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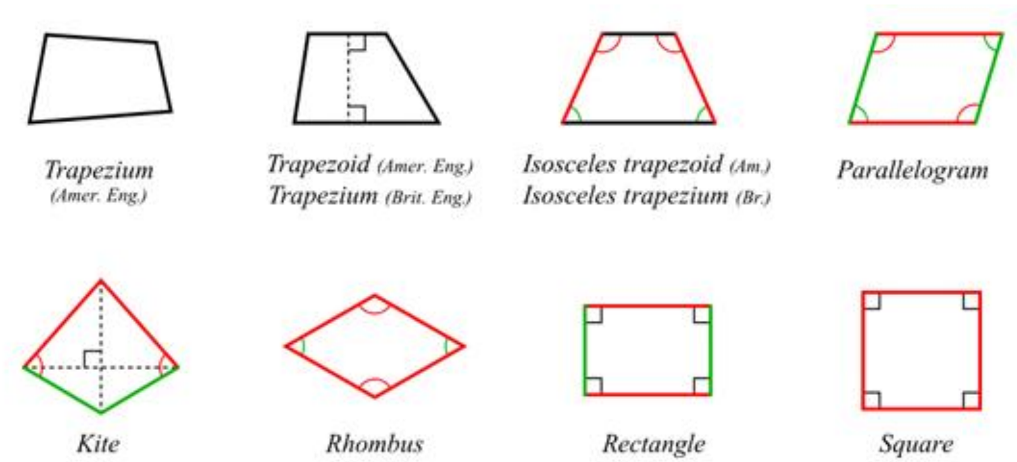
Summary of dynamics of the Penrose kite

Summary of dynamics of a Penrose kite: N = 4

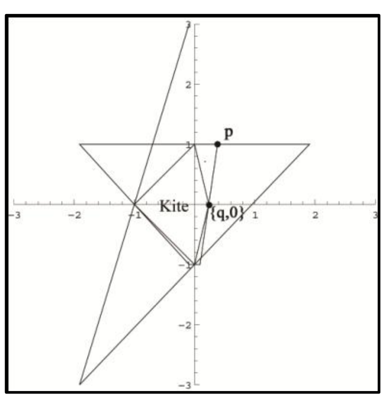
The following is the definition of a Kite from Wikipedia:

- Kite: two pairs of adjacent sides are of equal length. This implies that one diagonal divides the kite into congruent triangles, and so the angles between the two pairs of equal sides are equal in measure. It also implies that the diagonals are perpendicular. (It is common, especially in the discussions on plane tessellations, to refer to the concave quadrilateral with these properties as a *dart* or *arrowhead*, with term kite being restricted to the convex shape.)

