Chapter 5

Superfluidity

Superfluidity is closely related to Bose-Einstein condensation. In a phenomenological level, superfluid can flow through narrow capillaries or slits without dissipating energy. Superfluid does not possess the shear viscosity. The superfluid of liquid ^{4}He, below the so-called λ-point, was discovered by Kapitza [1] and independently by Allen and Misener [2]. Soon after Landau explained that if the excitation spectrum satisfies certain criteria, the motion of the fluid does not cause the energy dissipation [3]. These Landau criteria are met by the Bogoliubov excitation spectrum associated with the Bose-Einstein condensate consisting of an interacting Bose gas and thus establish the first connection between superfluidity and BEC. The connection between the two phenomena is further established in a deeper level through the relationship between irrotationality of the superfluid and the global phase of the BEC order parameter. This is the first subject of this chapter. The second subject of this chapter is the rotational properties of the irrotational superfluid, with special focus on the quantized vortices. We will conclude this chapter with the concept of superfluidity and BEC in a uniform 2D system, known as the Berezinskii-Kousterlitz-Thouless phase transition.

5.1 Landau's criteria of superfluidity

Landau's theory of superfluids is based on the Galilean transformation of energy and momentum. Let E and $\mathbf{P}$ be the energy and momentum of the fluid in a reference frame K. If we try to express the energy and momentum of the same fluid but in a moving frame K', which has a relative velocity $\mathbf{V}$ with respect to a reference frame K, we have the following relations:

$$\mathbf{P}' \equiv \mathbf{P} - M\mathbf{V}, \tag{5.1}$$

$$\begin{aligned} E' \equiv \frac{|\mathbf{P}'|^2}{2M} &= \frac{1}{2M}|\mathbf{P} - M\mathbf{V}|^2 \\ &= E - \mathbf{P}\cdot\mathbf{V} + \frac{1}{2}M|\mathbf{V}|^2, \end{aligned} \tag{5.2}$$

where $E = \frac{|\mathbf{P}'|^2}{2M}$ and M is the total mass of the fluid.

We first consider a fluid at zero temperature, in which all particles are in the ground state and flowing along a capillary at constant velocity $\mathbf{v}$. If the fluid is viscous, the motion will produce dissipation of energy via friction with the capillary wall and decrease of the kinetic energy. We assume that such dissipative processes take place through the creation of elementary excitation, which is the Bogoliubov quasi-particle for the case of an interacting Bose gas. Let us first describe this process in the reference frame K which, rather confusingly, moves with the same velocity v of the fluid. In this reference frame, the fluid is at rest. If a single elementary excitation with a momentum $\mathbf{p}$ appears in the fluid, the total energy of the fluid in the reference frame K is $E_0 + \varepsilon(\mathbf{p})$, where E_0 and $\varepsilon(\mathbf{p})$ are the ground state energy and the elementary excitation energy. Let us move to the moving frame K', in which the fluid moves with a velocity $\mathbf{v}$ but the capillary is at rest. In this moving frame K' which moves with the velocity $-\mathbf{v}$ with respect to the fluid, the energy and momentum of the fluid are given, setting $\mathbf{V} = -\mathbf{v}$ in (5.2) and (5.1), by

$$\mathbf{P}' = \mathbf{p} + M\mathbf{v}. \tag{5.3}$$

$$E' = E_0 + \varepsilon(\mathbf{p}) + \mathbf{p}\cdot\mathbf{v} + \frac{1}{2}M|\mathbf{v}|^2, \tag{5.4}$$

The above results indicate that the changes in energy and momentum caused by the appearance of one elementary excitation are $\varepsilon(\mathbf{p}) + \mathbf{p}\cdot\mathbf{v}$ and $\mathbf{p}$, respectively.

Spontaneous creation of elementary excitations, i.e. energy dissipation, can occur if and only if such a process is energetically favorable. This means if the energy of an elementary excitation, in the moving frame K' where the capillary is at rest, so that a thermal equilibrium condition is satisfied, is negative:

$$\varepsilon(\mathbf{p}) + \mathbf{p}\cdot\mathbf{v} < \mathbf{0}, \tag{5.5}$$

the dissipation of energy occurs. The above condition is satisfied when $|\mathbf{v}| > \frac{\varepsilon(\mathbf{p})}{|\mathbf{p}|}$ and $\mathbf{p}\cdot\mathbf{v} < \mathbf{0}$, i.e. when the elementary excitation has the momentum $\mathbf{p}$ opposite to the fluid velocity $\mathbf{v}$ and the fluid velocity $|\mathbf{v}|$ exceeds the critical value,

$$v_c = \min_{\mathbf{p}} \frac{\varepsilon(\mathbf{p})}{|\mathbf{p}|}, \tag{5.6}$$

where the minimum is calculated over all the values of $\mathbf{p}$. If instead the fluid velocity $\mathbf{v}$ is smaller than (5.6), then no elementary excitation will be spontaneously formed. Thus, the Landau's criteria of superfluidity is summarized as the relative velocity between the fluid and the capillary is smaller than the critical value, $v < v_c$.

