Supersymmetry Basics

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1. Prelude

This is the first piece of a series of notes on supersymmetry. We intend to present the very basics of supersymmetry that is needed to explore more advanced topics. This includes the construction and representations of super-Poincaré algebra in 4 dimensions, the superspace formulation of $N = 1$ supersymmetric theories, the Wess-Zumino model, and the super gauge theory. To fully show the power of superspace formulation, we also introduce the path integral quantization of $N = 1$ theories, super-Feynman rules for supergraphs. Finally we use these result to derive the perturbative nonrenormalization theorem for superpotential.

Derivations are made quite explicit, and some mid-steps are kept so that the results can be reproduced easily. However, this does not mean every single detail is presented since that would obviously affect the main line of development.

There are already a world of introductory books, reviews, lecture notes on supersymmetry. In preparing this note, we find those ones listed in [1–6] quite useful.

For any text on supersymmetry, notation is a big issue. In this note and succeeding ones, we mainly follow the convention of Wess & Bagger [1], since the book is quite standard and its notations are widely used. In particular, we use mostly plus $(-, +, +, +)$ metric for Minkowski spacetime. For spinorial index summation, we use northwest-southeast rule for undotted indices and southwest-northeast rule for dotted indices. More details of our conventions and some useful relations are listed in the appendix.

2. Supersymmetry Algebras

Supersymmetry algebra is almost the unique nontrivial extension of the relativistic spacetime symmetry algebra, mixing with internal symmetries. The algebra, by definition, has a $\mathbb{Z}_2$ graded structure. According to this structure, all generators are classified into two categories, which we will call bosonic and fermionic, respectively. If we collectively denote bosonic generators by B and fermionic generators by F, then the $\mathbb{Z}_2$ graded structure manifests itself through the following brackets,

$$[B, B] \sim B, \quad [B, F] \sim F, \quad \{F, F\} \sim B, \tag{1}$$

where the square bracket is antisymmetric with its two arguments, while the curly bracket is symmetric. In practice we always assume these brackets are realized as commutators or anticommutators. That is, we have

$$[B_1, B_2] = B_1B_2 - B_2B_1, \quad \{F_1, F_2\} = F_1F_2 + F_2F_1. \tag{2}$$

Therefore the super-Jacobi's identity follows trivially. By super-Jacobi's identity we refer to a natural extension of ordinary Jacobi's identity with the graded structure (1). More explicitly, the identity reads,

$$\begin{aligned}
0 &= [[B_1, B_2], B_3] + [[B_3, B_1], B_2] + [[B_2, B_3], B_1],\\
0 &= [[F, B_1], B_2] + [[B_2, F], B_1] + [[B_1, B_2], F],\\
0 &= [\{F_1, F_2\}, B] + \{[B, F_1], F_2\} - \{[F_2, B], F_1\},\\
0 &= [\{F_1, F_2\}, F_3] + [\{F_3, F_1\}, F_2] + [\{F_2, F_3\}, F_1].
\end{aligned} \tag{3}$$

Attention must be paid when using the identity with two odd generators since there is an "extra" minus sign before the last term in the third line.

We are going to find the most general supersymmetry algebra including the Poincaré algebra as a subalgebra in 4 dimensions. One can also consider supersymmetric extension in other spacetime dimensions and with other spacetime symmetries, e.g., conformal symmetry and (anti-)de Sitter symmetry. However, we will restrict ourselves within Poincaré algebra in 4 dimensions in the current note, and leave possible extensions to the future.

To begin with, we write down all commutators of Poincaré symmetry,

$$\boxed{\begin{aligned}
[P_m, P_n] &= 0,\\
[J_{mn}, P_\ell] &= -\,\mathrm{i}(\eta_{n\ell}P_m - \eta_{m\ell}P_n),\\
[J_{mn}, J_{pq}] &= -\,\mathrm{i}(\eta_{mq}J_{np} - \eta_{mp}J_{nq} - \eta_{nq}J_{mp} + \eta_{np}J_{mq}).
\end{aligned}} \tag{4}$$

The super algebra to be determined should include these generators, together with some fermionic generators, which we denoted by Q^M, as well as some bosonic (and thus internal, according to Coleman-Mandula theorem) generators, denoted by T^a. The task is to find all commutation relations among these generators.

Firstly, according to Coleman-Mandula theorem, T^a must be internal, and thus be closed within themselves. Without loss of generality, we assume they are Hermitian, so that

$$[J_{mn}, T_a] = [P_m, T^a] = 0, \quad [T^a, T^b] = \mathrm{i}f^{ab}{}_cT^c. \tag{5}$$

Secondly, the $\mathbb{Z}_2$ graded structure of the superalgebra requires that the commutator between one fermionic generator and one bosonic generator to be of the following form:

$$\begin{aligned}
[J_{mn}, Q^M] &= (b_{mn})^M{}_NQ^N,\\
[P_m, Q^M] &= (b_m)^M{}_NQ^N,\\
[T^a, Q^M] &= (t^a)^M{}_NQ^N.
\end{aligned}$$

This shows that Q's form finite dimensional representations of Lorentz group, translation group, and the internal group, with representation matrices b_{mn}, b_m, and t^a, respectively. One may apply Jacobi's identities of (Q,J,J), (Q,P,P), and (Q,T,T) to see this point more clearly.

As a finite dimensional representation of Lorentz algebra, the fermionic generators Q's always have the form $Q^A_{\alpha_1\cdots\alpha_{2p};\dot\beta_1\cdots\dot\beta_{2q}}$. Here the undotted and dotted labels correspond to left-chiral and right-chiral components in the Lorentz algebra $\mathfrak{so}(3,1)\cong\mathfrak{su}(2)_L+\mathfrak{su}(2)_R$, respectively, with $p,q=0,\frac{1}{2},1,\frac{3}{2},\cdots$, and A is the index other than Lorentz indices. Now, the anticommutator between the highest weight components of both Q and its complex conjugation Q^* must be a bosonic operator in representation $(p+q,p+q)$. However, it is only P_m, among all bosonic generators, is in this form, namely $(\frac{1}{2},\frac{1}{2})$, so we conclude that $p+q=\frac{1}{2}$. Therefore Q must be a spinorial generator in irrep $(\frac{1}{2},0)$ or $(0,\frac{1}{2})$. We fix Q to be in $(\frac{1}{2},0)$ without loss of generality, and write it as Q^A_α. Then Q^* must be in $(0,\frac{1}{2})$, which we denote as $\overline{Q}_{\dot\alpha A}$. Then, we have,

$$\begin{aligned}[][J_{mn},Q^A_\alpha]=&-\mathrm{i}(\sigma_{mn})_\alpha{}^\beta Q^A_\beta,\\ [J_{mn},\overline{Q}^{\dot\alpha}_A]=&-\mathrm{i}(\bar\sigma_{mn})^{\dot\alpha}{}_{\dot\beta}\overline{Q}^{\dot\beta}_A.\end{aligned}\tag{6}$$

It then follows immediately that $\{Q,\overline{Q}\}$, which carries $(\frac{1}{2},\frac{1}{2})$ representation of Lorentz group, must be of the form

$$\{Q^A_\alpha,\overline{Q}_{\dot\beta B}\}=2X^A{}_B(\sigma^m)_{\alpha\dot\beta}P_m.$$

The factor 2 is conventional. Now taking the Hermite conjugation reveals that $X^A{}_B$ is hermitian. Together with the fact that $\{Q,\overline{Q}\}$ is positive definite, we see that $X^A{}_B$ can always be diagonalized to identity $\delta^A{}_B$ by a linear redefinition of Q^A_α and $\overline{Q}_{\dot\beta B}$. Thus we have,

$$\{Q^A_\alpha,\overline{Q}_{\dot\beta B}\}=2\delta^A{}_B(\sigma^m)_{\alpha\dot\beta}P_m.\tag{7}$$

The commutators $[T,Q]$ and $[T,\overline{Q}]$ are also easy to determine. The A,B-indices in Q^A_α and $\overline{Q}_{\dot\beta B}$ are actually labels for representation matrices of internal symmetry generated by T^a, namely,

$$[T^a,Q^A_\alpha]=(t^a)^A{}_BQ^B_\alpha,\quad[T^a,\overline{Q}_{\dot\beta A}]=(t^{a*})_A{}^B\overline{Q}_{\dot\beta B}.\tag{8}$$

Applying $(T,Q,\overline{Q})$ identity, it is easy to see $(t^a)^A{}_B=(t^{a*})_B{}^A$, i.e., the matrix t^a is Hermitian.

Next we consider $[P,Q]$ commutators. Lorentz invariance requires that,

$$\begin{aligned}[][P_m,Q^A_\alpha]&=b^{AB}(\sigma_m)_{\alpha\dot\beta}\overline{Q}^{\dot\beta}_B,\\ [P_m,\overline{Q}^{\dot\alpha}_A]&=(b^*)_{AB}(\bar\sigma_m)^{\dot\alpha\beta}Q^B_\beta.\end{aligned}$$

To see what the coefficient b^{AB} can be, we apply Jacobi's identity of (P, P, Q), which implies,

$$b^{AB}(b^*)_{BC}(\sigma_{mn})_\alpha{}^\beta Q^C_\beta = 0, \tag{9}$$

which shows that $bb^* = 0$. To find further conditions on b, consider the (P, Q, Q) identity, in which we need $\{Q, Q\}$ bracket. Though lack of a explicit form, Lorentz structure requires that

$$\{Q^A_\alpha, Q^B_\beta\} = \epsilon_{\alpha\beta}Z^{AB} + Y^{AB}(\sigma^{mn})_\alpha{}^\beta\epsilon_{\beta\gamma}J_{mn}.$$

Then, $(P_\ell, Q^A_\alpha, Q^B_\beta)$ identity gives,

$$\begin{aligned}0 =& -\mathrm{i}Y^{AB}(\sigma^{mn}\epsilon)_{\alpha\beta}(\eta_{m\ell}P_n - \eta_{n\ell}P_m)\\ &+ 2b^{BA}(\sigma_\ell)_{\beta\dot\beta}(\sigma^m\epsilon)_\alpha{}^{\dot\beta}P_m - 2b^{AB}(\sigma^\ell)_{\alpha\dot\alpha}(\sigma^m\epsilon)_\beta{}^{\dot\alpha}P_m,\end{aligned}$$

which, after contracted with $\epsilon^{\alpha\beta}$, gives $b^{AB} = b^{BA}$. Thus, $bb^* = 0$ implies that $bb^\dagger = 0$, and so $b = 0$. That is,

$$[P_m, Q^A_\alpha] = [P_m, Q_{\dot\beta A}] = 0, \tag{10}$$

Substituting this back to the (P, Q, Q) identity further implies $Y^{AB} = 0$. So we have $\{Q^A_\alpha, Q^B_\beta\} = \epsilon_{\alpha\beta}Z^{AB}$. Here Z^{AB} are some bosonic generators carrying no Lorentz indices, and thus must be internal, can be expressible in terms of T^a. So we write,

$$Z^{AB} = a^{AB}_a T^a. \tag{11}$$

Furthermore, the (T, Q, Q) identity gives $[T, Z] \sim Z$, meaning that Z^{AB} form an invariant subalgebra of internal symmetry; the $(Q, Q, \overline{Q})$ identity gives $[Z, \overline{Q}] = 0$, and thus $[Z, Z] \sim [\{Q, Q\}, Z] = 0$. So the invariant algebra formed by Z^{AB} is Abelian. Thus we have,

$$[Z^{AB}, \text{everything}] = 0. \tag{12}$$

Thus Z^{AB} are called central charges of the super-algebra. Substituting this back to (T, Q, Q) identity, we get,

$$(t^a)^A{}_B a^{BC}_b = -a^{AB}_b(t^{a*})_B{}^C. \tag{13}$$

That means the coefficients a^{AB}_a intertwine the representation t^a with its conjugation t^{a*}. Thus the central charges can exist only for groups admitting such an intertwining relation. Up to now, all the (anti-)commutators of super-algebra have been nearly determined. However, the structure of the internal symmetry generated by T^a can be further restricted. In fact, it can be shown that, for N species of fermionic generators Q^A_α, the internal symmetry group is $U(N)$ if there is no central charges, and $Sp(N)$, if there is one central charges. See Chapter 2 of [3] for a general discussion. We will also briefly mention this again in next section.

In summary, we list the super extension of 4 dimensional Poincaré algebra, which we will refer to as super-Poincaré algebra, as follows,

$$\boxed{\begin{aligned}\{Q^A_\alpha,\overline{Q}_{\dot\beta B}\}&=2\delta^A{}_B(\sigma^m)_{\alpha\dot\beta}P_m,\\ \{Q^A_\alpha,Q^B_\beta\}&=\epsilon_{\alpha\beta}Z^{AB},\\ \{\overline{Q}_{\dot\alpha A},\overline{Q}_{\dot\beta B}\}&=-\,\epsilon_{\dot\alpha\dot\beta}Z^\dagger_{AB},\\ [P_m,Q^A_\alpha]&=[P_m,\overline{Q}_{\dot\alpha A}]=0,\\ [J_{mn},Q^A_\alpha]&=-\,\mathrm{i}(\sigma_{mn})_\alpha{}^\beta Q^A_\beta,\\ [J_{mn},\overline{Q}^{\dot\alpha}_A]&=-\,\mathrm{i}(\bar\sigma_{mn})^{\dot\alpha}{}_{\dot\beta}\overline{Q}^{\dot\beta}_A,\\ [T^a,Q^A_\alpha]&=(t^a)^A{}_BQ^B_\alpha,\\ [T^a,\overline{Q}_{\dot\beta A}]&=(t^{a*})_A{}^B\overline{Q}_{\dot\beta B},\\ [P_m,T^a]&=[J_{mn},T^a]=0,\\ [T^a,T^b]&=\mathrm{i}f^{ab}{}_cT^c,\\ [Z^{AB},\text{everything}]&=0,\end{aligned}} \tag{14}$$

together with the the Poincaré algebra (4).

3. Representations of Supersymmetry Algebras

In this section we discuss the unitary representation of super-Poincaré algebra on Hilbert space. The strategy is the same with the ordinary Poincaré group, namely the method of induced representation. For this purpose, we need the Casimir operators of super-Poincaré group.

Casimir operators of super-Poincaré algebra. Casimir operators commute with all symmetry generators, so their eigenvalues are same for states in an irreducible representation. Hence they are useful to classify irreducible representations. Recall that Poincaré algebra has two Casimir operators, the momentum squared, $P^2=P_mP^m$, and the square of Pauli-Lubański operator, namely $W^2=W_mW^m$ with $W_m=\frac{1}{2}\epsilon_{mnpq}P^nJ^{pq}$. The eigenvalues of these two Casimir operators of an irrep define its mass m^2 and spin J^2 (helicity s^2 for massless states).

In super-Poincaré algebra, it is easy to see that P^2 is still a Casimir operator, while W^2 no longer is, because one can show that $[W^2,Q]\neq 0$. This implies that, within an irrep of super-Poincaré algebra, each state will have the same mass, but their spin can be different. In fact, in the case of $N=1$ algebra, the second Casimir operator is given by C^2, defined via,

$$\begin{aligned} C^2 &= \tfrac{1}{2} C_{mn} C^{mn}, \\ C_{mn} &= C_m P_n - C_n P_m, \\ C_m &= W_m + \tfrac{1}{4} Q^\alpha (\sigma_m)_{\alpha\dot\beta} \overline{Q}^{\dot\beta}. \end{aligned} \tag{15}$$

Here we check that C^2 does commute with susy generators Q and $\overline{Q}$.

Induced representation. According to the method of induced representation, we consider two categories of representations, with mass $m = 0$ and $m > 0$, and choose a representative momentum vector for each of them. The tachyonic case $m < 0$ is impossible because it contradicts with the semi-positive definiteness of $\{Q, \overline{Q}\}$. For massless case, we choose $q^m = (E, 0, 0, E)$, and for massive case, we choose $q^m = (m, 0, 0, 0)$. The next step is to find the *little group* in each case, i.e., the subgroup of the superPoincaré that leaves the representative momentum intact. Finally, one find irreducible representations for little group, which are also required to be finite dimensional, and boost them by momentum operators to representations of whole super-Poincaré. Below we study the massless and massive cases, with the procedure outlined here, respectively.

3.1 Massless supermultiplets

As mentioned above, for massless states we choose the representative momentum to be $q^m = (E, 0, 0, E)$. In non-susy case, the little group is given by an $ISO(2)$ subgroup of Lorentz group $SO(3,1)$, i.e., the subgroup generated by $\{B_1, B_2, J\}$, defined via,

$$B_1 = J_{10} - J_{13}, \quad B_2 = J_{20} - J_{23}, \quad J = J_{12}, \tag{16}$$

which satisfies the commutation relations,

$$[B_1, B_2] = 0, \quad [J, B_1] = \mathrm{i} B_2, \quad [J, B_2] = -\mathrm{i} B_1. \tag{17}$$

Clearly this is isomorphic to Galilean group in 2 dimensions. Since we are looking for finite dimensional representations of the little group, the two translations $B_{1,2}$ should be represented trivially, with $B_{1,2}|q\rangle = 0$. Then, we can choose the eigenvalue λ of the only remaining generator L to label a representation, $L|q,\lambda\rangle = \lambda|q,\lambda\rangle$, where λ is real number and is called the helicity of the state. We note that λ must be integer or half-integer as in non-susy case, due to the double connectness of Lorentz group. In writing these equations we keep other possible labels of states implicit.

Now let's study the action of susy generators Q and $\overline{Q}$ on the state $|q,\lambda\rangle$. We act $\{Q, \overline{Q}\}$ on state $|q,\lambda\rangle$,

$$\{Q^A_\alpha, \overline{Q}_{\dot\beta B}\}|q,\lambda\rangle = 2\delta^A{}_B \begin{pmatrix} 2E & 0 \\ 0 & 0 \end{pmatrix} |q,\lambda\rangle.$$

This implies that $0 = \langle q,\lambda|\{Q_2^A,\overline{Q}_{\dot{2}}^A\}|q,\lambda\rangle = \big|Q_2^A|q,\lambda\rangle\big|^2 + \big|Q_{\dot{2}}^A|q,\lambda\rangle\big|^2$, thus $Q_2^A|q,\lambda\rangle = Q_{\dot{2}}^A|q,\lambda\rangle = 0$.

Next, consider the action of $\{Q,Q\}$,

$$\{Q_\alpha^A, Q_\beta^B\}|q,\lambda\rangle = \epsilon_{\alpha\beta}Z^{AB}|q,\lambda\rangle,$$

which must vanish, since the nonvanishing of $\epsilon_{\alpha\beta}$ requires one of two indices takes value 2, so the left side must contain a Q_2 generator that makes the expression vanish. As a result, we have $Z^{AB}|q,\lambda\rangle = 0$, namely, *central charges vanish for massless supermultiplets*.

