambda\mathbf{I}_k)^k\,\mathbf{e}_i=(\mathbf{A}-\lambda\mathbf{I}_k)^{k-i}(\mathbf{A}-\lambda\mathbf{I}_k)^i\,\mathbf{e}_i$ $=(\mathbf{A}-\lambda\mathbf{I}_k)^{k-i}\,\mathbf{0}_k=\mathbf{0}_k$ for all i, $1\le i\le k$. Now, for $1\le i\le k$, the i^{th} column of $(\mathbf{A}-\lambda\mathbf{I}_k)^k$ equals $(\mathbf{A}-\lambda\mathbf{I}_k)^k\,\mathbf{e}_i$. Hence, we have shown that every column of $(\mathbf{A}-\lambda\mathbf{I}_k)^k$ is zero, and so $(\mathbf{A}-\lambda\mathbf{I}_k)^k=\mathbf{O}_k$.

(4) Note for all the parts below that two matrices in Jordan canonical form are similar to each other if and only if they contain the same Jordan blocks, rearranged in any order.

(a) $\begin{bmatrix}[2]&0&0\\0&[2]&0\\0&0&[-1]\end{bmatrix}$, $\begin{bmatrix}[2]&0&0\\0&[-1]&0\\0&0&[2]\end{bmatrix}$, $\begin{bmatrix}[-1]&0&0\\0&[2]&0\\0&0&[2]\end{bmatrix}$

$\begin{bmatrix}\begin{bmatrix}2&1\\0&2\end{bmatrix}&\begin{matrix}0\\0\end{matrix}\\\begin{matrix}0&0\end{matrix}&[-1]\end{bmatrix}$, $\begin{bmatrix}[-1]&\begin{matrix}0&0\end{matrix}\\\begin{matrix}0\\0\end{matrix}&\begin{bmatrix}2&1\\0&2\end{bmatrix}\end{bmatrix}$

The three matrices on the first line are all similar to each other. The two matrices on the second line are similar to each other, but not to those on the first line.

(c) Using the quadratic formula, $x^2+6x+10$ has roots $-3+i$ and $-3-i$, each having multiplicity 1;
$\begin{bmatrix}[-3+i]&0\\0&[-3-i]\end{bmatrix}$ and $\begin{bmatrix}[-3-i]&0\\0&[-3+i]\end{bmatrix}$, which are similar to each other

(d) $x^4+4x^3+4x^2=x^2(x+2)^2$;
$\begin{bmatrix}\begin{bmatrix}0&1\\0&0\end{bmatrix}&\begin{matrix}0&0\\0&0\end{matrix}\\\begin{matrix}0&0\\0&0\end{matrix}&\begin{bmatrix}-2&1\\0&-2\end{bmatrix}\end{bmatrix}$, $\begin{bmatrix}\begin{bmatrix}-2&1\\0&-2\end{bmatrix}&\begin{matrix}0&0\\0&0\end{matrix}\\\begin{matrix}0&0\\0&0\end{matrix}&\begin{bmatrix}0&1\\0&0\end{bmatrix}\end{bmatrix}$, which are similar to each other but to none of the others;

$\begin{bmatrix}[0]&0&\begin{matrix}0&0\end{matrix}\\0&[0]&\begin{matrix}0&0\end{matrix}\\\begin{matrix}0\\0\end{matrix}&\begin{matrix}0\\0\end{matrix}&\begin{bmatrix}-2&1\\0&-2\end{bmatrix}\end{bmatrix}$, $\begin{bmatrix}[0]&\begin{matrix}0&0\end{matrix}&0\\\begin{matrix}0\\0\end{matrix}&\begin{bmatrix}-2&1\\0&-2\end{bmatrix}&\begin{matrix}0\\0\end{matrix}\\0&\begin{matrix}0&0\end{matrix}&[0]\end{bmatrix}$, $\begin{bmatrix}\begin{bmatrix}-2&1\\0&-2\end{bmatrix}&\begin{matrix}0&0\\0&0\end{matrix}\\\begin{matrix}0&0\\0&0\end{matrix}&\begin{matrix}[0]&0\\0&[0]\end{matrix}\end{bmatrix}$,

which are similar to each other but to none of the others;

$\begin{bmatrix}\begin{bmatrix}0&1\\0&0\end{bmatrix}&\begin{matrix}0&0\\0&0\end{matrix}\\\begin{matrix}0&0\\0&0\end{matrix}&\begin{matrix}[-2]&0\\0&[-2]\end{matrix}\end{bmatrix}$, $\begin{bmatrix}[-2]&\begin{matrix}0&0\end{matrix}&0\\\begin{matrix}0\\0\end{matrix}&\begin{bmatrix}0&1\\0&0\end{bmatrix}&\begin{matrix}0\\0\end{matrix}\\0&\begin{matrix}0&0\end{matrix}&[-2]\end{bmatrix}$, $\begin{bmatrix}\begin{matrix}[-2]&0\\0&[-2]\end{matrix}&\begin{matrix}0&0\\0&0\end{matrix}\\\begin{matrix}0&0\\0&0\end{matrix}&\begin{bmatrix}0&1\\0&0\end{bmatrix}\end{bmatrix}$,

which are similar to each other but to none of the others;

$$\begin{bmatrix} [-2] & 0 & 0 & 0 \\ 0 & [-2] & 0 & 0 \\ 0 & 0 & [0] & 0 \\ 0 & 0 & 0 & [0] \end{bmatrix}, \begin{bmatrix} [-2] & 0 & 0 & 0 \\ 0 & [0] & 0 & 0 \\ 0 & 0 & [-2] & 0 \\ 0 & 0 & 0 & [0] \end{bmatrix}, \begin{bmatrix} [-2] & 0 & 0 & 0 \\ 0 & [0] & 0 & 0 \\ 0 & 0 & [0] & 0 \\ 0 & 0 & 0 & [-2] \end{bmatrix},$$

$$\begin{bmatrix} [0] & 0 & 0 & 0 \\ 0 & [-2] & 0 & 0 \\ 0 & 0 & [-2] & 0 \\ 0 & 0 & 0 & [0] \end{bmatrix}, \begin{bmatrix} [0] & 0 & 0 & 0 \\ 0 & [-2] & 0 & 0 \\ 0 & 0 & [0] & 0 \\ 0 & 0 & 0 & [-2] \end{bmatrix}, \begin{bmatrix} [0] & 0 & 0 & 0 \\ 0 & [0] & 0 & 0 \\ 0 & 0 & [-2] & 0 \\ 0 & 0 & 0 & [-2] \end{bmatrix},$$

which are similar to each other but to none of the others

(f) $x^4 - 3x^2 - 4 = \left(x^2 - 4\right)\left(x^2 + 1\right) = (x-2)(x+2)(x-i)(x+i)$;
There are 24 possible Jordan forms. Because each eigenvalue has algebraic multiplicity 1, all of the Jordan blocks have size 1×1. Hence, any Jordan canonical form matrix with these blocks is diagonal with the 4 eigenvalues 2, -2, i, and $-i$ on the main diagonal. The 24 possibilities result from all of the possible orders in which these 4 eigenvalues can appear on the diagonal. All 24 possible Jordan canonical form matrices are similar to each other because they all have the same twenty-four 1×1 Jordan blocks, only in a different order.

