

SUMMARY OF VECTOR AND TENSOR NOTATION

-Bird, Stewart and Lightfoot "Transport Phenomena"

-Bird, Armstrong and Hassager "Dynamics of Polymeric Liquids"

The Physical quantities encountered in the theory of transport phenomena can be categorised into:

- **Scalars** (temperature, energy, volume, and time)
- **Vectors** (velocity, momentum, acceleration, force)
- **Second-order tensors** (shear stress or momentum flux tensor)

While for scalars only one type of multiplication is possible, for vectors and tensors several kinds are possible which are:

- single dot \cdot
- double dot $:$
- cross \times

The following types of parenthesis will also be used to denote the results of various operations.

$(\) = \text{scalar}$ $(\underline{u} \cdot \underline{w}), (\underline{\sigma} : \underline{\tau})$

$[\] = \text{vector}$ $[\underline{u} \times \underline{w}], [\underline{\tau} \cdot \underline{u}]$

$\{ \ } = \text{tensor}$ $\{\underline{\sigma} \cdot \underline{\tau}\}$

The multiplication signs can be interpreted as follows:

<u>Multiplication sign</u>	<u>Order of Result</u>
None	Σ
\times	$\Sigma-1$
\cdot	$\Sigma-2$
$:$	$\Sigma-4$

Scalars can be interpreted as 0th order tensors, and vectors as first order tensors.

Examples:

$s\underline{\tau}$ order is $0+2=2$ which is a 2nd order tensor

$\underline{u}\underline{w}$ order is $1+1-1=1$ which is a vector

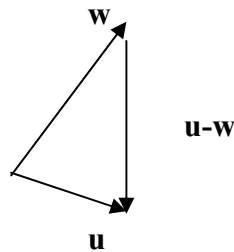
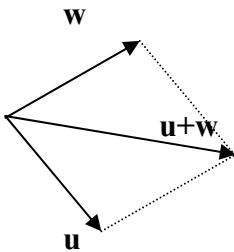
$\underline{\sigma}:\underline{\tau}$ order is $2+2-4=0$ which is a scalar

Definition of a Vector: A vector is defined as a quantity of a given magnitude and direction.

$|u|$ is the magnitude of the vector u

Two vectors are equal when their magnitudes are equal and when they point in the same direction.

Addition and Subtraction of Vectors:



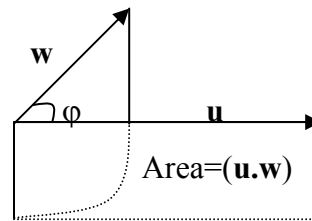
Dot Product of two Vectors:

$$(u \cdot w) = |u| |w| \cos(\varphi)$$

commutative $(u \cdot v) = (v \cdot u)$

not associative $(u \cdot v)w \neq u(v \cdot w)$

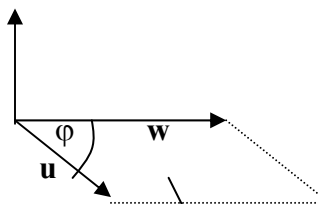
distributive $(u \cdot [v + w]) = (u \cdot v) + (u \cdot w)$



Cross Product of two Vectors:

$$[uxw] = |u| |w| \sin(\varphi) \mathbf{n}$$

where \mathbf{n} is a vector (unit magnitude) normal to the plane containing u and w and pointing in the direction that a right-handed screw will move if we turn u toward w by the shortest route.



Area of this equals the length of $[uxw]$

not commutative $[uxw] = -[wxu]$

not associative $[u \times [v \times w]] \neq [[u \times v] \times w]$

distributive $[[u + v] \times w] = [u \times w] + [v \times w]$

VECTOR OPERATIONS FROM AN ANALYTICAL VIEWPOINT

Define rectangular co-ordinates: $1, 2, 3 \rightarrow x, y, z$ respectively

Many formulae can be expressed more compactly in terms of the kronecker delta δ_{ij} and the alternating unit tensor ε_{ijk} , which are defined as:

$$\delta_{ij} = 1 \quad \text{if} \quad i=j$$

$$\delta_{ij} = 0 \quad \text{if} \quad i \neq j$$

and

$$\varepsilon_{ijk} = 1 \quad \text{if} \quad ijk = 123, 231, 312$$

$$\varepsilon_{ijk} = -1 \quad \text{if} \quad ijk = 321, 132, 213$$

$$\varepsilon_{ijk} = 0 \quad \text{if} \quad \text{any two indices are alike}$$

We will use the following definitions, which can be easily proved:

$$\sum_j \sum_k \varepsilon_{ijk} \varepsilon_{hjk} = 2 \delta_{ih}$$

and

$$\sum_k \varepsilon_{ijk} \varepsilon_{mnk} = \delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}$$

The determinant of a three-by-three matrix may be written as:

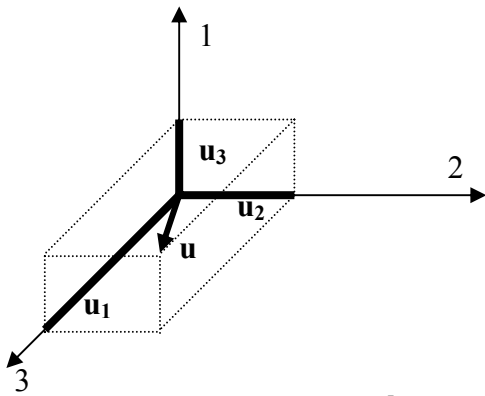
$$\begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{vmatrix} = \sum_i \sum_j \sum_k \varepsilon_{ijk} \alpha_{1i} \alpha_{2j} \alpha_{3k}$$

DEFINITION OF A VECTOR AND ITS MAGNITUDE: THE UNIT VECTORS

A vector \mathbf{u} can be defined completely by giving the magnitudes of its projections u_1 , u_2 , and u_3 on the co-ordinate axis 1, 2, and 3 respectively. Thus one may write

$$\mathbf{u} = \delta_1 u_1 + \delta_2 u_2 + \delta_3 u_3 = \sum_{i=1}^3 \delta_i u_i$$

where δ_1 , δ_2 , and δ_3 are the unit vectors in the direction of the 1, 2 and 3 axes respectively. The following identities between the vectors can be proven readily:



$$\delta_1 \cdot \delta_1 = \delta_2 \cdot \delta_2 = \delta_3 \cdot \delta_3 = 1$$

$$\delta_1 \cdot \delta_2 = \delta_2 \cdot \delta_3 = \delta_3 \cdot \delta_1 = 0$$

$$\delta_1 \times \delta_1 = \delta_2 \times \delta_2 = \delta_3 \times \delta_3 = 0$$

$$[\delta_1 \times \delta_2] = \delta_3 \quad [\delta_2 \times \delta_3] = \delta_1 \quad [\delta_3 \times \delta_1] = \delta_2$$

$$[\delta_2 \times \delta_1] = -\delta_3 \quad [\delta_3 \times \delta_2] = -\delta_1 \quad [\delta_1 \times \delta_3] = -\delta_2$$

All these relations can be summarized as:

$$(\delta_i \cdot \delta_j) = \delta_{ij}$$

$$[\delta_i \times \delta_j] = \sum_{k=1}^3 \varepsilon_{ijk} \delta_k$$

Addition of vectors:

$$\underline{u} + \underline{w} = \sum_i \delta_i u_i + \sum_i \delta_i w_i = \sum_i \delta_i (u_i + w_i)$$

Multiplication of a Vector by a Scalar:

$$s \underline{u} = s [\sum_i \delta_i u_i] = \sum_i \delta_i (s u_i)$$

Dot Product:

$$\begin{aligned} (\underline{u} \cdot \underline{w}) &= [\sum_i \delta_i u_i] \cdot [\sum_j \delta_j w_j] = \sum_i \sum_j (\delta_i \cdot \delta_j) u_i w_j = \\ &= \sum_i \sum_j \delta_{ij} u_i w_j = \sum_i u_i w_i \end{aligned}$$

Cross Product:

$$\begin{aligned} [\underline{u} \times \underline{w}] &= [(\sum_j \delta_j u_j) \times (\sum_k \delta_k w_k)] \\ &= \sum_j \sum_k [\delta_j \times \delta_k] u_j w_k = \sum_i \sum_j \sum_k \varepsilon_{ijk} \delta_i u_j w_k \\ &= \begin{vmatrix} \delta_1 & \delta_2 & \delta_3 \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \end{aligned}$$

Proofs of Identities (Example):

Prove the following identity

$$\underline{u} \times [\underline{v} \times \underline{w}] = \underline{v} (\underline{u} \cdot \underline{w}) - \underline{w} (\underline{u} \cdot \underline{v})$$

This identity will be proven for the i-component, so the summation \sum_i will be dropped out for the

sake of simplicity.

$$\begin{aligned}
 (\mathbf{u} \times [\mathbf{v} \times \mathbf{w}])_i &= \sum_j \sum_k \epsilon_{ijk} u_j [\mathbf{v} \times \mathbf{w}]_k = \sum_j \sum_k \epsilon_{ijk} u_j [\sum_l \sum_m \epsilon_{klm} v_l w_m] = \\
 &= \sum_j \sum_k \sum_l \sum_m \epsilon_{ijk} \epsilon_{klm} u_j v_l w_m \\
 &= \sum_j \sum_k \sum_l \sum_m \epsilon_{ijk} \epsilon_{lmk} u_j v_l w_m \\
 &= \sum_j \sum_l \sum_m (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) u_j v_l w_m \\
 &= \sum_j \sum_l \sum_m \delta_{il} \delta_{jm} u_j v_l w_m - \sum_j \sum_l \sum_m \delta_{im} \delta_{jl} u_j v_l w_m
 \end{aligned}$$

set $l=i$ in the first term and $m=i$ in the second term

$$= v_i \sum_j \sum_m \delta_{jm} u_j w_m - w_i \sum_j \sum_l \delta_{jl} u_j v_l$$

set $m=j$ in the first term and $l=j$ in the second term

$$\begin{aligned}
 &= v_i \sum_j u_j w_j - w_i \sum_j u_j v_j \\
 &= v_i (\mathbf{u} \cdot \mathbf{w}) - w_i (\mathbf{u} \cdot \mathbf{v}) \\
 &= \mathbf{v} (\mathbf{u} \cdot \mathbf{w}) - \mathbf{w} (\mathbf{u} \cdot \mathbf{v})
 \end{aligned}$$

VECTOR DIFFERENTIAL OPERATIONS

Define first the **del** operator, which is a vector

$$\nabla = \delta_1 \frac{\partial}{\partial x_1} + \delta_2 \frac{\partial}{\partial x_2} + \delta_3 \frac{\partial}{\partial x_3} = \sum_i \delta_i \frac{\partial}{\partial x_i}$$

The Gradient of a Scalar Field:

$$\nabla s = \delta_1 \frac{\partial s}{\partial x_1} + \delta_2 \frac{\partial s}{\partial x_2} + \delta_3 \frac{\partial s}{\partial x_3} = \sum_i \delta_i \frac{\partial s}{\partial x_i}$$

not commutative	$\nabla s \neq s \nabla$
not associative	$(\nabla r)s \neq \nabla(rs)$
distributive	$\nabla(r+s) = \nabla r + \nabla s$

The Divergence of a Vector Field:

$$\begin{aligned} (\nabla \cdot \mathbf{u}) &= \left[\sum_i \delta_i \frac{\partial}{\partial x_i} \right] \cdot \left[\sum_j \delta_j u_j \right] = \sum_i \sum_j [\delta_i \cdot \delta_j] \frac{\partial}{\partial x_i} u_j \\ &= \sum_i \sum_j \delta_{ij} \frac{\partial}{\partial x_i} u_j = \sum_i \frac{\partial u_i}{\partial x_i} \end{aligned}$$

not commutative	$(\nabla \cdot \mathbf{u}) \neq (\mathbf{u} \cdot \nabla)$
not associative	$(\nabla \cdot \mathbf{s}) \mathbf{u} \neq (\nabla \mathbf{s} \cdot \mathbf{u})$
distributive	$\nabla \cdot (\mathbf{u} + \mathbf{w}) = (\nabla \cdot \mathbf{u}) + (\nabla \cdot \mathbf{w})$

The Curl of a Vector Field:

$$\begin{aligned}
 [\nabla \times \mathbf{u}] &= \left[\left[\sum_j \delta_j \frac{\partial}{\partial x_j} \right] \times \left[\sum_k \delta_k \mathbf{u}_k \right] \right] = \sum_j \sum_k [\delta_j \times \delta_k] \frac{\partial}{\partial x_j} \mathbf{u}_k \\
 &= \begin{vmatrix} \delta_1 & \delta_2 & \delta_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{vmatrix} \\
 &= \delta_1 \left(\frac{\partial \mathbf{u}_3}{\partial x_2} - \frac{\partial \mathbf{u}_2}{\partial x_3} \right) + \delta_2 \left(\frac{\partial \mathbf{u}_1}{\partial x_3} - \frac{\partial \mathbf{u}_3}{\partial x_1} \right) + \delta_3 \left(\frac{\partial \mathbf{u}_2}{\partial x_1} - \frac{\partial \mathbf{u}_1}{\partial x_2} \right)
 \end{aligned}$$

$[\nabla \times \mathbf{u}] = \text{curl}(\mathbf{u}) = \text{rot}(\mathbf{u})$ It is distributive but not commutative or associative.

The Laplacian Operator:

The Laplacian of a scalar is:

$$(\nabla \cdot \nabla s) = \sum_i \frac{\partial^2}{\partial x_i^2} s = \frac{\partial^2 s}{\partial x_1^2} + \frac{\partial^2 s}{\partial x_2^2} + \frac{\partial^2 s}{\partial x_3^2}$$

The Laplacian of a vector is:

$$\nabla^2 \mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - [\nabla \times [\nabla \times \mathbf{u}]]$$

The Substantial Derivative of a Scalar Field:

If \mathbf{u} is assumed to be the local fluid velocity then:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\mathbf{u} \cdot \nabla)$$

The substantial derivative for a scalar is:

$$\frac{Ds}{Dt} = \frac{\partial s}{\partial t} + \sum_i u_i \frac{\partial s}{\partial x_i}$$

The substantial derivative for a vector is:

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = \sum_i \delta_i \left(\frac{\partial \mathbf{u}_i}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u}_i \right)$$

This expression is only to be used for rectangular co-ordinates. For all co-ordinates:

$$(\mathbf{u} \cdot \nabla) \mathbf{u} = \frac{1}{2} (\mathbf{u} \cdot \nabla) \mathbf{u} - [\mathbf{u} \times [\nabla \times \mathbf{u}]]$$

SECOND - ORDER TENSORS

A vector \mathbf{u} is specified by giving its three components, namely u_1 , u_2 , and u_3 . Similarly, a second-order tensor $\boldsymbol{\tau}$ is specified by giving its **nine** components.

$$\boldsymbol{\tau} = \begin{vmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{vmatrix}$$

The elements τ_{11} , τ_{22} , and τ_{33} are called diagonal while all the others are the non-diagonal elements of the tensor. If $\tau_{12}=\tau_{21}$, $\tau_{31}=\tau_{13}$, and $\tau_{32}=\tau_{23}$ then the tensor is **symmetric**. The **transpose** of $\boldsymbol{\tau}$ is defined as:

$$\boldsymbol{\tau}^* = \begin{vmatrix} \tau_{11} & \tau_{21} & \tau_{31} \\ \tau_{12} & \tau_{22} & \tau_{32} \\ \tau_{13} & \tau_{23} & \tau_{33} \end{vmatrix}$$

If $\boldsymbol{\tau}$ is symmetric then $\boldsymbol{\tau}=\boldsymbol{\tau}^*$.

Dyadic Product of Two Vectors:

This is defined as follows:

$$\mathbf{u}\mathbf{w} = \begin{vmatrix} u_1 w_1 & u_1 w_2 & u_1 w_3 \\ u_2 w_1 & u_2 w_2 & u_2 w_3 \\ u_3 w_1 & u_3 w_2 & u_3 w_3 \end{vmatrix}$$

Unit Tensor:

$$\delta = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

The components of the unit tensor are δ_{ij} (kronecker delta for $i,j=1,3$)

Unit Dyads:

These are just the dyadic products of unit vectors, $\delta_m \delta_n$ in which $m,n=1,2,3$.

$$\begin{aligned} \delta_1 \delta_1 &= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} & \delta_1 \delta_2 &= \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \\ \delta_1 \delta_3 &= \begin{vmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} & \delta_3 \delta_2 &= \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} \end{aligned}$$

Thus, a tensor can be represented as:

$$\underline{\underline{\tau}} = \sum_i \sum_j \delta_i \delta_j \tau_{ij}$$

and the dyadic product of two vectors as:

$$\underline{\underline{u w}} = \sum_i \sum_j \delta_i \delta_j u_i w_j$$

Also note the following identities:

$$(\delta_i \delta_j \cdot \delta_k \delta_l) = \delta_{il} \delta_{jk} \quad \text{scalar}$$

$$[\delta_i \delta_j \cdot \delta_k] = \delta_i \delta_{jk} \quad \text{vector}$$

$$[\delta_i \cdot \delta_j \delta_k] = \delta_{ij} \delta_k \quad \text{vector}$$

$$\delta_i \delta_j \cdot \delta_k \delta_l = \delta_{jk} \delta_i \delta_l \quad \text{tensor}$$

Addition of Tensors:

$$\begin{aligned} \sigma + \tau &= \sum_i \sum_j \delta_i \delta_j \sigma_{ij} + \sum_i \sum_j \delta_i \delta_j \tau_{ij} \\ &= \sum_i \sum_j \delta_i \delta_j (\sigma_{ij} + \tau_{ij}) \end{aligned}$$

Multiplication of a Tensor by a Scalar:

$$s \underline{\underline{\tau}} = s \sum_i \sum_j \delta_i \delta_j \tau_{ij} = \sum_i \sum_j \delta_i \delta_j (s \tau_{ij})$$

Double Dot Product of Two Tensors:

$$\sigma : \tau = (\sum_i \sum_j \delta_i \delta_j \sigma_{ij}) : (\sum_k \sum_l \delta_k \delta_l \tau_{kl})$$

$$= \sum_i \sum_j \sum_k \sum_l (\delta_i \delta_j \cdot \delta_k \delta_l) \sigma_{ij} \tau_{kl}$$

$$= \sum_i \sum_j \sum_k \sum_l \delta_{il} \delta_{jk} \sigma_{ij} \tau_{kl}$$

set $l=i$ and $k=j$ to simplify to:

$$= \sum_i \sum_j \sigma_{ij} \tau_{ji} \quad \text{which is a scalar } 2+2-4 = 0$$

Dot Product of Two Tensors:

$$\sigma \cdot \tau = (\sum_i \sum_j \delta_i \delta_j \sigma_{ij}) \cdot (\sum_k \sum_l \delta_k \delta_l \tau_{kl})$$

$$= \sum_i \sum_j \sum_k \sum_l (\delta_i \delta_j \cdot \delta_k \delta_l) \sigma_{ij} \tau_{kl}$$

$$= \sum_i \sum_j \sum_k \sum_l (\delta_{jk} \delta_i \delta_l) \sigma_{ij} \tau_{kl}$$

$$= \sum_i \sum_l \delta_i \delta_l (\sum_j \sigma_{ij} \tau_{jl})$$

Vector Product (or Dot Product) of a Tensor with a Vector:

$$[\tau \cdot \mathbf{u}] = [(\sum_i \sum_j \delta_i \delta_j \tau_{ij}) \cdot (\sum_k \delta_k \mathbf{u}_k)]$$

$$= \sum_i \sum_j \sum_k \delta_i \delta_{jk} \tau_{ij} \mathbf{u}_k$$

$$= \sum_i \delta_i (\sum_j \tau_{ij} \mathbf{u}_j)$$

Differential Operations:

$$\begin{aligned}
[\nabla \cdot \boldsymbol{\tau}] &= \left[\left(\sum_i \delta_i \frac{\partial}{\partial x_i} \right) \right] \cdot \left[\sum_j \sum_k \delta_j \delta_k \tau_{jk} \right] \\
&= \sum_i \sum_j \sum_k \left[\delta_i \cdot \delta_j \delta_k \right] \frac{\partial}{\partial x_i} \tau_{jk} \\
&= \sum_i \sum_j \sum_k \delta_{ij} \delta_k \frac{\partial}{\partial x_i} \tau_{jk} \\
&= \sum_k \delta_k \left(\sum_i \frac{\partial}{\partial x_i} \tau_{ik} \right)
\end{aligned}$$

Some other identities which can readily be proven are:

$$\mathbf{w} \cdot \nabla \mathbf{u} = \sum_i \sum_k \delta_k w_i \frac{\partial}{\partial x_i} u_k$$

$$\boldsymbol{\tau} : \nabla \mathbf{u} = \sum_i \sum_j \tau_{ij} \frac{\partial}{\partial x_j} u_i$$

INTEGRAL THEOREMS FOR VECTORS AND TENSORS

Gauss - Ostrogradskii Divergence Theorem :

If V is a closed region in space surrounded by a surface S then

$$\iiint_V (\nabla \cdot \mathbf{u}) dV = \iint_S (\mathbf{n} \cdot \mathbf{u}) dS = \iint_S (\mathbf{u} \cdot \mathbf{n}) dS = \iint_S u_n dS$$

where \mathbf{n} is the outwardly directed normal vector.

$$\iiint_V \nabla s dV = \iint_S \mathbf{n} s dS$$

where s is a scalar quantity.

$$\iiint_V [\nabla \cdot \boldsymbol{\tau}] dV = \iint_S [\mathbf{n} \cdot \boldsymbol{\tau}] dS$$

where $\boldsymbol{\tau}$ is a tensor.

The Stokes Curl Theorem:

If S is a surface bounded by a close curve C , then:

$$\iint_S ([\nabla \times \mathbf{u}] \cdot \mathbf{n}) dS = \oint_C (\mathbf{u} \cdot \mathbf{t}) dC$$

where \mathbf{t} is the tangential vector in the direction of the integration and \mathbf{n} is the unit vector normal to S in the direction that a right-handed screw would move if its head were twisted in the direction of integration along C .

The Leibnitz Formula for Differentiating a Triple Integral:

$$\frac{d}{dt} \iiint_V s \, dV = \iiint_V \frac{\partial s}{\partial t} \, dV + \iint_S s (u_s \cdot n) \, dS$$

where u_s is the velocity of any surface element, and s is a scalar quantity which can be a function of position and time i.e., $s=s(x,y,z,t)$. Keep in mind that $V=V(t)$ and $S=S(t)$.

If the surface of the volume is moving with the local fluid velocity ($u_s=u$), then

$$\frac{d}{dt} \iiint_V \rho s \, dV = \iiint_V \rho \frac{Ds}{Dt} \, dV$$

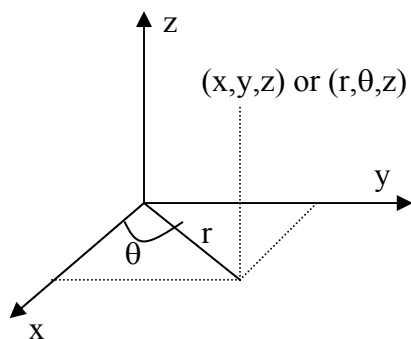
where ρ is the fluid density.

CURVILINEAR COORDINATES

Thus far, we have considered only rectangular co-ordinates x , y and z . However, many times in fluid mechanics it is more convenient to work with **curvilinear** co-ordinates. The two most common curvilinear co-ordinate systems are the **cylindrical** and the **spherical**. In this development, we are interested in knowing how to write various differentials, such as ∇s , $[\nabla \times \mathbf{v}]$, and $(\boldsymbol{\tau} : \nabla \mathbf{v})$ in curvilinear co-ordinates. It turns out that two are the useful tools in doing this

- The expression for ∇ in the curvilinear co-ordinates.
- The spatial derivatives of the unit vectors in curvilinear co-ordinates.

Cylindrical Coordinates



There a point is located by giving the values of r , θ , and z instead of x , y , and z which is the case for the Cartesian co-ordinates. From simple geometry one may derive the following expressions between these two systems of co-ordinates. These are:

$$x = r \cos \theta \quad r = +\sqrt{x^2 + y^2}$$

$$y = r \sin \theta \quad \theta = \arctan(y/x)$$

$$z = z \quad z = z$$

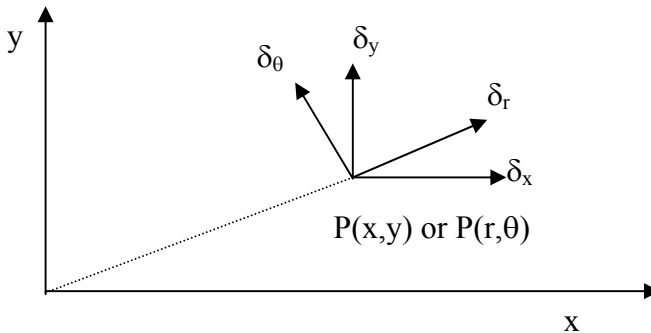
To convert derivatives with respect to x , y , and z into derivatives with respect to r , θ , and z , the "chain"rule of differentiation is used. Thus one may derive

$$\frac{\partial}{\partial x} = (\cos \theta) \frac{\partial}{\partial r} + \left(-\frac{\sin \theta}{r} \right) \frac{\partial}{\partial \theta} + (0) \frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial y} = (\sin \theta) \frac{\partial}{\partial r} + \left(\frac{\cos \theta}{r} \right) \frac{\partial}{\partial \theta} + (0) \frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial z} = (0) \frac{\partial}{\partial r} + (0) \frac{\partial}{\partial \theta} + (1) \frac{\partial}{\partial z}$$

With these relations, derivatives of any scalar functions with respect to x , y and z can be expressed in terms of derivatives with respect to r , θ and z . Now we turn our attention to the interrelationship between the unit vectors. We note those in the Cartesian coordinates as δ_x , δ_y , and δ_z and those in the cylindrical coordinates as δ_r , δ_θ , and δ_z . To see how these are related consider the Figure below where it can be seen that as the point P is moving in the (x,y) plane the directions of δ_r , δ_θ change. Elementary trigonometrical arguments lead to the following relations:



$$\delta_r = (\cos \theta) \delta_x + (\sin \theta) \delta_y + (0) \delta_z$$

$$\delta_\theta = (-\sin \theta) \delta_x + (\cos \theta) \delta_y + (0) \delta_z$$

$$\delta_z = (0) \delta_x + (0) \delta_y + (1) \delta_z$$

These can be solved for δ_x , δ_y , and δ_z to result

$$\delta_x = (\cos \theta) \delta_r + (-\sin \theta) \delta_\theta + (0) \delta_z$$

$$\delta_y = (\sin \theta) \delta_r + (\cos \theta) \delta_\theta + (0) \delta_z$$

$$\delta_z = (0) \delta_r + (0) \delta_\theta + (1) \delta_z$$

Vectors and tensors can be decomposed into components in all systems of co-ordinates just as with

respect to rectangular co-ordinates discussed previously. For example:

$$[\mathbf{v} \times \mathbf{w}] = \delta_r (v_\theta w_z - v_z w_\theta) + \delta_\theta (v_z w_r - v_r w_z) + \delta_z (v_r w_\theta - v_\theta w_r)$$

$$(\boldsymbol{\sigma} \cdot \boldsymbol{\tau}) = \delta_r \delta_r (\sigma_{rr} \tau_{rr} + \sigma_{r\theta} \tau_{\theta r} \tau_{\theta r} + \sigma_{rz} \tau_{zr}) + \delta_r \delta_\theta (\cdot) + \delta_r \delta_z (\cdot) + \dots$$

Spherical Coordinates

The spherical co-ordinates are related to rectangular by the following relations

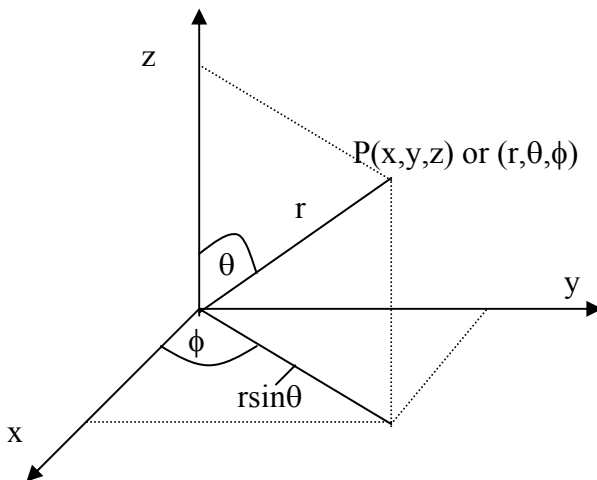


Figure: A Spherical system of co-ordinates

$$x = r \sin \theta \cos \phi \quad r = +\sqrt{x^2 + y^2 + z^2}$$

$$y = r \sin \theta \sin \phi \quad \theta = \arctan(\sqrt{x^2 + y^2}/z)$$

$$z = r \cos \theta \quad \phi = \arctan(y/z)$$

The derivative operators are as follows:

$$\frac{\partial}{\partial x} = (\sin \theta \cos \phi) \frac{\partial}{\partial r} + \left(-\frac{\cos \theta \cos \phi}{r} \right) \frac{\partial}{\partial \theta} + \left(-\frac{\sin \phi}{r \sin \theta} \right) \frac{\partial}{\partial \phi}$$

$$\frac{\partial}{\partial y} = (\sin \theta \sin \phi) \frac{\partial}{\partial r} + \left(\frac{\cos \theta \sin \phi}{r} \right) \frac{\partial}{\partial \theta} + \left(\frac{\cos \phi}{r \sin \theta} \right) \frac{\partial}{\partial \phi}$$

$$\frac{\partial}{\partial z} = (\cos \theta) \frac{\partial}{\partial r} + \left(-\frac{\sin \theta}{r} \right) \frac{\partial}{\partial \theta} + (0) \frac{\partial}{\partial \phi}$$

The relations between the unit vectors are:

$$\delta_r = (\sin \theta \cos \phi) \delta_x + (\sin \theta \sin \phi) \delta_y + (\cos \theta) \delta_z$$

$$\delta_\theta = (\cos \theta \cos \phi) \delta_x + (\cos \theta \sin \phi) \delta_y + (-\sin \theta) \delta_z$$

$$\delta_\phi = (-\sin \phi) \delta_x + (\cos \phi) \delta_y + (0) \delta_z$$

These can be solved for δ_x , δ_y , and δ_z to result:

$$\delta_x = (\sin \theta \cos \phi) \delta_r + (\cos \theta \cos \phi) \delta_\theta + (-\sin \phi) \delta_\phi$$

$$\delta_y = (\sin \theta \sin \phi) \delta_r + (\cos \theta \sin \phi) \delta_\theta + (\cos \phi) \delta_\phi$$

$$\delta_z = (\cos \theta) \delta_r + (-\sin \theta) \delta_\theta + (0) \delta_\phi$$

Some example operations in spherical co-ordinates are:

$$\begin{aligned} (\sigma : \tau) = & \sigma_{rr} \tau_{rr} + \sigma_{r\theta} \tau_{r\theta} + \sigma_{r\phi} \tau_{r\phi} + \sigma_{\theta r} \tau_{r\theta} + \sigma_{\theta\theta} \tau_{\theta\theta} + \sigma_{\theta\phi} \tau_{\theta\phi} + \\ & + \sigma_{\phi r} \tau_{r\phi} + \sigma_{\phi\theta} \tau_{\theta\phi} + \sigma_{\phi\phi} \tau_{\phi\phi} \end{aligned}$$

$$(\mathbf{u} \cdot [\mathbf{v} \times \mathbf{w}]) = \begin{vmatrix} u_r & u_\theta & u_\phi \\ v_r & v_\theta & v_\phi \\ w_r & w_\theta & w_\phi \end{vmatrix}$$

These examples tell us that the relations (not involving ∇ !) discussed earlier can be written in terms of spherical components.

DIFFERENTIAL OPERATIONS IN CURVILINEAR COORDINATES

The operator ∇ will now be derived in cylindrical and spherical co-ordinates.

Cylindrical:

The following relations can be obtained by differentiating the relations between the unit vectors in the cylindrical co-ordinates with those in the Cartesian ones.

$$\begin{aligned} \frac{\partial}{\partial r} \delta_r &= 0 & \frac{\partial}{\partial r} \delta_\theta &= 0 & \frac{\partial}{\partial r} \delta_z &= 0 \\ \frac{\partial}{\partial \theta} \delta_r &= -\delta_\theta & \frac{\partial}{\partial \theta} \delta_\theta &= \delta_r & \frac{\partial}{\partial \theta} \delta_z &= 0 \\ \frac{\partial}{\partial z} \delta_r &= 0 & \frac{\partial}{\partial z} \delta_\theta &= 0 & \frac{\partial}{\partial z} \delta_z &= 0 \end{aligned}$$

The definition of ∇ in Cartesian co-ordinates is:

$$\nabla = \delta_x \frac{\partial}{\partial x} + \delta_y \frac{\partial}{\partial y} + \delta_z \frac{\partial}{\partial z}$$

Substituting δ_x , δ_y , and δ_z in terms of δ_r , δ_θ , and δ_z and simplifying we obtain ∇ for cylindrical co-ordinates, that is:

$$\nabla = \delta_r \frac{\partial}{\partial r} + \delta_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \delta_z \frac{\partial}{\partial z}$$

Spherical Coordinates

The following relations can be obtained by differentiating the relations between the unit vectors in the spherical coordinates with those in the Cartesian ones.

$$\begin{aligned} \frac{\partial}{\partial r} \delta_r &= 0 & \frac{\partial}{\partial r} \delta_\theta &= 0 & \frac{\partial}{\partial r} \delta_\phi &= 0 \\ \frac{\partial}{\partial \theta} \delta_r &= \delta_\theta & \frac{\partial}{\partial \theta} \delta_\theta &= -\delta_r & \frac{\partial}{\partial \theta} \delta_\phi &= 0 \\ \frac{\partial}{\partial \phi} \delta_r &= \delta_\phi \sin \theta & \frac{\partial}{\partial \phi} \delta_\theta &= \delta_\phi \cos \theta & \frac{\partial}{\partial \phi} \delta_z &= -\delta_r \sin \theta - \delta_\theta \cos \theta \end{aligned}$$

The definition of ∇ in Cartesian co-ordinates is:

$$\nabla = \delta_x \frac{\partial}{\partial x} + \delta_y \frac{\partial}{\partial y} + \delta_z \frac{\partial}{\partial z}$$

Substituting δ_x , δ_y , and δ_z in terms of δ_r , δ_θ , and δ_ϕ , and simplifying we obtain ∇ for spherical coordinates, that is:

$$\nabla = \delta_r \frac{\partial}{\partial r} + \delta_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \delta_\phi \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi}$$

For more details see:

1. R.B. Bird, W.E. Stewart and E.N. Lightfoot, "Transport Phenomena," Wiley, New York, 1960.
2. R.B. Bird, R.C. Armstrong and O. Hassager, "Dynamics of Polymeric Liquids," Vol.1, "Fluid Mechanics," Wiley, New York, 1977.

TABLE A.7-1
SUMMARY OF DIFFERENTIAL OPERATIONS INVOLVING THE ∇ -OPERATOR
IN RECTANGULAR COORDINATES^a (x, y, z)

$(\nabla \cdot v) = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$	(A)
$(\nabla^2 s) = \frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} + \frac{\partial^2 s}{\partial z^2}$	(B)
$(\tau : \nabla v) = \tau_{xx} \left(\frac{\partial v_x}{\partial x} \right) + \tau_{yy} \left(\frac{\partial v_y}{\partial y} \right) + \tau_{zz} \left(\frac{\partial v_z}{\partial z} \right) + \tau_{xy} \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right) + \tau_{yz} \left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} \right) + \tau_{zx} \left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z} \right)$	(C)
$\left\{ \begin{array}{l} [\nabla s]_x = \frac{\partial s}{\partial x} \\ [\nabla s]_y = \frac{\partial s}{\partial y} \\ [\nabla s]_z = \frac{\partial s}{\partial z} \end{array} \right.$	(D)
	(E)
	(F)
$\left\{ \begin{array}{l} [\nabla \times v]_x = \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \\ [\nabla \times v]_y = \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \\ [\nabla \times v]_z = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \end{array} \right.$	(G)
	(H)
	(I)
$\left\{ \begin{array}{l} [\nabla \cdot \tau]_x = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \\ [\nabla \cdot \tau]_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} \\ [\nabla \cdot \tau]_z = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \end{array} \right.$	(J)
	(K)
	(L)
$\left\{ \begin{array}{l} [\nabla^2 v]_x = \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \\ [\nabla^2 v]_y = \frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \\ [\nabla^2 v]_z = \frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \end{array} \right.$	(M)
	(N)
	(O)
$\left\{ \begin{array}{l} [v \cdot \nabla v]_x = v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \\ [v \cdot \nabla v]_y = v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \\ [v \cdot \nabla v]_z = v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \end{array} \right.$	(P)
	(Q)
	(R)

^a Operations involving the tensor τ are given for symmetrical τ only.

TABLE A.7-2
SUMMARY OF DIFFERENTIAL OPERATIONS INVOLVING THE ∇ -OPERATOR
IN CYLINDRICAL COORDINATES* (r, θ, z)

$$(\nabla \cdot \mathbf{v}) = \frac{1}{r} \frac{\partial}{\partial r} (rv_r) + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} \quad (A)$$

$$(\nabla^2 s) = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial s}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 s}{\partial \theta^2} + \frac{\partial^2 s}{\partial z^2} \quad (B)$$

$$\begin{aligned} (\boldsymbol{\tau} : \nabla \mathbf{v}) &= \tau_{rr} \left(\frac{\partial v_r}{\partial r} \right) + \tau_{\theta\theta} \left(\frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right) + \tau_{zz} \left(\frac{\partial v_z}{\partial z} \right) \\ &+ \tau_{r\theta} \left[r \frac{\partial}{\partial r} \left(\frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right] + \tau_{\theta z} \left(\frac{1}{r} \frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z} \right) \\ &+ \tau_{rz} \left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z} \right) \end{aligned} \quad (C)$$

$$\left\{ \begin{array}{l} [\nabla s]_r = \frac{\partial s}{\partial r} \quad (D) \\ [\nabla s]_\theta = \frac{1}{r} \frac{\partial s}{\partial \theta} \quad (E) \\ [\nabla s]_z = \frac{\partial s}{\partial z} \quad (F) \end{array} \right. \quad \left\{ \begin{array}{l} [\nabla \times \mathbf{v}]_r = \frac{1}{r} \frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z} \quad (G) \\ [\nabla \times \mathbf{v}]_\theta = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} \quad (H) \\ [\nabla \times \mathbf{v}]_z = \frac{1}{r} \frac{\partial}{\partial r} (rv_\theta) - \frac{1}{r} \frac{\partial v_r}{\partial \theta} \quad (I) \end{array} \right.$$

$$\left\{ \begin{array}{l} [\nabla \cdot \boldsymbol{\tau}]_r = \frac{1}{r} \frac{\partial}{\partial r} (r\tau_{rr}) + \frac{1}{r} \frac{\partial}{\partial \theta} \tau_{r\theta} - \frac{1}{r} \tau_{\theta\theta} + \frac{\partial \tau_{rz}}{\partial z} \quad (J) \\ [\nabla \cdot \boldsymbol{\tau}]_\theta = \frac{1}{r} \frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial r} + \frac{2}{r} \tau_{r\theta} + \frac{\partial \tau_{\theta z}}{\partial z} \quad (K) \\ [\nabla \cdot \boldsymbol{\tau}]_z = \frac{1}{r} \frac{\partial}{\partial r} (r\tau_{rz}) + \frac{1}{r} \frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z} \quad (L) \end{array} \right.$$

$$\left\{ \begin{array}{l} [\nabla^2 \mathbf{v}]_r = \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (rv_r) \right) + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2} \quad (M) \\ [\nabla^2 \mathbf{v}]_\theta = \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (rv_\theta) \right) + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2} \quad (N) \\ [\nabla^2 \mathbf{v}]_z = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \quad (O) \end{array} \right.$$

$$\left\{ \begin{array}{l} [\mathbf{v} \cdot \nabla \mathbf{v}]_r = v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} \quad (P) \\ [\mathbf{v} \cdot \nabla \mathbf{v}]_\theta = v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z \frac{\partial v_\theta}{\partial z} \quad (Q) \\ [\mathbf{v} \cdot \nabla \mathbf{v}]_z = v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \quad (R) \end{array} \right.$$

* Operations involving the tensor $\boldsymbol{\tau}$ are given for symmetrical $\boldsymbol{\tau}$ only.