By looking at the Bogoliubov excitation spectrum in the previous chapter 4, one can easily conclude that the weakly interacting Bose gas at zero temperature satisfies the Landau's criteria of superfluidity and that the critical velocity is given by the sound velocity as shown in Fig. 5.1(a). Strongly interacting fluids such as liquid ^{4}He also fulfil the Landau criteria but in this case the critical velocity is smaller than the sound velocity due to the complicated excitation spectrum, as shown in Fig. 5.1(b). It is easily understood that the critical velocity decreases with the decrease in the particle-particle interaction and disappears in the limit of an ideal gas because $v_c = \min_p \frac{\varepsilon(p)}{|p|} = 0$ for $\varepsilon(p) = \frac{p^2}{2m}$. The particle-particle interaction is a crucial requirement in the appearance of superfluidity.

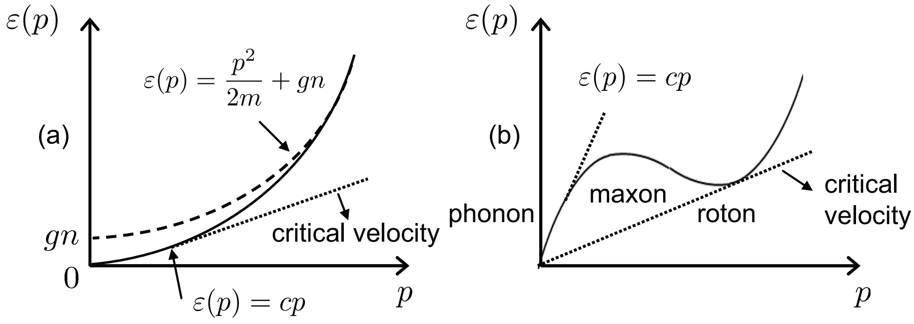

Figure 5.1: (a) The excitation spectrum of a weakly interacting Bose gas, for which the critical velocity is equal to the sound velocity, $v_c = c$. (b) The excitation spectrum of a strongly interacting Bose liquid, for which the critical velocity is smaller than the sound velocity, $v_c < c$.

5.2 Superfluidity at finite temperatures

Let us next consider a uniform Bose-Einstein condensed fluid at a finite temperature. We assume the thermodynamic properties of the system are described by the Bogoliubov quasi-particles in thermal equilibrium distributions. According to the above argument, no new excitations can be created directly by the condensate due to the motion of the superfluid with respect to the capillary. However, the quasi-particles are excited thermally and the fluid associated with the quasi-particles is not superfluid but normal fluid. These elementary excitations can collide with the capillary walls and dissipate their energies and momenta. Thus, we have the two fluid components at a finite temperature: a superfluid without viscosity and a normal fluid with viscosity. Collisions establish thermodynamic equilibrium in the normal fluid in the frame where the capillary is at rest (capillary frame).

If the energy and momentum of the quasi-particle are $\varepsilon(\mathbf{p})$ and $\mathbf{p}$ in the frame where the superfluid is at rest (superfluid frame), the energy of the same quasi-particle in the capillary frame becomes $\varepsilon(\mathbf{p}) + \mathbf{p} \cdot \mathbf{v_s}$, where $\mathbf{v_s}$ is the relative velocity of the superfluid and the capillary. The Bogoliubov quasi-particles obey the thermal equilibrium distribution in the capillary frame (not in the superfluid frame). Thus, the quasi-particle population is given by

$$N_p = \frac{1}{\exp\left[\frac{\varepsilon(\mathbf{p}) + \mathbf{p} \cdot \mathbf{v_s}}{k_B T}\right] - 1}. \tag{5.7}$$

If $\varepsilon(\mathbf{p}) + \mathbf{p} \cdot \mathbf{v}_s > 0$, i.e. $|\mathbf{v}_s| < \min_{\mathbf{p}} \frac{\varepsilon(\mathbf{p})}{|\mathbf{p}|}$, the quasi-particle population N_p is positive for all values of p. Therefore, we can conclude the coexistence of the two fluids becomes possible. Notice that the condition for the positive N_p, $v_s < \min \frac{\varepsilon(\mathbf{p})}{|\mathbf{p}|}$, is identical to the Landau's criteria of superfluidity. Figure 5.2 shows the quasi-equilibrium population N_p for a positive superfluid velocity $\mathbf{v}_s > 0$. In conclusion, co-existence of the superfluid and normal fluid is possible at finite temperatures if the Landau's criteria (5.6) is satisfied.