(6) (a) Consider the matrix

$$\mathbf{B} = \begin{bmatrix} -9 & 5 & 8 \\ -4 & 3 & 4 \\ -8 & 4 & 7 \end{bmatrix}.$$

We find a Jordan canonical form for $\mathbf{B}$. You can quickly calculate that $p_{\mathbf{B}}(x) = x^3 - x^2 - x + 1 = (x-1)^2(x+1)$. Thus, the eigenvalues for $\mathbf{B}$ are $\lambda_1 = 1$ and $\lambda_2 = -1$. We must find the sizes of the Jordan blocks corresponding to these eigenvalues and a fundamental sequence of generalized eigenvectors corresponding to each block. The following is one possible solution. (Other answers are possible if, for example, the eigenvalues are taken in a different order or if different generalized eigenvectors are chosen.)

The Cayley-Hamilton Theorem tells us that $p_{\mathbf{B}}(\mathbf{B}) = (\mathbf{B} - \mathbf{I}_3)^2 (\mathbf{B} + \mathbf{I}_3) = \mathbf{O}_3$.

Step A: We begin with the eigenvalue $\lambda_1 = 1$.

Step A1: Let

$$\mathbf{D} = (\mathbf{B} + \mathbf{I}_3) = \begin{bmatrix} -8 & 5 & 8 \\ -4 & 4 & 4 \\ -8 & 4 & 8 \end{bmatrix}.$$

Then $(\mathbf{B} - \mathbf{I}_3)^2 \mathbf{D} = \mathbf{O}_3$.

Step A2: Next, we search for the *smallest* positive integer k such that $(\mathbf{B} - \mathbf{I}_3)^k \mathbf{D} = \mathbf{O}_3$. Now,

$$\mathbf{B} - \mathbf{I}_3 = \begin{bmatrix} -10 & 5 & 8 \\ -4 & 2 & 4 \\ -8 & 4 & 6 \end{bmatrix},$$

and so,

$$(\mathbf{B} - \mathbf{I}_3)\,\mathbf{D} = \begin{bmatrix} -4 & 2 & 4 \\ -8 & 4 & 8 \\ 0 & 0 & 0 \end{bmatrix} \neq \mathbf{O}_3,$$

while, as we have seen, $(\mathbf{B} - \mathbf{I}_3)^2 \mathbf{D} = \mathbf{O}_3$. Hence, $k = 2$.

Step A3: We choose a maximal linearly independent subset of the columns of $(\mathbf{B} - \mathbf{I}_3)^{k-1} \mathbf{D} =$

$(\mathbf{B}-\mathbf{I}_3)\,\mathbf{D}$ to get as many linearly independent generalized eigenvectors as possible. Since all of the columns are multiples of the first, the first column alone suffices. Thus, $\mathbf{v}_{11}=[-4,-8,0]$. We decide whether to simplify $\mathbf{v}_{11}$ (by multiplying every entry by $-\frac{1}{4}$) after $\mathbf{v}_{12}$ is determined.

Step A4: Next, we work backwards through the products of the form $(\mathbf{B}-\mathbf{I}_3)^{k-j}\,\mathbf{D}$ for j running from 2 up to k, choosing the same column in which we found the generalized eigenvector $\mathbf{v}_{11}$. Because $k=2$, the only value we need to consider here is $j=2$. Hence, we let $\mathbf{v}_{12}=[-8,-4,-8]$, the first column of $(\mathbf{B}-\mathbf{I}_3)^{(2-2)}\,\mathbf{D}=\mathbf{D}$. Since the entries of $\mathbf{v}_{12}$ are exactly divisible by -4, we simplify the entries of both $\mathbf{v}_{11}$ and $\mathbf{v}_{12}$ by multiplying by $-\frac{1}{4}$. Hence, we will use $\mathbf{v}_{11}=[1,2,0]$ and $\mathbf{v}_{12}=[2,1,2]$.

Steps A5 and A6: Now by construction, $(\mathbf{B}-\mathbf{I}_3)\,\mathbf{v}_{12}=\mathbf{v}_{11}$ and $(\mathbf{B}-\mathbf{I}_3)\,\mathbf{v}_{11}=\mathbf{0}$. Since the total number of generalized eigenvectors we have found for λ_1 equals the algebraic multiplicity of λ_1, which is 2, we can stop our work for λ_1. We therefore have a fundamental sequence $\{\mathbf{v}_{11},\mathbf{v}_{12}\}$ of generalized eigenvectors corresponding to a 2×2 Jordan block associated with $\lambda_1=1$ in a Jordan canonical form for $\mathbf{B}$.

To complete this example, we still must find a fundamental sequence of generalized eigenvectors corresponding to $\lambda_2=-1$. We repeat Step A for this eigenvalue.

Step A1: Let

$$\mathbf{D}=(\mathbf{B}-\mathbf{I}_3)^2=\begin{bmatrix}16&-8&-12\\0&0&0\\16&-8&-12\end{bmatrix}.$$

Then $(\mathbf{B}+\mathbf{I}_3)\,\mathbf{D}=\mathbf{O}_3$.

Step A2: Next, we search for the *smallest* positive integer k such that $(\mathbf{B}+\mathbf{I}_3)^k\,\mathbf{D}=\mathbf{O}_3$. However, it is obvious here that $k=1$.