The remaining generators of the little group that have nontrivial action on $|q,\lambda\rangle$ include Q_1^A, $Q_{\dot{1}A}$, and J. They form the following brackets,

$$\begin{aligned}&\{Q_1^A,\overline{Q}_{\dot{1}B}\} = 4E\delta^A{}_B, && \{Q_1^A,Q_1^B\} = \{\overline{Q}_{\dot{1}A},\overline{Q}_{\dot{1}B}\} = 0,\\ &[J,Q_1^A] = -\tfrac{1}{2}Q_1^A, && [J,Q_{\dot{1}A}] = +\tfrac{1}{2}Q_{\dot{1}A}.\end{aligned} \tag{18}$$

The first line tells us that the normalized operators $a^A = \frac{1}{\sqrt{2E}}Q_1^A$ and $a_A^\dagger = \frac{1}{\sqrt{2E}}Q_{\dot{1}A}$ generate a Clifford algebra. The second line shows that the action of rasing operator $a_A^\dagger$ or lowering operator a^A increases or decreases the helicity of the state by 1/2. Therefore, the supermultiplet can be built by firstly defining the Clifford vacuum $|\Omega(q,\lambda)\rangle$ by $a^A|\Omega(q,\lambda)\rangle = 0$ for $A = 1,\cdots,N$, and then acting raising operators on it. Since the raising operators are all anticommute, this procedure must be terminated at some state.

To see this in more detail we first consider the simple example of $N = 1$. It's easy to see that $N = 1$ massless supermultiplet consists of two states only, namely, $|\Omega(q,\lambda)\rangle$ and $a^\dagger|\Omega(q,\lambda)\rangle$. They have helicities λ and $\lambda + 1/2$, respectively. However, in order to correctly represent massless particles with two states of opposite helicities, we need to add another massless supermultiplet containing helicities $-\lambda - 1/2$ and $-\lambda$ states. As a result, the union of these two supermultiplets describes a complex scalar and a massless Marojana fermion, each of which has two states.

Then consider the general case of N susy generators. Now the Clifford algebra (18) admits an $U(N)$ automorphism, given by the transformation,

$$Q_1^A \to U^A{}_B Q_1^B, \quad \overline{Q}_{\dot{1}A} \to \overline{Q}_{\dot{1}B}(U^\dagger)^B{}_A, \quad U \in U(N). \tag{19}$$

Thus the N copies of raising operators form the fundamental representation of $U(N)$ while N lowering operators form the corresponding conjugate representation. When acting the raising operators to the Clifford vacuum of helicity λ, we will get $\binom{N}{n}$ states of helicity $\lambda + \frac{n}{2}$. The procedure is terminated at a single state of helicity $\lambda + \frac{1}{2}N$. Then we get $\sum\limits_{n=0}^{N}\binom{N}{n} = 2^N$ states in total, i.e., the supermultiplet obtained in this way has dimension 2^N. However, as discussed for the example of $N = 1$, an additional supermultiplet with opposite helicity contents is usually needed to form a complete representation for massless particles. The only exception to this helicity doubling rule is the self-conjugate supermultiplet, where states come in pairs with opposite helicities.

As an explicit example, an $N = 2$ massless supermultiplet consists of following 4 states,

$$|\Omega(q,\lambda)\rangle, \quad a_A^\dagger|\Omega(q,\lambda)\rangle, \quad \tfrac{1}{\sqrt{2}}a_1^\dagger a_2^\dagger|\Omega(q,\lambda)\rangle.$$

They have helicities λ, $\lambda+1/2$, and $\lambda+1$, and are $SU(2)$ singlet, doublet, and singlet, respectively. The supermultiplet is self conjugate when $\lambda=-1/2$, in which case no additional supermultiplet is needed. When $\lambda\neq -1/2$, we still need another supermultiplet to complete the representation for massless particles.

Now it is easy to see that if we require a rigid susy theory, then we can have at most $N = 4$, for a supermultiplet of $N > 4$ must involve state of helicity> 1. The only known consistent theory of spin-3/2 and spin-2 is supergravity, which needs local susy rather than rigid susy. Similarly, we would exclude massless susy theories with $N > 8$ because such theories must involves massless particle with spin> 2, and it seems impossible to introduce consistent interactions for such high spin massless particle in 4 dimensional relativistic quantum field theory [7,8]. For this reason we call $N = 4$ theory the maximally extended Yang-Mills theory and $N = 8$ theory the maximally extended supergravity.

3.2 Massive supermultiplets

For massive states we take the representative momentum to be $q^m = (m, 0, 0, 0)$. Then the little group, besides the internal and fermionic parts, is an $SO(3)$ subgroup of the Lorentz group, generated by $J_i = \frac{1}{2}\epsilon_{ijk}J^{jk}$ with $i,j,k = 1,2,3$. Thus a state can be labeled by the eigenvalues of P_m, $J^2 = J_iJ_i$, and J_3, which we write as $|q,j,j_3\rangle$. Then we may examine the action of $\{Q,\overline{Q}\}$ and $\{Q,Q\}$, as did for massless case. Now we distinguish two cases with and without central charges.

Without central charges. In this case we have $\{Q,Q\} = \{\overline{Q},\overline{Q}\} = 0$, and the action of $\{Q,\overline{Q}\}$ on state $|q,j,j_3\rangle$ is given by

$$\{Q_\alpha^A,\overline{Q}_{\dot\beta B}\}|q,j,j_3\rangle = 2\delta^A{}_B\begin{pmatrix} m & 0\\ 0 & m\end{pmatrix}|q,j,j_3\rangle. \tag{20}$$

Thus we find the normalized operators $a_\alpha^A = \frac{1}{\sqrt{2}}Q_\alpha^A$ and $a_{\alpha A}^\dagger = \frac{1}{\sqrt{2}}Q_{\dot\alpha A}$ still form a Clifford algebra,

$$\{Q_\alpha^A,\overline{Q}_{\dot\beta B}\} = 2m\delta^A{}_B\delta_{\alpha\dot\beta}, \quad \{Q_\alpha^A,Q_\beta^B\} = \{\overline{Q}_{\dot\alpha A},\overline{Q}_{\dot\beta B}\} = 0. \tag{21}$$

This time we have $2N$ raising and $2N$ lowering operators, 2 times many as massless case. To see the supermultiplet formed in this case, we firstly consider the $N = 1$ case.

The Clifford vacuum $|\Omega(q,j,j_3)\rangle$ with quantum numbers indicated is defined through $a_\alpha|\Omega(q,j,j_3)\rangle = 0$. Then a massive supermultiplet can be formed to be

$$|\Omega(q,j,j_3)\rangle, \quad a_1^\dagger|\Omega(q,j,j_3)\rangle, \quad a_1^\dagger|\Omega(q,j,j_3)\rangle, \quad \tfrac{1}{\sqrt{2}}a_1^\dagger a_2^\dagger|\Omega(q,j,j_3)\rangle. \tag{22}$$

Recall that Q_α^A belongs to $(\frac{1}{2},0)$ and $\overline{Q}_{\dot\alpha A}$ belongs to $(0,\frac{1}{2})$, so both of them are spinor under little group $SO(3)$. Then, the spin of above states can be found by usual summation rule of angular momentum. More explicitly, suppose $j\neq 0$, then two states of $a_A^\dagger|\Omega(q,j,j_3)\rangle$ have spin $j\pm 1/2$, and $\frac{1}{\sqrt{2}}a_1^\dagger a_2^\dagger|\Omega(q,j,j_3)\rangle$ has spin j, since here a_1 and a_2 anticommute. On the other hand, when $j=0$, both of $a_A^\dagger|\Omega(q,j,j_3)\rangle$ have spin 1/2.

When there are $N>1$ copies of susy generators, the Clifford algebra (22) has an obvious automorphism $SU(2)\otimes U(N)$ where $SU(2)$ is simply the space rotation and operates on spinorial indices, and $U(N)$ operates on indices A. But this is not the largest group. Actually, the algebra (22) is invariant under a larger group $SO(4N)$, which contains $SU(2)\otimes U(N)$ as a subgroup. To make this manifest, we redefine the raising and lowering operators as,

$$\begin{aligned} \Gamma^A &= \tfrac{1}{\sqrt{2}}\big(a_1^A + a_{1A}^\dagger\big), & \Gamma^{N+A} &= \tfrac{1}{\sqrt{2}}\big(a_2^A + a_{2A}^\dagger\big), \\ \Gamma^{2N+A} &= \tfrac{\mathrm{i}}{\sqrt{2}}\big(a_1^A - a_{1A}^\dagger\big), & \Gamma^{3N+A} &= \tfrac{\mathrm{i}}{\sqrt{2}}\big(a_2^A - a_{2A}^\dagger\big), \end{aligned} \tag{23}$$

with 1 and 2 spinorial indices and $A=1,\cdots,N$. Then the $4N$ Hermitian operators Γ^r form the following bracket,

$$\{\Gamma^r,\Gamma^s\} = \delta^{rs}, \quad (r,s=1,\cdots,4N) \tag{24}$$

which is clearly $SO(4N)$ invariant. Now we can still define the Clifford vacuum via $a_\alpha^A|\Omega(m,j,j_3)\rangle = 0$ for all α and all A. Then, acting raising operators $a_{\alpha A}^\dagger$ on Clifford vacuum, we will finally get 2^{2N} states, forming a spinor representation of $SO(4N)$. This representation can be decomposed into two irreducible representations of dimension 2^{2N-1}, corresponding to bosonic and fermionic parts.

Besides $SU(2)\otimes U(N)$ mentioned above, the automorphism group $SO(4N)$ also contains another subgroup $SU(2)\otimes USp(2N)$. This subgroup is important in that states of the same spin form an irreducible representation of $USp(2N)$.

With central charges. When central charges are present, we can write super-brackets acting on a state $|q,j,j_3\rangle$ associated with representative momentum $q^m=(m,0,0,0)$ as,

$$\{Q_\alpha^A,\overline{Q}_{\dot\beta B}\} = 2m\delta^A{}_B\delta_{\alpha\dot\beta}, \quad \{Q_\alpha^A,Q_\beta^B\} = \epsilon_{\alpha\beta}Z^{AB}, \quad \{\overline{Q}_{\dot\alpha A},\overline{Q}_{\dot\beta B}\} = -\epsilon_{\dot\alpha\dot\beta}Z_{AB}^\dagger \tag{25}$$

Note that according to convention of [1], $Z_{AB} = -Z^{AB}$. The central charge Z^{AB} is antisymmetric with its two indices and commutes with everything. So we are free to bring it to the following standard form $\widetilde{Z}^{AB} = U^A{}_B U^C{}_D Z^{CD}$, by unitary rotations $U \in U(N)$,

$$\begin{aligned} &\widetilde{Z} = \mathrm{diag}\big(Z_1\epsilon, \cdots, Z_{N/2}\epsilon\big), && (N \text{ even}) \\ &\widetilde{Z} = \mathrm{diag}\big(Z_1\epsilon, \cdots, Z_{N/2-1}\epsilon, 0\big), && (N \text{ odd}) \end{aligned} \tag{26}$$

where Z_i's are numbers and $\epsilon = \mathrm{i}\tau_2$ is 2×2 antisymmetric matrix with $\epsilon^{12} = 1$. Now, we perform the same unitary rotation $\widetilde{Q}^A_\alpha = U^A{}_B Q^B_\alpha$ on susy generators Q and similarly on $\overline{Q}$, and decompose the indices $A = (a, I)$ with $a = 1, 2$ and $I = 1, \cdots, N/2$. Then the brackets (25) can be rewritten as,

$$\begin{aligned} \{\widetilde{Q}^{aI}_\alpha, \widetilde{\overline{Q}}_{\dot\beta bJ}\} &= 2M\delta^a{}_b \delta^I{}_J \delta_{\alpha\dot\beta}, \\ \{\widetilde{Q}^{aI}_\alpha, \widetilde{Q}^{bJ}_\beta\} &= \epsilon_{\alpha\beta}\epsilon^{ab}\delta^{IJ} Z_J, \\ \{\widetilde{\overline{Q}}_{\dot\alpha aI}, \widetilde{\overline{Q}}_{\beta bJ}\} &= -\,\epsilon_{\dot\alpha\dot\beta}\epsilon_{ab}\delta_{IJ} Z_J, \end{aligned} \tag{27}$$

To further simplify these anticommutators, we define new operators a^I_α and b^I_α, as follows,

$$\begin{aligned} a^I_\alpha &= \tfrac{1}{\sqrt{2}}\big(\widetilde{Q}^{1I}_\alpha + \epsilon_{\alpha\dot\beta}\widetilde{\overline{Q}}{}^{2I}_{\dot\beta}\big), \\ b^I_\alpha &= \tfrac{1}{\sqrt{2}}\big(\widetilde{Q}^{1I}_\alpha - \epsilon_{\alpha\dot\beta}\widetilde{\overline{Q}}{}^{2I}_{\dot\beta}\big). \end{aligned} \tag{28}$$

Then, the anticommutators read,

$$\begin{aligned} \{a^I_\alpha, a^J_\beta\} &= \{b^I_\alpha, b^J_\beta\} = \{a^I_\alpha, b^J_\beta\} = 0, \\ \{a^I_\alpha, a^{J\dagger}_\beta\} &= \delta_{\alpha\beta}\delta^{IJ}(2m + Z_J), \\ \{b^I_\alpha, b^{J\dagger}_\beta\} &= \delta_{\alpha\beta}\delta^{IJ}(2m - Z_J). \end{aligned} \tag{29}$$

Now this is again a Clifford algebra with N raising and lowering operators when $Z_I < 2m$, and the supermultiplet can be formed in a similar way as described before, and it has dimension 2^{2N}. However, once some Z_I's are equal to $2m$, the corresponding $\{b, b^\dagger\}$ brackets vanish, and the dimension of the Clifford algebra decreases. In particular, if $Z_I = 2m$ for all $I = 1, \cdots, N/2$, all b's should be removed from the Clifford algebra, and the remaining a's can generate a supermultiplet of dimension 2^N, which is the same with the corresponding massless supermultiplet. Such supermultiplet is usually called short multiplet, and it describes the so-called *BPS saturated states*. The Clifford algebra (29) still admits the automorphism group $USp(2N)$ provided al $Z_I < 2m$. Once a central charge saturates the BPS bound, the automorphism group becomes $USp(N)$, or $USp(N+1)$ for N odd.

4. $N = 1$ Superspace and Superfields

4.1 Superspace

Superspace formulation is a convenient way to realize both Poincaré and supersymmetry transformation as coordinate transformations. In this way the supersymmetry are kept manifest during every step of derivations. Thus it is very useful when quantizing a supersymmetric theory. One can make an analogy with the covariant formulation of special relativity. The superspace language to a supersymmetric theory, comparing with the components field description, is what the covariant formulation of special relativity, e.g., x^μ, A_μ, comparing with the components description, e.g., $(t, \vec{x})$ or $(\phi, \vec{A})$.

For $N = 1$ supersymmetry, superspace formulation is an elegant way to derive all renormalizable, and some imporantant non-renormalizable theories. The path integral quantization based on superspace formulation is also useful when deriving important $N = 1$ nonrenormalization theorems. Although these theorem can also be derived more elegantly by applying holomorphy arguments, the diagrammatic proof by using superspace Feynman rules is conceptually more straightforward.

A formal construction of superspace formulation is to make use of coset construction, realizing the superspace as a coset space. In general, for a group G with a subgroup H, the coset space G/H consists of equivalent classes of the identification $\sim$, with

$$g_1 \sim g_2 \qquad \text{iff} \qquad g_2^{-1} g_1 \in H.$$

A quite remarkable fact is that the 4 dimensional spacetime itself, can already be identified as a coset, namely Poincaré/Lorentz. This identification means not only the correct dimensionality, but also the correct transformation rules of coset coordinates under a general Poincaré transformation. That is, the Lorentz subgroup is realized linearly in a vector representation while the remaining translations are realized nonlinearly.

With this prototypical example in mind, it is easy to guess that a natural realization of superspace is the coset SuperPoincaré/Lorentz. This is indeed the case. Now we elaborate this idea. A general element $g_0 \in$SuperPoincaré can be written as

$$\boxed{g_0 = \exp\big(-\mathrm{i}a^m P_m + \mathrm{i}\xi^\alpha Q_\alpha + \mathrm{i}\bar{\xi}_{\dot\alpha}\overline{Q}^{\dot\alpha}\big)\exp\big(\tfrac{1}{2}\mathrm{i}\omega^{mn}J_{mn}\big).} \tag{30}$$

Then we can choose the representative element in each equivalent class to be the one with $\omega^{mn} = 0$. The points in coset space can now be parameterized by the coordinates $z^M = (x^m, \theta^\alpha, \bar\theta_{\dot\alpha})$ and be written as $\exp(\mathrm{i}z \cdot K)$. Here we define the superspace inner product as $z \cdot K = -x^m P_m + \theta^\alpha Q_\alpha + \bar\theta_{\dot\alpha}\overline{Q}^{\dot\alpha}$, with $K_M = (P_m, Q_\alpha, \overline{Q}^{\dot\alpha})$.