TABLE A.7-3

SUMMARY OF DIFFERENTIAL OPERATIONS INVOLVING THE ∇ -OPERATOR
IN SPHERICAL COORDINATES* (r, θ, ϕ)

$$(\nabla \cdot v) = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (v_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} \quad (A)$$

$$(\nabla^2 s) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial s}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial s}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 s}{\partial \phi^2} \quad (B)$$

$$\begin{aligned} (\tau : \nabla v) &= \tau_{rr} \left(\frac{\partial v_r}{\partial r} \right) + \tau_{\theta\theta} \left(\frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right) \\ &+ \tau_{\phi\phi} \left(\frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta \cot \theta}{r} \right) \\ &+ \tau_{r\theta} \left(\frac{\partial v_\theta}{\partial r} + \frac{1}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} \right) + \tau_{r\phi} \left(\frac{\partial v_\phi}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{v_\phi}{r} \right) \\ &+ \tau_{\theta\phi} \left(\frac{1}{r} \frac{\partial v_\phi}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} - \frac{\cot \theta}{r} v_\phi \right) \end{aligned} \quad (C)$$

$$\left\{ \begin{array}{l} [\nabla s]_r = \frac{\partial s}{\partial r} \\ [\nabla s]_\theta = \frac{1}{r} \frac{\partial s}{\partial \theta} \\ [\nabla s]_\phi = \frac{1}{r \sin \theta} \frac{\partial s}{\partial \phi} \end{array} \right. \quad \begin{array}{l} (D) \\ (E) \\ (F) \end{array}$$

$$\left\{ \begin{array}{l} [\nabla \times v]_r = \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (v_\phi \sin \theta) - \frac{1}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} \\ [\nabla \times v]_\theta = \frac{1}{r \sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{1}{r} \frac{\partial}{\partial r} (r v_\phi) \\ [\nabla \times v]_\phi = \frac{1}{r} \frac{\partial}{\partial r} (r v_\theta) - \frac{1}{r} \frac{\partial v_r}{\partial \theta} \end{array} \right. \quad \begin{array}{l} (G) \\ (H) \\ (I) \end{array}$$

TABLE A.7-3 (continued)

$$\left\{ \begin{aligned} [\nabla \cdot \tau]_r &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \tau_{rr}) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\tau_{r\theta} \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial \tau_{r\phi}}{\partial \phi} - \frac{\tau_{\theta\theta} + \tau_{\phi\phi}}{r} & (J) \\ [\nabla \cdot \tau]_\theta &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \tau_{r\theta}) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\tau_{\theta\theta} \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial \tau_{\theta\phi}}{\partial \phi} \\ &\quad + \frac{\tau_{r\theta}}{r} - \frac{\cot \theta}{r} \tau_{\phi\phi} & (K) \\ [\nabla \cdot \tau]_\phi &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \tau_{r\phi}) + \frac{1}{r} \frac{\partial \tau_{\theta\phi}}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \tau_{\phi\phi}}{\partial \phi} + \frac{\tau_{r\phi}}{r} + \frac{2 \cot \theta}{r} \tau_{\theta\phi} & (L) \\ [\nabla^2 v]_r &= \nabla^2 v_r - \frac{2v_r}{r^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} - \frac{2v_\theta \cot \theta}{r^2} - \frac{2}{r^2 \sin \theta} \frac{\partial v_\phi}{\partial \phi} & (M) \\ [\nabla^2 v]_\theta &= \nabla^2 v_\theta + \frac{2}{r^2} \frac{\partial v_r}{\partial r} - \frac{v_\theta}{r^2 \sin^2 \theta} - \frac{2 \cos \theta}{r^2 \sin^2 \theta} \frac{\partial v_\phi}{\partial \phi} & (N) \\ [\nabla^2 v]_\phi &= \nabla^2 v_\phi - \frac{v_\phi}{r^2 \sin^2 \theta} + \frac{2}{r^2 \sin \theta} \frac{\partial v_r}{\partial r} + \frac{2 \cos \theta}{r^2 \sin^2 \theta} \frac{\partial v_\theta}{\partial \phi} & (O) \\ [v \cdot \nabla v]_r &= v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r} & (P) \\ [v \cdot \nabla v]_\theta &= v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2 \cot \theta}{r} & (Q) \\ [v \cdot \nabla v]_\phi &= v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_r}{r} + \frac{v_\theta v_\phi \cot \theta}{r} & (R) \end{aligned} \right.$$

^a Operations involving the tensor τ are given for symmetrical τ only.

We now use the relations given in Eqs. A.7-1, 2, and 3 to evaluate the derivatives of the unit vectors. This gives

$$\begin{aligned}
 (\nabla \cdot \mathbf{v}) &= (\delta_r \cdot \delta_r) \frac{\partial v_r}{\partial r} + (\delta_r \cdot \delta_\theta) \frac{\partial v_\theta}{\partial r} + (\delta_r \cdot \delta_z) \frac{\partial v_z}{\partial r} \\
 &\quad + (\delta_\theta \cdot \delta_r) \frac{1}{r} \frac{\partial v_r}{\partial \theta} + (\delta_\theta \cdot \delta_\theta) \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + (\delta_\theta \cdot \delta_z) \frac{1}{r} \frac{\partial v_z}{\partial \theta} \\
 &\quad + \frac{v_r}{r} (\delta_\theta \cdot \delta_\theta) + \frac{v_\theta}{r} (\delta_\theta \cdot \{-\delta_r\}) \\
 &\quad + (\delta_z \cdot \delta_r) \frac{\partial v_r}{\partial z} + (\delta_z \cdot \delta_\theta) \frac{\partial v_\theta}{\partial z} + (\delta_z \cdot \delta_z) \frac{\partial v_z}{\partial z} \quad (\text{A.7-18})
 \end{aligned}$$

Since $(\delta_r \cdot \delta_r) = 1$, $(\delta_r \cdot \delta_\theta) = 0$, etc., the latter simplifies to

$$(\nabla \cdot \mathbf{v}) = \frac{\partial v_r}{\partial r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z} \quad (\text{A.7-19})$$

which is the same as Eq. A of Table A.7-2.

b. Next we examine the dyadic product $\nabla \mathbf{v}$:

$$\begin{aligned}
 \nabla \mathbf{v} &= \left\{ \delta_r \frac{\partial}{\partial r} + \delta_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \delta_z \frac{\partial}{\partial z} \right\} \{ \delta_r v_r + \delta_\theta v_\theta + \delta_z v_z \} \\
 &= \delta_r \delta_r \frac{\partial v_r}{\partial r} + \delta_r \delta_\theta \frac{\partial v_\theta}{\partial r} + \delta_r \delta_z \frac{\partial v_z}{\partial r} \\
 &\quad + \delta_\theta \delta_r \frac{1}{r} \frac{\partial v_r}{\partial \theta} + \delta_\theta \delta_\theta \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \delta_\theta \delta_z \frac{1}{r} \frac{\partial v_z}{\partial \theta} \\
 &\quad + \delta_\theta \delta_\theta \frac{v_r}{r} - \delta_\theta \delta_r \frac{v_\theta}{r} \\
 &\quad + \delta_z \delta_r \frac{\partial v_r}{\partial z} + \delta_z \delta_\theta \frac{\partial v_\theta}{\partial z} + \delta_z \delta_z \frac{\partial v_z}{\partial z} \\
 &= \delta_r \delta_r \frac{\partial v_r}{\partial r} + \delta_r \delta_\theta \frac{\partial v_\theta}{\partial r} + \delta_r \delta_z \frac{\partial v_z}{\partial r} \\
 &\quad + \delta_\theta \delta_r \left(\frac{1}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} \right) + \delta_\theta \delta_\theta \left(\frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right) + \delta_\theta \delta_z \frac{1}{r} \frac{\partial v_z}{\partial \theta} \\
 &\quad + \delta_z \delta_r \frac{\partial v_r}{\partial z} + \delta_z \delta_\theta \frac{\partial v_\theta}{\partial z} + \delta_z \delta_z \frac{\partial v_z}{\partial z} \quad (\text{A.7-20})
 \end{aligned}$$

Hence the rr -component is $\partial v_r / \partial r$, the $r\theta$ -component is $\partial v_\theta / \partial r$, etc.

INTRODUCTION - FUNDAMENTAL CONCEPTS

Definition of a Fluid

All forms of matter can be classified in terms of their physical appearance or phase into three classes: solids, liquids and gases. Liquids and gases are called **fluids**. A fluid is defined as "a substance that under the action of an infinitesimal force deforms permanently and continuously.

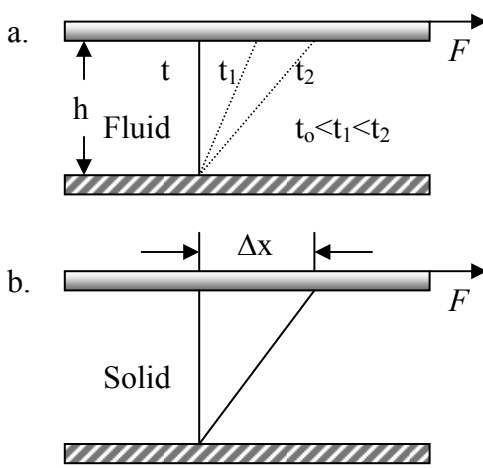


Fig 1

Consider that the space between two parallel plates is occupied by a fluid (Fig. 1a). If a small force is applied on the upper plate, then the plate will move with a constant velocity, V . As a result, the fluid will deform permanently and continuously. If instead a solid is placed in the space between the two plates and the same force is applied, the solid will be deformed by a certain amount, indicated by the displacement, Δx , in Fig. 1b. This displacement will remain there as long as the force is not removed. The above behaviour is customarily indicated by plotting $(F/A$ -shear stress vs dh/h -rate of strain) for fluids and $(F/A$ vs $\Delta x/h$ -strain) for solids.

Continuum Hypothesis

It is possible to study the flow of gases and liquids (fluids) from the molecular point of view by writing the appropriate equations for each molecule and taking into account all molecular interactions. However, mathematically this is a very complex problem and very impractical for most engineering applications. It is possible to describe many flow problems without a detailed knowledge of molecular motions and interactions, by introducing the *continuum hypothesis*. According to this we assume that at every point in the region occupied by a deformable material the state of that material can be described in terms of the velocity components, V_i , and material properties such as T , p , ρ , and μ .

However, we know that matter consists of discrete molecules. The precise location defined

by the coordinates, x , y , and z may correspond in reality to a point within a molecule or to a point in the open space between molecules. In the former case density is very high and in the latter density is zero. T and p have also no meaning in either case, as they are associated with statistical averages involving many molecules. However, for practical purposes, it is still possible to make use of the concept of a continuum as long as there exist a volume size that is sufficiently small that spatial derivatives can be defined but also sufficiently large that there are enough molecules to give averages that converge to unique values of the field variables. See Fig.2, which utilises the density to illustrate the concept of the continuum.

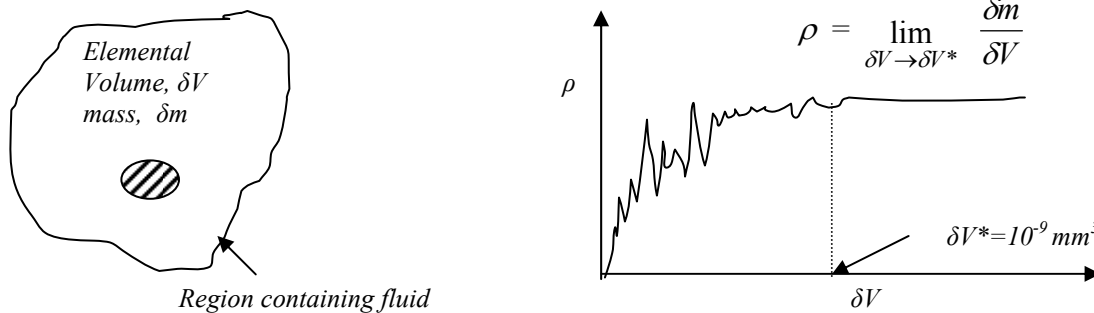


Fig.2

A criterion used to evaluate the validity of the continuum approach is based on Knudsen number, Kn /(mean free path of molecules)/(characteristic length of flow), so that

$Kn < 0.01$ continuum approach valid

$Kn > 0.1$ must use statistical approach

At intermediate values, we can sometimes use continuum equations with modified boundary conditions involving a relaxation of the no-slip boundary condition. For an ideal gas the mean free path is proportional to T/p .

Compressibility

Compressibility defines the ability of a fluid to change its density under the action of pressure. It is defined as the inverse of the bulk modulus, that is

$$\beta = \frac{1}{E} = -\frac{1}{\tilde{v}} \frac{d\tilde{v}}{dP} = \frac{1}{\rho} \frac{d\rho}{dp}$$

where β is the compressibility, \tilde{v} is the specific volume and ρ is the density of the fluid. For liquids the bulk modulus is very high (water at 20°C, $E=2,140,000$ kPa), so that the change of density with pressure is negligible. In analysis of flows, liquids are treated as incompressible fluids. For gases at low speeds compared to the speed of sound the density changes are also small. Thus for gas flow a useful measure of the role of compressibility is the Mach number defined as:

$$Ma \equiv \frac{U}{C}$$

where U is the characteristic velocity of problem and C is the velocity of sound in the fluid. For an ideal gas, the velocity of sound is given by:

$$C = \sqrt{\frac{\gamma RT}{M}}$$

where M is the molecular weight of the gas, γ is the specific gravity, and R is the gas constant. If $Ma < 0.3$ one may neglect the density changes occurring due to compressibility effects.

Finally note that density gradients may also arise from temperature gradients (viscous heating) and composition in situations where heat and mass transfer are occurring (taking place).

No-Slip Boundary Condition

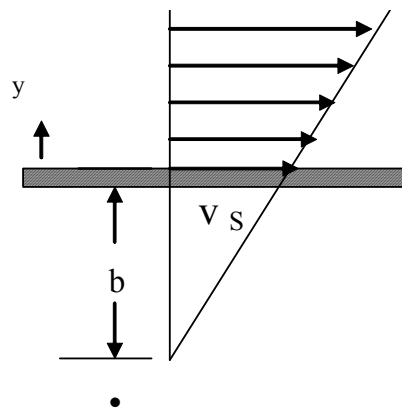
Experiments have shown that a fluid adjacent to a solid interface cannot slip relative to the surface. This is true no matter how small the viscosity is. This was basically concluded from the fact that the use of the no-slip boundary condition has led to predictions, which agreed very well with experimental observations. Thus,

$$\mathbf{V}_{\text{fluid}} = \mathbf{V}_{\text{wall}}$$

which implies no relative motion between the wall and the fluid. The no-slip boundary condition for Newtonian fluids was the subject of some controversy among nineteenth-century theoreticians who tried to formulate such slip laws (for example Navier proposed such a model). It was rather difficult to accept the no-slip condition for fluids that do not "wet" adjacent solid surfaces like water on wax. However, the wetting phenomenon is related to the surface tension which has absolutely nothing to do with the no-slip condition.

Gases at extremely low pressures do not obey the no-slip condition and are the subject of a special field of study called rarefied gas dynamics. In addition some rheologically complex fluids exhibit a slip at the wall under certain conditions. For example, molten polyethylenes of high molecular weight have been found to slip when the wall shear stress exceeds a critical value usually about 0.1 MPa. For example for the case of a passive polymer/wall interface (no interaction between the polymer and solid surface), de Gennes [C.R. Acad.Sci. Paris serie B, **288**, 219-222 (1979)] proposed an interfacial rheological law in terms of an extrapolation length, b , as follows (inspired by Navier),

$$v_S = b \left[\frac{dv_y}{dy} \right]_{y=0} = b \dot{\gamma}_w = \left[\frac{b}{\mu} \right] \sigma_w$$



where v_S is the slip velocity, $\dot{\gamma}_w$ is the shear rate at the wall (the slope of the velocity profile at the interface, see Figure above), and μ is the viscosity of the melt at $\dot{\gamma}_w$.

Surface Tension

At the interface between a liquid and a gas, or between two immiscible liquids, forces develop which relate the anisotropy of the interactions between liquid molecules in the case of a liquid-liquid interface.

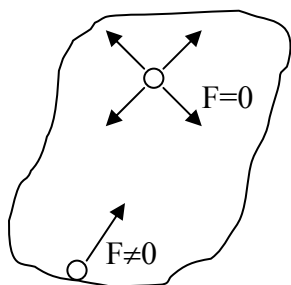


Fig.3

For molecules in the interior (bulk), interactions are isotropic and the net force on each liquid molecule vanishes. This is not the case for molecules at the interface. These are attracted more in the interior of the liquid than by gas molecules such that a nonzero net force results.

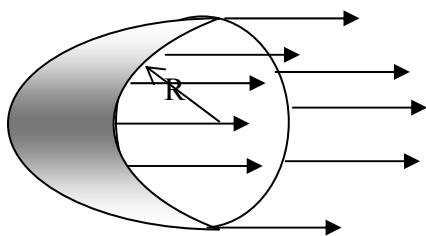


Fig. 4

As a result of these forces a small amount of mercury forms an almost spherical droplet or a small amount of water forms a spherical droplet on a waxed surface. If such a droplet is cut half, there, there is the action of a force per unit length (surface tension) and this is balanced from the pressure force. If $\Delta p = p_B - p_A$ where p_B : interior pressure, p_A : exterior pressure, the net pressure force is:

$$\Delta p \pi R^2 = 2 \pi R \sigma \quad \text{or} \quad \Delta p = \frac{2 \sigma}{R}$$

This pressure difference is called the **capillary pressure**, which is due to the surface tension. Note

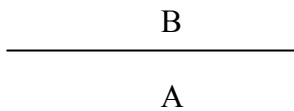
that the pressure inside a drop is greater than the pressure outside the drop.

These principles can also be generalised for 2-dimensional surfaces to produce the Young-Laplace equation of capillary

$$\Delta p = \sigma \left(\frac{1}{R_1} + \frac{1}{R_2} \right)$$

where R_1 , and R_2 are the two principal radii of curvature of the 2-dimensional surface.

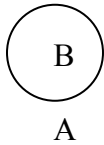
(1) For a plane surface: $R_1 = R_2 = \infty$



$\Delta p = 0$ or $p_A = p_B$ (For plane interfaces there is no pressure jump).

(2) For a sphere:

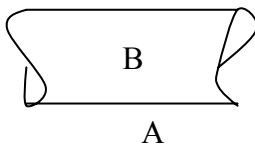
$$R_1 = R_2 = R$$



$\Delta p = 2\sigma/R$ or $p_B - p_A = 2\sigma/R$ (For this case there is a pressure jump from the inside to outside).

(3) For a cylinder:

$$R_1 = 0 \text{ and } R_2 = R$$



$p_B - p_A = \sigma/R$ (For cylindrical interfaces there is also a pressure jump from the inside to outside).

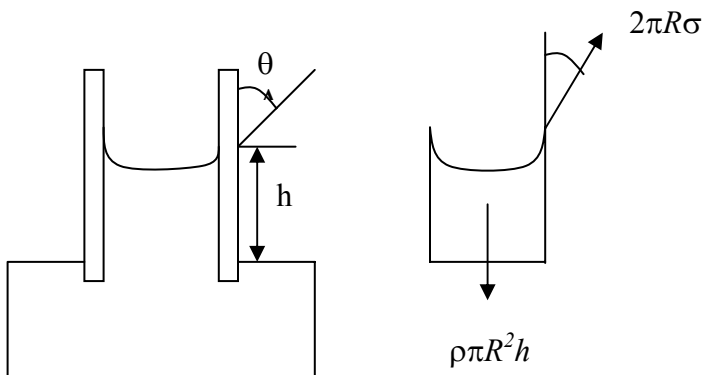
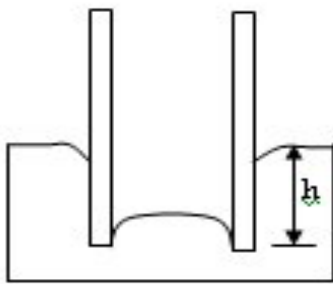


Fig. 5: Rise of liquid in a capillary tube



I

Fig. 5b: fall of liquid in a capillary tube

$$\rho g \pi R^2 h = 2 \pi R \sigma \cos \theta \quad \text{or} \quad h = \frac{2 \sigma \cos \theta}{\rho R g}$$

Measuring h , and θ , one may use this equation as a method to determine the surface tension.

Surface tension plays a significant role in a diversity of small-scale slow flows, as well as in immiscible liquids under equilibrium.

Liquid volumes tend to attain spherical shapes that exhibit the minimum surface-to-volume ratio, the more so the higher their surface tension.

- Movement of liquids through soil and other porous media, flow of thin films, formation of drops and bubbles, breakage of liquid jets.
- Formation and stabilisation of thin films, also surface tension controls levelling and spreading of liquids on substrates with application to spray coating or painting.
- Enhanced oil recovery. Crude oil is trapped in underwater porous natural reservoirs, confined between impermeable rock layers).

KINEMATICS

Kinematics comes from the Greek word **kinesis**, which means motion. It is defined as the science that deals with the study of motion without making reference to the forces that cause motion.

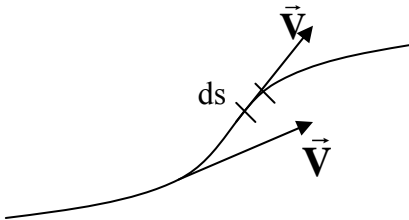
It is essential for:

- The development of a quantitative theory of Fluid Mechanics.
- The interpretation of data obtaining using various visualisation experimental methods.

Streamlines, Pathlines, Streaklines and Timelines

Four generally different types of curves are considered in the study of fluid motion: the streamlines, pathlines, streaklines and timelines. The curves describe various aspects of fluid motion.

Streamline: It is a line in space that is everywhere tangent to the velocity vector at every instant of time.



Consider the velocity \underline{V} at some point with components $\underline{V} = (u, v, w) = (v_x, v_y, v_z)$, and an infinitesimal arc length along the streamline $\underline{ds} = (dx, dy, dz)$. The velocity, \underline{V} , at that point is parallel to \underline{ds} , so that $\underline{V} \times \underline{ds} = \underline{0}$. From this, one may derive the following equation for the streamline.

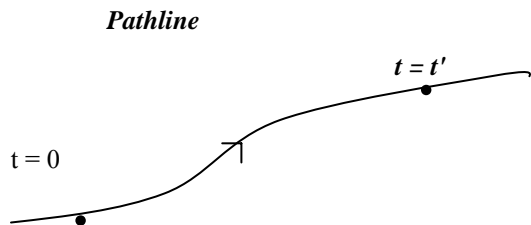
Fig. 6

$$\frac{dx}{V_x} = \frac{dy}{V_y} = \frac{dz}{V_z}$$

Note that the form of this parametric equation is $f(x, y, z) = 0$.

Pathline: It is the actual path traversed by a given fluid particle. The position of this line depends on the particle selected and the time interval over which this line is traversed by the particle. The

equations for the pathline are as follows:



$$\frac{dx_i}{dt} = v_i(x_1, x_2, x_3) \quad \text{for } i = 1, 2, 3$$

Integrating these equations one may obtain the parametric equations for the pathline:

$$x_1 = x_1(t) \quad x_2 = x_2(t) \quad x_3 = x_3(t)$$

A pathline may be identified by a fluid with a luminous dye injected instantaneously at one point and take a long exposure photograph (shutter open).

Streakline: It is the line joining the temporary location of all the particles that have passed through a given point in a flow field. A plume of smoke or dye injected at one point gives a streakline. Fig. 7 below illustrates pathlines and streaklines for an unsteady flow. Note that for a steady state flow all streamlines, pathlines and streaklines coincide.

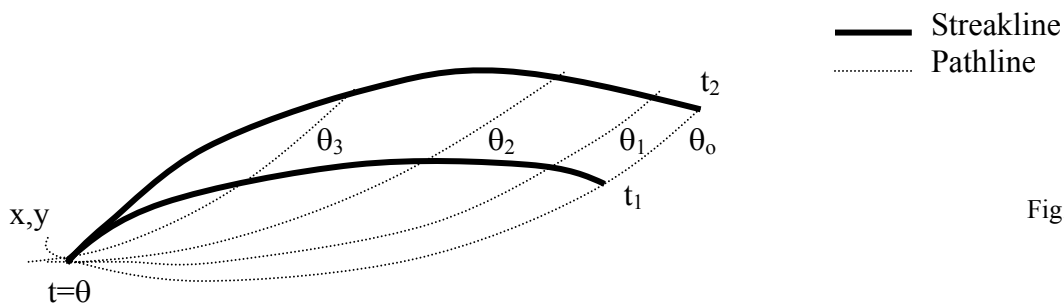


Fig. 7

Timeline: At time $t=t_0$ a set of fluid particles is marked and the subsequent behaviour of the lines thus formed is monitored.

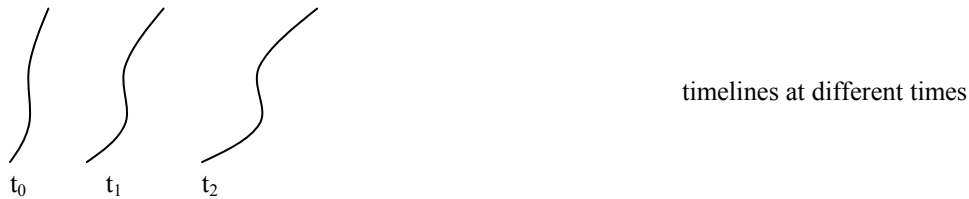


Fig. 8

EULERIAN VERSUS LANGRAGIAN POINTS OF VIEW

Two approaches are possible for the study of fluid motion, namely the **Langrangian** and the **Eulerian** approach. The Langrangian approach is based on an analysis of the motion of a particular collection of matter (particles). For each of the particles the following two fundamental principles can be applied:

- Conservation of mass (mass of the body cannot change with time)

$$\frac{dm}{dt} = 0$$

- Newton's second law of motion (The rate of change of the momentum of the particle is equal to the sum of all forces acting on that particle).

$$m \frac{dV}{dt} = \sum F_i$$

This description is not very convenient to analyse fluid motion and it is used mainly in particle mechanics. In continuum mechanics this method requires the description of motion of a large number of particles and the mathematical problem to solve becomes cumbersome. For deformable

materials the **Eulerian** approach is more convenient where our focus of interest is generally a fixed region of space through which the material moves, rather than a particular body of material. We are interested to determine ρ , \mathbf{V} , T and p at various positions in the space (field variables). For example, it is the function $p(x, y, z, t)$ that is of interest (Eulerian approach), rather than how the pressure of a particular fluid particle changes as a function of time (Langrangian approach).

To transform the above two equations (conservation of mass and momentum) from the Langrangian to the Eulerian point of view, we need two tools:

1. The material or substantial derivative operator.
2. The Reynolds transport theorem.

THE MATERIAL DERIVATIVE

Let us consider a fluid property or a field variable ϕ , which is a function of position and time, that is:

$$\phi = \phi(x, y, z, t)$$

we wish to derive an expression that relates the rate of change of ϕ with time, for the particular fluid element that happens to be located at (x, y, z) at the time t . This can be found as follows. The total derivative is

$$d\phi = \frac{\partial\phi}{\partial x} dx + \frac{\partial\phi}{\partial y} dy + \frac{\partial\phi}{\partial z} dz + \frac{\partial\phi}{\partial t} dt$$

and dividing by dt , the total derivative now becomes

$$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial x} \frac{dx}{dt} + \frac{\partial\phi}{\partial y} \frac{dy}{dt} + \frac{\partial\phi}{\partial z} \frac{dz}{dt} + \frac{\partial\phi}{\partial t}$$

Note that $dx/dt=v_x$, $dy/dt=v_y$, and $dz/dt=v_z$, then

$$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial x} v_x + \frac{\partial\phi}{\partial y} v_y + \frac{\partial\phi}{\partial z} v_z + \frac{\partial\phi}{\partial t}$$

or in a vector notation

$$\frac{d\phi}{dt} = \underline{\mathbf{V}} \cdot \nabla \phi + \frac{\partial\phi}{\partial t}$$

To distinguish the Eulerian time rate of change from the Lagrangian one, some authors use the symbol D/Dt , i.e.

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{\mathbf{V}} \cdot \nabla$$

This derivative is usually called the material or substantial derivative. If the field variable is the velocity itself $\underline{\mathbf{V}}$, then the acceleration in the Eulerian frame is:

$$\frac{D\mathbf{V}}{Dt} = \frac{\partial\mathbf{V}}{\partial x} v_x + \frac{\partial\mathbf{V}}{\partial y} v_y + \frac{\partial\mathbf{V}}{\partial z} v_z + \frac{\partial\mathbf{V}}{\partial t}$$

The partial derivative $\partial\mathbf{V}/\partial t$ is the acceleration in the Lagrangian frame. Note for steady flow $\partial\mathbf{V}/\partial t=0$, however, $D\mathbf{V}/Dt$ is not always zero for steady flows. For example in a converging channel, where the **convective acceleration terms** are not zero, in spite of the fact that the **local** or **temporal** acceleration is zero.

THE REYNOLDS TRANSPORT THEOREM

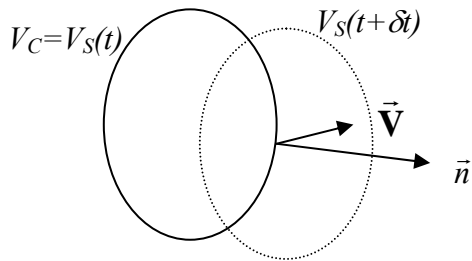


Fig. 9

Suppose that our control volume $V_C = V_S(t)$ at time t is the one in Figure 9. Due to the bulk motion this moves and deforms, so that after a time interval of dt takes a new shape $V_C = V_S(t + dt)$. We wish to

find a relationship between the rate of change of a volume integral over a moving system consisting of particular fluid elements, and operators involving an integral over a fixed volume in space. In other words **express a time derivative following a fluid body (Langrangian frame) in terms of field variables described in the Eulerian frame.** Thus, we are interested to express derivatives of the following form,

$$\frac{d}{dt} \int_{V_S(t)} \phi(x, y, z, t) dV$$

in terms of derivatives involving an integral over a fixed volume in space. Note that if $\phi = \rho$ the above integral gives the rate of change of the mass in the control volume, and if $\phi = \rho \mathbf{V}$ then the integral represents the rate of change of the momentum of the fluid in the control volume. Therefore this type of integral will be very useful to use in deriving the equation of fluid mechanics.

Because V_S is a function of time, we cannot simply move the derivative inside the integral and replace $V_S(t)$ by V_C . To do this, one may use the definition of derivative, that is

$$\frac{d}{dt} \int_{V_S(t)} \phi(x, y, z, t) dV = \lim_{\delta t \rightarrow 0} \left[\frac{\int_{V_S(t+\delta t)} \phi(t+\delta t) dV - \int_{V_S(t)} \phi(t) dV}{\delta t} \right]$$

Add and subtract in the above equation

$$\frac{1}{\delta t} \int_{V_S(t)} \phi(t + \delta t) dV$$

Thus the right-hand side becomes

$$\lim_{\delta t \rightarrow 0} \left(\frac{1}{\delta t} \left[\int_{V_S(t+\delta t)} \phi(t + \delta t) dV - \int_{V_S(t)} \phi(t + \delta t) dV \right] + \frac{1}{\delta t} \left[\int_{V_S(t)} \phi(t + \delta t) dV - \int_{V_S(t)} \phi(t) dV \right] \right)$$

However,

$$\lim_{\delta t \rightarrow 0} \left(\frac{1}{\delta t} \left[\int_{V_S(t)} \phi(t + \delta t) dV - \int_{V_S(t)} \phi(t + \delta t) dV \right] \right) = \int_{V_C} \left(\frac{\partial \phi}{\partial t} \right) dV$$

Because the limits are the same and as $\delta t \rightarrow 0$ then $V_S(t) \rightarrow V_C$. This explains why we have replaced $V_S(t)$ with V_C in the above integral, or we can say that at time t : $V_S(t) = V_C$.

Also,

$$\int_{V_S(t+\delta t)} \phi(t + \delta t) dV - \int_{V_S(t)} \phi(t + \delta t) dV = \int_{V_S(t+\delta t) - V_S(t)} \phi(t + \delta t) dV$$

However,

$$dV = (\mathbf{n} \cdot \mathbf{V}) \delta t \, dA$$

Thus, the volume integral becomes a surface integral

$$\int_{V_S(t+\delta t) - V_S(t)} \phi(t + \delta t) dV = \int_{A_S(t)} \phi(t + \delta t) (\mathbf{n} \cdot \mathbf{V}) \delta t \, dA$$

Dividing by δt and taking the limit

$$\lim_{\delta t \rightarrow 0} \left[\frac{1}{\delta t} \int_{A_S(t)} \phi(t + \delta t) (\mathbf{n} \cdot \mathbf{V}) \delta t \, dA \right] = \int_{A_C} \phi(t) (\mathbf{n} \cdot \mathbf{V}) \, dA$$

As $\delta t \rightarrow 0$, then $A_S \rightarrow A_C$.

Using now the Gauss theorem

$$\int_{A_C} \phi(t) (\mathbf{n} \cdot \mathbf{V}) \, dA = \int_{V_C} \nabla \cdot (\phi \mathbf{V}) \, dV$$

Thus

$$\frac{d}{dt} \int_{V_S(t)} \phi(t) \, dV = \int_{V_C} \left[\frac{\partial \phi}{\partial t} + \nabla \cdot (\phi \mathbf{V}) \right] \, dV$$

This is the **Reynolds transport theorem**.

THE CONTINUITY EQUATION

The law of conservation of mass tells us that the mass of a particular collection of material particles cannot change. From the Lagrangian point of view this can be expressed mathematically as:

$$\frac{d}{dt} \int_{V_S(t)} \rho dV = 0$$

where $V_S(t)$ is a function of time due to the motion of the fluid. Using the Reynolds transport theorem and substituting ρ for ϕ , then the above equation becomes

$$\int_V \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) \right] dV = 0$$

or

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0$$

or using index notation

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v_i)}{\partial x_i} = 0$$

This is the continuity equation for a Cartesian Co-ordinate System. For a fluid with constant density

$$\nabla \cdot \mathbf{V} = 0 \quad \text{or} \quad \frac{\partial v_i}{\partial x_i} = 0$$

Other Co-ordinate Systems

1. Cylindrical

$$\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (\rho r v_r) + \frac{1}{r} \frac{\partial}{\partial \theta} (\rho v_\theta) + \frac{\partial}{\partial z} (\rho v_z) = 0$$

2. Spherical

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} (\rho r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\rho v_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} (\rho v_\phi) = 0$$

STREAM FUNCTIONS

For two dimensional and axisymmetric flows, the continuity can be used to show that the complete velocity field can be described in terms of a single, scalar field variable, which is called streamfunction, $\psi(x, y, t)$. In this development, we will consider only the case of constant-density flow.

2-D Case

In a 2-D flow the velocity components are: $v_x(x, y)$, and $v_y(x, y)$. Thus for steady state, incompressible flow the continuity reduces to

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \quad \text{or} \quad \frac{\partial v_x}{\partial x} = -\frac{\partial v_y}{\partial y}$$

This implies the existence of a scalar function, $\psi(x, y)$ whose total differential is:

$$d\psi = v_x dy - v_y dx \quad \text{or} \quad d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy$$

It can be seen that:

$$v_x = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v_y = -\frac{\partial \psi}{\partial x}$$

where $\psi(x, y)$ is called the **Langrange stream function**.

Consider a line along which ψ is a constant:

$$d\psi = 0 = v_x dy - v_y dx \quad \text{or} \quad \left(\frac{dy}{dx} \right)_\psi = \frac{v_y}{v_x}$$

This is the Equation for the **streamline**. Thus, streamlines are lines of constant ψ .

Axisymmetric Flow

In these type of flows no velocity gradients exist in the θ direction. Thus the continuity equation for steady incompressible flow reduces to

$$\frac{1}{r} \frac{\partial}{\partial r} (r v_r) + \frac{\partial}{\partial z} (v_z) = 0$$

This can be rearranged to give

$$\frac{\partial}{\partial r} (r v_r) = - \frac{\partial}{\partial z} (r v_z) = 0$$

This implies

$$d\psi = r v_z dr - r v_r dz$$

Thus

$$v_z = \frac{1}{r} \frac{\partial \psi}{\partial r} \quad \text{and} \quad v_r = - \frac{1}{r} \frac{\partial \psi}{\partial z}$$

where $\psi(r,z)$ is called the **Stokes stream function**.

in **spherical coordinates**, the Stokes stream function is defined by

$$v_r = \frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta} \quad \text{and} \quad v_\theta = - \frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r}$$

THE MOMENTUM EQUATION

Newton's second law of motion is:

$$\frac{d}{dt}(m \mathbf{V}) = \Sigma \mathbf{F}$$

or using index notation,

$$\frac{d}{dt}(m v_i) = \Sigma \mathbf{F}$$

If this equation is applied for a fluid system consisting of a particular set of fluid elements then one may write:

$$\frac{d}{dt} \int_{V_S(t)} \rho v_i dV = \Sigma \mathbf{F}$$

where $V_S(t)$ is the volume of space occupied by the fluid system at time t . This equation is based on a Lagrangian description of the flow (follow the behaviour of a particular set of fluid elements over a period of time). As discussed previously, in fluid mechanics we prefer a description in terms of field variables. In other words we focus our attention at a specific space in the flow field and calculate the field variables such as p , \mathbf{V} , and ρ . This is the Eulerian description. The above equation can be transformed by using the Reynolds transport theorem. Thus,

$$\frac{d}{dt} \int_{V_S(t)} \rho v_i dV = \int_V \left[\frac{\partial (\rho v_i)}{\partial t} + \frac{\partial}{\partial x_j} (\rho v_i v_j) \right] dV$$

Expanding terms, one may write

$$\frac{d}{dt} \int_{V_s(t)} \rho v_i dV = \int_V \left[\rho \frac{\partial v_i}{\partial t} + v_i \frac{\partial \rho}{\partial t} + v_i \frac{\partial (\rho v_j)}{\partial x_j} + \rho v_j \frac{\partial v_i}{\partial x_j} \right] dV$$

The second and third terms are

$$v_i \left[\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v_j)}{\partial x_j} \right]$$

From continuity this is zero. Now, using the notation for the material or substantial derivative one may write

$$\rho \frac{\partial v_i}{\partial t} + \rho v_j \frac{\partial v_i}{\partial x_j} = \rho \frac{D v_i}{D t}$$

Then Newton's second law in the Eulerian form is

$$\int_V \rho \frac{D \mathbf{V}}{D t} dV = \Sigma \mathbf{F} \quad \text{or} \quad \int_V \rho \frac{D \mathbf{V}}{D t} dV = \mathbf{F}$$

There are two types of force that can act on the system, **body** and **surface** forces. Thus,

$$\mathbf{F} = \mathbf{F}_b + \mathbf{F}_s$$

THE BODY FORCE

The body force acts directly on the whole mass of the fluid element. This can be gravitational or electromagnetic (conducting fluid). In this development we will only consider the gravitational body force. The gravitational force per unit mass, \mathbf{G} , is derivable from a potential, $\mathbf{g}h$, where \mathbf{g} is the acceleration of gravity and h is the vertical distance above a reference plane.

$$\mathbf{G} = -\nabla(\mathbf{g}h)$$

Assume \mathbf{g} constant (not a function of h) for practical fluid mechanics problems

$$\mathbf{G} = -\mathbf{g} \nabla h \quad \text{or} \quad G_i = -g \frac{\partial h}{\partial x_i}$$

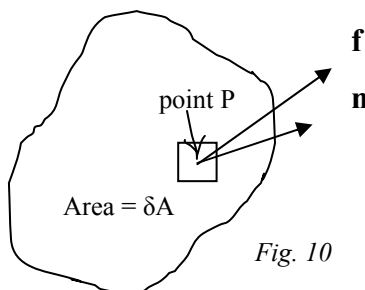
Thus, the body force can be expressed as:

$$\mathbf{F}_b = \int_V \rho \mathbf{G} dV = -\int_V \rho \mathbf{g} \nabla h dV$$

THE SURFACE FORCE (For more details see R. Aris "Vectors, Tensors and the Basic Equations of Fluid Mechanics".

Represent the surface force, \mathbf{F}_s , in terms of a volume integral over some function of a field variable. This field must describe the state of stress at a point (force per unit area).

Consider a surface element of the field system.



\mathbf{f} = surface force vector acting on the element

\mathbf{n} = outer-directed unit vector at point P

Cauchy's stress principle asserts that $\mathbf{f}/\delta A$, tends towards a limit as $\delta A \rightarrow 0$. This limit is called the stress vector.

$$\lim_{\delta A \rightarrow 0} \left(\frac{\mathbf{f}}{\delta A} \right) = \mathbf{t}^{(n)}$$

The orientation of the surface is specified by giving the vector \mathbf{n} .

For local equilibrium the Newton's law of action and reaction applies.

$$\mathbf{t}^{(-n)} = -\mathbf{t}^{(n)}$$

Thus,

$$\mathbf{F}_s = \int_A \mathbf{t}^{(n)} dA$$

However, since the components of \mathbf{t} depend on \mathbf{n} , it appears that to describe completely the state of stress at point P, it is necessary to give the values of the components for every possible orientation of the surface, i.e. for every direction of the normal vector \mathbf{n} .

Fortunately, this is not the case. If the components of \mathbf{t} are known for any three, perpendicular, unit normal vectors, the components of \mathbf{t} can be found for any other direction of the unit normal vector.

