Calculus 3
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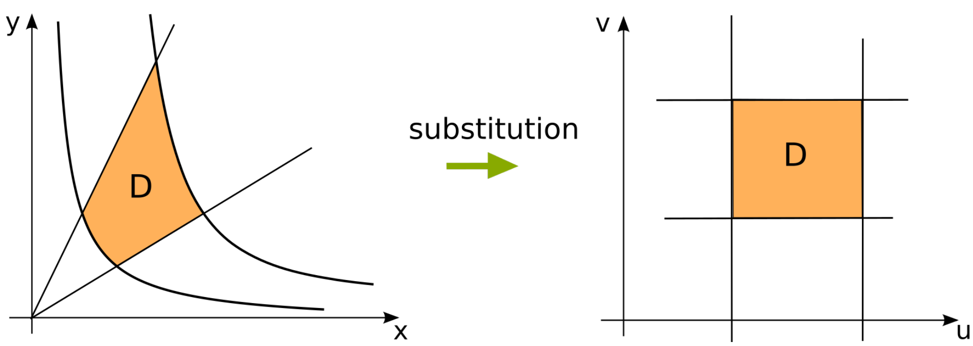
Substitution for Double and Triple Intrgrals. Cylindrical and Spherical Coordinates

General substitution for double integrals.

We have seen many examples in which a region in xy-plane has more convenient representation in polar coordinates than in xy-parametrization. In general, say that two new parameters, u and v, represent the region better than the parameters x and y. In cases like that, one can transform the region in xy-plane to a region in uv-plane by the **substitution**

$$x = g(u, v) \quad y = h(u, v).$$



Thus, a substitution is just a convenient **reparametrization** of a surface when the parameters x and y are changed to u and v. When evaluating the integral $\int\int_D f(x,y)dxdy$ using substitution, the area element $dA = dxdy$ becomes $|J|dudv$ where **the Jacobian determinant** J is given by

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}.$$

Thus,

$$\int\int_D f(x,y)\ dx\ dy = \int\int_D f(x(u,v), y(u,v))\ |J| du\ dv$$

Note that in one-dimensional case, the Jacobian determinant is simply the derivative of the substitution $u = u(x)$ solved for x so that $x = x(u) \Rightarrow dx = x'(u)du$.

Jacobian for polar coordinates. The polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ can be considered as a substitution in which $u = r$ and $v = \theta$. Thus, $x_r = \cos\theta, x_\theta = -r\sin\theta$ and $y_r = \sin\theta, y_\theta = r\cos\theta$. The Jacobian is

$$J = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

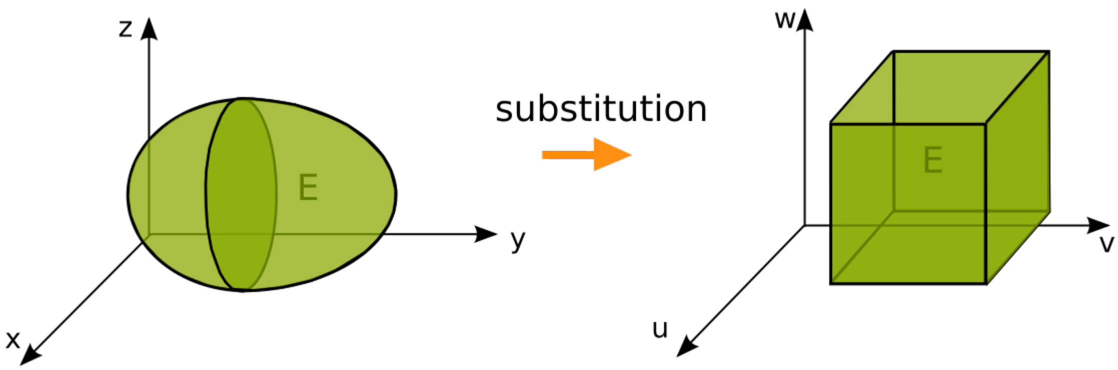
This explains the presence of r in the integrals of the section on Polar Coordinates.

$$\int\int_D f(x,y)dxdy = \int\int f(r\cos\theta, r\sin\theta)\ r\ dr\ d\theta.$$

General substitution for triple integrals.