SUSY transformations of super-coordinates. According to the spirit of coset construction, the superPoincaré transformation of superspace coordinates are determined by the left group action. Thus we consider the action of g_0 on the point $e^{z\cdot K}$ from left,

$$g_0 e^{\mathrm{i}z\cdot K} = e^{\mathrm{i}z'\cdot K} e^{\mathrm{i}\omega'^{mn}J_{mn}/2}. \tag{31}$$

Note that the coset point $e^{\mathrm{i}z\cdot K}$ will be shifted away from our chosen parameterization, and develop a J_{mn} term, after the left g_0 action, as shown above. Then, the coordinate transformation for small group action can be found from this equation by applying the Hausdorff's formula $e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\cdots}$,

$$\begin{aligned} g_0 e^{\mathrm{i}z\cdot K} \simeq &\exp\Big\{ -\mathrm{i}(a+x)^m P_m + \mathrm{i}(\xi+\theta)Q + \mathrm{i}(\bar\xi+\bar\theta)\overline{Q} + \tfrac{1}{2}\mathrm{i}\omega^{mn}J_{mn} \\ &\quad - \tfrac{1}{2}\big(\xi^\alpha\bar\theta^{\dot\beta} + \bar\xi^{\dot\beta}\theta^\alpha\big)\{Q_\alpha,\overline{Q}_{\dot\beta}\} + \tfrac{1}{4}\omega^{mn}x^\ell[J_{mn},P_\ell] \\ &\quad - \tfrac{1}{2}\omega^{mn}\theta^\alpha[J_{mn},Q_\alpha] - \tfrac{1}{2}\omega^{mn}\theta_{\dot\alpha}[J_{mn},Q^{\dot\alpha}]\Big\} \\ = &\exp\Big\{ -\mathrm{i}\big(x^m + a^m + \mathrm{i}\theta\sigma^m\bar\xi - \mathrm{i}\xi\sigma^m\bar\theta + \tfrac{1}{2}\omega^{mn}x_n\big)P_m + \tfrac{1}{2}\mathrm{i}\omega^{mn}J_{mn} \\ &\quad + \mathrm{i}\big(\theta^\alpha + \xi^\alpha - \tfrac{\mathrm{i}}{2}\omega^{mn}\theta^\beta(\sigma_{mn})_\beta{}^\alpha\big)Q_\alpha \\ &\quad + \mathrm{i}\big(\bar\theta_{\dot\alpha} + \bar\xi_{\dot\alpha} - \tfrac{\mathrm{i}}{2}\omega^{mn}\bar\theta_{\dot\beta}(\bar\sigma_{mn})^{\dot\beta}{}_{\dot\alpha}\big)Q^{\dot\alpha}\Big\} \end{aligned} \tag{32}$$

In above derivation we use the relation such as $[\xi Q,\bar\theta\overline{Q}] = \xi^\alpha\bar\theta^{\dot\beta}\{Q_\alpha,\overline{Q}_{\dot\beta}\}$, as well as the shorthand notation $\xi\sigma^m\bar\theta = \xi^\alpha(\sigma^m)_{\alpha\dot\beta}\bar\theta^{\dot\beta}$. Then we get the infinitesimal transformation rules of super-coordinates,

$$\boxed{\begin{aligned} &x^m \to x^m + a^m + \mathrm{i}\theta\sigma^m\bar\xi - \mathrm{i}\xi\sigma^m\bar\theta + \tfrac{1}{2}\omega^{mn}x_n, \\ &\theta^\alpha \to \theta^\alpha + \xi^\alpha - \tfrac{\mathrm{i}}{2}\omega^{mn}\theta^\beta(\sigma_{mn})_\beta{}^\alpha, \\ &\bar\theta_{\dot\alpha} \to \bar\theta_{\dot\alpha} + \bar\xi_{\dot\alpha} - \tfrac{\mathrm{i}}{2}\omega^{mn}\bar\theta_\beta(\bar\sigma_{mn})^{\dot\beta}{}_{\dot\alpha}. \end{aligned}} \tag{33}$$

It is worth noting that the transformation rule above becomes exact even for finite parameters $(a^m,\xi^\alpha,\bar\xi_{\dot\alpha})$ if we turn off the rotation by setting $\omega^{mn}=0$, because all higher order commutators vanish in the Hausdorff's formula quoted above.

As expected, the Lorentz rotation acts linearly on all coset coordinates, while spacetime translation and supersymmetry transformation are nonlinearly realized as supertranslations. A special point is that supersymmetry also leaves a footstep on commuting coordinate x^m, due to the nonvanishing bracket $\{Q_\alpha,\overline{Q}_{\dot\beta}\}$. This can be understood as a sort of "noncommuting coordinates", and it will distort the geometry structure of the superspace from the trivial one.

Geometric structure. As a coset space, the $N=1$ superspace has certain geometric structure, described by its vielbein and spin connection. When speaking of these geometric structures, we should distinguish the "general curved" indices for coset coordinates from the "local flat" indices. We use $(\cdot)^M = (\cdot)^{(m,\mu,\dot\mu)}$ as "curved" indices and

$(\cdot)^A = (\cdot)^{(a,\alpha,\dot\alpha)}$ as "flat" indices. Then, in the coset construction, the vielbein 1-form E^A and spin connection 1-form Ω^{mn} are defined as components of the Maurer-Cartan 1-form $-\mathrm{i}e^{-\mathrm{i}z\cdot K}\mathrm{d}e^{\mathrm{i}z\cdot K}$, via,

$$-\mathrm{i}e^{-\mathrm{i}z\cdot K}\mathrm{d}e^{\mathrm{i}z\cdot K} = E^A K_A + \tfrac{1}{2}\Omega^{mn}J_{mn}. \tag{34}$$

To find these components, we use the formula

$$\mathrm{d}e^X = e^X \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)!}\mathrm{ad}_X^n\big(\mathrm{d}X\big),$$

where $\mathrm{ad}_X(Y) \equiv [X,Y]$. Then, taking $X = -\mathrm{i}x^m P_m + \theta Q + \bar\theta\bar Q$, we have

$$\begin{aligned}
&-\mathrm{i}e^{\mathrm{i}x^m P_m - \mathrm{i}\theta Q - \mathrm{i}\bar\theta\bar Q}\mathrm{d}e^{-\mathrm{i}x^m P_m + \mathrm{i}\theta Q + \mathrm{i}\bar\theta\bar Q}\\
&= -\mathrm{d}x^m P_m - \mathrm{i}\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)!}\mathrm{ad}^n_{(-\mathrm{i}\theta Q - \mathrm{i}\bar\theta\bar Q)}\big(\mathrm{i}\mathrm{d}\theta Q + \mathrm{i}\mathrm{d}\bar\theta\bar Q\big)\\
&= \big[-\mathrm{d}x^a + \mathrm{i}\theta\sigma^a(\mathrm{d}\bar\theta) - \mathrm{i}(\mathrm{d}\theta)\sigma^a\bar\theta\big]P_a + (\mathrm{d}\theta)Q + (\mathrm{d}\bar\theta)\bar Q.
\end{aligned} \tag{35}$$

From this 1-form we see that the spin connection vanishes, and the components of vielbein can be read off from $E^A = \mathrm{d}z^M E_M{}^A$, and be written as

$$\boxed{E_M{}^A = \begin{pmatrix} e_m{}^a & e_m{}^\alpha & e_{m\dot\alpha} \\ e_\mu{}^a & e_\mu{}^\alpha & e_{\mu\dot\alpha} \\ e^{\dot\mu a} & e^{\dot\mu\alpha} & e^{\dot\mu}{}_{\dot\alpha}\end{pmatrix} = \begin{pmatrix} \delta_m{}^a & 0 & 0 \\ -\mathrm{i}(\sigma^a)_{\mu\dot\nu}\bar\theta^{\dot\nu} & \delta_\mu{}^\alpha & 0 \\ -\mathrm{i}\theta^\rho(\sigma^a)_{\rho\dot\nu}\epsilon^{\dot\nu\dot\mu} & 0 & \delta^{\dot\mu}{}_{\dot\alpha}\end{pmatrix}.} \tag{36}$$

We can also define the inverse vielbein $E_A{}^M$ as usual, through either $E_M{}^A E_A{}^N = \delta_M{}^N$ or $E_A{}^M E_M{}^B = \delta_A{}^B$. Explicitly, we have

$$\boxed{E_A{}^M = \begin{pmatrix} e_a{}^m & e_a{}^\mu & e_{a\dot\mu} \\ e_\alpha{}^m & e_\alpha{}^\mu & e_{\alpha\dot\mu} \\ e^{\dot\alpha m} & e^{\dot\alpha\mu} & e^{\dot\alpha}{}_{\dot\mu}\end{pmatrix} = \begin{pmatrix} \delta_a{}^m & 0 & 0 \\ \mathrm{i}(\sigma^m)_{\alpha\dot\beta}\bar\theta^{\dot\beta} & \delta_\alpha{}^\mu & 0 \\ \mathrm{i}\theta^\gamma(\sigma^m)_{\gamma\dot\beta}\epsilon^{\dot\beta\dot\alpha} & 0 & \delta^{\dot\alpha}{}_{\dot\mu}\end{pmatrix}.} \tag{37}$$

With the veirbein and spin connection known, we can find further geometric objects on superspace. A very important one is the covariant derivative, defined through $\mathrm{D}_A = E_A{}^M(\partial_M + \frac{1}{2}\omega_M{}^{mn}J_{mn})$. Note that $\omega_M{}^{mn} = 0$, then it is easy to find,

$$\boxed{\begin{aligned}
&\mathrm{D}_a = \partial_a,\\
&\mathrm{D}_\alpha = \frac{\partial}{\partial\theta^\alpha} + \mathrm{i}(\sigma^m)_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_m,\\
&\bar{\mathrm{D}}^{\dot\alpha} = \frac{\partial}{\partial\bar\theta_{\dot\alpha}} + \mathrm{i}\theta^\gamma(\sigma^m)_{\gamma\dot\beta}\epsilon^{\dot\beta\dot\alpha}\partial_m.
\end{aligned}} \tag{38}$$

In practice, it is also convenient to define a antichiral covariant derivative with lower index, namely,

$$\boxed{\bar{\mathrm{D}}_{\dot\alpha} \equiv \epsilon_{\dot\alpha\dot\beta}\bar{\mathrm{D}}^{\dot\beta} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - \mathrm{i}\theta^\beta(\sigma^m)_{\beta\dot\alpha}\partial_m,} \tag{39}$$

where $\partial/\partial\bar\theta^{\dot\alpha} = (\partial/\partial\bar\theta_{\dot\beta})\epsilon_{\dot\beta\dot\alpha}$. We define $\bar{\mathrm{D}}_{\dot\alpha}$ with an extra minus sign to match the convention of Wess & Bagger [1].

With this expression, we can easily prove following useful supercommutators,

$$\boxed{\begin{aligned}&[\partial_m, \mathrm{D}_\alpha] = [\partial_m, \bar{\mathrm{D}}_{\dot\alpha}] = 0,\\ &\{\mathrm{D}_\alpha, \mathrm{D}_\beta\} = \{\bar{\mathrm{D}}_{\dot\alpha}, \bar{\mathrm{D}}_{\dot\beta}\} = 0,\\ &\{\mathrm{D}_\alpha, \bar{\mathrm{D}}_{\dot\beta}\} = -2\mathrm{i}(\sigma^m)_{\alpha\dot\beta}\partial_m.\end{aligned}} \tag{40}$$

As an example, we prove the last one for an arbitrary functions f on superspace,

$$\begin{aligned}\{\mathrm{D}_\alpha, \bar{\mathrm{D}}_{\dot\beta}\}f =\ & \big(\partial_\alpha + \mathrm{i}(\sigma^m)_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_m\big)\big(-\partial_{\dot\beta} - \mathrm{i}\theta^\beta(\sigma^n)_{\beta\dot\beta}\partial_n\big)f\\ &+ \big(-\partial_{\dot\beta} - \mathrm{i}\theta^\beta(\sigma^n)_{\beta\dot\beta}\partial_n\big)\big(\partial_\alpha + \mathrm{i}(\sigma^m)_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_m\big)f\\ =& -\big(\partial_\alpha\partial_{\dot\beta} + \partial_{\dot\beta}\partial_\alpha\big)f + (\sigma^m)_{\alpha\dot\alpha}(\sigma^n)_{\beta\dot\beta}\big(\bar\theta^{\dot\alpha}\theta^\beta + \theta^\beta\bar\theta^{\dot\alpha}\big)\partial_m\partial_n f\\ &- 2\mathrm{i}(\sigma^m)_{\alpha\dot\beta}\partial_m f - \mathrm{i}(\sigma^m)_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_m\partial_{\dot\beta}f + \mathrm{i}\theta^\beta(\sigma^m)_{\beta\dot\beta}\partial_m\partial_\alpha f\\ &- \mathrm{i}\theta^\beta(\sigma^m)_{\beta\dot\beta}\partial_\alpha\partial_m f + \mathrm{i}(\sigma^m)_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_{\dot\beta}\partial_m f\\ =& -2\mathrm{i}(\sigma^m)_{\alpha\dot\beta}\partial_m f.\end{aligned}$$

From (40) we see that the commutation relations of these covariant derivatives agree with the SUSY algebra, with the identification of $P_m = -\mathrm{i}\partial_m$.

4.2 Superfields

Superfields are functions defined on the superspace. The SUSY transformations of the supercoordinates induce a corresponding SUSY transformation on superfields. The simplest superfield is the scalar superfield $\phi(z)$, which by definition is invariant under such a SUSY transformation, namely,

$$\phi'(z) = \phi(z'), \tag{41}$$

where the induced SUSY transformation on ϕ has been defined in the passive way, which is contrary to the conventions for x-space fields. This convention actually derives from the SUSY transformations for component fields which has become the standard one in SUSY community. With this convention, the infinitesimal SUSY transformation on a scalar superfield $\phi(z)$ is given by

$$\delta\phi(z) \equiv \phi'(z) - \phi(z) = \phi(z') - \phi(z) = \delta z^M\partial_M\phi(z). \tag{42}$$

From discussion above we know that the coordinate variation δz^M is further induced by the left action of a small superPoincaré group element $\delta g^I = (a^m, \xi^\alpha, \bar{\xi}_{\dot\alpha}, \omega^{mn})$. Thus we may further write $\delta z^M \partial_M = \delta g^I f_I{}^M \partial_M \equiv \delta g^I X_I$. By definition, $X_I = f_I{}^M \partial_M$ is nothing but the Killing vector associated with SUSY transformations on the superspace. Then from (33) we find that

$$f_I{}^M = \begin{pmatrix} \delta_m{}^\ell & 0 & 0 \\ -\mathrm{i}(\sigma^\ell)_{\alpha\dot\beta}\theta^{\dot\beta} & \delta_\alpha{}^\beta & 0 \\ -\mathrm{i}\theta^\gamma(\sigma^\ell)_{\gamma\dot\beta}\epsilon^{\dot\beta\dot\alpha} & 0 & \delta^{\dot\alpha}{}_{\dot\beta} \\ \frac{1}{2}(x_m\delta_n^\ell - x_n\delta_m^\ell) & -\mathrm{i}\theta^\beta(\sigma_{mn})_\beta{}^\alpha & -\mathrm{i}\bar\theta_{\dot\beta}(\bar\sigma_{mn})^{\dot\beta}{}_{\dot\alpha} \end{pmatrix}. \tag{43}$$

Therefore, the Killing vector $X_I = f_I{}^M \partial_M = (X_m, X_\alpha, \overline{X}^{\dot\alpha}, X_{mn})$ is given by

$$\boxed{\begin{aligned} X_m &= \partial_m, \\ X_\alpha &= \frac{\partial}{\partial\theta^\alpha} - \mathrm{i}(\sigma^m)_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_m, \\ \overline{X}^{\dot\alpha} &= \frac{\partial}{\partial\bar\theta_{\dot\alpha}} - \mathrm{i}\theta^\gamma(\sigma^m)_{\gamma\dot\beta}\epsilon^{\dot\beta\dot\alpha}\partial_m, \\ X_{mn} &= \tfrac{1}{2}(x_m\partial_n - x_n\partial_m) - \mathrm{i}\theta^\beta(\sigma_{mn})_\beta{}^\alpha\frac{\partial}{\partial\theta^\alpha} - \mathrm{i}\bar\theta_{\dot\beta}(\bar\sigma_{mn})^{\dot\beta}{}_{\dot\alpha}\frac{\partial}{\partial\bar\theta_{\dot\alpha}}. \end{aligned}} \tag{44}$$

As always, the index of Killing vector $X^{\dot\alpha}$ can be lowered according to $X_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}X^{\dot\beta}$. Then, all the Killing vectors X_I also form a closed algebra. In particular, we have

$$\boxed{\begin{aligned} [X_m, X_\alpha] &= [X_m, \overline{X}_{\dot\beta}] = 0, \\ \{X_\alpha, X_\beta\} &= \{\overline{X}_{\dot\alpha}, \overline{X}_{\dot\beta}\} = 0, \\ \{X_\alpha, \overline{X}_{\dot\beta}\} &= 2\mathrm{i}(\sigma^m)_{\alpha\dot\beta}X_m. \end{aligned}} \tag{45}$$

Note that there is an overall sign difference on the right hand side of these commutators, comparing with original susy algebra. This is due to our passive interpretation of susy action. On the other hand, had we define the group action on the coset space by right action (rather than left action), the Killing vectors would be given exactly by the covariant derivatives introduced in (38).

It is illuminating to display the susy transformation of a scalar field in component form at this stage. By expanding into components, a scalar superfield $\phi(z)$ can be written as,

$$\boxed{\begin{aligned} F(x^\mu, \theta^\alpha, \theta_{\dot\alpha}) = f(x) + \theta\phi(x) + \bar\theta\bar\chi(x) + \theta^2 m(x) + \bar\theta^2 n(x) + \theta\sigma^m\bar\theta v_m(x) \\ + \theta^2\bar\theta\bar\lambda(x) + \bar\theta^2\theta\psi(x) + \theta^2\bar\theta^2 d(x). \end{aligned}} \tag{46}$$

Thus, the SUSY transformation on component fields, defined by $\delta F(z) = \delta f(x) + \theta^\alpha\delta\chi_\alpha(x) + \cdots$, can be worked out by acting on each component field the operator