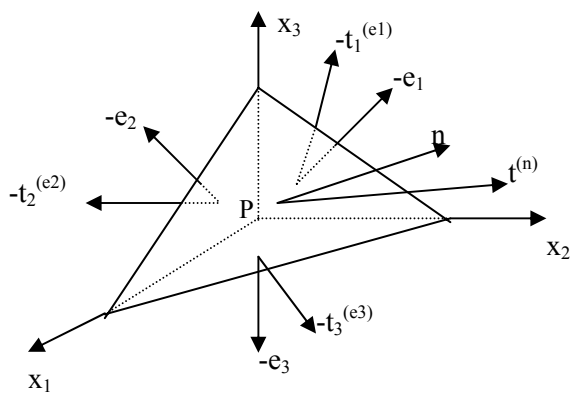


Fig. 11

Consider the small tetrahedron. Sides 1, 2, and 3 are perpendicular to the co-ordinate axes, while the fourth has an area δA .

The net surface force acting on this element is:

$$\mathbf{F}_s = \mathbf{t}^{(n)} \delta A - \mathbf{t}_i^{(e_1)} \delta A_1 - \mathbf{t}_2^{(e_2)} \delta A_2 - \mathbf{t}_3^{(e_3)} \delta A_3$$

The minus signs are coming from:

$$\mathbf{t}_i^{(-e_i)} = -\mathbf{t}_i^{(e_i)}$$

The areas can be expressed as: $\delta A_i = \mathbf{n}_i \delta A$

Thus, we may write

$$\mathbf{F}_s = \left| \mathbf{t}^{(n)} - \mathbf{n}_i \mathbf{t}_i^{(e_i)} \right| \delta A$$

If the tetrahedral fluid element shrink in volume toward a point P, then the net surface force will approach 0 (principle of local equilibrium). Thus, the above equation simplifies to

$$\mathbf{t}^{(n)} = \mathbf{n}_i \mathbf{t}_i^{(e_i)}$$

Thus, in order to describe completely the state of stress at a point in a continuum, we must specify the components of the three stress vectors (9 components). Therefore,

$$\mathbf{F}_s = \int_A \mathbf{n}_i \mathbf{t}_i^{(e_i)} dA$$

or defining the **stress tensor**, σ_{ij} and using dyadic notation

$$\mathbf{F}_s = \int_A (\mathbf{n} \cdot \boldsymbol{\sigma}) dA$$

$$\boldsymbol{\sigma} = \begin{vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{vmatrix} \quad \text{symmetric tensor}$$

σ_{ij} i indicates the co-ordinate axis that is normal to the face the stress acts on.

j indicates the direction in which the force is acting.

Using the divergence theorem of Gauss

$$\mathbf{F}_s = \int_A (\mathbf{n} \cdot \boldsymbol{\sigma}) dA = \int_V \nabla \cdot \boldsymbol{\sigma} dV$$

where

$$\nabla \cdot \boldsymbol{\sigma} = \frac{\partial \sigma_{ij}}{\partial x_i}$$

CAUCHY'S EQUATION

Substituting \mathbf{F}_s and \mathbf{F}_B in Newton's second law written in a Eulerian frame, we obtain:

$$\int_V \rho \frac{DV}{Dt} dV = - \int_V \rho \mathbf{g} \nabla h dV + \int_V (\nabla \bullet \boldsymbol{\sigma}) dV$$

or

$$\int_V \left(\rho \frac{DV}{Dt} + \rho \mathbf{g} \nabla h - \nabla \bullet \boldsymbol{\sigma} \right) dV = 0$$

or

$$\rho \frac{DV}{Dt} = - \rho \mathbf{g} \nabla h + (\nabla \bullet \boldsymbol{\sigma})$$

This is called **Cauchy's equation**. In the following pages, this equation is given in expanded form in Cartesian, Cylindrical and Spherical coordinates.

CAUCHY'S EQUATION IN RECTANGULAR COORDINATES (x, y, z)

x - component

$$\rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} - \rho g \frac{\partial h}{\partial x}$$

y - component

$$\rho \left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) = \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} - \rho g \frac{\partial h}{\partial y}$$

z - component

$$\rho \left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) = \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} - \rho g \frac{\partial h}{\partial z}$$

CAUCHY'S EQUATION IN CYLINDRICAL COORDINATES (r, θ, z)

r – component

$$\begin{aligned} \rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} \right) = \\ = \frac{1}{r} \frac{\partial (r \sigma_{rr})}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} - \frac{\sigma_{\theta\theta}}{r} + \frac{\partial \sigma_{rz}}{\partial z} - \rho g \frac{\partial h}{\partial r} \end{aligned}$$

θ – component

$$\begin{aligned} \rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z \frac{\partial v_\theta}{\partial z} \right) = \\ = \frac{1}{r^2} \frac{\partial (r^2 \sigma_{r\theta})}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} - \frac{\partial \sigma_{\theta z}}{\partial z} - \rho g \frac{\partial h}{\partial \theta} \end{aligned}$$

z – component

$$\begin{aligned} \rho \left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) = \\ = \frac{1}{r} \frac{\partial (r \sigma_{rz})}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} - \rho g \frac{\partial h}{\partial z} \end{aligned}$$

CAUCHY'S EQUATION IN SPHERICAL COORDINATES (r, θ, ϕ)

r – component

$$\begin{aligned} \rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r} \right) = \\ = \frac{1}{r^2} \frac{\partial (r^2 \sigma_{rr})}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial (\sigma_{r\theta} \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \sigma_{r\phi}}{\partial \phi} - \frac{\sigma_{\theta\theta} + \sigma_{\phi\phi}}{r} - \rho g \frac{\partial h}{\partial r} \end{aligned}$$

θ – component

$$\begin{aligned} \rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2 \cot \theta}{r} \right) = \\ = \frac{1}{r^2} \frac{\partial (r^2 \sigma_{r\theta})}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial (\sigma_{\theta\theta} \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \sigma_{\theta\phi}}{\partial \phi} + \frac{\sigma_{r\theta}}{r} - \frac{\cot \theta}{r} \sigma_{\phi\phi} - \rho g \frac{\partial h}{\partial \theta} \end{aligned}$$

ϕ – component

$$\begin{aligned} \rho \left(\frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_r}{r} + \frac{v_\theta v_\phi}{r} \cot \theta \right) = \\ = \frac{1}{r^2} \frac{\partial (r^2 \sigma_{r\phi})}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\phi}}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \sigma_{\phi\phi}}{\partial \phi} + \frac{\sigma_{r\phi}}{r} + \frac{2 \cot \theta}{r} \sigma_{\theta\phi} - \rho g \frac{\partial h}{\partial \phi} \end{aligned}$$

ROLE OF RHEOLOGY IN FLUID MECHANICS

The Equations we have developed so far, the continuity and the Cauchy's equations are not sufficient to solve a boundary value problem involving the motion of a deformable material. We need an additional relationship that describes how the material deforms when under stress. In other words additional equations relating the stress tensor components to deformation rates are required. These type of relations are called **rheological equations of state** or simply **constitutive equations**.

Rheological constitutive equations are material dependent and must be determined by experiment or from a valid molecular theory. Here in this course, we will consider a constitutive equation for an incompressible fluid that is inelastic, has no yield stress and whose structure is time- and deformation rate-independent. Such a fluid is called a **Newtonian Fluid**. Single-phase liquids of low molecular weight are usually Newtonian fluids, i.e., most gases, water, glycerine, etc.,

The Viscous stress: For a fluid at rest, all the components of the stress tensor are not zero. For such a fluid, the stress tensor is isotropic, and its components are:

$$\begin{vmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{vmatrix} \quad \text{or} \quad \sigma = -p \mathbf{I}$$

or

$$\sigma_{ij} = -p \delta_{ij} \quad \text{where } \delta_{ij} \text{ is the Kronecker delta}$$

Thus there is a contribution to the stress tensor that is not related to the motion of the fluid. This contribution is thus not relevant to the rheological constitutive equation. To account for this, we define a "**viscous stress**" ("**extra stress**") as follows:

$$\boldsymbol{\sigma} = \boldsymbol{\tau} - p \mathbf{I}$$

Thus in a fluid at rest the viscous stress, $\boldsymbol{\tau}$, is 0. Thus, now Cauchy's equation can be rewritten in terms of the viscous stresses as follows:

$$\rho \frac{DV}{Dt} = -\nabla p + \nabla \cdot \boldsymbol{\tau} - \rho \mathbf{g} \nabla h$$

In the following pages the Cauchy's equation is written in terms of viscous stresses in Cartesian, Cylindrical and Spherical coordinates. To solve this equation along with the continuity, expressions for the viscous stresses are needed in terms of the kinematics of flow (as discussed before rheological equations of state or simply constitutive equations).

The rheological constitutive equations can be classified according to the type of mechanical behaviour described by the equation. Specifically, the viscous stress at time, t , $\boldsymbol{\tau}(t)$, in a material element may depend on one or more of the following features of deformation history of that material element.

Deformation at time t

Purely elastic material

Rate of deformation at time t

Purely viscous material

Deformation at past times t' (where $-\infty < t' < t$)

(a) Material exhibiting structural time-dependency

(b) Viscoelastic material

In this course of fluid mechanics we will be concerned only with the case of purely viscous material with focus on the Newtonian fluid.

CAUCHY'S EQUATION IN RECTANGULAR COORDINATES (x, y, z)

x - component

$$\rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) = - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} - \rho g \frac{\partial h}{\partial x}$$

y - component

$$\rho \left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) = - \frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} - \rho g \frac{\partial h}{\partial y}$$

z - component

$$\rho \left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) = - \frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} - \rho g \frac{\partial h}{\partial z}$$

CAUCHY'S EQUATION IN CYLINDRICAL COORDINATES (r, θ, z)

r – component

$$\begin{aligned} \rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} \right) = \\ = - \frac{\partial p}{\partial r} + \frac{1}{r} \frac{\partial (r \tau_{rr})}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} - \frac{\tau_{\theta\theta}}{r} + \frac{\partial \tau_{rz}}{\partial z} - \rho g \frac{\partial h}{\partial r} \end{aligned}$$

θ – component

$$\begin{aligned} \rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z \frac{\partial v_\theta}{\partial z} \right) = \\ = - \frac{1}{r} \frac{\partial p}{\partial \theta} + \frac{1}{r^2} \frac{\partial (r^2 \tau_{r\theta})}{\partial r} + \frac{1}{r} \frac{\partial \tau_{\theta\theta}}{\partial \theta} - \frac{\partial \tau_{\theta z}}{\partial z} - \rho g \frac{\partial h}{\partial \theta} \end{aligned}$$

z – component

$$\begin{aligned} \rho \left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) = \\ = - \frac{\partial p}{\partial z} + \frac{1}{r} \frac{\partial (r \tau_{rz})}{\partial r} + \frac{1}{r} \frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z} - \rho g \frac{\partial h}{\partial z} \end{aligned}$$

CAUCHY'S EQUATION IN SPHERICAL COORDINATES (r, θ, ϕ)

r – component

$$\begin{aligned} \rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r} \right) = \\ = - \frac{\partial p}{\partial r} + \frac{1}{r^2} \frac{\partial (r^2 \tau_{rr})}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial (\tau_{r\theta} \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \tau_{r\phi}}{\partial \phi} - \frac{\tau_{\theta\theta} + \tau_{\phi\phi}}{r} - \rho g \frac{\partial h}{\partial r} \end{aligned}$$

θ – component

$$\begin{aligned} \rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2 \cot \theta}{r} \right) = \\ = - \frac{1}{r} \frac{\partial p}{\partial \theta} + \frac{1}{r^2} \frac{\partial (r^2 \tau_{r\theta})}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial (\tau_{\theta\theta} \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \tau_{\theta\phi}}{\partial \phi} + \frac{\tau_{r\theta}}{r} - \frac{\cot \theta}{r} \tau_{\phi\phi} - \rho g \frac{\partial h}{\partial \theta} \end{aligned}$$

ϕ – component

$$\begin{aligned} \rho \left(\frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_r}{r} + \frac{v_\theta v_\phi}{r} \cot \theta \right) = \\ = - \frac{1}{r \sin \theta} \frac{\partial p}{\partial \phi} + \frac{1}{r^2} \frac{\partial (r^2 \tau_{r\phi})}{\partial r} + \frac{1}{r} \frac{\partial \tau_{\theta\phi}}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \tau_{\phi\phi}}{\partial \phi} + \frac{\tau_{r\phi}}{r} + \frac{2 \cot \theta}{r} \tau_{\theta\phi} - \rho g \frac{\partial h}{\partial \phi} \end{aligned}$$

ANALYSIS OF DEFORMATION

In this section, we seek a relationship to describe the deformation undergoing by elements in flowing fluids and then relate those with the stress tensor to result constitutive equations. It is noted that such deformation measures should be symmetric tensors in order to be able to relate them with the stress tensor, which is also symmetric.

Consider the system depicted in the Figure below undergoing deformation. To describe this deformation, we first seek the motion of point P relative to C. The difference between the x-component of the velocity v_x at points P and C, which are considered to be very close, is:

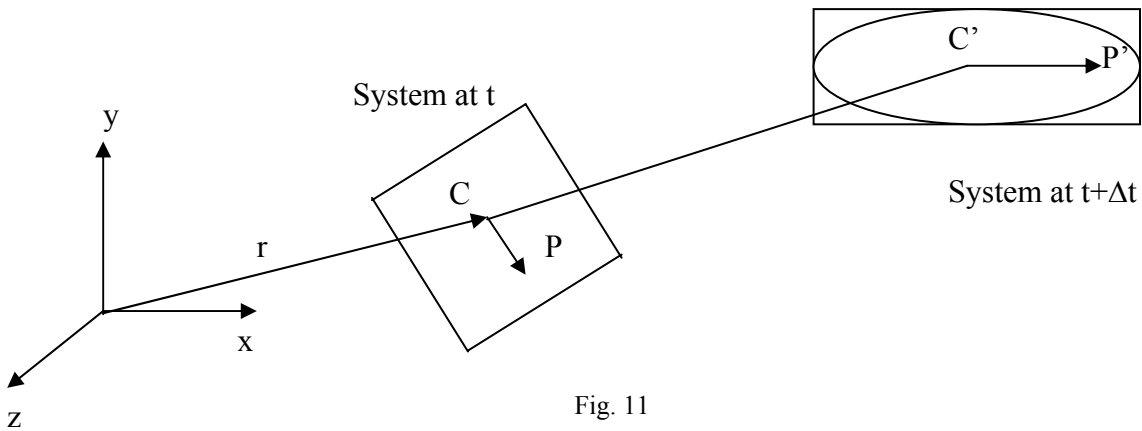


Fig. 11

$$d v_x = \frac{\partial v_x}{\partial x} dx + \frac{\partial v_x}{\partial y} dy + \frac{\partial v_x}{\partial z} dz$$

In general using index notation to also include the other two components,

$$d v_i = \frac{\partial v_i}{\partial x_j} dx_j$$

The motion of P relative to C depends, therefore, on the nine components dv_i/dx_j and may be written as:

$$d\mathbf{V} = \begin{pmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} & \frac{\partial v_x}{\partial z} \\ \frac{\partial v_y}{\partial x} & \frac{\partial v_y}{\partial y} & \frac{\partial v_y}{\partial z} \\ \frac{\partial v_z}{\partial x} & \frac{\partial v_z}{\partial y} & \frac{\partial v_z}{\partial z} \end{pmatrix} \cdot d\mathbf{r}$$

or

$$d\mathbf{V} = \nabla \mathbf{V} \cdot d\mathbf{r}$$

where $\nabla \mathbf{V}$ is the gradient of the velocity tensor (not a symmetric tensor), which is not a very good choice to be related with the stress tensor (symmetric tensor). As an example consider simple shear:

The velocity components are:

$$\begin{aligned} v_1 &= \dot{\gamma} x_2 \\ v_2 &= v_3 = 0 \end{aligned}$$

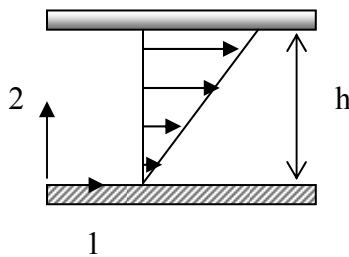


Fig 12

Then

$$\nabla \mathbf{V} = \begin{pmatrix} 0 & \dot{\gamma} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which is not symmetric. Also this tensor is **not** zero for rigid body rotation where clearly there is no

deformation.

The relative motion described by the previous equations results from the combined effects of **rotation** and **deformation**. Rotation is not a result of shear stresses but is only due to the normal stresses. Since a relation is sought between the stress tensor and the rate of deformation, the effect of rotation should be eliminated from the above relation.

From rigid body theory, one may prove that the rotational velocity $d\mathbf{V}_{\text{rot}}$ is given by

$$d\mathbf{V}_{\text{rot}} = \begin{vmatrix} 0 & \omega_{12} & \omega_{13} \\ \omega_{21} & 0 & \omega_{23} \\ \omega_{31} & \omega_{32} & 0 \end{vmatrix} \cdot d\mathbf{r}$$

where

$$\omega_{ij} = \frac{1}{2} \left(\frac{\partial v_j}{\partial x_i} - \frac{\partial v_i}{\partial x_j} \right)$$

is the **rotation tensor** (not symmetric) which describes the rotation of fluid elements in flows. Two times of this tensor gives the **vorticity tensor**, ζ , that is $\zeta = 2\omega$.

Having calculated the rotation contribution to the motion of the fluid elements, one may write:

$$d\mathbf{V}_{\text{def}} = d\mathbf{V} - d\mathbf{V}_{\text{rot}}$$

or

$$dV_{def} = \begin{vmatrix} \frac{\partial v_x}{\partial x} & \frac{1}{2} \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right) & \frac{1}{2} \left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \right) \\ \frac{1}{2} \left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} \right) & \frac{\partial v_y}{\partial y} & \frac{1}{2} \left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} \right) \\ \frac{1}{2} \left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z} \right) & \frac{1}{2} \left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} \right) & \frac{\partial v_z}{\partial z} \end{vmatrix} \cdot dr$$

The above tensor is called the rate of deformation tensor, $\dot{\boldsymbol{\gamma}}$, which is a symmetric tensor as should be and can be put in the following form

$$\dot{\gamma}_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)$$

or

$$\dot{\boldsymbol{\gamma}} = \frac{1}{2} (\nabla \mathbf{V} + \nabla \mathbf{V}^T)$$

where $\nabla \mathbf{V}^T$ is the transpose of the velocity gradient tensor.

As an example consider simple shear as obtained by means of a sliding plate rheometer (see above Fig.12). The shear rate, $\dot{\boldsymbol{\gamma}}$, is defined as the ratio of the velocity of the upper plate to the gap spacing between the plates. The rate of deformation tensor for this flow is:

$$\dot{\boldsymbol{\gamma}} = \begin{vmatrix} 0 & \dot{\gamma} & 0 \\ \dot{\gamma} & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

which is clearly a symmetric tensor. It can be proved that this tensor is equal to zero for solid body rotation as it should be (no deformation). Thus, the tensor $\dot{\boldsymbol{\gamma}}$ is a good choice for measuring the deformation rate of fluid elements.

In the following pages the components of rate-of-deformation tensor are given in Cartesian, Cylindrical and Spherical co-ordinates.

COMPONENTS OF $\dot{\boldsymbol{\gamma}}$ IN CARTESIAN COORDINATES (x, y, z)

$$\dot{\gamma}_{xx} = \frac{\partial v_x}{\partial x}$$

$$\dot{\gamma}_{xy} = \dot{\gamma}_{yx} = \frac{1}{2} \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right)$$

$$\dot{\gamma}_{xz} = \dot{\gamma}_{zx} = \frac{1}{2} \left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \right)$$

$$\dot{\gamma}_{yy} = \frac{\partial v_y}{\partial y}$$

$$\dot{\gamma}_{yz} = \dot{\gamma}_{zy} = \frac{1}{2} \left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} \right)$$

$$\dot{\gamma}_{zz} = \frac{\partial v_z}{\partial z}$$

COMPONENTS OF $\dot{\boldsymbol{\gamma}}$ IN CYLINDRICAL COORDINATES (r, θ, z)

$$\dot{\gamma}_{rr} = \frac{\partial v_r}{\partial r}$$

$$\dot{\gamma}_{r\theta} = \dot{\gamma}_{\theta r} = \frac{1}{2} \left(r \frac{\partial (v_\theta/r)}{\partial r} + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right)$$

$$\dot{\gamma}_{rz} = \dot{\gamma}_{zr} = \frac{1}{2} \left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z} \right)$$

$$\dot{\gamma}_{\theta\theta} = \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}$$

$$\dot{\gamma}_{\theta z} = \dot{\gamma}_{z\theta} = \frac{1}{2} \left(\frac{\partial v_\theta}{\partial z} + \frac{1}{r} \frac{\partial v_z}{\partial \theta} \right)$$

$$\dot{\gamma}_{zz} = \frac{\partial v_z}{\partial z}$$

COMPONENTS OF $\dot{\boldsymbol{\gamma}}$ IN SPHERICAL COORDINATES (r, θ, ϕ)

$$\dot{\gamma}_{rr} = \frac{\partial v_r}{\partial r}$$

$$\dot{\gamma}_{r\theta} = \dot{\gamma}_{\theta r} = \frac{1}{2} \left(r \frac{\partial (v_\theta/r)}{\partial r} + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right)$$

$$\dot{\gamma}_{r\phi} = \dot{\gamma}_{\phi r} = \frac{1}{2} \left(\frac{1}{r \sin \theta} \frac{\partial v_r}{\partial \phi} + r \frac{\partial (v_\phi/r)}{\partial r} \right)$$

$$\dot{\gamma}_{\phi\phi} = \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta \cot \theta}{r}$$

$$\dot{\gamma}_{\theta\phi} = \dot{\gamma}_{\phi\theta} = \frac{1}{2} \left(\frac{\sin \theta}{r} \frac{\partial (v_\phi/\sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} \right)$$

$$\dot{\gamma}_{\theta\theta} = \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}$$

THE NEWTONIAN FLUID

Newtonian fluid is defined as the one, which satisfies the following relationship

$$\tau_{ij} = 2 \mu \dot{\gamma}$$

or in index notation

$$\tau_{ij} = \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)$$

which implies that the stress tensor in such a fluid is proportional to the rate-of-deformation tensor with the coefficient μ to be the viscosity of the fluid. This is **Newton's law of viscosity**. In the following pages the components of the stress tensor for an incompressible Newtonian fluid are given.

COMPONENTS OF $\boldsymbol{\tau}$ IN CARTESIAN COORDINATES (x, y, z)

$$\tau_{xx} = 2\mu \frac{\partial v_x}{\partial x}$$

$$\tau_{xy} = \tau_{yx} = \mu \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right)$$

$$\tau_{xz} = \tau_{zx} = \mu \left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \right)$$

$$\tau_{yy} = 2\mu \frac{\partial v_y}{\partial y}$$

$$\tau_{yz} = \tau_{zy} = \mu \left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} \right)$$

$$\tau_{zz} = 2\mu \frac{\partial v_z}{\partial z}$$

COMPONENTS OF $\boldsymbol{\tau}$ IN CYLINDRICAL COORDINATES (r, θ, z)

$$\tau_{rr} = 2 \mu \frac{\partial v_r}{\partial r}$$

$$\tau_{r\theta} = \tau_{\theta r} = \mu \left(r \frac{\partial (v_\theta / r)}{\partial r} + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right)$$

$$\tau_{rz} = \tau_{zr} = \mu \left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z} \right)$$

$$\tau_{\theta\theta} = 2 \mu \left(\frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right)$$

$$\tau_{\theta z} = \tau_{z\theta} = \mu \left(\frac{\partial v_\theta}{\partial z} + \frac{1}{r} \frac{\partial v_z}{\partial \theta} \right)$$

$$\tau_{zz} = 2 \mu \frac{\partial v_z}{\partial z}$$

COMPONENTS OF $\boldsymbol{\tau}$ IN SPHERICAL COORDINATES (r, θ, ϕ)

$$\tau_{rr} = 2\mu \frac{\partial v_r}{\partial r}$$

$$\tau_{r\theta} = \tau_{\theta r} = \mu \left(r \frac{\partial (v_\theta/r)}{\partial r} + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right)$$

$$\tau_{r\phi} = \tau_{\phi r} = \mu \left(\frac{1}{r \sin \theta} \frac{\partial v_r}{\partial \phi} + r \frac{\partial (v_\phi/r)}{\partial r} \right)$$

$$\tau_{\phi\phi} = 2\mu \left(\frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta \cot \theta}{r} \right)$$

$$\tau_{\theta\phi} = \tau_{\phi\theta} = \mu \left(\frac{\sin \theta}{r} \frac{\partial (v_\phi/\sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} \right)$$

$$\tau_{\theta\theta} = 2\mu \left(\frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right)$$

NON-NEWTONIAN FLUIDS

Fluids, which do not obey Newton's law of viscosity, are called non-Newtonian fluids. Gases and single-phase low molecular weight liquids are Newtonian fluids (water, glycerine, and ethanol).

However, many other commercially important materials are non-Newtonian fluids (slurries, paints, foodstuffs, and molten polymers). Some of these fluids simply exhibit non-linear viscous effects, while others show effects of "memory" which are related to their viscoelastic behaviour. In general phenomena exhibited by non-Newtonian fluids include:

- i. Shear rate dependent viscosity (non-linear viscous effect)
- ii. Yield stress
- iii. Time-dependent structure
- iv. Elasticity

Shear rate dependent viscosity:

This is the simplest case of non-Newtonian behaviour, where the viscosity depends on the rate deformation. The most common type of behaviour is when the viscosity decreases with an increase of the shear rate. Materials exhibiting such behaviour are said to be "**shear-thinning**" or "**pseudoplastic**". Less commonly encountered behaviour is the increase of viscosity with increase of the shear rate ("**shear-thickening**" or "**dilatant**"). Molten polymers are shear-thinning fluids, while some concentrated suspensions behave as shear-thickening ones. Figure 13 illustrates the behaviour of a Newtonian, shear-thinning, shear thickening and a plastic fluid (see below) in a shear stress versus shear rate plot. Note that the viscosity is given by the local slope of the curve.

To model such behaviour the "power-law" viscosity model is frequently employed, that is:

$$\tau = \eta \dot{\gamma}$$

where η is the non-Newtonian viscosity given by:

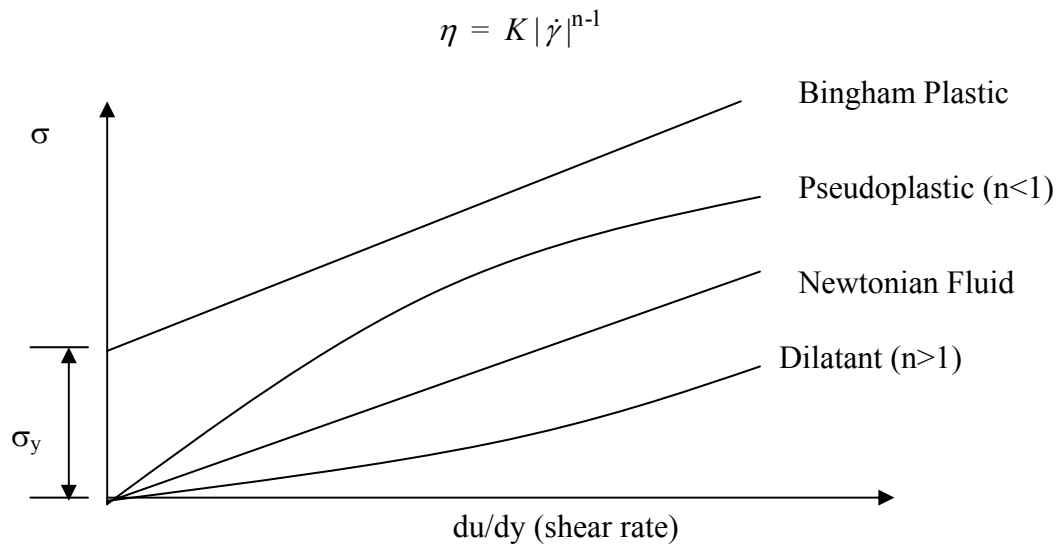


Fig.13

where K is the consistency index, n is the power-law exponent and $|\dot{\gamma}|$ is the magnitude of the rate-of-deformation tensor given by $\sqrt{(\dot{\gamma} : \dot{\gamma})}$. The different types of behaviours are obtained as follows:

- $n=1$ Newtonian
- $n>1$ Dilatant or shear-thickening
- $n<1$ Pseudoplastic or shear-thinning

Yield Stress and Plasticity:

The existence of yield stress is frequently encountered in the rheological behaviour of concentrated suspensions. This is some critical value of the shear stress below, which the material does not flow. For shear stresses greater than this critical stress the material may behave as a Newtonian, pseudoplastic or dilatant. Materials exhibiting such a behaviour are said to be "**plastic**".

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_y + \eta_p \dot{\boldsymbol{\gamma}}$$

The simplest type of plasticity is the one which follows the **Bingham** model, that is:

where $\boldsymbol{\sigma}_y$ is the yield stress and η_p is the viscosity. In other words, a Bingham fluid is a Newtonian fluid with a yield stress.

Time-dependent structure:

All types of materials involved in our previous discussion have a time-independent structure. In other words, if a constant shear rate or stress is applied, their structure as well as their viscosity do not change with time (time-independent) once a steady-state is obtained. This is in contrast with the behaviour of some concentrated suspensions whose structure changes with time and as a result their viscosity changes with time as well. When the viscosity decreases with time the material is said to be "**thixotropic**". The opposite behaviour is rheopexy and the material following this behaviour is said to be "**rheoplectic**".

Viscoelasticity:

Materials that exhibit viscous resistance to deformation and elasticity are said to be **viscoelastic** materials. Such a behaviour is time-dependent and the stress at some time t depends on the past deformation history that the fluid elements were subjected to. This effect is also known as "**memory**" effect.

In start up and cessation of steady shear of a Newtonian fluid the stress builds up to its steady-state value and to zero respectively, instantaneously. This is not the case with a viscoelastic material. In start up of steady shear, the stress is time dependent and only approaches its steady value after a significant period of time which depends on the rate of shear. This behaviour is different from that of a material exhibiting structural time dependency where all the energy is dissipated. In the viscoelastic materials some of the energy is stored. Thus, in the cessation of steady shear the shear stress decays to zero again after a significant period of time which also depends on the past deformation history. This also implies that the material will also exhibit partial recoil in order to relax.

THE NAVIER-STOKES EQUATIONS

We start with **Cauchy's** equation written in terms of viscous stresses, that is:

$$\rho \frac{D\mathbf{V}}{Dt} = -\rho \mathbf{g} \nabla h - \nabla p + \nabla \cdot \boldsymbol{\tau}$$

Using index notation this can be written as:

$$\rho \frac{D v_j}{Dt} = -\rho g \frac{\partial h}{\partial x_j} - \frac{\partial p}{\partial x_j} + \frac{\partial \tau_{ij}}{\partial x_i}$$

The extra or viscous stress is given by the definition of an incompressible Newtonian fluid is:

$$\tau_{ij} = \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)$$

Combining the last two equations to eliminate τ_{ij} ,

$$\rho \frac{D v_j}{Dt} = -\rho g \frac{\partial h}{\partial x_j} - \frac{\partial p}{\partial x_j} + \frac{\partial}{\partial x_i} \left[\mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \right]$$

Assuming constant viscosity, μ

$$\rho \frac{D v_j}{Dt} = -\rho g \frac{\partial h}{\partial x_j} - \frac{\partial p}{\partial x_j} + \left[\mu \left(\frac{\partial^2 v_i}{\partial x_i \partial x_j} + \frac{\partial^2 v_j}{\partial x_i \partial x_i} \right) \right]$$

The first term in the parenthesis can be written as:

$$\frac{\partial}{\partial x_j} \left(\frac{\partial v_i}{\partial x_i} \right)$$

But for an incompressible fluid, the continuity equation tells us that the **quantity in the parenthesis is zero**.

Thus for a Newtonian fluid with constant density and viscosity we can write:

$$\rho \frac{D v_j}{D t} = -\rho g \frac{\partial h}{\partial x_j} - \frac{\partial p}{\partial x_j} + \mu \frac{\partial}{\partial x_i} \left(\frac{\partial v_i}{\partial x_i} \right) \quad j = 1, 2, 3$$

Using vector notation:

$$\rho \frac{D \mathbf{V}}{D t} = -\rho g \nabla h - \nabla p + \mu \nabla^2 \mathbf{V}$$

or if the Laplacian is written in a standard vectorial form then

$$\rho \frac{D \mathbf{V}}{D t} = -\rho g \nabla h - \nabla p + \mu \nabla \times (\nabla \times \mathbf{V})$$

These are the Navier-Stokes Equations for constant fluid density and viscosity. They constitute a system of three non-linear second order partial differential equations. Together with the continuity equation they form a set of four equations which is complete for incompressible flows, in principle they are sufficient to solve for the four dependent variables, P , v_x , v_y , and v_z for a Cartesian system of coordinates.

The Navier-Stokes equations also require initial and boundary conditions. The proper boundary conditions for the velocity on a solid boundary are:

$$v_n = v_t = 0$$

where v_n is the normal component of the velocity relative to the solid boundary and v_t is the tangential exponent. These conditions are known as the **no-penetration** and **no-slip** viscous boundary conditions respectively. If there are free surfaces involved in the flow additional boundary conditions are also required to solve the problem. Finally the pressure which is also a dependent variable, requires boundary conditions too.

Finally, if the flow is fully enclosed by solid boundaries, the only role of gravity force is to increase the pressure by an amount equal to the static head, ρgh . In this case the pressure and the gravity force terms in the Navier-Stokes equations can be combined by defining the **hydrodynamic pressure**, that is:

$$P \equiv p + \rho g h$$

and the N-S equations can be rewritten as:

$$\rho \frac{D v_j}{D t} = - \frac{\partial P}{\partial x_j} + \mu \frac{\partial}{\partial x_i} \left(\frac{\partial v_i}{\partial x_i} \right) \quad j = 1, 2, 3$$

In the following pages the Navier-Stokes equations for an incompressible fluid of constant viscosity are given in Cartesian, Cylindrical and Spherical coordinates.

NAVIER-STOKES EQUATIONS IN RECTANGULAR COORDINATES (x, y, z), $\rho, \mu = \text{constant}$

x - component

$$\rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) = - \frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right) - \rho g \frac{\partial h}{\partial x}$$

y - component

$$\rho \left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) = - \frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right) - \rho g \frac{\partial h}{\partial y}$$

z - component

$$\rho \left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) = - \frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right) - \rho g \frac{\partial h}{\partial z}$$

NAVIER-STOKES EQUATIONS IN CYLINDRICAL COORDINATES (r, θ , z), $\rho, \mu = \text{constant}$

r – component

$$\begin{aligned} & \rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} \right) = \\ & = - \frac{\partial p}{\partial r} + \mu \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial (r v_r)}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} \right] - \rho g \frac{\partial h}{\partial r} \end{aligned}$$

θ – component

$$\begin{aligned} & \rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z \frac{\partial v_\theta}{\partial z} \right) = \\ & = - \frac{1}{r} \frac{\partial p}{\partial \theta} + \mu \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial (r v_\theta)}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} \right] - \rho g \frac{\partial h}{\partial \theta} \end{aligned}$$

z – component

$$\begin{aligned} & \rho \left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) = \\ & = - \frac{\partial p}{\partial z} + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right] - \rho g \frac{\partial h}{\partial z} \end{aligned}$$

NAVIER-STOKES EQUATIONS IN SPHERICAL COORDINATES (r, θ, ϕ), $\rho, \mu = \text{constant}$

r – component

$$\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r} \right) = -\frac{\partial p}{\partial r} - \rho g \frac{\partial h}{\partial r} +$$

$$+ \mu \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial v_r}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial v_r}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_r}{\partial \phi^2} - \frac{2 v_r}{r^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} - \frac{2 v_\theta \cot \theta}{r^2} - \frac{2}{r^2 \sin \theta} \frac{\partial v_\phi}{\partial \phi} \right]$$

θ – component

$$\rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2 \cot \theta}{r} \right) = -\frac{1}{r} \frac{\partial p}{\partial \theta} - \rho g \frac{\partial h}{\partial \theta}$$

$$+ \mu \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial v_\theta}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial v_\theta}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_\theta}{\partial \phi^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2 \sin^2 \theta} - \frac{2 \cos \theta}{r^2 \sin^2 \theta} \frac{\partial v_\phi}{\partial \phi} \right]$$

ϕ – component

$$\rho \left(\frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_r}{r} + \frac{v_\theta v_\phi}{r} \cot \theta \right) = -\frac{1}{r \sin \theta} \frac{\partial p}{\partial \phi} - \rho g \frac{\partial h}{\partial \phi}$$

$$+ \mu \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial v_\phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial v_\phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_\phi}{\partial \phi^2} - \frac{v_\phi}{r^2 \sin^2 \theta} + \frac{2}{r^2 \sin^2 \theta} \frac{\partial v_r}{\partial \phi} + \frac{2 \cos \theta}{r^2 \sin^2 \theta} \frac{\partial v_\theta}{\partial \phi} \right]$$

THE VISCOSITY

Viscosity is perhaps the most important property of flowing Newtonian fluids because it completely characterises their rheological behaviour. Gases in general do not offer any significant resistance to flow and therefore may be treated as fluids of zero viscosity. The viscosity of liquids is a function of composition, pressure and temperature. It increases with increase of the molecular weight and with increase of pressure, more and more steeply as the molecular weight increases.

An equation frequently employed to model the pressure dependency of viscosity is the following,

$$\mu_p = \mu_0 \exp(\beta p)$$

where μ_p is the viscosity at pressure p , μ_0 is the viscosity at ambient pressure and β is the pressure dependency coefficient of viscosity. Under common processing pressure conditions, the viscosity changes very little. For example at 30°C, the viscosity of toluene changes from 5220 μP to 8120 μP when the pressure changes from 0.1 MPa to 63.5 MPa.

Temperature has a much stronger effect on the viscosity of fluids. The viscosity of gases in general increases with increase of temperature while that of liquids decreases. This is because the relative roles of collision and intermolecular forces are different in these two states of matter. In gases momentum is transferred through molecular collisions. Thus, an increase of temperature increases the number of molecular collisions which increases the resistance to flow and as a result gases at a higher temperature appear to have a higher viscosity. In liquids the molecular collisions are overshadowed by the effects of interacting fields among the closely packed liquid molecules. An increase of temperature in general increases the free volume in liquids and in general decreases molecular collisions and interaction intermolecular forces. These effects are reflected upon a decrease of the viscosity of liquids.

An Arrhenius type equation is frequently used to model the effect of T on the viscosity of liquids, that is:

$$\frac{\mu_T}{\mu_{T_0}} = \exp\left[\frac{E}{R}\left(\frac{1}{T} - \frac{1}{T_0}\right)\right]$$

where μ_T is the viscosity at temperature T , μ_0 is the viscosity at temperature T_0 and E is an activation energy

for viscosity.

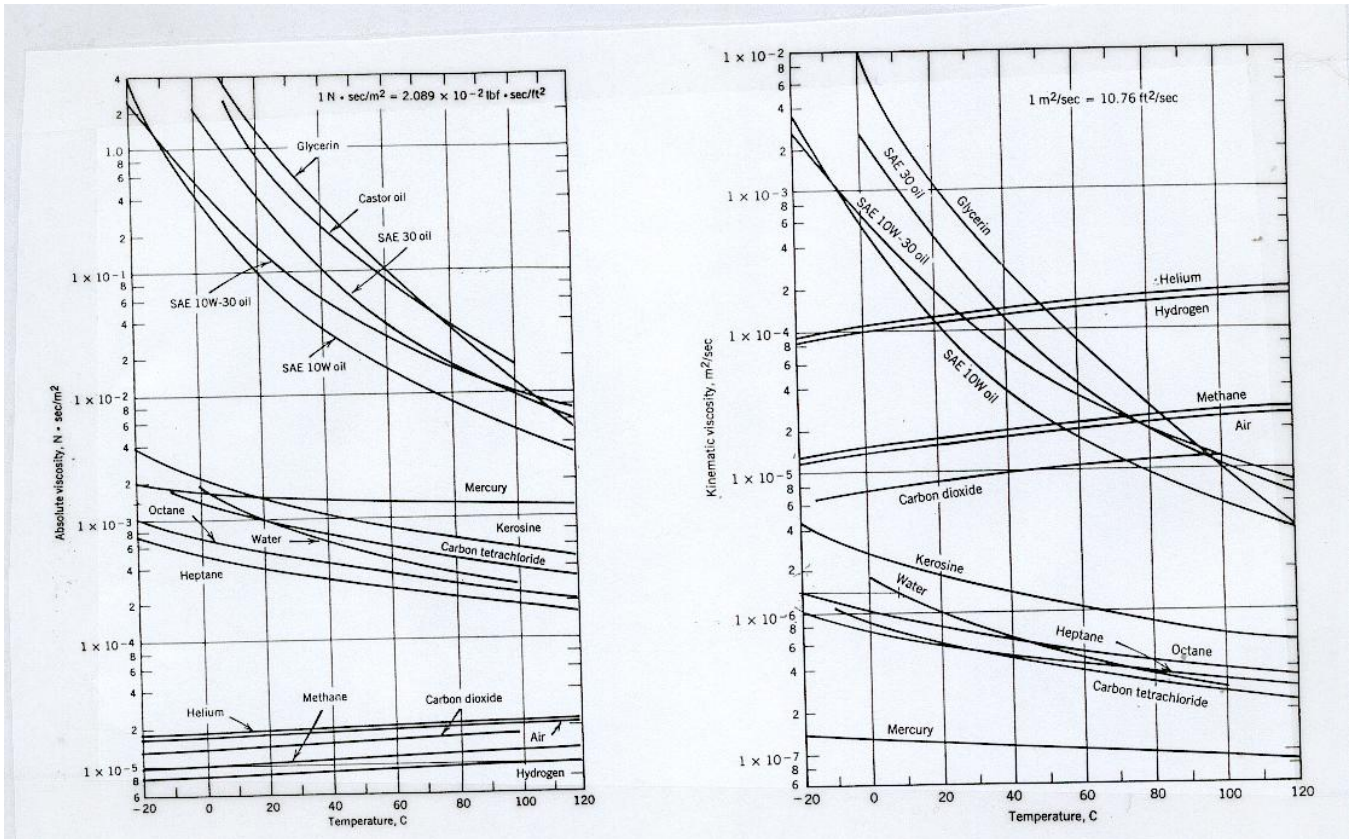
$$\mu = \frac{b T^{1/2}}{1 + S/T}$$

For gases the effect of T on their viscosity is modelled through Sutherland correlation as follows: where *b* and *S* are empirical constants.