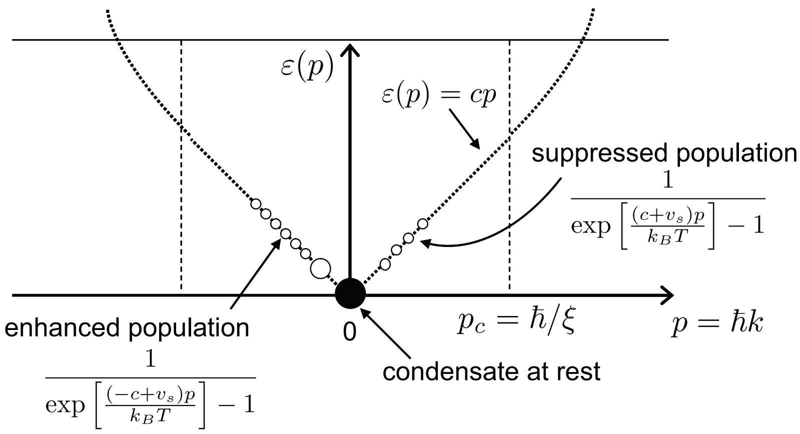

Figure 5.2: The quasi-particle population N_p in the superfluid frame. In this frame, the condensate (superfluid) is at rest ($p = 0$), while the quasi-particle population becomes asymmetric, more population in a negative $\mathbf{p}$ region and less population in a positive $\mathbf{p}$ region due to the friction with the capillary wall which moves with a velocity $-\mathbf{v_s}$ with respect to the superfluid.

5.3 Superfluid velocity and phase of order parameter

We have established one important connection between BEC and superfluidity, which is the relationship between the Landau's criteria for the critical velocity for superfluidity and the sound velocity determined by the Bogoliubov excitation spectrum. In this section we will develop one more important connection between BEC and superfluidity, which addresses the relationship of the superfluid velocity v_s and the phase S of BEC order parameter.

The presence of a large number of particles in a ground state permits the introduction of the $c-$number order parameter $\psi_0(r,t)$ as discussed in Chapter 1. The quantum mechanical field operator $\hat{\psi}(r,t)$ satisfies the Heisenberg equation of motion

$$i\hbar\frac{d}{dt}\hat{\psi}(r,t) = \left[\hat{\psi}(r,t),\hat{\mathcal{H}}\right] = \left[-\frac{\hbar^2\nabla^2}{2m} + V_{\text{ext}}(r,t) + \int \hat{\psi}^+(r',t)V(r'-r)\hat{\psi}(r',t)dr'\right]\hat{\psi}(r,t). \tag{5.8}$$

We replace $\hat{\psi}(r,t)$ with $\psi_0(r,t)$ and use the effective soft potential V_{eff} for $V(r'-r)$ (see Chapter4). The $c-$number order parameter $\psi_0(r,t)$ varies slowly over the inter-particle interaction range, and so we can substitute r' for r in (5.8) and finally obtain

$$i\hbar\frac{d}{dt}\psi_0(r,t) = \left[-\frac{\hbar^2\nabla^2}{2m} + V_{\text{ext}}(r,t) + g|\psi_0(r,t)|^2\right]\psi_0(r,t). \tag{5.9}$$

This is the famous nonlinear Schrödinger equation, which is also referred to as Gross-Pitaevskii equation, and $g = \int V_{\text{eff}}(r)dr$ [4, 5]. Equation (5.9) plays a crucial role in both nonlinear optics and BEC physics. In the field of nonlinear optics, a photon (Maxwell field) acquires an effective mass via material's dispersion and interacts with each other via optical Kerr effect [6]. For example, optical solitons and modulational instability in nonlinear optical media are properly described by (5.9). Thus, it is not hard to anticipate the physics of BEC has many common features with the physics of nonlinear optics [6].

In the case of BEC, the local density of particles is related to the squared order parameter by

$$n(r,t) = |\psi_0(r,t)|^2, \tag{5.10}$$

and thus the total number of particles is equal to $N = \int |\psi_0(r)|^2 dr$. Here we assume the negligible quantum and thermal depletion, which is used already to derive (5.9). If we multiply (5.9) by $\psi_0^*(r,t)$ and subtract the complex conjugate of the resulting equation, we obtain the following continuity equation under superfluidity:

$$\frac{d}{dt}n(r,t) + \text{div}\,[j(r,t)] = 0 \tag{5.11}$$

where the particle current density is

$$j(r,t) = -\frac{i\hbar}{2m}\left(\psi_0^*\nabla\psi_0 - \psi_0\nabla\psi_0^*\right). \tag{5.12}$$

Here the Laplacian $\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}$, the divergent $\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$ and the gradient $\nabla U = \frac{\partial U}{\partial x}\mathbf{e_x} + \frac{\partial U}{\partial y}\mathbf{e_y} + \frac{\partial U}{\partial z}\mathbf{e_z}$ are defined in this way. From (5.11) it can be concluded that the Gross-Pitaevskii equation guarantees the conservation of the total particle number $N = \int n(r)dr$. If we express the $c-$number order parameter $\psi_0(r,t)$ in terms of its amplitude and phase,

$$\psi_0(r,t) = \sqrt{n(r,t)}e^{iS(r,t)}, \tag{5.13}$$

the particle current density (5.12) is rewritten as $j(r,t) = n(r,t)\frac{\hbar}{m}\nabla S(r,t)$. This result means that the superfluid velocity v_s of the condensate particle is related to the gradient of the phase S of order parameter:

$$v_s(r,t) = \frac{\hbar}{m}\nabla S(r,t). \tag{5.14}$$

The phase of the order parameter plays the role of a velocity potential and v_s is referred to as a velocity field.