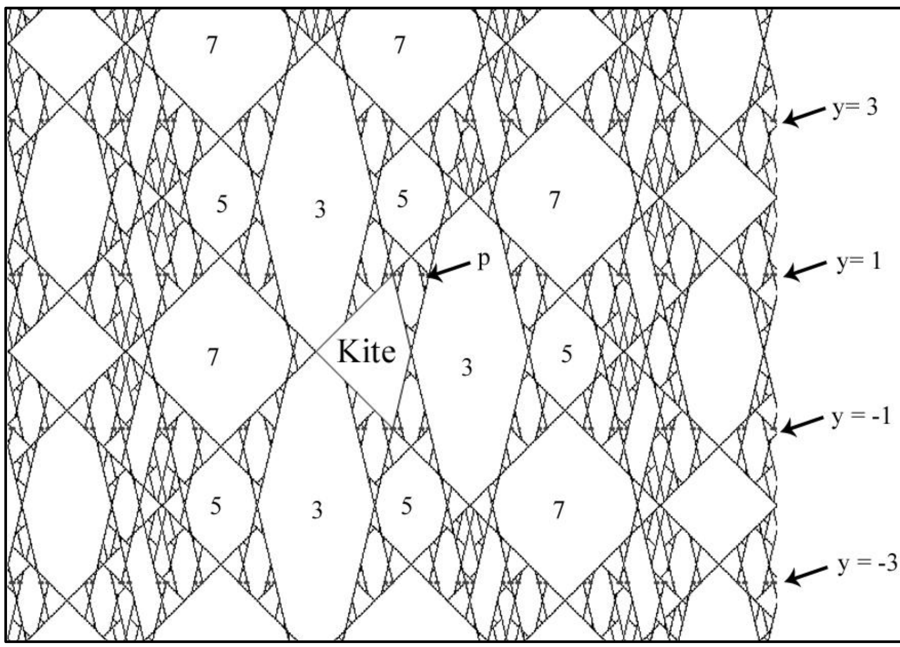


The kites form an affine family which is distinct from the trapezoids.A canonical Penrose Kite has vertices {{0,1},{q,0},{0,-1},{-1,0}} with $q \in (0,1)$ When q is irrational, it is called an irrational kite. Richard Schwartz has recently shown that every irrational kite has unbounded orbits, but initially he proved this result for the special case of $q = \sqrt{5} - 2$ as shown below. This is called a Penrose kite because Roger Penrose combined it with a 'dart' to define a non-periodic tiling of the plane.

For his initial point in the unbounded orbit, Schwartz used $p = \{(1-q)/2, 1\}$. The first 6 points in that orbit are shown below.



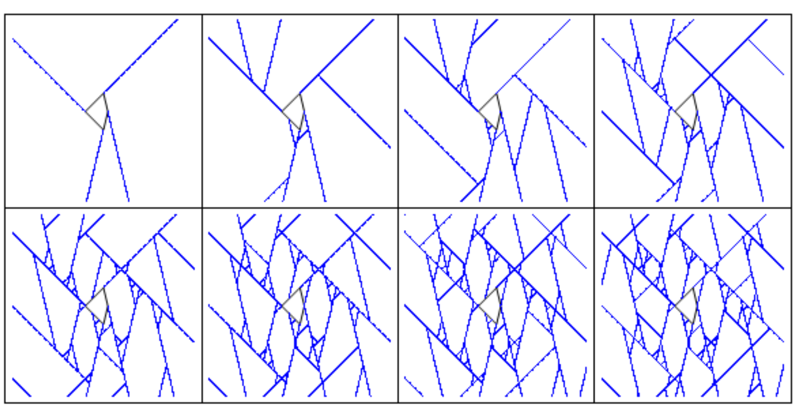
All the points in the orbit if p lie on a lattice of horizontal lines of the form y = k where k is an odd integer. They are woven through the web like thread in fabric and they form almost perfect Cantor sets. The numbers given here are the periods of prominent regions. Most regions have period doubling and these are the periods of the centers. The web is intricate but not fractal - so there is no obvious signs pointing to the complexity.



Using the notebook NonRegular.nb, set **Mc = {{0,1},{Sqrt[5]-2,,0},{0,-1},{-1,0}}**

To watch the web develop:
Gr[depth_]:=Show[Graphics[{AbsolutePointSize[1.0],poly[Mom],Blue,
Point[WebPoints[.02,15,depth]]},PlotRange->{{left,right},{bottom,top}}]];

GraphicsGrid[{{Gr[0],Gr[1],Gr[2],Gr[3]},{Gr[4],Gr[5],Gr[6],Gr[7]}},Frame->All]