$\mathbf{X} = \xi^\alpha X_\alpha + \bar{\xi}_{\dot{\alpha}} \overline{X}^{\dot{\alpha}} = \xi^\alpha \partial_\alpha + \bar{\xi}_{\dot{\alpha}} \partial^{\dot{\alpha}} - \mathrm{i}\xi\sigma^m\bar{\theta}\partial_m + \mathrm{i}\theta\sigma^m\bar{\xi}\partial_m.$

$$
\begin{aligned}
\mathbf{X}f &= -\,\mathrm{i}\big(\xi\sigma^m\bar{\theta} - \theta\sigma^m\bar{\xi}\big)\partial_m f,\\
\mathbf{X}\theta\phi &= \xi\phi - \mathrm{i}(\xi\sigma^m\bar{\theta})(\theta\partial_m\phi) + \mathrm{i}(\theta\sigma^m\bar{\xi})(\theta\partial_m\phi)\\
&= \xi\phi - \tfrac{\mathrm{i}}{2}(\theta\sigma^m\bar{\theta})(\xi\sigma^n\bar{\sigma}_m\partial_n\phi) + \tfrac{\mathrm{i}}{2}\theta^2\big[(\partial_m\phi)\sigma^m\bar{\xi}\,\big],\\
\mathbf{X}\bar{\theta}\bar{\chi} &= \bar{\xi}\bar{\chi} - \mathrm{i}(\xi\sigma^m\bar{\theta})(\bar{\theta}\partial_m\bar{\chi}) + \mathrm{i}(\theta\sigma^m\bar{\xi})(\bar{\theta}\partial_m\bar{\chi}),\\
&= \bar{\xi}\bar{\chi} + \tfrac{\mathrm{i}}{2}\bar{\theta}^2(\xi\sigma^m\partial_m\bar{\chi}) + \tfrac{\mathrm{i}}{2}(\theta\sigma^m\bar{\theta})\big[(\partial_m\bar{\chi})\bar{\sigma}_n\sigma^m\bar{\xi}\big],\\
\mathbf{X}\theta^2 m &= 2\xi\theta m - \mathrm{i}(\xi\sigma^m\bar{\theta})(\theta^2\partial_m m),\\
\mathbf{X}\bar{\theta}^2 n &= 2\bar{\xi}\bar{\theta}n + \mathrm{i}(\theta\sigma^m\bar{\xi})(\bar{\theta}^2\partial_m n),\\
\mathbf{X}(\theta\sigma^m\bar{\theta})v_m &= (\xi\sigma^m\bar{\theta} - \theta\sigma^m\bar{\xi})v_m - \mathrm{i}\big(\xi\sigma^n\bar{\theta} - \theta\sigma^n\bar{\xi}\big)(\theta\sigma^m\bar{\theta})\partial_n v_m\\
&= (\xi\sigma^m\bar{\theta} - \theta\sigma^m\bar{\xi})v_m - \tfrac{\mathrm{i}}{2}\theta^2\theta^\alpha(\sigma^m)_{\alpha\dot{\alpha}}\xi^\beta(\sigma^n)_{\beta\dot{\beta}}\epsilon^{\dot{\alpha}\dot{\beta}}\partial_n v_m\\
&\quad + \tfrac{\mathrm{i}}{2}\theta^2\epsilon^{\beta\alpha}(\sigma^n)_{\beta\dot{\beta}}\bar{\xi}^{\dot{\beta}}(\sigma^m)_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_n v_m,\\
\mathbf{X}\theta^2\bar{\theta}\bar{\lambda} &= 2(\xi\theta)(\bar{\theta}\bar{\lambda}) + \theta^2(\bar{\xi}\bar{\lambda}) - \mathrm{i}(\xi\sigma^m\bar{\theta})\theta^2(\bar{\theta}\partial_m\bar{\lambda})\\
&= (\theta\sigma^m\bar{\theta})(\bar{\lambda}\bar{\sigma}_m\xi) + \theta^2(\bar{\xi}\bar{\lambda}) + \tfrac{\mathrm{i}}{2}\theta^2\bar{\theta}^2(\xi\sigma^m\partial_m\bar{\lambda}),\\
\mathbf{X}\bar{\theta}^2\theta\psi &= \bar{\theta}^2(\xi\psi) + 2(\bar{\xi}\bar{\theta})(\theta\psi) + \mathrm{i}(\theta\sigma^m\bar{\xi})\bar{\theta}^2(\theta\partial_m\psi),\\
&= \bar{\theta}^2(\xi\psi) + (\theta\sigma^m\bar{\theta})(\bar{\xi}\bar{\sigma}_m\psi) + \tfrac{\mathrm{i}}{2}\theta^2\bar{\theta}^2\big[(\partial_m\psi)\sigma^m\bar{\xi}\,\big],\\
\mathbf{X}\theta^2\bar{\theta}^2 d &= 2\bar{\theta}^2(\xi\theta)d + 2\theta^2(\bar{\xi}\bar{\theta})d.
\end{aligned}
$$

In deriving these results one may find the relation $(\sigma^m)_{\alpha\dot{\alpha}}(\bar{\sigma}_n)^{\dot{\beta}\beta} = -2\delta_\alpha{}^\beta\delta_{\dot{\alpha}}{}^{\dot{\beta}}$ useful. Specifically, one can use it to derive more relations that can be put directly into use, such as,

$$
\begin{aligned}
(\xi\sigma^m\bar{\theta})(\theta\partial_m\phi) &= \tfrac{1}{2}(\theta\sigma^m\bar{\theta})(\xi\sigma^n\bar{\sigma}_m\partial_n\phi),\\
(\theta\phi)(\bar{\theta}\bar{\chi}) &= \tfrac{1}{2}(\theta\sigma^m\bar{\theta})(\bar{\chi}\bar{\sigma}_m\phi),
\end{aligned}
$$

Finally, we get

$$
\boxed{
\begin{aligned}
\delta f &= \xi\phi + \bar{\xi}\bar{\chi},\\
\delta\phi^\alpha &= 2\xi^\alpha m - \epsilon^{\alpha\beta}(\sigma^m)_{\beta\dot{\beta}}\bar{\xi}^{\dot{\beta}}\big(v_m - \mathrm{i}\partial_m f\big),\\
\delta\bar{\chi}_{\dot{\alpha}} &= 2\xi_{\dot{\alpha}}n + \xi^\alpha(\sigma^m)_{\alpha\dot{\alpha}}\big(v_m - \mathrm{i}\partial_m f\big),\\
\delta m &= \bar{\xi}\bar{\lambda} + \tfrac{\mathrm{i}}{2}\big[(\partial_m\phi)\sigma^m\bar{\xi}\,\big],\\
\delta n &= \xi\psi + \tfrac{\mathrm{i}}{2}\xi\sigma^m\partial_m\bar{\chi},\\
\delta v_m &= \bar{\lambda}\bar{\sigma}_m\xi + \bar{\xi}\bar{\sigma}_m\psi - \tfrac{\mathrm{i}}{2}\xi\sigma^n\bar{\sigma}_m\partial_n\phi + \tfrac{\mathrm{i}}{2}(\partial_m\bar{\chi})\bar{\sigma}_n\sigma^m\bar{\xi},\\
\delta\bar{\lambda}_{\dot{\alpha}} &= 2\bar{\xi}_{\dot{\alpha}}d - \mathrm{i}\xi^\alpha(\sigma^m)_{\alpha\dot{\alpha}}\partial_m m + \tfrac{\mathrm{i}}{2}\epsilon^{\beta\alpha}(\sigma^n)_{\beta\dot{\beta}}\bar{\xi}^{\dot{\beta}}(\sigma^m)_{\alpha\dot{\alpha}}\partial_n v_m,\\
\delta\psi^\alpha &= 2\xi^\alpha d + \mathrm{i}\epsilon^{\alpha\beta}(\sigma^m)_{\beta\dot{\beta}}\bar{\xi}^{\dot{\beta}}\partial_m m + \tfrac{\mathrm{i}}{2}\xi^\gamma(\sigma^n)_{\gamma\dot{\beta}}(\bar{\sigma}^m)^{\dot{\beta}\alpha}\partial_n v_m,\\
\delta d &= \tfrac{\mathrm{i}}{2}\big[\xi\sigma^m\partial_m\bar{\lambda} + (\partial_m\psi)\sigma^m\bar{\xi}\,\big].
\end{aligned}
}
\tag{47}
$$

We see that a scalar superfield contains x-space fields of spin-0, 1/2 and 1. It is simply the "general scalar supermultiplet" in the language of tensor calculus of rigid susy. We know that this supermultiplet is "reducible", in the sense that a subset of its components transform into themselves without on-shell condition. Therefore it may be possible to constrain some of components in this supermultiplet to zero in a susy-invariant way. In superspace formulation, these constraint can be put into an elegant form by acting covariant derivative D_M on the superfield ϕ. Here it is worth noting that the covariant derivative D_M is really "covariant" because they supercommute with all Killing vectors, namely, $[\mathrm{D}_M, X_I] = 0$. More explicitly, we have,

$$\boxed{\{\mathrm{D}_\alpha, X_\beta\} = \{\overline{\mathrm{D}}_{\dot\alpha}, \overline{X}_{\dot\beta}\} = \{\mathrm{D}_\alpha, \overline{X}_{\dot\beta}\} = \{\overline{\mathrm{D}}_{\dot\alpha}, X_\beta\} = 0.} \tag{48}$$

They can be proved directly from the definitions (38) and (44).

Now with a scalar superfield and covariant derivative in hand, we can construct new superfields. Rather than taking products of fields and covariant derivatives, we are actually more interested in applying constraint on a given superfield such as the general scalar superfield F studied above. This is because unconstrained superfields like F are usually reducible, in the sense that a subset of all component fields may transform within this subset closely. Therefore, we expect that appropriate constraint could help to pick up irreducible part from a general superfield. The constraint should of course be susy invariant, and be such that no x-space constraint is generated (in form of differential equations).

The general scalar superfield F, as we shall see, is a typical reducible superfield. However, it is not completely reducible, i.e., it can not be written as a "direct sum" of several irreducible superfields. Nevertheless, we will see that the most general renormalizable $N = 1$ susy theories in 4 dimensions can be constructed starting from two distinct ways of constraining the superfield F. The first one is given by the chiral constraint $\mathrm{D}_{\dot\alpha} F = 0$ and the second one is the reality constraint $F^\dagger = F$. The resulted superfields are called chiral superfield and vector superfield, respectively. In what follows we study these two types of constraints in turn.

5. $N = 1$ Chiral Theory

The super space formulation is an elegant and powerful tool to construct $N = 1$ rigid susy theories, in both of their classical and quantum forms. The classical action can be built as superspace integral over susy-invariant superfields and the supersymmetry is manifest during all derivations. When performing superspace integral, it is important to distinguish between chiral and unchiral superfields. This is because, the chiral superfields are constrained by chiral condition, thus its integral over the whole superspace yields zero result. Therefore, in building an action functional for susy theories, we should study two distinct possibilities. One is an integral of unchiral superfield F over the whole superspace, namely $\int \mathrm{d}z\, F(z)$, with $z=(x,\theta,\bar\theta)$ and $\mathrm{d}z=\mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar\theta$, and the other is an integral of chiral superfield Φ over a part of superspace only, namely $\int \mathrm{d}^4x\mathrm{d}^2\theta\Phi(z)+\text{h.c.}$.

5.1 Chiral superfields

The chiral superfield, by definition, is a scalar superfield Φ satisfying the chiral constraint $\mathrm{D}_\alpha\Phi=0$. To find the component form of the chiral superfield Φ, we note that the constraint is solved by $\Phi=\Phi(y_-,\theta)$ where $y_-^m=x^m+\mathrm{i}\theta\sigma^m\bar\theta$. That is, Φ can has arbitrary dependence on y_-^m and θ^α but is independent of $\bar\theta_{\dot\alpha}$. This can be seen easily from the fact that $\mathrm{D}_{\dot\alpha}y_-^m=0$ and $\mathrm{D}_{\dot\alpha}\theta^\alpha=0$. Then, we may immediately write,

$$\boxed{\begin{aligned}\Phi&=A(y_-)+\sqrt{2}\theta\psi(y_-)+\theta^2F(y_-)\\&=A(x)+\mathrm{i}\theta\sigma^m\bar\theta\partial_mA(x)+\tfrac{1}{4}\theta^2\bar\theta^2\Box A(x)\\&\quad+\sqrt{2}\theta\psi(x)-\tfrac{\mathrm{i}}{\sqrt{2}}\theta^2\big(\partial_m\psi(x)\big)\sigma^m\bar\theta+\theta^2F(x).\end{aligned}} \tag{49}$$

Similarly, we may consider the conjugate field $\Phi^\dagger$, which satisfies the constraint $\mathrm{D}_\alpha\Phi^\dagger=0$, which can be solved by $\Phi^\dagger=\Phi^\dagger(y_+,\bar\theta)$ with $y_+^m=x^m-\mathrm{i}\theta\sigma^m\bar\theta$. Then we find

$$\boxed{\begin{aligned}\Phi^\dagger&=A^*(y_+)+\sqrt{2}\bar\theta\bar\psi(y_+)+\bar\theta^2F^*(y_+)\\&=A^*(x)-\mathrm{i}\theta\sigma^m\bar\theta\partial_mA^*(x)+\tfrac{1}{4}\theta^2\bar\theta^2\Box A^*(x)\\&\quad+\sqrt{2}\bar\theta\bar\psi(x)+\tfrac{\mathrm{i}}{\sqrt{2}}\bar\theta^2\theta\sigma^m\partial_m\bar\psi(x)+\bar\theta^2F^*(x).\end{aligned}} \tag{50}$$

To find the susy transformations of component fields (A,ψ,F), it is useful to reexpress Killing vectors X_α and $\overline{X}_{\dot\alpha}$ in terms of new coordinates $(y_-,\theta,\bar\theta)$ as

$$\boxed{X_\alpha=\frac{\partial}{\partial\theta^\alpha},\quad \overline{X}_{\dot\alpha}=-\frac{\partial}{\partial\bar\theta^{\dot\alpha}}+2\mathrm{i}\theta^\alpha(\sigma^m)_{\alpha\dot\alpha}\frac{\partial}{\partial y^m}.} \tag{51}$$

Then, we have $\mathbf{X}=\xi^\alpha X_\alpha+\bar\xi_{\dot\alpha}X^{\dot\alpha}=\xi^\alpha\partial_\alpha+\bar\xi_{\dot\alpha}\partial^{\dot\alpha}+2\mathrm{i}\theta\sigma^m\bar\xi(\partial/\partial y^m)$, and thus,

$$\begin{aligned}\mathbf{X}A(y_-)&=2\mathrm{i}\theta\sigma^m\bar\xi\partial_mA(y_-),\\\mathbf{X}\theta\psi(y_-)&=\xi\psi(y_-)-\mathrm{i}\theta^2\big(\partial_m\psi(y_-)\big)\sigma^m\bar\xi,\\\mathbf{X}\theta^2F(y_-)&=2\xi\theta F(y_-).\end{aligned}$$

The susy transformation of component fields then follows directly, as

$$\boxed{\begin{aligned}\delta A&=\sqrt{2}\xi\psi,\\\delta\psi_\alpha&=\sqrt{2}\mathrm{i}(\sigma^m)_{\alpha\dot\alpha}\bar\xi^{\dot\alpha}\partial_mA+\sqrt{2}\xi_\alpha F,\\\delta F&=-\sqrt{2}\mathrm{i}(\partial_m\psi)\sigma^m\bar\xi.\end{aligned}} \tag{52}$$

The chiral superfield contains components of spin-0 and 1/2. It can be used to construct Wess-Zumino model. We will discuss this model and its possible extensions in next section.

5.2 Wess-Zumino model

The Wess-Zumino model may be the simplest rigid susy theory in 4 dimensions. In superspace formulation, it can be constructed with a single chiral superfield Φ satisfying $\overline{\mathrm{D}}_{\dot\alpha}\Phi = 0$ together with its complex conjugation. We firstly write down its action functional in superspace,

$$\boxed{S = \int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar\theta\,\Phi^\dagger\Phi + \int \mathrm{d}^4x\Big[\mathrm{d}^2\theta\left(\lambda\Phi + \frac{m}{2}\Phi^2 + \frac{g}{3}\Phi^3\right) + \text{h.c.}\Big].} \tag{53}$$

To find the component form of this action, we should work out the $\theta^2\bar\theta^2$-component of $\Phi^\dagger\Phi$ as well as the θ^2-component of Φ^2 and Φ^3. From (49) and (50), we see that

$$\begin{aligned}
\Phi^\dagger\Phi|_{\theta^2\bar\theta^2} &= \tfrac{1}{4}\big[A^*\Box A + (\Box A^*)A - 2(\partial_m A^*)(\partial^m A)\big] \\
&\quad + F^*F - \tfrac{\mathrm{i}}{2}\big(\psi\sigma^m\partial_m\bar\psi + \bar\psi\bar\sigma^m\partial_m\psi\big), \\
\Phi^2|_{\theta^2} &= 2AF - \psi\psi, \\
\Phi^3|_{\theta^2} &= 3A^2F - 3A\psi\psi.
\end{aligned}$$

Thus we get the Lagrangian in x-space to be

$$\boxed{\begin{aligned}\mathscr{L} &= A^*\Box A + F^*F - \mathrm{i}\psi\sigma^m\partial_m\bar\psi \\ &\quad + \Big[\lambda F + m\big(AF - \tfrac{1}{2}\psi\psi\big) + gA\big(AF - \psi\psi\big) + \text{h.c.}\Big].\end{aligned}} \tag{54}$$

Clearly, the Lagrangian contains two complex scalar A and F and a Weyl spinor ψ. Among them, A and ψ have canonical kinetic terms, while F is an auxiliary field without a kinetic term. Therefore, the model has $(2 + 2)$ off-shell bosonic degrees of freedom and 4 off-shell fermionic degrees of freedom. After applying equations of motion, we see that the numbers of both bosonic and fermionic states are 2. In fact, the equations of motion corresponding to F and F^* read,

$$F^* + \lambda + mA + gA^2 = 0, \qquad F + \lambda + mA^* + gA^{*2} = 0. \tag{55}$$

Substituting them back into the Lagrangian (54), we get

$$\begin{aligned}\mathscr{L} &= A^*\Box A - \mathrm{i}\psi\sigma^m\partial_m\bar\psi - \tfrac{1}{2}m(\psi\psi + \bar\psi\bar\psi) \\ &\quad - \mathscr{V}_F(A, A^*) - g\big(A\psi\psi + A^*\bar\psi\bar\psi\big),\end{aligned} \tag{56}$$

where the scalar potential,

$$\mathscr{V}_F(A, A^*) = F^*F = |\lambda + mA + gA^2|^2, \tag{57}$$

is nonnegative, with the global minimum $\mathscr{V}_F = 0$ reached by $F = 0$.

5.3 General chiral theories

Now we consider the effective action expressed in terms of some chiral superfields Φ^I and $\Phi^{\dagger I}$ only. In this case the most general renormalizable Lagrangian can be easily found on dimensional ground, and gives simply the Wess-Zumino model,

$$\mathscr{L} = \int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, \Phi^{\dagger I}\Phi^I + 2\mathrm{Re}\int \mathrm{d}^2\theta \Big(\lambda_I \Phi^I + \frac{1}{2} m_{IJ}\Phi^I\Phi^J + \frac{1}{3} g_{IJK}\Phi^I\Phi^J\Phi^K\Big). \tag{58}$$

Now, as a first step of generalization without restriction of renormalizability, we can write down the following Lagrangian, which is actually the most general non-derivative Lagrangian of chiral superfields,

$$\boxed{\mathscr{L} = \mathscr{L}_{\mathcal{K}} + \mathscr{L}_{\mathcal{W}} = \int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, \mathcal{K}(\Phi,\Phi^\dagger) + 2\,\mathrm{Re}\int \mathrm{d}^2\theta\, \mathcal{W}(\Phi),} \tag{59}$$

where $\mathcal{K}(\Phi,\Phi^\dagger)$ is an arbitrary function of both Φ^I and $\Phi^{\dagger I}$ but not their derivatives, and is called Kähler potential, $\mathcal{W}(\Phi)$ is an arbitrary function of Φ^I without derivatives, and is called superpotential.

Nonlinear σ model. We consider the Kähler potential $\mathcal{K}$ first. It is a natural extension of the quadratic term $\Phi^\dagger\Phi$, which provides correct kinetic term for each component field except for auxiliary one. In fact, the quadratic term is the lowest order among all nontrivial terms.

A quick observation is that the corresponding action is invariant under the following transformation,

$$\mathcal{K}(\Phi,\Phi^\dagger) \to \mathcal{K}(\Phi,\Phi^\dagger) + 2\,\mathrm{Re}\, f(\Phi), \tag{60}$$

where $f(\Phi)$ is an arbitrary function of Φ. This is obvious because $f(\Phi)$ is also chiral, and thus its $\theta^2\bar{\theta}^2$-term is a total derivative.