Step A3: Since $k-1=0$, $(\mathbf{B}+\mathbf{I}_3)^{k-1}\,\mathbf{D}=\mathbf{D}$. Hence, each nonzero column of $\mathbf{D}=(\mathbf{B}-\mathbf{I}_3)^2$ is a generalized eigenvector for $\mathbf{B}$ corresponding to $\lambda_2=-1$. In particular, the first column of $(\mathbf{B}-\mathbf{I}_3)^2$ serves nicely as a generalized eigenvector $\mathbf{v}_{21}$ for $\mathbf{B}$ corresponding to $\lambda_2=-1$. The other columns of $(\mathbf{B}-\mathbf{I}_3)^2$ are scalar multiples of the first column.

Steps A4, A5, and A6: No further work for λ_2 is needed here because $\lambda_2=-1$ has algebraic multiplicity 1, and hence only one generalized eigenvector corresponding to λ_2 is required. However, we can simplify $\mathbf{v}_{21}$ by multiplying the first column of $(\mathbf{B}-\mathbf{I}_3)^2$ by $\frac{1}{16}$, yielding $\mathbf{v}_{21}=[1,0,1]$. Thus, $\{\mathbf{v}_{21}\}$ is a fundamental sequence of generalized eigenvectors corresponding to the 1×1 Jordan block associated with $\lambda_2=-1$ in a Jordan canonical form for $\mathbf{B}$.

Thus, we have completed Step A for both eigenvalues.

Step B: Finally, we now have an ordered basis $(\mathbf{v}_{11},\mathbf{v}_{12},\mathbf{v}_{21})$ comprised of fundamental sequences of generalized eigenvectors for $\mathbf{B}$. Letting $\mathbf{P}$ be the matrix whose columns are these basis vectors, we find that

$$\begin{aligned}\mathbf{A} &= \mathbf{P}^{-1}\mathbf{B}\mathbf{P}=\begin{bmatrix}1&0&-1\\-2&1&2\\4&-2&-3\end{bmatrix}\begin{bmatrix}-9&5&8\\-4&3&4\\-8&4&7\end{bmatrix}\begin{bmatrix}1&2&1\\2&1&0\\0&2&1\end{bmatrix}\\ &= \begin{bmatrix}\begin{bmatrix}1&1\\0&1\end{bmatrix}&\begin{matrix}0\\0\end{matrix}\\\begin{matrix}0&0\end{matrix}&[-1]\end{bmatrix},\end{aligned}$$

which gives a Jordan canonical form for $\mathbf{B}$. Note that we could have used the generalized eigenvectors in the order $\mathbf{v}_{21},\mathbf{v}_{11},\mathbf{v}_{12}$ instead, which would have resulted in the other possible Jordan canonical form for $\mathbf{B}$,

$$\begin{bmatrix} [-1] & 0 & 0 \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$}} \\ 0 & & \end{bmatrix}.$$

(d) Consider the matrix

$$\mathbf{B} = \begin{bmatrix} 8 & 5 & 0 \\ -5 & -3 & -1 \\ 10 & 7 & 0 \end{bmatrix}.$$

We find a Jordan canonical form for $\mathbf{B}$. You can quickly calculate that $p_{\mathbf{B}}(x) = x^3 - 5x^2 + 8x - 6 = (x-3)(x-(1-i))(x-(1+i))$. Thus, the eigenvalues for $\mathbf{B}$ are $\lambda_1 = 3$, $\lambda_2 = 1-i$, and $\lambda_3 = 1+i$. Because each eigenvalue has geometric multiplicity 1, each of the three Jordan blocks will be 1×1 blocks. Hence, the Jordan canonical form matrix will be diagonal. We could just solve this problem using our method for diagonalization from Section 5.6 of the textbook. However, we shall use the Method of this section here. In what follows, we consider the eigenvalues in the order given above. (Other answers are possible if, for example, the eigenvalues are taken in a different order or if different eigenvectors are chosen.)

The Cayley-Hamilton Theorem tells us that

$$p_{\mathbf{B}}(\mathbf{B}) = (\mathbf{B} - 3\mathbf{I}_3)(\mathbf{B} - (1-i)\mathbf{I}_3)(\mathbf{B} - (1+i)\mathbf{I}_3) = \mathbf{O}_3.$$

Step A: We begin with the eigenvalue $\lambda_1 = 3$.
Step A1: Let

$$\mathbf{D} = (\mathbf{B} - (1-i)\mathbf{I}_3)(\mathbf{B} - (1+i)\mathbf{I}_3) = \begin{bmatrix} 25 & 15 & -5 \\ -25 & -15 & 5 \\ 25 & 15 & -5 \end{bmatrix}.$$

Then $(\mathbf{B} - 3\mathbf{I}_3)\mathbf{D} = \mathbf{O}_3$.
Step A2: Next, we search for the *smallest* positive integer k such that $(\mathbf{B} - 3\mathbf{I}_3)^k \mathbf{D} = \mathbf{O}_3$. Clearly, $k = 1$.
Step A3: We choose a maximal linearly independent subset of the columns of $(\mathbf{B} - 3\mathbf{I}_3)^{k-1}\mathbf{D} = \mathbf{D}$ to get as many linearly independent eigenvectors as possible. Since all of the columns are multiples of the first, the first column alone suffices. We can multiply this column by $\frac{1}{25}$, obtaining $\mathbf{v}_{11} = [1, -1, 1]$.
Steps A4, A5, and A6: No further work for λ_1 is needed here because $\lambda_1 = 3$ has algebraic multiplicity 1, and hence only one generalized eigenvector corresponding to λ_1 is required. Thus, $\{\mathbf{v}_{11}\}$ is a fundamental sequence of generalized eigenvectors corresponding to the 1×1 Jordan block associated with $\lambda_1 = 3$ in a Jordan canonical form for $\mathbf{B}$.

Thus, we have completed Step A for this eigenvalue. To complete this example, we still must find a fundamental sequences of generalized eigenvectors corresponding to λ_2 and λ_3. We now repeat Step A for $\lambda_2 = 1 - i$.
Step A1: Let

$$\mathbf{D} = (\mathbf{B} - 3\mathbf{I}_3)(\mathbf{B} - (1+i)\mathbf{I}_3) = \begin{bmatrix} 10-5i & 5-5i & -5 \\ -15+5i & -8+6i & 7+i \\ 5-10i & 1-7i & -4+3i \end{bmatrix}.$$

Then $(\mathbf{B} - (1-i)\mathbf{I}_3)\mathbf{D} = \mathbf{O}_3$.
Step A2: Next, we search for the *smallest* positive integer k such that $(\mathbf{B} - (1-i)\mathbf{I}_3)^k \mathbf{D} = \mathbf{O}_3$.