Just as for double integrals, a region over which a triple integral is being taken may have easier representation in another coordinate system, say in uvw-space, than in xyz-space. In cases like that, one can transform the region in xyz-space to a region in uvw-space by the **substitution**

$$x = x(u, v, w), \quad y = y(u, v, w), \text{ and } z = z(u, v, w).$$



When evaluating the integral $\int\int\int_E f(x, y, z)dxdydz$ using substitution, the volume element $dV = dxdydz$ becomes $|J|dudvdw$ where **the Jacobian determinant** J is given by

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

Thus, $$\int\int\int_E f(x, y, z)\, dx\, dy\, dz = \int\int\int_E f(x(u, v, w), y(u, v, w), z(u, v, w))\, |J| du\, dv\, dw$$

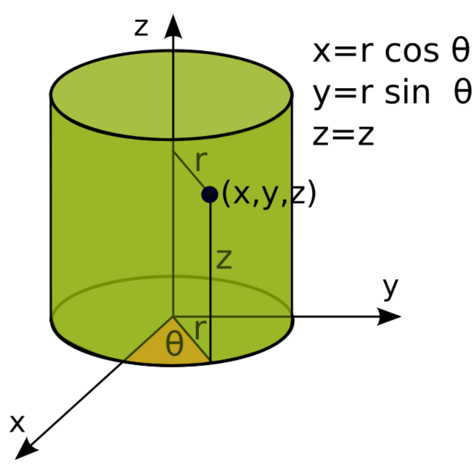
Two main examples of such substitution are **cylindrical and spherical coordinates**.

Cylindrical coordinates.

Recall that the cylinder $x^2 + y^2 = a^2$ can be parametrized by $x = a\cos\theta$, $y = a\sin\theta$ and $z = z$. Assuming now that the radius a is not constant and using the variable r to denote it just as in polar coordinates, we obtain the cylindrical coordinates

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}$$

Thus, x, y and r are related by $\qquad x^2+y^2 = r^2.$



The Jacobian of cylindrical coordinates is

$$J = \begin{vmatrix} x_r & x_\theta & x_z \\ y_r & y_\theta & y_z \\ z_r & z_\theta & z_z \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

Thus, when using cylindrical coordinates to evaluate a triple integral of a function $f(x, y, z)$ defined over a solid region E above the surface $z = g(x, y)$ and below the surface $z = h(x, y)$ with the projection D in the xy-plane. If the projection D has a representation in the polar coordinates $D = \{ (r, \theta) \mid \alpha \le \theta \le \beta, \;\; r_1(\theta) \le r \le r_2(\theta) \}$, then the triple integral

$$\int\int\int_E f(x, y, z)\; dx\; dy\; dz = \int_\alpha^\beta \left(\int_{r_1(\theta)}^{r_2(\theta)} \left(\int_{g(r,\theta)}^{h(r,\theta)} f(r\cos\theta, r\sin\theta, z)\; dz \right) r \;\; dr \right) d\theta$$

Spherical coordinates.

Besides cylindrical coordinates, another frequently used coordinates for triple integrals are **spherical coordinates.** Spherical coordinates are mostly used for the integrals over a solid whose definition involves spheres.

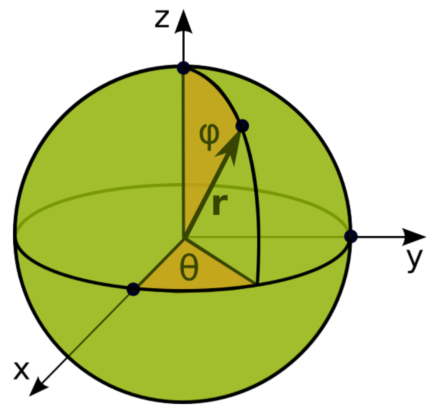
If $P = (x, y, z)$ is a point in space and O denotes the origin, let

- r denote the length of the vector $\overrightarrow{OP} = \langle x, y, z\rangle$, i.e. the distance of the point $P = (x, y, z)$ from the origin O. Thus,

$$x^2 + y^2 + z^2 = r^2;$$

- θ be the angle between the projection of vector $\overrightarrow{OP} = \langle x, y, z\rangle$ on the xy-plane and the vector $\overrightarrow{i}$ (positive x axis); and

- ϕ be the angle between the vector $\overrightarrow{OP}$ and the vector $\overrightarrow{k}$ (positive z-axis).