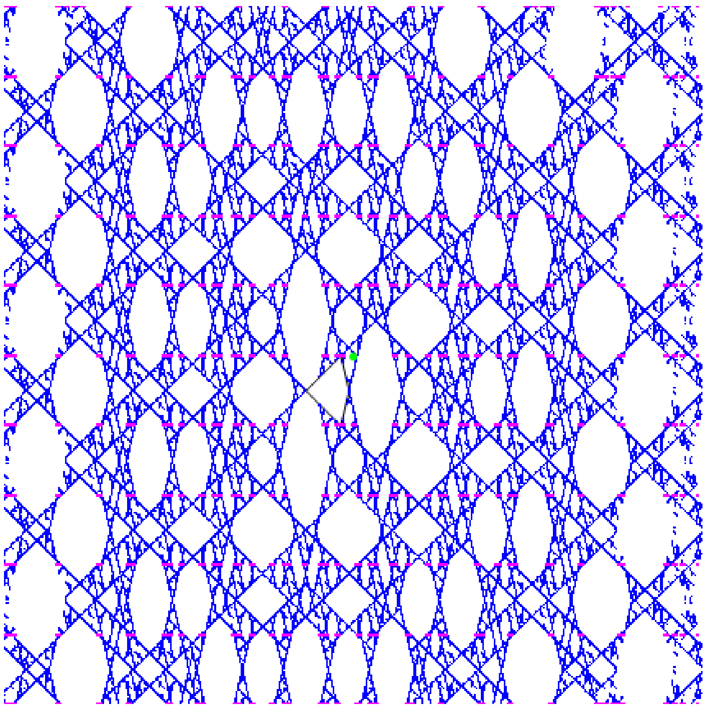


To see the Cantor thread:

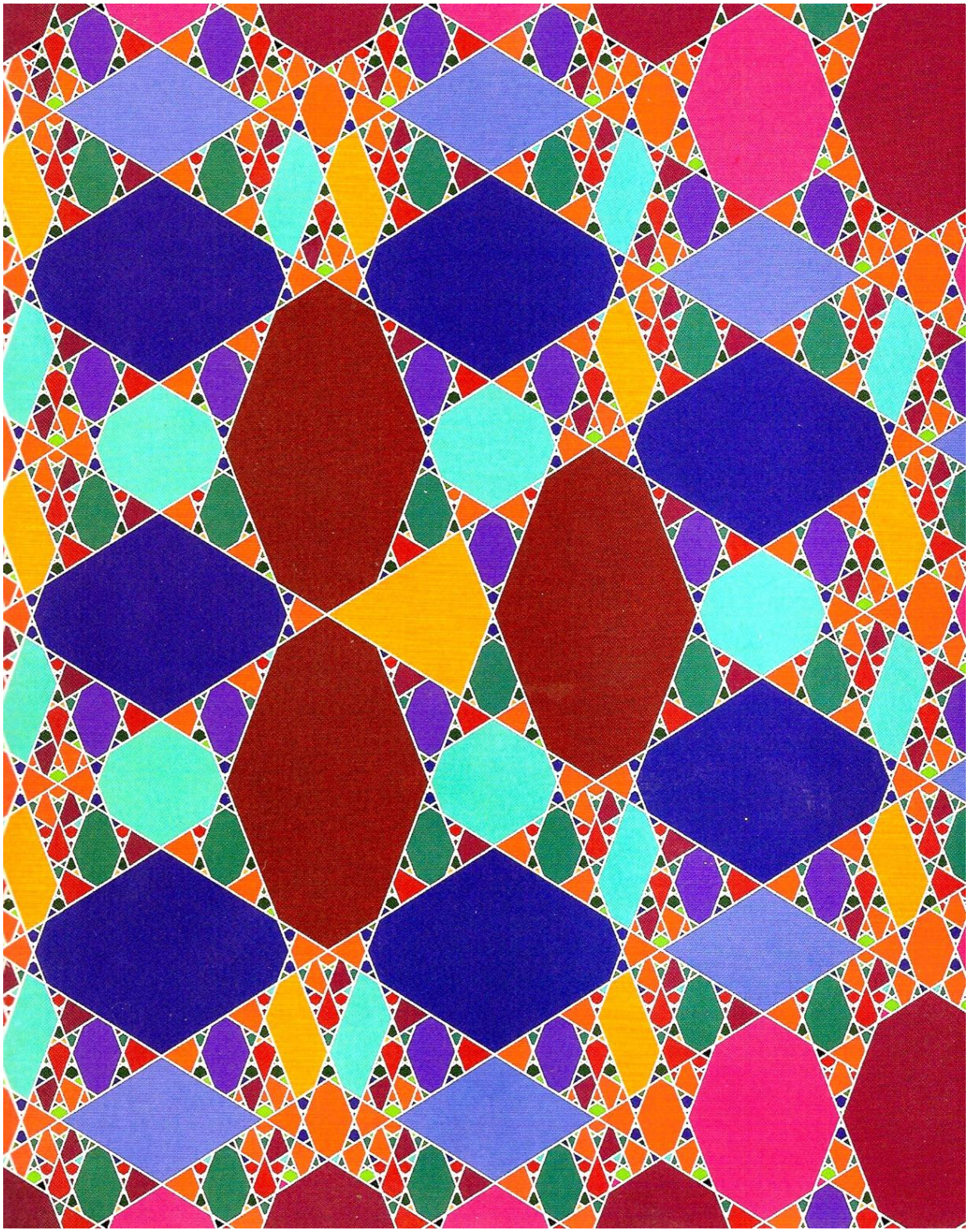
The unbounded orbit is **p = {1/2 (3-Sqrt[5]),1}; K = V[p,50000];**