Now we consider the component form of the Kähler potential term. Recall that a left-chiral superfield Φ has the following component:

$$\begin{aligned}\Phi^I =&\ A^I(x) + \mathrm{i}\theta\sigma^m\bar{\theta}\partial_m A^I(x) + \tfrac{1}{4}\theta^2\bar{\theta}^2\Box A^I(x)\\ &+ \sqrt{2}\theta\psi^I(x) - \tfrac{\mathrm{i}}{\sqrt{2}}\theta^2\big(\partial_m\psi^I(x)\big)\sigma^m\bar{\theta} + \theta^2 F^I(x).\end{aligned} \tag{61}$$

Then, the Kähler potential term can be expanded at the point (A, A^*) as,

$$\begin{aligned}\mathscr{L}_{\mathcal{K}} = \Big[&(\partial_I\mathcal{K})_|\widetilde{\Phi}^I + (\bar{\partial}_{\bar{I}}\mathcal{K})_|\widetilde{\Phi}^{\dagger\bar{I}} + (\partial_I\bar{\partial}_{\bar{J}}\mathcal{K})_|\widetilde{\Phi}^I\widetilde{\Phi}^{\dagger\bar{J}}\\ &+ \tfrac{1}{2}(\partial_I\partial_J\bar{\partial}_{\overline{K}}\mathcal{K})_|\widetilde{\Phi}^I\widetilde{\Phi}^J\widetilde{\Phi}^{\dagger\overline{K}} + \tfrac{1}{2}(\partial_I\bar{\partial}_{\bar{J}}\bar{\partial}_{\overline{K}}\mathcal{K})_|\widetilde{\Phi}^I\widetilde{\Phi}^{\dagger\bar{J}}\widetilde{\Phi}^{\dagger\overline{K}}\\ &+ \tfrac{1}{4}(\partial_I\partial_J\bar{\partial}_{\overline{K}}\bar{\partial}_{\bar{L}}\mathcal{K})_|\widetilde{\Phi}^I\widetilde{\Phi}^J\widetilde{\Phi}^{\dagger\overline{K}}\widetilde{\Phi}^{\dagger\bar{L}}\Big]_{\theta^2\bar{\theta}^2}.\end{aligned}$$

where $\widetilde{\Phi}^I = \Phi^I - A^I$, and the subscript " $|$ " in $(\partial_I K)_|$ and similar expressions means taking values at (A, A^*). Using

$$\begin{aligned}
\widetilde{\Phi}^I\widetilde{\Phi}^{\dagger J}|_{\theta^2\bar\theta^2} =& -\tfrac{1}{2}(\partial_m A^I)(\partial^m A^{*\bar J}) + F^I F^{*\bar J} \\
&- \tfrac{\mathrm{i}}{2}\big[\psi^I\sigma^m\partial_m\bar\psi^{\bar J} - (\partial_m\psi^I)\sigma^m\bar\psi^{\bar J}\big], \\
\widetilde{\Phi}^I\widetilde{\Phi}^J\widetilde{\Phi}^{\dagger\overline{K}}|_{\theta^2\bar\theta^2} =& \tfrac{\mathrm{i}}{2}\big[(\partial_m A^I)\psi^J\sigma^m\bar\psi^{\overline{K}} + (\partial_m A^J)\psi^i\sigma^m\bar\psi^{\overline{K}}\big] - \psi^I\psi^J F^{*\overline{K}}, \\
\widetilde{\Phi}^I\widetilde{\Phi}^J\widetilde{\Phi}^{\dagger\overline{K}}\widetilde{\Phi}^{\dagger\bar L}|_{\theta^2\bar\theta^2} =& (\psi^I\psi^J)(\bar\psi^{\overline{K}}\bar\psi^{\bar L}),
\end{aligned}$$

On the other hand, we may view Φ^I and $\Phi^{\dagger\bar I}$ as complex maps from spacetime manifold to a complex manifold with coordinates $(A^I, A^{*\bar I})$. From this viewpoint, we add bars for anti-holomorphic indices, and use the notation $\partial_I = \partial/\partial A^I$, $\bar\partial_{\bar I} = \partial/\partial A^{*\bar I}$. Then $\mathcal{K}(A, A^*)$ provides a Kähler metric on the target space by

$$g_{I\bar J} = \partial_I\bar\partial_{\bar J}\mathcal{K}(A, A^*). \tag{62}$$

Then we have,

$$\begin{aligned}
(\partial_I\partial_J\bar\partial_{\overline{K}}\mathcal{K})_| &= \partial_I g_{J\overline{K}} = g_{K\overline{K}}\Gamma^K_{IJ}, \\
(\partial_I\partial_J\bar\partial_{\overline{K}}\bar\partial_{\bar L}\mathcal{K})_| &= \partial_I\bar\partial_{\bar L} g_{J\overline{K}} = R_{J\overline{K}I\bar L} + \Gamma^K_{IJ} g_{K\bar I}\Gamma^{\bar I}_{\overline{K}\bar L}.
\end{aligned}$$

Thus the Kähler potential in the Lagrangian has the following component form,

$$\begin{aligned}
\mathscr{L}_{\mathcal{K}} = & g_{I\bar J}\Big[-(\partial_m A^I)(\partial_m A^{*\bar J}) + F^I F^{*\bar J} - \frac{\mathrm{i}}{2}\big(\psi^I\sigma^m(\overline{\mathcal{D}}_m\bar\psi)^{\bar J} - (\mathcal{D}_m\psi)^I\sigma^m\bar\psi^{\bar J}\big)\Big] \\
& - \frac{1}{2}g_{K\overline{K}}\big(\Gamma^K_{IJ}\psi^I\psi^J F^{*\overline{K}} + \Gamma^{\overline{K}}_{\bar I\bar J}\bar\psi^{\bar I}\bar\psi^{\bar J}F^K\big) + \frac{1}{4}\big(\partial_I\bar\partial_{\bar L} g_{J\overline{K}}\big)(\psi^I\psi^J)(\bar\psi^{\overline{K}}\bar\psi^{\bar L}),
\end{aligned} \tag{63}$$

where $(\mathcal{D}_m\psi)^I = \partial_m\psi^I + \Gamma^I_{JK}(\partial_m A^J)\psi^K$, $(\overline{\mathcal{D}}_m\bar\psi)^{\bar I} = \partial_m\bar\psi^{\bar I} + \Gamma^{\bar I}_{\bar J\overline{K}}(\partial_m A^{*\bar J})\bar\psi^{\overline{K}}$.

Then we consider the superpotential $\mathcal{W}(\Phi)$. To find the corresponding component form, we also expand it around A^I, which leads to,

$$\mathscr{L}_{\mathcal{W}} = 2\,\mathrm{Re}\Big[(\partial_I\mathcal{W})_| F^I - \tfrac{1}{2}(\partial_I\partial_J\mathcal{W})\psi^I\psi^J\Big]. \tag{64}$$

Then, combining (63) and (64), we can again solve the auxiliary field F^I from its equation of motion to be,

$$F^I = -g^{I\bar J}(\bar\partial_{\bar J}\mathcal{W}^\dagger)_| + \tfrac{1}{2}\Gamma^I_{JK}\psi^J\psi^K. \tag{65}$$

Substitute this solution back into the Lagrangian (63) and (64), we get,

$$\boxed{\begin{aligned}
\mathscr{L} = & -g_{I\bar J}\Big[(\partial_m A^I)(\partial_m A^{*\bar J}) + \frac{\mathrm{i}}{2}\Big(\psi^I\sigma^m(\overline{\mathcal{D}}_m\bar\psi)^{\bar J} - (\mathcal{D}_m\psi)^I\sigma^m\bar\psi^{\bar J}\Big)\Big] \\
& + \frac{1}{4}R_{J\overline{K}I\bar L}(\psi^I\psi^J)(\bar\psi^{\overline{K}}\bar\psi^{\bar L}) - g^{I\bar J}(\partial_I\mathcal{W})_|(\bar\partial_{\bar J}\mathcal{W}^\dagger)_| \\
& - \frac{1}{2}(\mathcal{D}_I\mathcal{D}_J\mathcal{W})_|\psi^I\psi^J - \frac{1}{2}(\overline{\mathcal{D}}_{\bar I}\overline{\mathcal{D}}_{\bar J}\mathcal{W})_|\bar\psi^{\bar I}\bar\psi^{\bar J}.
\end{aligned}} \tag{66}$$

6. $N = 1$ Gauge Theories

6.1 Vector superfields

The vector superfield V is actually a "real" general scalar superfield, satisfying the reality condition $V^\dagger = V$. From the discussion above we see that a general scalar superfield contains a complex vector component. The reality condition here constrains this vector field to be real also. Thus the vector superfield V may be used to construct gauge theories. Therefore, we need to introduce gauge transformation for vector component which should be consistent with supersymmetry. Meanwhile, the independent components in V are much more than needed. As we will show now, these two problems can be solved together in an elegant way by introducing super gauge transformation.

To this end, we note that the sum of a chiral superfield with its complex conjugate is a vector superfield. Now let $\mathrm{i}\Lambda$ be a chiral superfield, in which the factor i is conventional. Then, the super gauge transformation for the vector superfield V is defined to be

$$\boxed{V \to V + \mathrm{i}(\Lambda - \Lambda^\dagger).} \tag{67}$$

To see the effect of this transformation, we goes to the component field representation. For V, we write,

$$\begin{aligned} V = {} & C(x) + \mathrm{i}\theta\chi(x) - \mathrm{i}\bar\theta\bar\chi(x) + \tfrac{\mathrm{i}}{2}\theta^2\big[M(x) + \mathrm{i}N(x)\big] - \tfrac{\mathrm{i}}{2}\bar\theta^2\big[M^*(x) - \mathrm{i}N^*(x)\big] \\ & - \theta\sigma^m\bar\theta v_m(x) + \mathrm{i}\theta^2\bar\theta\big[\bar\lambda(x) + \tfrac{\mathrm{i}}{2}\bar\sigma^m\partial_m\chi(x)\big] - \mathrm{i}\bar\theta^2\theta\big[\lambda(x) + \tfrac{\mathrm{i}}{2}\sigma^m\partial_m\bar\chi(x)\big] \\ & + \tfrac{1}{4}\theta^2\bar\theta^2\big[D(x) + \tfrac{1}{2}\Box C(x)\big]. \end{aligned} \tag{68}$$

The reality condition requires that $C(x), M(x), N(x), D(x)$ and $v_m(x)$ to be real field. This parametrization for V is specially chosen so that $\lambda(x)$ and $D(x)$ are gauge invariant, as will be seen. For the gauge parameter superfield $\mathrm{i}\Lambda$, we can write it as $\mathrm{i}\Lambda = A(y_-) + \sqrt{2}\theta\psi(y_-) + \theta^2 F(y_-)$ as always. Then, clearly we have

$$\begin{aligned} \mathrm{i}(\Lambda - \Lambda^\dagger) = {} & A + A^* + \sqrt{2}(\theta\psi + \bar\theta\bar\psi) + \theta^2 F + \bar\theta^2 F^* + \mathrm{i}\theta\sigma^m\bar\theta\partial_m(A - A^*) \\ & + \tfrac{\mathrm{i}}{\sqrt{2}}\theta^2\bar\theta\bar\sigma^m\partial_m\psi + \tfrac{\mathrm{i}}{\sqrt{2}}\bar\theta^2\theta\sigma^m\partial_m\bar\psi + \tfrac{1}{4}\theta^2\bar\theta^2\Box(A + A^*). \end{aligned} \tag{69}$$

Then the gauge transformation (67) can be rewritten in component form, as

$$\begin{aligned} C \to C + A + A^*, \qquad & v_m \to v_m - \mathrm{i}\partial_m(A - A^*), \\ \chi^\alpha \to \chi^\alpha - \sqrt{2}\mathrm{i}\psi^\alpha, \qquad & \lambda^\alpha \to \lambda^\alpha, \\ M + \mathrm{i}N \to M + \mathrm{i}N - 2\mathrm{i}F, \qquad & D \to D. \end{aligned} \tag{70}$$

We see that the vector component v_m has the correct gauge transformation as required. The two higher components λ^α and D are indeed gauge invariant, while the three lower components C, χ^α, and $M_\mathrm{i}N$, can be gauged away completely. Therefore we can choose a gauge by using the degrees of freedom from $\mathrm{Re}\,A, \psi^\alpha, F$, such that V has the following form:

$$\boxed{V_{\mathrm{WZ}} = -\theta\sigma^m\bar{\theta}v_m(x) + \mathrm{i}\theta^2\bar{\theta}\bar{\lambda}(x) - \mathrm{i}\bar{\theta}^2\theta\lambda(x) + \tfrac{1}{2}\theta^2\bar{\theta}^2 D(x).} \tag{71}$$

This is the so-called Wess-Zumino gauge. Note that the gauge freedom in vector field v_m is not fixed, which corresponds to the freedom in $\mathrm{Im}\,A$.

A gauge invariant superfield W_α can be constructed from V via

$$\boxed{W_\alpha = -\tfrac{1}{4}\bar{\mathrm{D}}^2\mathrm{D}_\alpha V.} \tag{72}$$

It complex conjugate $\overline{W}_{\dot{\alpha}} = -\frac{1}{4}\mathrm{D}^2\bar{\mathrm{D}}_{\dot{\alpha}}V$ is of course also gauge invariant. Indeed, the gauge transformation of W_α is given by

$$\delta W_\alpha = -\tfrac{\mathrm{i}}{4}\bar{\mathrm{D}}^2\mathrm{D}_\alpha\Lambda = \tfrac{\mathrm{i}}{4}\bar{\mathrm{D}}^{\dot{\alpha}}\{\mathrm{D}_\alpha, \bar{\mathrm{D}}_{\dot{\alpha}}\}\Lambda = \tfrac{1}{2}(\sigma^m)_{\alpha\dot{\alpha}}\partial_m\bar{\mathrm{D}}^{\dot{\alpha}}\Lambda = 0. \tag{73}$$

The component form of W_α can be found directly from its definition. Since it is gauge invariant, we can begin with V_{WZ} in Wess-Zumino gauge. The calculation gets simpler in $(y_-, \theta, \bar{\theta})$ or $y_+, \theta, \bar{\theta}$ coordinates. Thus it is useful to keep in mind the covariant derivatives in these coordinates, given by,

$$\boxed{\begin{cases} \mathrm{D}_\alpha = \dfrac{\partial}{\partial\theta^\alpha} + 2\mathrm{i}(\sigma^m)_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\dfrac{\partial}{\partial y_-^m}, \\ \bar{\mathrm{D}}_{\dot{\alpha}} = -\dfrac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}, \end{cases} \qquad \text{in } (y_-, \theta, \bar{\theta}) \text{ coordinates}} \tag{74}$$

and,

$$\boxed{\begin{cases} \mathrm{D}_\alpha = \dfrac{\partial}{\partial\theta^\alpha}, \\ \bar{\mathrm{D}}_{\dot{\alpha}} = -\dfrac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - 2\mathrm{i}\theta^\alpha(\sigma^m)_{\alpha\dot{\alpha}}\dfrac{\partial}{\partial y_+^m}, \end{cases} \qquad \text{in } (y_+, \theta, \bar{\theta}) \text{ coordinates}} \tag{75}$$

So, begin with (71), we firstly go into $(y_+, \theta, \bar{\theta})$ coordinates to simplify D_α, then to $(y_-, \theta, \bar{\theta})$ to simplify $\bar{\mathrm{D}}^2$,

$$\begin{aligned} V_{\mathrm{WZ}} =& -\theta\sigma^m\bar{\theta}v_m(y_+) + \mathrm{i}\theta^2\bar{\theta}\bar{\lambda}(y_+) - \mathrm{i}\bar{\theta}^2\theta\lambda(y_+) \\ &+ \tfrac{1}{2}\theta^2\bar{\theta}^2\big[D(y_+) + \mathrm{i}\partial_m v^m(y_+)\big] \\ \Rightarrow \mathrm{D}_\alpha V_{\mathrm{WZ}} =& -(\sigma^m)_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}v_m(y_+) + 2\mathrm{i}\theta_\alpha\big(\bar{\theta}\bar{\lambda}(y_+)\big) - \mathrm{i}\bar{\theta}^2\lambda_\alpha(y_+) \\ &+ \theta_\alpha\bar{\theta}^2\big[D(y_+) + \partial_m v^m(y_+)\big] \\ =& -(\sigma^m)_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\big[v_m(y_-) - 2\mathrm{i}\theta\sigma^n\bar{\theta}\partial_n v_m(y_-)\big] \\ &+ 2\mathrm{i}\theta_\alpha\big[\bar{\theta}\bar{\lambda}(y_-) - 2\mathrm{i}(\theta\sigma^n\bar{\theta})\bar{\theta}\partial_n\bar{\lambda}(y_-)\big] \\ &- \mathrm{i}\bar{\theta}^2\lambda_\alpha(y_-) + \theta_\alpha\bar{\theta}^2\big[D(y_-) + \partial_m v^m(y_-)\big] \end{aligned}$$

$$\Rightarrow W_\alpha = \mathrm{i}(\sigma^m)_{\alpha\dot{\alpha}}(\theta\sigma^n\epsilon)^{\dot{\alpha}}\partial_n v_m(y_-) + \tfrac{1}{2}\theta^2(\sigma^n)_{\alpha\dot{\alpha}}\partial_n\bar{\lambda}^{\dot{\alpha}}(y_-)$$
$$-\mathrm{i}\lambda_\alpha(y_-) + \theta_\alpha\big[D(y_-) + \mathrm{i}\partial_m v^m(y_-)\big]$$