Clearly, $k = 1$.
Step A3: We choose a maximal linearly independent subset of the columns of $(\mathbf{B} - 3\mathbf{I}_3)^{k-1}\mathbf{D} = \mathbf{D}$ to get as many linearly independent eigenvectors as possible. Since all of the columns are multiples of the first, the first column alone suffices. (The second column is $(\frac{3}{5} - \frac{1}{5}i)$ times the first column, and the third column is $\left(-\frac{2}{5} - \frac{1}{5}i\right)$ times the first column. We can discover these scalars by dividing the first entry in each column by the first entry in the first column, and then verify these scalars are correct for the remaining entries.) We can multiply this column by $\frac{1}{5}$, obtaining $\mathbf{v}_{21} = [2 - i, -3 + i, 1 - 2i]$.
Steps A4, A5, and A6: No further work for λ_2 is needed here because $\lambda_2 = 1 - i$ has algebraic multiplicity 1, and hence only one generalized eigenvector corresponding to λ_2 is required. Thus, $\{\mathbf{v}_{21}\}$ is a fundamental sequence of generalized eigenvectors corresponding to the 1×1 Jordan block associated with $\lambda_2 = 1 - i$ in a Jordan canonical form for $\mathbf{B}$.

Thus, we have completed Step A for this eigenvalue. To complete this example, we still must find a fundamental sequence of generalized eigenvectors corresponding to λ_3. We now repeat Step A for $\lambda_3 = 1 + i$.
Step A1: Let

$$\mathbf{D} = (\mathbf{B} - 3\mathbf{I}_3)(\mathbf{B} - (1 - i)\mathbf{I}_3) = \begin{bmatrix} 10 + 5i & 5 + 5i & -5 \\ -15 - 5i & -8 - 6i & 7 - i \\ 5 + 10i & 1 + 7i & -4 - 3i \end{bmatrix}.$$

Then $(\mathbf{B} - (1 + i)\mathbf{I}_3)\mathbf{D} = \mathbf{O}_3$.
Step A2: Next, we search for the *smallest* positive integer k such that $(\mathbf{B} - (1 + i)\mathbf{I}_3)^k\mathbf{D} = \mathbf{O}_3$. Clearly, $k = 1$.
Step A3: We choose a maximal linearly independent subset of the columns of $(\mathbf{B} - (1 + i)\mathbf{I}_3)^{k-1}\mathbf{D} = \mathbf{D}$ to get as many linearly independent eigenvectors as possible. Since all of the columns are multiples of the first, the first column alone suffices. (The second column is $(\frac{3}{5} + \frac{1}{5}i)$ times the first column, and the third column is $\left(-\frac{2}{5} + \frac{1}{5}i\right)$ times the first column.) We can multiply this column by $\frac{1}{5}$, obtaining $\mathbf{v}_{31} = [2 + i, -3 - i, 1 + 2i]$.
Steps A4, A5, and A6: No further work for λ_3 is needed here because $\lambda_3 = 1 + i$ has algebraic multiplicity 1, and hence only one generalized eigenvector corresponding to λ_3 is required. Thus, $\{\mathbf{v}_{31}\}$ is a fundamental sequence of generalized eigenvectors corresponding to the 1×1 Jordan block associated with $\lambda_3 = 1 + i$ in a Jordan canonical form for $\mathbf{B}$.

Thus, we have completed Step A for all three eigenvalues.
Step B: Finally, we now have an ordered basis $(\mathbf{v}_{11}, \mathbf{v}_{21}, \mathbf{v}_{31})$ comprised of fundamental sequences of generalized eigenvectors for $\mathbf{B}$. Letting $\mathbf{P}$ be the matrix whose columns are these basis vectors, we find that

$$\begin{aligned} \mathbf{A} &= \mathbf{P}^{-1}\mathbf{B}\mathbf{P} = \frac{1}{2}\begin{bmatrix} 10 & 6 & -2 \\ -1 - 2i & -1 - i & i \\ -1 + 2i & -1 + i & -i \end{bmatrix}\begin{bmatrix} 8 & 5 & 0 \\ -5 & -3 & -1 \\ 10 & 7 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 - i & 2 + i \\ -1 & -3 + i & -3 - i \\ 1 & 1 - 2i & 1 + 2i \end{bmatrix} \\ &= \begin{bmatrix} [3] & 0 & 0 \\ 0 & [1 - i] & 0 \\ 0 & 0 & [1 + i] \end{bmatrix}, \end{aligned}$$

which gives a Jordan canonical form for $\mathbf{B}$.

(g) We are given that $p_{\mathbf{B}}(x) = x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1 = (x + 1)^5$. Therefore, the only eigenvalue for $\mathbf{B}$ is $\lambda = -1$. (Other answers are possible if, for example, different generalized eigenvectors are chosen or they are used in a different order.)

Step A: We begin by finding generalized eigenvectors for $\lambda = -1$.

Step A1: $p_{\mathbf{B}}(\mathbf{B}) = (\mathbf{B} + \mathbf{I}_5)^5\,\mathbf{I}_5 = \mathbf{O}_5$. We let $\mathbf{D} = \mathbf{I}_5$.

Step A2: We calculate as follows:

$$(\mathbf{B} + \mathbf{I}_5)\,\mathbf{D} = (\mathbf{B} + \mathbf{I}_5) = \begin{bmatrix} -2 & -2 & -1 & 5 & 3 \\ -8 & 3 & 7 & -2 & 1 \\ 0 & -3 & -3 & 6 & 3 \\ -4 & 1 & 3 & 0 & 1 \\ 0 & -2 & -2 & 4 & 2 \end{bmatrix}$$

and $(\mathbf{B} + \mathbf{I}_5)^2\,\mathbf{D} = (\mathbf{B} + \mathbf{I}_5)^2 = \mathbf{O}_5$. Hence $k = 2$.