The conversion equations are $\underline{x = r\cos\theta\sin\phi \quad y = r\sin\theta\sin\phi \quad z = r\cos\phi.}$

The Jacobian determinant can be computed to be $\boxed{J = r^2\sin\phi.}$ Thus,

$$dx\; dy\; dz = r^2\; \sin\phi\; dr\; d\phi\; d\theta.$$

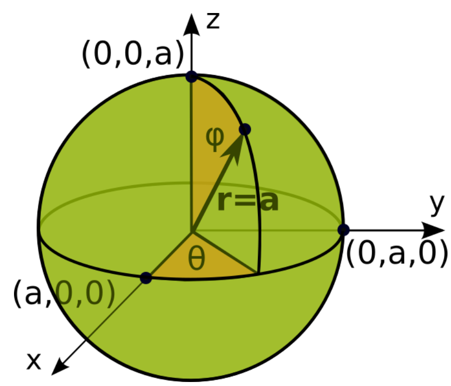
Note that the angle θ is the same in cylindrical and spherical coordinates.

Note that the distance r is *different in cylindrical and in spherical coordinates.*

	Meaning of r	Relation to x, y, z
Cylindrical	distance from (x, y, z) to z-axis	$x^2 + y^2 = r^2$
Spherical	distance from (x, y, z) to the origin	$x^2 + y^2 + z^2 = r^2$

Spherical coordinates parametrization of a sphere. If a is a positive constant and a point (x, y, z) is on the sphere centered at the origin of radius a, then the coordinates satisfy the equation

$$x^2 + y^2 + z^2 = a^2.$$



So, the distance from the origin r is exactly a for every such point. In other words, r is *constant and equal to* a. Thus, the equation of the sphere in spherical coordinates become simple and short

$$r = a$$

and the equations $\quad x = a\cos\theta\sin\phi, \quad y = a\sin\theta\sin\phi, \quad z = a\cos\phi$
parametrize the sphere. When these equations are substituted in the expression $x^2 + y^2 + z^2$, it simplifies to a^2 (you should convince yourself of this fact).

Practice problems.

1. Evaluate the triple integral
   (a) $\int\int\int_E \sqrt{x^2+y^2}\, dx\, dy\, dz$ where E is the region that lies between the cylinders $x^2+y^2 = 1$ and $x^2+y^2 = 4$ and between the xy-plane and the plane $z = x+3$.
   (b) $\int\int\int_E (x^2+y^2+z^2)\, dx\, dy\, dz$ where E is the unit ball $x^2+y^2+z^2 \le 1$.
   (c) $\int\int\int_E z\ dx\ dy\ dz$ where E is the region between the spheres $x^2+y^2+z^2 = 1$ and $x^2+y^2+z^2 = 4$ in the first octant.

2. Find the volume of the solid enclosed by the paraboloids $z = x^2+y^2$ and $z = 36 - 3x^2 - 3y^2$.

3. Find the volume of the ellipsoid $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{25} = 1$ by using the transformation $x = 2u$, $y = 3v$ $z = 5w$.

4. Determine the bounds (in spherical coordinates) for the following regions between the spheres $x^2+y^2+z^2 = 1$ and $x^2+y^2+z^2 = 4$.
   (a) The region between the two spheres and above the xy-plane.
   (b) The region between the two spheres and to the right of the xz-plane.
   (c) The region between the two spheres and in front of the yz-plane.

5. Use the given substitution to evaluate the integral.
   (a) $\int\int_D (3x+4y)\, dx\, dy$ where D is the region bounded by the lines $y = x$, $y = x-2$, $y = -2x$, and $y = 3-2x$. The substitution $x = \frac{1}{3}(u+v)$, $y = \frac{1}{3}(v-2u)$ transforms the region to a rectangle $0 \le u \le 2$ and $0 \le v \le 3$.
   (b) $\int\int_D xy\, dx\, dy$ where D is the region in the first quadrant bounded by the curves $y = x$, $y = 3x$, $y = \frac{1}{x}$, and $y = \frac{3}{x}$. The substitution $x = \frac{u}{v}$, $y = v$ transforms the region into a region with bounds $1 \le u \le 3$ and $\sqrt{u} \le v \le \sqrt{3u}$.

Solutions.