Then, using the relation,

$$(\sigma^n)_{\alpha\dot{\alpha}}(\sigma^m)_{\beta\dot{\beta}} - (\sigma^m)_{\alpha\dot{\alpha}}(\sigma^n)_{\beta\dot{\beta}} = 2(\sigma^{nm}\epsilon)_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} + 2(\epsilon\bar{\sigma}^{nm})_{\dot{\alpha}\dot{\beta}}\epsilon_{\alpha\beta},$$
$$(\sigma^n)_{\alpha\dot{\alpha}}(\sigma^m)_{\beta\dot{\beta}} + (\sigma^m)_{\alpha\dot{\alpha}}(\sigma^n)_{\beta\dot{\beta}} = -\eta^{nm}\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} + 4(\sigma^{\ell n}\epsilon)_{\alpha\beta}(\epsilon\bar{\sigma}^{\ell m})_{\dot{\alpha}\dot{\beta}},$$

we finally reach the following expression for W_α, and similarly for $\overline{W}_{\dot{\alpha}}$,

$$\boxed{\begin{aligned} W_\alpha &= -\mathrm{i}\lambda_\alpha(y_-) + \theta_\alpha D(y_-) - \mathrm{i}(\sigma^{mn}\theta)_\alpha v_{mn}(y_-) + \theta^2(\sigma^m)_{\alpha\dot{\alpha}}\partial_m\bar{\lambda}^{\dot{\alpha}}(y_-), \\ \overline{W}^{\dot{\alpha}} &= \mathrm{i}\bar{\lambda}^{\dot{\alpha}}(y_+) + \bar{\theta}^{\dot{\alpha}}D(y_+) + \mathrm{i}(\bar{\sigma}^{mn}\bar{\theta})^{\dot{\alpha}}v_{mn}(y_+) + \bar{\theta}^2(\bar{\sigma}^m)^{\dot{\alpha}\alpha}\partial_m\lambda_\alpha(y_+), \end{aligned}} \tag{76}$$

where $v_{mn} = \partial_m v_n - \partial_n v_m$ is the ordinary field strength. Note that W_α and $\overline{W}_{\dot{\alpha}}$, are spinorial superfield and are chiral (antichiral), namely, $\overline{\mathrm{D}}_{\dot{\alpha}}W_\alpha = 0$, $\mathrm{D}_\alpha\overline{W}_{\dot{\alpha}} = 0$, as are obvious from their definition. However, W_α and $\overline{W}_{\dot{\alpha}}$ are not the most general spinorial (anti)chiral superfield, i.e., they satisfy certain constraint. The constraint is actually from the reality condition for $V^\dagger = V$, which implies that

$$\mathrm{D}^\alpha W_\alpha = \overline{\mathrm{D}}_{\dot{\alpha}}\overline{W}^{\dot{\alpha}}. \tag{77}$$

A proof for this relation is straightforward,

$$\mathrm{D}^\alpha W_\alpha = \tfrac{1}{4}\big(\{\mathrm{D}^\alpha, \overline{\mathrm{D}}^{\dot{\alpha}}\} - \overline{\mathrm{D}}^{\dot{\alpha}}\mathrm{D}^\alpha\big)\big(\{\overline{\mathrm{D}}_{\dot{\alpha}}, \mathrm{D}_\alpha\} - \mathrm{D}_\alpha\overline{\mathrm{D}}_{\dot{\alpha}}\big)V$$
$$= \tfrac{1}{4}\overline{\mathrm{D}}^{\dot{\alpha}}\mathrm{D}^2\overline{\mathrm{D}}_{\dot{\alpha}}V = \overline{\mathrm{D}}_{\dot{\alpha}}\overline{W}^{\dot{\alpha}}.$$

In fact, one may begin directly with a constraint chiral spinorial superfield W_α and to find its component form (76). The strategy we present here, however, is to firstly solve the constraint (77) by an unconstrained vector superfield V through (72). The unconstrained solution V, then, is said to be the prepotential. This is fully in parallel with the case of ordinary Maxwell theory, where one can begin with either the vector potential A_m subjected to gauge transformation, or the field strength F_{mn} subjected to Bianchi identity. Indeed, the superfield W_α can be thought as a susy generalization of field strength. In next section we will study how to use it to construct super Abelian gauge theory as well as its non-Abelian generalization.

6.2 Super-Maxwell theory

Before studying the general case of non-Abelian gauge theory, we firstly take the Abelian gauge theory as a warming-up exercise, which we refer to as super-Maxwell theory.

As we learned in last section, an susy invariant action functional for gauge field can be built from the super field strength W_α defined in (72). In fact, a proper action is given by

$$\boxed{S=\int \mathrm{d}^4x\Big[\mathrm{d}^2\theta\,\frac{1}{4}W^\alpha W_\alpha+\text{h.c.}\Big].} \tag{78}$$

To find the component form, we evaluate θ^2-component of $W^\alpha W_\alpha$ as follows,

$$W^\alpha W_\alpha|_{\theta^2}=-2\mathrm{i}\lambda\sigma^m\partial_m\bar{\lambda}-\tfrac{1}{2}v_{mn}v^{mn}+\tfrac{\mathrm{i}}{2}v_{mn}\widetilde{v}^{mn}+D^2,$$

where $\widetilde{v}_{mn}=\frac{1}{2}\epsilon_{mnpq}v^{pq}$ is the dual field strength. The term $v_{mn}\widetilde{v}^{mn}$ has no effect in Abelian gauge theory since it is a total derivative while the Abelian gauge theory is topologically trivial. Thus the component form of the Lagrangian is

$$\boxed{\mathscr{L}=-\frac{1}{4}v_{mn}v^{mn}+\frac{1}{2}D^2-\mathrm{i}\lambda\sigma^m\partial_m\bar{\lambda}.} \tag{79}$$

Therefore a pure (without matter field) supersymmetric Abelian gauge theory contains a free vector boson v_m (photon) together with its susy partner λ (photino), which is a free, massless and $U(1)$-neutral Majorana fermion. The counting of degrees of freedom goes as #boson$=1+3$ and #fermion$=4$ for off-shell states, and #boson=#fermion$=2$ for on-shell states. We note that when counting off-shell states, the gauge freedom in photon should be excluded because the theorem of equal bosonic and fermionic states is established for gauge invariant states.

Here we briefly touch on the susy theory for a massive vector field. The theory is clearly not gauge invariant. Thus the mass term m^2V^2 should be built from the original form of V in (68). The $\theta^2\bar{\theta}^2$-component of V^2 then involves not only v_m, λ, and D, but also M, N, C, and χ. This is easy to be understood since an extra polarization state for massive vector particle needs corresponding susy counterpart.

Now we study the matter couplings of the super Maxwell theory. By matter we mean some chiral superfields Φ_i. If these fields are charged under local $U(1)$ gauge symmetry with $U(1)$ charge t_i, their gauge transformation should be given by

$$\Phi_i\to e^{-\mathrm{i}t_i\Lambda}\Phi_i,\quad \Phi_i^\dagger\to e^{\mathrm{i}t_i\Lambda^\dagger}\Phi_i^\dagger. \tag{80}$$

Clearly the original bilinear form $\Phi^\dagger\Phi$ in Wess-Zumino model is not gauge invariant. Recall that the vector superfield V transforms under local $U(1)$ according to (67), then a gauge invariant bilinear constructed from Φ_i and its complex conjugation is given by $\Phi_i^\dagger e^{t_iV}\Phi_i$. Then, a general renormalizable $U(1)$ gauge theory has the following action,

$$\begin{aligned}S=&\int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\,\Phi_i^\dagger e^{t_iV}\Phi_i+\int\mathrm{d}^4x\Big[\mathrm{d}^2\theta\,\frac{1}{4}W^\alpha W_\alpha+\text{h.c.}\Big]\\&+\int\mathrm{d}^4x\Big[\mathrm{d}^2\theta\Big(\frac{1}{2}m_{ij}\Phi_i\Phi_j+\frac{1}{3}g_{ijk}\Phi_i\Phi_j\Phi_k\Big)+\text{h.c.}\Big].\end{aligned} \tag{81}$$

The couplings m_{ij} and g_{ijk} should vanish for those terms with $t_i+t_j \neq 0$ or $t_i+t_j+t_k \neq 0$, respectively. The integrand in the second line of this action is conventionally referred to as superpotential. To find the component form of the gauge invariant kinetic term for matter fields, we can evaluate V in Wess-Zumino gauge. Then,

$$\begin{aligned}\Phi^\dagger e^{tV}\Phi|_{\theta^2\bar\theta^2} = {}& F^*F + A^*\Box A - \mathrm{i}\bar\psi\bar\sigma^m\partial_m\psi \\ & + tv_m\big(\tfrac{1}{2}\bar\psi\bar\sigma^m\psi + \tfrac{\mathrm{i}}{2}A^*\partial^m A - \tfrac{\mathrm{i}}{2}A\partial^m A^*\big) \\ & - \tfrac{\mathrm{i}}{\sqrt{2}}t\big(A\bar\lambda\bar\psi - A^*\lambda\psi\big) + \tfrac{1}{2}\big(tD - \tfrac{1}{2}t^2 v_m v^m\big)A^*A. \end{aligned} \tag{82}$$

SQED. As an prototypical example, we write down the Lagrangian of super quantum electrodynamics (SQED) in its superfield form. By construction, SQED is a $U(1)$ gauge theory with two chiral superfield $\Phi_\pm$ with charge ± 1. The gauge transformation of them are given by $\Phi_\pm \to e^{\mp\mathrm{i}e\Lambda}\Phi_\pm$ which we rescale by the gauge coupling e. Then, the action reads,

$$\begin{aligned} S = {}& \int \mathrm{d}^4x\Big(\mathrm{d}^2\theta\,\frac{1}{4}W^\alpha W_\alpha + \text{h.c.}\Big) + \int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar\theta\,\Phi^\dagger_\pm e^{\pm eV}\Phi_\pm \\ & + m\int \mathrm{d}^4x\big(\mathrm{d}^2\theta\,\Phi_+\Phi_- + \mathrm{d}^2\bar\theta\,\Phi^\dagger_+\Phi^\dagger_-\big). \end{aligned} \tag{83}$$

Therefore, SQED contains a massless photon, which is a vector gauge particle, a photino, the susy counterpart of photon, which is a massless Majorana fermion without $U(1)$ charge. Furthermore, SQED contains two complex scalar fields (selectron) with $U(1)$ charges ± 1, as well as two Weyl spinors (electron and positron) with charges ± 1 also. They can be gathered into a single Dirac spinor. Of course, electron (positron) and selectron have equal mass.

6.3 Non-Abelian gauge theories

Now we come to the non-Abelian theory with gauge group G with the corresponding algebra given by $[T^a, T^b] = \mathrm{i}t^{abc}T^c$. The generators T^a are chosen to be Hermitian and are normalized as $\operatorname{tr}(T_aT_b) = k\delta_{ab}$ with $k > 0$. The gauge transformation in this case is most straightforward for matter fields, which is simply given by $\Phi \to e^{-\mathrm{i}\Lambda}\Phi$ and $\Phi^\dagger \to \Phi^\dagger e^{\mathrm{i}\Lambda^\dagger}$, where $\Lambda = \Lambda^a T^a$. Then, in order that $\Phi^\dagger e^{2V}\Phi$ remains gauge invariant, we see that the gauge transformation for the vector superfield $V = V^aT^a$ should be generalized to,

$$e^{2V} \to e^{-\mathrm{i}\Lambda^\dagger}e^{2V}e^{\mathrm{i}\Lambda}. \tag{84}$$

Here we write e^{2V} rather than e^V to recover the correct field normalization in the Lagrangian. After expanding this transformation rule we see that at the leading order, $V \to V + \frac{\mathrm{i}}{2}(\Lambda - \Lambda^\dagger) + \cdots$, thus one can still use Wess-Zumino gauge in non-Abelian case. The full expression for this transformation rule at the linear order in Λ reads,

$$V \to V + \mathrm{i}\,\mathrm{ad}_V(\Lambda + \Lambda^\dagger) + \mathrm{i}\,\mathrm{ad}_V\big[\coth\big(\mathrm{ad}_V\big)(\Lambda - \Lambda)^\dagger\big] + \mathcal{O}(\Lambda^2). \tag{85}$$

The gauge covariant super field strength can be defined as,

$$\boxed{W_\alpha = -\frac{1}{8}\overline{\mathrm{D}}^2\big(e^{-2V}\mathrm{D}_\alpha e^{2V}\big),} \tag{86}$$

Then W_α transforms under local G as,

$$\begin{aligned} W_\alpha \to &-\tfrac{1}{8}\overline{\mathrm{D}}^2\big[e^{-\mathrm{i}\Lambda}e^{-2V}e^{\mathrm{i}\Lambda^\dagger}\mathrm{D}_\alpha\big(e^{-\mathrm{i}\Lambda^\dagger}e^{2V}e^{\mathrm{i}\Lambda}\big)\big] \\ =&\, e^{-\mathrm{i}\Lambda}W_\alpha e^{\mathrm{i}\Lambda} - \tfrac{1}{8}e^{-\mathrm{i}\Lambda}\overline{\mathrm{D}}^2\mathrm{D}_\alpha e^{\mathrm{i}\Lambda} \\ =&\, e^{-\mathrm{i}\Lambda}W_\alpha e^{\mathrm{i}\Lambda}. \end{aligned} \tag{87}$$

As claimed, W_α is gauge covariant. The component form of W_α can also be easily found in Wess-Zumino gauge in which $V_{\mathrm{WZ}}^3 = 0$. Then we have,

$$\begin{aligned} e^{2V_{\mathrm{WZ}}} =&\, 1 - 2\theta\sigma^m\bar\theta v_m(y_+) + 2\mathrm{i}\theta^2\bar\theta\bar\lambda(y_+) - 2\mathrm{i}\bar\theta^2\theta\lambda(y_+) \\ &+ \theta^2\bar\theta^2\big[D(y_+) + \mathrm{i}\partial_m v^m(y_+) - v_m(y_+)v^m(y_+)\big], \end{aligned} \tag{88}$$

and finally we find that,

$$\boxed{\begin{aligned} W_\alpha =& -\mathrm{i}\lambda_\alpha(y_-) + \theta_\alpha D(y_-) - \mathrm{i}(\sigma^{mn}\theta)_\alpha v_{mn}(y_-) \\ &+ \theta^2(\sigma^m)_{\alpha\dot\alpha}\nabla_m\bar\lambda^{\dot\alpha}(y_-), \\ \overline{W}^{\dot\alpha} =&\, \mathrm{i}\bar\lambda^{\dot\alpha}(y_+) + \bar\theta^{\dot\alpha}D(y_+) + \mathrm{i}(\bar\sigma^{mn}\bar\theta)^{\dot\alpha}v_{mn}(y_+) \\ &+ \bar\theta^2(\bar\sigma^m)^{\dot\alpha\alpha}\nabla_m\lambda_\alpha(y_+), \end{aligned}} \tag{89}$$

where $v_{mn} = \partial_m v_n - \partial_n v_m + \mathrm{i}[v_m, v_n]$ is the non-Abelian field strength, $\widetilde{v}_{mn} = \frac{1}{2}\epsilon_{mnpq}v^{pq}$ is the dual field strength, and $\nabla_m\bar\lambda^{\dot\alpha} = \partial_m\bar\lambda^{\dot\alpha} + \mathrm{i}[v_m, \bar\lambda^{\dot\alpha}]$ is the gauge covariant derivative for $\bar\lambda^{\dot\alpha}$. With these expressions, we can evaluate the bilinear term $W^\alpha W_\alpha$ as we did for Abelian theory,

$$W^\alpha W_\alpha|_{\theta^2} = -2\mathrm{i}\lambda\sigma^m\partial_m\bar\lambda - \tfrac{1}{2}v_{mn}v^{mn} + \tfrac{\mathrm{i}}{2}v_{mn}\widetilde{v}^{mn} + D^2,$$

The topological term $v_{mn}\tilde{v}^{mn}$ could be important for non-Abelian gauge group. To keep track of this term in the action, we introduce a complex coefficient τ, defined to be

$$\boxed{\tau = \frac{\theta_{\mathrm{YM}}}{2\pi} + \frac{4\pi\mathrm{i}}{g^2},} \tag{90}$$

where g is the gauge coupling and θ_{YM} is the topological angle. Then, a well-defined action can be written as

$$\boxed{S = \frac{\tau}{16\pi\mathrm{i}k}\int \mathrm{d}^4x\mathrm{d}^2\theta\,\mathrm{tr}\,W^\alpha W_\alpha + \text{h.c.}.} \tag{91}$$

The definition for τ and the choice of imaginary part in the action is conventional in literature, and k is arises from normalization of generators through $\mathrm{tr}\,(T_aT_b) = k\delta_{ab}$. Then, we find the component form of the Lagrangian to be

$$\boxed{\mathscr{L} = \frac{1}{k}\,\mathrm{tr}\left[-\frac{1}{4g^2}v_{mn}v^{mn} + \frac{1}{2g^2}D^2 - \frac{\mathrm{i}}{g^2}\lambda\sigma^m\partial_m\bar{\lambda} + \frac{\theta_{\mathrm{YM}}}{32\pi^2}v_{mn}\tilde{v}^{mn}\right].} \tag{92}$$

To include matter superfield is also straightforward. For a chiral superfield Φ_i lying in some representation of G with matrix T^a_{ij}, the kinetic term is given by $\Phi_i^\dagger(e^{2V})_{ij}\Phi_j$. In components, we get,

$$\begin{aligned}\Phi^\dagger e^{2V}\Phi|_{\theta^2\bar{\theta}^2} = {}& \tfrac{1}{4}\big[A^*\Box A + (\Box A^*)A\big] - \tfrac{1}{2}(\partial_m A^*)(\partial^m A)\\ &+ \mathrm{i}A^* v_m\partial^m A - \mathrm{i}(\partial_m A^*)v^m A + A^*\big(D - v_m v^m\big)A\\ &- \tfrac{\mathrm{i}}{2}\big[\bar{\psi}\bar{\sigma}^m\partial_m\psi - (\partial_m\bar{\psi})\bar{\sigma}^m\psi\big] + \bar{\psi}\bar{\sigma}^m v_m\psi + F^*F\\ &+ \sqrt{2}\mathrm{i}\big(A^*\lambda\psi - \bar{\psi}\bar{\lambda}A\big).\end{aligned} \tag{93}$$

Then, upon integration by parts, we get the Lagrangian for matter field as

$$\boxed{\begin{aligned}\mathscr{L} = {}& -(\nabla_m A)^\dagger(\nabla^m A) - \mathrm{i}\bar{\psi}\bar{\sigma}^m\nabla_m\psi + F^*F\\ &+ A^*DA + \sqrt{2}\mathrm{i}(A^*\lambda\psi - \bar{\psi}\bar{\lambda}A),\end{aligned}} \tag{94}$$

where $\nabla_m A = (\partial_m + \mathrm{i}v_m)A$ and $\nabla_m\psi = (\partial_m + \mathrm{i}v_m)\psi$, and v should be understood as matrix under the representation to which the matter fields belong. Now, combining the Lagrangian for gauge fields (92) and for matter fields (94), we see that the auxiliary field D, when substituted by its field equation $D^a = -g^2A^*T^aA$, generates additional term in the scalar potential, given by

$$\mathscr{V}_D(A, A^*) = \frac{1}{2g^2}D^aD^a = \frac{g^2}{2}(A^*T^aA)^2, \tag{95}$$

which, just like $\mathscr{V}_F$ in (57), also contributes a nonnegative term to the scalar potential, as required by supersymmetry.