Step A3: Each nonzero column of $(\mathbf{B} + \mathbf{I}_5)\,\mathbf{D}$ is a generalized eigenvector for $\mathbf{B}$ corresponding to $\lambda = -1$. We let $\mathbf{v}_{11} = [-2, -8, 0, -4, 0]$, the first column of $(\mathbf{B} + \mathbf{I}_5)\,\mathbf{D}$. (We do not divide by 2 here since not all of the entries of the vector $\mathbf{v}_{12}$ calculated in the next step are divisible by 2.) The second column of $(\mathbf{B} + \mathbf{I}_5)\,\mathbf{D}$ is not a scalar multiple of $\mathbf{v}_{11}$, so we let $\mathbf{v}_{21} = [-2, 3, -3, 1, -2]$, the second column of $(\mathbf{B} + \mathbf{I}_5)\,\mathbf{D}$. Row reduction shows that the remaining columns of $(\mathbf{B} + \mathbf{I}_5)\,\mathbf{D}$ are linear combinations of the first two (see the adjustment step, below).

Step A4: Next, we let $\mathbf{v}_{12} = [1, 0, 0, 0, 0]$ and $\mathbf{v}_{22} = [0, 1, 0, 0, 0]$, the first and second columns of $\mathbf{D}$ $(= \mathbf{I}_5)$, respectively. Thus, $(\mathbf{B} + \mathbf{I}_5)\,\mathbf{v}_{12} = \mathbf{v}_{11}$, and $(\mathbf{B} + \mathbf{I}_5)\,\mathbf{v}_{22} = \mathbf{v}_{21}$. This gives us our first two fundamental sequences of generalized eigenvectors, $\{\mathbf{v}_{11}, \mathbf{v}_{12}\}$ and $\{\mathbf{v}_{21}, \mathbf{v}_{22}\}$ corresponding to two 2×2 Jordan blocks for $\lambda = -1$.

Steps A5 and A6: We have two fundamental sequence of generalized eigenvectors consisting of a total of 4 vectors. But, since the algebraic multiplicity of λ is 5, we must still find another generalized eigenvector. We do this by making an adjustment to the matrix $\mathbf{D}$.

Recall that each column of $(\mathbf{B} + \mathbf{I}_5)\,\mathbf{D}$ is a linear combination of $\mathbf{v}_{11}$ and $\mathbf{v}_{21}$. In fact, the row reduction we did in Step A3 shows that

$$\begin{array}{rlcrcrcrcr} & 1^{st} \text{ column of } (\mathbf{B} + \mathbf{I}_5)\,\mathbf{D} & = & 1\mathbf{v}_{11} & + & 0\mathbf{v}_{21} & = & f_1\mathbf{v}_{11} & + & g_1\mathbf{v}_{21} \\ & 2^{nd} \text{ column of } (\mathbf{B} + \mathbf{I}_5)\,\mathbf{D} & = & 0\mathbf{v}_{11} & + & 1\mathbf{v}_{21} & = & f_2\mathbf{v}_{11} & + & g_2\mathbf{v}_{21} \\ & 3^{rd} \text{ column of } (\mathbf{B} + \mathbf{I}_5)\,\mathbf{D} & = & -\frac{1}{2}\mathbf{v}_{11} & + & 1\mathbf{v}_{21} & = & f_3\mathbf{v}_{11} & + & g_3\mathbf{v}_{21} \\ & 4^{th} \text{ column of } (\mathbf{B} + \mathbf{I}_5)\,\mathbf{D} & = & -\frac{1}{2}\mathbf{v}_{11} & + & (-2)\mathbf{v}_{21} & = & f_4\mathbf{v}_{11} & + & g_4\mathbf{v}_{21} \\ \text{and} & 5^{th} \text{ column of } (\mathbf{B} + \mathbf{I}_5)\,\mathbf{D} & = & -\frac{1}{2}\mathbf{v}_{11} & + & (-1)\mathbf{v}_{21} & = & f_5\mathbf{v}_{11} & + & g_5\mathbf{v}_{21} \end{array}$$

where the f_i's and g_i's represent the respective coefficients of $\mathbf{v}_{11}$ and $\mathbf{v}_{21}$ for each column of $(\mathbf{B} + \mathbf{I}_5)\,\mathbf{D}$. We create a new matrix $\mathbf{H}$ whose i^{th} column is $f_i\mathbf{v}_{12} + g_i\mathbf{v}_{22}$. Thus,

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then, since $(\mathbf{B} + \mathbf{I}_5)\mathbf{v}_{12} = \mathbf{v}_{11}$ and $(\mathbf{B} + \mathbf{I}_5)\mathbf{v}_{22} = \mathbf{v}_{21}$, we get that $(\mathbf{B} + \mathbf{I}_5)\,\mathbf{H} = (\mathbf{B} + \mathbf{I}_5)\,\mathbf{D}$. Let $\mathbf{D}_1 = \mathbf{D} - \mathbf{H} = \mathbf{I}_5 - \mathbf{H}$. Clearly, $(\mathbf{B} + \mathbf{I}_5)\,\mathbf{D}_1 = \mathbf{O}_5$. We now revisit Steps A2 through A6 using the matrix $\mathbf{D}_1$ instead of $\mathbf{D}$. The purpose of this adjustment to the matrix $\mathbf{D}$ is to attempt to eliminate the effects of the two fundamental sequences of length 2, thus unmasking shorter fundamental sequences.

Step A2: We have

$$\mathbf{D}_1 = \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -1 & 2 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and $(\mathbf{B}+\mathbf{I}_5)\,\mathbf{D}_1 = \mathbf{O}_5$. Hence $k = 1$.

Step A3: We look for new generalized eigenvectors among the columns of $\mathbf{D}_1$. We must choose columns of $\mathbf{D}_1$ that are not only linearly independent of each other, but also of our previously computed generalized eigenvectors (at the first level). We check this using the Independence Test Method by row reducing

$$\left[\begin{array}{rr|rrr} -2 & -2 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -8 & 3 & -1 & 2 & 1 \\ 0 & -3 & 1 & 0 & 0 \\ -4 & 1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 & 1 \end{array}\right],$$

where the first two columns are $\mathbf{v}_{11}$ and $\mathbf{v}_{21}$, and the other columns are the nonzero columns of $\mathbf{D}_1$. The resulting reduced row echelon form matrix is

$$\left[\begin{array}{rr|rrr} 1 & 0 & 0 & -\frac{1}{4} & -\frac{1}{8} \\ 0 & 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & -\frac{3}{2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Hence, the third column of $\mathbf{D}_1$ is linearly independent of $\mathbf{v}_{11}$ and $\mathbf{v}_{21}$, and the remaining columns of $\mathbf{D}_1$ are linear combinations of $\mathbf{v}_{11}$, $\mathbf{v}_{21}$, and the third column of $\mathbf{D}_1$. Therefore, we let $\mathbf{v}_{31} = [1, -2, 2, 0, 0]$, which is 2 times the 3^{rd} column of $\mathbf{D}_1$.