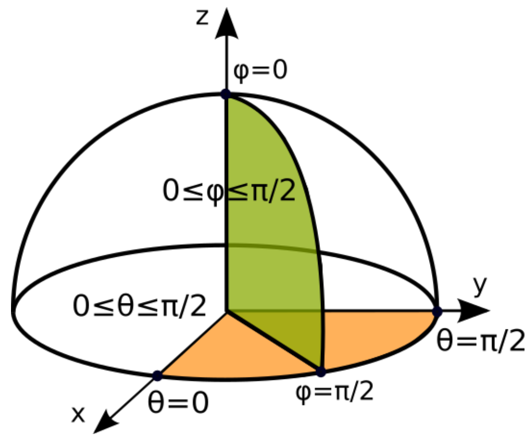
1. (a) Use cylindrical coordinates. The plane $z = x+3$ is the z-upper bound and the xy-plane $z = 0$ is the z-lower bound. The bounds for r and θ are determined as when working with polar coordinates: the region between the circles $x^2+y^2 = 1$ and $x^2+y^2 = 4$ can be described by $0 \le \theta \le 2\pi$ and $1 \le r \le 2$. Since $x = r\cos\theta$, the plane $z = x+3$ becomes $z = r\cos\theta + 3$. Thus, the integral is

$$\int\int\int_E \sqrt{x^2+y^2}\, dx\, dy\, dz = \int_0^{2\pi}\int_1^2\int_0^{r\cos\theta+3} \sqrt{r^2}\; r\; dr\, d\theta\, dz = \int_0^{2\pi} d\theta \int_1^2 r^2\, dr \int_0^{r\cos\theta+3} dz =$$

$$\int_0^{2\pi} d\theta \int_1^2 r^2\, dr\, (r\cos\theta + 3) = \int_0^{2\pi} d\theta \int_1^2 (r^3\cos\theta + 3r^2)\, dr = \int_0^{2\pi} d\theta \; \left(\frac{r^4}{4}\cos\theta + 3\frac{r^3}{3}\right)\Big|_1^2 =$$

$\int_0^{2\pi} d\theta\, (\frac{15}{4}\cos\theta + 7) = 14\pi.$

(b) Use spherical coordinates. The function $x^2+y^2+z^2$ is r^2 and $dV = dxdydz$ is $r^2\sin\phi dr d\theta d\phi$. The bounds for the unit sphere are $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$, and $0 \le r \le 1$. Thus, we have $\int\int\int_E (x^2+y^2+z^2)\, dx\, dy\, dz = \int_0^{2\pi}\int_0^{\pi}\int_0^1 r^2 r^2 \sin\phi\, dr\, d\theta\, d\phi = \int_0^{2\pi} d\theta \int_0^{\pi}\sin\phi d\phi \int_0^1 r^4\, dr = 2\pi(-\cos\phi)|_0^{\pi}\, \frac{r^5}{5}\Big|_0^1 = 2\pi(2)\frac{1}{5} = \frac{4\pi}{5}.$



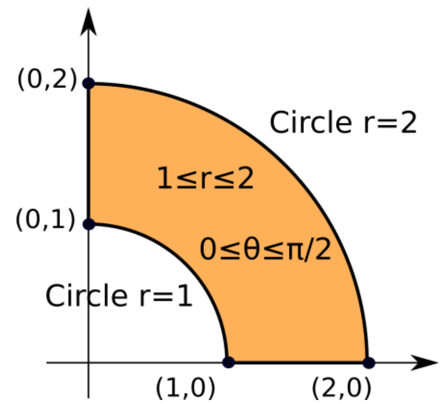
(c) Use spherical coordinates. The function z is $r\cos\phi$ and $dV = dxdydz$ is $r^2\sin\phi dr d\theta d\phi$. Since the region is in the first octant, $0 \le \phi \le \frac{\pi}{2}$. The bounds for r and θ can be determined from the intersection with xy-plane on the figure on the right. Hence, $0 \le \theta \le \frac{\pi}{2}$ and the bounds for r are determined by the radii of the spheres, so $1 \le r \le 2$. Thus,

$$\int\int\int_E z\, dx\, dy\, dz =$$

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_1^2 r\cos\phi\, r^2\sin\phi dr\, d\theta\, d\phi =$$

$$\int_0^{\pi/2} d\theta \int_0^{\pi/2}\cos\phi\sin\phi d\phi \int_1^2 r^3 dr =$$