Finally the ratio of the viscosity to density is the **kinematic viscosity**, *v*, defined by

$$v \equiv \frac{\mu}{\rho}$$

As explained later this kinematic viscosity is also a vorticity transfer coefficient, which determines how fast a shear signal propagates into fluids. The two Figures below show the viscosity and kinematic viscosity of some selected gases and liquids as a function of *T*.



THE EULER EQUATIONS

Substitution of $\mu=0$ in the Navier-Stokes equations reduces them to a form known as the Euler equations:

$$\rho \frac{D \mathbf{V}}{Dt} = -\rho \mathbf{g} \nabla h - \nabla p$$

These equations were developed earlier than the Navier-Stokes equations. It is noted that these equations may be considered as an approximation and because are first order equations cannot satisfy both boundary conditions applied to the Navier-Stokes ones. These are recommended to be used away from solid boundaries where viscous effects are minimal. In these areas the assumption $\mu=0$ is a fair estimate.

THE NAVIER-STOKES EQUATIONS IN TWO-DIMENSIONAL FLOWS - THE STREAM FUNCTION

Consider a 2-D flow in the (x, y) plane and no velocity component in the z -direction. Thus:

$$v_z = 0 \quad \text{and} \quad \frac{\partial (\cdot)}{\partial z} = 0$$

The equation of continuity and the Navier-Stokes equations can be simplified as:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

$$\rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} \right) = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} \right) - \rho g \frac{\partial h}{\partial x}$$

$$\rho \left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial z} \right) = -\frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} \right) - \rho g \frac{\partial h}{\partial y}$$

Cross-differentiating the last two equations and subtracting one from the other, we have two equations, with two unknowns, namely v_x and v_y . Since the flow is also **2-D** we can make use of the **Lagrange stream function** to further reduce the number of unknowns. The final result is a **fourth-order** partial differential equation

$$\frac{\partial \nabla^4 \psi}{\partial t} - \frac{\partial \psi}{\partial y} \frac{\partial \nabla^2 \psi}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \nabla^2 \psi}{\partial y} = \frac{\mu}{\rho} \nabla^4 \psi$$

METHODS OF SOLUTION

The Navier-Stokes are non-linear partial differential equations and there is no general method to obtain an analytical solution. In other words there is no existence theorem for a solution. Thus, each problem in fluid mechanics must be carefully formulated as to geometry and proper boundary conditions. Then a method to attack the problem should be chosen in order to get a solution. The obtained solution, which depends on the method, falls in one of the following categories:

1. Exact Solution: Such solutions in general are possible to obtain under special cases such as:

- A. If the BC's are independent of time and the starting transient is not of interest, the solution is assumed to be independent of time;
- B. If the BC's have a certain symmetry, it is assumed that the solution will also have this symmetry;
- C. Assumptions that there are no "end effects" i.e., no variations of velocity in the downstream directions.

Such solutions are **true, exact** solutions to the equations of motion.

2. Approximate Solution: If the simplifications of the type described above do not reduce the system of equations to one that can be solved directly, simplifying approximations may be justified under certain circumstances.

- A. If the Reynolds number is very low (much less than 1), the inertia terms in the N-S equations

may be neglected.

B. Flow at high Reynolds number very near a solid boundary, certain simplifying approximations lead to the Boundary layer equations.

It is noted that these types of approximations are coming from a method known as **ordering analysis** and examples will be discussed later. The solutions arising from such procedures are **approximate solutions, not exact solutions.**

3. Numerical Solution: In the most general case, neither an exact nor an approximate analytical solution is possible. In this case, a numerical solution must be sought using:

- Finite differences

- Finite elements

This is the domain of **Computational Fluid Mechanics.**

FLUID STATICS

Hydrostatics is the branch of fluid mechanics which considers static fluids, i.e., fluids at rest with respect to any co-ordinate system. Therefore, the Lagrangian approach is sufficient to describe fluids at rest once the co-ordinate system is attached to the fluid. For fluids at rest or moving as “rigid bodies” (again the subject of study of hydrostatics) there is no deformation undergone by the fluid. In other words all viscous normal and shear stresses are set to zero. In this case, one may write the equations that describe these fluids (i.e. pressure distribution, effects of acceleration as:

$$\rho \frac{d\mathbf{V}}{dt} = -\rho g \nabla h - \nabla p \quad \text{or} \quad \rho \mathbf{a} = -\rho g \nabla h - \nabla p \quad (1)$$

Using index notation this can be written as:

$$\rho \frac{d v_j}{dt} = -\rho g \frac{\partial h}{\partial x_j} - \frac{\partial p}{\partial x_j} \quad \text{or} \quad \rho a_i = -\rho g \frac{\partial h}{\partial x_j} - \frac{\partial p}{\partial x_j} \quad (2)$$

where \mathbf{a} is the acceleration vector. For a Cartesian system of co-ordinates (x, y, z) this equation takes the following form:

$$\begin{aligned} \rho a_x &= -\frac{\partial p}{\partial x} + \rho g_x \\ \rho a_y &= -\frac{\partial p}{\partial y} + \rho g_y \\ \rho a_z &= -\frac{\partial p}{\partial z} + \rho g_z \end{aligned} \quad (3)$$

For a fluid at rest with the acceleration zero and the direction of g in the negative z direction these

equations become:

$$\frac{\partial p}{\partial x} = 0$$

$$\frac{\partial p}{\partial y} = 0$$

$$\frac{\partial p}{\partial z} = -\rho g$$
(4)

These equations tell us that that the pressure in a fluid at rest changes in the vertical direction.

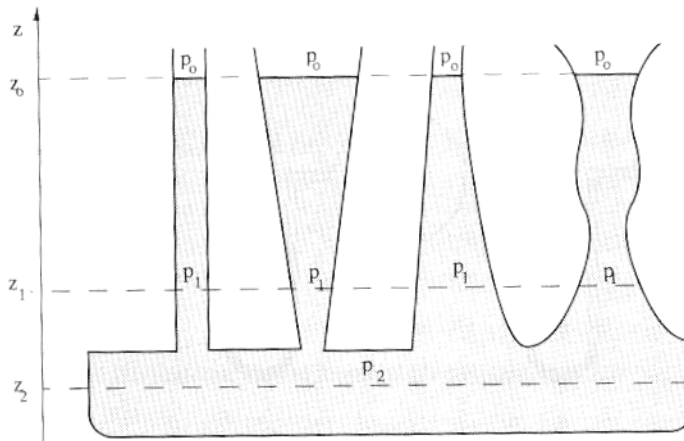


Figure 1. Pressure in an incompressible fluid.

Simply in Figure 1, by integrating equation 4, it can be said that:

$$p = p_o + \rho g(z_o - z) \quad \text{§}$$

Sometimes one is interested in the pressures excess above that of the atmosphere. This is called the *gage pressure*, as distinguished from the *absolute pressure* p .

Using the same principles on the manometers (combinations of tubes using different liquids) depicted on Figure 2, one may easily derive the following relationships, where p_A is the pressure in the thalamus.

- (a) $p_A = p_o + \rho gh$
- (b) $p_o = p_A + \rho gh$
- (c) $p_o = \rho g(z_A - z_o)$ where the pressure within the capillary is zero.

$$(d) \quad p_o = \rho g(z_o - z_A) = \rho g \sin(L)$$

$$(e) \text{ and } (f) \quad p_A = p_o + \sum_{i=1}^n \rho_i g(z_{i+1} - z_i)$$

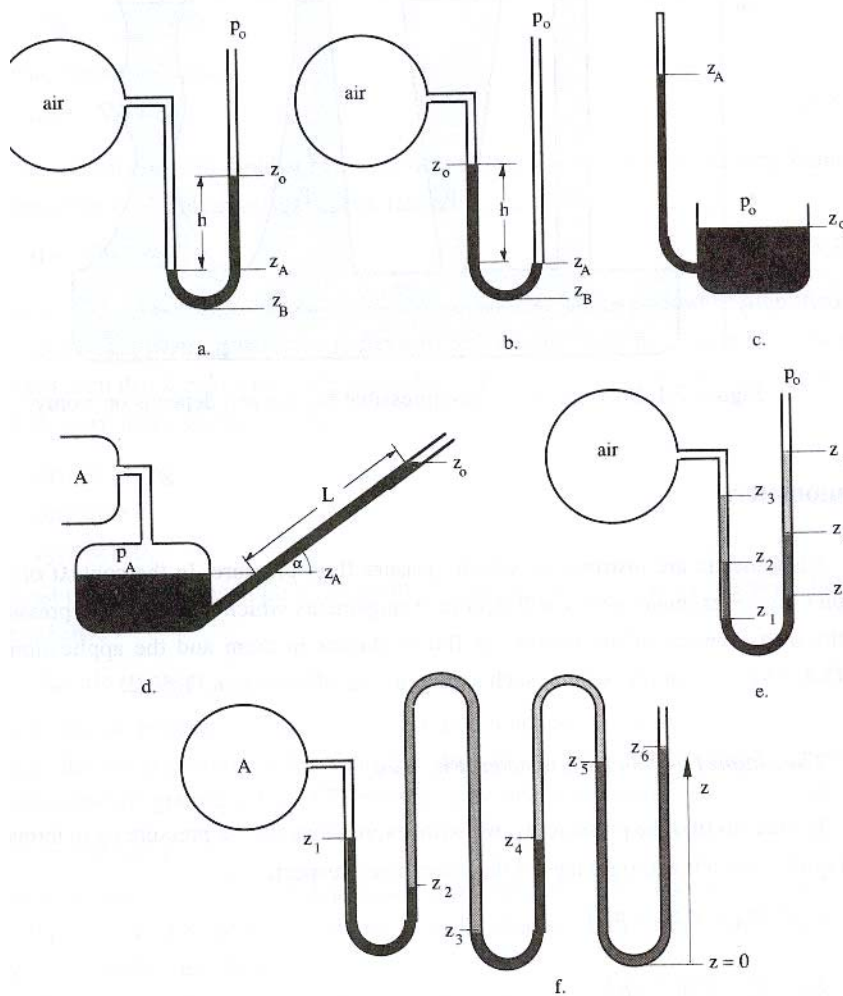


Figure 2: Manometers

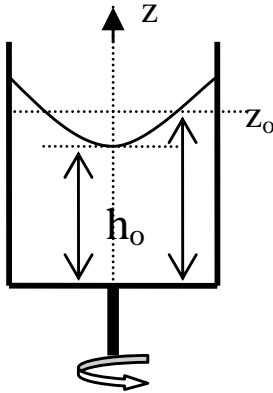
EQUATION OF HYDROSTATICS IN ACCELERATING FRAMES OF REFERENCE

Considering that the variation of g is negligible with elevation (good assumption), Equation 1 can be rewritten as:

$$\rho \mathbf{a} = -\rho \mathbf{g} - \nabla p \quad \text{or} \quad 0 = -\nabla p + \rho(\mathbf{g} - \mathbf{a})$$

This is exactly the same as the equation of hydrostatics except for the body force term which is now **$\mathbf{g}-\mathbf{a}$** .

Example: A cylindrical bucket, originally filled with water to a level h , rotates about its axis of symmetry with an angular velocity ω , as shown in Figure 3. After sometime the water rotates like a rigid body. Find the pressure distribution in the fluid and the shape of the free surface, $z_o = f(r)$.



The equations in cylindrical co-ordinates can be written as:

$$0 = -\frac{\partial p}{\partial r} + \rho\omega^2 r$$

$$0 = -\frac{1}{r} \frac{\partial p}{\partial \theta}$$

$$0 = -\frac{\partial p}{\partial z} - \rho g$$

where the second equation implies that pressure is not a function of θ . The other two equations can be integrated to yield.

Figure 3. Rotating bucket.

$$\left\{ \begin{array}{l} p = \frac{1}{2} \rho \omega^2 r^2 + f_1(z) \\ p = -\rho g z + f_2(r) \end{array} \right\} \quad \text{or} \quad p = \frac{1}{2} \rho \omega^2 r^2 - \rho g z + c$$

The constant c is evaluated from $p(0, h_o) = p_o$. Thus, once c is evaluated and substituted back into the pressure equation:

$$p(r, z) = p_o + \frac{1}{2} \rho \omega^2 r^2 - \rho g z + \rho g h_o$$

The equation for the free surface can be found from the condition that $p(r, z_o) = p_o$. Thus,

$$z_o = h_o + \frac{\omega^2 r^2}{2g}$$

The value of h_o can be found from the original volume

$$V = \pi R^2 h = \int_0^R z_o 2\pi r dr = \int_0^R \left(h_o + \frac{\omega^2 r^2}{4g} \right) 2\pi r dr ,$$

Integrating

$$\pi R^2 h = \pi R^2 \left(h_o + \frac{\omega^2 R^2}{4g} \right)$$

Finally,

$$h_o = h - \frac{\omega^2 R^2}{4g}$$

FORCES ACTING ON SUBMERGED SURFACES

Surfaces in contact with a fluid are called *submerged surfaces*. The force acting on an element of a submerged surface dS is then:

$$d\mathbf{F} = -p\mathbf{n}dS$$

\mathbf{n} being the element's outer normal unit vector. The total force acting on the surface S is:

$$\mathbf{F} = -\int_S p\mathbf{n}dS = -\int_S (p_o + \rho gh)\mathbf{n}dS$$

where h is the depth of the fluid below $p=p_o$.

Considering a Cartesian system of coordinates where $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ are the unit vectors in (x, y, z) , then:

$$F_x = -\int_S p dS_x = -\int_S (p_o + \rho gh) dS_x$$

$$F_y = -\int_S p dS_y = -\int_S (p_o + \rho gh) dS_y$$

$$F_z = -\int_S p dS_z = -\int_S (p_o + \rho gh) dS_z$$

where S_x , S_y , and S_z are the projections of S on the x , y and z planes respectively.

FORCE ON A SUBMERGED PLANE SURFACE

Consider a submerged plane surface which coincides with the z -plane, i.e. $ndS_z = kdS$. See Figure 4 below.

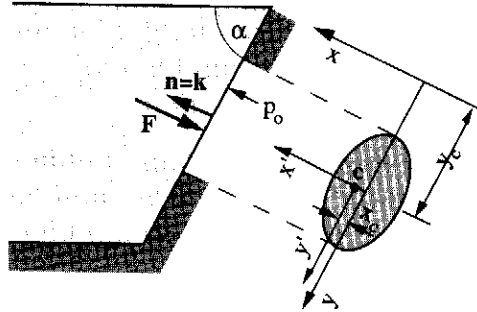


Figure 4. Force on a submerged plane surface with its centroid at C.

Then the resultant force is acting in the z -directions with:

$$F_z = -\int_S p dS = -\int_S \rho g h dS$$

Substituting $h = y \sin \alpha$,

$$F_z = -\rho g \sin \alpha \int_S y dS = -\rho g y_C S \sin \alpha$$

$$F_z = -p_C S$$

where y_C is the y -coordinate of the centroid of the area S , which is defined by,

$$y_C = \frac{1}{S} \int_S y dS$$

and $p_C = \rho g y_C \sin \alpha$ is the hydrostatic pressure at y_C . This method is useful for surfaces with known centroids.

The conditions to calculate the resultant force represented by the distributed forces correctly and its point of application are:

1. Equal these forces in magnitude and direction (already done above)
2. The resultant has its point of application such that its moment about any axis parallel to the x -coordinate axis equals the total moment of the distributed forces about the same axis; and

3. The resultant has its point of application such that its moment about any axis parallel to the y-coordinate axis equals the total moment of the distributed forces about the same axis. Condition (2) determines y_F and condition (3) determines x_F . Therefore, condition (2) can be written as:

$$F_z y_F = -\int_S y dF = -\rho g \sin \alpha \int_S y^2 dS$$

The surface integral is the second moment of the area or the moment of inertia of the area with respect to the x-axis, i.e. I_{xx} . This moment can be related to the moment about any axis parallel to x-axis and passing through the centroid through the Steiner's theorem.

$$I_{xx} = \int_S y^2 dS = I_{x'x'} + S y_C^2$$

Combining the above equations, we can obtain y_F , as:

$$y_F = \frac{1}{y_C S} \int_S y^2 dS = \frac{I_{x'x'}}{y_C S} + y_C$$

Some moments for standard shapes are tabulated.

For a plane not symmetrical with respect to the y-axis x_F may be found in a similar way from:

$$F_z x_F = \int_S x dF = -\rho g \sin \alpha \int_S xy dS = -\rho g \sin \alpha I_{xy}$$

Steiner's theorem is used again:

$$I_{xy} = I_{x'y'} + S x_C y_C$$

This yields

$$x_F = \frac{1}{y_C S} \int_S xy dS = \frac{I_{x'y'}}{y_C S} + x_C$$

where the x-coordinate of the area centroid is defined by

$$x_C = \frac{1}{S} \int_S x dS$$

EXAMPLE (a) A vertical plate AA'B'B is set under water of density ρ as shown in Figure 5.

Find the resultant force, its direction and its point of application, y_F .

(b) A large plate is now shown in the same Figure. Find the resultant force, its direction and its point of application, y_F .

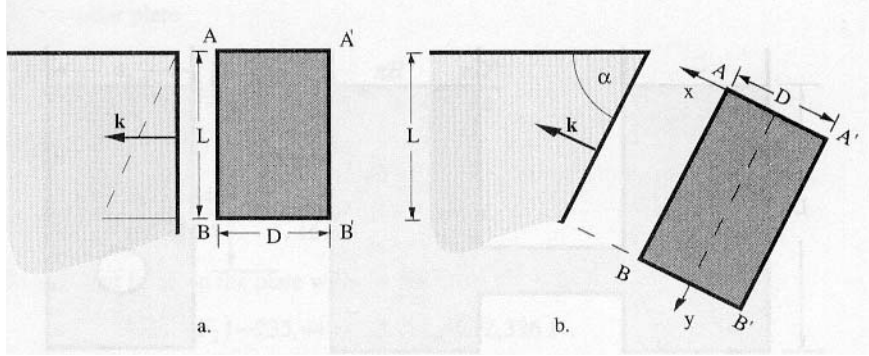


Figure 5. Rectangular plate under water.

(a) The centroid of the plate is $L/2$ deep.

$$F_z = -\rho g \sin \alpha \int_S y dS = -\rho g y_c S \sin \alpha \quad \text{or} \quad F = -k\rho g \frac{L}{2} DL = -k\rho g \frac{DL^2}{2}$$

The moment of inertia of the plate about its centroid is

$$I_{x'x'} = 2 \int_0^{L/2} Dy^2 dy = \frac{2}{3} D \left(\frac{L}{2} \right)^3 = \frac{1}{12} DL^3$$

Then

$$y_F = \frac{I_{x'x'}}{y_c S} + y_c = \frac{\frac{1}{12} DL^3}{\frac{L}{2} LD} + \frac{L}{2} = \frac{2}{3} L$$

(b) The resultant force is:

$$F = -k\rho g \frac{L}{2} D \frac{L}{\sin \alpha} = -k\rho g \frac{DL^2}{2 \sin \alpha}$$

Substituting the moment of inertia and the y_c ,

$$I_{x'x'} = \frac{1/12 DL^3}{\sin^3 \alpha}, \quad \text{and} \quad y_c = \frac{L}{2 \sin \alpha}$$

we can obtain

$$y_F = \frac{2L}{3 \sin a}$$

The point of application is again $2/3$ of the plate and at $2/3$ of the maximal depth.

EXAMPLE: Two reservoirs A and B are filled with water and connected by a pipe (see Figure 6). Find the resultant force and the point of application of this force on the partition with the pipe.

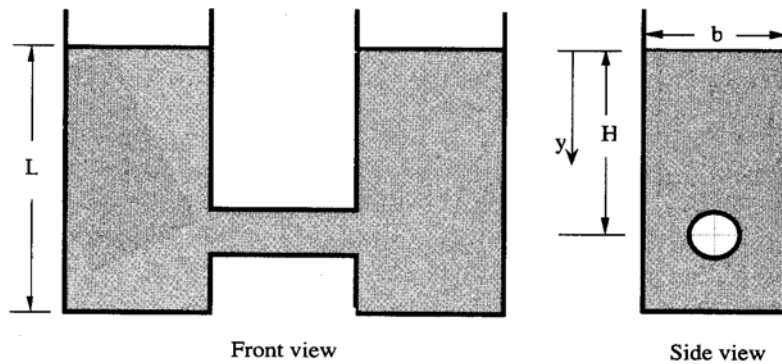


Figure 6: two reservoirs connected by a pipe.

Let the subscripts 1 and 2 refer to the plate and to the circle. The sought force is the difference between that of the rectangular plate and that of the circle.

$$-F_1 = L \times b \times \rho g \frac{L}{2} = \frac{\rho g L^2 b}{2}$$

The point of application is at $2/3L$.

The force on the circular pipe the size of the hole is:

$$-F_2 = \rho g H \frac{\pi D^2}{4}$$

For the circular plate

$$I_{x'x'} = \frac{1}{2} I_r = \frac{1}{2} \int_0^R r^2 2\pi r dr = \frac{\pi D^4}{64}$$

The point of application

$$y_{F2} = \frac{\pi D^4 / 64}{\pi D^2 H / 4} + H + \frac{D^2}{16H} + b$$

The resultant force is:

$$-F_{12} = -(F_1 - F_2)$$

The point of application is found from:

$$y_F F_{12} = y_{F1} F_1 - y_{F2} F_2$$

with the formulas all derived above.

COMPLETELY SUBMERGED BODIES

The force acting on a completely submerged body can be calculated by:

$$\mathbf{F} = -\int_S p \mathbf{n} dS = -\int_V \nabla p dV = -k \int_V \frac{dp}{dz} dV = k \int_V \rho g dV = k \rho g V$$

Therefore there is only one force acting on a completely submerged body and this is in the vertical direction. This is also known as *Archimedes Principle* and the resultant force as the *buoyancy force*.. In deriving the above formula the Gauss theorem was utilized that transforms a surface integral into a volume one.

The centre of buoyancy of a submerged body can also be calculated by:

$$x_B \rho g V = \int_V x \rho g dV \quad \text{or} \quad x_B = \frac{1}{V} \int_V x dV$$

Similarly

$$y_B = \frac{1}{V} \int_V y dV \quad \text{and} \quad z_B = \frac{1}{V} \int_V z dV$$

The same analysis can also be performed for a floating body to calculate the buoyancy force.

EXAMPLE: Ice at $-10\text{ }^{\circ}\text{C}$ has the density $\rho_i = 998.15\text{ kg/m}^3$. A 1,000-ton spherical iceberg floats at sea as shown in Figure 7. The salty water has a density of $\rho_s = 1,025\text{ kg/m}^3$. By how much does the tip of the iceberg stick out of the water?

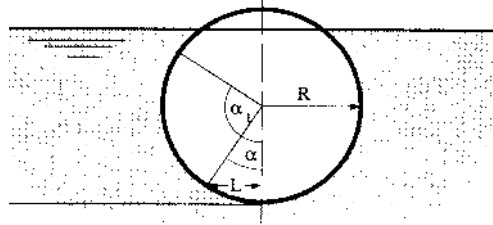


Figure 7: Spherical iceberg

The volume of the iceberg is $V = 10^6\text{ kg}/998.15 = 1001.9\text{ m}^3$ which corresponds to a sphere of $R = 6.207\text{ m}$. The volume of sea water it must displace by Archimede's principle, is:

$$V_w = 10^6 / 1,025 = 975.6\text{ m}^3$$

From the Figure we can derive the following geometrical relations:

$$z = R(1 - \cos \alpha) \quad dz = R \sin \alpha d\alpha$$

$$L = R \sin \alpha \quad dV = \pi L^2 dz = \pi R^3 \sin^3 \alpha d\alpha$$

$$V = \int_0^{\alpha_1} \pi R^3 \sin^3 \alpha d\alpha = \pi R^3 \int_0^{\alpha_1} (1 - \cos^2 \alpha) \sin \alpha d\alpha = 975.6\text{ m}^3$$

Solving we obtain:

$$\alpha_1 = 144^\circ \quad z_1 = 11.229\text{ m}$$

The part sticking out of the water will be:

$$2R - z_1 = 1.185\text{ m}$$

FORCES ON GATES AND SUBMERGED BOUNDARIES

The methodology developed so far can be used to calculate forces on gates as well as on submerged surfaces in general. Consider the three shapes of gates indicated on figure 8. Calculate the resultant force of the water pressure exerted on each gate and its line of application such as the correct moments results.

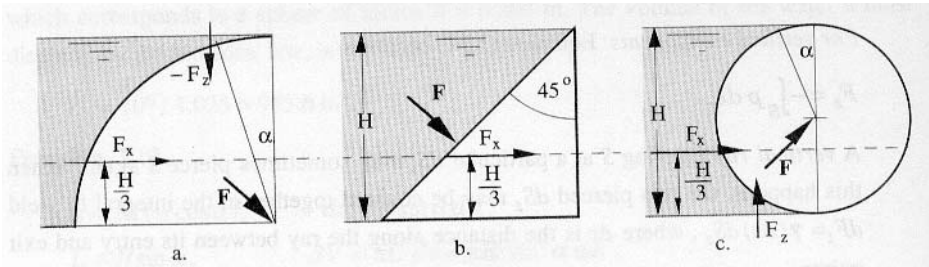


Figure 8: Shapes of gates

Solution

For gate a

$$p = \rho gh = \gamma H(1 - \cos \alpha).$$

Using Eq. (3.18), we have

$$dF = -p n dS = -p n H d\alpha,$$

where $\mathbf{n} = \mathbf{r}$; hence

$$dF_x = -p H \sin \alpha d\alpha = \gamma H^2 (1 - \cos \alpha) \sin \alpha d\alpha,$$

$$F_x = \int_0^{\pi/2} \gamma H^2 (1 - \cos \alpha) \sin \alpha d\alpha = \gamma H^2 \left[1 - \frac{1}{2} \right] = \frac{1}{2} \gamma H^2,$$

$$\begin{aligned} z_c F_x &= \int_0^{\pi/2} z \gamma H^2 (1 - \cos \alpha) \sin \alpha d\alpha = H^3 \gamma \int_0^{\pi/2} (\cos \alpha - \cos^2 \alpha) \sin \alpha d\alpha \\ &= H^3 \gamma \left[\frac{1}{2} - \frac{1}{3} \right] = \frac{1}{6} H^3 \gamma, \quad z_c = H/3, \end{aligned}$$

$$-dF_z = p H \cos \alpha d\alpha = \gamma H^2 (1 - \cos \alpha) \cos \alpha d\alpha,$$

$$-F_z = \int_0^{\pi/2} \gamma H^2 (\cos \alpha - \cos^2 \alpha) d\alpha = \gamma H^2 \left[1 - \frac{\pi}{4} \right] = 0.2146 \gamma H^2,$$

$$(-F_z)(-x_c) = \int_0^{\pi/2} \gamma H^3 (\cos \alpha - \cos^2 \alpha) \sin \alpha d\alpha = \gamma H^3 \left[\frac{1}{2} - \frac{1}{3} \right] = \frac{\gamma H^3}{6},$$

$$(-x_c) = 0.7766 H,$$

$$|\mathbf{F}| = \left| \sqrt{F_x^2 + F_z^2} \right| = \left| \sqrt{\left(\frac{1}{2}\gamma H^2\right)^2 + (0.2146\gamma H^2)^2} \right|$$

$$= 0.5441\gamma H^2 = 0.5441 \times 9,810 \times 4^2 = 85,403 \text{ N}$$

and

$$\alpha_F = \arctan \left| \frac{F_x}{F_z} \right| = \arctan \left| \frac{\frac{1}{2}}{0.2146} \right| = 66.7^\circ.$$

For gate b, we have from Example 3.4

$$z_c = \frac{H}{3},$$

$$F = H\sqrt{2} \times \gamma \times \frac{1}{2}H = 0.7071\gamma H^2 = 110,986 \text{ N}.$$

The point of application of that force is at $H/3$, and its direction is perpendicular to the inclined plane.

For gate c

$$p = \frac{1}{2}\gamma H(1 - \cos \alpha),$$

$$dF_x = p \frac{H}{2} \sin \alpha d\alpha = \frac{1}{4}\gamma H^2(1 - \cos \alpha) \sin \alpha d\alpha,$$

$$F_x = \int_0^\pi \frac{1}{4}\gamma H^2(1 - \cos \alpha) \sin \alpha d\alpha = \frac{1}{4}\gamma H^2[2 - 0] = \frac{1}{2}\gamma H^2,$$

$$z_c F_x = \int_0^\pi \frac{1}{4}z \gamma H^2(1 - \cos \alpha) \sin \alpha d\alpha = \frac{1}{8}\gamma H^3 \int_0^\pi (1 + \cos \alpha)(1 - \cos \alpha) \sin \alpha d\alpha$$

$$= \frac{1}{8}\gamma H^3 \int_0^\pi (1 - \cos^2 \alpha) \sin \alpha d\alpha = \frac{1}{8}\gamma H^3 \left[2 - \frac{2}{3} \right] = \frac{1}{6}\gamma H^3.$$

Hence

$$z_x = \frac{z_c F_x}{F_x} = \frac{H}{3}.$$

The force acting upward, F_z , is found by Archimedes' law as the buoyancy force acting on half the cylinder:

$$F_z = \gamma V = \gamma \left(\frac{\pi H^2 \times 1}{4 \times 2} \right) = 0.3927\gamma H^2.$$

We also know from solid mechanics that the center of gravity of half a circle is at

$$-x_c = \frac{4}{3\pi} r = \frac{2}{3\pi} H = 0.2122 H,$$

$$|\mathbf{F}| = \left| \sqrt{F_x^2 + F_z^2} \right| = \left| \sqrt{\left(\frac{1}{2}\gamma H^2\right)^2 + (0.3927\gamma H^2)^2} \right|$$

$$= 0.6358\gamma H^2 = 0.6358 \times 9,810 \times 4^2 = 99,792 \text{ N}$$

and finally

$$\alpha_F = 360 - \arctan \left| \frac{F_x}{F_z} \right| = 308^\circ.$$

HYDRODYNAMIC STABILITY

Two forces are acting on floating bodies, gravity and buoyancy. A static equilibrium is obtained when both forces are acting on the same line (fig. 9a,c). A moment may appear which tends to increase the roll angle α , in which case the situation is called unstable; or the moment may tend to decrease α and diminish the roll. When no moment appears the situation is denoted stable. When the center of gravity is lower than that of buoyancy the situation is usually stable.

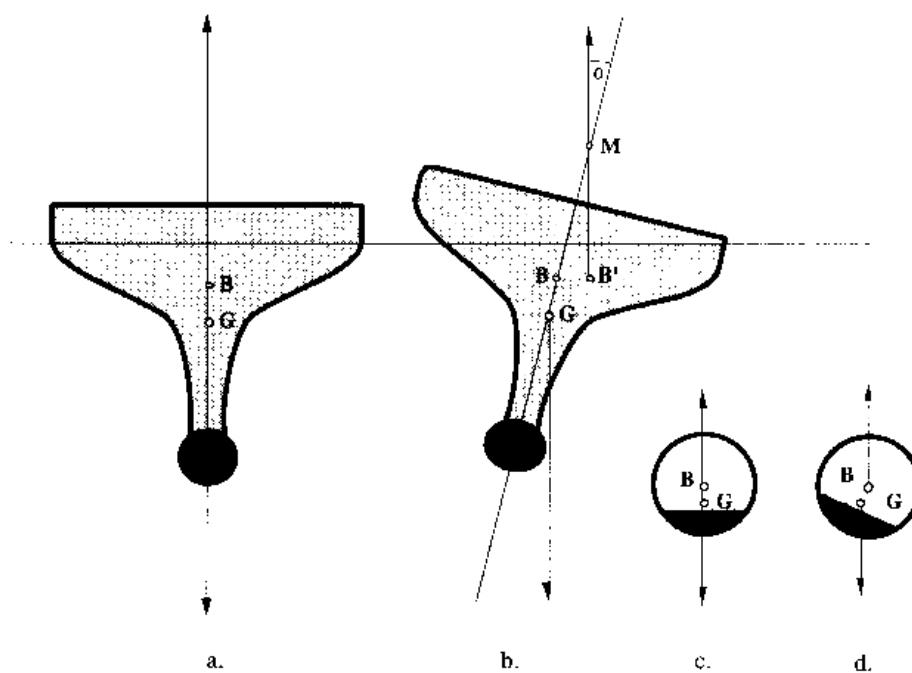


Figure 9: A submarine and a sail boat with the centers of gravity and buoyancy.

Figure 10 below shows stable and unstable equilibria and how the moments work to cause instabilities.

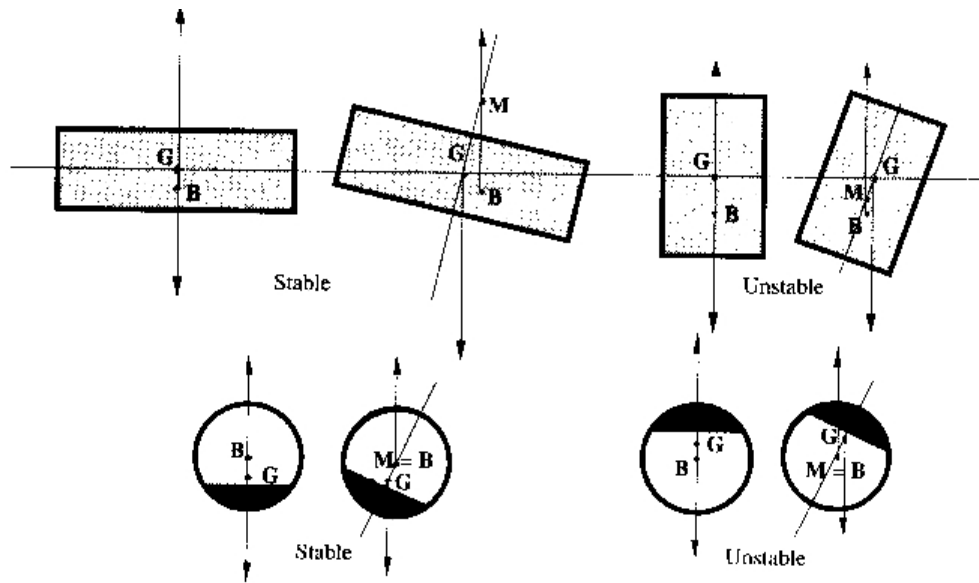


Figure 10: Stable and unstable equilibria

MECHANICS OF INTERFACES

A force balance on the interface S of two immiscible liquids gives (surface density negligible otherwise gravity and acceleration should be included)

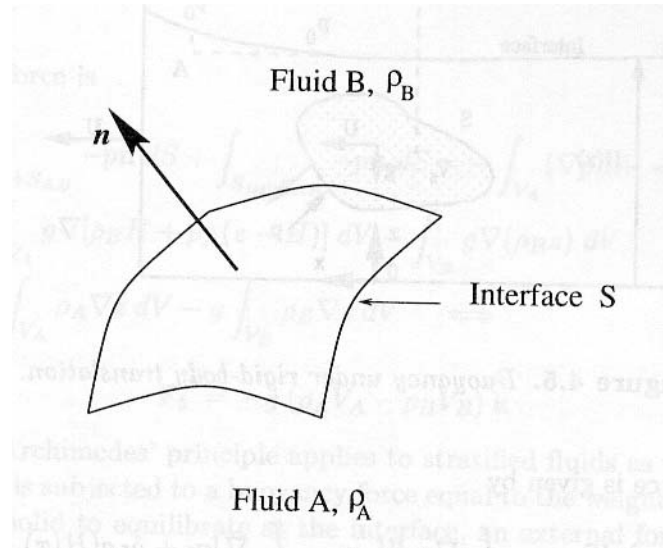


Figure 11: The interface between two immiscible fluids

$$\mathbf{n} \cdot (\boldsymbol{\sigma}_B - \boldsymbol{\sigma}_A) + \nabla_{||} \sigma + n 2H\sigma = 0$$

where \mathbf{n} is the unit normal vector to the interface pointing from liquid B to A, σ is the surface tension and $\boldsymbol{\sigma}_A$ and $\boldsymbol{\sigma}_B$ are the stress tensor written across the interface for the two fluids. Note that the gradient operator is defined in terms of local coordinates (\mathbf{n}, \mathbf{t}) that is normal and tangential to the interface.

$$\nabla_{||} = \mathbf{t} \frac{\partial}{\partial t} + \mathbf{n} \frac{\partial}{\partial n}$$

The surface tension gradient which is present with surfactants and with nonisothermal interfaces is responsible for shear stress discontinuities which often cause flow in thin films. In the absence of these two effects, the gradient is zero.

$$\nabla_{||} \sigma = 0$$

Thus, the equation of the interface is reduced to:

$$\mathbf{n} \cdot (\boldsymbol{\sigma}_B - \boldsymbol{\sigma}_A) + n 2H\sigma = 0$$

The two components of this equation are those in the normal and tangential directions respectively:

$$\text{Normal: } \mathbf{n} \cdot [\mathbf{n} \cdot (\boldsymbol{\sigma}_B - \boldsymbol{\sigma}_A) + n2H\boldsymbol{\sigma}] = (P_B - P_A) + (\tau_{m,A} - \tau_{m,B}) - 2H\sigma = 0$$

$$\text{Tangential: } \mathbf{t} \cdot [\mathbf{n} \cdot (\boldsymbol{\sigma}_B - \boldsymbol{\sigma}_A) + n2H\boldsymbol{\sigma}] = \tau_{m,A} - \tau_{m,B} = 0$$

The mean curvature, $2H$, of a surface is necessary in order to account for the role of surface tension that gives rise to *normal stress discontinuities*.

INTERFACES IN STATIC EQUILIBRIUM

Under no flow conditions, stresses are zero and therefore the equation for the interface reduces to the Young-Laplace equation:

$$\nabla p = 2H\sigma$$

This equation governs the configuration of interfaces under gravity and surface tension effects. This equation most of the time is solved numerically to find the shape of the interface. For interfaces and free surfaces with general configuration, the mean curvature $2H$ can be expressed as:

$$2H\mathbf{n} = \frac{d\mathbf{t}}{ds}$$

where \mathbf{t} and \mathbf{n} are the tangent and normal vectors respectively and s is the arc length.

For a *cylindrically symmetric* surface (translational symmetry with constant curvature, see Figure 12), the surface wave (shape) can be described by

$$z = z(x)$$

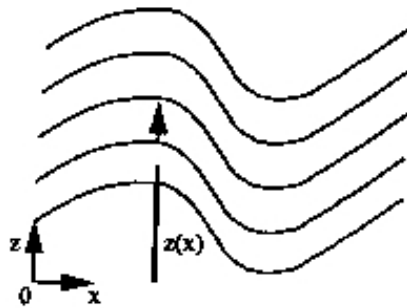


Figure 12: Cylindrically symmetric interface

The mean curvature is:

$$2H = \frac{z_{xx}}{\sqrt{(1+z_x^2)^3}}$$

For *rotationally symmetric* surface (surface generated by rotating a rigid curve, see Figure 13), the surface can be described by:

$$z = z(r)$$

The mean curvature then becomes:

$$2H = \frac{z_r}{r(1+z_r^2)^{1/2}} + \frac{z_{rr}}{(1+z_r^2)^{3/2}} = \frac{1}{2} \frac{d}{dr} \left(\frac{rz_r}{\sqrt{1+z_r^2}} \right)$$

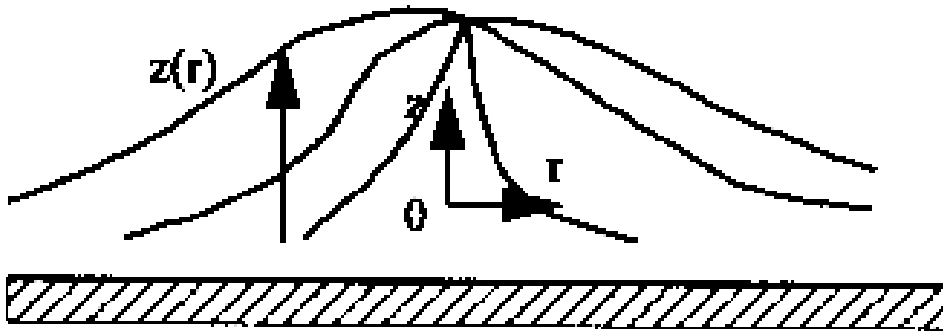


Figure 13: Rotationally symmetric surface

The same interface can be alternatively described by:

$$r = r(z)$$

in which case:

$$2H = \frac{1}{r(1+r_z^2)^{1/2}} - \frac{r_{zz}}{(1+r_z^2)^{3/2}}$$

For static interfaces as described above, the Young-Laplace equation applies

$$2H = \frac{P_B - P_A}{\sigma}$$

For **planar interfaces** $H=0$ and there is no jump in pressure across the interface.