Inserting (5.13) into (5.9), one can derive on explicit equation for the phase $S(r,t)$;

$$\hbar\frac{d}{dt}S + \left(\frac{1}{2}mv_s^2 + V_{\text{ext}} + gn - \frac{\hbar^2}{2m\sqrt{n}}\nabla^2\sqrt{n}\right) = 0. \tag{5.15}$$

The particle continuity equation (5.11) and the phase equation (5.15) provide a closed set of coupled equations, which is fully equivalent to the Gross-Pitaevskii equation (5.8). The term containing the gradient of the particle density in (5.15) implements the Heisenberg uncertainty relationship between particle number and phase, and is called "quantum pressure".

The stationary solution of (5.9) has the form of

$$\psi_0(r,t) = \psi_0(r)\exp(-i\mu t), \tag{5.16}$$

where $\hbar\mu = \frac{\partial E}{\partial N}$ is the chemical potential and the total energy of the system E is

$$E = \int \left(\frac{\hbar^2}{2m} |\nabla\psi_0|^2 + V_{\text{ext}}(r)|\psi_0|^2 + \frac{g}{2}|\psi_0(r)|^4 \right) dr. \tag{5.17}$$

Using (5.16) in (5.9), one obtains the time independent Gross-Pitaevskii equation

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V_{\text{ext}}(r) - \hbar\mu + g|\psi_0(r)|^2 \right) \psi_0(r) = 0. \tag{5.18}$$

The solution of (5.18) with the lowest energy defines the order parameter of the ground state, which is a real function. The excited states are, however, complex functions, the quantized vortex state being the most famous example of such excited states.

For a uniform gas, in the absence of the external potential $V_{\text{ext}}(r) = 0, \nabla^2\psi_0 = 0$ and (5.17) is reduced to $E = \frac{1}{2}gn^2V$ and the chemical potential is equal to $\hbar\mu = gn$. These results are identical to those of the Bogoliubov theory, developed in Chapter 4 in the case of the weakly interacting uniform Bose gas.

5.4 Quantized vortices in superfluids

The story of quantized vortices provides an important insight into the problem of rotations in superfluids. Quantized vortices were first predicted by Onsager [7] and Feynman [8]. In fact, it is well known that a superfluid cannot rotate. In usual rigid systems, the tangential velocity corresponding to a rotation is given by $\mathbf{v} = \mathbf{\Omega} \times \mathbf{r}$, where $\mathbf{\Omega}$ is the angular velocity vector and r is the distance vector from the origin of rotation. Such a system has a diffused vorticity $\text{curl}(\mathbf{v}) = \nabla \times (\mathbf{\Omega} \times \mathbf{r})\, 2\mathbf{\Omega} \neq 0$. On the other hand, the superfluid velocity given by (5.14) satisfies

$$\text{curl}(\mathbf{v_s}) = \frac{\hbar}{m}\nabla \times \nabla S. \tag{5.19}$$

which means a superfluid turns out to be irrotational and is thus expected to rotate in a completely different way from a rigid rotator. Here the rotation is defined by $\nabla \times \mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \mathbf{e_x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \mathbf{e_y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \mathbf{e_z}$

Let us consider a superfluid confined in a macroscopic cylinder of radius R and length L (Fig. 5.3). The solution of the Gross-Pitaevskii equation for a rotation around the $z-$axis of the cylinder is

$$\psi_0(r) = e^{is\varphi}|\psi_0(r)|, \tag{5.20}$$

where we have used the cylindrical coordinates (r, φ and z) and $|\psi_0(r)| = \sqrt{n(r)}$. Due to the symmetry of the problem, the modulus of the order parameter $|\psi_0(r)|$ depends only on the radial variable r. The parameter s should be an integer to ensure that the order parameter $\psi_0(r)$ is single valued. From (5.14) and (5.20), the tangential velocity is

$$v_s = \frac{\hbar}{m}|\nabla S| = \frac{\hbar}{m}\frac{1\partial}{r\partial\varphi}S = \frac{\hbar}{m}\frac{s}{r}. \tag{5.21}$$

This result is completely different from the tangential velocity $\mathbf{v} = \mathbf{\Omega} \times \mathbf{r}$ of the rigid rotator whose modulus increases with r (Fig. 5.4). At large distances from the $z-$axis, the tangential velocity v_s of the superfluid approaches to zero and the irrotationality of

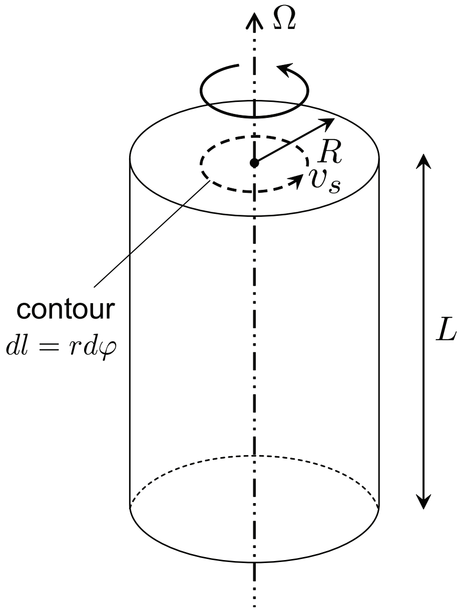

Figure 5.3: A cylindrical vessel containing a superfluid. The circulation of the velocity v_s is quantized in units of h/m and is independent of the radius of the contour.

superfluids (5.19) is satisfied. The circulation of the tangential velocity over a closed contour around the $z-$axis is given by

$$\oint v_s dl = 2\pi s \frac{\hbar}{m}, \tag{5.22}$$

which is quantized in units of h/m, independent of the radius of the contour. This is called "Onsager-Feynman quantization condition".

