Summer Packet BC Calculus 2015-16

Name:

As we continue our journey through Calculus, there are certain skills that you learned last year, which must be remembered or reviewed. If you do not recall these skills, you will find that you will consistently get problems incorrect next year. Finding derivatives and integrals are the staple of Calculus classes. Therefore, you must know all the basic rules of differentiation and integration. This summer assignment is designed to help you review/relearn those topics and to re-familiarize yourself with the AP exam.

I cannot stress enough how important it is to make sure you recall all the concepts covered on this summer assignment. Take your time to relearn differentiation, integration, and trigonometric topics you may have forgotten. If you do not feel confident with your answers to these questions or you need any guidance, please study your notes from your AB class, or reference the internet. If you do not fully understand the packet, it will make your Calculus BC experience much more difficult. I do not want you to do poorly in this class because of a lack of prior knowledge.

Please **do not wait until the last minute** to complete the summer assignment. There are a lot of problems to solve and I want to make sure you take your time understanding each of them. At the same time, please **do not finish the whole packet in the beginning of the summer**. The point of this assignment is to refresh your memory of skills that are needed in the course and if you do all the work early, you may forget again before school starts.

The packet is due on the first day of school, no exceptions. If you forget to bring it with you, I will consider you unprepared for class and it will result in a zero for the class prep grade for that day. The entire packet will be be graded for completeness and correctness.

There will be an in-class test on the material from the assignment on the 2nd class. I expect your work to be shown neatly on separate sheets of paper to be turned in without the packet (graphing paper is preferred for the graphs) and answers clearly labeled. As an alternative to paper, you may use Notability (built in graph “paper’!) on your iPad. The electronic version of this packet is at **www.mathorama.com** (choose Calculus from the side bar), as well as showbie.com **Class code DKJFZ**

The heading, i.e. your name, class, period, assignment title, and information, must appear on every assignment. Follow correct form when answering questions. Legibility and understandability are important. If I cannot find each answer or if your work is illegible, you will receive no credit for the problem.

Keep in mind that half of the AP exam does not allow a calculator to be used, so don’t rely too heavily on one. If you find yourself relying on outside help to get through these exercises, (needing to work with others, or use the internet to search to learn or relearn topics), you probably need to work more problems than just these here. People lift weights not because the weights need lifting, but because it helps the lifter– just so, these problems don’t need to be solved by you because they need solving— rather, they need solving to help you.

I think the **Khan Academy iPad App** is a wonderful thing. It awards badges to encourage you, remembers to re-ask you questions you miss, and has links to relevant hints and videos. To add me as a coach, use my email cthiel@sfhs.net. To join my **AP Calc BC Practice Differentials** class on KhanAcademy.org, use **Class code CA85JA** and to join my **AP Calc BC Summer Practice Integrals** class use **Class code DKKP7T**

I think the **GeoGebra iPad App** is a wonderful thing as well. There are some great tutorials on how to use it on YouTube.

I look forward to working with each of you in the fall. Good luck and have a great summer!

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Based on six years of reading... the six most common errors I saw were the following

(1) Rounding too much on intermediate steps. If you need to find an endpoint for an integral, do not round it to three decimal places since that will most likely ruin your final answer. Store the whole number in the calculator, or at least use four or five decimal places.

(2) Unnecessary simplification. Leave an answer such as 4* 5* 9^(1/2) or sin(pi/3) or sqrt(5^2 - 1), rather than risking a boneheaded algebra or calculator mistake. If you do not need the decimal number, do not find it.

(3) Applied problems answered without context. If you are asked about electricity usage, express your answer in terms of electricity usage, and be sure to include UNITS.

(4) Impersonal pronouns. Do not "it" all over your paper. Do not write, "It is increasing because it is positive," write "the function is increasing because its derivative is positive". If there is more than one function present, do not write "the function" or "the derivative". IDENTIFY OBJECTS.

(5) Answering part of a question correctly in the wrong question space. If, while solving 4a, you find the answer to part 4c, either recopy the work and answer into the space for 4c, or better, write "The answer is ... , see the work in part 4a."

(6) Working by hand those computations that your calculator is supposed to do. I cannot tell you how many times I saw students try to evaluate a definite integral such as intregral(0 to 4, sin(e^x)) dx by hand, when a handful of calculator keystrokes would have produced the required numerical answer. Remember the valid uses of a calculator include finding numerical intersection points and the numerical value of a definite integral.

My favorite instruction: READ the PROBLEM. Do the work to answer the question asked. READ THE PROBLEM AGAIN. Did you forget to answer anything or to show anything requested?

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AP CALCULUS
Stuff you **MUST** Know Cold

Curve sketching and analysis

$y = f(x)$ must be continuous at each:

critical point: $\frac{dy}{dx} = 0$ or undefined.

local minimum :
$\frac{dy}{dx}$ goes (-,0,+) or (-,und,+)
or $\frac{d^2y}{dx^2} > 0$.

local maximum :
$\frac{dy}{dx}$ goes (+,0,-) or (+,und,-)
or $\frac{d^2y}{dx^2} < 0$.

pt of inflection : concavity changes.
$\frac{d^2y}{dx^2}$ goes (+,0,-),(-,0,+),
(+,und,-), or (-,und,+)

Basic Derivatives

$$\frac{d}{dx}(x^n) = nx^{n-1}$$
$$\frac{d}{dx}(\sin x) = \cos x$$
$$\frac{d}{dx}(\cos x) = -\sin x$$
$$\frac{d}{dx}(\tan x) = \sec^2 x$$
$$\frac{d}{dx}(\cot x) = -\csc^2 x$$
$$\frac{d}{dx}(\sec x) = \sec x \tan x$$
$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$
$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$
$$\frac{d}{dx}(e^x) = e^x$$

More Derivatives

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$
$$\frac{d}{dx}\left(\cot^{-1} x\right) = \frac{-1}{1+x^2}$$
$$\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{|x|\sqrt{x^2-1}}$$
$$\frac{d}{dx}\left(\csc^{-1} x\right) = \frac{-1}{|x|\sqrt{x^2-1}}$$
$$\frac{d}{dx}(a^x) = a^x \ln a$$
$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

Differentiation Rules

Chain Rule
$$\frac{d}{dx}[f(u)] = f'(u)\frac{du}{dx}$$
$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Product Rule
$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + \frac{du}{dx}v$$

Quotient Rule
$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2}$$

"PLUS A CONSTANT"

The Fundamental Theorem of Calculus

$$\int_a^b f(x)dx = F(b) - F(a)$$
where $F'(x) = f(x)$.

Corollary to FTC

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)dt = f\left(b(x)\right)b'(x) - f\left(a(x)\right)a'(x)$$

Intermediate Value Theorem

If the function $f(x)$ is continuous on $[a,b]$, then for any number c between $f(a)$ and $f(b)$, there exists a number d in the open interval (a,b) such that $f(d) = c$.

Rolle's Theorem

If the function $f(x)$ is continuous on $[a,b]$, the first derivative exist on the interval (a,b), and $f(a) = f(b)$; then there exists a number $x = c$ on (a,b) such that
$$f'(c) = 0.$$

Mean Value Theorem

If the function $f(x)$ is continuous on $[a,b]$, and the first derivative exists on the interval (a,b), then there exists a number $x = c$ on (a,b) such that
$$f'(c) = \frac{f(b) - f(a)}{b-a}.$$

Theorem of the Mean Value

If the function $f(x)$ is continuous on $[a,b]$ and the first derivative exist on the interval (a,b), then there exists a number $x = c$ on (a,b) such that
$$f(c) = \frac{\int_a^b f(x)dx}{(b-a)}.$$
This value $f(c)$ is the "average value" of the function on the interval $[a,b]$.

Trapezoidal Rule

$$\int_a^b f(x)dx = \frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$

Solids of Revolution and friends

Disk Method
$$V = \pi\int_a^b [R(x)]^2\,dx$$

Washer Method
$$V = \pi\int_a^b \left([R(x)]^2 - [r(x)]^2\right)dx$$

Shell Method(no longer on AP)
$$V = 2\pi\int_a^b r(x)h(x)dx$$

ArcLength
$$L = \int_a^b \sqrt{1 + [f'(x)]^2}dx$$

Surface of revolution (No longer on AP)
$$S = 2\pi\int_a^b r(x)\sqrt{1 + [f'(x)]^2}dx$$

Distance, velocity and acceleration

velocity $= \frac{d}{dt}$(position).

acceleration $= \frac{d}{dt}$(velocity).

velocity vector $= \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$.

speed $= |v| = \sqrt{(x')^2 + (y')^2}$.

$$\text{Distance} = \int_{\text{initial time}}^{\text{final time}} |v|dt = \int_{t_0}^{t_f} \sqrt{(x')^2 + (y')^2}dt$$

average velocity $=$
$$\frac{\text{final position} - \text{initial position}}{\text{total time}}.$$

Integration by Parts

$$\int u dv = uv - \int v du$$

Integral of Log

$$\int \ln x dx = x \ln x - x + \text{C}.$$

Taylor Series

If the function f is "smooth" at $x = a$, then it can be approximated by the n^{th} degree polynomial

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Maclaurin Series

A Taylor Series about $x = 0$ is called Maclaurin.