W1= WebPoints[.02,12,200]; box[{0,0},10];

Show[Graphics[{AbsolutePointSize[1.0],poly[Mom],Blue,Point[W1], Magenta,AbsolutePointSize[3.0], Point[K]},PlotRange->{{left,right},{bottom,top}}]]



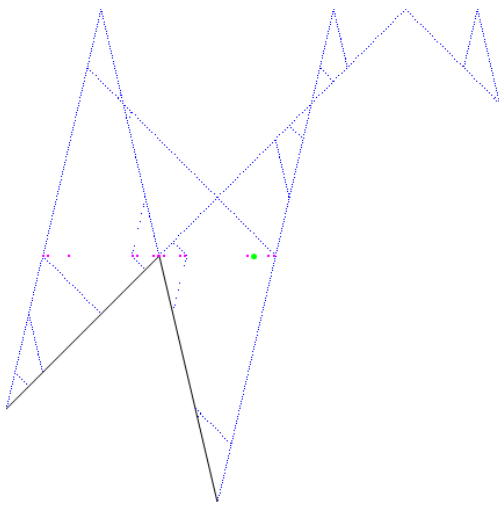
Compare this with the graphic below from page 417 of the *Math Book* by Clifford Pickover. In the accompanying article, Pickover discusses the Tangent Map and the proof by Richard Schwartz.

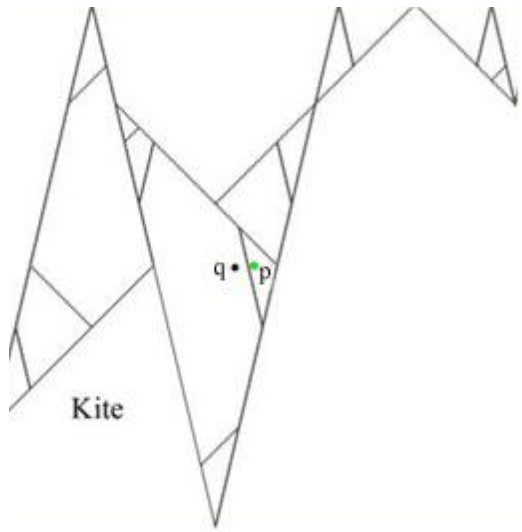


To zoom in on the region around p use a finer web and generate the first 500,000 points in the orbit.

K = V[p,500000]; W1= WebPoints[.01, 6, 1000]; box[p,1]

Show[Graphics[{AbsolutePointSize[1.0],poly[Mom],Blue,Point[W1], Magenta,AbsolutePointSize[3.0], Point[K], AbsolutePointSize[5.0], Green, Point[p]},PlotRange->{{left,right},{bottom,top}}]] (*on the left below*)





Because this orbit is unbounded, the neighborhood around p will always be unresolved, but it must be identical to the neighborhood of each point in the orbit. In this case the neighborhood consists of intervals because the points all lie on the line y = k for k odd. The case shown above is k = 1.