Substituting (5.20) into (5.18), one obtains the following equation for the modulus of the order parameter:

$$-\frac{\hbar^2}{2m}\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}|\psi_0|\right) + \frac{\hbar^2 s^2}{2mr^2}|\psi_0| + g|\psi_0|^3 - \hbar\mu|\psi_0| = 0. \tag{5.23}$$

We assume the solution for the above equation has the form

$$|\psi_0| = \sqrt{n} f(\eta), \tag{5.24}$$

where $\eta = r/\xi$ and $\xi = \hbar/\sqrt{2mgn}$ is the healing length introduced in the previous chapter. The real function $f(\eta)$ then satisfies the equation

$$\frac{1}{\eta}\frac{d}{d\eta}\left(\eta\frac{df}{d\eta}\right) + \left(1 - \frac{s^2}{\eta^2}\right) f - f^3 = 0, \tag{5.25}$$

with the constraint $f(\infty) = 1$. This is because the density must approach its unperturbed value n and hence $|\psi_0| = \sqrt{n}$ where the superfluid velocity v_s approaches to zero at $\eta \to \infty$. In Fig. 5.5 the function $f(\eta)$ is shown for the values of $s = 1$ and $s = 2$. As $\eta \to 0$, the

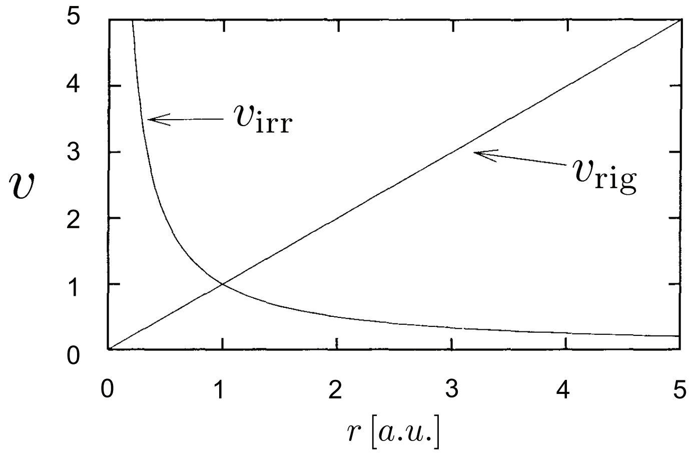

Figure 5.4: Tangential velocity field for irrotational ($v_{\rm irr}$) and rigid rotational ($v_{\rm rig}$)flow. The irrotational velocity field diverges like $1/r$ as $r \to 0$. Here r and v are measured in arbitrary units (a.u.)

function $f(\eta)$ tends to zero as $f \sim \eta^{|s|}$, so that the superfluid density $n(r) = |\psi_0(r)|^2$ tends to zero on the axis of the vortex. The perturbation of the density exists in a spatial region of the healing length ξ from the vortex line ($z-$axis). Of course, the ground state solution corresponding to $s = 0$ has the uniform density $f(\eta) = 1$.

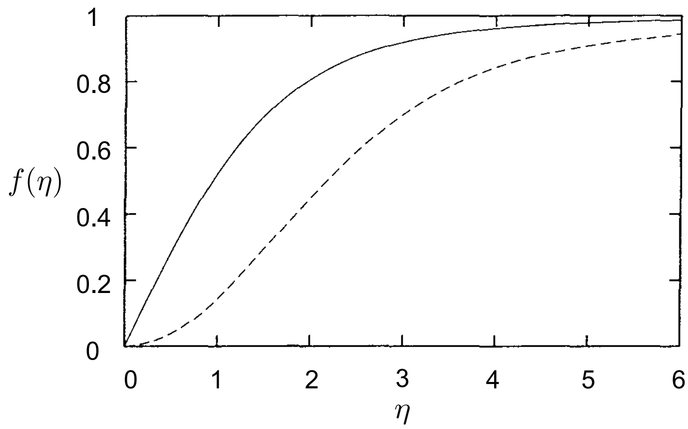

Figure 5.5: Vortical solutions ($s = 1$, solid line; $s = 2$, dashed line) of the Gross-Pitaevskii equation as a function of the radial coordinate $\eta = r/\xi$. The density of the gas is given by $n(r) = nf^2$, where n is the density of the uniform gas.