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

Lagrange Error Bound

If $P_n(x)$ is the n_{th} degree Taylor polynomial of $f(x)$ about c and $\left|f^{(n+1)}(t)\right| \leq M$ for all t between x and c, then

$$|f(x) - P_n(x)| \leq \frac{M}{(n+1)!} |x-c|^{n+1}$$

Alternating Series Error Bound

If $S_N = \sum_{k=1}^{N} (-1)^n a_n$ is the N^{th} partial sum of a convergent alternating series, then

$$|S_\infty - S_N| \leq |a_{N+1}|$$

Euler's Method

If given that $\frac{dy}{dx} = f(x, y)$ and that the solution passes through (x_0, y_0),

$$y(x_0) = y_0$$

$$\vdots$$

$$y(x_n) = y(x_{n-1}) + f(x_{n-1}, y_{n-1}) \cdot \Delta x$$

In other words:

$$x_{\text{new}} = x_{\text{old}} + \Delta x$$

$$y_{\text{new}} = y_{\text{old}} + \left.\frac{dy}{dx}\right|_{(x_{\text{old}}, y_{\text{old}})} \cdot \Delta x$$

Ratio Test

The series $\sum_{k=0}^{\infty} a_k$ converges if

$$\lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| < 1.$$

If limit equals 1, you know nothing.

Polar Curves

For a polar curve $r(\theta)$, the **Area** inside a "leaf" is

$$\int_{\theta 1}^{\theta 2} \frac{1}{2} [r(\theta)]^2 d\theta,$$

where $\theta 1$ and $\theta 2$ are the "first" two times that $r = 0$.
The **slope** of $r(\theta)$ at a given θ is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{d}{d\theta}[r(\theta)\sin\theta]}{\frac{d}{d\theta}[r(\theta)\cos\theta]}$$

l'Hopital's Rule

If $\dfrac{f(a)}{g(a)} = \dfrac{0}{0}$ or $= \dfrac{\infty}{\infty}$,
then $\displaystyle\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$.

Topics to Study

Elementary Functions

Properties of Functions
A function f is defined as a set of all ordered pairs (x, y), such that for each element x, there corresponds exactly one element y.

The domain of f is the set x.
The range of f is the set y.

Combinations of Functions
If $f(x) = 3x + 1$ and $g(x) = x^2 - 1$
a) the sum $f(x) + g(x) = (3x + 1) + (x^2 - 1) = x^2 + 3x$
b) the difference $f(x) - g(x) = (3x + 1) - (x^2 - 1) = -x^2 + 3x + 2$
c) the product $f(x)g(x) = (3x + 1)(x^2 - 1) = 3x^3 + x^2 - 3x - 1$
d) the quotient $f(x)/g(x) = (3x + 1)/(x^2 - 1)$
e) the composite $(f \circ g)(x) = f(g(x)) = 3(x^2 - 1) + 1 = 3x^2 - 2$

Inverse Functions
Functions f and g are inverses of each other if

$f(g(x)) = x$ for each x in the domain of g
$g(f(x)) = x$ for each x in the domain of f

The inverse of the function f is denoted f^{-1}.
To find f^{-1}, switch x and y in the original equation and solve the equation for y in terms of x.

Exercise: If $f(x) = 3x + 2$, then $f^{-1}(x) =$
(A) $\frac{1}{3x+2}$
(B) $\frac{x}{3} - 2$
(C) $3x - 2$
(D) $\frac{1}{2}x + 3$
(E) $\frac{x-2}{3}$

The answer is E. $x = 3y + 2$
$3y = x - 2$
$y = \frac{x-2}{3}$

Even and Odd Functions
The function $y = f(x)$ is even if $f(-x) = f(x)$.
Even functions are symmetric about the y-axis (e.g. $y = x^2$)
The function $y = f(x)$ is odd if $f(-x) = -f(x)$.
Odd functions are symmetric about the origin (e.g. $y = x^3$)

Exercise: If the graph of $y = 3^x + 1$ is reflected about the y-axis, then an equation of the reflection is $y =$

(A) $3^x - 1$
(B) $\log_3 (x - 1)$
(C) $\log_3 (x + 1)$
(D) $3^{-x} + 1$
(E) $1 - 3^x$

The answer is D. The reflection of $y = f(x)$ in the y-axis is $y = f(-x)$

Periodic Functions

You should be familiar with the definitions and graphs of these trigonometric functions: sine, cosine, tangent, cotangent, secant, and cosecant

Exercise: If $f(x) = \sin(\tan^{-1} x)$, what is the range of f?

(A) $(-\pi/2, \pi/2)$
(B) $[-\pi/2, \pi/2]$
(C) $(0, 1]$
(D) $(-1, 1)$
(E) $[-1, 1]$

The answer is D. The range of $\sin x$ is (E), but the points at which $\sin x = \pm 1$ $(\pi/2 + k\pi)$, $\tan^{-1} x$ is undefined. Therefore, the endpoints are not included.

Note: The range is expressed using interval notation:

$(a,b) \Leftrightarrow a < x < b$

$[a,b] \Leftrightarrow a \leq x \leq b$

Zeros of a Function

These occur where the function $f(x)$ crosses the x-axis. These points are also called the roots of a function.

Exercise: The zeros of $f(x) = x^3 - 2x^2 + x$ is

(A) 0, -1
(B) 0, 1
(C) -1
(D) 1
(E) -1, 1

The answer is B. $f(x) = x(x^2 - 2x + 1) = x(x - 1)^2$

Properties of Graphs
You should review the following topics:
a) Intercepts
b) Symmetry
c) Asymptotes
d) Relationships between the graph of
$y = f(x)$ and $y = kf(x)$
$y = f(kx)$
$y - k = f(x - h)$
$y = |f(x)|$
$y = f(|x|)$

Limits

Properties of Limits
If b and c are real numbers, n is a positive integer, and the functions f and g have limits as $x \rightarrow c$, then the following properties are true.

1. Scalar multiple: $\lim\limits_{x\to c}[b(f(x))] = b[\lim\limits_{x\to c} f(x)]$
2. Sum or difference: $\lim\limits_{x\to c}[f(x) \pm g(x)] = \lim\limits_{x\to c} f(x) \pm \lim\limits_{x\to c} g(x)$
3. Product: $\lim\limits_{x\to c}[f(x)g(x)] = [\lim\limits_{x\to c} f(x)][\lim\limits_{x\to c} g(x)]$
4. Quotient: $\lim\limits_{x\to c}[f(x)/g(x)] = [\lim\limits_{x\to c} f(x)]/[\lim\limits_{x\to c} g(x)]$, if $\lim\limits_{x\to c} g(x) \neq 0$

One-Sided Limits
$\lim\limits_{x\to a+} f(x)$ x approaches c from the right
$\lim\limits_{x\to a-} f(x)$ x approaches c from the left

Limits at Infinity
$\lim\limits_{x\to +\infty} f(x) = L$ or $\lim\limits_{x\to -\infty} f(x) = L$
The value of $f(x)$ approaches L as x increases/decreases without bound.
$y = L$ is the <u>horizontal asymptote</u> of the graph of f.

Some Nonexistent Limits
$\lim\limits_{x\to 0}\frac{1}{x^2}$ $\lim\limits_{x\to 0}\frac{|x|}{x}$ $\lim\limits_{x\to 0}\sin\frac{1}{x}$

Some Infinite Limits
$\lim\limits_{x\to 0}\frac{1}{x^2} = \infty$ $\lim\limits_{x\to 0+}\ln x = -\infty$

Exercise: What is $\lim_{x \to 0} \frac{\sin x}{x}$?

(A) 1
(B) 0
(C) ∞
(D) $\frac{\pi}{2}$
(E) The limit does not exist.

The answer is A. You should memorize this limit.

Continuity

Definition

A function f is continuous at c if:

1. $f(c)$ is defined
2. $\lim_{x \to c} f(x)$ exists
3. $\lim_{x \to c} f(x) = f(c)$

Graphically, the function is continuous at c if a pencil can be moved along the graph of $f(x)$ through $(c, f(c))$ without lifting it off the graph.