Steps A4, A5, and A6: Because $k = 1$, the generalized eigenvector $\mathbf{v}_{31}$ for $\lambda = -1$ corresponds to a 1×1 Jordan block. This gives the sequence $\{\mathbf{v}_{31}\}$, and we do not need to find more vectors for this sequence.

We now have the following three fundamental sequences of generalized eigenvectors for $\lambda = -1$: $\{\mathbf{v}_{11}, \mathbf{v}_{12}\}$,$\{\mathbf{v}_{21}, \mathbf{v}_{22}\}$ and $\{\mathbf{v}_{31}\}$. Since we have now found 5 generalized eigenvectors for $\mathbf{B}$ corresponding to $\lambda = -1$, and since the algebraic multiplicity of λ is 5, we are finished with Step A.

Step B: We now have the ordered basis $(\mathbf{v}_{11}, \mathbf{v}_{12}, \mathbf{v}_{21}, \mathbf{v}_{22}, \mathbf{v}_{31})$ for $\mathbb{C}^5$ consisting of 3 sequences of generalized eigenvectors for $\mathbf{B}$. If we let

$$\mathbf{P} = \begin{bmatrix} -2 & 1 & -2 & 0 & 1 \\ -8 & 0 & 3 & 1 & -2 \\ 0 & 0 & -3 & 0 & 2 \\ -4 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \end{bmatrix},$$

the matrix whose columns are the vectors in this ordered basis. Using this, we obtain the following

Jordan canonical form for $\mathbf{B}$:

$$\begin{aligned}
\mathbf{A} &= \mathbf{P}^{-1}\mathbf{B}\mathbf{P} \\
&= \frac{1}{8}\begin{bmatrix} 0 & 0 & 0 & -2 & -1 \\ 8 & 0 & -4 & -4 & -4 \\ 0 & 0 & 0 & 0 & -4 \\ 0 & 8 & 8 & -16 & -8 \\ 0 & 0 & 4 & 0 & -6 \end{bmatrix}\begin{bmatrix} -3 & -2 & -1 & 5 & 3 \\ -8 & 2 & 7 & -2 & 1 \\ 0 & -3 & -4 & 6 & 3 \\ -4 & 1 & 3 & -1 & 1 \\ 0 & -2 & -2 & 4 & 1 \end{bmatrix}\begin{bmatrix} -2 & 1 & -2 & 0 & 1 \\ -8 & 0 & 3 & 1 & -2 \\ 0 & 0 & -3 & 0 & 2 \\ -4 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \end{matrix} & [-1] \end{bmatrix}.
\end{aligned}$$

(7) (b) Consider the matrix

$$\mathbf{B} = \begin{bmatrix} 4 & -4 & -7 \\ 3 & -5 & -13 \\ -1 & 3 & 8 \end{bmatrix}.$$

We find a Jordan canonical form for $\mathbf{B}$. You can quickly calculate that $p_{\mathbf{B}}(x) = x^3 - 7x^2 + 16x - 12 = (x-2)^2(x-3)$. Thus, the eigenvalues for $\mathbf{B}$ are $\lambda_1 = 2$ and $\lambda_2 = 3$. We must find the sizes of the Jordan blocks corresponding to these eigenvalues and a fundamental sequence of generalized eigenvectors corresponding to each block. Now, the Cayley-Hamilton Theorem tells us that $p_{\mathbf{B}}(\mathbf{B}) = (\mathbf{B} - 2\mathbf{I}_3)^2(\mathbf{B} - 3\mathbf{I}_3) = \mathbf{O}_3$.

Step A: We begin with the eigenvalue $\lambda_1 = 2$.

Step A1: Let

$$\mathbf{D} = (\mathbf{B} - 3\mathbf{I}_3) = \begin{bmatrix} 1 & -4 & -7 \\ 3 & -8 & -13 \\ -1 & 3 & 5 \end{bmatrix}.$$

Then $(\mathbf{B} - 2\mathbf{I}_3)^2\mathbf{D} = \mathbf{O}_3$.

Step A2: Next, we search for the *smallest* positive integer k such that $(\mathbf{B} - 2\mathbf{I}_3)^k\mathbf{D} = \mathbf{O}_3$. Now,

$$(\mathbf{B} - 2\mathbf{I}_3)\mathbf{D} = \begin{bmatrix} -3 & 3 & 3 \\ -5 & 5 & 5 \\ 2 & -2 & -2 \end{bmatrix} \neq \mathbf{O}_3,$$

while, as we have seen, $(\mathbf{B} - 2\mathbf{I}_3)^2\mathbf{D} = \mathbf{O}_3$. Hence, $k = 2$.

Step A3: We choose a maximal linearly independent subset of the columns of $(\mathbf{B} - 2\mathbf{I}_3)^{k-1}\mathbf{D} = (\mathbf{B} - 2\mathbf{I}_3)\mathbf{D}$ to get as many linearly independent generalized eigenvectors as possible. Since all of the columns are multiples of the first, the first column alone suffices. Thus, $\mathbf{u}_{11} = [-3, -5, 2]$.

Step A4: Next, we work backwards through the products of the form $(\mathbf{B} - 2\mathbf{I}_3)^{k-j}\mathbf{D}$ for j running from 2 up to k, choosing the same column in which we found the generalized eigenvector $\mathbf{u}_{11}$. Because $k = 2$, the only value we need to consider here is $j = 2$. Hence, we let $\mathbf{u}_{12} = [1, 3, -1]$, the first column of $(\mathbf{B} - 2\mathbf{I}_3)^{(2-2)}\mathbf{D} = \mathbf{D}$.