For cylinders:
$$2H = \frac{1}{R_1} + \frac{1}{\infty} = \frac{1}{R}$$

For spheres:
$$2H = \frac{1}{R_1} + \frac{1}{R_2} = \frac{2}{R}$$

Which means that the pressure is smaller inside the cylinder or sphere compared to outside.

MEASUREMENT OF SURFACE TENSION

1. Wilhelmy method: A Plate of known dimensions S , L and h and density ρ_s is being pulled from a liquid of density ρ_B and surface tension σ in contact with air of density ρ_A .

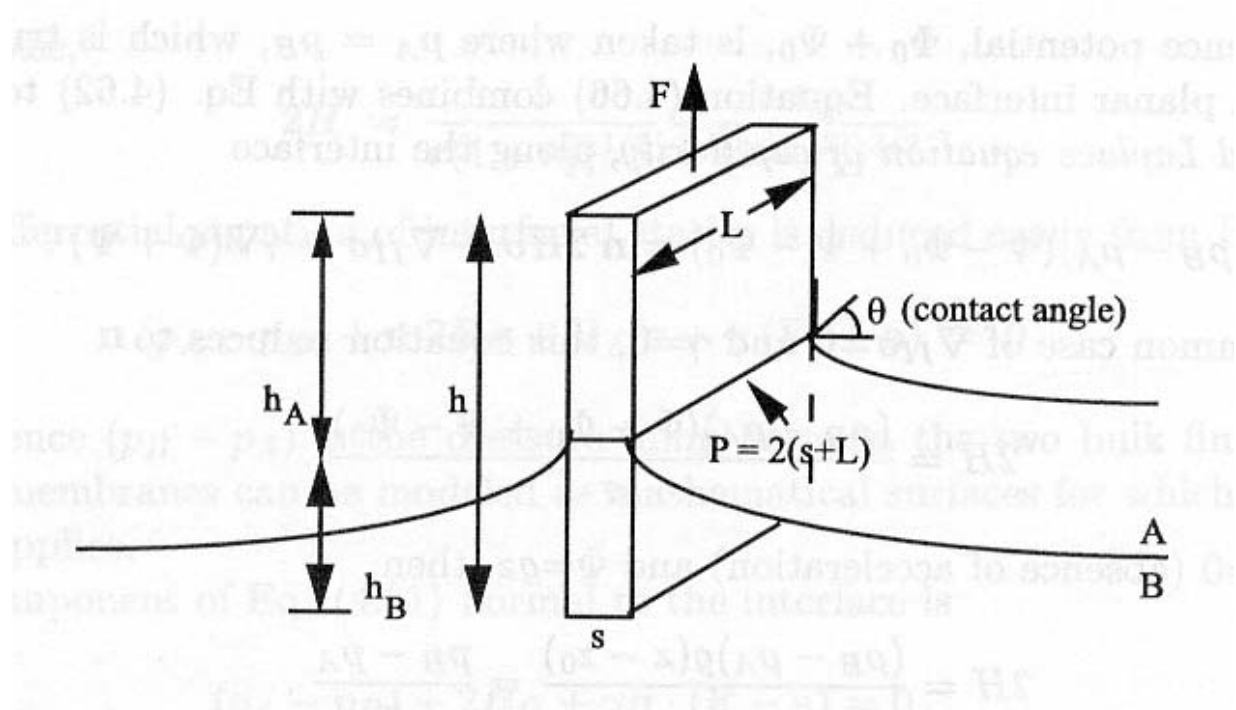


Figure 14: The Wilhelmy plate method

The net force exerted by fluid A on the submerged part is (buoyancy):

$$F_A = -\rho_A g h_A SL$$

The net force exerted by fluid B on the submerged part is (buoyancy):

$$F_B = -\rho_B g h_B SL$$

The surface tension force on the plate is (pulling downwards):

$$F_\sigma = \sigma P \cos \theta$$

The weight of the plate is:

$$W = \rho_s g V = \rho_s g (h_A + h_B) SL$$

The total force balance thus gives

$$F_A + F_B + F_\sigma + W = F \quad \text{or} \quad F = gSL[h_A(\rho_s - \rho_A) + h_B(\rho_s - \rho_B)] + \sigma P \cos \theta$$

Since everything is known, σ can be calculated.

An improved method is when the plate is completely immersed in fluid A. In this case the force, F_o becomes equal to:

$$F_o = \sigma P \cos \theta + (\rho_s - \rho_A)ghSL$$

The surface tension can be easily calculated without knowing the densities of fluids.

2. Capillary rise on a vertical wall

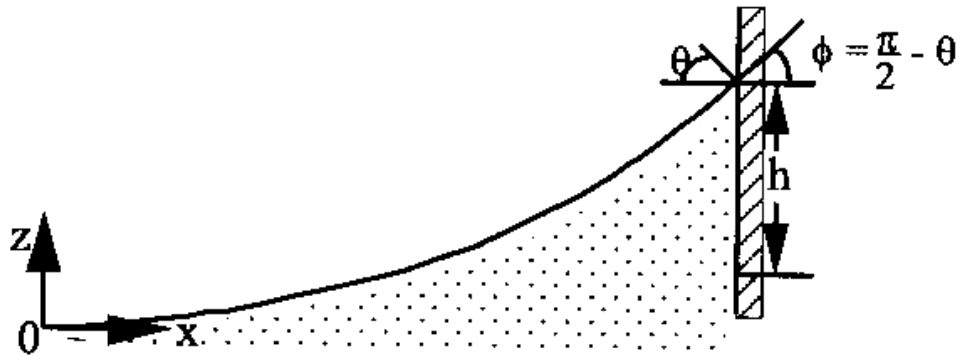


Figure 15: Capillary rise

The Young-Laplace equation in the presence of gravity is:

$$\frac{\sigma}{R} = -g\Delta(\rho z)$$

From differential geometry:

$$\frac{1}{R} = \frac{d\phi}{ds}$$

where ϕ is the local inclination and s the arclength. Also from differential geometry:

$$\frac{dz}{ds} = \sin \phi \quad \text{and} \quad \frac{dx}{ds} = \cos \phi$$

Therefore,

$$\frac{d\phi}{ds} = \frac{d\phi}{dz} \frac{dz}{ds} = \frac{d\phi}{dz} \sin \phi = -\frac{g\Delta\rho}{\sigma} z$$

Integration gives

$$\cos \phi = -\frac{g\Delta\rho}{2\sigma} z^2 + c$$

If we let $\cos \phi = 1$ at $z=0$, then

$$\cos \phi - 1 = -2 \sin^2(\phi/2) = -\frac{g\Delta\rho}{2\sigma} z^2$$

which yields

$$z = \pm 2 \sqrt{\frac{\sigma}{g\Delta\rho}} \sin\left(\frac{\phi}{2}\right)$$

The meniscus intersects the wall at a contact angle θ and a height h above the free surface.

Therefore,

$$h = 2 \sqrt{\frac{\sigma}{g\Delta\rho}} \sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$$

If $\theta < \pi/2$, h is positive; if $\theta > \pi/2$, h is negative

3. Interfacial tension by sessile drop

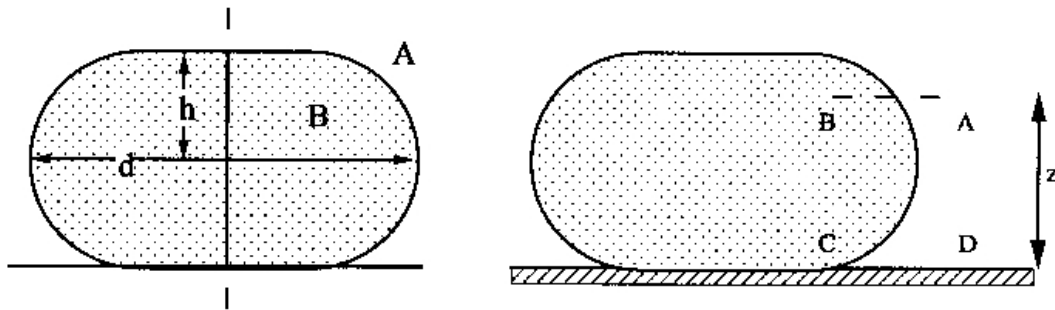


Figure 16: Axisymmetric sessile drop

Consider the points marked as A, B, C and D in Figure 16.

Across the meniscus we have

$$p_B - p_A = 2H\sigma$$

From hydrostatics:

$$p_D - p_A = \rho_A g z$$

$$p_C - p_B = \rho_B g z$$

$$p_C = p_D$$

which yield

$$p_B - p_A = -(\rho_B - \rho_A) g z$$

Combining the above equations we obtain the equation for the interfacial tension.

$$\sigma = \frac{g z}{2H} (\rho_A - \rho_B)$$

The surface of the droplet is given by $z = z(x, y)$. For rotationally symmetric interfaces of this dependence:

$$2H = \frac{z_{,xx}(1 + z_{,yy}^2) - 2z_{,x}z_{,y}z_{,xy} + 2z_{,yy}(1 + z_{,x}^2)}{(1 + z_{,x}^2 + z_{,y}^2)^{3/2}}$$

The solution (numerical) of this contains two constants, which can be determined by:

$$x = y = 0, \quad z = z_{\max} \quad \text{or} \quad \partial z / \partial r = 0$$

The description is completed by:

$$r = \pm d/2, \quad z_r \rightarrow \infty, \quad z = z_{\max} - h,$$

which determines h.

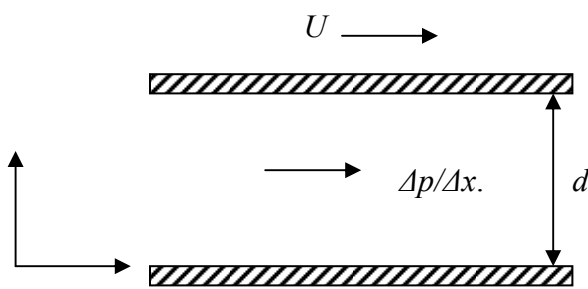
EXACT SOLUTIONS OF THE NAVIER-STOKES EQUATIONS

As discussed before there are a few cases where a exact solution to the Navier-Stokes equations can be found by integrating them analytically. Some of these will be presented in this chapter and some others will be given in the assignments.

TIME - INDEPENDENT FLOWS

Plane Poiseuille Flow

Two parallel plates with a gap d between them in the y -direction are shown in the Figure below. The flow field is extended infinitely in the x and z directions, so that end effects can be neglected.



The fluid flows in the x positive direction under the action of a pressure drop $\Delta p / \Delta x$. In addition, the upper plate moves with a constant velocity U in the x -positive direction. Determine the velocity profile, the maximum velocity, its location and the volume flow rate per unit width of the channel.

Neglecting the transients of the flow, this is an one-dimensional flow. One may clearly assume that $v_z=0$ and that from an intuitive guess one may take $v_y=0$. This last guess is pursued until either a solution is found or if a solution cannot be found then it is dropped.

Using the continuity for steady-state incompressible flow,

$$\frac{\partial v_x}{\partial x} = 0$$

which tells us that $v_x=v_x(y)$. Thus, whatever the velocity is at some x -coordinate, it repeats itself for other x values. Such a flow is called **fully developed**. Now considering the x -component of the Navier-Stokes equations and using the simplifications discussed above, one may write.

$$0 = -\frac{\partial p}{\partial x} + \rho g_x + \mu \frac{\partial^2 v_x}{\partial y^2}$$

The y -component simplifies to:

$$0 = -\frac{\partial p}{\partial y} + \rho g_y$$

we have taken $-\rho g(\partial h/\partial x) = \rho g_x$ and $-\rho g(\partial h/\partial y) = \rho g_y$. Furthermore, from the definition of coordinates $g_x=0$, and $g_y=g$. Thus, the y -component tells us that there exists a pressure gradient in the y -direction due to gravity. The x -component can be integrated twice to result,

$$v_x = \frac{\Delta p}{\Delta x} \frac{y^2}{2\mu} + C_1 y + C_2$$

The gradient $\partial p/\partial x$ has been substituted with $\Delta p/\Delta x$, which is a constant. This can be inferred from inspecting the viscous and pressure terms of the x -component of the Navier-Stokes equations (see above).

The boundary conditions to be used in order to evaluate the constant C_1 and C_2 are:

$$y=0, \quad v_x=0$$

$$y=d, \quad v_x=U$$

Evaluating the constant C_1 , and C_2 , one may obtain:

$$v_x = \left(-\frac{\Delta p}{\Delta x} \right) \frac{d^2}{2\mu} \left[\frac{y}{d} - \left(\frac{y}{d} \right)^2 \right] + U \left(\frac{y}{d} \right)$$

The special case $U=0$ results in

$$v_x = \left(-\frac{\Delta p}{\Delta x} \right) \frac{d^2}{2\mu} \left[\frac{y}{d} - \left(\frac{y}{d} \right)^2 \right]$$

which is known as plane Poiseuille flow, while the case of $\Delta p/\Delta x = 0$ results in

$$v_x = U \frac{y}{d}$$

which is known as simple shear flow or plane Couette flow. Several velocity profiles for all these cases are given in the Figure below.

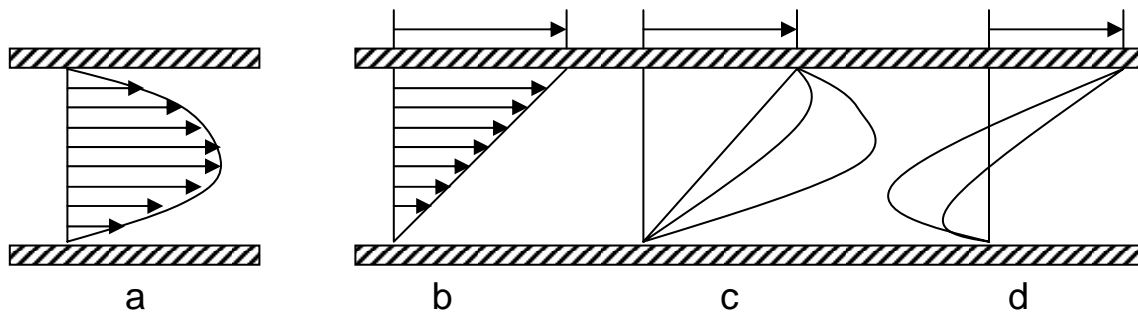


Figure: a.Plane Couette flow b.Shear Flow c.Flows with $\Delta P/\Delta x < 0$ and d. $\Delta P/\Delta x > 0$.

Maximum Velocity: The maximum velocity, $v_{x, \max}$ and its location, y_0 can be inferred from the condition, $dv_x/dy=0$ which results in,

$$y_0 = \frac{d}{2} \left(1 + \frac{2\mu U}{(-\Delta p/\Delta x)d^2} \right)$$

and

$$v_{x, \max} = \left(-\frac{\Delta p}{\Delta x} \right) \frac{d^2}{2\mu} \left[\frac{1}{4} - \left(\frac{\mu U}{(-\Delta p/\Delta x)d^2} \right)^2 \right] + U \left[\frac{1}{2} + \frac{\mu U}{(-\Delta p/\Delta x)d^2} \right]$$

For plane Poiseuille flow these expressions become:

$$y_0 = \frac{d}{2}$$

and

$$v_{x, \max} = \left(-\frac{\Delta p}{\Delta x} \right) \frac{d^2}{8 \mu}$$

The volume flow rate per unit width, Q/w (w is the width) can be obtained from:

$$Q/w = \int_0^d v_x dy = \left(-\frac{\Delta p}{\Delta x} \right) \frac{d^3}{12 \mu} + \frac{U d}{2}$$

The average velocity is defined as the flow rate divided by the cross sectional area where in this case may be simplified to:

$$\bar{v}_x = \frac{Q}{d} = \left(-\frac{\Delta p}{\Delta x} \right) \frac{d^2}{12 \mu} + \frac{U}{2}$$

Flow in a round tube

Consider a round tube with the z -coordinate as its axis of symmetry. Determine the velocity profile, the maximum velocity, flow rate and average velocity for steady, fully developed flow of an incompressible Newtonian fluid. Considering one-dimensional flow with $v_\theta = v_r = 0$, the continuity equation reduces to:

$$\frac{\partial v_z}{\partial z} = 0$$

which is the condition for fully developed flow. The Navier-Stokes equations can be simplified as:

$$0 = \frac{\partial p}{\partial r}$$

$$0 = \frac{l}{r} \frac{\partial p}{\partial \theta}$$

$$0 = -\frac{\partial p}{\partial z} + \frac{\mu}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right)$$

Note that the effect of gravity has been neglected. In general the effect of gravity can always be neglected except in these cases where it is the primary driving force for flow. The first two equations indicate that p is a function of z only, and in a manner similar to that in the flow between two parallel plates, one may show that

$$\frac{\partial p}{\partial z} = \text{const} = \frac{\Delta p}{\Delta z}$$

Integrating twice the third equation results into:

$$v_z = \frac{l}{4\mu} \left(\frac{\Delta p}{\Delta z} \right) r^2 + C_1 + C_2 \ln r$$

The boundary conditions to be applied in order to evaluate the two constants of integration are:

$$B.C.1: \quad v_z = 0 \quad \text{at} \quad r = R$$

$$B.C.2: \quad \frac{dv_z}{dr} = 0 \quad \text{at} \quad r = 0$$

Evaluating the constants of integration, the following velocity profile results.

$$v_z = -\left(\frac{\Delta p}{\Delta z}\right) \frac{R^2}{4\mu} \left[1 - \left(\frac{r}{R}\right)^2 \right]$$

The minus sign indicated that the flow is in the direction of decreasing pressure. The maximum velocity occurs at the centerline where $r = 0$, so that

$$v_{z,\max} = -\left(\frac{\Delta p}{\Delta z}\right) \frac{R^2}{4\mu}$$

The volume flow rate can be calculated from:

$$Q = \int_0^{2\pi} \int_0^R v_z r dr d\theta = \frac{\pi R^4}{8\mu} \left(-\frac{\Delta p}{\Delta z} \right)$$

Finally the average velocity is defined as:

$$\bar{v}_z = \frac{\int_0^{2\pi} \int_0^R v_z r dr d\theta}{\int_0^{2\pi} \int_0^R r dr d\theta} = -\frac{\Delta p}{\Delta z} \frac{R^2}{8\mu}$$

Simple Shear Flow

The Sliding Plate Rheometer

The flow generated between two parallel plates is referred to as simple shear (see schema below):

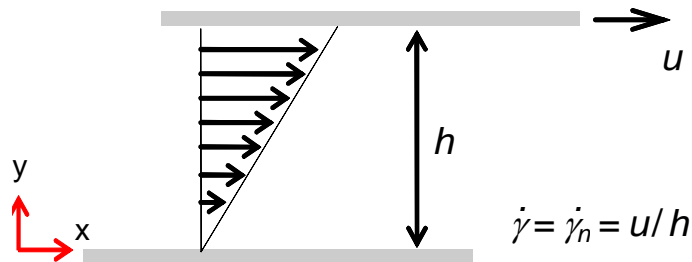


Figure: Velocity profile in simple shear (sliding plate rheometer)

Simplifying the Navier-Stokes equations for this flow

-no pressure gradient

-1-D flow (only v_x which depends on y)

$$\frac{\partial^2 v_x}{\partial y^2} = 0 \quad \text{or} \quad v_x = c_1 y + c_2$$

Applying $v_x = u$ at $y = h$ and $v_x = 0$ at $y = 0$

$$v_x = \frac{u}{h} y \quad \text{or} \quad v_x = \dot{\gamma} y$$

Note that the velocity profile does not depend on the type of fluid used.

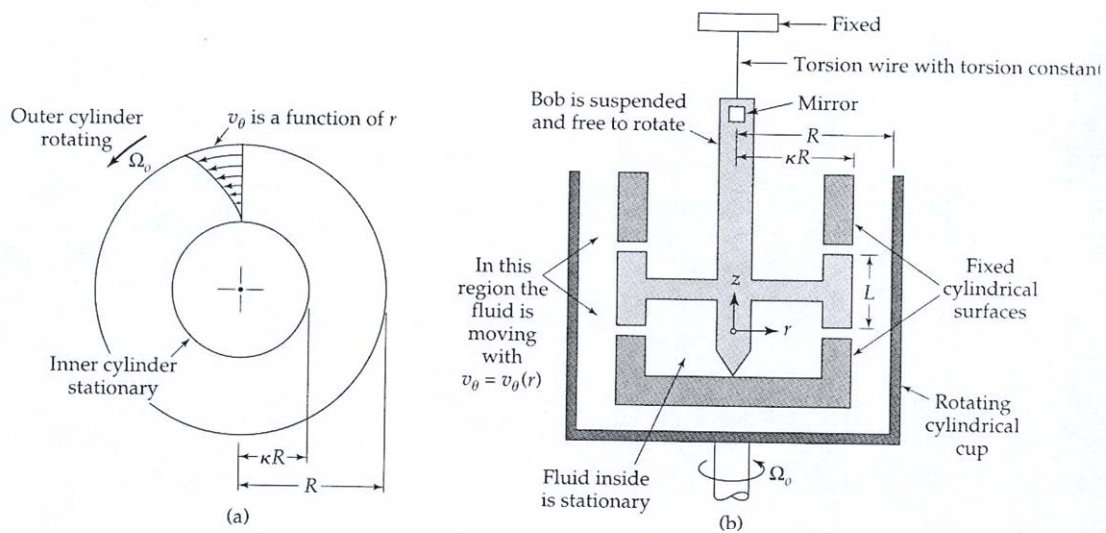
This is the simplest flow that can be generated in a lab to measure the viscosity of fluids. Using a constant velocity that results a constant shear rate would generate a constant force (constant shear stress at the wall) and thus the viscosity can be obtained by:

$$\mu = \frac{\sigma}{\dot{\gamma}}$$

Torsional Flow between Two concentric Cylinders

Couette Viscometer

The Figure below shows a schematic of a Couette instrument, where the fluid is placed in the gap. Then the cap is rotated. The viscosity causes the bob to turn until the torque produced by the momentum transferred equals the product of the torsion constant k_1 and the angular displacement θ_b of the bob.



Reasonable postulates are: $V_\theta = v_\theta(r)$, $V_r = V_z = 0$, $p = p(r, z)$. We expect p to be a function of z due to gravity and a function of r due to centrifugal acceleration

Continuity All terms are zero

$$r\text{-component} \quad -\rho \frac{v_\theta^2}{r} = -\frac{\partial p}{\partial r}$$

$$\theta\text{-component} \quad 0 = \frac{d}{dr} \left(\frac{1}{r} \frac{d}{dr} (r v_\theta) \right)$$

$$z\text{-component} \quad 0 = -\frac{\partial p}{\partial z} - \rho g$$

Integrate the second equation and use

$$\text{B.C.1: at } r=\kappa R \quad v_\theta = 0$$

B.C.2: at $r=R$ $v_{\theta} = \Omega_0 R$

The velocity profile is:

$$v_{\theta} = \Omega_0 R \frac{\left(\frac{r}{\kappa R} - \frac{\kappa R}{r} \right)}{\left(\frac{1}{\kappa} - \kappa \right)}$$

The shear stress distribution is:

$$\tau_{r\theta} = -\mu r \frac{d}{dr} \left(\frac{v_{\theta}}{r} \right) = -2\mu\Omega_0 \left(\frac{R}{r} \right)^2 \left(\frac{\kappa^2}{1 - \kappa^2} \right)$$

The torque acting on the inner cylinder is

$$T_z = (-\tau_{r\theta}) \Big|_{r=\kappa R} \cdot 2\pi\kappa R L \cdot \kappa R = 4\pi\mu\Omega_0 R^2 L \left(\frac{\kappa^2}{1 - \kappa^2} \right)$$

It is important to know when turbulent flow starts since the above analysis is valid only for laminar flow

Flow between two Parallel Discs

Parallel Plate Rheometer

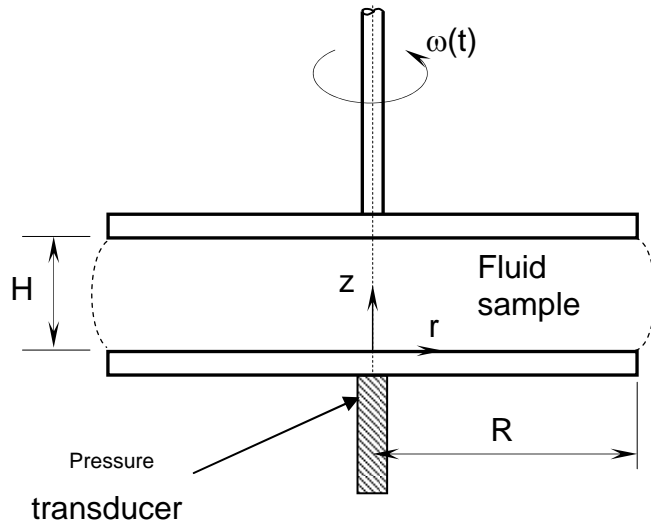


Figure: Parallel plate rheometer

The two plates are mounted on a common axis of symmetry, and the sample is inserted in the space between them. The upper plate is rotated at a specified angular velocity $\omega(t)$ and as a result the sample is subjected to shear. The motion of the upper plate is programmed, and the resulting torque, M , is measured (constant strain rheometers). Analysis of this flow gives the velocity to be:

$$v_{\theta} = \frac{r\Omega z}{H} \quad \text{so that the shear rate depends on the radial position}$$

$$\dot{\gamma} = \frac{R\Omega}{H} \frac{r}{R} \quad \text{which is a non-uniform flow?}$$

However it is still possible to relate the viscosity of a fluid with the torque needed to rotate the disk with a prescribed rotational speed.

Capillary Rheometer

The most widely used type of melt rheometer is the capillary rheometer. This device consists of a reservoir, or barrel and a plunger or piston that causes the fluid to flow through the capillary die of known diameter, D , and length, L . The quantities normally measured are the flow rate, Q , (related to the piston speed) and the driving pressure, ΔP , (related to force on the piston that is measured by means of a load cell).

Capillary rheometers are used primarily to determine the viscosity in the shear rate range of 5 to $1,000 \text{ s}^{-1}$. To calculate the viscosity, one must know the wall shear stress and the wall shear rate.

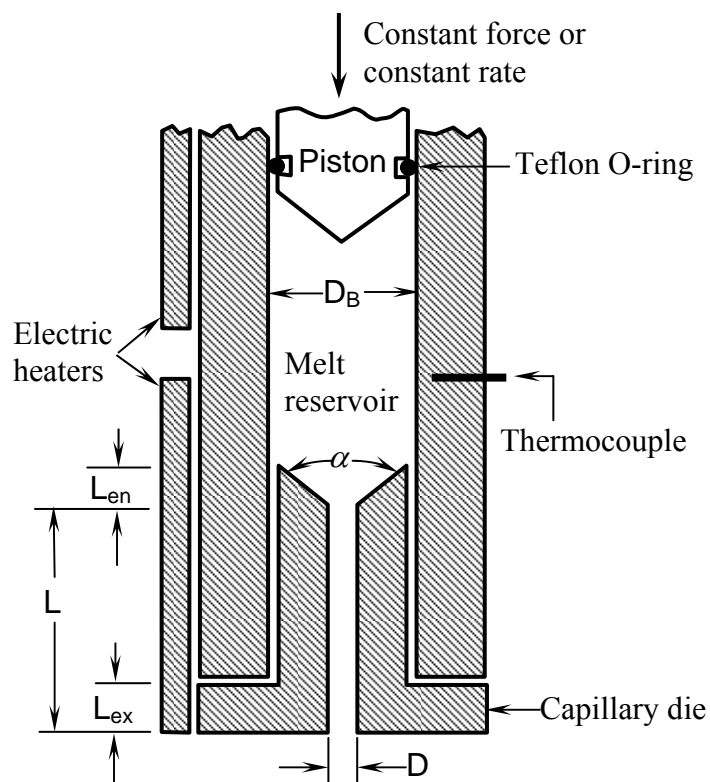


Figure: Capillary rheometer

For the steady flow of an incompressible fluid in a tube of diameter R , driven by a pressure gradient dp/dz , a force balance on a cylindrical element of the fluid gives:

$$\sigma_{rz}(r) = \frac{r}{2} \left(\frac{dp}{dz} \right)$$

When the flow is fully-developed over length L , the absolute value of the shear stress at the wall σ_w

is:

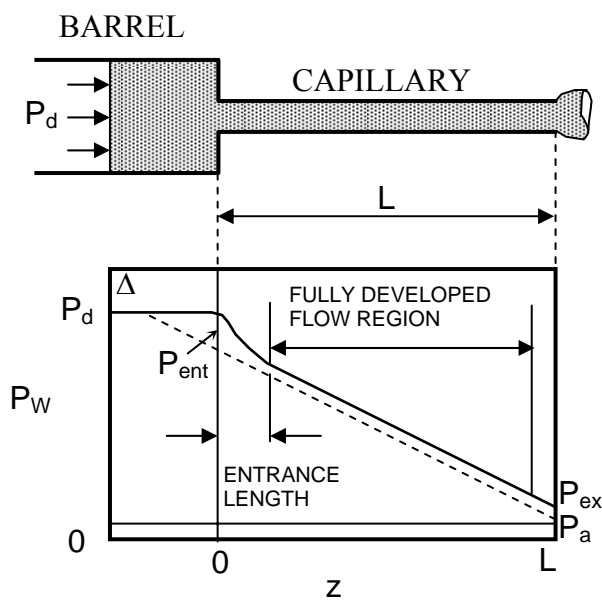
$$\sigma_w \equiv -\sigma_{rz}|_{r=R} = \frac{-\Delta p \cdot R}{2L} = \frac{-\Delta p \cdot D}{4L}$$

where ΔP is the pressure drop over the length of tube.

The magnitude of the wall shear rate, $\dot{\gamma}_w$, for a Newtonian fluid can be calculated as:

$$\dot{\gamma}_w = \left. \frac{\partial v_z}{\partial r} \right|_{r=R} = \frac{32Q}{\pi D^3}$$

The pressure drop must be corrected for the additional pressure required for the fluid to pass



through the contraction between the

barrel and the capillary. One can see

that there is a significant pressure

drop near the entrance of the die,

Δp_{ent} . And a small at the end Δp_{ex} .

The total pressure correction for exit

and entrance regions is called the end

pressure, Δp_{end} , that is,

$$\Delta p_{end} = \Delta p_{ex} + \Delta p_{ent}$$

The true wall shear stress is then obtained as:

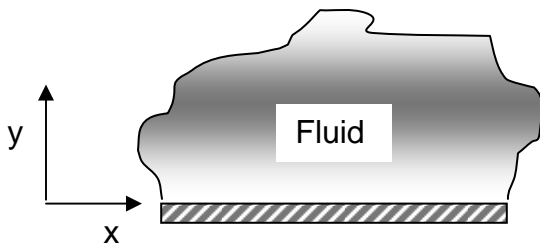
$$\sigma_w = \frac{(\Delta p - \Delta p_{end})}{4(L/D)}$$

The Δp_{end} can be calculated using a capillary of $L/D=0$.

TIME-DEPENDENT FLOWS

The Rayleigh Problem (Stokes first problem)

Consider an infinite flat plate with an infinite domain of fluid on its upper side. The fluid and the plate are at rest. At the time $t=0$ the plate is impulsively set into motion with the velocity U and continues to move at that speed. Determine the velocity profile as a function of time. This problem is known as the Stokes first Problem, and again a solution is sought in which $v_y=0$ everywhere.



This again is a one-dimensional problem where there are no pressure gradients. The equation to be solved is ($v_x=v_x(y)$, $v_y=v_z=0$):

$$\frac{\partial v_x}{\partial t} = \frac{\mu}{\rho} \frac{\partial^2 v_x}{\partial y^2}$$

subject to the following initial and boundary conditions respectively:

$$v_x(y, 0) = 0, \quad v_x(\infty, t) = 0, \quad v_x(0, t) = U$$

To solve this problem a method is used which is called a **similarity transformation**. According to this, a new independent variable η is sought in the form

$$\eta = B y t^n$$

The above equation can be transformed as follows:

$$\frac{d^2 v_x}{d\eta^2} - \frac{n}{\eta} \left(\frac{y^2}{vt} \right) \frac{d v_x}{d\eta} = 0$$

This equation should be expressed in such a way so that no y or t should appear. We therefore choose η such that the combination y^2/t is proportional to η^2 and select a convenient B such that the equation now reduces to:

$$\frac{d^2 v_x}{d\eta^2} - 2\eta \frac{d v_x}{d\eta} = 0$$

Comparison of the last two equations requires:

$$\eta = \frac{y}{2\sqrt{vt}}, \quad n = -\frac{1}{2}, \quad B = \frac{1}{2\sqrt{v}}$$

The boundary conditions in terms of η are

$$v_x = U \quad \text{at} \quad \eta = 0$$

$$v_x = 0 \quad \text{at} \quad \eta \rightarrow \infty$$

Writing the equation as

$$\frac{v_x''}{v_x'} = -2\eta$$

and integrating

$$\frac{dv_x}{d\eta} = C_1 e^{-\eta^2}$$

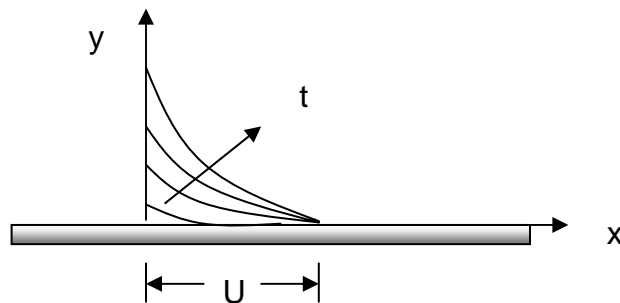
Another integration results to

$$v_x = C_1 \int e^{-\eta^2} d\eta + C_2$$

Applying the two boundary conditions results in the final solution

$$v_x = U [1 - \operatorname{erf} \eta] = U \left[1 - \operatorname{erf} \left(\frac{y}{2\sqrt{\nu t}} \right) \right]$$

which is the similarity solution. Similarity because there are infinite pairs of (y,t) which give the same η which, in turn, uniquely defines v_x . Some velocity profiles are sketched in the Figure below.



Unsteady Laminar Flow Between Two Parallel Plates

Resolve the previous problem with a wall at $y=h$. This flow system has a steady-state limit, whereas the previous did not.

The equation is the same:

$$\frac{\partial v_x}{\partial t} = \frac{\mu}{\rho} \frac{\partial^2 v_x}{\partial y^2}$$

with

I.C.: at $t \leq 0$, $v_x = 0$ for all y

B.C.1: at $y = 0$, $v_x = U$ for all $t > 0$

B.C.2: at $y = h$, $v_x = 0$ for all $t > 0$

Using: $\phi = v_x/U$; $\eta = y/b$; $\tau = \nu t/b^2$, it becomes:

$$\frac{\partial \phi}{\partial \tau} = \frac{\partial^2 \phi}{\partial \eta^2}$$

This system has a finite solution at infinite time. This is:

$$\phi_\infty = 1 - \eta$$

Therefore, we seek a solution of the form:

$$\phi(\eta, \tau) = \phi_\infty(\eta) - \phi_t(\eta, \tau)$$

where the last part is the transient part of the solution which dies out with time. Substituting this into the original equation and boundary conditions gives:

$$\frac{\partial \phi_t}{\partial \tau} = \frac{\partial^2 \phi_t}{\partial \eta^2}$$

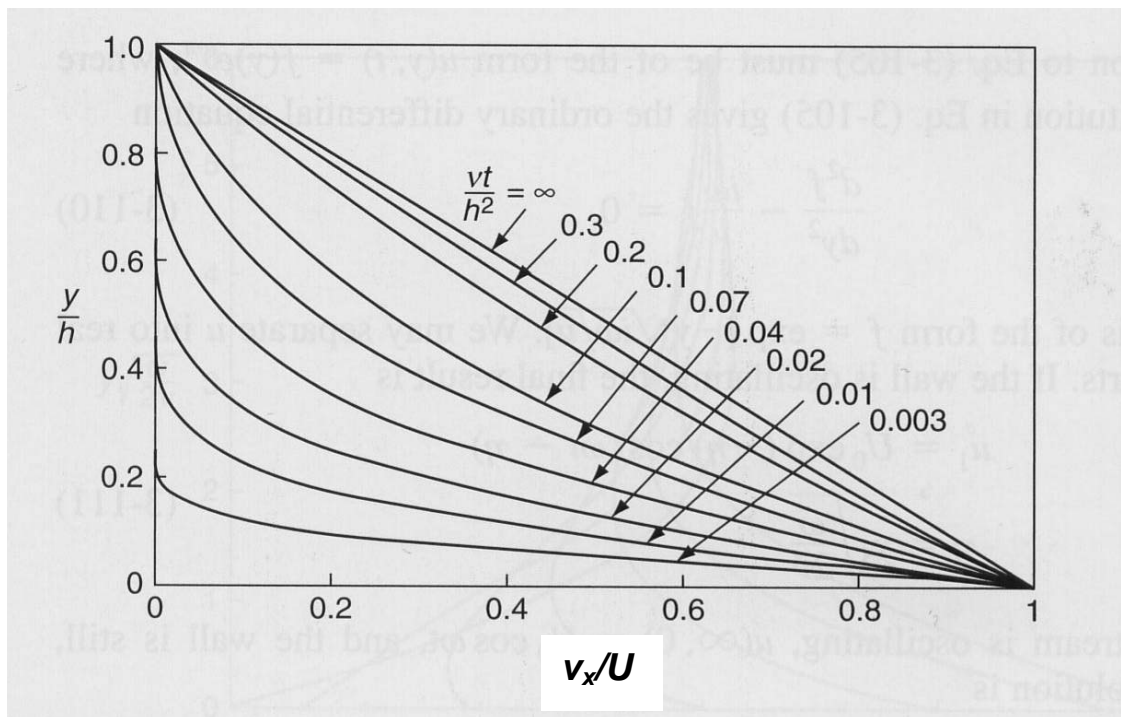
with $\phi_t = \phi_\infty$ at $\tau = 0$, $\phi_t = 0$ at $\eta = 0$ and 1 . The equation can be solved by using the “method of separation of variables”. According to this a solution is sought of the form

$\phi_t(\eta, \tau) = f(\eta)g(\tau)$. Substitute to get

$$\frac{1}{g} \frac{\partial g}{\partial \tau} = \frac{1}{f} \frac{\partial^2 f}{\partial \eta^2} = c^2$$

Thus we obtain two equations which can be solved to result.

$$v_x/U = \left(1 - \frac{y}{h}\right) - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \exp(-n^2 \pi^2 \nu t / h^2) \sin \frac{n\pi y}{h}$$



Starting Flow in a Circular Pipe

Suppose that the fluid in a long pipe is at rest at $t=0$. at which time a constant pressure gradient dp/dz is applied. An axial flow will commence which gradually approaches the steady state Poiseuille flow

$$v_z = -\left(\frac{\Delta p}{\Delta z}\right) \frac{R^2}{4\mu} \left[1 - \left(\frac{r}{R}\right)^2 \right] = v_{z,\max} \left[1 - \left(\frac{r}{R}\right)^2 \right]$$

This problem was solved by Szymanski in 1932. The boundary conditions are:

Initial condition: $v_z(r, 0) = 0$

No slip condition: $v_z(R, t) = 0$

Symmetry: $\left. \frac{\partial v_z}{\partial z} \right|_{r=0} = 0$

The solution is given in terms of the Bessel function J_0 and is expressed as follows:

$$\frac{v_z}{v_{z,\max}} = \left(1 - \left(\frac{r}{R}\right)^2 \right) - \sum_{n=1}^{\infty} \frac{8J_0(\lambda_n r/R)}{\lambda_n^3 J_1(\lambda_n)} \exp\left(-\lambda_n^2 \frac{\nu t}{r_o^2}\right)$$

In the next page the solution is plotted and the first ten roots of the Bessel function are listed in a table.