Exercise:

If $\begin{cases} f(x) = \dfrac{3x^2 + x}{2x} & \text{for } x \neq 0 \\ f(0) = k, & \end{cases}$

and if f is continuous at $x = 0$, then $k =$

(A) -3/2
(B) -1
(C) 0
(D) 1
(E) 3/2

The answer is E. $\lim_{x \to 0} f(x) = 3/2$

Intermediate Value Theorem

If f is continuous on $[a, b]$ and k is any number between $f(a)$ and $f(b)$, then there is at least one number c between a and b such that $f(c) = k$.

Differential Calculus

Definition

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

and if this limit exists

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

If f is differentiable at $x = c$, then f is continuous at $x = c$.

Differentiation Rules

General and Logarithmic Differentiation Rules

1. $\frac{d}{dx}[cu] = cu'$
2. $\frac{d}{dx}[u \pm v] = u' \pm v'$ sum rule
3. $\frac{d}{dx}[uv] = uv' + vu'$ product rule
4. $\frac{d}{dx}[\frac{u}{v}] = \frac{vu' - uv'}{v^2}$ quotient rule
5. $\frac{d}{dx}[c] = 0$
6. $\frac{d}{dx}[u^n] = nu^{n-1}u'$ power rule
7. $\frac{d}{dx}[x] = 1$
8. $\frac{d}{dx}[\ln u] = \frac{u'}{u}$
9. $\frac{d}{dx}[e^u] = e^u u'$
10. $\frac{d}{dx}[f(g(x))] = f'(g(x))\, g'(x)$ chain rule

Derivatives of the Trigonometric Functions

1. $\frac{d}{dx}[\sin u] = (\cos u)u'$
2. $\frac{d}{dx}[\csc u] = -(\csc u \cot u)u'$
3. $\frac{d}{dx}[\cos u] = -(\sin u)u'$
4. $\frac{d}{dx}[\sec u] = (\sec u \tan u)u'$
5. $\frac{d}{dx}[\tan u] = (\sec^2 u)u'$
6. $\frac{d}{dx}[\cot u] = -(\csc^2 u)u'$

Derivatives of the Inverse Trigonometric Functions

1. $\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}}$
2. $\frac{d}{dx}[\text{arccsc } u] = \frac{-u'}{|u|\sqrt{u^2-1}}$
3. $\frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$
4. $\frac{d}{dx}[\text{arcsec } u] = \frac{u'}{|u|\sqrt{u^2-1}}$
5. $\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2}$
6. $\frac{d}{dx}[\text{arccot } u] = \frac{-u'}{1+u^2}$

Implicit Differentiation

Implicit differentiation is useful in cases in which you cannot easily solve for y as a function of x.

Exercise: Find $\frac{dy}{dx}$ for $y^3 + xy - 2y - x^2 = -2$

$$\frac{dy}{dx}[y^3 + xy - 2y - x^2] = \frac{dy}{dx}[-2]$$
$$3y^2\frac{dy}{dx} + (x\frac{dy}{dx} + y) - 2\frac{dy}{dx} - 2x = 0$$
$$\frac{dy}{dx}(3y^2 + x - 2) = 2x - y$$
$$\frac{dy}{dx} = \frac{2x - y}{3y^2 + x - 2}$$

Higher Order Derivatives

These are successive derivatives of $f(x)$. Using prime notation, the second derivative of $f(x)$, $f''(x)$, is the derivative of $f'(x)$. The numerical notation for higher order derivatives is represented by:

$$f^{(n)}(x) = y^{(n)}$$

The second derivative is also indicated by $\frac{d^2y}{dx^2}$.

Exercise: Find the third derivative of $y = x^5$.

$$y' = 5x^4$$
$$y'' = 20x^3$$
$$y''' = 60x^2$$

Derivatives of Inverse Functions

If $y = f(x)$ and $x = f^{-1}(y)$ are differentiable inverse functions, then their derivatives are reciprocals:

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

Logarithmic Differentiation

It is often advantageous to use logarithms to differentiate certain functions.

1. Take ln of both sides
2. Differentiate
3. Solve for y'
4. Substitute for y
5. Simplify

Exercise: Find $\frac{dy}{dx}$ for $y = \left(\frac{x^2+1}{x^2-1}\right)^{1/3}$

$$\ln y = \tfrac{1}{3}[\ln(x^2 + 1) - \ln(x^2 - 1)]$$
$$\frac{y'}{y} = \frac{1}{3}\left[\frac{1}{x^2+1} - \frac{1}{x^2-1}\right]$$

$$y' = \frac{-2}{(x^2+1)(x^2-1)}\left(\frac{x^2-1}{x^2+1}\right)^{1/3}$$
$$y' = \frac{-2}{(x^2+1)^{4/3}(x^2-1)^{2/3}}$$

Mean Value Theorem
If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

L'Hôpital's Rule
If lim $f(x)/g(x)$ is an indeterminate of the form 0/0 or ∞/∞, and if lim $f'(x)/g'(x)$ exists, then

$$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$$

The indeterminate form $0 \cdot \infty$ can be reduced to 0/0 or ∞/∞ so that L'Hôpital's Rule can be applied.

Note: L'Hôpital's Rule can be applied to the four different indeterminate forms of ∞/∞: ∞/∞, $(-\infty)/\infty$, $\infty/(-\infty)$, and $(-\infty)/(-\infty)$

Exercise: What is $\lim_{x \to 0} \frac{\sin x + 1}{x}$?
(A) 2
(B) 1
(C) 0
(D) ∞
(E) The limit does not exist.

The answer is B. $\lim_{x \to 0} \frac{\cos x}{1} = 1$

Tangent and Normal Lines
The derivative of a function at a point is the slope of the tangent line. The normal line is the line that is perpendicular to the tangent line at the point of tangency.

Exercise: The slope of the normal line to the curve $y = 2x^2 + 1$ at (1, 3) is
(A) -1/12
(B) -1/4
(C) 1/12
(D) 1/4
(E) 4

The answer is B. $y' = 4x$
$y = 4(1) = 4$
slope of normal = -1/4

Extreme Value Theorem
If a function $f(x)$ is continuous on a closed interval, then $f(x)$ has both a maximum and minimum value in the interval.

Curve Sketching

Situation	Indicates
$f'(c) > 0$	f increasing at c
$f'(c) < 0$	f decreasing at c
$f'(c) = 0$	horizontal tangent at c
$f'(c) = 0, f'(c^-) < 0, f'(c^+) > 0$	relative minimum at c
$f'(c) = 0, f'(c^-) > 0, f'(c^+) < 0$	relative maximum at c
$f'(c) = 0, f''(c) > 0$	relative minimum at c
$f'(c) = 0, f''(c) < 0$	relative maximum at c
$f'(c) = 0, f''(c) = 0$	*further investigation required*
$f''(c) > 0$	concave upward
$f''(c) < 0$	concave downward
$f''(c) = 0$	*further investigation required*
$f''(c) = 0, f''(c^-) < 0, f''(c^+) > 0$	point of inflection
$f''(c) = 0, f''(c^-) > 0, f''(c^+) < 0$	point of inflection
$f(c)$ exists, $f'(c)$ does not exist	possibly a vertical tangent; possibly an absolute max. or min.

Newton's Method for Approximating Zeros of a Function

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

To use Newton's Method, let x_1 be a guess for one of the roots. Reiterate the function with the result until the required accuracy is obtained.

Optimization Problems
Calculus can be used to solve practical problems requiring maximum or minimum values.

<u>Exercise</u>: A rectangular box with a square base and no top has a volume of 500 cubic inches. Find the dimensions for the box that require the least amount of material.

Let V = volume, S = surface area, x = length of base, and h = height of box

$V = x^2h = 500$
$S = x^2 + 4xh = x^2 + 4x(500/x^2) = x^2 + (2000/x)$
$S' = 2x - (2000/x^2) = 0$
$2x^3 = 2000$

$x = 10, h = 5$
Dimensions: 10 x 10 x 5 inches

Rates-of-Change Problems

Distance, Velocity, and Acceleration

$y = s(t)$ position of a particle along a line at time t

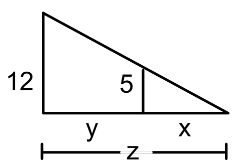
$v = s'(t)$ instantaneous velocity (rate of change) at time t

$a = v'(t) = s''(t)$ instantaneous acceleration at time t

Related Rates of Change

Calculus can be used to find the rate of change of two or more variable that are functions of time t by differentiating with respect to t.

Exercise: A boy 5 feet tall walks at a rate of 3 feet/sec toward a streetlamp that is 12 feet above the ground.
a) What is the rate of change of the tip of his shadow?
b) What is the rate of change of the length of his shadow?