The energy of the quantized vortex E_v is calculated as $E(s \neq 0) - E(s = 0)$, where the total energy E of the superfluid is given by (5.17), where the order parameter is given by (5.20). In terms of the dimensionless function $f(\eta)$, one obtains

$$E_v \equiv E(s \neq 0) - E(s = 0) = \frac{L\pi\hbar^2 n}{m} \int_0^{R/\xi} \left[\left(\frac{df}{d\eta} \right)^2 + \frac{s^2}{\eta^2} f^2 + \frac{1}{2} \left(f^2 - 1 \right)^2 \right] \eta d\eta. \quad (5.26)$$

For the $s \neq 0$ vortex, the excitation energy is

$$E_v = L\pi n \frac{\hbar^2 s^2}{m} ln \left(\frac{R}{r_c} \right), \tag{5.27}$$

where r_c is the size of the vortex core determined by (5.25). For $s = 1$, r_c is equal to $\xi/1.46$. The main contribution to E_v comes from the kinetic energy (the first term of the integrand in (5.17). The energy E_v increases with the size of the sample as L ln R. However, the relative energy $E_v/E(s = 0)$ with respect to the ground state energy is vanishingly small in a large sample, since $E(s = 0)$ increases as $V = L\pi R^2$.

The above result has been derived in the laboratory (rest) frame. If a cylinder is rotating with angular velocity Ω around $z-$axis, we must evaluate the energy in a rotating frame, in which a cylinder is at rest and thus thermal equilibrium is expected to take place, using the Galilean transformations, the excess energy in the rotating frame is written as

$$E'_v = E_v - \mathbf{\Omega} \cdot L_z, \tag{5.28}$$

where L_z is the angular momentum of the superfluid in the laboratory frame. The order parameter (5.20) is an eigenstate of the angular momentum with an eigenvalue of $l_z = s\hbar$ per particle, so that the vortex solution ($s \neq 0$) carries a total angular momentum equal to $L_z = Ns\hbar$, where $N = n\pi R^2 L$ is the total number of particles, while the ground state ($s = 0$) carries no total angular momentum $L_z = 0$. It is then easy to see that in the rotating frame the vortex solution with $\mathbf{\Omega} \cdot \mathbf{L_z} > 0$ becomes energetically more favorable compared to the ground state with $L_z = 0$ if the angular velocity Ω is large enough. On the other hand, at a low enough angular velocity, the superfluid part remains at rest while the normal component is brought into rotation. The critical angular velocity Ω_c for the vortex solution with $s \geq 1$ to be preferred is given by

$$\Omega_c = \frac{E_v}{L_z} = \frac{\hbar(2s-1)}{mR^2} ln \left(\frac{R}{r_c} \right). \tag{5.29}$$

With the increase in the angular velocity Ω, the higher-order vortex solution with $s \geq 2$ acquires also a lower energy than the ground state with $s = 0$. Since a superfluid cannot rotate in a rigid way, the rotation will eventually be realized through the creation of quantized vortices.

Notice that the dependence of (5.27) on r_c is logarithmic and hence the energy of the vortex depends very weakly on the actual value of the core size. Since the angular momentum L_z is proportional to s and the energy E_v is proportional to s^2, the vortices with $s \geq 2$ are energetically unstable. At a very large angular velocity, the state with multiple $s = 1$ vortices is preferred to the state with single $s \geq 2$ vortex. Since the number of $s = 1$ vortices created in this system, N_v, should be equal to s in (5.29), the vortex density per unit area in a very large angular velocity is given by

$$n_v \equiv \frac{N_v}{\pi R^2} = \frac{m}{\pi \hbar} \Omega = \frac{m}{h} \Omega, \tag{5.30}$$

where $ln\,(R/r_c) \simeq 1$ in the limit of large s is used.

It is worth stressing that vortices can exist as a stationary configuration only in a superfluid. In the presence of a small viscosity, the vortex will diffuse from the $z-$axis and eventually spreads over the whole volume of the cylinder, which is indistinguishable from a rigid rotator.

Quantized vortices in atomic BEC have been experimentally observed by use of a suitable rotating modulation of the trap to stir the condensate [9, 10]. Above the critical angular velocity (5.29) one observes the formation of the vortex, as shown in Fig. 5.6. At sufficiently high angular velocities, array of more vortices are formed in a triangular lattice, which is similar to those is superconductors [11].

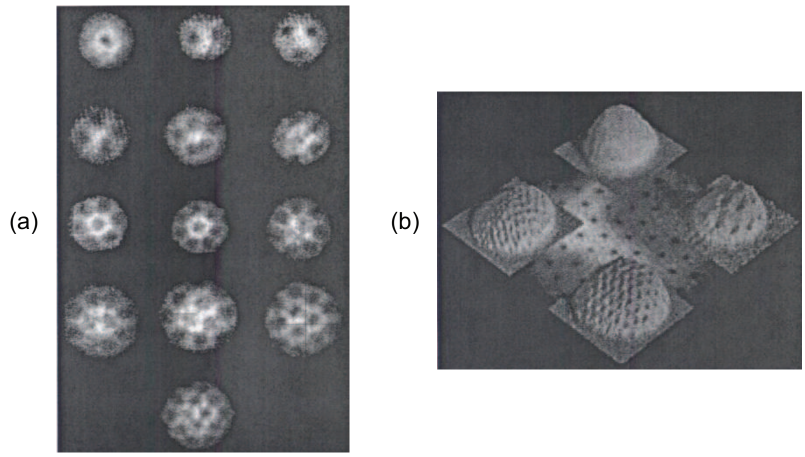

Figure 5.6: The observed array of quantized vortices in the ENS experiment (a) and in the MIT experiment (b).