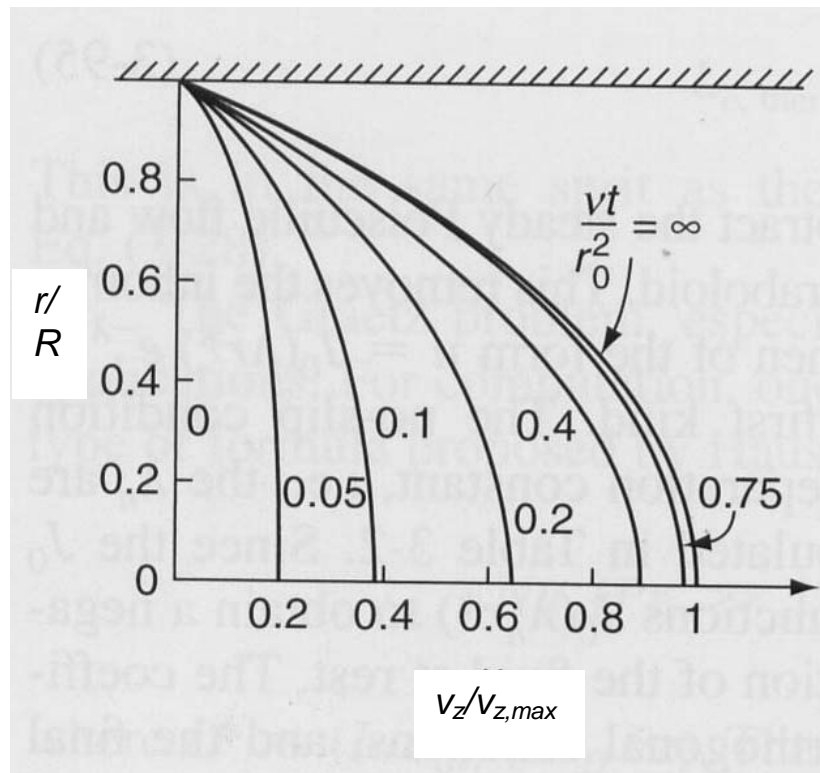
The above equation implies: Flows with small diameter and large viscosity will develop rapidly. At a dimensionless time of 0.75 (see next page) the profile approaches almost its steady state shape. For air 0.75 translates to $t=1.25$ s, whereas for SAE 30 oil this is 0.06 s for identical conditions.

First ten roots of the Bessel function J_0^\dagger

n	λ_n	$J_1(\lambda_n)$
1	2.4048	0.5191
2	5.5201	-0.3403
3	8.6537	0.2715
4	11.7915	-0.2325
5	14.9309	0.2065
6	18.0711	-0.1877
7	21.2116	0.1733
8	24.3525	-0.1617
9	27.4935	0.1522
10	30.6346	-0.1442

† For $n > 10$:

$$\lambda_n = \frac{(4n-1)\pi}{4} \quad J_1(\lambda_n) \approx (-1)^{n+1} \left(\frac{2}{\pi\lambda_n} \right)^{1/2}$$



OTHER EXACT SOLUTIONS

- Flow in an Axisymmetric Annulus
- Flow between Rotating Concentric Cylinders
- Flow in a pipe starting from rest
- The flow near an oscillating flat plate; Stokes second problem
- Stagnation in plane flow (Hiemenz flow)
- Two dimensional non-steady stagnation flow
- Stagnation in three-dimensional flow
- Flow near a rotating disk
- Flow in convergent and divergent channels

For more details see: H. Schlichting, "Boundary Layer Theory," 7th ed., McGraw-Hill, New York, 1979, Chapter 5.

DIMENSIONAL ANALYSIS AND SIMILARITY

All the equations derived so far are dimensional. This means that their various terms have physical dimensions. Before solving these equations a system of units should be adopted. However, using the parameters of the problem to normalise the dependent and independent variables, the Navier-Stokes equations can be written in terms of dimensionless variables, and thus the equations can be solved without making reference to a particular system of physical units.

Apart from this, this dedimensionalisation is of interest for several other reasons:

- Solving the dimensionless equations the obtained solution is in a generalised form. For example all solutions for the fully developed flow in a circular tube can be shown to reduce to a single solution.
- Using the principles of dimensional analysis, the experimental results can be generalised by making use of dimensionless variables, thus substantially reducing the number of experiments. For example in an experimental study, instead of establishing the relative importance of each of the independent variables on the dependent variable, the variables are grouped into dimensionless groups and then the relative importance of these dimensionless groups is studied on the group which includes the dependent variable.

Dimensionless form of the Navier-Stokes Equations:

To write the Navier-Stokes equations in dimensionless form, the parameters of the problem should be used in order to normalise the dependent and independent variables. These parameters include the physical properties of the fluid i.e. density, ρ , and viscosity, μ , geometric variables such as some characteristic length, D , and other parameters which may arise from the boundary conditions, which could be some characteristic velocity, U .

Using these characteristic variables, we define the dimensionless variables as follows:

$$v_i^* \equiv \frac{v_i}{U} \quad x_i^* \equiv \frac{x_i}{D} \quad t^* \equiv \frac{tU}{D} \quad h^* \equiv \frac{h}{D} \quad p^* \equiv \frac{p}{\rho U^2}$$

Introducing these into the Navier-Stokes written in index form, we obtain

$$\frac{\rho U^2}{D} \frac{D \mathbf{v}_i^*}{D t^*} = - \frac{\rho U^2}{D} \frac{\partial p^*}{\partial x_i^*} - \rho g \frac{\partial h^*}{\partial x_i^*} + \frac{U \mu}{D^2} \frac{\partial}{\partial x_j^*} \left(\frac{\partial \mathbf{v}_i^*}{\partial x_j^*} \right)$$

Now we collect all the coefficients to the right-hand side of the equation to form dimensionless groups

$$\frac{D \mathbf{v}_i^*}{D t^*} = - \frac{\partial p^*}{\partial x_i^*} - \frac{g D}{U^2} \frac{\partial h^*}{\partial x_i^*} + \frac{\mu}{D U \rho} \frac{\partial}{\partial x_j^*} \left(\frac{\partial \mathbf{v}_i^*}{\partial x_j^*} \right)$$

We define

$$\frac{U^2}{g D} \equiv \text{Froude number (Fr)}$$

$$\frac{\rho U D}{\mu} \equiv \text{Reynolds number (Re)}$$

Thus the Navier-Stokes can be written as:

$$\frac{D \mathbf{v}_i^*}{D t^*} = - \frac{\partial p^*}{\partial x_i^*} - \frac{1}{Fr} \frac{\partial h^*}{\partial x_i^*} + \frac{1}{Re} \frac{\partial}{\partial x_j^*} \left(\frac{\partial \mathbf{v}_i^*}{\partial x_j^*} \right)$$

Similarly the continuity for an incompressible fluid can be put in a dimensionless form as follows:

$$\frac{\partial v_i^*}{\partial x_i^*} = 0$$

These equations tell us that the solution depends on the two dimensionless groups, Fr and Re , and of course a particular set of boundary conditions which depend on the problem. Thus, one may tabulate the solution of these equations as a function of Fr and Re in a generalised form. It is noted, however, that for flows without any free surfaces the role of gravity is only to increase the pressure. In such cases, if one introduces a modified pressure ($P_{mod} = p + \rho gh$), then he can eliminate the term which involves the Froude number. Therefore, in flows through closed conduits the solution depends only on the Reynolds number.

For flows with a free surface, the surface tension, σ , may be important particularly if the free surface is curved. When the fluid whose flow is being analysed is a liquid and the fluid on the other side is a gas, then we can assume:

$$\eta_{liq} \ll \eta_{gas}$$

$$\rho_{liq} \ll \rho_{gas}$$

The proper boundary conditions for the interface are:

$$\sigma_t = 0$$

and

$$\sigma_n \equiv \tau_n - p = -p_\alpha + \sigma \left(\frac{1}{R_1} + \frac{1}{R_2} \right)$$

where σ_t , and σ_n are the tangential and normal stresses respectively, σ is the surface tension of the liquid, p_a is the pressure in the gas phase and R_1 , and R_2 are the radii of curvature. If this boundary condition is made dimensionless, a new dimensionless group appears as a coefficient, that is,

$$\text{Weber number } (We) \equiv \frac{\rho U^2 D}{\sigma}$$

The dimensionless numbers can be interpreted in terms of ratios of the various forces involved in fluid flow. Thus,

$$Re = \frac{\text{Inertia force}}{\text{Viscous force}}$$

$$Fr = \frac{\text{Inertia force}}{\text{Gravity force}}$$

$$We = \frac{\text{Inertia force}}{\text{Surface force}}$$

The term "inertia" force is to be understood as a measure of the magnitude of the rate of change of momentum (mass x acceleration). It is not actually a force.

Dimensional Analysis

As discussed before the purpose of dimensional analysis is to reduce the number of variables and group them in dimensionless form. This significantly reduces the number of experiments required to complete an experimental study as well as helps in establishing empirical models to describe experimental results.

Example: Suppose that the force required holding a particular body immersed in a free stream of fluid is known to be depended on:

$$F = f(L, V, \rho, \mu)$$

Analytical solution is not possible to obtain, thus we must find F experimentally. In general it takes about 10 points to well define a curve.

To address the effect of L, we need to perform experiments with 10 different values of L. For each L, we need 10 fluids with different density while constant viscosity, 10 fluids with different viscosity while same density and 10 different values of fluid velocity, which means 10^4 experiments. At \$5/experiment and 1/2 hr each of them, one may understand the money and time required. Another problem is finding liquids with different density, although same viscosity and vice versa. However, using dimensional analysis,

$$\frac{F}{\rho V^2 L^2} = f\left(\frac{\rho VL}{\mu}\right)$$

or

$$c_F = f(Re)$$

where c_F is the force coefficient (in a slightly different form is called the drag coefficient). The problem has now been reduced to studying the effect of the second dimensionless group, $\rho VL/\mu$, on the first dimensionless group $F/\rho V^2 L^2$. Thus, performing 10 **only** experiments, we can establish the form of the function f.

Buckingham PI Theorem

Given a physical problem in which the dependent parameter is a function of $n-1$ independent parameters, we may express this as follows:

$$q_1 = f(q_2, q_3, \dots, q_n)$$

In the previous example the dependent parameter is F , thus q_1 , while the independent parameters are D, V, ρ and μ , with $n=5$ in this case.

We can define now a new functional:

$$q_1 - f(q_2, q_3, \dots, q_n) = g(q_1, q_2, \dots, q_n) = 0$$

Therefore we have a functional of n parameters (including dependent and independent parameters). Then the n parameters may be grouped into $n-m$ independent dimensionless groups or ratios, where m is equal to the minimum number of independent variables required to specify the dimensions of all parameters q_1, q_2, \dots, q_n . In other words

$$\Pi_1 = G(\Pi_2, \Pi_3, \dots, \Pi_{n-m})$$

Note: A Π parameter is not independent if it can be formed from a product or quotient (combination) of the other parameters of the problem. For example,

$$\Pi_5 = \frac{2 \Pi_1}{\Pi_2 \Pi_3} \quad \text{or} \quad \Pi_6 = \frac{\Pi_1^{4/5}}{\Pi_4^3}$$

Π_5 , and Π_6 are not independent dimensionless groups.

Determining the Π groups

The drag force on a smooth sphere depends on the relative velocity between the fluid and the sphere, V , the sphere diameter, D , the fluid density, ρ , and the fluid viscosity, μ . Obtain a set of dimensionless groups that can be used to correlate experimental data.

GIVEN:

$$F = f(\rho, V, D, \mu)$$

One dependent variable: F

Four independent variables: ρ, μ, V, D

S1: List all parameters involved

$$F \quad \rho \quad \mu \quad V \quad D$$

S2: Select a system of fundamental dimensions

M, L, t (Mass, Length, time)

S3: List the dimensions of all parameters

$$F \rightarrow \frac{ML}{t^2}, \quad V \rightarrow \frac{L}{t}, \quad D \rightarrow L, \quad \rho \rightarrow \frac{M}{L^3}, \quad \mu \rightarrow \frac{M}{Lt}$$

From this we find that $r=3$ (three primary dimensions)

S4: Select from the list of parameters a number of repeating parameters which is equal to the number of primary dimensions, $r=m$. Since we have 5 parameters the selection is not uniquely defined. However, certain rules apply to the procedure of selection.

(i) Do not select the dependent variable

(ii) Do not select two variables, which can result from the same fundamental dimensions. For example, if we have L and L^3 , select only one.

In this case we select ρ , V , and D .

S5: Set up dimensionless groups. We have to set up $n-m=5-3=2$ groups

$$\rho^a V^b D^c F = M^0 L^0 t^0$$

or

$$\left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b L^c \frac{M L}{t^2} = M^0 L^0 t^0$$

$$M: \quad a + 1 = 0$$

$$L: \quad -3a + b + c + 1 = 0$$

$$t: \quad -b - 2 = 0$$

Solving this system of equations we may get: $a=-1$, $b=-2$, $c=-2$. Thus the first dimensionless group is:

$$\Pi_1 = \rho^{-1} V^{-2} D^{-2} F = \frac{F}{\rho V^2 D^2}$$

Similarly setting:

$$\rho^a V^b D^c \mu = M^0 L^0 t^0$$

we can get:

$$\Pi_2 = \frac{\mu}{\rho V D}$$

Thus we end up with:

$$\frac{F}{\rho V^2 L^2} = f \left(\frac{\rho VL}{\mu} \right)$$

S6: Check to see that each group obtained is dimensionless.

Modelling

As discussed before, dimensional analysis is a valuable tool in modelling experimental data. In other words is a tool valuable in establishing empirical correlations for experimental findings. From dimensional analysis we obtain

$$\Pi_1 = f(\Pi_2, \Pi_3, \dots, \Pi_{n-m})$$

With sufficient testing, the model data will reveal the desired dimensionless function between variables. This ensures complete similarity between model and prototype.

Formal Statement: Flow conditions for a model test are completely similar if all relevant dimensionless parameters have the same corresponding values for model and prototype provided also that the boundary conditions are the same in both model and prototype.

However, complete similarity is very difficult to attain. Engineering literature speaks of particular types of similarity.

1. **Geometric Similarity:** A model and prototype are geometrically similar if and only if all body dimensions in all three co-ordinates have the same linear-scale ratio. This is tantamount to say that the initial and boundary conditions are the same in both model and prototype.
2. **Kinematic Similarity:** The motions of two systems are kinematically similar if homologous particles lie at homologous points at homologous times.
3. **Dynamic Similarity:** It exists when model and prototype have the same length-scale ratio, time-

scale ratio, and force-scale ratio. Geometric similarity is a first requirement; otherwise proceed no further. Then dynamic similarity exists, simultaneously with kinematic similarity, if model and prototype force and pressure coefficients are identical. For example for **incompressible flow**:

a. With no free surface: Reynolds number equal

b. With a free surface: If all dimensionless groups describing the problem are equal, Re , Fr , and We .

In both the above cases geometric similarity is presumed.

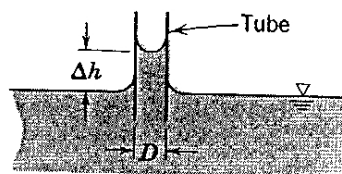
EXAMPLE: CAPILLARY RISE-USE OF DIMENSIONAL MATRIX

When a small tube is dipped into a pool of liquid, surface tension causes a *meniscus* to form at the free surface, which is elevated or depressed depending on the contact angle at the liquid-solid-gas interface. Experiments indicate that the magnitude of this *capillary effect*, Δh , is a function of the tube diameter, D , liquid specific weight, γ , and surface tension, σ . Determine the number of independent Π parameters that can be formed and obtain a set.

EXAMPLE PROBLEM 7.3

GIVEN: $\Delta h = f(D, \gamma, \sigma)$

FIND: (a) Number of independent Π parameters.
(b) Evaluate one set.



Liquid
(Specific weight = γ
Surface tension = σ)

SOLUTION:

(Circled numbers refer to steps in the procedure for determining dimensionless Π parameters.)

- ① Δh D γ σ $n = 4$ parameters
 ② Choose primary dimensions (use both M, L, t and F, L, t dimensions to illustrate the problem in determining m)

③ (a) M, L, t

$$\begin{array}{cccc} \Delta h & D & \gamma & \sigma \\ L & L & \frac{M}{L^2 t^2} & \frac{M}{t^2} \end{array}$$

$r = 3$ primary dimensions

(b) F, L, t

$$\begin{array}{cccc} \Delta h & D & \gamma & \sigma \\ L & L & \frac{F}{L^3} & \frac{F}{L} \end{array}$$

$r = 2$ primary dimensions

Thus we ask, "Is m equal to r ?" Let us check the dimensional matrix to find out. The dimensional matrix is

	Δh	D	γ	σ
M	0	0	1	1
L	1	1	-2	0
t	0	0	-2	-2

	Δh	D	γ	σ
F	0	0	1	1
L	1	1	-3	-1

The rank of a matrix is equal to the order of its largest nonzero determinant.

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & -2 & 0 \\ 0 & -2 & -2 \end{vmatrix} = 0 - (1)(-2) + (1)(-2) = 0$$

$$\begin{vmatrix} -2 & 0 \\ -2 & -2 \end{vmatrix} = 4 \neq 0 \quad \therefore m = 2 \\ m \neq r$$

- ④ $m = 2$. Choose D, γ as repeating parameters.
 ⑤ $n - m = 2$ dimensionless groups will result.

$$\Pi_1 = D^a \gamma^b \Delta h \\ = (L)^a \left(\frac{M}{L^2 t^2} \right)^b (L) = M^0 L^0 t^0$$

$$\left. \begin{array}{l} M: b + 0 = 0 \\ L: a - 2b + 1 = 0 \\ t: -2b + 0 = 0 \end{array} \right\} \begin{array}{l} b = 0 \\ a = -1 \end{array}$$

$$\text{Therefore, } \Pi_1 = \frac{\Delta h}{D}$$

$$\Pi_2 = D^c \gamma^d \sigma \\ = (L)^c \left(\frac{M}{L^2 t^2} \right)^d \frac{M}{t^2} = M^0 L^0 t^0$$

$$\left. \begin{array}{l} M: d + 1 = 0 \\ L: c - 2d = 0 \\ t: -2d - 2 = 0 \end{array} \right\} \begin{array}{l} d = -1 \\ c = -2 \end{array}$$

$$\text{Therefore, } \Pi_2 = \frac{\sigma}{D^2 \gamma}$$

- ⑥ Check, using F, L, t dimensions

$$\Pi_1 = \frac{\Delta h}{D} : \frac{L}{L} = [1]$$

$$\Pi_2 = \frac{\sigma}{D^2 \gamma} : \frac{F}{L} \frac{1}{L^2} \frac{L^3}{F} = [1]$$

$$\begin{vmatrix} 1 & 1 \\ -3 & -1 \end{vmatrix} = -1 + 3 = 2 \neq 0$$

$$\therefore m = 2 \\ m = r$$

- $m = 2$. Choose D, γ as repeating parameters.
 $n - m = 2$ dimensionless groups will result.

$$\Pi_1 = D^e \gamma^f \Delta h \\ = (L)^e \left(\frac{F}{L^3} \right)^f L = F^0 L^0 t^0$$

$$\left. \begin{array}{l} F: f = 0 \\ L: e - 3f + 1 = 0 \end{array} \right\} e = -1$$

$$\text{Therefore, } \Pi_1 = \frac{\Delta h}{D}$$

$$\Pi_2 = D^g \gamma^h \sigma \\ = (L)^g \left(\frac{F}{L^3} \right)^h \frac{F}{L} = F^0 L^0 t^0$$

$$\left. \begin{array}{l} F: h + 1 = 0 \\ L: g - 3h - 1 = 0 \end{array} \right\} \begin{array}{l} h = -1 \\ g = -2 \end{array}$$

$$\text{Therefore, } \Pi_2 = \frac{\sigma}{D^2 \gamma}$$

- Check, using M, L, t dimensions

$$\Pi_1 = \frac{\Delta h}{D} : \frac{L}{L} = [1]$$

$$\Pi_2 = \frac{\sigma}{D^2 \gamma} : \frac{M}{t^2} \frac{1}{L^2} \frac{L^2 t^2}{M} = [1]$$

Therefore, both systems of dimensions yield the same dimensionless Π parameters. The predicted functional relationship is

$$\Pi_1 = f(\Pi_2) \quad \text{or} \quad \frac{\Delta h}{D} = f\left(\frac{\sigma}{D^2 \gamma}\right)$$

This result is reasonable on physical grounds. The fluid is static; one would not expect time to be an important dimension.

The purpose of this problem is to illustrate use of the dimensional matrix to determine the required number of repeating parameters.

■ DIMENSIONLESS GROUPS OF SIGNIFICANCE IN FLUID MECHANICS

Over the years, several hundred different dimensionless groups that are important in engineering have been identified. Following tradition, each such group has been given the name of a prominent scientist or engineer, usually the one who pioneered its use. Several are so fundamental and occur so frequently in fluid mechanics that we should take time to learn their definitions. Understanding their physical significance also gives insight into the phenomena we study.

Forces encountered in flowing fluids include those due to inertia, viscosity, pressure, gravity, surface tension, and compressibility. The ratio of any two forces will be dimensionless. We have previously shown that the inertia force is proportional to $\rho V^2 L^2$. To facilitate forming ratios of forces, we can express each of the remaining forces as follows:

$$\text{Viscous force} = \tau A \propto \mu \frac{du}{dy} A \propto \mu \frac{V}{L} L^2 \propto \mu V L$$

$$\text{Pressure force} = (\Delta p) A \propto (\Delta p) L^2$$

$$\text{Gravity force} = m g \propto g \rho L^3$$

$$\text{Surface tension force} = \sigma L$$

$$\text{Compressibility force} = E_v A \propto E_v L^2$$

Inertia forces are important in most fluid mechanics problems. The ratio of the inertia force to each of the other forces listed above leads to five fundamental dimensionless groups encountered in fluid mechanics.

In the 1880s, Osborne Reynolds, the British engineer, studied the transition between laminar and turbulent flow regimes in a tube. He discovered that the parameter (later named after him)

$$Re = \frac{\rho \bar{V} D}{\mu} = \frac{\bar{V} D}{\nu}$$

is a criterion by which the flow regime may be determined. Later experiments have shown that the *Reynolds number* is a key parameter for other flow cases as well. Thus, in general,

$$Re = \frac{\rho V L}{\mu} = \frac{V L}{\nu}$$

where L is a characteristic length descriptive of the flow field geometry. The Reynolds number is the ratio of inertia forces to viscous forces. "Large" Reynolds number flows generally are turbulent. Flows in which the inertia forces are "small" compared to viscous forces are characteristically laminar flows.

In aerodynamic and other model testing, it is convenient to present pressure data in dimensionless form. The ratio

$$Eu = \frac{\Delta p}{\frac{1}{2} \rho V^2}$$

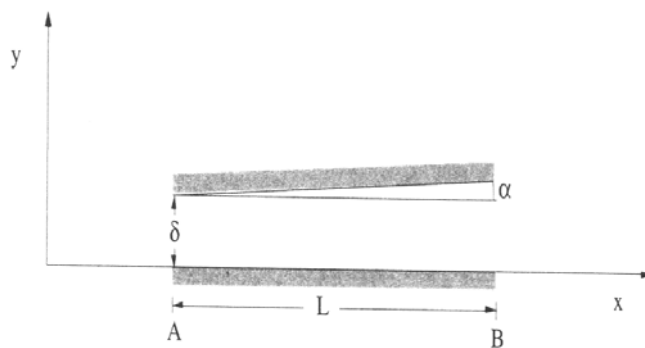
FLOWS WITH NEGLIGIBLE ACCELERATION

The non-linear terms in the Navier-Stokes equations result from the acceleration of the fluid. These terms contribute the most difficulty in the solution of the equation. For fully-developed incompressible flows in conduits of constant cross section these non-linear terms disappear and as a result the equations are easily solved.

There are some cases where the acceleration is not identical to zero, but still the inertia terms may be neglected without presenting a serious error. The question is what are these types of flows (flows with negligible acceleration) and how we identify them. There are at least two such families of flows: flows in **narrow gaps** and **creeping** flows.

Flow in Narrow Gaps

Consider the two dimensional flow of an incompressible fluid in a narrow gap between the two plates shown in the Figure. For the sake of simplicity and without loss of



generality the plates are assumed to be flat and the gap width to be slightly diverging to the direction of flow. The lower plate is set stationary and the upper plate may be inclined to it with the small angle α . We make the following assumptions to assure the small degree of convergence.

$$\frac{\delta}{L} \ll 1, \quad \alpha < \frac{\delta}{L}$$

The mean velocity in the x -direction is defined as:

$$\bar{v}_x = \frac{1}{\delta} \int_0^{\delta} v_x dy$$

Conservation of mass requires,

$$\bar{v}_{x,A} \delta = \bar{v}_{x,B} (\delta + L \alpha)$$

Thus,

$$\left| \frac{\partial v_x}{\partial x} \right| \approx \left| \frac{\partial \bar{v}_x}{\partial x} \right| \approx \frac{\bar{v}_{x,B} - \bar{v}_{x,A}}{L} = \bar{v}_{x,B} \frac{\alpha}{\delta}$$

From the equation of continuity

$$\frac{\partial v_x}{\partial x} = - \frac{\partial v_y}{\partial y}$$

Therefore, combining

$$\left| \frac{\partial v_y}{\partial y} \right| \approx \left| \frac{\bar{v}_{x,B} \alpha}{\delta} \right|$$

The boundary conditions for v_y are

$$v_y = 0 \quad \text{at } y = 0 \quad \text{and} \quad \text{at } y = \delta$$

The largest value v_y can attain is in the vicinity of the middle of the gap, and this could be approximately

$$\frac{\partial v_y}{\partial y} \approx \frac{v_y - 0}{\delta/2} \quad \text{or} \quad |v_y| \approx \frac{\delta}{2} \left| \frac{\partial v_y}{\partial y} \right| \approx \left| \frac{\bar{v}_{x,B} \alpha}{2} \right| \ll |\bar{v}_{x,B}|$$

Therefore the y-component of the velocity may be neglected in comparison with the x-component.

Furthermore in the B-cross section

$$\left| \frac{\partial v_x}{\partial y} \right|_{x=B} \approx \frac{\bar{v}_{x,B}}{\delta}$$

Therefore,

$$\frac{\left| \frac{\partial v_x / \partial x}{\partial v_x / \partial y} \right|}{\left| \frac{\partial v_x / \partial x}{\partial v_x / \partial y} \right|} \approx \alpha < \frac{\delta}{L} \ll 1$$

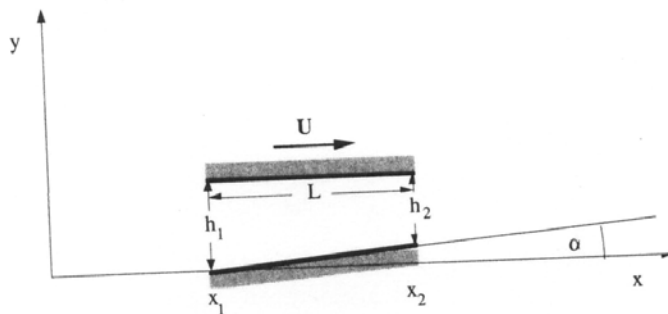
The x-derivative of v_x is negligible compared to its y-derivative. Using these two simplifications, the **approximate** form of the Navier-Stokes equations for two dimensional gap flows becomes,

$$0 = -\frac{dp}{dx} + \mu \frac{d^2 v_x}{d y^2}$$

This can be easily integrated to result the fully developed velocity profile illustrated previously. However, the pressure drop is not a constant quantity any longer. There is a way to determine how pressure changes with the axial length. This will be illustrated in the next section.

Reynolds Lubrication Theory

An important application of flows in narrow gaps (flows with negligible acceleration) is in Reynolds lubrication theory. This theory yields the forces which appear in bearings and other lubricating sliding surfaces provided that acceleration forces may be neglected.



The x component of the momentum for the flow in the Figure besides is

$$0 = -\frac{dp}{dx} + \mu \frac{d^2 v_x}{d y^2}$$

Once this solved for fully developed flow yields

$$v_x = \frac{1}{2\mu} \left(\frac{dp}{dx} \right) y[y - h(x)] + U \frac{y}{h(x)}$$

Assuming a wide bearing, then the mass flow between the plates is conserved

$$\dot{m} = \int_0^{h(x)} \rho v_x dy = \frac{1}{2} \rho U h(x) - \frac{\rho}{12\mu} \left(\frac{dp}{dx} \right) h^3(x) = \text{const} = \frac{1}{2} \rho U h_0$$

where, h_0 is a parameter defining the mass flow rate to be determined later. Solving for the pressure drop

$$\frac{dp}{dx} = 6\mu U \frac{h - h_0}{h^3}$$

The lower plate is inclined to the horizontal by the angle

$$\alpha \cong \frac{dh}{dx}$$

Thus, $\frac{dp}{dx} = \frac{dp}{dh} \cdot \frac{dh}{dx} = \alpha \frac{dp}{dh}$. Substitute into the pressure drop equation yields

$$\frac{dp}{dh} = 6\mu \frac{U}{\alpha} \frac{h - h_0}{h^3}$$

Integration and satisfaction of the boundary conditions that $p=p_0$ at h_1 and h_2 yields the values of the constant of integration and h_0 .

$$p - p_0 = 6\mu \frac{U}{\alpha} \frac{(h - h_1)(h - h_2)}{(h_1 + h_2)h^2}$$

The mass flow rate can now be calculated from

$$\dot{m} = \frac{1}{2} \rho U h_0 = \frac{1}{2} \rho U \frac{2h_1 h_2}{h_1 + h_2}$$

The lift force \mathbf{L} per unit width (force of separation) acting on the upper plate is

$$L = \int_{x_1}^{x_2} (p - p_0) dx = \frac{6\mu U L^2}{(k-1)^2 h_2^2} \left[\ln k - \frac{2(k-1)}{k+1} \right]$$

where $k = h_1 / h_2 > 1$ and $L = x_2 - x_1 = \frac{1}{\alpha} (h_2 - h_1)$

The drag force **D** per unit width acting on the upper plate is:

$$D = \int_{x_1}^{x_2} \mu \left(\frac{dv_x}{dy} \right)_{y=h} dx = \frac{2\mu UL}{(k-1)h_2} \left[2 \ln k - \frac{3(k-1)}{k+1} \right]$$

To maximize lift (applications in floating magnetic readers) let $dL/dk=0$, that results $k=2.2$

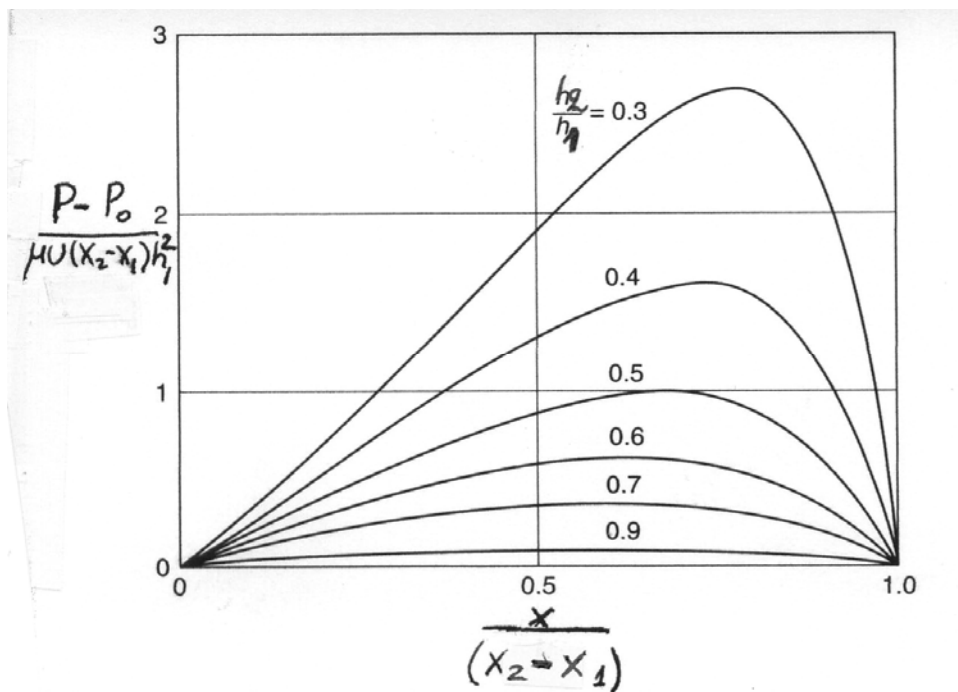
Which corresponds to:

$$L=0.4\mu UR \quad D=1.2\mu UR \quad D/L=3/R$$

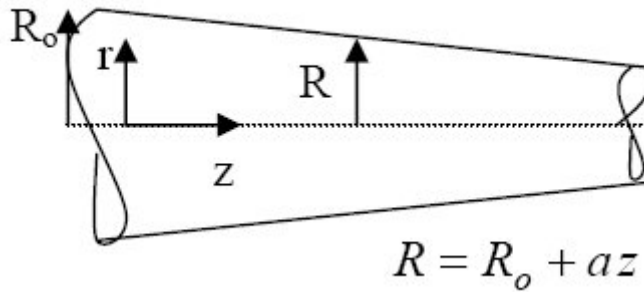
with

$$R=2L/(h_1+h_2)$$

The pressure in between the edges of the bearing might reach very high values this prevents the two surfaces from touching. The figure below plots some pressure profiles. It can be seen that the maximum pressure depends heavily on the degree of contraction.



Flow in a Slightly Tapered Tube



The taper of the tube will require a flow in the radial direction and an acceleration in the axial direction. Assume that flow maintains axial symmetry and that $v_z = v_z(r, z)$, $v_r = v_r(r, z)$, $v_\theta = 0$.

The Reynolds number for this flow is defined as:

$$Re = \alpha \left(\frac{\rho R_0 V}{\mu} \right)$$

Where V is the average velocity at a given cross section where R is given by $R = R_0 + az$.

The equations of continuity and those of motion in the r and z direction can be simplified as follows:

Continuity

$$\frac{1}{r} \frac{\partial}{\partial r} (r v_r) + \frac{\partial v_z}{\partial z} = 0$$

r-component

$$\rho \left(v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z} \right) = - \frac{\partial p}{\partial r} + \mu \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial (r v_r)}{\partial r} \right) + \frac{\partial^2 v_r}{\partial z^2} \right]$$

z-component

$$\rho \left(v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} \right) = - \frac{\partial p}{\partial z} + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) + \frac{\partial^2 v_z}{\partial z^2} \right]$$

This set of equations is difficult to be solved. However, we can perform an **ordering analysis** to simplify it. We estimate the order of magnitude of the various velocities and derivatives. (Define an average velocity $V \equiv Q/\pi R_L^2$).

$$v_z \approx O(V) \approx O(Q/\pi R_L^2)$$

We can calculate the order of v_r from continuity

$$\frac{\partial v_z}{\partial z} \approx O\left[\left(\frac{Q}{\pi R_L^2} - \frac{Q}{\pi R_o^2}\right)/L\right] \approx O\left[V\left(1 - \left(\frac{R_L}{R_o}\right)^2\right)/L\right] \text{ expected to be small}$$

because $(R_o - R_L)/L \ll 1$. If U now denotes the order of v_r then,

$$\frac{1}{r}(r v_r) \approx O\left(\frac{U}{R_L}\right)$$

Finally, from continuity one may estimate U in terms of V , that is

$$U \equiv V\left(\frac{R_L}{L}\right)\left[1 - \left(\frac{R_L}{R_o}\right)^2\right]$$

Similarly all the terms in the Navier-Stokes equations can be analyzed in this way to find their order of magnitude. Using the approximations,

$$v_r \text{ is small}$$

$$\frac{\partial v_z}{\partial z} \ll \frac{\partial v_z}{\partial r}$$

$$\frac{\partial v_r}{\partial r} \text{ is also very small}$$

It can be shown that for slightly tapered tubes the equations can be reduced to only one equation in the z-direction, that is:

$$0 = -\frac{\partial p}{\partial z} + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) \right]$$

This was solved previously to result the velocity profile. Integrating the velocity profile to calculate the volume flow rate we can get

$$Q = \frac{\pi R^4}{8 \mu} \left(-\frac{dp}{dz} \right)$$

Note that $R=R(z)$ and during integration to obtain this equation, we hold z constant. The process of adapting locally the results for a uniform geometry to a slowly varying geometry is known as **lubrication approximation**.

Now express the change in R as,

$$R = R_o + (R_L - R_o) (z/L)$$

Also note that,

$$\frac{dp}{dz} = \frac{dp}{dR} \frac{dR}{dz} = \frac{dp}{dR} \frac{R_L - R_o}{L}$$

Thus,

$$Q = \frac{\pi R^4}{8 \mu} \left(- \frac{dp}{dR} \right) \left(\frac{R_L - R_o}{L} \right)$$

Note that Q is constant for all z (and hence all R). Solving this ODE for the pressure distribution as a function of R_L , we get:

$$\begin{aligned} Q &= \frac{3 \pi}{8 \mu} \frac{p_o - p_L}{L} \frac{R_o - R_L}{R_L^3 - R_o^3} = \\ &= \frac{\pi (p_o - p_L) R_o^4}{8 \mu L} \left[1 - \frac{1 + (R_L/R_o) + (R_L/R_o)^2 - 3(R_L/R_o)^3}{1 + (R_L/R_o) + (R_L/R_o)^2} \right] \end{aligned}$$

Note that the final result may be expressed as the Haagen-Poiseuille result multiplied by a correction factor.

For more details on this particular problem see: Bird, R.C. Armstrong and O. Hassager, Dynamics of Polymeric Liquids, "Vol. 1, "Fluid Mechanics", Wiley, New York, 1987.

Creeping Flows

The Navier-Stokes equations for steady flows not involving free surfaces can be written in dimensionless form as follows:

$$Re \frac{D\mathbf{v}_i^*}{Dt^*} = -\frac{\partial P^*}{\partial x_i^*} + \frac{\partial}{\partial x_j^*} \left(\frac{\partial \mathbf{v}_i^*}{\partial x_j^*} \right)$$

where P^* stands for the modified pressure which also contains the effect of gravity. For very small values of the Reynolds number $Re/\rho VD/\mu \ll 1$ this equation simplifies to:

$$\frac{\partial \mathbf{v}_i^*}{\partial t^*} = -\frac{\partial P^*}{\partial x_i^*} + \frac{\partial}{\partial x_j^*} \left(\frac{\partial \mathbf{v}_i^*}{\partial x_j^*} \right)$$

or in vector notation (dimensional)

$$\frac{\partial \mathbf{V}}{\partial t} = -\nabla P + \mu \nabla^2 \mathbf{V}$$

The flows described by these equations are called Stokes creeping flows. The equations are linear and possess some properties, which are useful in their solution.

Taking the divergence of this equation,

$$\nabla \cdot \left(\frac{\partial \mathbf{V}}{\partial t} \right) = \nabla \cdot (\nabla^2 P - \mu \nabla (\nabla^2 \mathbf{V}))$$

or it may be rewritten as:

$$\frac{\partial}{\partial t} (\nabla \cdot \mathbf{V}) = \nabla^2 P + \mu \nabla^2 (\nabla \cdot \mathbf{V})$$

From continuity $\nabla \cdot \mathbf{V} = 0$ for incompressible flow. Therefore,

$$\nabla^2 P = 0$$

Thus the pressure in creeping flows is a harmonic function, i.e., it satisfies the Laplace equation. In addition, it can be shown that (how?):

$$\nabla^2 (\nabla \times \mathbf{V}) = 0$$

This is a more useful form since most of the time the boundary conditions are specified in terms of velocities. The solutions to Stokes' equations possess the following interesting properties:

1. For start-up flows, the velocity distribution reaches steady state instantly.
2. All flows are "kinematically reversible". This means that if the velocities in the boundary conditions are suddenly reversed in sign, all fluid particles will flow back along the same streamline they were following before the reversal. In other words, the streamlines are the same for forward and backward flow.

Creeping Flow past a Sphere

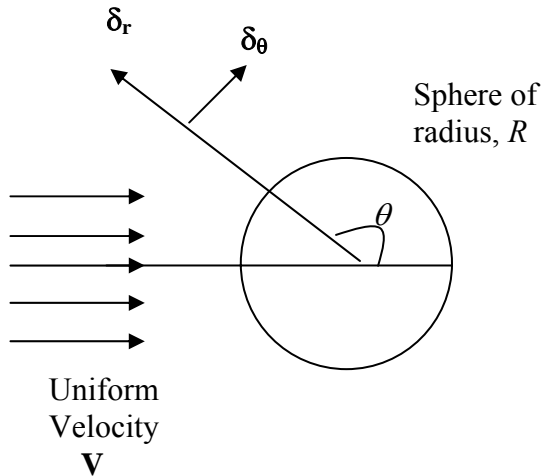
The low Reynolds number flow around a sphere is an important problem in classical fluid mechanics. The sphere has a radius of R and a Newtonian incompressible fluid with a uniform velocity V flows around the sphere. The fluid has a density ρ and a viscosity μ .