In any web we can identify the exact point in the step sequence where the neighboring regions differ In the example below, the step sequence of p and the step sequence of q are identical until the 39th place and therefore the boundary line between them is part of the level-39 web but not the level-38 web.

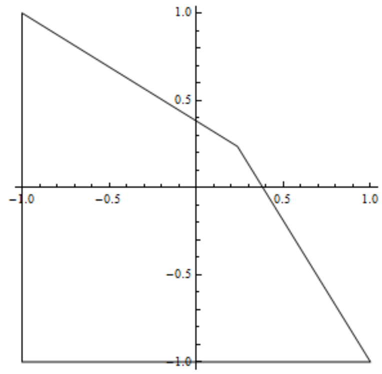
Step sequence for p: 1, 1, 1, 2, 1, 2, 2, 1, 2, 2, 2, 1, 2, 2, 1, 3, 1, 2, 2, 2, 1, 3, 1, 2, 2, 1, 3, 1, 2, 2, 2, 2, 2, 1, 3, 1, 2, 2, 1, ...

Step sequence for q: 1, 1, 1, 2, 1, 2, 2, 1, 2, 2, 2, 1, 2, 2, 1, 3, 1, 2, 2, 2, 1, 3, 1, 2, 2, 1, 3, 1, 2, 2, 2, 2, 2, 1, 3, 1, 2, 2, **2**, ...

Below is an 'upright' version of the Kite, rotated by Pi/4 and scaled:

Mc={{-2+Sqrt[5],-2+Sqrt[5]},{1,-1},{-1,-1},{-1,1}};

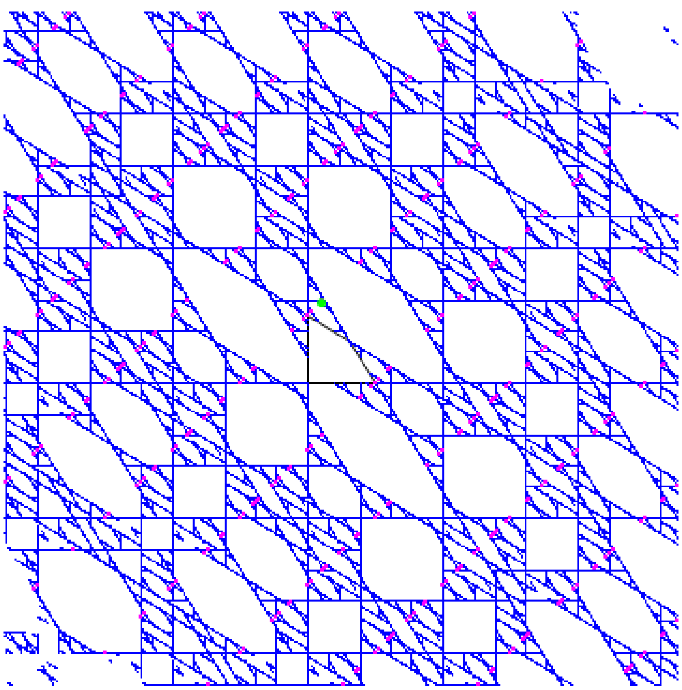
Graphics[poly[Mc], Axes->True]



The unbounded orbit is now at **p = {1/2 (1-Sqrt[5]),1/2 (5-Sqrt[5])};** as shown in green below and the 'thread' from its orbit has an angle of Pi/4.