5.5 Berezinskii-Kosterlitz-Thouless (BKT) phase transition and quantized vortex-pairs

In uniform two-dimensional systems, conventional off-diagonal long range order is destroyed by thermal fluctuations at any finite temperature and BEC cannot occur in contrast to the three dimensional case [12, 13]. This is the Mermin-Wayner-Hohenberg theorem. However, the two-dimensional system can form a quasi-condensate and become superfluid below a finite critical temperature, which is referred to as the BKT phase transition [14, 15]. Figure 5.7 shows a conjectured phase diagram of a two-dimensional weakly interacting Bose gas [16]. With decreasing a temperature below the certain point T_{MF}, the mocroscopic occupation in the ground state, i.e. the mean field is formed but with free vortices as fundamental excitations. With further decrease in a temperature below the BKT transition point T_{BKT}, the superfluid order is formed through the pairing of vortices with opposite circulation. By forming such a bound vortex-pair, the long range phase fluctuation is suppressed and the topological order can appear in the system. Finally at a lower temperature T_{BEC}, the crossover from the superfluid regime to the true BEC phase without vortex-pairs is formed. This scenario behind Fig. 5.7 is still open for questions and must be tested by experimental studies.

Figure 5.8(a) shows the proliferation of free vortices above the transition temperature T_{BKT} in a two-dimensional Bose gas system [17]. The observed onset of free vortex

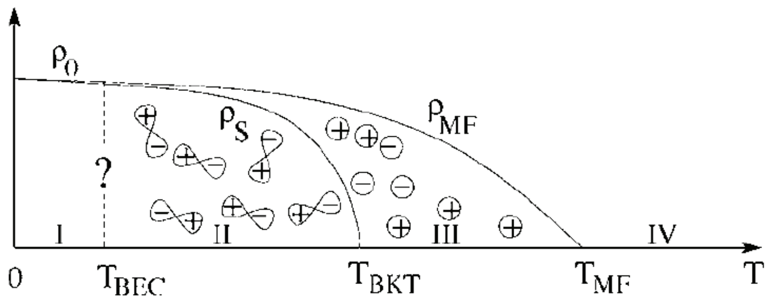

Figure 5.7: Schematic phase diagram of a two-dimensional trapped weakly interacting dilute Bose gas. ρ_0 stands for the density of the condensate, ρ_s is the superfluid density, $\rho_{\rm MF}$ is the mean-field density, which can be estimated perturbatively. $T_{\rm BEC}$ is the crossover temperature from the superfluid regime to the true Bose-Einstein condensation phase. $T_{\rm BKT}$ is the critical temperature of the Kosterlitz-Thouless transition. $T_{\rm MF}$ marks the critical region of mean field formation.

proliferation with increasing temperature coincides with the loss of quasi-long-range coherence [17]. These observations provide experimental hint for the BKT phase transition.

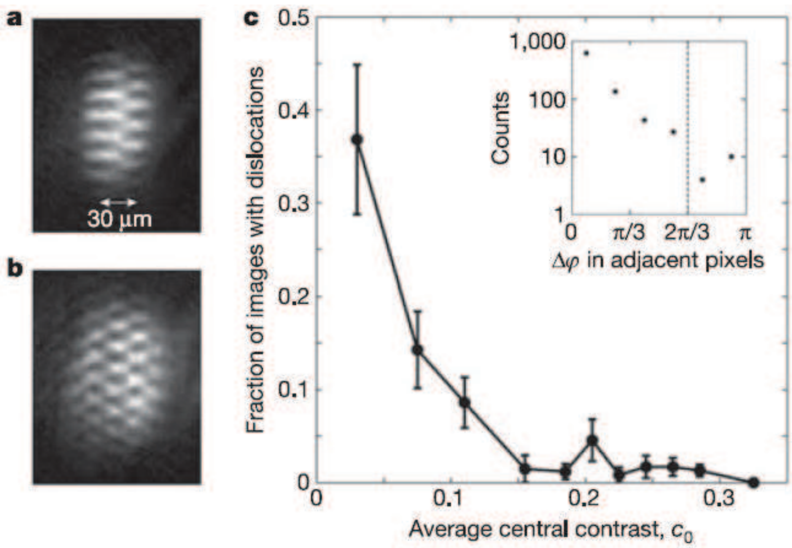

Figure 5.8: Proliferation of free vortices at high temperature (low interference fringe contrast) [17].

More recently a bound pair of vortices with opposite circulations has been directly observed in an exciton-polariton condensate [18]. Figure 5.9 shows the observed phase distribution and interference pattern of the vortex-pair, which is compared to the theoretical prediction. It is demonstrated that the phase disturbance is indeed localized by forming a bound pair and the quasi-long-range order is established.