$\frac{5}{x} = \frac{12}{z}$ $\frac{x+y}{x} = \frac{12}{5}$

$z = \frac{12}{5}x$ $x = \frac{5}{7}y$

$\frac{dx}{dt} = \frac{5}{7}\left(\frac{dy}{dt}\right)$ $\frac{dz}{dt} = \frac{12}{5}\left(\frac{dx}{dt}\right)$

$\frac{dx}{dt} = \frac{5}{7}(3)$ $\frac{dz}{dt} = \frac{12}{5}\left(\frac{15}{7}\right)$

b) $= \frac{15}{7}$ ft/sec a) $= \frac{36}{7}$ ft/sec

Note: the answers are independent of the distance from the light.

Exercise: A conical tank 20 feet in diameter and 30 feet tall (with vertex down) leaks water at a rate of 5 cubic feet per hour. At what rate is the water level dropping when the water is 15 feet deep?

$V = \frac{1}{3}\pi r^2 h$ $\frac{dv}{dt} = \frac{1}{9}\pi h^2 \frac{dh}{dt}$

$\frac{r}{h} = \frac{10}{30}$ $5 = \frac{1}{9}\pi h^2 \frac{dh}{dt}$

$r = \frac{1}{3}h$ $\frac{dh}{dt} = \frac{45}{\pi h^2}$

$V = \frac{1}{27}\pi h^3$ $\frac{dh}{dt} = \frac{1}{5\pi}$ ft/hr

Integral Calculus

Indefinite Integrals

Definition: A function $F(x)$ is the antiderivative of a function $f(x)$ if for all x in the domain of f,

$$F'(x) = f(x)$$

$$\int f(x)\,dx = F(x) + C, \text{ where } C \text{ is a constant.}$$

Basic Integration Formulas

General and Logarithmic Integrals

1. $kf(x)\,dx = k\,f(x)\,dx$
2. $\int [f(x) \pm g(x)]\,dx = f(x)\,dx \pm g(x)\,dx$
3. $\int k\,dx = kx + C$
4. $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C,\ n \neq -1$
5. $\int e^x\,dx = e^x + C$
6. $\int a^x\,dx = \dfrac{a^x}{\ln a} + C,\ a > 0,\ a \neq 1$
7. $\int \dfrac{dx}{x} = \ln |x| + C$

Trigonometric Integrals

1. $\int \sin x\,dx = -\cos x + C$
2. $\int \cos x\,dx = \sin x + C$
3. $\int \sec^2 x\,dx = \tan x + C$
4. $\int \csc^2 x\,dx = -\cot x + C$
5. $\int \sec x \tan x\,dx = \sec x + C$
6. $\int \csc x \cot x\,dx = -\csc x + C$
7. $\int \tan x\,dx = -\ln |\cos x| + C$
8. $\int \cot x\,dx = \ln |\sin x| + C$
9. $\int \sec x\,dx = \ln |\sec x + \tan x| + C$
10. $\int \csc x\,dx = -\ln |\csc x + \cot x| + C$
11. $\int \dfrac{dx}{\sqrt{a^2 - x^2}} = \arcsin \dfrac{x}{a} + C$
12. $\int \dfrac{dx}{a^2 + x^2} = \dfrac{1}{a} \arctan \dfrac{x}{a} + C$
13. $\int \dfrac{dx}{x\sqrt{x^2 - a^2}} = \dfrac{1}{a} \operatorname{arcsec} \dfrac{x}{a} + C$

Integration by Substitution

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C$$

If $u = g(x)$, then $du = g'(x)\,dx$ and $\int f(u)\,du = F(u) + C$

Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

Distance, Velocity, and Acceleration (*on Earth*)

$a(t) = s''(t) = -32$ ft/sec^2

$v(t) = s'(t) = \int s''(t)\,dt = \int -32\,dt = -32t + C_1$

  at $t = 0$, $v_0 = v(0) = (-32)(0) + C_1 = C_1$

$s(t) = \int v(t)\,dt = \int (-32t + v_0)\,dt = -16t^2 + v_0 t + C_2$

Separable Differential Equations

It is sometimes possible to separate variables and write a differential equation in the form $f(y)\,dy + g(x)\,dx = 0$ by integrating:

$$\int f(y)\,dy + \int g(x)\,dx = C$$

Exercise: Solve for $\frac{dy}{dx} = \frac{-2x}{y}$

$2x\,dx + y\,dy = 0$

$x^2 + \frac{y^2}{2} = C$

Applications to Growth and Decay

Often, the rate of change or a variable y is proportional to the variable itself.

$\dfrac{dy}{dt} = ky$ separate the variables

$\dfrac{dy}{y} = k\,dt$ integrate both sides

$\ln |y| = kt + C_1$

$y = Ce^{kt}$ Law of Exponential Growth and Decay

Exponential growth when $k > 0$

Exponential decay when $k < 0$

Definition of the Definite Integral

The definite integral is the limit of the Riemann sum of f on the interval $[a, b]$

$$\lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)\,dx$$

Properties of Definite Integrals

1. $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
2. $\int_a^b kf(x)\,dx + k\int_a^b f(x)\,dx$
3. $\int_a^a f(x)\,dx = 0$
4. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
5. $\int_b^a f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$
6. If $f(x) \le g(x)$ on $[a, b]$, then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$

Approximations to the Definite Integral

Riemann Sums

$\int_a^b f(x)dx = S_n = \sum_{i=1}^{n} f(x_i)\Delta x$

Trapezoidal Rule

$\int_a^b f(x)dx \approx [\frac{1}{2} f(x_0) + f(x_1) + f(x_2) + ... + f(x_{n-1}) + \frac{1}{2} f(x_n)] \dfrac{b-a}{n}$

The Fundamental Theorem of Calculus

If f is continuous on $[a, b]$ and if $F' = f$, then

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

The Second Fundamental Theorem of Calculus

If f is continuous on an open interval I containing a, then for every x in the interval,

$$\frac{d}{dx} \int_a^x f(t)\, dt = f(x)$$

Area Under a Curve

If $f(x) \geq 0$ on $[a, b]$	$A = \int_a^b f(x)\, dx$
If $f(x) \leq 0$ on $[a, b]$	$A = -\int_a^b f(x)\, dx$
If $f(x) \geq 0$ on $[a, c]$ and $f(x) \leq 0$ on $[c, b]$	$A = \int_a^c f(x)\, dx - \int_c^b f(x)\, dx$

Exercise

The area enclosed by the graphs of $y = 2x^2$ and $y = 4x + 6$ is:

(A) 76/3

(B) 32/3

(C) 80/3

(D) 64/3

(E) 68/3

The answer is D. Intersection of graphs:

$2x^2 = 4x + 6$

$2x^2 - 4x + 6 = 0$

$x = -1, 3$

$A = \int_{-1}^{3} 4x + 6 - 2x^2$

$= (2x^2 + 6x - \dfrac{2x^3}{3})\Big|_{-1}^{3}$

$= 18 + 18 - 18 - (2 - 6 + 2/3)$

$= 64/3$

Average Value of a Function on an Interval

$$\frac{1}{b-a} \int_a^b f(x)\, dx$$

Volumes of Solids with Known Cross Sections

1. For cross sections of area $A(x)$, taken perpendicular to the x-axis:

$V = \int_a^b A(x)\, dx$

2. For cross sections of area $A(y)$, taken perpendicular to the y-axis:

$V = \int_a^b A(y)\, dy$

Volumes of Solids of Revolution: Disk Method

$V = \int_a^b \pi\, r^2\, dx$

Rotated about the x-axis: $V = \int_a^b \pi\, [f(x)]^2\, dx$

Rotated about the y-axis: $V = \int_a^b \pi\, [f(y)]^2\, dy$

Volumes of Solids of Revolution: Washer Method

$V = \int_a^b \pi\, (r_o^2\, dx - r_i^2\,)\, dx$

Rotated about the x-axis: $V = \int_a^b \pi\, [(f_1(x))^2 - (f_2(x))^2]\, dx$

Rotated about the y-axis: $V = \int_a^b \pi\, [(f_1(y))^2 - (f_2(y))^2]\, dy$

<u>Exercise</u>: Find the volume of the region bounded by the y-axis, $y = 4$, and $y = x^2$ if it is rotated about the line $y = 6$.