- a. Find the velocity field for the flow around the sphere.
- b. Obtain an expression for the drag force

The boundary conditions far from the sphere can be expressed as:

$$v_r \rightarrow -V \cos \theta, \quad \text{for } r \rightarrow \infty$$

$$v_\theta \rightarrow V \sin \theta, \quad \text{for } r \rightarrow \infty$$



At the surface of the sphere the conditions are:

$$v_r = 0, \quad \text{for } r = R$$

$$v_\theta = 0, \quad \text{for } r = R$$

This problem can be solved with the use of the streamfunction in spherical co-ordinates for axisymmetrical flow with no ϕ -dependence. The velocity components can be expressed as:

$$v_r = -\frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta}$$

$$v_\theta = \frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r}$$

The boundary conditions far from the sphere become

$$\psi \rightarrow \frac{1}{2} V r^2 \sin^2 \theta, \quad \text{for } r \rightarrow \infty$$

In view of this condition, we assume for ψ

$$\psi = f(r) \sin^2 \theta$$

The equation for ψ becomes

$$\nabla^4 \psi = 0$$

This can be solved (see Bird et al., Dynamics of Polymeric Liquids) to yield.

$$\psi = -V R^2 \left[\frac{1}{2} \left(\frac{r}{R} \right)^2 - \frac{3}{4} \left(\frac{r}{R} \right) + \frac{1}{4} \left(\frac{R}{r} \right) \right] \sin^2 \theta$$

$$v_r = V \left[1 - \frac{3}{2} \left(\frac{R}{r} \right) + \frac{1}{2} \left(\frac{R}{r} \right)^3 \right] \cos \theta$$

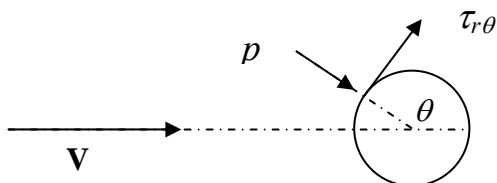
$$v_\theta = -V \left[1 - \frac{3}{4} \left(\frac{R}{r} \right) - \frac{1}{4} \left(\frac{R}{r} \right)^3 \right] \sin \theta$$

To find the drag force on the sphere we need also the pressure distribution. This can be obtained by substituting the velocity components in both the r and θ components of the equation of motion.

Solving these two equations will result (neglect the effect of gravity):

$$P = \frac{3}{2} \mu V \frac{R}{r^2} \cos \theta - P_o$$

Thus the total drag force can be calculated (Force exerted by the fluid on the sphere in the horizontal direction or direction of flow)



$$- p \cos \theta - \tau_{r\theta} \sin \theta$$

$$\tau_{r\theta} = \mu \left(\frac{1}{r} \frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} \right) = \frac{\mu V \sin \theta}{r} \left(\frac{3R^3}{2r^3} \right)$$

Thus

$$F = \int_0^\pi \tau_{r\theta}|_{r=R} \sin \theta \, dA + \int_0^\pi p|_{r=R} \cos \theta \, dA$$

where $dA = 2\pi r^2 \sin \theta d\theta$. The final result is:

$$F = 4\pi \mu V R + 2\pi \mu V R = 6\pi \mu V R$$

which is Stokes law.

Drag Coefficient: This coefficient is defined as:

$$C_D = \frac{F}{\frac{1}{2} \rho V^2 (PR.AR.)} = \frac{F}{\frac{1}{2} \rho V^2 \pi R^2}$$

where *PR.AR.* stands for projected area which for a sphere becomes the area of a circle, that is πR^2 .

Thus,

$$C_D = \frac{24}{Re}, \quad \text{where } Re \equiv \frac{DV\rho}{\mu}$$

Skin Friction Coefficient: This is defined as:

$$C_f = \frac{\text{skin friction force (shear force)}}{\frac{1}{2} \rho V^2 \pi R^2} = \frac{16}{Re}$$

Thus we have contributions from skin friction (shear force) and pressure or form drag force.

A Sphere Moving in a Stationary Fluid

For a sphere moving in a stationary fluid to the right, we can use superposition to determine the stream function, since the governing equations are linear.

- First we change the direction of the flow by introducing a minus sign
- We superpose a uniform flow to the right, which "stops the fluid and moves the sphere". The Stokes streamfunction for uniform flow to the right is:

$$\psi = V r^2 \sin^2 \theta / 2$$

and the new streamfunction is thus:

$$\psi = V r^2 \sin^2 \theta \left(\frac{3R}{4r} - \frac{R^3}{4r^3} \right)$$

The total drag force on the sphere is again the same as previously:

$$F_D = 6 \mu V \pi R$$

This result agrees very well with experimental data for $Re < 0.1$, and the error is only a few percent up to $Re=1$. This is very surprising, because Stokes equation is only valid when the Reynolds number is much less than 1. However, drag depends on the flow near the body, and this is where the ratio of inertia to viscous forces is smallest. Far from the sphere the assumption that $Re \ll 1$ becomes locally incorrect and the predicted velocity distribution becomes increasingly inaccurate as r increases.

OSEEN'S SOLUTION

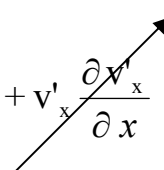
Oseen (1910) proposed a method for linearizing the inertia terms of the N-S equation for the case where the acceleration is not neglected entirely but is still assumed to be quite small. He proposed to write v_x as the main flow velocity plus a perturbation.

$$v_x = V + v'_x$$

Thus the non-linear term becomes

$$v_x \frac{\partial v_x}{\partial x} = (V + v'_x) \frac{\partial (V + v'_x)}{\partial x} = V \frac{\partial v'_x}{\partial x} + v'_x \frac{\partial v'_x}{\partial x}$$

neglect



Oseen proposed to neglect the last term. In this way we end up with a linearized approximation valid when the perturbation is small. When the drag coefficient, C_D is calculated using Oseen's solution, the result is:

$$C_D = \frac{24}{Re} \left(1 + \frac{3}{16} Re \right)$$

This is a good approximation up to Re numbers of about 2.

References

1. J.M. Dealy "Foundations of Fluid mechanics", Course notes, McGill University, 1994.
2. D. Pnueli, and C. Gutfinger, Fluid Mechanics, Cambridge University Press, Cambridge 1992
3. R.B. Bird, W.E. Stewart, and E.N. Lightfoot, Transport Phenomena, John Wiley, 2nd edition, NY 2002.
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HIGH REYNOLDS NUMBER FLOWS

Regions Far from Boundaries

The Navier-Stokes equations in dimensionless form and using the modified pressure are:

$$\frac{D v_i^*}{D t^*} = -\frac{\partial P^*}{\partial x_i^*} + \frac{1}{Re} \frac{\partial}{\partial x_j^*} \left(\frac{\partial v_i^*}{\partial x_j^*} \right)$$

For very high Reynolds number the last term may be omitted and the following equations are recovered in a vectorial dimensional form,

$$\rho \frac{D \mathbf{V}}{D t} = -\nabla P - \rho \mathbf{g} \nabla h$$

or in index notation

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = -g \frac{\partial h}{\partial x_i} - \frac{1}{\rho} \frac{\partial P}{\partial x_i}$$

which are the **Euler's equations**. Flows of constant density fluid, which obey these equations, are called "**ideal fluid flows**".

Internal flows of high Reynolds number are normally turbulent and Euler's equations are not useful for such flows. These equations are mostly useful for external flows, i.e., flows around an immersed body, such as aerofoil, flows around the wing of an aircraft. Also these equations are useful for the calculation of dp/dx as will be explained later.

As one approach the surface of the immersed body, the Reynolds number locally becomes smaller and smaller and there the Euler's equations are not useful. In such regions the viscous terms are becoming important. Thus, there is a layer in fact extremely thin where the viscous effects are important and there the Navier-equations should be solved. This layer known as the boundary layer will be the subject of study in the next chapter. Outside of this layer (free stream flow), viscous

effects are negligible and there one may use the Euler equations to determine the velocity patterns.

Therefore for external flows, one may proceed as follows:

- i. Solve the ideal flow problem as if the boundary layer were not present, i.e., use the Euler equations
2. Take the ideal flow solution for $y=0$, i.e., at the wall as the outer boundary conditions at $y=\delta$ for solution of the flow in the boundary layer, where δ is the thickness of the boundary layer.

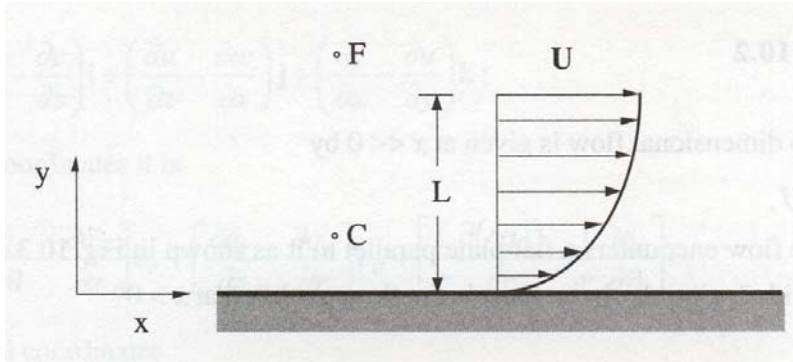


Figure: Point F is far away from boundary (Euler equation valid) whereas point C close to boundary (Euler equation invalid)

Boundary Conditions

To solve the Navier-Stokes equations we used as boundary conditions at a solid boundary the assumptions of no-slip and of impenetrable wall. In other words, the tangential and normal components of the velocity at a solid wall are zero

$$v_t = 0 \quad v_n = 0 \quad \text{at solid boundaries}$$

This is the case because the Navier-Stokes are second order PDE's. However, the Euler's equations are first order and therefore one has to drop one of the two. The no slip boundary condition is dropped, since the zero normal velocity defines the solid boundary. Thus,

$$v_n = 0$$

is to be used with the Euler's equations.

Irrotational Motion

The Euler's equations can also be written as

$$\frac{\partial \mathbf{V}}{\partial t} - \mathbf{V} \times \nabla \times \mathbf{V} = -\nabla \left[\frac{p}{\rho} + \frac{1}{2} \mathbf{V}^2 + \mathbf{g}h \right]$$

Taking the curl of this equation zeros the right hand side, thus $\nabla \times$

$$\frac{\partial}{\partial t} (\nabla \times \mathbf{V}) - \nabla \times (\mathbf{V} \times \nabla \times \mathbf{V}) = 0$$

Define a vorticity vector $\boldsymbol{\zeta} = \nabla \times \mathbf{V}$ and then the above equation becomes

$$\frac{\partial \boldsymbol{\zeta}}{\partial t} - \nabla \times (\mathbf{V} \times \boldsymbol{\zeta}) = 0$$

One way to satisfy this equation is by $\boldsymbol{\zeta} = 0$. Flow for which $\boldsymbol{\zeta} = 0$ are **irrotational flows**. The vorticity tensor equals twice the angular velocity defined previously. Hence, the term irrotational flow, which implies flows with no angular velocity. The Figure below illustrates nicely the difference between rotational and irrotational flows and indeed provides some physical insight into the character of these types of flows.



Figure. Rotational vs. irrotational flow

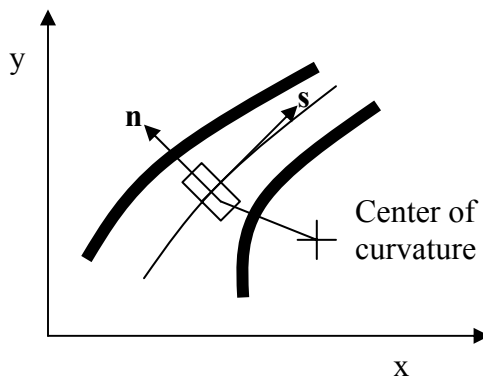
If now we return to the original Euler's equation and assume **irrotational steady flow**, then

$$\nabla \left[\frac{p}{\rho} + \frac{1}{2} \mathbf{V}^2 + \mathbf{g} h \right] = 0$$

which implies

$$\frac{p}{\rho} + \frac{1}{2} \mathbf{V}^2 + g h = \text{const.}$$

This equation is the **"strong" Bernoulli equation** which holds for steady, inviscid and irrotational flows in the whole domain.



It is noted that there is also the **"weak" Bernoulli equation**, which holds along streamlines for ideal fluid flow (no irrotational). This can be derived if the Euler's equation is rewritten for a coordinate, s , that lies along a streamline, so that ds represent an infinitesimal distance along the streamline (see Figure besides).

$$v_j \frac{\partial v_i}{\partial x_j} = v_s \frac{\partial v_s}{\partial s} = \frac{1}{2} \frac{\partial}{\partial s} (v_s^2) = \frac{1}{2} \frac{d}{ds} (v^2)$$

Thus the Euler's equation can be written for a steady flow

$$\frac{1}{2} \frac{dV^2}{ds} = -g \frac{dh}{ds} - \frac{1}{\rho} \frac{dp}{ds}$$

Integrating and rearranging

$$\frac{p}{\rho} + \frac{1}{2} \mathbf{V}^2 + gh = \text{const.}$$

which is the same as the above Bernoulli's equation, but this time was derived for an ideal (inviscid) steady flow and only holds along a streamline. Thus, the constant in this equation may vary from streamline to streamline while the one in the strong Bernoulli equation is a constant for the entire flow field.

In the direction normal to a streamline one may derive the following equation:

$$\frac{1}{\rho} \frac{\partial p}{\partial n} = \frac{\mathbf{V}^2}{R}$$

where R is the radius of curvature of the streamline and $\alpha_n = -\mathbf{V}^2/R$ is the centripetal acceleration. The above equation tells us that there is an increase of pressure in the outwardly normal direction to the streamline.

The Circulation Γ

The circulation is defined as the counterclockwise line integral around any closed contour in the field flow, of the tangential component of the velocity vector

$$\Gamma = \oint \mathbf{v}_T \, ds = \oint \mathbf{V} \cdot d\mathbf{s}$$

From Stoke's theorem

$$\Gamma = \oint \mathbf{V} \cdot d\mathbf{s} = \int_A (\nabla \times \mathbf{V}) \cdot \mathbf{n} \, dA$$

or

$$\Gamma = \int_A (\boldsymbol{\zeta} \cdot \mathbf{n}) \, dA$$

A flow in which the vorticity is everywhere zero is said to be irrotational as discussed before.

Clearly the circulation is also zero for any contour in such a flow.

Kelvin's Theorem (Conservation of Circulation)

Starting from Euler's equations one can show that for ideal fluid flow, for a contour that follows fluid elements circulation is conserved, that is

$$\frac{D\Gamma}{Dt} = 0$$

This equation is valid for any contour, no matter how small. Thus this implies that the vorticity of a fluid element can never change in an ideal fluid flow.

Corrolary: If the upstream flow is irrotational, it must remain irrotational at all downstream points (principle of persistence of irrotationality).

POTENTIAL FLOW (Irrotational Flow)

Velocity Potential

From vector calculus, if ϕ is a scalar, then

$$\text{curl}(\text{grad } \phi) = \nabla \times (\nabla \phi) = 0$$

For irrotational flow the condition is: $\nabla \times \mathbf{V} = \mathbf{0}$. Thus for every irrotational flow there must exist a scalar field, $\phi(x, y, z)$ whose gradient is equal to the velocity vector,

$$\mathbf{V} = \nabla \phi$$

where ϕ is called the "velocity potential". In Cartesian this is:

$$v_i = \frac{\partial \phi}{\partial x_i}$$

and in cylindrical,

$$v_r = \frac{\partial \phi}{\partial r}, \quad v_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta}, \quad v_z = \frac{\partial \phi}{\partial z}$$

Substituting these into the continuity for incompressible flows results in

$$\nabla^2 \phi = 0 \quad \text{or} \quad \frac{\partial}{\partial x_i} \left(\frac{\partial \phi}{\partial x_i} \right) = 0$$

which is the Laplace equation. Thus the velocity potential is a harmonic function and solutions can obtain easily due to the linearity of this equation. A proper set of boundary conditions must guarantee no normal flow relative to rigid surfaces. This Laplace equation is really a direct integration of the equation of continuity. Euler's equation is still used through Bernoulli equation to evaluate the pressure. Thus the momentum and continuity equations become coupled.

Since the Laplace equation is linear, superposition of solutions is permissible, and elaborate flows may be constructed by superposition of simpler ones. It is noted, however, that Bernoulli's equation is not linear. Therefore, the pressure in a flow obtained by superposition of two flows is not a sum of the pressures of the two partial flows.

Two-Dimensional Irrotational Flows:

We now consider two-dimensional irrotational flows. The equation for the definition of the velocity potential ($\mathbf{V}=\nabla\phi$) implies:

$$v_x = \frac{\partial\phi}{\partial x} \quad \text{and} \quad v_y = \frac{\partial\phi}{\partial y}$$

The corresponding equations for the streamfunction are:

$$v_x = \frac{\partial\psi}{\partial y} \quad \text{and} \quad v_y = -\frac{\partial\psi}{\partial x}$$

Using the continuity one may also prove that in two-dimensional irrotational flow the streamfunction is also harmonic. Furthermore,

$$v_x = \frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} \quad \text{and} \quad v_y = \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}$$

or in cylindrical coordinates

$$v_r = \frac{\partial\phi}{\partial r} = \frac{1}{r} \frac{\partial\psi}{\partial\theta} \quad \text{and} \quad v_\theta = \frac{1}{r} \frac{\partial\phi}{\partial\theta} = -\frac{\partial\psi}{\partial r}$$

But these are the Cauchy-Riemann condition that guarantee the existence of an analytic function, $f(z)$, of a complex variable, z , where

$$z = x + iy$$

or in polar coordinates

$$z = r(\cos\theta + i\sin\theta) = r e^{i\theta}$$

Thus for every irrotational flow there exists an analytic function, $f(z)$, that is related to the velocity potential and streamfunction as follows:

$$f(z) = \phi(x, y) + i\psi(x, y)$$

where $f(z)$ is called the "complex potential". Since $f(z)$ is analytic, its derivative exists:

$$w(z) = \frac{df}{dz} = \frac{df}{dx} = \phi_x + i\psi_x = v_x - i v_y$$

This function is called the "complex velocity". In polar coordinates this is

$$w(z) = (v_r - i v_\theta) e^{-i\theta}$$

Given the complex potential, it is a very concise way of describing the potential flow. Since both ϕ and ψ satisfy linear differential equations, their solutions are superposable. Thus $f(z)$ is superposable. In other words, if $f_1(z)$ and $f_2(z)$ are complex potentials, f_1+f_2 is also a complex potential that describes some irrotational flow field.

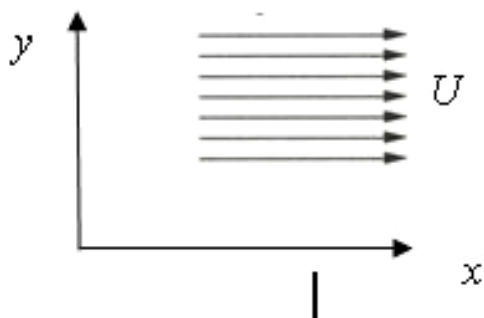
The family of $\phi=\text{const}$ lines and that of $\psi=\text{const}$ lines intersect orthogonally. This is easily shown by noting that

$$\nabla \phi = i v_x + j v_y, \quad \nabla \psi = -i v_y + j v_x, \quad \nabla \phi \cdot \nabla \psi = 0$$

EXAMPLES

Uniform Flow in the x-direction

The components of the velocity are:



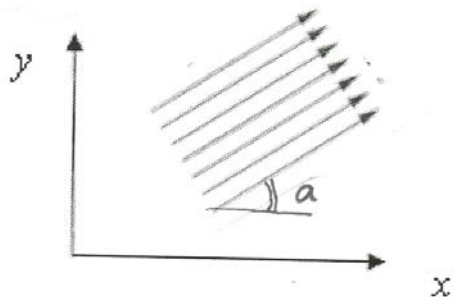
$$v_x = U \quad v_y = 0$$

Then

$$\phi = Ux \quad \psi = Uy \quad f(z) = U(x + iy) = Uz$$

Uniform flow at an angle α with the x-direction

The components of the velocity are:



$$v_x = U \cos \alpha \quad v_y = U \sin \alpha \quad v_z = 0$$

Then

$$\phi = Ux \cos \alpha \quad \psi = Uy \sin \alpha$$

and

$$w(z) = v_x - i v_y = U \cos \alpha - i U \sin \alpha = U (\cos \alpha - i \sin \alpha) = U e^{-i\alpha}$$

and finally integrating

$$f(z) = U e^{-i\alpha} z$$

Source or Sink Flow

For a two-dimensional source axisymmetric flow (line source) the only non-zero component is the radial, that is (from continuity)

$$v_r = \frac{Q}{2\pi r}, \quad v_\theta = 0$$

where Q is the "strength" of the sink i.e., fluid flow per unit length of tube if such a radial flow is visualised to be emitted through the wall of a long slender tube made of porous material. Q is taken positive for a source and negative for a sink. It is noted that such a velocity profile does not satisfy continuity at the origin.

Integrating the velocity profile, the velocity potential may be obtained, that is

$$\phi = \frac{Q}{2\pi} \ln r \quad \text{because} \quad v_r = \frac{\partial \phi}{\partial r}$$

and the stream function

$$\psi = \frac{Q}{2\pi} \theta$$

Thus, the complex potential is:

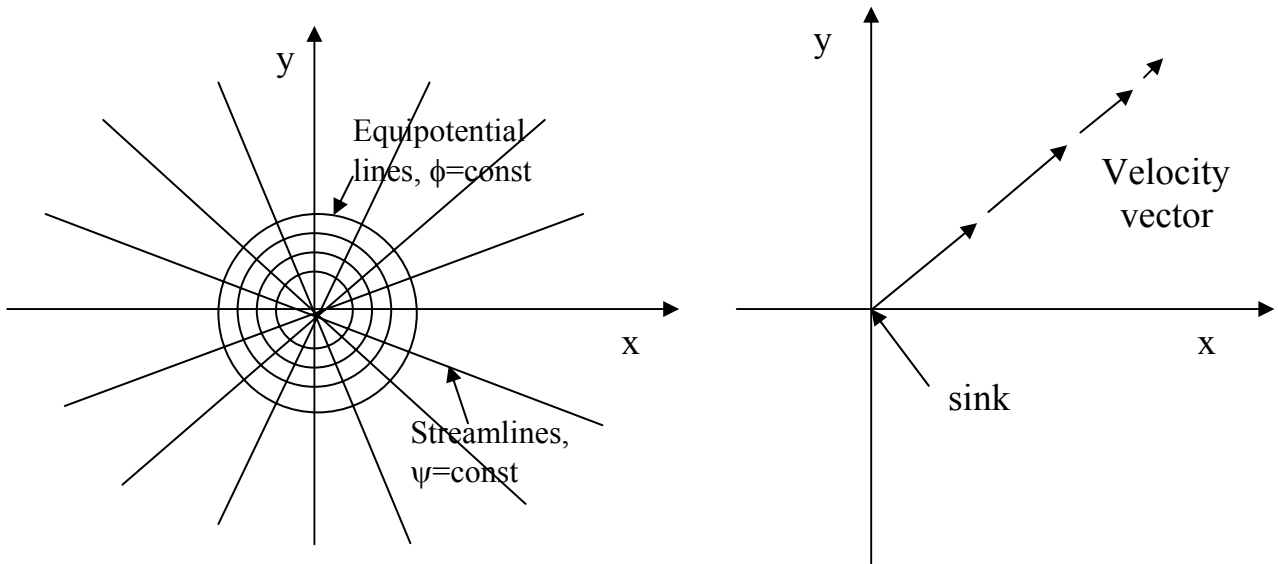
$$f(z) = \frac{Q}{2\pi} \ln(z) \quad \text{because} \quad \ln z = \ln r + i\theta$$

For a source located at z_0 rather than at the origin

$$f(z) = \frac{Q}{2\pi} \ln(z - z_0)$$

For a sink replace Q with $-Q$ in all the above equations. The schematic below illustrates streamlines and equipotential lines as well as the velocity field for a 2-D source flow. Note that the streamlines

are orthogonal to the equipotential lines.



Potential Vortex

In potential vortex flow the radial velocity component v_r is zero and only v_θ exists. From the continuity,

$$\frac{1}{r} \frac{\partial (r v_\theta)}{\partial r} = 0$$

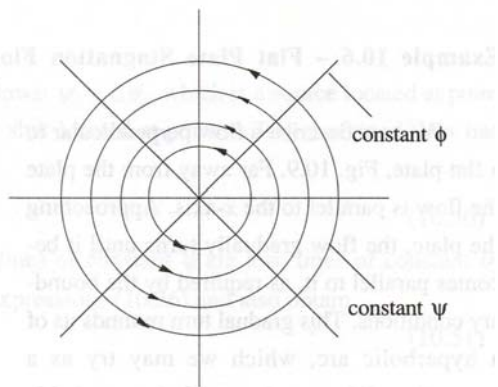
Integration yields

$$v_\theta = \frac{C}{r} + f(\theta)$$

and because nothing depends on $\theta, f=0$. Thus

$$v_\theta = \frac{C}{r}$$

The velocity potential and streamfunctions are as follows:



$$\phi = C\theta, \quad \psi = -C \ln r$$

Comparison with the source flow shows that ϕ and ψ lines have simply changed their roles. The constant C is related to the circulation around the vortex as shown below

$$\Gamma = \oint_C \mathbf{V} \cdot ds = \int \left(\frac{C}{r} \right) \delta_\theta \cdot (\delta_r dr + \delta_\theta r d\theta) = \int_0^{2\pi} C d\theta = 2\pi C$$

Hence

$$C = \frac{\Gamma}{2\pi}$$

and

$$\phi = \frac{\Gamma}{2\pi} \theta, \quad \psi = -\frac{\Gamma}{2\pi} \ln r$$

The complex potential is:

$$f(z) = \phi + i\psi = \frac{\Gamma}{2\pi} \theta - \frac{\Gamma i}{2\pi} \ln r = -\frac{\Gamma i}{2\pi} \ln z$$

For a vortex located at z_0 , rather than at the origin:

$$f(z) \equiv -\frac{\Gamma i}{2\pi} \ln(z - z_0)$$

There is no contradiction between $\Gamma \neq 0$ and the irrotationality of this flow. The circulation along any closed path not linked with the origin is zero. The singular point at the origin contributes the Γ value, and the same Γ is obtained along any contour linked with the origin.

Flow over Streamlined Bodies

1. Around a 2-D half body

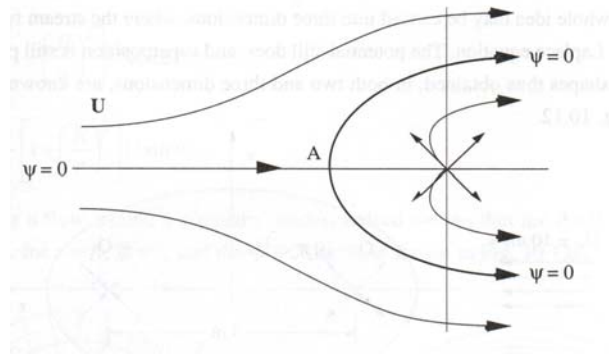
First we consider the superposition of a two-dimensional source flow and a parallel flow. In this way we describe the flow past a 2-D streamlined "half body". This combined flow has the following velocity potential and streamfunction

$$\phi = \frac{Q}{2\pi} \ln r + U r \cos \theta, \quad \psi = \frac{Q}{2\pi} \theta + U r \sin \theta$$

and the following velocity components

$$v_r = \frac{Q}{2\pi r} + U \cos \theta, \quad v_\theta = -U \sin \theta$$

For this flow we may recognise the need for the stagnation point A. Inspecting we see that for $\theta=\pi$ and $r=Q/2\pi U$ both velocity components vanish, and thus a stagnation point is obtained.

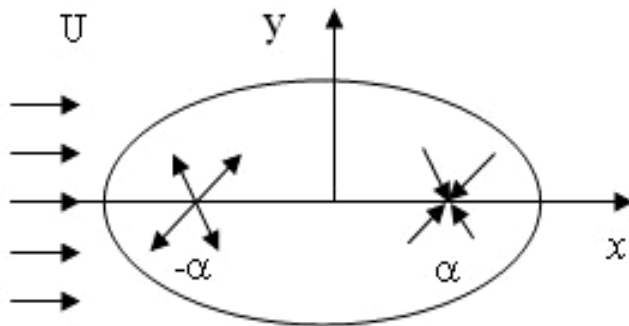


The same conclusion could have been reached by the argument that the zero streamline, $\psi=0$, could not pass through the source and therefore it must split. At the splitting point the velocity vector has more than one direction, and therefore its magnitude must be zero.

We also note that the streamline $\psi=0$ encircling the source cannot close again. Far to the right the flow becomes parallel again, but the splitting streamline does not close because if it did, the output of the source would have nowhere to go.

2. Source plus sink plus uniform flow

If a sink is superposed with a source and a parallel flow the zero streamline of the previous case can be closed again. In this way, the result represents a finite solid two-dimensional body immersed in the flow.



One possible arrangement is shown in the figure besides. The shapes resulted are known as the Rankine ovals. The complex potential in a case where the source is at position $x = -\alpha$ and the sink is at $x = \alpha$ is:

$$f(z) = Uz + \frac{Q}{2\pi} \ln \left(\frac{z + \alpha}{z - \alpha} \right)$$

3. Flow around a Cylinder

If we let the source and the sink move toward each other in the flow described previously, the body will become more circular. To obtain the flow around a true circular cylinder, the source and the sink must be at the same location, i.e., α must go to zero. But then this simplifies to a parallel flow. To avoid this, we imagine a process where as $2\alpha \rightarrow 0$, $Q \rightarrow \infty$. We wish Q to increase at such a rate that the limit for the complex potential is well defined and finite. In other words we let:

$$\lim_{\alpha \rightarrow 0, Q \rightarrow \infty} 2\alpha Q = m$$

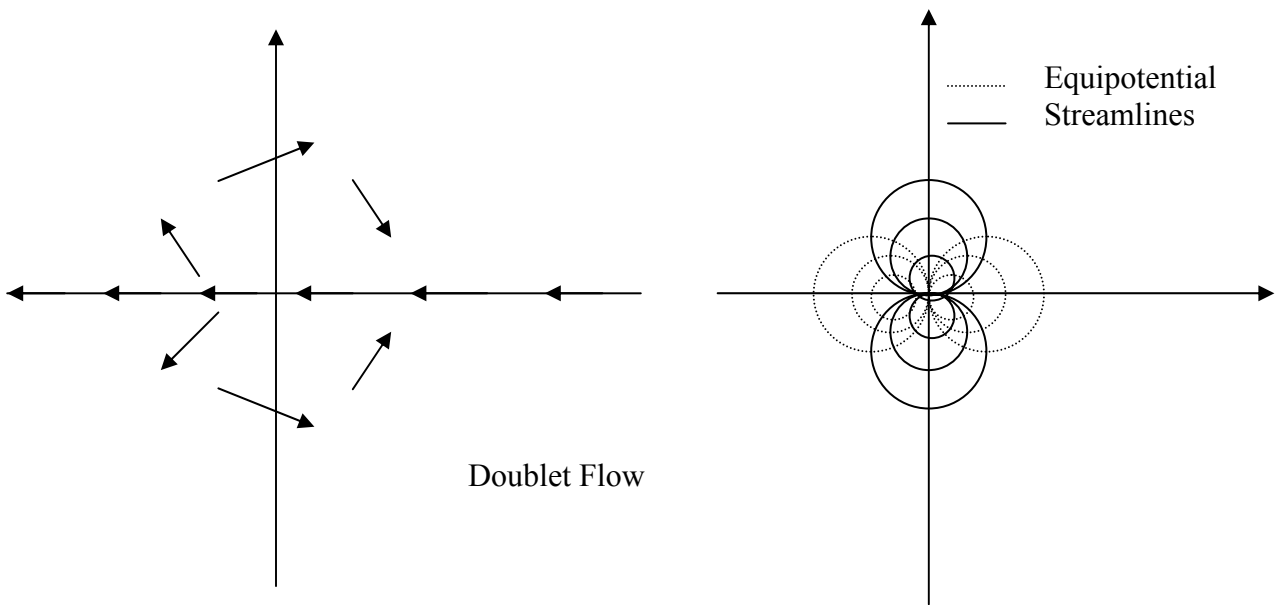
By taking the limit, the result can be written as:

$$f(z) = Uz + \frac{\lambda}{z}$$

where λ is the strength of the doublet (source+sink), $\lambda = m/2\pi$.

But what is the radius of the cylinder? This can be found by finding the location of the stagnation point, the point where $v=0$ on the axis of symmetry. The result is that the radius is $\sqrt{\lambda}U$. Thus the complex potential can be written as:

$$f(z) = U \left(z + \frac{\alpha^2}{z} \right)$$



4. Superposition of Vortex Flow on the Flow around a Stationary Cylinder

If we superpose vortex flow on the flow around a stationary cylinder, we get a flow in which there is a tangential velocity at the wall of the cylinder, as would occur if the cylinder were rotating. The complex potential in this case can easily be written by using the superposability principle of $f(z)$. Thus

$$f(z) = U \left(z + \frac{\alpha^2}{z} \right) - \frac{\Gamma i}{2\pi} \ln \left(\frac{z}{\alpha} \right)$$

The sign on the vortex is such that the cylinder is rotating in the counterclockwise direction.

Also the arbitrary constant is chosen to give $\psi=0$ on the surface of the cylinder. It can be shown that for negative circulation there is positive (upward) lift on the cylinder. To calculate this lift one needs the pressure distribution around the cylinder. This can be obtained by using Bernoulli equation.

$$\frac{P_\infty}{\rho} + \frac{U^2}{2} = \left(\frac{P}{\rho} + \frac{v_\theta^2}{2} \right)_{r=\alpha}$$

The tangential component of the velocity from the complex potential is:

$$v_\theta = -\frac{\Gamma}{2\pi r} - \left[1 + \left(\frac{\alpha}{r} \right)^2 \right] U \sin \theta$$

and the radial one is:

$$v_r = U \cos \theta \left[1 - \left(\frac{\alpha}{r} \right)^2 \right]$$

Thus, from Bernoulli the pressure distribution is:

$$P_\alpha = P_\infty + \frac{\rho U^2}{2} [1 - 4 \sin^2 \theta] - \frac{\rho}{2} \left(\frac{\Gamma}{2\pi \alpha} \right)^2 - \rho \frac{\Gamma U}{\pi \alpha} \sin \theta$$

The first two terms are symmetrical so that contribute nothing on the lift force. The last yields a lift

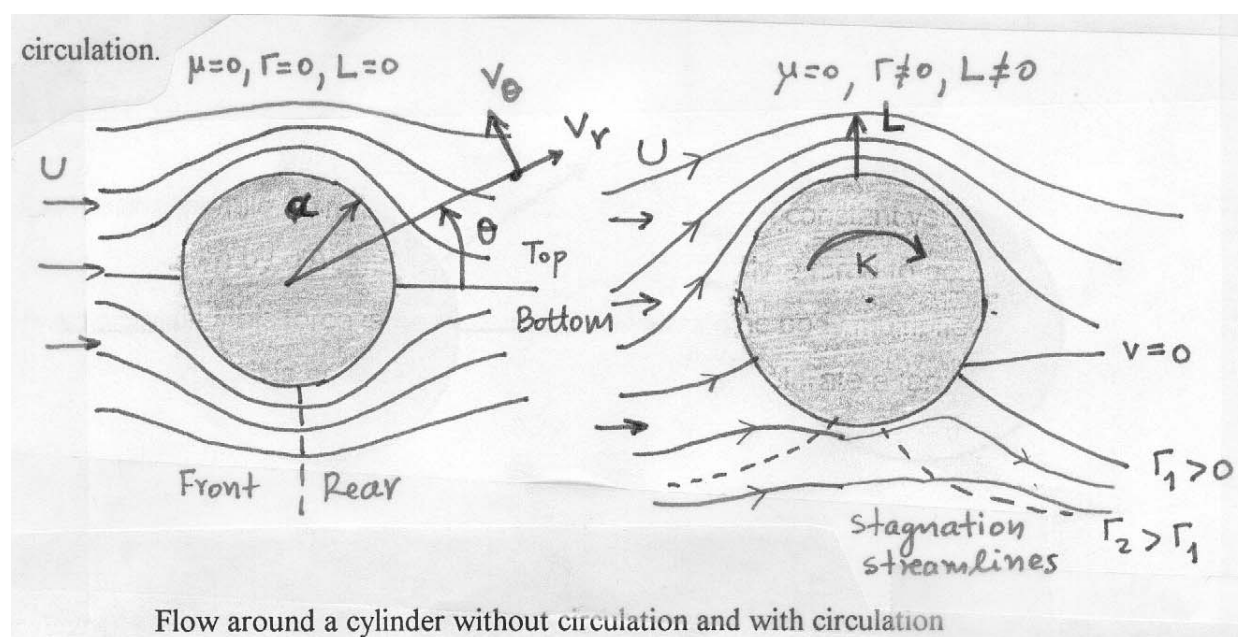
$$dL = -P_\alpha b ds \sin \theta = -P_\alpha b \alpha \sin \theta d\theta = \rho \frac{\Gamma U b}{\pi} \sin^2 \theta d\theta$$

force, that is ($\sin\theta$ comes from projection of the pressure force in the vertical direction)

where b is the length of the cylinder, P_α is the pressure on the surface, ds is some differential surface area on the cylinder and θ is the angle defined as in the unit trigonometric circle. Integrating,

$$L = \rho b U \Gamma$$

The presence of this lift is called the Magnus effect, has some very important implications in aerodynamics. Also it helps to explain why spin on a golf ball can cause it to veer off to the side and how a curve ball can be thrown. The Figure below illustrates the velocity patterns with and without circulation.



DRAG ON BODIES - d'Alambert's Paradox

Using the ideal fluid theory to calculate the drag force on immersed bodies gives the surprising result of zero. While it is obvious that there would be no viscous drag predicted by this theory, it is well known that the principle source of drag at high Reynolds numbers is form drag i.e., pressure drag.

The great French mathematician d'Alambert showed in 1752 that drag predicted by ideal fluid theory is zero. This paradox threw some doubt on the validity of Euler's equations. It was not until 1904 that Prandtl resolved this issue by developing the boundary layer theory.

The role of viscosity is "dual".

- First viscosity plays a direct role in drag as the mechanism of skin friction or viscous drag.
- However, even though it operates only in a thin layer near the body, it also affects the pressure distribution in such a way that form drag occurs.

ADDED MASS - (Hydrodynamic mass)

While there is no drag for a body moving at constant velocity in an ideal fluid at rest, as shown by d'Alambert, it is necessary to apply a force to accelerate a body in an ideal fluid, and this force is greater than the mass of the body multiplied by its acceleration. This is because it is necessary at the same time to accelerate a large body of fluid surrounding the body. This additional force is usually accounted for by use of the concept of the "added mass".

$$M_a = -\frac{\rho}{U^2} \int_A \phi \frac{\partial \phi}{\partial n} ds$$

where the surface integral is taken over the surface of the body, ds is the infinitesimal surface, and n is the spatial co-ordinate normal to the surface.

Thus the total force required to accelerate the body is:

$$F = (M_a + M_{body}) \frac{dU}{dt}$$

HIGH REYNOLDS NUMBER FLOWS

2. Regions close to Boundaries - The Boundary Layer Theory

In this chapter we still consider high Reynolds number flows, but this time our focus will be concentrated in regions close to solid boundaries where viscous effects become important. As also discussed in the previous chapter for high Reynolds number flows we have two solutions.

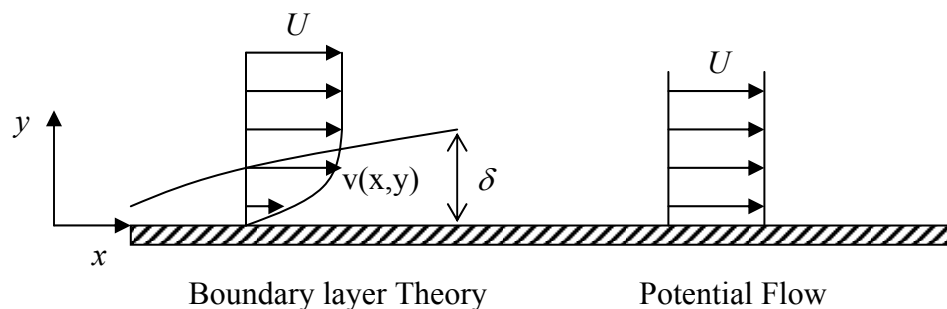
-For regions far from solid boundaries the Euler's equations apply and if the flow is also irrotational we can use the much simpler Laplace equation.

-However, there always be a thin layer near a solid boundary wall where this solution is not valid, because the local Reynolds number is small in that region. In addition the Euler's equations do not satisfy the no-slip boundary condition, an observation well documented experimentally. Because this boundary layer is thin, it makes possible a simplification of the Navier-Stokes that is similar to that used in lubrication flows.

To match the two solutions, it is assumed that the solution to the potential flow problem at the boundary ($y=0$) gives an acceptable approximation of the velocity and pressure distributions at the outer limit of the boundary layer. For example consider a flow in the x -direction parallel to a flat plate, as shown in the figure below. The potential flow for this example has already been solved,

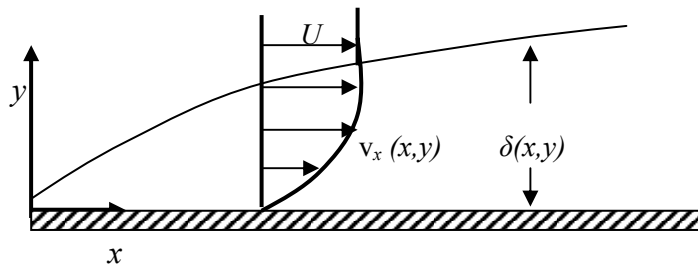
$$\phi = U x, \quad v_x = U = \text{const.}$$

which implies a uniform velocity profile. The real velocity profile matches the no-slip boundary condition at the solid boundary as shown schematically below. Comparing the two velocity profiles, we see that the solution to the potential flow problem is correct only far from the solid wall.