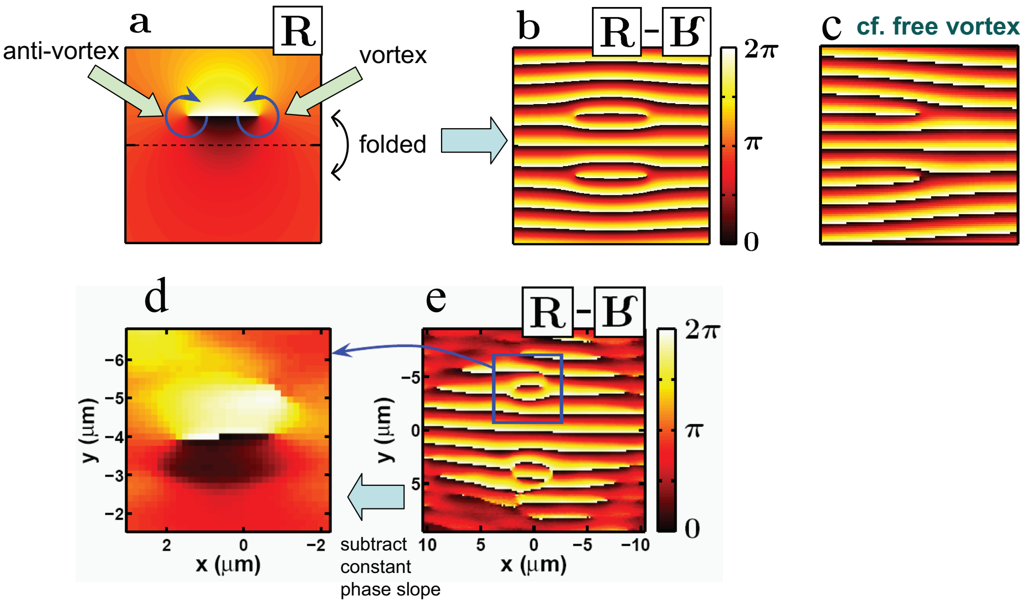

Figure 5.9: Theoretical phase distribution (a) and interference pattern (b) of a vortex-pair, compared to the observed phase distribution (d) and interference pattern (e). The interference pattern for a free vortex is shown in (c) for comparison [18].

Bibliography

[1] P. Kapitza. Viscosity of liquid helium below the λ-point. *Nature*, 141:74, 1938.

[2] J. F. Allen and A. D. Misener. Flow of liquid helium II. *Nature*, 141:75–75, 1938.

[3] L. D. Landau. The theory of superfluidity of helium II. *J. Phys. USSR*, 5:71, 1941.

[4] E. P. Gross. Structure of quantized vortex. *Nuovo Cimento*, 20:454, 1961.

[5] A. Peres. Vortex lines in an imperfect Bose gas. *Soviet Physics JETP-USSR*, 13:451, 1961.

[6] H. A. Haus. *Electromagnetic Noise and Quantum Optical Measurements.* Springer, Berlin, 2000.

[7] L. Onsager. Statistical hydrodynamics. *Nuovo Cimento*, 6:279–287, 1949.

[8] R. P. Feynman. *Progress in Low Temperature Physics, Vol. 1.* North-Holland, Amsterdam, 1955.

[9] M. R. Matthews, B. P. Anderson, P. C. Haljan, D. S. Hall, C. E. Wieman, and E. A. Cornell. Vortices in a bose-einstein condensate. *Phys. Rev. Lett.*, 83:2498–2501, 1999.

[10] F. Chevy, K. W. Madison, V. Bretin, and J. Dalibard. Interferometric detection of a single vortex in a dilute bose-einstein condensate. *Phys. Rev. A*, 64:031601R, 2001.

[11] A. A. Abrikosov. On the magnetic properties of superconductors of the second group. *Zh. Exp. Teor. Fiz.*, 32:1442, 1957.

[12] N. D. Mermin and H. Wagner. Absence of ferromagnetism or antiferromagnetism in one- or 2- dimensional isotropic Heisenberg models. *Phys. Rev. Lett.*, 17:1133, 1966.

[13] P. C. Hohenberg. Existence of long-range order in one and two dimensions. *Phys. Rev.*, 158:383–386, 1967.

[14] V. L. Berezinskii. Destruction of long-range order in one-dimensional and two-dimensional systems possessing a continuous symmetry group ii. *Sov. Phys. JETP*, 34:610, 1972.

[15] J. M. Kosterlitz and D. J. Thouless. Ordering, metastability and phase-transitions in 2 dimensional systems. *J. Phys. C*, 6:1181, 1973.

[16] A. Posazhennikova. Colloquium: Weakly interacting, dilute bose gases in 2d. *Rev. Mod. Phys.*, 78:1111, 2006.

[17] Z. Hadzibabic, P. Kruger, M. Cheneau, B. Battelier, and J. Dalibard. Berezinskii-kosterlitz-thouless crossover in a trapped atomic gas. *Nature*, 441:1118–1121, 2006.

[18] G. Roumpos, M. D. Fraser, A. Löfler, S. Höfling, A. Forchel, and Y. Yamamoto. Single vortex-antivortex pair in an exciton polariton condensate. 2010. arXiv:1005.1897v1.