W1= WebPoints[.02,15,500]; box[{0,0},10]; K = V[p, 50000];

Show[Graphics[{AbsolutePointSize[1.0],poly[Mom],Magenta,Blue,AbsolutePointSize[1.0], Point[W1],AbsolutePointSize[2.0], Magenta, Point[K], AbsolutePointSize[5.0], Green, Point[p]},PlotRange->{{left,right},{bottom,top}},AspectRatio->Automatic]]

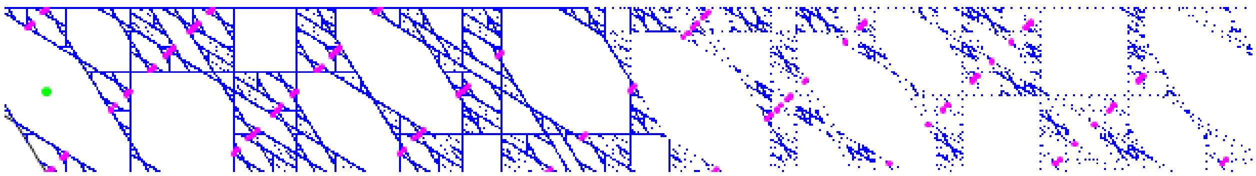


This enables us to easily generate Pinwheel maps or strips to track the progress of the web. Our primary strip will include the Dads, so

top = 3; bottom = -1; left = 0; right = arbitrary;

We can probe the initial part of this strip be extending the web:

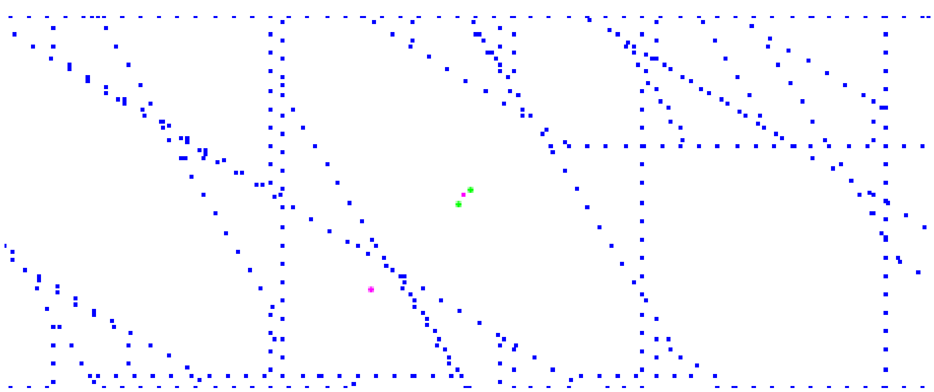
W2= WebPoints[.2,50,300];



Mom is just visible on the left. The magenta points are part of the orbit of the unbounded point p. The green dot is the center of 'Dad' at {1,1}. The period is 3.These centers can be found because of period doubling. The remaining points inside Dad have period 6.

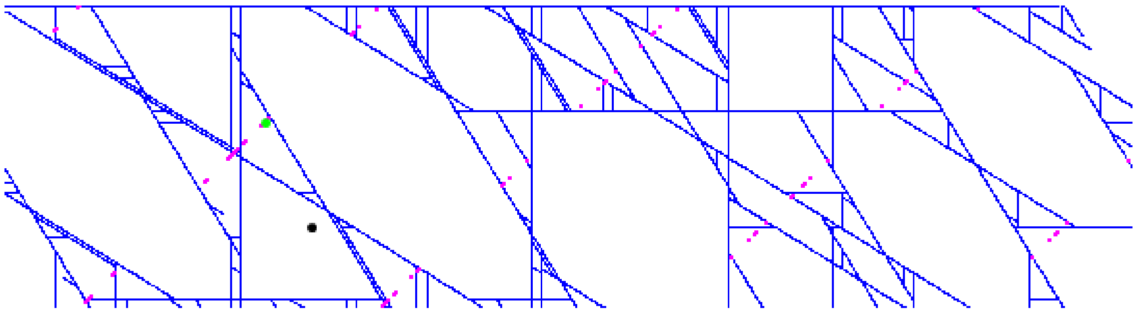
The strip below runs from -1 to 90. It seems that the initial part of the web is 'almost' periodic. The green dot at {84,0} shows the start of a new cycle.