PRANDTL'S BOUNDARY LAYER THEORY

A procedure for simplifying the N-S equations for the special case of boundary layer flow was originally developed by Ludwig Prandtl, a professor at the University of Göttingen in Germany, to solve problems in aerodynamics. The resulting equations and techniques constitute "Prandtl's boundary layer theory". We will illustrate the theory for the simplest two-dimensional case, the flow over a surface.



δ is only a function of x .

v_x is a function of x , and y .

v_y is a function of x , and y .

For this problem we have from the Navier-Stokes equations for plane flow:

Continuity:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

direction x:

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{\mu}{\rho} \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} \right)$$

direction y:

$$v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \frac{\mu}{\rho} \left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} \right)$$

The boundary conditions are: absence of slip between the fluid and the wall, i.e. $v_x=v_y=0$ for $y=0$, and $v_x=U$, $v_y=0$ for $y \rightarrow \infty$.

Because the boundary layer, δ , is thin, Prandtl proposed for $y < \delta$

$$\frac{\partial^2 v_x}{\partial x^2} \gg \frac{\partial^2 v_x}{\partial y^2}$$

which means that the streamwise diffusion of momentum is negligible compared to the transverse diffusion of momentum. From continuity one may estimate the order of magnitude of $\partial v_y / \partial y$ which is the same as that of $\partial v_x / \partial x$. With similar arguments the y-component of the Navier-Stokes simplifies to (see also H. Schlichting, *Boundary Layer Theory*, 7th ed., McGraw-Hill, New York, 1979):

$$\frac{\partial p}{\partial y} = 0 \quad \text{for } y < \delta$$

which tells us that the pressure is not a function of the y direction, but it only depends on the x-direction. Thus, one may use the results from potential theory to calculate the pressure distribution. The Bernoulli equation is:

$$\frac{p}{\rho} + g h + \frac{U^2}{2} = \text{const.}$$

Differentiating with respect to x, we get

$$-\frac{1}{\rho} \frac{dp}{dx} = U \frac{dU}{dx}$$

Using these simplifications, one may write the continuity and the Navier-Stokes equations as:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = U \frac{dU}{dx} + \nu \frac{\partial^2 v_x}{\partial y^2}$$

The boundary conditions are:

$$v_x = v_y = 0 \quad \text{at} \quad y = 0$$

$$v_x = U(x) \quad \text{at} \quad y = \delta$$

Using the streamfunction these two equations can be reduced to a single equation

$$\frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} = -\frac{1}{\rho} \frac{dP}{dx} + \nu \frac{\partial^3 \psi}{\partial y^3}$$

THE BOUNDARY LAYER THICKNESS

Within the boundary layer thickness the velocity, $v_x(x,y)$ increases from the value of 0 at the solid boundary and approaches $U(x)$ asymptotically. Thus, it is impossible to indicate a boundary-layer thickness in an unambiguous way. However, it is convenient to define some measures for the boundary layer. These are the following:

1. The boundary layer thickness, δ : This thickness is also referred to as the ninety-nine percent boundary layer thickness. This is defined as the distance from the wall at which the velocity component, $v_x(x,y)$ approaches 99% of the value of $U(x)$. Thus,

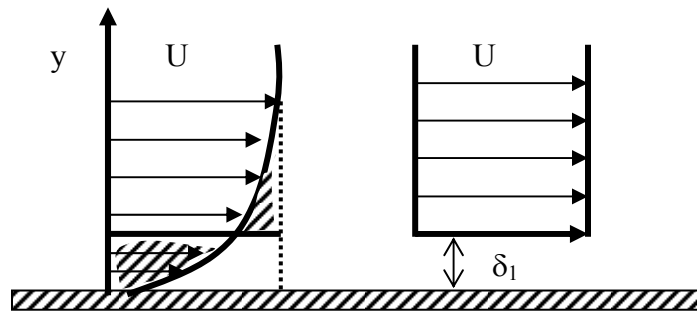
$$\delta(0.99) \equiv \text{value at which } v_x \equiv 0.99 U$$

2. The displacement thickness, δ_1 : The presence of the boundary layer reduces the total mass flux in the x-direction, and this reduction can be expressed in terms of a characteristic distance, δ_1 (displacement thickness).

$$\int_0^h \rho U dy - \int_0^h \rho v_x dy = \int_0^{\delta_1} \rho U dy$$

The quantity h is sufficiently large that the entire boundary layer is included within it. If the density is uniform then this expression may be simplified to:

$$\delta_1 = \int_0^h \left(1 - \frac{v_x}{U} \right) dy$$



Displacement thickness

3. The momentum thickness, δ_2 : The total momentum flux is also reduced by the presence of the boundary layer, and this reduction can also be used to define a distance scale, δ_2 , called the momentum thickness.

$$\int_0^h \rho v_x U dy - \int_0^h \rho v_x^2 dy = \int_0^{\delta_2} \rho U^2 dy$$

For a fluid with uniform density this reduces to:

$$\delta_2 = \int_0^h \frac{v_x}{U} \left(1 - \frac{v_x}{U} \right) dy$$

Note that δ_1 , and δ_2 would not change much, if the upper limit of the integral, h , is replaced with the 99% boundary layer thickness δ . This is because the term in the parenthesis, it takes values from 0.01 (at $y=\delta$) and less (at $y>\delta$). In fact it approaches asymptotically 0 for $y>\delta$.

THE BOUNDARY LAYER ON A FLAT PLATE AT ZERO INCIDENCE

For such a case, the solution to the potential flow problem is that the velocity is uniform in the x-direction (direction of flow)

$$U(x, y=0) \equiv U(x) = U = \text{const.}$$

From Bernoulli

$$\frac{dp}{dx} = 0$$

Thus, the Boundary Layer equations can be simplified to:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \nu \frac{\partial^2 v_x}{\partial y^2}$$

and the boundary conditions are:

$$v_x = v_y = 0 \quad \text{at} \quad y = 0$$

$$v_x = U \quad \text{at} \quad y = \infty$$

In terms of the streamfunction:

$$\frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} = \nu \frac{\partial^3 \psi}{\partial y^3}$$

and the boundary conditions become:

$$\frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial x} = 0 \quad \text{at} \quad y = 0$$

$$\frac{\partial \psi}{\partial y} = U(x) \quad \text{at} \quad y = \infty$$

This problem was studied by Blasius, a doctoral student of Prandtl. He found that a similarity variable exists which can be used to transform the problem into an ODE. Specifically he assumed that:

$$\frac{\psi}{\sqrt{\nu x U}} = f \left(y \sqrt{\frac{U}{\nu x}} \right)$$

or

$$\psi = \sqrt{\nu x U} \quad f(\eta)$$

where

$$\eta \equiv y \sqrt{\frac{U}{\nu x}}$$

In terms of velocity this becomes:

$$\frac{v_x}{U} = f'(\eta)$$

This implies that the velocity profiles at various x-locations are "self-similar". Using this transformation into the partial differential equation for the streamfunction yields:

$$f''' + \frac{1}{2} f f'' = 0$$

with

$$f = f' = 0 \quad \text{at} \quad \eta = 0$$

$$f' = 1 \quad \text{at} \quad \eta = \infty$$

This ODE must be solved numerically. The velocity profile is given by:

$$v_x = \frac{\partial \psi}{\partial y} \sqrt{\nu x U} \left(\frac{d f}{d \eta} \right) \left(\frac{\partial \eta}{\partial y} \right) = U f'(\eta)$$

Fluid Mechanics books tabulate values of $f'(\eta)$. The Table below summarises some values:

η	f''	f'	f
0	0.332	0	0
1	0.323	0.330	0.166
2	0.267	0.630	0.650
3	0.161	0.846	1.397
4	0.0642	0.956	2.306
5	0.0159	0.992	3.283
6	0.0024	0.999	4.280
∞	0	1.000	∞

The wall shear stress is:

$$\sigma_w = \mu \left(\frac{dv_x}{dy} \right)_{y=0} = \mu U f'' \left(\frac{dv_x}{dy} \right)_{y=0} = \sqrt{\frac{\rho \mu}{x}} U^{3/2} (f'')_{\eta=0} = 0.332 \sqrt{\frac{\rho U^3 \mu}{x}}$$

or

$$\sigma_w = \frac{0.332 \rho U^2}{\sqrt{Re_x}}$$

where Re_x is the Reynolds number defined as $Re_x \equiv Ux/\nu$. For a plate of width b and length L , we have for the drag force, F_D , on one side:

$$F_D = \int_0^L \sigma_w b dx = \frac{0.664 \rho U^2 b L}{\sqrt{Re_L}}$$

where Re_L is the Reynolds number defined as $Re_L/UL/\nu$. Thus the local skin friction coefficient, C_f , for this case becomes:

$$C_f = \frac{\sigma_w}{\frac{1}{2}\rho U^2} = \frac{0.664}{\sqrt{Re_x}}$$

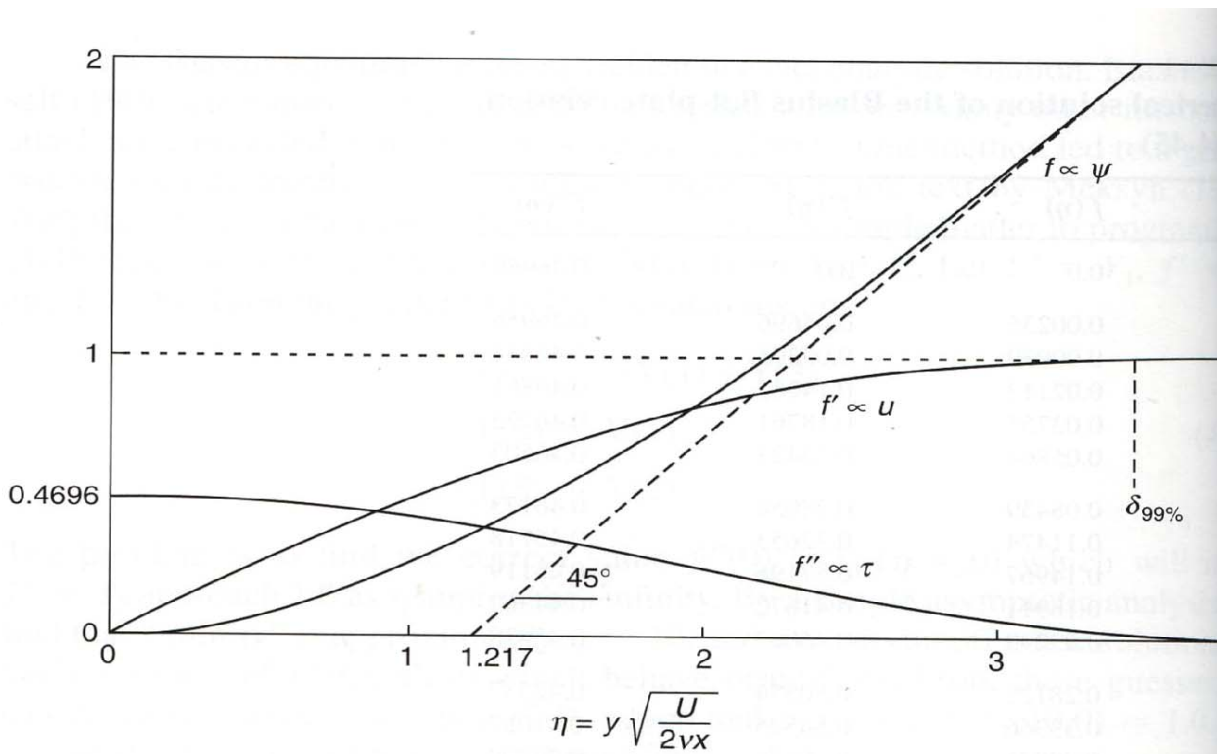
while the average skin friction coefficient over a length L , \bar{C}_f is:

$$\bar{C}_f = \frac{1.328}{\sqrt{Re_L}}$$

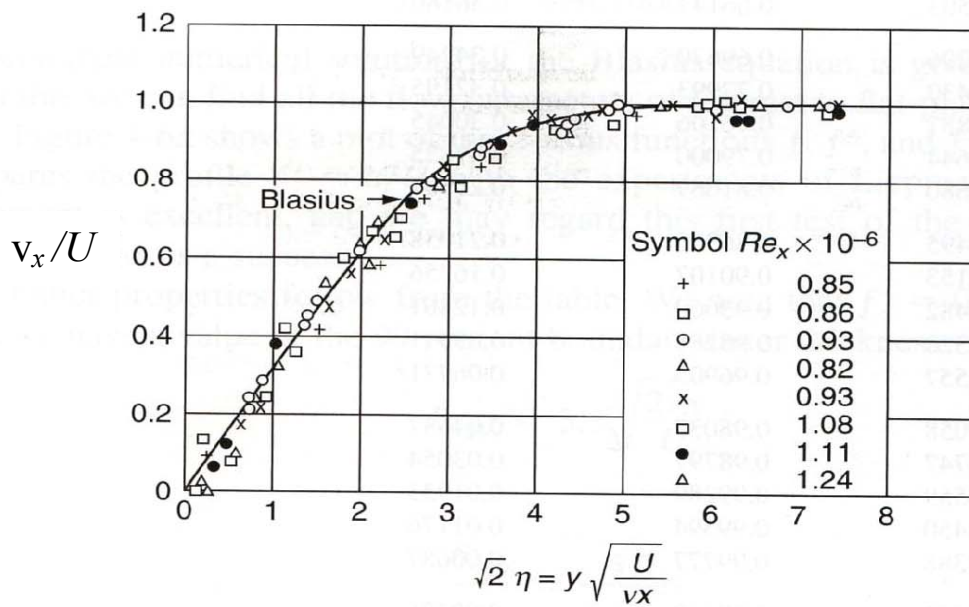
Also the following relations can be derived:

$$\delta(0.99) = \frac{5x}{\sqrt{Re_x}} \quad \delta_1 = \frac{1.73x}{\sqrt{Re_x}} \quad \delta_2 = \frac{0.664x}{\sqrt{Re_x}}$$

The figure below shows the Blasius solution in graphical form, as well as comparison of the solution with experimental measurements. It can be seen that the agreement is excellent.



(a)



(b)

Figure (adopted from White , Viscous Fluid Flow, 2006) (a) numerical solution of Blasius for the flat plate boundary layer and (b) comparison with experiments.

OTHER SIMILARITY SOLUTIONS

Falkner and Skan (1931) carried out an analysis of the boundary layer equations to find all similarity solutions that could be expressed in the form (similar to a generalized Blasius solution).

$$v_x = U(x) f'(\eta) \quad \text{where} \quad \eta \equiv \frac{y}{\xi(x)}$$

They found that such similarity solutions exist when (see White, 2006 for a proof):

$$\frac{\xi}{\nu} \frac{d}{dx} (U \xi) = \alpha \quad \text{and} \quad \frac{\xi^2}{\nu} \frac{dU}{dx} = \beta$$

where α and β are constants. In fact β is a measure of the pressure gradient dp/dx . Substituting these into the Navier-Stokes in terms of streamfunction the following ODE is obtained.

$$f''' + \alpha f f'' + \beta [1 - f'^2] = 0$$

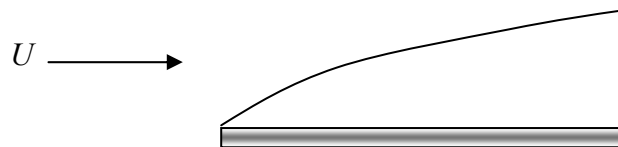
Special cases from these equation may be obtained:

1. Flow over a flat Plate

$$\alpha = 1/2$$

$$\beta = 0$$

$$\xi = \sqrt{(\nu x/U)}$$



This gives the Blasius solution

2. Flow over a Wedge

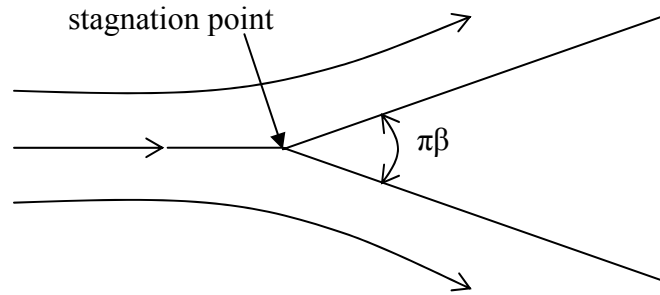
$$\alpha = 1$$

$\pi\beta$ = wedge angle

$$U(x) = c x^{\beta/(2-\beta)}$$

and

$$\xi(x) = \sqrt{\frac{v(2-\beta)}{c}} x^{(1-\beta)/(2-\beta)}$$



3. Other Flows

These include stagnation 2-D and 3-D, flow around a corner, flows in convergent and divergent channels etc. For more details see Schlichting (1979).

VON KARMAN - POHLHAUSEN INTEGRAL METHOD

Up to this point we have seen some of the exact solutions to the boundary layer equations. Exact in a sense that the equations are solved exactly irrespectively of the method used, analytic or numerical. However, there are situations where exact solutions cannot be found (except full numerical solution of the PDE's). For these cases an approximate method due to Von Karman and Pohlhausen can be used. The approximation is that the boundary layer equation is not satisfied pointwise but rather on the average over the region.

Todor Von Karman, a native Budapest was a research assistant at Gottingen. His research had to do with beam stability, but he became interested in the work on boundary layers that was going on there. He and Pohlhausen developed independently an approximate technique for solving the boundary layer equations which is discussed in this section.

According to the method, the differential boundary layer equations are first transformed into an integral equation. This stage is exact. Then we proceed to satisfy this integral equation by the selection of an appropriate velocity profile inside the boundary layer. This stage introduces the approximation into the method, because satisfying the integral relation is a necessary condition, but not a sufficient condition.

Consider again the boundary layer equation

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} - U \frac{dU}{dx} = \frac{\mu}{\rho} \frac{\partial^2 v_x}{\partial y^2}$$

Integrate the above equation with respect to y from $y=0$ to $y=h > \delta$

$$\int_0^h \left[v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} - U \frac{dU}{dx} \right] dy = \frac{1}{\rho} \int_0^h \mu \frac{\partial^2 v_x}{\partial y^2} dy$$

The right hand side integral is:

$$\frac{1}{\rho} \int_0^h \mu \frac{\partial^2 v_x}{\partial y^2} dy = \frac{1}{\rho} \int_0^h \frac{\partial}{\partial y} \left(\mu \frac{\partial v_x}{\partial y} \right) dy = \frac{1}{\rho} \int_0^h \frac{\partial \tau}{\partial y} dy = \frac{1}{\rho} \tau \Big|_0^h = -\frac{1}{\rho} \tau_w$$

Consider the second term on left and integrate by parts to get

$$\int_0^h v_y \frac{\partial v_x}{\partial y} dy = v_x v_y \Big|_0^h - \int_0^h v_x \frac{\partial v_y}{\partial y} dy$$

From continuity

$$\frac{\partial v_x}{\partial x} = -\frac{\partial v_y}{\partial y}$$

or

$$v_y = -\int_0^h \frac{\partial v_x}{\partial x} dy$$

and

$$v_x v_y|_{y=h} = -U \int_0^h \left(\frac{\partial v_x}{\partial x} \right) dy = - \int_0^h U \frac{\partial v_x}{\partial x} dy$$

Thus the term being considered is:

$$\int_0^h v_y \frac{\partial v_x}{\partial y} dy = - \int_0^h U \frac{\partial v_x}{\partial x} dy + \int_0^h v_x \frac{\partial v_x}{\partial x} dy$$

The first term on the right

$$\int_0^h U \frac{\partial v_x}{\partial x} dy = \int_0^h \frac{\partial (v_x U)}{\partial x} dy - \int_0^h v_x \frac{\partial U}{\partial x} dy$$

Using all the above

$$\int_0^h \left[2 v_x \frac{\partial v_x}{\partial x} - \frac{\partial (v_x U)}{\partial x} + v_x \frac{dU}{dx} - U \frac{dU}{dx} \right] dy = - \frac{\tau_w}{\rho}$$

But

$$2 v_x \frac{\partial v_x}{\partial x} = \frac{\partial (v_x^2)}{\partial x}$$

Multiply by (-1) and rearrange

$$\int_0^h \left[\frac{\partial (v_x U)}{\partial x} + \frac{\partial (v_x^2)}{\partial x} + (U - v_x) \frac{dU}{dx} \right] dy = \frac{\tau_w}{\rho}$$

or

$$\frac{\partial}{\partial x} \int_0^h v_x (U - v_x) dy + \frac{dU}{dx} \int_0^h (U - v_x) dy = \frac{\tau_w}{\rho}$$

Using

$$\int_0^h (U - v_x) dy = U \delta_1$$

and

$$\int_0^h v_x (U - v_x) dy = U^2 \delta_2$$

Finally we get

$$\frac{d}{dx} (U^2 \delta_2) + \delta_1 U \frac{dU}{dx} = \frac{\tau_w}{\rho}$$

This is the momentum integral equation of Von Karman and Pohlhausen. If we have considered an unsteady flow then this equation would have become:

$$\frac{d}{dt} (U \delta_1) + \frac{d}{dx} (U^2 \delta_2) + \delta_1 U \frac{dU}{dx} = \frac{\tau_w}{\rho}$$

For steady flow the momentum integral equation can be rewritten as:

$$\frac{\tau_w}{\rho U^2} = \frac{d \delta_2}{dx} + \frac{2 \delta_2}{U} \frac{dU}{dx} + \frac{\delta_1}{U} \frac{dU}{dx}$$

Up to this point this equation is still exact. The approximation comes in when a velocity profile, $v_x = f(y/\delta)$, is assumed to evaluate the various terms. The procedure is as follows:

- Assume a reasonable form for the velocity profile, $v_x = f(y/\delta)$, parabolic or higher order. This profile should meet the following basic criteria

i. Continuity of $v_x(y)$

$$v_x = U \quad \text{at } y = \delta$$

ii. Continuity of shear stress

$$\partial v_x / \partial y = 0 \quad \text{at } y = \delta$$

iii. Zero second derivative of $v_x(y)$ at $y=0$ because i.e., for a flat plate

$$\frac{\partial p}{\partial x} = \frac{dU}{dx} = 0$$

From momentum at $y=0$

$$\frac{1}{\rho} \frac{\partial p}{\partial x} = \nu \left(\frac{\partial^2 v_x}{\partial y^2} \right)_{y=0} = 0$$

Example: Consider flow over a flat plate, and make a very naive assumption regarding the form of the velocity distribution:

$$v_x = \frac{U y}{\delta(x)}$$

Note that this crude approximation satisfies only the first and third criterion. Using the definitions of δ , δ_1 , and δ_2 then calculate:

$$\delta_1 = \frac{\delta}{2}, \quad \delta_2 = \frac{\delta}{6}, \quad \tau_w = \mu \frac{U}{\delta}$$

Substituting into the momentum integral equation

$$\frac{U^2}{6} \frac{d\delta}{dx} = \frac{\nu U}{\delta} \quad \text{with } \delta=0 \text{ at } x=0$$

$$\delta(x) = \frac{2\sqrt{3} x}{\sqrt{Re_x}}$$

and

$$v_x(x, y) = \frac{yU\sqrt{Re_x}}{2\sqrt{3} x}$$

One may now obtain

$$\delta(0.99) = \frac{3.43 x}{\sqrt{Re_x}}$$

$$\delta_2 = \frac{1.732 x}{\sqrt{Re_x}}$$

$$C_D = \frac{1.156}{\sqrt{Re_x}}$$

The corresponding exact coefficients are 5, 1.73 and 1.328 respectively. Thus a very crude approximation leads to correct functional forms and values of the constants that are of the right order of magnitude.

BOUNDARY LAYER SEPARATION

The boundary layer near a flat plate in parallel flow and at zero incidence is particularly simple, because the static pressure remains constant in the whole field of flow. Note also that the pressure gradient in the direction of flow is governed by the mainstream potential flow through Bernoulli's equation. In cases where there is an "adverse" pressure gradient, i.e. $dp/dx > 0$ a phenomenon referred to the boundary layer separation may occur. According to this a reversal of flow in the boundary layer near the wall may occur.

To explain the very important phenomenon of boundary layer separation let us consider the flow about a blunt body, e.g. about a circular cylinder as shown in the Figure below. In the frictionless flow, the fluid particles are accelerated on the upstream half from D to E, and decelerated on the downstream half from E to F. Hence the pressure decreases from D to E and increases from E to F. Because from E to F the pressure gradient is "adverse" a separation in the boundary layer occurs which is accompanied by a flow reversal. Also below a schematic diagram (magnification of the surface of the cylinder) illustrates more comprehensively the phenomenon in terms of velocity profiles.

The fact that separation occurs only in decelerated flow ($dp/dx > 0$) can be easily inferred from a consideration of the relation between the pressure gradient dp/dx and the

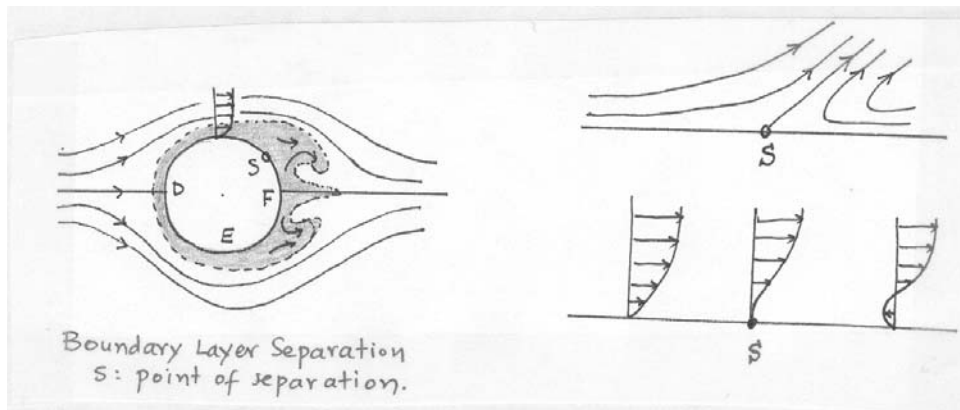


Figure: Boundary layer separation

velocity distribution, v_x , with the aid of the boundary layer equations. Evaluating the momentum equation at the wall ($v_x = v_y = 0$) leads to:

$$\mu \left(\frac{\partial^2 v_x}{\partial y^2} \right)_{y=0} = \frac{d p}{d x}$$

In the neighbourhood of the wall the curvature of the velocity profile depends only on the pressure gradient, and the curvature of the velocity profile at the wall changes its sign with the pressure gradient. For flow with decreasing pressure gradient ($dP/dx < 0$) this relation tells us that $(\partial^2 v_x / \partial y^2)_{wall} < 0$ and therefore $(\partial^2 v_x / \partial y^2) < 0$ over the whole domain. In the region of pressure increase ($dP/dx > 0$) we have $(\partial^2 v_x / \partial y^2)_{wall} > 0$ and since at distances far from the wall $(\partial^2 v_x / \partial y^2) > 0$ the velocity profile always exhibits a point of inflexion in the region where separation exists. From the Figure below one may infer that the condition for the onset of separation is:

$$\left(\frac{\partial v_x}{\partial y} \right)_{y=0} = 0$$

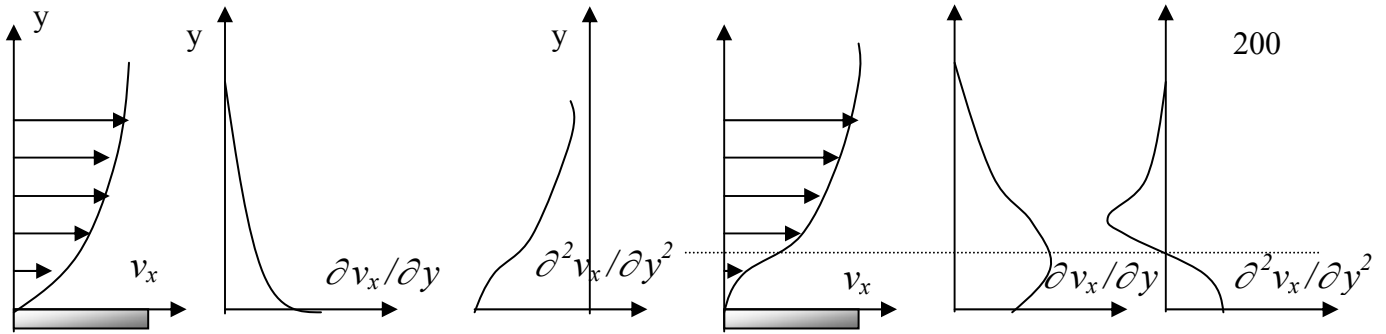
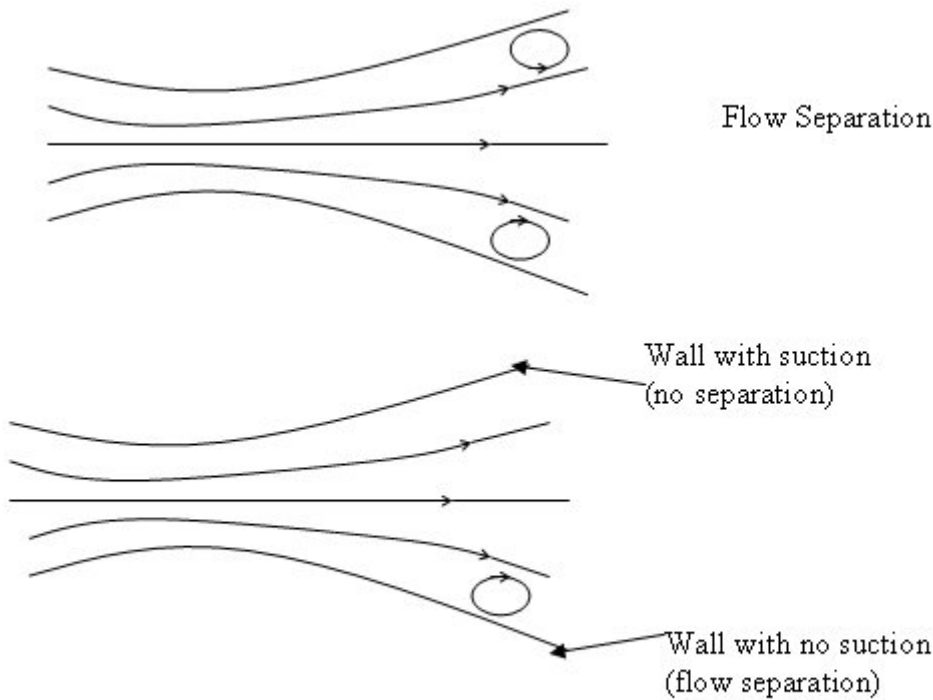


Fig. Velocity distribution in a boundary layer with pressure decrease

Fig. Velocity distribution in a boundary layer with pressure increase; PI=point of inflexion

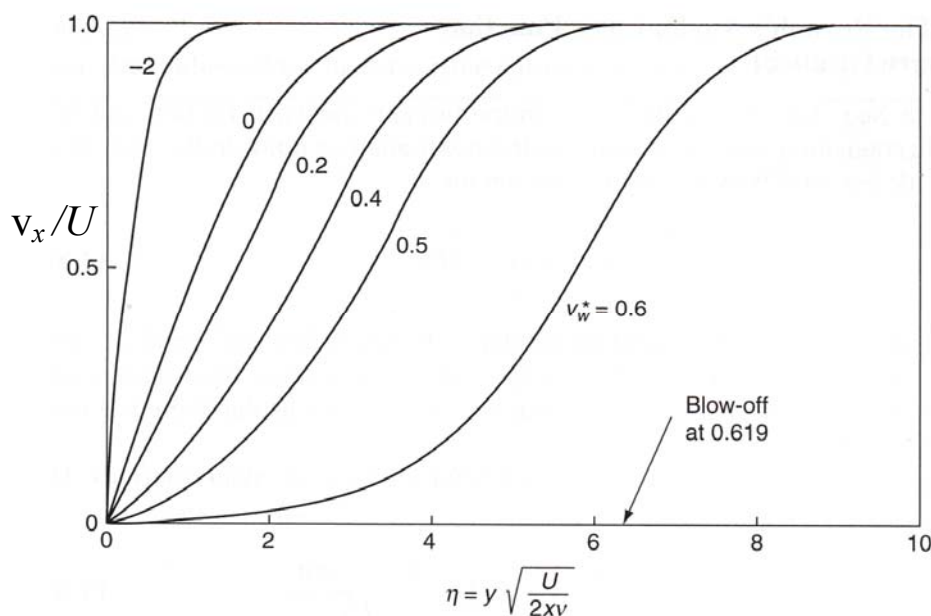
Note: Separation decreases C_D (drag coefficient), while no separation increases C_D (Applications: design of planes, cars, aerodynamics etc).



THE FLAT PLATE WITH WALL SUCTION OR BLOWING

As discussed above boundary layer separation decreases drag in general. To impose or prevent separation, one may alter the boundary conditions at the wall by imposing a nonzero wall velocity in the transverse direction, $v_y \ll U$, with positive (blowing) or negative (suction). The streamwise wall velocity $v_{x,wall}=0$. This has practical applications in mass transfer, drying, ablation, transpiration cooling and boundary layer control (already discussed). The wall velocity, v_y , at the wall where $\eta = y\sqrt{U/2x\nu}$ is equal to zero, is: $v_{y,wall} = -f(0)\sqrt{\nu U/2x}$. Note that this velocity component at the wall is allowed to vary in such a fashion that a similarity solution exists. Therefore suction and blowing can be simulated by a nonzero value of the Blasius stream function, $f(0)$. This problem was solved by Schlichting and Bussmann (1943) subject to the following conditions: $f'(0) = 0, f'(\infty) = 1, f(0) \neq 0$. The results are strongly dependent on the suction-

blowing parameter, $v_w^* = \frac{v_{y,wall}}{U} \sqrt{\text{Re}_x} = \frac{-f(0)}{\sqrt{2}}$. The figure below summarizes the results.



Suction thins the boundary layer and greatly increases wall slope (friction, heat transfer). These profiles are very stable and delay transition to turbulence. Blowing thickens the boundary layer and makes profile S-shaped and prone to transition to turbulence. At

$v_w^* = 0.619$, the solution yields $\partial v_x / \partial y = 0$ at the wall. The boundary layer is said to be blown off and the heat transfer and friction are zero.

FREE-SHEAR FLOWS

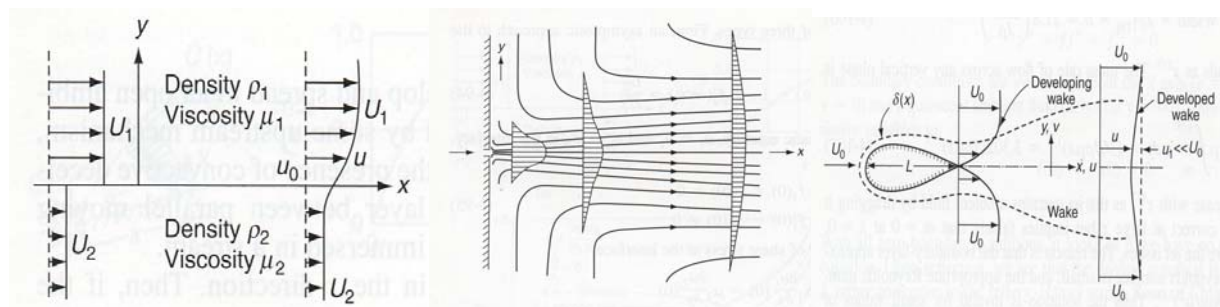
Shear free layers are unaffected by walls and develop and spread in an open ambient fluid. They possess velocity gradients, created by some upstream mechanism that they try to smooth out by viscous diffusion in the presence of convective deceleration. Examples are (1) free-shear layer between parallel moving streams (2) jet i.e. injection of a fluid through a small opening into a still ambient fluid and (3) wake behind a body immersed in a stream. Jets and wakes are unstable and are more likely in practice to be turbulent than laminar.

For such flows if the Reynolds number is large most of the boundary-layer approximations are valid and for 2-D flows these are:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \quad (\text{continuity})$$

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \nu \frac{\partial^2 v_x}{\partial y^2} \quad (x\text{-momentum equation})$$

Note that in solving these equations there are no walls to enforce a no-slip condition. In most of these flows, just downstream from the disturbance the velocity profiles are non-similar and developing. These will be similar further downstream. In our brief discussion we will discuss similar solutions for the shear layer of shear layers of two different streams. For all the other cases and many more, see White (2006).



(a) Shear Layers

(b) Plane Laminar Jet

(c) Plane Laminar Wake

The Free-Shear Layer between Two Different Streams

The figure below shows the flow problem under consideration. The discontinuity in the velocity profiles is smoothed out by viscosity into an S-shaped shear layer between the two. The case $U_2=0$ corresponds to boundary layer flow over a flat plate.

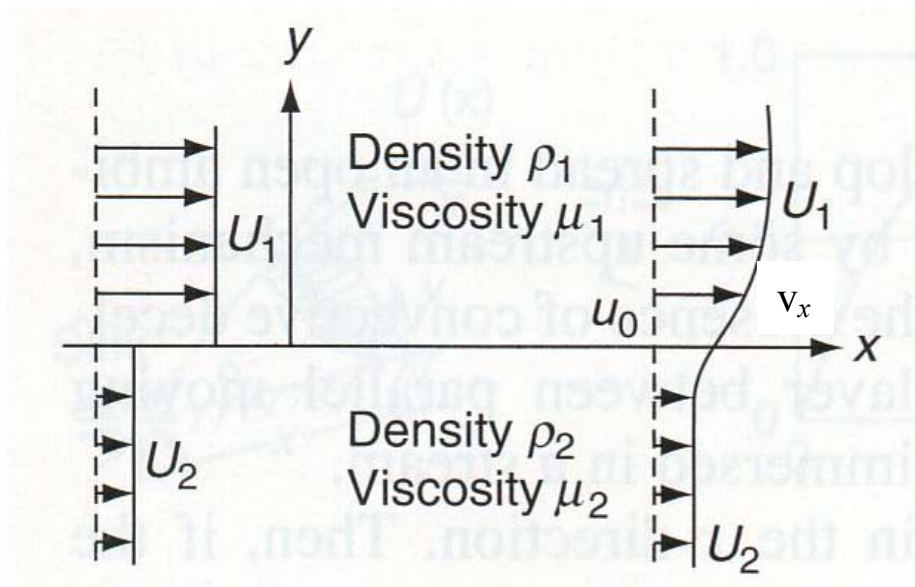


Figure: Velocity distribution between two parallel streams of different properties

The equations for solving this problem are defined in terms of the following Blasius-type similarity variables:

$$\eta_j = y \sqrt{\frac{U_1}{2x\nu_j}} \quad f'_j = \frac{v_{x,j}}{U_1} \quad \text{for } j=1,2$$

Substitute into the equations for shear free flows, Blasius type of equations can be developed, these are:

$$f_j''' + f_j f_j'' = 0 \quad j = 1,2$$

The boundary conditions require equality of velocities at the interface, equality of shear stress at the interface and asymptotic approach to the free stream velocities at infinite distance. The solution is plotted below. Note the Blasius solution for flow over a flat plate and the development of the S-shaped velocity profile.

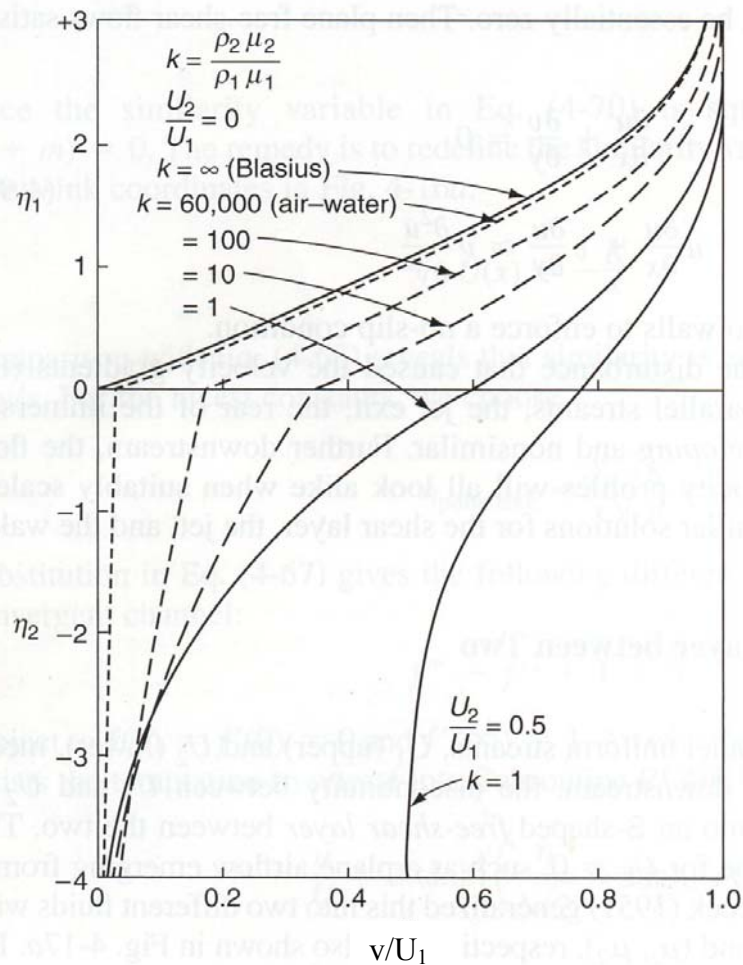


Figure: Velocity profiles

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