$$\frac{\pi}{2}\,\frac{1}{2}\,\frac{r^4}{4}\Big|_1^2 = \frac{15\pi}{16}$$

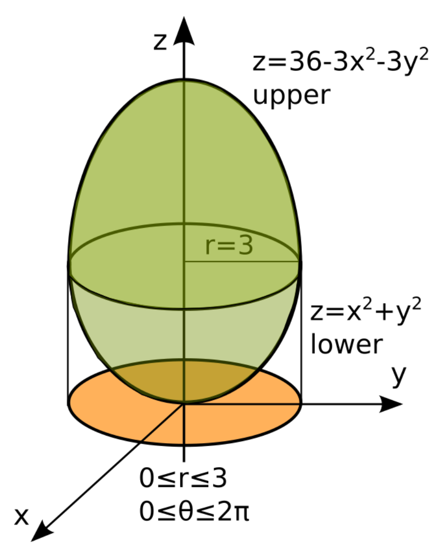


2. Use cylindrical coordinates. The paraboloids have the equations $z = x^2+y^2 = r^2$ and $z = 36 - 3x^2 - 3y^2 = 36 - 3r^2$. The first is the lower z-bound and the second is the upper (see the figure below). The bounds for θ are $0 \le \theta \le 2\pi$.

The paraboloids intersect in a circle. The projection of the circle in xy-plane determines the r-bounds. The intersection is when $36 - 3r^2 = r^2 \Rightarrow 36 = 4r^2 \Rightarrow 9 = r^2 \Rightarrow r = 3$ (the negative solution is not relevant). Thus, the r-bounds are $0 \le r \le 3$. The volume is

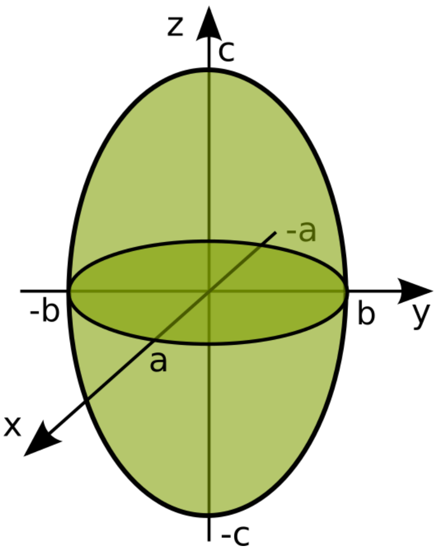
$$V = \int\int\int\, dxdydz = \int_0^{2\pi}\int_0^3\int_{r^2}^{36-3r^2} r\, dr\, d\theta\, dz =$$

$$\int_0^{2\pi} d\theta \int_0^3 r\, dr(36-3r^2-r^2) = 2\pi(18r^2 - r^4)\Big|_0^3 = 162\pi.$$



3. The substitution $x = 2u$, $y = 3v$ and $z = 5w$ converts the ellipsoid into a sphere of radius 1.

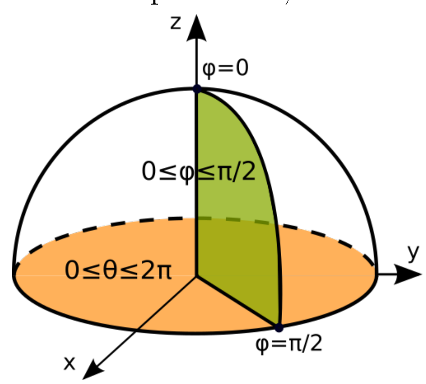
   The Jacobian of this substitution is $J = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{vmatrix} = 30$. Thus, the volume is $V = \int\int\int \; dx\,dy\,dz = \int\int\int 30 \;\, du\,dv\,dw$. Since the integral is taken over a inside of the sphere, use the spherical coordinates. The Jacobian is $r^2 \sin\phi$ so $dudvdw = r^2 \sin\phi drd\theta d\phi$. Since the radius is 1 and we are integrating over entire sphere, the bounds are $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \pi$, and $0 \leq r \leq 1$. Thus, the volume is $V = \int\int\int 30 \; du\,dv\,dw = \int\int\int 30 \; r^2 \sin\phi \; dr\,d\phi\,d\theta =$



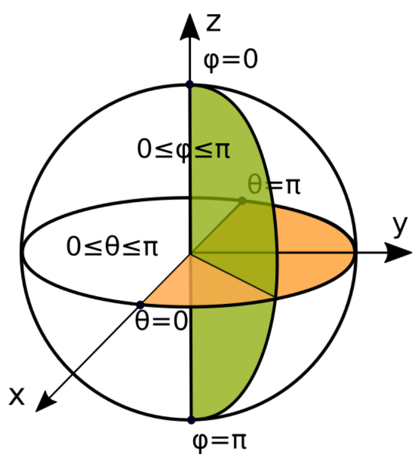
$$30 \int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi\, d\phi \int_0^1 r^2 dr = \; 30 \; 2\pi \, (-\cos\phi)|_0^{\pi} \; \frac{r^3}{3}\bigg|_0^1 = 120\pi\frac{1}{3} = 40\pi.$$