$\pi \int_0^2 [(x^2 - 6)^2 - (4 - 6)^2\,]dx$

$= \dfrac{192\pi}{5}$ cubic units

Volumes of Solids of Revolution: Cylindrical Shell Method

$V = \int_a^b 2\pi\, rh\, dr$

Rotated about the x-axis: $V = 2\pi \int_a^b xf(x)\, dx$

Rotated about the y-axis: $V = 2\pi \int_a^b yf(y)\, dy$

Some Useful Formulas

$\log_a x = \dfrac{\log x}{\log a}$

$\sin^2 x + \cos^2 x = 1$
$1 + \tan^2 x = \sec^2 x$
$1 + \cot^2 x = \csc^2 x$

$\sin 2x = 2 \sin x \cos x$
$\cos 2x = \cos^2 x - \sin^2 x$

$\sin^2 x = ½(1- \cos 2\theta)$
$\cos^2 x = ½(1+ \cos 2\theta)$

Volume of a right circular cylinder = $\pi r^2 h$
Volume of a cone = $\frac{1}{3}\pi r^2 h$
Volume of a sphere = $\frac{4}{3}\pi r^3$

Calculator Tips and Programs

Your calculator will serve as an extremely useful tool if you take advantage of all of its functions. We will base all of the following tips and programs on the TI-82, which most calculus students use today.

On the AP Calculus Exam, you will need your calculator for Part B of Section I and for Section II. You will need to know how to do the following:
1. simple calculations
2. find the intersection of two graphs
3. graph a function and be able to find properties listed under *Elementary Functions* in the *Topics to Study* section (e.g. domain, range, asymptotes)

Here are some of the functions available that you should know how to use:

In the **CALC** menu:
1. calculate the value of a function at x = c
2. calculate the roots of a function
3. find the minimum of a function
4. find the maximum of a function
5. find the point of intersection of two functions
6. find the slope of the tangent at (x, y)
7. find the area under the curve from a to b

In the **MATH MATH** menu:

6. find the minimum of a function
fMin(*expression***,** *variable***,** *lower***,** *upper***)**

7. find the maximum of a function
fMax(*expression***,** *variable***,** *lower***,** *upper***)**

8. find the numerical derivative at a given value
nDeriv(*expression***,** *variable***,** *value***)**

9. find the numerical integral of an expression
fnInt(*expression***,** *variable***,** *lower***,** *upper***)**

0. calculate the root of an expression
solve(*expression***,** *variable***,** *guess***, {***lower***,** *upper***})**

Calculator Programs

One of the easiest programs to create is one that will solve for f(x). You can also run the program multiple times to find other values for the same function.

```
PROGRAM: SOLVE
: Input X
: 3x^2 + 2 → X          [type your function here and place → X at the end]
: Disp X
```

Here is a program to solve for a quadratic equation:

```
PROGRAM: QUADRAT
:Input "A? ", A
:Input "B? ", B
:Input "C? ", C
:(- B + √ (B² - 4AC) ) / 2A → D
:(- B - √ (B² - 4AC) ) / 2A → E
:B² - 4AC → F
:ClrHome
:Disp "+ EQUALS"
:Disp D
:Disp "- EQUALS"
:Disp E
:Disp "B² - 4AC EQUALS"
:Disp F
```

To Run: Enter a, b, and c for $ax^2 + bx + c$.
"+ EQUALS" and "- EQUALS" give the roots of the equation

Here is a program that will use the trapezoidal rule to approximate a definite integral:

```
PROGRAM: TRAP
:ClrHome
:Input "F(X) IN QUOTES:", Y0
:Input "START(A):", A
:Input "END(B):", B
:Input "NO. OF DIV. (N):", N
:(B - A) / N → D
:0 → S
:For (X, A, B, D)
:S + Y0 → S
:End
:A → X : Y0 → F
:B → X : Y0 → L
:D * (-F + S - L) → A
:ClrHome
:Disp "EST AREA="
:Disp A
```

AP Calculus AB
Exam Review Sheet A - Session 1

Name: ________________________

1) A container has the shape of an open right circular cone. The height of the container is 10 cm and the diameter of the opening is 10 cm. Water in the container is evaporating so that its depth h is changing at the constant rate of $-\dfrac{3}{10}$ cm/hr. (The volume of a cone is $V = \dfrac{1}{3}\pi r^2 h$.)

a) Find the volume V of water in the container when $h = 5$ cm. Indicate units of measure.

b) Find the rate of change of the volume of water in the container, with respect to time, when $h - 5$ cm. Indicate units of measure.

c) Show that the rate of change of the volume of water in the container due to evaporation is directly proportional to the exposed surface area of the water. What is the constant of proportionality?

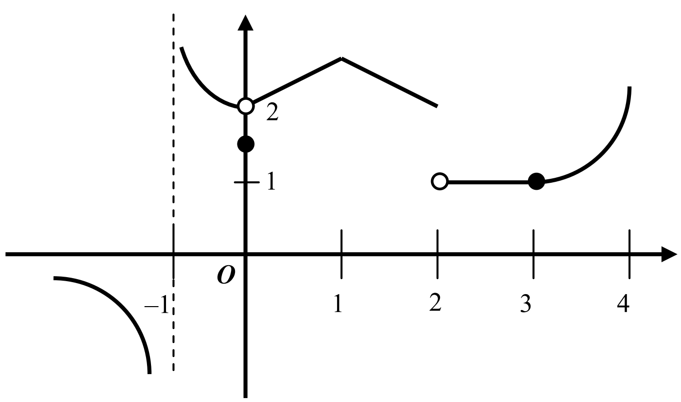
2) What is $\lim_{h\to 0}\dfrac{\cos\left(\dfrac{3\pi}{2}+h\right)-\cos\left(\dfrac{3\pi}{2}\right)}{h}$?

$a)$ 1 $\qquad b)$ $\dfrac{\sqrt{2}}{2}$ $\qquad c)$ 0 $\qquad d)$ -1 $\qquad e)$ The limit does not exist.

---

3) What is $\lim_{x\to\infty}\dfrac{x^2-4}{2+x-4x^2}$?

$a)$ -2 $\qquad b)$ $-\dfrac{1}{4}$ $\qquad c)$ $\dfrac{1}{2}$ $\qquad d)$ 1 $\qquad e)$ The limit does not exist.

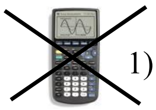


4) The graph of a function f is shown above. If $\lim_{x\to b} f(x)$ exists and f is not continuous at b, then $b =$

$a)$ -1 $\qquad b)$ 0 $\qquad c)$ 1 $\qquad d)$ 2 $\qquad e)$ 3

AP Calculus AB Name: ________________________

Exam Review Sheet A - Session 2



1) Consider the curve given by $x^2 + 4y^2 = 7 + 3xy$.

a) Show that $\frac{dy}{dx} = \frac{3y - 2x}{8y - 3x}$

b) Show that there is a point P with x-coordinate 3 at which the line tangent to the curve at P is horizontal. Find the y-coordinate of P.

c) Find the value of $\frac{d^2y}{dx^2}$ at the point P found in part (b). Does the curve have a local maximum, a local minimum, or neither at the point P? Justify your answer.

2) If $f'(x) = \sin\left(\dfrac{\pi e^x}{2}\right)$ and $f(0)=1$, then $f(2)=$

$a)\ -1.819 \qquad b)\ -0.843 \qquad c)\ -0.819 \qquad d)\ 0.157 \qquad e)\ 1.157$

3) The solution to the differential equation $\dfrac{dy}{dx} = \dfrac{x^3}{y^2}$, where $y(2)=3$, is

$a)\ y = \sqrt[3]{\dfrac{3}{4}x^4} \qquad b)\ y = \sqrt[3]{\dfrac{3}{4}x^4} + \sqrt[3]{15} \qquad c)\ y = \sqrt[3]{\dfrac{3}{4}x^4} + 15$

$d)\ y = \sqrt[3]{\dfrac{3}{4}x^4 + 5} \qquad e)\ y = \sqrt[3]{\dfrac{3}{4}x^4 + 15}$

---

x	1.1	1.2	1.3	1.4
$f(x)$	4.18	4.38	4.56	4.74

4) Let f be a function such that $f''(x) < 0$ for all x in the closed interval [1, 2]. Selected values of f are shown in the table above. Which of the following must be true about $f'(1.2)$?

$a)\ f'(1.2) < 0 \qquad b)\ 0 < f'(1.2) < 1.6 \qquad c)\ 1.6 < f'(1.2) < 1.8$

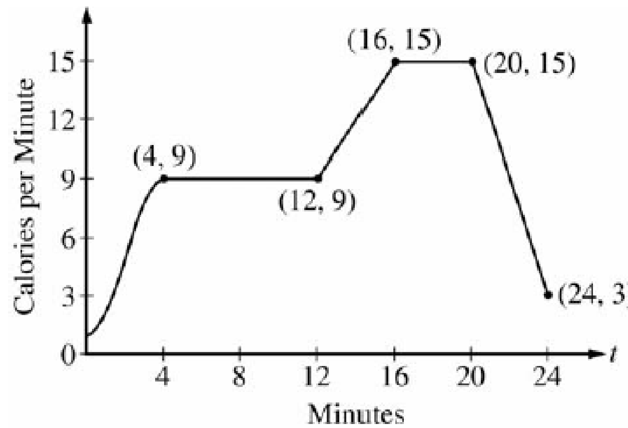
$d)\ 1.8 < f'(1.2) < 2.0 \qquad e)\ f'(1.2) > 2.0$

AP Calculus AB
Exam Review Sheet A - Session 3

Name: ________________________

1)

Question 4



The rate, in calories per minute, at which a person using an exercise machine burns calories is modeled by the function f. In the figure above, $f(t) = -\frac{1}{4}t^3 + \frac{3}{2}t^2 + 1$ for $0 \le t \le 4$ and f is piecewise linear for $4 \le t \le 24$.