To find the exact cycle length, use the new DadRight as shown below. We will find its center and subtract {1,1} to obtain the local origin. To find the center, pick an arbitrary point p1 inside DadRight and iterate. K = V[p1,2000]; Period[K]. In this case the period is 246 so it will return in 123 iterations: p2 = K[[124]]. These are the two small green dots below. The center is half-way between at c1 ≈{85.068883707497,1.068883707497}. In the plot below we can see that DadRight (and DadLeft) are a little higher than the earlier Dads whose edges were on the boundaries. This error term is the same in x and y.

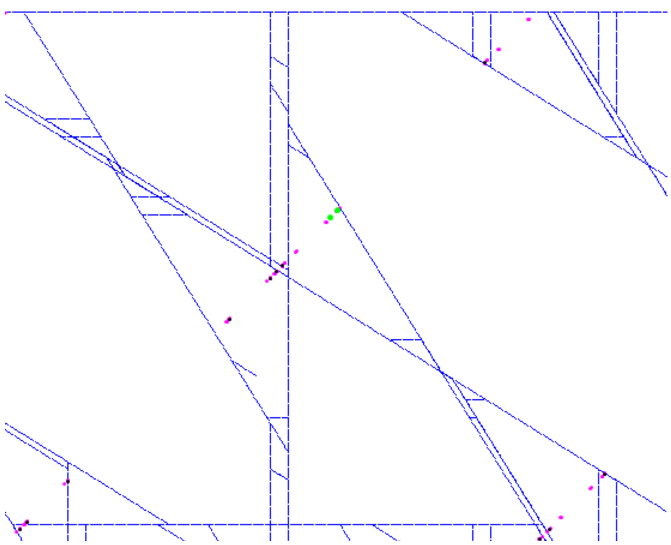


This means that the surrogate origin2 is at c1-{1,1} ≈{84.06888370749726,0.06888370749726}. The new origin is shown above in magenta. The web period for the first cycle is about 84.Recall that the N5Kite also had a semi-periodic web with period about 299.)

Now we can test a surrogate non-periodic point p1 = p + origin2 .This point has period 10362 so it is not in the orbit of p, but the plot below shows that it is perfectly aligned with the lattice of points from the orbit of p. The green dot is p1 and the magenta points are points in the orbit of p obtained from 20 million iterations. (The black dot is origin2.)



Since the slope of this lattice line is 1, we can choose test points close to p1with this same slope. We know that the immediate neighbors of p1 will have period 10362 but this periodic neighborhood is small and p2 = p1 + {.053,.053}≈ {83.50384971874738, 1.5038497187473712} has an orbit with a long period,or no period. This point is shown below in green slightly above p1. The first points in its orbit are shown below in black. They seem to track the unbounded orbit of the point p (in magenta) and it is possible that they are part of this orbit.



The orbit of p must always return to the neighborhood of each point in its orbit, so its unbounded path involves long cycles of 'semi-periodic' behavior. If it is tracked for long enough on any computer, the program would declare it to be periodic - no matter what test was used. To prove that it was unbounded, Richard Schwartz did not track its orbit, but instead used an algebraic description of the orbit called an Arithmetic Graph. Arithmetic Graphs and Pinwheel maps are discussed in the section on Projections.

In his monograph *Outer billiards on kites* , Richard Schwartz shows that the Tangent Map has unbounded orbits on any irrational kite - that is any kite wih a single irrational vertex. He conjectures that this is 'generic' for large classes of polygons.

References:

R. Schwartz, *Unbounded orbits for outer billiards.* I. J. Mod. Dyn. **1** (2007), 371–424.

R. Schwartz, Outer billiards on kites, Ann. of Math. Studies, 171, Princeton Univ. Press, Princeton, New Jersey, 2009, xiv+306 pp., ISBN 978-0-691-14249-4

R. Schwartz, Outer Billiards and the Pinwheel Map. arXiv:1004.3025.

R. Schwartz, Outer Billiards, Arithmetic Graphs, and the Octagon. arXiv:1006.2782.

S. Tabachnikov, On the dual billiard problem. Adv. Math. 115 (1995), 221–249. MR1354670

Cliffford Pickover, The Math Book , Sterling Publishing, 2009: ISBN 978-1-4027-5796-9