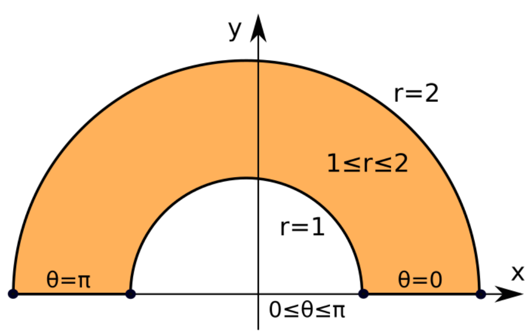
4. Since the radius of the first sphere is 1 and the radius of the second sphere is 2, the r-bounds are $1 \leq r \leq 2$ for all three parts.

   (a) Note that the values of θ are 0 to 2π because the projection in the xy plane is entire region between two circles. The bounds for ϕ are 0 to $\frac{\pi}{2}$ (see the figure on the right).

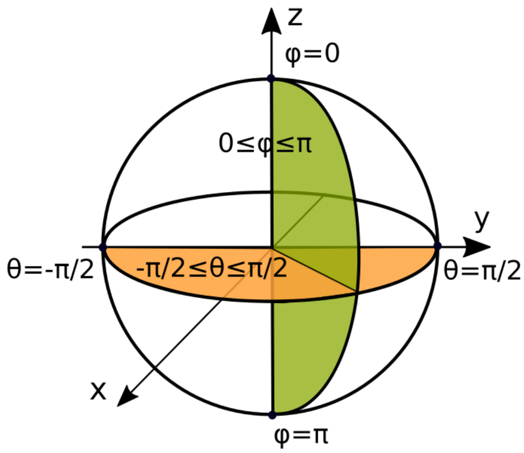


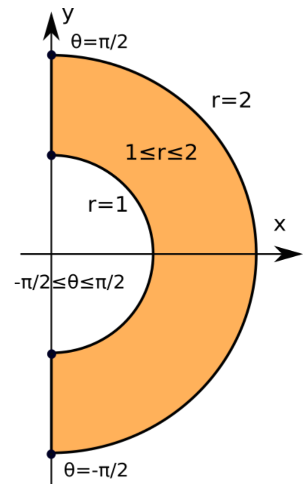
   (b) The right side of the xz-plane $y = 0$ corresponds to $y > 0$. Hence, the projection in xy-plane is *above* the x-axis. So, the values of θ are 0 to π. The bounds for ϕ are 0 to π as the figure below illustrates.





(c) The front of the yz-plane $x = 0$ corresponds to $x > 0$. Hence, the projection in xy-plane is *to the right* of the y-axis. So, the values of θ are $\frac{-\pi}{2}$ to $\frac{\pi}{2}$. The bounds for ϕ are 0 to π as the figure below illustrates.





5. (a) Calculate the Jacobian $J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{-2}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{9} + \frac{2}{9} = \frac{1}{3}$. $\int\int_D (3x + 4y)\ dx\ dy = \int_0^2 \int_0^3 (u + v + \frac{4}{3}(v - 2u))\ \frac{1}{3} du\ dv = \frac{1}{3}\int_0^2 \left(uv + \frac{v^2}{2} + \frac{4v^2}{6} - \frac{8uv}{3}\right)\Big|_0^3\ du = \frac{1}{3}\int_0^2 (3u + \frac{9}{2} + 6 - 8u)\ du = \frac{1}{3}(6 + 9 + 12 - 16) = \frac{11}{3}$

(b) The Jacobian is $J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{1}{v} & \frac{-u}{v^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{v}$. $\int\int_D xy\ dx\ dy = \int_1^3 \int_{\sqrt{u}}^{\sqrt{3u}} \frac{u}{v}\, v\, \frac{1}{v} du\, dv = \int_1^3 u du\ \ln v\Big|_{\sqrt{u}}^{\sqrt{3u}} = \int_1^3 u du\ \ln\sqrt{3} = \ln\sqrt{3}\frac{u^2}{2}\Big|_1^3 = 4\ln\sqrt{3} = 2\ln 3 = 2.197$