Steps A5 and A6: Now by construction, $(\mathbf{B} - 2\mathbf{I}_3)\mathbf{u}_{12} = \mathbf{u}_{11}$ and $(\mathbf{B} - 2\mathbf{I}_3)\mathbf{u}_{11} = \mathbf{0}$. Hence, $\{\mathbf{u}_{11}, \mathbf{u}_{12}\}$ forms a fundamental sequence of generalized eigenvectors for $\lambda_1 = 2$. Since the total number of generalized eigenvectors we have found for λ_1 equals the algebraic multiplicity of λ_1, which is 2, we can stop our work for λ_1. We therefore have a fundamental sequence $\{\mathbf{u}_{11}, \mathbf{u}_{12}\}$ of generalized eigenvectors corresponding to a 2×2 Jordan block associated with $\lambda_1 = 2$ in a Jordan canonical form for $\mathbf{B}$.

To complete this example, we still must find a fundamental sequence of generalized eigenvectors corresponding to $\lambda_2 = 3$. We repeat Step A for this eigenvalue.

Step A1: Let

$$\mathbf{D} = (\mathbf{B} - 2\mathbf{I}_3)^2 = \begin{bmatrix} -1 & -1 & -4 \\ -2 & -2 & -8 \\ 1 & 1 & 4 \end{bmatrix}.$$

Then $(\mathbf{B} - 3\mathbf{I}_3)\,\mathbf{D} = \mathbf{O}_3$.

Step A2: Next, we search for the *smallest* positive integer k such that $(\mathbf{B} - 3\mathbf{I}_3)^k\,\mathbf{D} = \mathbf{O}_3$. However, it is obvious here that $k = 1$.

Step A3: Since $k - 1 = 0$, $(\mathbf{B} - 3\mathbf{I}_3)^{k-1}\,\mathbf{D} = \mathbf{D}$. Hence, each nonzero column of $\mathbf{D} = (\mathbf{B} - 2\mathbf{I}_3)^2$ is a generalized eigenvector for $\mathbf{B}$ corresponding to $\lambda_2 = 3$. In particular, the first column of $(\mathbf{B} - 2\mathbf{I}_3)^2$ serves nicely as a generalized eigenvector $\mathbf{u}_{21} = [-1, -2, 1]$ for $\mathbf{B}$ corresponding to $\lambda_2 = 3$. The other columns of $(\mathbf{B} - 3\mathbf{I}_3)^2$ are scalar multiples of the first column.

Steps A4, A5 and A6: No further work for λ_2 is needed here because $\lambda_2 = 3$ has algebraic multiplicity 1, and hence only one generalized eigenvector corresponding to λ_2 is required. Thus, $\{\mathbf{u}_{21}\}$ is a fundamental sequence of generalized eigenvectors corresponding to the 1×1 Jordan block associated with $\lambda_2 = 3$ in a Jordan canonical form for $\mathbf{B}$.

Thus, we have completed Step A for both eigenvalues.

Step B: Finally, we now have an ordered basis $(\mathbf{u}_{11}, \mathbf{u}_{12}, \mathbf{u}_{21})$ comprised of fundamental sequences of generalized eigenvectors for $\mathbf{B}$. Letting $\mathbf{Q}$ be the matrix whose columns are these basis vectors, we find that

$$\begin{aligned} \mathbf{A} &= \mathbf{Q}^{-1}\mathbf{B}\mathbf{Q} = \begin{bmatrix} -1 & 0 & -1 \\ -1 & 1 & 1 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 4 & -4 & -7 \\ 3 & -5 & -13 \\ -1 & 3 & 8 \end{bmatrix} \begin{bmatrix} -3 & 1 & -1 \\ -5 & 3 & -2 \\ 2 & -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & [3] \end{bmatrix}, \end{aligned}$$

which gives a Jordan canonical form for $\mathbf{B}$.

(10) (a) $(\mathbf{A} + a\mathbf{I}_n)(\mathbf{A} + b\mathbf{I}_n) = (\mathbf{A} + a\mathbf{I}_n)\mathbf{A} + (\mathbf{A} + a\mathbf{I}_n)(b\mathbf{I}_n)$
$= \mathbf{A}^2 + a\mathbf{I}_n\mathbf{A} + \mathbf{A}(b\mathbf{I}_n) + (a\mathbf{I}_n)(b\mathbf{I}_n)$
$= \mathbf{A}^2 + a\mathbf{A} + b\mathbf{A} + ab\mathbf{I}_n = \mathbf{A}^2 + b\mathbf{A} + a\mathbf{A} + ba\mathbf{I}_n$
$= \mathbf{A}^2 + b\mathbf{I}_n\mathbf{A} + \mathbf{A}(a\mathbf{I}_n) + (b\mathbf{I}_n)(a\mathbf{I}_n)$
$= (\mathbf{A} + b\mathbf{I}_n)\mathbf{A} + (\mathbf{A} + b\mathbf{I}_n)(a\mathbf{I}_n) = (\mathbf{A} + b\mathbf{I}_n)(\mathbf{A} + a\mathbf{I}_n)$.

(15) Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{P}$, and $q(x)$ be as given in the statement of the problem. We proceed using induction on k, the degree of the polynomial q.

Base Step: Suppose $k = 0$. Then $q(x) = c$ for some $c \in \mathbb{C}$. Hence, $q(\mathbf{A}) = c\mathbf{I}_n = q(\mathbf{B})$. Therefore, $\mathbf{P}^{-1}q(\mathbf{A})\mathbf{P} = \mathbf{P}^{-1}(c\mathbf{I}_n)\mathbf{P} = c\mathbf{P}^{-1}\mathbf{I}_n\mathbf{P} = c\mathbf{P}^{-1}\mathbf{P} = c\mathbf{I}_n = q(\mathbf{B})$.

Inductive Step: Suppose that $s(\mathbf{B}) = \mathbf{P}^{-1}s(\mathbf{A})\mathbf{P}$ for every k^{th} degree polynomial $s(x)$. We need to prove that $q(\mathbf{B}) = \mathbf{P}^{-1}q(\mathbf{A})\mathbf{P}$, where $q(x)$ has degree $k + 1$.