(a) Find $f'(22)$. Indicate units of measure.

(b) For the time interval $0 \le t \le 24$, at what time t is f increasing at its greatest rate? Show the reasoning that supports your answer.

(c) Find the total number of calories burned over the time interval $6 \le t \le 18$ minutes.

(d) The setting on the machine is now changed so that the person burns $f(t) + c$ calories per minute. For this setting, find c so that an average of 15 calories per minute is burned during the time interval $6 \le t \le 18$.

2) Oil is leaking from a tanker at the rate of $R(t)=2,000e^{-0.2t}$ gallons per hour, where t is measured in hours. How much oil leaks out of the tanker from time $t=0$ to $t=10$?

a) 54 gallons *b)* 271 gallons *c)* 865 gallons
d) 8,647 gallons *e)* 14,778 gallons

---

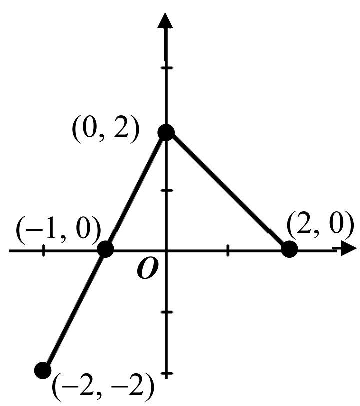
3) $\int (x-1)\sqrt{x}\,dx =$

$a)\ \frac{3}{2}\sqrt{x}-\frac{1}{\sqrt{x}}+C$ $b)\ \frac{2}{3}x^{\frac{3}{2}}+\frac{1}{2}x^{\frac{1}{2}}+C$ $c)\ \frac{1}{2}x^2-x+C$

$d)\ \frac{2}{5}x^{\frac{5}{2}}-\frac{2}{3}x^{\frac{3}{2}}+C$ $e)\ \frac{1}{2}x^2+2x^{\frac{3}{2}}-x+C$

---

4)



Graph of f

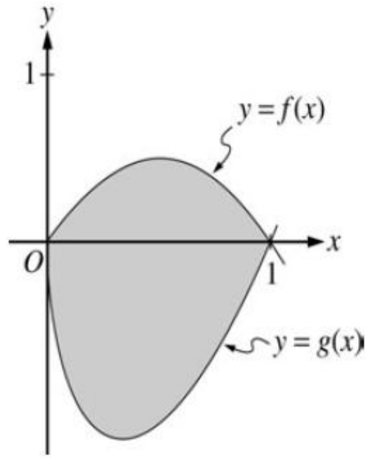
The graph of the function f shown above consists of two line segments. If g is the function defined by $g(x)=\int_0^x f(t)\,dt$, then $g(-1)=$

$a)\ -2$ $b)\ -1$ $c)\ 0$ $d)\ 1$ $e)\ 2$

AP Calculus AB
Exam Review Sheet A - Session 4

Name: ________________________

1)



Let f and g be the functions given by $f(x) = 2x(1-x)$ and $g(x) = 3(x-1)\sqrt{x}$ for $0 \le x \le 1$. The graphs of f and g are shown un the figure above.

a) Find the area of the shaded region enclosed by the graphs of f and g.

b) Find the volume of the solid generated when the shaded region enclosed by the graphs of f and g is revolved about the horizontal line $y = 2$.

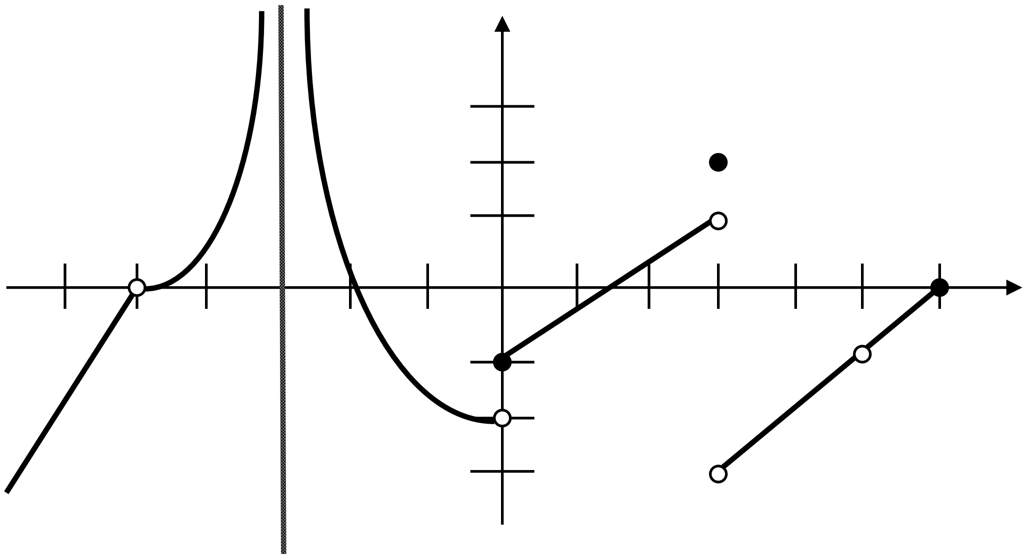
c) Let h be the function given by $h(x) = kx(1-x)$ for $0 \le x \le 1$. For each $k > 0$, the region (not shown) enclosed by the graphs of h and g is the base of a solid with square cross sections perpendicular to the x-axis. There is a value of k for which the volume of this solid is equal to 15. Write, but to not solve, an equation involving an integral expression that could be used to find the value of k.

Name___________________________

AP Calculus BC **Summer Review Packet (Limits & Derivatives)**

Limits

1. Answer the following questions using the graph of f(x) given below.



(a) Find $f(0)$

(b) Find $f(3)$

(c) Find $\lim\limits_{x \to -5} f(x)$

(d) Find $\lim\limits_{x \to 0^+} f(x)$

(e) Find $\lim\limits_{x \to 3^-} f(x)$

(f) Find $\lim\limits_{x \to -3^+} f(x)$

(g) List all x-values for which f(x) has a removable discontinuity. Explain what section(s) of the definition of continuity is (are) violated at these points.

(h) List all x-values for which $f(x)$ has a nonremovable discontinuity. Explain what section(s) of the definition of continuity is (are) violated at these points.

In problems 2-10, find the limit (if it exists) using analytic methods (i.e. without using a calculator).

2. $\lim\limits_{x \to -2} \dfrac{3x^2 + 21x + 30}{x^3 + 8}$

3. $\lim\limits_{x \to \pi/6} \dfrac{1 - \cos^2 x}{4x}$

4. $\lim\limits_{x \to 4} \dfrac{\sqrt{x-3} - 1}{x - 4}$

5. $\lim\limits_{x \to 0} \dfrac{[1/(x+1)] - 1}{x}$

6. $\lim_{x \to 0} \frac{\left[1/\sqrt{1+x}\right]-1}{x}$

7. $\lim_{\theta \to 0} \frac{\sin 6\theta^3}{7\theta}$

8. $\lim_{t \to 0} \frac{\sin^2 3t^2}{t^3}$

9. $\lim_{x \to 6^-} \frac{|6x-36|}{6-x}$

10. $\lim_{\Delta x \to 0} \frac{\sin((\pi/6)+\Delta x)-(1/2)}{\Delta x}$

Hint: $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

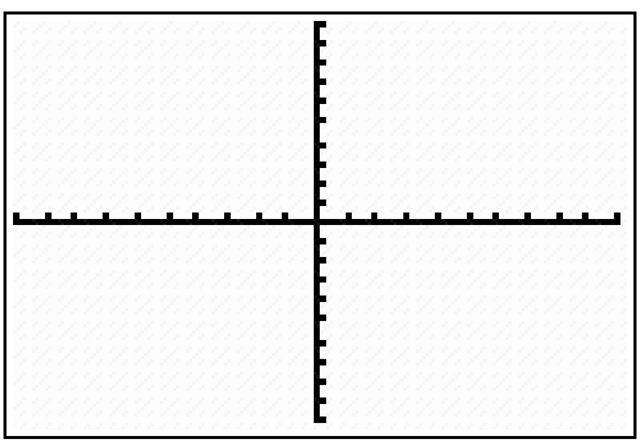
11. **Suppose** $f(x)=\begin{cases} \dfrac{\sqrt{2x+1}-\sqrt{3}}{x-1}, x\geq 0 \\ 4x^2+k, x<0 \end{cases}$.