7. Path Integral Quantization

7.1 Superspace path integral

Now we introduce path integral quantization in superspace formalism. The basic idea is the same with the ordinary path integral formulation. We begin with the action functional, which is an integral over the whole superspace for nonchiral fields and a part of the superspace for chiral fields. In general, we may write

$$S = \int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\,\mathscr{L}[V,\Phi,\Phi^\dagger] + \Big(\int \mathrm{d}^4x\mathrm{d}^2\theta\,\mathscr{L}[\Phi] + \mathrm{h.c.}\Big), \tag{96}$$

where Φ is a chiral superfield and V is a vector superfield. The most general renormalizable $N = 1$ rigid susy theories can be expressed solely in terms of Φ, $\Phi^\dagger$, and V.

To find a corresponding quantum theory for this classical action, we should define the partition function $Z[J]$, from which we can find the generating functional $G[J]$ for connected Green's functions, as well as the 1PI effective action $\Gamma[\phi_{\text{cl}}]$. All these quantities can be defined as usual. In particular, we can still write down a perturbative expansion for $Z[J]$. Suppose the action S can be decomposed into a free part and an interacting part, $S = S_{\text{free}} + S_{\text{int.}}$, then,

$$\begin{aligned} Z[J] = &\, Z[0]\int \mathcal{D}\Phi\mathcal{D}\Phi^\dagger\mathcal{D}V \exp\big[\mathrm{i}S + \mathrm{i}V\cdot J_V + \mathrm{i}(\Phi\cdot J_\Phi + \text{h.c})\big] \\ = &\exp\bigg(\mathrm{i}S_{\text{int}}\Big[\frac{\delta}{\mathrm{i}\delta J_V},\frac{\delta}{\mathrm{i}\delta J_\Phi},\frac{\delta}{\mathrm{i}\delta J_{\Phi^\dagger}}\Big]\bigg)Z_{\text{free}}[J], \end{aligned} \tag{97}$$

where $V\cdot J_V = \int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar\theta\, VJ_V$, $\Phi\cdot J_\Phi = \int \mathrm{d}^4x\mathrm{d}^2\theta\,\Phi J_\Phi$, and,

$$Z_{\text{free}}[J] = Z[0]\exp\bigg(-\frac{\mathrm{i}}{2}\int \mathrm{d}z_1\mathrm{d}z_2 J_i(z_1)G_{ij}(z_1,z_2)J_j(z_2)\bigg), \tag{98}$$

with $i,j = V,\Phi,\Phi^\dagger$. The two-point Green's function $G_{ij}(z_1,z_2)$ can be found by inverting the coefficient (matrix) of the quadratic term in the free action. Now we are going to find them for both chiral and vector superfields. We first consider the chiral superfield, in the context of Wess-Zumino model.

Wess-Zumino model. We recall that the free action, including the source term, for a chiral field Φ is given by

$$S_{\text{free}} = \int \mathrm{d}z\,\Phi^\dagger\Phi + \int \mathrm{d}^4x\bigg[\mathrm{d}^2\theta\,\Big(J\Phi + \frac{m}{2}\Phi^2\Big) + \mathrm{d}^2\bar\theta\,\Big(\Phi^\dagger J^\dagger + \frac{m}{2}(\Phi^\dagger)^2\Big)\bigg]. \tag{99}$$

The situation is complicated by the the chiral integration $\mathrm{d}^2\theta$ because in order to carry out the path integral, we want that the action integral is performed over the whole superspace. The trick to convert the chiral integral to the whole superspace integral is to make use of the projector $P_\pm$, defined by

$$P_+ = \frac{\bar{\mathrm{D}}^2\mathrm{D}^2}{16\Box}, \qquad\qquad P_- = \frac{\mathrm{D}^2\bar{\mathrm{D}}^2}{16\Box}. \tag{100}$$

Clearly for a general superfield F, P_+F is chiral and P_-F is antichiral. Moreover it is easy to see that $P_+\Phi = \Phi$ and $P_+J = J$ because both Φ and J are chiral. Then, with the relation $\int \mathrm{d}^2\theta = -\frac{1}{4}\mathrm{D}^2$, we rewrite the free action of Wess-Zumino multiplet as

$$\begin{aligned} S_{\text{free}} = &\int \mathrm{d}z\,\Phi^\dagger\Phi + \bigg[\int \mathrm{d}^4x\mathrm{d}^2\theta\,\Big(\Phi P_+J + \frac{m}{2}\Phi P_+\Phi\Big) + \text{h.c.}\bigg] \\ = &\int \mathrm{d}z\bigg[\Phi^\dagger\Phi - \Big(J\frac{\mathrm{D}^2}{4\Box}\Phi + J^\dagger\frac{\bar{\mathrm{D}}^2}{4\Box}\Phi^\dagger\Big) - \frac{m}{2}\Big(\Phi\frac{\mathrm{D}^2}{4\Box}\Phi + \Phi^\dagger\frac{\bar{\mathrm{D}}^2}{4\Box}\Phi^\dagger\Big)\bigg] \end{aligned}$$

$$= \int \mathrm{d}z \left[\frac{1}{2}(\Phi, \Phi^\dagger) \begin{pmatrix} -\frac{m\mathrm{D}^2}{4\Box} & 1 \\ 1 & -\frac{m\bar{\mathrm{D}}^2}{4\Box} \end{pmatrix} \begin{pmatrix} \Phi \\ \Phi^\dagger \end{pmatrix} - (\Phi, \Phi^\dagger) \begin{pmatrix} \frac{\mathrm{D}^2}{4\Box} J \\ \frac{\bar{\mathrm{D}}^2}{4\Box} J^\dagger \end{pmatrix} \right]. \tag{101}$$

Then, we find the free part of the generating functional $W_{\text{free}}[J] = -\mathrm{i}\log Z_{\text{free}}[J]$ to be

$$\begin{aligned} W_{\text{free}}[J] =& -\frac{1}{2}\int \mathrm{d}z \left(\tfrac{\mathrm{D}^2}{4\Box} J, \tfrac{\bar{\mathrm{D}}^2}{4\Box} J^\dagger \right) \begin{pmatrix} -\frac{m\mathrm{D}^2}{4\Box} & 1 \\ 1 & -\frac{m\bar{\mathrm{D}}^2}{4\Box} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\mathrm{D}^2}{4\Box} J \\ \frac{\bar{\mathrm{D}}^2}{4\Box} J^\dagger \end{pmatrix} \\ =& -\frac{1}{2}\int \mathrm{d}z \left(\tfrac{\mathrm{D}^2}{4\Box} J, \tfrac{\bar{\mathrm{D}}^2}{4\Box} J^\dagger \right) \begin{pmatrix} \frac{m\bar{\mathrm{D}}^2}{4(\Box-m^2)} & 1 + \frac{m^2\bar{\mathrm{D}}^2\mathrm{D}^2}{16\Box(\Box-m^2)} \\ 1 + \frac{m^2\mathrm{D}^2\bar{\mathrm{D}}^2}{16\Box(\Box-m^2)} & \frac{m\mathrm{D}^2}{4(\Box-m^2)} \end{pmatrix} \begin{pmatrix} \frac{\mathrm{D}^2}{4\Box} J \\ \frac{\bar{\mathrm{D}}^2}{4\Box} J^\dagger \end{pmatrix} \\ =& -\frac{1}{2}\int \mathrm{d}z \left(J, J^\dagger \right) \begin{pmatrix} \frac{m\mathrm{D}^2}{4\Box(\Box-m^2)} & \frac{1}{\Box-m^2} \\ \frac{1}{\Box-m^2} & \frac{m\bar{\mathrm{D}}^2}{4\Box(\Box-m^2)} \end{pmatrix} \begin{pmatrix} J \\ J^\dagger \end{pmatrix}. \end{aligned} \tag{102}$$

In deriving this result we have repeatedly used the definition of projectors $P_\pm$ as well as the relations such as $P_+ J = J$. We also note that one is allowed to perform integration by parts within super-integral with either spacetime or covariant super derivatives. For instance, we have $\int \mathrm{d}z\, F\mathrm{D}_\alpha G = -\int \mathrm{d}z\, G\mathrm{D}_\alpha F$ for arbitrary superfields F and G.

Now we obtained the needed 2-point functions, or propagators, to be

$$\langle \Phi(z_1)\Phi^\dagger(z_2)\rangle = \frac{\mathrm{i}}{16}\frac{\bar{\mathrm{D}}_1^2\mathrm{D}_1^2}{\Box - m^2}\delta(z_1 - z_2), \quad \langle \Phi(z_1)\Phi(z_2)\rangle = \frac{\mathrm{i}}{4}\frac{m\bar{\mathrm{D}}_1^2}{\Box - m^2}\delta(z_1 - z_2). \tag{103}$$

To find the Feynman rules for vertices, we consider an example of 3-point vertex Φ^3, for which we need to compute,

$$\begin{aligned} & \exp\left(\mathrm{i}S_{\text{int.}}\left[\frac{\delta}{\mathrm{i}\delta J}, \frac{\delta}{\mathrm{i}\delta J^\dagger} \right] \right) Z_{\text{free}}[J] \\ & \supset \frac{\mathrm{i}g}{3}\int \mathrm{d}^4x\mathrm{d}^2\theta_z \left[\frac{\delta}{\mathrm{i}\delta J(z)} \right]^3 \mathrm{i}J(z_1)\mathrm{i}J(z_2)\mathrm{i}J(z_3) \\ & = 2\mathrm{i}g\int \mathrm{d}^4x\mathrm{d}^2\theta_z \left(\frac{-\bar{\mathrm{D}}_z^2}{4} \right)^3 \delta(z - z_1)\delta(z - z_2)\delta(z - z_3) \\ & = 2\mathrm{i}g\int \mathrm{d}z \left(\frac{-\bar{\mathrm{D}}_z^2}{4} \right)^2 \delta(z - z_1)\delta(z - z_2)\delta(z - z_3). \end{aligned}$$

As can be seen clearly, each action of the functional derivative $\delta/\delta J(z)$ on $J(z_i)$ yields a factor of $(-\bar{\mathrm{D}}^2/4)\delta(z - z_i)$. However, at the end of the calculation, one of the three factors is used to convert the chiral super-integral to an integral over whole superspace. Therefore, when evaluating super-Feynman diagrams, one should assign each n-point chiral vertex with $(n-1)$ factors of $-\bar{\mathrm{D}}^2/4$. Furthermore, when computing 1PI diagrams, the external legs should be amputated, namely, we multiply the inverse propagator to each external leg, and take an integration. Thus one more $-\bar{\mathrm{D}}^2/4$ is used to extend this integral to the whole superspace. As a result, when computing 1PI diagrams, one should assign each vertex connecting I internal lines $(I - 1)$ factors of $-\bar{\mathrm{D}}^2/4$.

To get the momentum space super-Feynman rules, we need to convert all results above into momentum space. This is straightforward as it goes exactly the same with the ordinary field theories. We only make a nearly trivial remark that only the commuting coordinates x^m need Fourier transformations. Then, after going through a standard procedure, we reach the following *super-Feynman rules for Wess-Zumino model* in momentum space.

1. The propagators are given by
$$\begin{pmatrix} \langle\Phi(z_1)\Phi(z_2)\rangle & \langle\Phi(z_1)\Phi^\dagger(z_2)\rangle \\ \langle\Phi^\dagger(z_1)\Phi(z_2)\rangle & \langle\Phi^\dagger(z_1)\Phi^\dagger(z_2)\rangle \end{pmatrix} = \frac{-\mathrm{i}}{p^2+m^2}\begin{pmatrix} -\frac{\mathrm{D}^2}{4p^2} & 1 \\ 1 & -\frac{\bar{\mathrm{D}}^2}{4p^2} \end{pmatrix}\delta^4(\theta_1-\theta_2).$$
2. For each chiral, or antichiral vertex to which I internal lines are attached, an integral of $2\mathrm{i}g\int \mathrm{d}^4\theta$ and $(I-1)$ factors of $-\bar{\mathrm{D}}^2/4$, or $-\mathrm{D}^2/4$, are assigned, respectively.
3. For each independent loop an integral $\int\frac{\mathrm{d}^4k}{(2\pi)^4}$ is associated.
4. Usual combinatoric factors are understood.

Super Yang-Mills theory. To derive the superspace Feynman rules for super Yang-Mills theory, we take the topological angle $\theta_{\mathrm{YM}}=0$ since it does not affect perturbation theory. We further rescale the vector superfield[1] according to $V\to gV$ so that the coefficient before the quadratic term in V does not contain g. Then, expanding the action, we get,
$$\begin{aligned} S=&\frac{1}{2g^2k}\mathrm{Re}\int \mathrm{d}^4x\mathrm{d}^2\theta\,\mathrm{tr}\,W^\alpha W_\alpha \\ =&-\frac{1}{32g^2k}\mathrm{Re}\int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar\theta\,\mathrm{tr}\left(e^{-2gV}\mathrm{D}^\alpha e^{2gV}\right)\bar{\mathrm{D}}^2\left(e^{-2gV}\mathrm{D}_\alpha e^{2gV}\right) \\ =&\frac{1}{8}\int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar\theta\left[V^a\mathrm{D}^\alpha\bar{\mathrm{D}}^2\mathrm{D}_\alpha V^a+\mathcal{O}(V^3)\right]. \end{aligned} \tag{104}$$
The higher order terms are irrelevant, thus we do not bother to write all them down. However, note that the action is gauge invariant, which allows us to perform calculation in Wess-Zumino gauge. Moreover, we know that in Wess-Zumino gauge, V_{WZ} begins from the $\theta^A\theta^{\dot\beta}$ term. Therefore the expansion above will be terminated at finite order.

To quantize the theory, we need a gauge fixing term and corresponding ghost term. The gauge fixing is given by
$$-\frac{\alpha}{8}\int \mathrm{d}z\,(\mathrm{D}^2V^a)(\bar{\mathrm{D}}^2V^a), \tag{105}$$

[1] The original definition for V is said to be in holomorphic normalization, and the new convention taken here is said to be canonical normalization. Note that such a field rescaling leaves nontrivial footprint in the path integral measure, and gives rise to what is called a holomorphic anomaly. This anomaly implies that the gauge coupling constants defined in holomorphic and canonical normalization will run differently.

with α the gauge fixing parameter. Then the quadratic term of the prepotential is given by

$$S_{\text{free}} = \frac{1}{8}\int \mathrm{d}z \left[V^a \mathrm{D}^\alpha \overline{\mathrm{D}}^2 \mathrm{D}_\alpha V^a - \alpha(\mathrm{D}^2 V^a)(\overline{\mathrm{D}}^2 V^a) \right]. \tag{106}$$

7.2 Nonrenormalization theorems

A great thing about susy theories is that certain quantities do not receive radiative correction, or, they are not renormalized by quantum effect. Such statement in susy theories are collectively called nonrenormalization theorems. Now, with the super-Feynman rules derived above, we prove a nonrenormalization theorem for superpotential of chiral superfields.

Theorem. *The 1PI effective action $\Gamma[\Phi, \Phi^\dagger]$ of the Wess-Zumino model (53) can be represented as a single integral over the whole anticommuting variables, namely,*

$$\begin{aligned} \Gamma[\Phi, \Phi^\dagger] = & \sum_n \int \mathrm{d}^4 x_1 \cdots \mathrm{d}^4 x_n \int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, G(x_1, \cdots, x_n) \\ & \times \prod_{k=1}^n F_k^{(n)}[\Phi(x_k, \theta, \bar{\theta}), \Phi^\dagger(x_k, \theta, \bar{\theta})], \end{aligned} \tag{107}$$

where $G_n(x_1, \cdots, x_n)$'s are translational invariant functions, and $F_k^{(n)}[\Phi, \Phi^\dagger]$'s are local functionals of chiral superfield and its complex conjugate, namely, it contains at most a polynomial of derivatives of superfields.

Proof. To prove this theorem, let us consider an arbitrary loop with n vertices in an arbitrary 1PI diagram. According to the super-Feynman rules for the Wess-Zumino model, the part expressed in spinorial coordinate has the following structure,

$$\begin{aligned} \int & \mathrm{d}^4\theta_1 \cdots \mathrm{d}^4\theta_n\, (\mathrm{D}_1^2)^{\ell_1} (\overline{\mathrm{D}}_1^2)^{k_1} \delta^4(\theta_1 - \theta_2) \\ & \times (\mathrm{D}_2^2)^{\ell_2} (\overline{\mathrm{D}}_2^2)^{k_2} \delta^4(\theta_2 - \theta_3) \cdots (\mathrm{D}_n^2)^{\ell_n} (\overline{\mathrm{D}}_n^2)^{k_n} \delta^4(\theta_n - \theta_1) \end{aligned}$$

where ℓ_i and k_i $(i = 1, \cdots, n)$ take values of 0 or 1, and the corresponding derivative factors arise from the vertices, while the δ-functions $\delta^4(\theta_i - \theta_j)$ come from propagators. The derivatives D_i^2 and $\overline{\mathrm{D}}_i^2$ can always be adjust to the order shown above by using the relations $\overline{\mathrm{D}}^2\mathrm{D}^2\overline{\mathrm{D}}^2 = 16\Box\overline{\mathrm{D}}^2$ and $\mathrm{D}^2\overline{\mathrm{D}}^2\mathrm{D}^2 = 16\Box\mathrm{D}^2$. Now, the θ-integration can be performed one by one, in the following way,

$$\begin{aligned} \int & \mathrm{d}^4\theta_1 \cdots \mathrm{d}^4\theta_n\, (\mathrm{D}_1^2)^{\ell_1} (\overline{\mathrm{D}}_1^2)^{k_1} \delta^4(\theta_1 - \theta_2) \\ & \times (\mathrm{D}_2^2)^{\ell_2} (\overline{\mathrm{D}}_2^2)^{k_2} \delta^4(\theta_2 - \theta_3) \cdots (\mathrm{D}_n^2)^{\ell_n} (\overline{\mathrm{D}}_n^2)^{k_n} \delta^4(\theta_n - \theta_1) \end{aligned}$$

$$\sim \int \mathrm{d}^4\theta_2 \cdots \mathrm{d}^4\theta_n \, (\mathrm{D}_2^2)^{\ell_2'} (\bar{\mathrm{D}}_2^2)^{k_2'} \delta^4(\theta_2 - \theta_3)$$
$$\times \cdots \times (\mathrm{D}_n^2)^{\ell_n} (\bar{\mathrm{D}}_n^2)^{k_n} \delta^4(\theta_n - \theta_1)\big|_{(\theta_1,\bar\theta_1)=(\theta_2,\bar\theta_2)}$$
$$\sim \int \mathrm{d}^4\theta_n \, (\mathrm{D}_2^2)^{\ell_n'} (\bar{\mathrm{D}}_2^2)^{k_n'} \delta^4(\theta_n - \theta_1)\big|_{(\theta_1,\bar\theta_1)=(\theta_n,\bar\theta_n)}.$$

This final expression vanishes unless $\ell_n' = k_n' = 1$, in which case the above expression yields a single factor $16\int \mathrm{d}^4\theta_n$. Thus we see that for an arbitrary loop in a 1PI diagram, the spinorial structure can be represented by a single integral $\mathrm{d}^4\theta$. Now we repeat this process to all loops in a 1PI diagram and to all 1PI diagrams, we see finally that the 1PI effective action itself can be represented as a single integral over the whole spinorial coordinates $\int \mathrm{d}^4\theta$. **QED.**

Now we observe that the superpotential, $\int \mathrm{d}^2\theta(\frac{1}{2}m\Phi^2 + \frac{1}{3}g\Phi^3) + \text{h.c.}$, cannot be represented as a whole-superspace integral $\int \mathrm{d}^4\theta$. Therefore, we immediately reach the corollary that *the superpotential does not renormalize.*

Another important corollary is that *a field configuration preserving supersymmetry at the classical level does not receive perturbative quantum corrections, i.e., perturbative quantum effects do not break supersymmetry.* To see this, we only need to note that the effective potential is the x-space independent part of the action, which must vanishes for a supersymmetric configuration. Now, recall that the classical potential is given by $\mathscr{V} = |F|^2 + \frac{1}{2g^2}D^2$, then $\mathscr{V} = 0$ implies that auxiliary fields F and D vanish. On the other hand, the quantum correction to the effective potential, according to the nonrenormalization theorem above, must have the form,

$$\int \mathrm{d}^4x \mathrm{d}^4\theta \, F(V_{\text{cl.}}, \Phi_{\text{cl.}}, \Phi_{\text{cl.}}^\dagger),$$

where the subscript "cl." means the classical configuration. However, all these field configuration cannot have θ-dependence, since their spinorial component must vanish as required by Poincaré symmetry, and their auxiliary components also must vanish as discussed above. Then we immediately see that perturbative quantum correction must vanish for such field configurations.