Suppose that $q(x) = a_{k+1}x^{k+1} + a_kx^k + \cdots + a_1x + a_0$. Then $q(\mathbf{A}) = a_{k+1}\mathbf{A}^{k+1} + a_k\mathbf{A}^k + \cdots + a_1\mathbf{A} + a_0\mathbf{I}_n = (a_{k+1}\mathbf{A}^k + a_k\mathbf{A}^{k-1} + \cdots + a_1\mathbf{I}_n)\mathbf{A} + a_0\mathbf{I}_n = s(\mathbf{A})\mathbf{A} + a_0\mathbf{I}_n$, where $s(x) = a_{k+1}x^k + a_kx^{k-1} + \cdots + a_1$. Similarly, $q(\mathbf{B}) = s(\mathbf{B})\mathbf{B} + a_0\mathbf{I}_n$ for the same polynomial $s(x)$. Therefore,

$\mathbf{P}^{-1}q(\mathbf{A})\mathbf{P} = \mathbf{P}^{-1}\left(s(\mathbf{A})\mathbf{A} + a_0\mathbf{I}_n\right)\mathbf{P} = \left(\mathbf{P}^{-1}s(\mathbf{A})\mathbf{A} + \mathbf{P}^{-1}\left(a_0\mathbf{I}_n\right)\right)\mathbf{P} = \mathbf{P}^{-1}s(\mathbf{A})\mathbf{A}\mathbf{P} + \mathbf{P}^{-1}\left(a_0\mathbf{I}_n\right)\mathbf{P}$ $= \mathbf{P}^{-1}s(\mathbf{A})\left(\mathbf{P}\mathbf{P}^{-1}\right)\mathbf{A}\mathbf{P} + a_0\mathbf{P}^{-1}\mathbf{I}_n\mathbf{P} = \left(\mathbf{P}^{-1}s(\mathbf{A})\mathbf{P}\right)\left(\mathbf{P}^{-1}\mathbf{A}\mathbf{P}\right) + a_0\mathbf{P}^{-1}\mathbf{P} = s(\mathbf{B})\left(\mathbf{P}^{-1}\mathbf{A}\mathbf{P}\right) + a_0\mathbf{P}^{-1}\mathbf{P}$ (by the inductive hypothesis) $= s(\mathbf{B})\mathbf{B} + a_0\mathbf{I}_n = q(\mathbf{B})$.

(16) (a) We compute the (i,j) entry of $\mathbf{A}^2$. First consider the case in which $i \leq k$ and $j \leq k$. The (i,j) entry of $\mathbf{A}^2 = a_{i1}a_{1j} + \cdots + a_{ik}a_{kj} + a_{i(k+1)}a_{(k+1)j} + \cdots + a_{i(k+m)}a_{(k+m)j} = \left((i,j) \text{ entry of } \mathbf{A}_{11}^2\right) + \left((i,j) \text{ entry of } \mathbf{A}_{12}\mathbf{A}_{21}\right)$.

If $i > k$ and $j \leq k$, then the (i,j) entry of $\mathbf{A}^2 = a_{i1}a_{1j} + \cdots + a_{ik}a_{kj} + a_{i(k+1)}a_{(k+1)j} + \cdots + a_{i(k+m)}a_{(k+m)j} = ((i,j) \text{ entry of } \mathbf{A}_{21}\mathbf{A}_{11}) + ((i,j) \text{ entry of } \mathbf{A}_{22}\mathbf{A}_{21})$.

The other two cases are handled similarly.

(19) (g) The number of Jordan blocks having size *exactly* $k \times k$ is the number having size at least $k \times k$ minus the number of size at least $(k+1) \times (k+1)$. By part (f), $r_{k-1}(\mathbf{A}) - r_k(\mathbf{A})$ gives the total number of Jordan blocks having size at least $k \times k$ corresponding to λ. Similarly, the number of Jordan blocks having size at least $(k+1) \times (k+1)$ is $r_k(\mathbf{A}) - r_{k+1}(\mathbf{A})$. Hence the number of Jordan blocks having size exactly $k \times k$ equals $(r_{k-1}(\mathbf{A}) - r_k(\mathbf{A})) - (r_k(\mathbf{A}) - r_{k+1}(\mathbf{A}))$ $= r_{k-1}(\mathbf{A}) - 2r_k(\mathbf{A}) + r_{k+1}(\mathbf{A})$.

(20) (a) False. For example, any matrix in Jordan canonical form that has more than one Jordan block can not be similar to a matrix with a single Jordan block. This follows from the uniqueness statement in Theorem 1. The matrix $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ is a specific example of such a matrix.

(b) True. This is stated in the first paragraph of the subsection entitled "Finding A Jordan Canonical Form."

(c) False. The geometric multiplicity of an eigenvalue equals the dimension of the eigenspace corresponding to that eigenvalue, *not* the dimension of the generalized eigenspace. For a specific example, consider the matrix $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, whose only eigenvalue is $\lambda = 0$. The eigenspace for λ is spanned by $\mathbf{e}_1$, and so the dimension of the eigenspace, which equals the geometric multiplicity of $\lambda = 0$, is 1. However, the generalized eigenspace for $\mathbf{A}$ is $\{\mathbf{v} \in \mathbb{C}^2 | (\mathbf{A} - 0\mathbf{I}_2)^k\mathbf{v} = \mathbf{0}\}$ for some positive integer k. But $\mathbf{A} - 0\mathbf{I}_2 = \mathbf{A}$, so $(\mathbf{A} - 0\mathbf{I}_2)^2 = \mathbf{O}_2$. Thus, the generalized eigenspace for $\mathbf{A}$ is $\mathbb{C}^2$, which has dimension 2. (Note, by the way, that $p_{\mathbf{A}}(x) = x^2$, and so the eigenvalue 0 has algebraic multiplicity 2.)

(d) False. According to Theorem 1, a Jordan canonical form for a matrix is unique, except for the order in which the Jordan blocks appear on the main diagonal. However, if there are two or more Jordan blocks, and if they are all identical, then all of the Jordan canonical form matrices are the same, since changing the order of the blocks in such a case does not change the matrix. One example is the Jordan canonical form matrix $\mathbf{I}_2$, which has two identical 1×1 Jordan blocks.

(e) False. For a simple counterexample, suppose $\mathbf{A} = \mathbf{J} = \mathbf{I}_2$, and let $\mathbf{P}$ and $\mathbf{Q}$ be any two distinct nonsingular matrices, such as $\mathbf{I}_2$ and $-\mathbf{I}_2$.

(f) True. Theorem 1 asserts that there is a nonsingular matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is in Jordan canonical form. The comments just before Example 5 show that the columns of $\mathbf{P}$ are a basis for $\mathbb{C}^n$ consisting of generalized eigenvectors for $\mathbf{A}$. Since these columns span $\mathbb{C}^n$, every vector in $\mathbb{C}^n$ can be expressed as a linear combination of these column vectors.