(a) For what value of *k* will *f* be piecewise continuous at *x* = 0? Explain why this is true using one-sided limits. (**Hint: A function is continuous at**

***x* =*c* if (1) *f*(*c*) exists, (2)** $\lim\limits_{x\to c} f(x)$ **exists, and (3)** $\lim\limits_{x\to c} f(x)=f(c)$ **.)**

(b) Using the value of *k* that you found in part (a), **accurately** graph *f* below. Approximate the value of $\lim\limits_{x\to 1} f(x)$

$\lim\limits_{x\to 1} f(x) =$ ________________



(c) Rationalize the numerator to find $\lim\limits_{x\to 1} f(x)$ analytically.

12. **Analytically** determine the values of *b* and *c* such that the function *f* is continuous on the entire real number line. **See the hint given in problem 11.**

$$f(x)=\begin{cases} x+1, 1<x<3 \\ x^2+bx+c, x<1 \text{ or } x>3 \end{cases}$$

In problem 13, circle the correct answer and explain why the answer is the correct one.

13. If $f(x)=x^3+x-3$, and if *c* is the only real number such that *f(c)* = 0, then by the Intermediate Value Theorem, *c* is necessarily between

(A) -2 and -1

(B) -1 and 0

(C) 0 and 1

(D) 1 and 2

(E) 2 and 3

Hint: The Intermediate Value Theorem states that if *f* is a continuous function on the interval [a, b] and *k* is any number between *f(a)* and *f(b)*, then there must exist at least one number $c\in[a,b]$ such that *f(c) = k.*

<u>Derivatives</u>

In problems 1 & 2, find the derivative of the function by using the limit definition of the derivative.

1. $f(x) = x^3 - 2x + 3$

2. $f(x) = \dfrac{x+1}{x-1}$

In problems 3-14, find the derivative of the given function using the power, product, quotient, and/or chain rules.

3. $f(x) = (3x^2 + 7)(x^2 - 2x + 3)$

4. $f(x) = \sqrt{x}\sin x$

5. $f(t) = t^3 \cos t$

6. $f(x) = \dfrac{x^2 + x - 1}{x^2 - 1}$

7. $f(x) = \dfrac{x^4 + x}{\tan^2 x}$

8. $f(x) = 3x^2 \sec^3 x$

9. $f(x) = 3x\csc x + x\cot x$

10. $f(x) = \left(\dfrac{x+5}{x^2 - 6x}\right)^2$

11. $f(x) = (x^3 - 2)^{3/2}(5x^2 + 1)^{5/2}$

12. $f(x) = x^3 \cot^4(7x)$

13. $f(x) = 5\sin^2(\sqrt{3x^4 + 1})$

Problems continue on the next page.

In problems 14 & 15, find an equation of the tangent line to the graph of *f* at the indicated point *P*.

14. $f(x)=\frac{1+\cos x}{1-\cos x}, P\left(\frac{\pi}{2},1\right)$

15. $f(x)=\left(x^2-1\right)^{2/3}, P(3,4)$

In problems 16 & 17, find the second derivative of the given function.

16. $f(x)=(4x^2-3x)^{3/2}$

17. $h(x)=x^3\cos(\pi x)$

In problem 18, use the position function $s(t) = -16t^2 + v_0 t + s_0$ for free-falling objects.

18. A ball is thrown straight down from the top of a 220-foot tall building with an initial velocity of -22 feet per second.

(a) Determine the average velocity of the ball on the interval [1, 2].

(b) Determine the instantaneous velocity of the ball at $t = 3$.

(c) Determine the time t at which the average velocity on [0, 2] equals the instantaneous velocity.

(d) What is the velocity of the ball when it strikes the ground?

In problem 19-24, circle the correct answer and explain why the answer is the correct one.

19. $\lim_{h\to 0}\dfrac{\cos\left(\dfrac{\pi}{6}+h\right)-\cos\left(\dfrac{\pi}{6}\right)}{h}=$

(A) Does not exist

(B) $\dfrac{1}{2}$

(C) $-\dfrac{1}{2}$

(D) $\dfrac{\sqrt{3}}{2}$

(E) $-\dfrac{\sqrt{3}}{2}$

20. Let *f* and *g* be differentiable functions with values for *f(x), g(x), f'(x), and g'(x)* shown below for *x* = 1 and *x* = 2.

x	***f(x)***	***g(x)***	***f'(x)***	***g'(x)***
1	4	-4	12	-8
2	5	1	-6	4

Find the value of the derivative of $f(x) \bullet g(x)$ at *x* = 1.

(A) -96

(B) -80

(C) -48

(D) -32

(E) 0

21. Let $f(x) = \begin{cases} 3x^2 + 4, x < 1 \\ x^3 + 3x, x \geq 1 \end{cases}$. Which of the following is true?

I. *f(x)* is continuous at *x* = 1

II. *f(x)* is differentiable at *x* = 1

III. $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^+} f(x)$

(A) I only

(B) II only

(C) III only

(D) I and III only

(E) II and III only

22. The equation of the line tangent to the curve $f(x)=\dfrac{kx+8}{k+x}$ at *x* = -2 is *y* = *x* + 4. What is the value of *k*?

(A) -3

(B) -1

(C) 1

(D) 3

(E) 4

23. An equation of the line <u>normal</u> to the curve $y=\sqrt[3]{x^2-1}$ at the point where *x* = 3 is

(A) $y+12x=38$

(B) $y-4x=10$

(C) $y+2x=4$

(D) $y+2x=8$

(E) $y-2x=-4$

Hint: A <u>normal line</u> to a curve at a point is perpendicular to the tangent line to the curve at the same point.

24. If the nth derivative of *y* is denoted as $y^{(n)}$ and $y=-\sin x$, then $y^{(14)}$ is the same as

(A) *y*

(B) $\frac{dy}{dx}$

(C) $\frac{d^2y}{dx^2}$

(D) $\frac{d^3y}{dx^3}$

(E) None of the above

(1) Let $f(x) = \dfrac{x}{|x^2 - 9|}$.

(a) Find the domain of f.

(b) Write an equation for each vertical asymptote of the graph of f.

(c) Write an equation for each horizontal asymptote of the graph of f.

(d) Is f odd, even, or neither? Justify your answer.

(e) Find all values of x for which f is discontinuous and classify each discontinuity as removable or non-removable.

(2) Let $f(x) = \begin{cases} x^2 - a^2x & \text{if } x < 2, \\ 4 - 2x^2 & if x \geq 2. \end{cases}$

(a) Find the $\lim\limits_{x \to 2^-} f(x)$.

(b) Find the $\lim\limits_{x \to 2^+} f(x)$.

(c) Find all values of a that make f continuous at 2. Justify your answer.

(3) Let $f(x) = \dfrac{x^3 - 2x^2 + 1}{x^2 + 3}$.

(a) Find all zeros of f.

(b) Determine $\lim_{x \to \infty} f(x)$.

(4) Suppose that a function f and its first derivative have the following values at $x = 0$ and $x = 1$.

x	$f(x)$	$f'(x)$
0	9	-2
1	-3	4

Find the first derivative of the following functions at the given value of x.

(a) $3f(x), x = 1$

(b) $xf(x),\ x = 1$

(c) $x^2 f(x),\ x = 1$

(d) $\dfrac{f(x)}{x},\ x = 1$

(e) $\dfrac{f(x)}{x^2 + 2},\ x = 0$

(f) $f(x) \cdot f(x),\ x = 0$

(5) A particle moves along the x-axis so that at any time $t \geq 0$, its position is given by $x(t) = t^3 - 12t + 5$.

(A) Find the velocity of the particle at any time t.
(B) Find the acceleration of the particle at any time t.
(C) Find all values of t for which the particle is at rest.
(D) Find the speed of the particle when its acceleration is zero.
(E) Is the particle moving toward the origin or away from the origin when $t = 3$. Justify you answer.

(6) The function f is differentiable for all real numbers and satisfied the conditions in the table below. The function g is defined by $g(x) = \dfrac{f(x)}{f(x) - 3}$.

x	$f(x)$	$f'(x)$
2	-1	3
4	3	5

(a) Write an equation of the line tangent to the graph of f at $x = 4$.
(b) Is f continuous at $x = 3$? Justify your answer.
(c) Is there a zero of f in the interval $[2, 4]$? Justify your answer.
(d) Find $g'(2)$.
(e) Explain why g is not differentiable at $x = 4$.