One may be wondering if the nonrenormalization theorem would imply that the super Yang-Mills action (91) does not renormalize, since it is also a chiral integration. However, this is not true since the action contains derivatives, which can be used to convert the action to an integral over the whole superspace, as we did in (104). As a result, the vector superfield does receive wave function renormalization.

A. Notations and Useful Relations

$$\sigma^m = (\sigma^0, \sigma^i), \quad \bar\sigma^m = (\sigma^0, -\sigma^i). \tag{108}$$

$$\sigma^0=\begin{pmatrix}-1&0\\0&-1\end{pmatrix},\quad \sigma^1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma^2=\begin{pmatrix}0&-\mathrm{i}\\\mathrm{i}&0\end{pmatrix},\quad \sigma^3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{109}$$

$$\begin{aligned}(\sigma^{mn})_\alpha{}^\beta&=\tfrac14\big[(\sigma^m)_{\alpha\dot\alpha}(\bar\sigma^n)^{\dot\alpha\beta}-(\sigma^n)_{\alpha\dot\alpha}(\bar\sigma^m)^{\dot\alpha\beta}\big],\\(\bar\sigma^{mn})^{\dot\alpha}{}_{\dot\beta}&=\tfrac14\big[(\bar\sigma^m)^{\dot\alpha\alpha}(\sigma^n)_{\alpha\dot\beta}-(\bar\sigma^n)^{\dot\alpha\alpha}(\sigma^m)_{\alpha\dot\beta}\big].\end{aligned} \tag{110}$$

$$\gamma^m=\begin{pmatrix}0&\mathrm{i}\sigma^m\\\mathrm{i}\bar\sigma^m&0\end{pmatrix}. \tag{111}$$

$$(\sigma^m)_{\alpha\dot\alpha}(\bar\sigma_m)^{\dot\beta\beta}=-2\delta_\alpha^\beta\delta_{\dot\alpha}^{\dot\beta},\quad (\sigma^m)_{\alpha\dot\alpha}(\sigma_m)_{\beta\dot\beta}=-2\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}. \tag{112}$$

$$(\sigma^{mn}\sigma_{mn})_\alpha{}^\beta=-\tfrac12\delta_\alpha^\beta,\quad (\sigma^{mn})_\alpha{}^\beta(\bar\sigma_{mn})^{\dot\alpha}{}_{\dot\beta}=0. \tag{113}$$

$$\begin{aligned}[\sigma_{mn},\sigma_{pq}]&=\tfrac14\big(\eta_{mp}\sigma_{nq}-\eta_{mq}\sigma_{np}-\eta_{np}\sigma_{mq}+\eta_{nq}\sigma_{mp}\big),\\[\bar\sigma_{mn},\bar\sigma_{pq}]&=\tfrac14\big(\eta_{mp}\bar\sigma_{nq}-\eta_{mq}\bar\sigma_{np}-\eta_{np}\bar\sigma_{mq}+\eta_{nq}\bar\sigma_{mp}\big).\end{aligned} \tag{114}$$

Thus we see that $4\mathrm{i}\sigma_{mn}$ and $4\mathrm{i}\bar\sigma_{mn}$ are left-spinor and right-spinor representations of Lorentz generators J_{mn}, respectively.

B. Complex Geometry

In this appendix we review some basics of complex geometry relevant to main text, following [9]. We begin with the most crucial concept in complex analysis, namely a holomorphic (analytic) map. A complex valued function $f:\mathbb{C}^m\to\mathbb{C}$ is said to be holomorphic, if $f=f_1+\mathrm{i}f_2$ satisfies the Cauchy-Riemann relations for each $z^\mu=x^\mu+\mathrm{i}y^\mu$ $(1\le\mu\le m)$,

$$\frac{\partial f_1}{\partial x^\mu}=\frac{\partial f_2}{\partial y^\mu},\quad \frac{\partial f_2}{\partial x^\mu}=-\frac{\partial f_1}{\partial y^\mu}. \tag{115}$$

Similarly, a map $(f^1,\cdots,f^n):\mathbb{C}^m\to\mathbb{C}^n$ is holomorphic if each f^λ $(1\le\lambda\le n)$ is holomorphic.

Complex manifold. The complex manifold is defined to be a topological space $\mathcal{M}$, endowed with a set of pairs $\{U_i,\varphi_i\}$, such that $\{U_i\}$ is an open cover of $\mathcal{M}$, and φ_i is a homeomorphism from U_i to an open neighbourhood of $\mathbb{C}^m$, such that, for any U_i and U_j with $U_i\cap U_j\neq\varnothing$, the map $\varphi_j\circ\varphi_i^{-1}:\varphi_i(U_i\cap U_j)\to\varphi_j(U_i\cap U_j)$ is holomorphic. Then, $\mathcal{M}$ has complex dimension m, denoted as $\dim_{\mathbb{C}}\mathcal{M}=m$. As a manifold, it's real dimension is $2m$. From definition, $\mathcal{M}$ is also a differentiable manifold, ensured by its analytic property.

One can define the holomorphic map $f:\mathcal{M}\to\mathcal{N}$ between two complex manifolds $\mathcal{M}$ and $\mathcal{N}$ with $\dim_{\mathbb{C}}\mathcal{M}=m$ and $\dim_{\mathbb{C}}\mathcal{N}=n$, by requiring the corresponding map between two local charts, as two open neighbourhoods in $\mathbb{C}^m$ and $\mathbb{C}^n$, to be holomorphic. Then, a holomorphic function of $\mathcal{M}$ is a holomorphic map $f : \mathcal{M} \to \mathbb{C}$. The set of holomorphic function on $\mathcal{M}$ is denoted by $\mathcal{O}(\mathcal{M})$. It can be proved that *any holomorphic function on a compact complex manifold is a constant.*

On a given point p on $\mathcal{M}$ covered by local coordinate patch $z^\mu = x^\mu + \mathrm{i}y^\mu$, the tangent space $T_p\mathcal{M}$ is spanned by $2m$ vectors $\{\partial/\partial x^\mu, \partial/\partial y^\mu; \mu = 1, \cdots, m\}$. Now, let's define a complexified tangent space $T_p\mathcal{M}^{\mathbb{C}}$, spanned, with complex coefficients, by $2m$ vectors $\partial/\partial z^\mu = \frac{1}{2}(\partial/\partial x^\mu + \mathrm{i}\partial/\partial y^\mu)$ and $\partial/\partial \bar{z}^\mu = \frac{1}{2}(\partial/\partial x^\mu - \mathrm{i}\partial/\partial y^\mu)$. We can similarly define the complexified cotangent space as a complex vector space spanned by $\mathrm{d}z^\mu = \mathrm{d}x^\mu + \mathrm{i}\,\mathrm{d}y^\mu$ and $\mathrm{d}\bar{z}^\mu = \mathrm{d}x^\mu - \mathrm{i}\,\mathrm{d}y^\mu$. These two basis are dual to each other, namely,

$$\begin{aligned} \Big\langle \mathrm{d}z^\mu, \frac{\partial}{\partial z^\nu} \Big\rangle &= \Big\langle \mathrm{d}\bar{z}^\mu, \frac{\partial}{\partial \bar{z}^\nu} \Big\rangle = \delta^\mu{}_\nu, \\ \Big\langle \mathrm{d}z^\mu, \frac{\partial}{\partial \bar{z}^\nu} \Big\rangle &= \Big\langle \mathrm{d}\bar{z}^\mu, \frac{\partial}{\partial z^\nu} \Big\rangle = 0. \end{aligned} \tag{116}$$

The *almost complex structure* J is a real $(1,1)$-tensor field which is a linear map $J_p : T_p\mathcal{M} \to T_p\mathcal{M}$ at a given point $p \in \mathcal{M}$, and acts as $J_p(\partial/\partial x^\mu) = \partial/\partial y^\mu$ and $J_p(\partial/\partial y^\mu) = -\partial/\partial x^\mu$. This tensor can be extended to $T_p\mathcal{M}^{\mathbb{C}}$, with the action $J_p(\partial/\partial z) = \mathrm{i}\partial/\partial z^\mu$ and $J_p(\partial/\partial \bar{z}) = -\mathrm{i}\partial/\partial \bar{z}$. Thus in this basis, we have,

$$J_p = \mathrm{i}\,\mathrm{d}z^\mu \otimes \frac{\partial}{\partial z^\mu} - \mathrm{i}\,\mathrm{d}\bar{z}^\mu \otimes \frac{\partial}{\partial \bar{z}^\mu}. \tag{117}$$

So we can define a projector $P_\pm = \frac{1}{2}(I \mp \mathrm{i}J_p)$, such that $T_p\mathcal{M}^{\mathbb{C}}$ decomposes into two linear spaces, namely $T_p\mathcal{M}^\pm = \{P_\pm Z; Z \in T_p\mathcal{M}^{\mathbb{C}}\}$. Clearly, we have $T_p\mathcal{M}^+ = \mathrm{span}\{\partial/\partial z^\mu\}$ and $T_p\mathcal{M}^- = \mathrm{span}\{\partial/\partial \bar{z}^\mu\}$. Elements in these two subspaces are called holomorphic and anti-holomorphic vectors, respectively.

Hermitian manifold. Now, let $\mathcal{M}$ be a complex manifold of $\dim_{\mathbb{C}} \mathcal{M} = m$, and with a Riemannian metric g as a differentiable manifold. We can extend g, $\forall Z = X + \mathrm{i}Y$, $W = U + \mathrm{i}V \in T_p\mathcal{M}^{\mathbb{C}}$, as,

$$g_p(Z, W) = g_p(X, U) - g_p(Y, V) + \mathrm{i}\big(g_p(X, V) + g_p(Y, U)\big).$$

If g satisfies $g_p(J_pX, J_pY) = g_p(X, Y)$ at each point $p \in \mathcal{M}$, it is said to be a *Hermitian metric*, and the pair $(\mathcal{M}, g)$ is a *Hermitian manifold*. Any complex manifold with Riemannian metric allows a Hermitian metric. In fact, with a given Riemannian metric g and almost complex structure J, it is easy to show that the metric $\hat{g}$ defined via $\hat{g}_p(X, Y) = \frac{1}{2}\big[g_p(X, Y) + g_p(J_pX, J_pY)\big]$ is a Hermitian metric.

With an Hermitian metric, a vector X at $p \in \mathcal{M}$ is orthogonal to J_pX. Furthermore, in the basis $\{\partial/\partial z^\mu, \partial/\partial \bar{z}^\mu\}$ of $T_p\mathcal{M}^{\mathbb{C}}$, we define the component of Hermitian metric g as, $g_{\mu\nu} = g(\partial^\mu, \partial^\mu)$, $g_{\bar{\mu}\bar{\nu}} = g(\bar{\partial}^\mu, \bar{\partial}^\nu)$, $g_{\mu\bar{\nu}} = g(\partial^\mu, \bar{\partial}^\nu)$ etc, then it's easy to show that

$g_{\mu\nu} = g_{\bar{\mu}\bar{\nu}} = 0$. Thus g has the following form,

$$g = g_{\mu\bar{\nu}} \mathrm{d}z^\mu \otimes \mathrm{d}\bar{z}^\nu + g_{\bar{\mu}\nu} \mathrm{d}\bar{z}^\mu \otimes \mathrm{d}z^\nu. \tag{118}$$

One can introduce a covariant derivative compatible with the Hermitian metric g, on a Hermitian manifold, the corresponding connection, called *Hermitian connection*, has the components $\Gamma^\lambda_{\mu\nu} = g^{\lambda\bar{\lambda}}\partial_\mu g_{\nu\bar{\lambda}}$, $\Gamma^{\bar{\lambda}}_{\bar{\mu}\bar{\nu}} = g^{\bar{\lambda}\lambda}\partial_{\bar{\mu}} g_{\lambda\bar{\nu}}$, and all others with mixed indices vanish. This follows from the metric compatibility $0 = \nabla_\kappa g_{\mu\bar{\nu}} = \partial_k g_{\mu\bar{\nu}} - \Gamma^\lambda_{\kappa\mu} g_{\lambda\bar{\nu}}$ and $0 = \nabla_{\bar{\kappa}} g_{\mu\bar{\nu}} = \partial_{\bar{\kappa}} g_{\mu\bar{\nu}} - \Gamma^{\bar{\lambda}}_{\bar{\kappa}\bar{\nu}} g_{\mu\bar{\lambda}}$. By direct calculation, one can show that the almost complex structure J satisfies $\nabla_\mu J = \nabla_{\bar{\mu}} J = 0$.

From Hermitian connection, we can introduce *torsion* T and *curvature* R of a Hermitian manifold, defined via,

$$\begin{aligned} T(X,Y) &= \nabla_X Y - \nabla_Y X - [X,Y], \\ R(X,Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \end{aligned} \tag{119}$$

The only nonvanishing components of T are $T^\lambda{}_{\mu\nu}$ and $T^{\bar{\lambda}}{}_{\bar{\mu}\bar{\nu}} = (T^\lambda{}_{\mu\nu})^*$, with,

$$T^\lambda{}_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} = g^{\bar{\lambda}\lambda}(\partial_\mu g_{\nu\bar{\lambda}} - \partial_\nu g_{\mu\bar{\lambda}}), \tag{120}$$

and the only nonvanishing components of R are $R^\kappa{}_{\lambda\bar{\mu}\nu} = -R^\kappa{}_{\lambda\nu\bar{\mu}}$ and $R^{\bar{\kappa}}{}_{\bar{\lambda}\mu\bar{\nu}} = -R^{\bar{\kappa}}{}_{\bar{\lambda}\bar{\nu}\mu}$, with $R^{\bar{\kappa}}{}_{\bar{\lambda}\mu\bar{\nu}} = (R^\kappa{}_{\lambda\bar{\mu}\nu})^*$, and,

$$R^\kappa{}_{\lambda\bar{\mu}\nu} = \partial_{\bar{\mu}}\Gamma^\kappa_{\nu\lambda} = \partial_{\bar{\mu}}(g^{\bar{\lambda}\kappa}\partial_\nu g_{\lambda\bar{\lambda}}). \tag{121}$$

Given a Hermitian manifold $(\mathcal{M}, g)$, the tensor field Ω defined via $\Omega_p(X,Y) = g_p(J_p X, Y)$, $\forall X, Y \in T_p\mathcal{M}$ and $\forall p \in \mathcal{M}$ is antisymmetric with its two arguments, and thus is a two-form, called the *Kähler form* of the Hermitian metric g. It has the expression $\Omega = \mathrm{i} g_{\mu\bar{\nu}} \mathrm{d}z^\mu \wedge \mathrm{d}\bar{z}^\nu$ and thus is a real form, $\bar{\Omega} = \Omega$. Furthermore, it is also invariant under the action of J, namely $\Omega(JX, JY) = \Omega(X,Y)$. It can be proved that $\underbrace{\Omega \wedge \cdots \wedge \Omega}_{m \text{ times}}$ is a nowhere vanishing $2m$-form where $m = \dim_{\mathbb{C}} \mathcal{M}$, and can be used as a volume element. So, *a complex manifold is orientable.*

Kähler Manifold. If the Kähler form Ω of a Hermitian metric g on a Hermitian manifold $(\mathcal{M}, g)$ is closed, $\mathrm{d}\Omega = 0$, then g is called a *Kähler metric*, and $(\mathcal{M}, g)$ is called a *Kähler manifold*. Not all complex manifold admits a Kähler metric. It can be proved that an Hermitian manifold $(\mathcal{M}, g)$ is a Kähler manifold if and only if the almost complex structure J satisfies $\nabla_\mu J = 0$ where ∇_μ is the covariant derivative with (torsion free) Levi-Civita connection.

The condition $\mathrm{d}\Omega = 0$ can be written in $(z, \bar{z})$ basis as $(\partial + \bar{\partial})\mathrm{i} g_{\mu\bar{\nu}} \mathrm{d}z^\mu \wedge \mathrm{d}\bar{z}^\nu = 0$, from which one can deduce,

$$\frac{\partial g_{\mu\bar{\nu}}}{\partial z^\lambda} = \frac{\partial g_{\lambda\bar{\nu}}}{\partial z^\mu}, \quad \frac{\partial g_{\mu\bar{\nu}}}{\partial \bar{z}^\lambda} = \frac{\partial g_{\mu\bar{\lambda}}}{\partial \bar{z}^\nu}. \tag{122}$$

Obviously, if the Hermitian metric g is given locally (on a coordinate patch U) by $g_{\mu\bar{\nu}} = \partial_\mu \partial_{\bar{\nu}} \mathcal{K}$ with $\mathcal{K}$ a function on U, it will naturally satisfy the condition above, and thus is a Kähler metric. Conversely, any Kähler metric can be locally written in this form. The function $\mathcal{K}$ is called *Kähler potential*.

It can be proved that *any complex submanifold of a Kähler manifold is again a Kähler manifold*.

From the condition (122) we see that *Kähler metric is torsion free*.

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