(7) Let $f(x) = \dfrac{\cos x}{\cos x - 2}$ for $-2\pi \leq x \leq 2\pi$.

(a) Sketch a graph of f in the window $[-2\pi, 2\pi]$ by $[-2, 2]$.

(b) Find $f'(x)$.

(c) Find all values in the domain of f for which $f'(x) = 0$.

(d) Use information obtained from parts (a) and (c) to find the range of f.

(8) A function f and its first and second derivatives are defined for all real numbers, and it is given that $f(0) = 2, f'(0) = 3$, and $f''(0) = -1$.

(a) Define a function g by $g(x) = e^{kx} + f(x)$, where k is a constant. Find $g'(0)$ and $g''(0)$ in terms of k. Show your work.

(b) Define a function h by $h(x) = \cos(bx)f(x)$, where b is a constant. Find $h'(x)$ and write an equation for the line tangent to the graph of h at $x = 0$.

(9) Let $\dfrac{e^x + e^{-x}}{2}$.

(a) Find $\dfrac{dy}{dx}$.

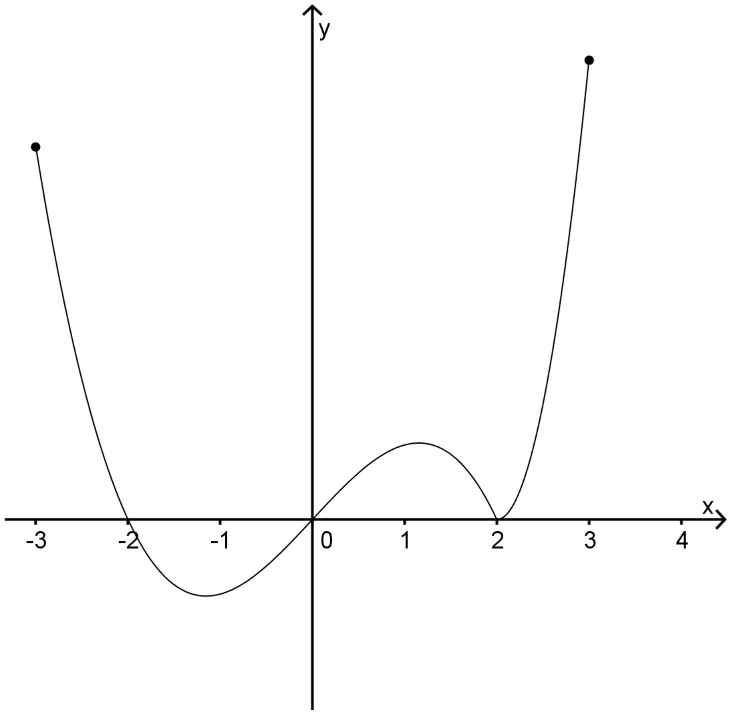
(b) Find $\dfrac{d^2y}{dx^2}$.

(c) Find an equation of the line tangent to the curve at $x = 1$.

(d) Find an equation of the line normal (perpendicular to the tangent line) to the curve at $x = 1$.

(e) Find any points where the tangent line is horizontal.

(10) The accompanying figure shows the graph of the derivative of a function f. The domain of f is the closed interval $[-3.3]$.



(a) For what values of x in the open interval $(-3, 3)$ does f have a relative maximum? Justify your answer.

(b) For what values of x in the open interval $(-3, 3)$ does f have a relative minimum? Justify your answer.

(c) For what values of x is the graph of f concave up? Justify your answer.

(d) Suppose $f(-3) = 0$. Sketch a possible graph of f on the domain $[-3, -3]$.

(11) The volume V of a cone $\left(V = \dfrac{1}{3}\pi r^2 h\right)$ is increasing at the rate of 4π cubic inches per second. At the instant when the radius of the cone is 2 inches, its volume is 8π cubic inches and the radius is increasing at $1/3$ inch per second.

(a) At the instant when the radius of the cone is 2 inches, what is the rate of change of the area of its base.

(b) At the instant when the radius of the cone is 2 inches, what is the rate of change of its height h.

(c) At the instant when the radius of the cone is 2 inches, what is the instantaneous rate of change of the area of its base with respect to its height h?

(12) The rate at which water flows out of a pipe is given by a differentiable function R of time t. The table below records the rate at 4-hour intervals for a 24-hour period.

t (hours)	$R(t)$ (gallons per hour)
0	9.6
4	10.3
8	10.9
12	11.1
16	10.9
20	10.5
24	9.6

(a) Use the Trapezoidal Rule with 6 subdivisions of equal length to approximate $\int_0^{24} R(t)\, dt$. Explain the meaning of your answer in terms of water flow, using correct units.

(b) Is there some time t between 0 and 24 such that $R'(t) = 0$? Justify your answer.

(c) Suppose the rate of water flow is approximated by $Q(t) = 0.01(950 + 25x - x^2)$. Use $Q(t)$ to approximate the average rate of water flow during the 24-hour period. Indicate units of measure.

(13) Let f be a differentiable function with the following properties:

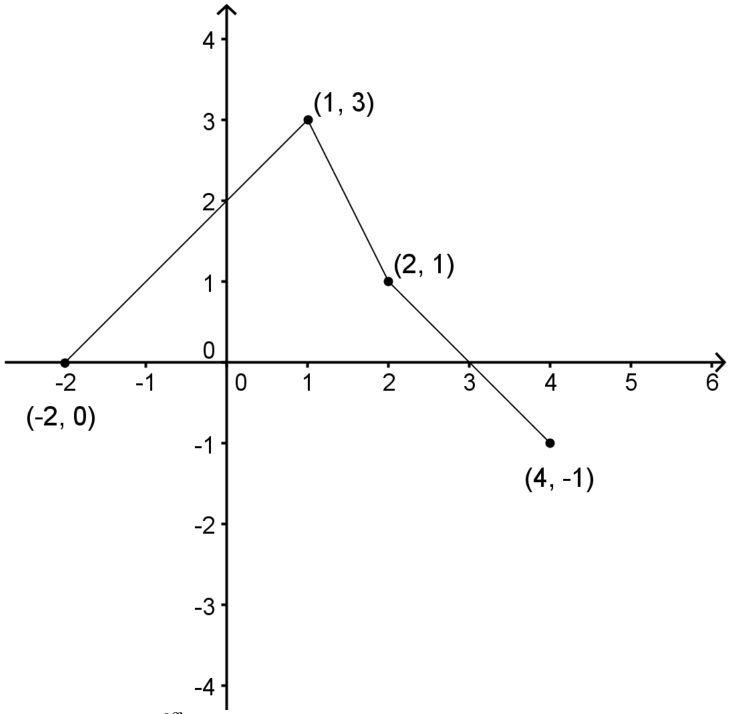
i. $f'(x) = ax^2 + bx$

ii. $f'(1) = -6$ and $f''(1) = 6$

iii. $\int_1^2 f(x)\, dx = 14$

Find $f(x)$. Show your work.

(14) The graph of the function f, consisting of three line segments, is shown below.



Let $g(x) = \int_1^x f(t)\ dt$.

i. Compute $g(4)$ and $g(-2)$.
ii. Find the instantaneous rate of change of g, with respect to x at $x = 2$.
iii. Find the absolute minimum value of g on the closed interval $[-2, 4]$. Justify your answer.
iv. The second derivative of g is not defined at $x = 1$ and $x = 2$. Which of these values are x-coordinates of points of inflection of the graph of g. Justify your answer.

(15) A population P of wolves at time t years ($t \geq 0$) is increasing at a rate directly proportional to $600 - P$, where the constant of proportionality is k.

(a) If $P(0) = 200$, find $P(t)$ in terms of t and k.

(b) If $P(2) = 500$, find k.

(c) $\lim_{t\to\infty} P(t)$.

(16) Let $v(t)$ be the velocity, in feet per second, of a skydiver at time t seconds, $t \geq 0$. After her parachute opens, her velocity satisfies the differential equation $dv/dt = -2(v + 17)$, with initial condition $v(0) = -47$.

(a) Use separation of variable to find an expression for v in terms of t, where t is measured in seconds.

(b) Terminal velocity is defined as $\lim_{t\to\infty} v(t)$. Find the terminal velocity of the skydiver to the nearest foot per second.

(c) It is safe to land when her speed is 20 feet per secongbh d. At what time t does she reach this speed?

(17) Let R be the region in the first quadrant enclosed by the $y-$axis and the graphs of $y = 2 + \sin x$ and $y = \sec x$.

(a) Find the area of R.

(b) Find the volume of the solid generated when R is revolved about the x-axis.

(c) Find the volume of the solid whose base is R and whose cross sections cut by planes perpendicular to the x-axis are squares.