Introduction to Optimization

LEARNING OBJECTIVES

After completing this supplement you should be able to

1. Recognize decision-making situations that may benefit from an optimization modeling approach.
2. Formulate algebraic models for linear programming problems.
3. Develop spreadsheet models for linear programming problems.
4. Use Excel's Solver Add-In to solve linear programming problems.
5. Interpret the results of models and perform basic sensitivity analysis.

SUPPLEMENT OUTLINE

Everyone, almost daily, solves optimization problems in informal ways by using mental models. If you have ever looked at a map when planning a trip and decided upon a route to your destination to try to minimize either distance or time (or even to maximize scenic or recreational benefits), you have solved an optimization problem. If you have ever been faced with too much school work (i.e., studying for final exams and completing final projects) and a fixed amount of time, you have undoubtedly solved an optimization problem that seeks to allocate the time available so that you would, perhaps, maximize your grades in the various courses.

Whereas informal optimization is done nearly every day by individuals, organizations carry out formal optimization to assist in such decisions as product mix, pricing, scheduling, routing and logistics, supply chain management, facility location analysis, and financial planning & asset management. Many of these topics are either directly related to operations management or have strong connections to it. Optimization as a decision support tool has applications across all functional areas of an organization.

INTRODUCTION

This chapter provides an introduction to optimization models and solution approaches. Optimization is a major field within the discipline of Management Science. The emphasis is on developing appropriate mathematical models to describe situations, implementing these models in a spreadsheet, using a spreadsheet-based solver to solve the optimization problems, and using human intelligence and judgment to interpret the results. We emphasize a particular type of optimization problem, called Linear Programming problems (or linear optimization). In Linear Programming (LP) problems, all of the relationships among the variables are linear.

Think briefly about the allocation of study time mentioned in the opening paragraphs. In an optimization mindset, there is an **objective** you want to either maximize or minimize, and there may be **constraints** within which you need to operate. There are also specific quantities, called **decision variables,** over which you have control. Therefore, this is termed a **constrained optimization problem.** A verbal statement of the study time problem might be that you want to maximize your grade point average. Constraints are a limited total amount of time to study, and a desire to pass every course. Decision variables are the amounts of time allocated to each course. A more structured statement of the problem is

Maximize the objective:	Grade point average
Subject to the constraints:	Stay within available study time
	Pass each course with a grade of at least C
Decision variables:	Amount of time to spend on each course

▶ **Objective**
The quantity to be maximized or minimized.

▶ **Constraints**
Limitations or requirements that must be satisfied.

▶ **Decision variables**
Quantities under the control of the decision maker.

▶ **Constrained optimization problem**
A mathematical model in which one is trying to maximize or minimize some quantity, while satisfying a set of constraints.

Our purpose is not to develop a formal mathematical model of the study time example. However, the actual process of thinking about a problem in a structured way often leads to insights you can use to make a better decision. You would probably think about your current grade in each course, the remaining requirements of the course, and the amount of study time needed in order to at least pass each course. You might also think about how much time you would need to spend in order to get the best possible grade in each course. This structured thinking often leads to a better overall decision, even if a formal mathematical model is never developed. Even though this example is fairly simple, there are multiple factors to consider when ultimately making the decision. For example, if you spend the time required to get the best grade possible in a particular course, does that jeopardize your ability to pass another class?

The advantage of formal optimization modeling is that it can simultaneously consider the effects of alternate decisions to produce the best overall decision according to the objective. Humans are capable of considering a few factors simultaneously. As the size and complexity of a problem increase, humans cannot adequately keep in mind all of the effects of a decision. Large-scale optimization software can handle thousands or even millions of decision variables and constraints. Human intelligence is still needed in the problem definition, or formulation, stage of problem solving, and in the interpretation of the results of the computer solution. Computer algorithms are best suited to taking a well-formulated problem definition and making the necessary computations to produce the best solution to that problem. In this chapter we focus on problem formulation, spreadsheet model development, solution with Solver, an Excel add-in, and interpretation of the results.

The steps involved in solving optimization problems are shown in Figure B-1. These steps should be looked at as a guide.

Steps Involved in Solving Optimization Problems

- Understand the problem, perhaps by drawing a diagram which represents the problem
- Write a problem formulation in words, including decision variables, objective function, and constraints
- Write the algebraic formulation of the problem.
  - Define the decision variables
  - Write the objective function
  - Write the constraints
- Develop a spreadsheet model
- Set up the Solver settings and solve the problem
- Examine the results and make corrections to the model
- Analyze and interpret the results

FIGURE B-1

Steps involved in solving optimization problems

DJJ Enterprises manufactures automotive parts. Two of these parts are camshafts and gears. Camshafts earn a profit of $25 per unit and gears earn $18 per unit. Three major resources are utilized in the production process: steel, labor, and machine time. It takes 5 lbs of steel to make a camshaft, and 8 lbs to make a gear. Camshafts require 1 hour of labor; gears require 4 hours. It takes 3 hours machine time per camshaft, and 2 hours per gear. For the current planning period, 5000 lbs steel, 1500 hours labor, and 1000 hours machine time are available. DJJ would like to maximize profit during the current planning period, within allowable resources.

EXAMPLE B.1

Product Mix Decision

ALGEBRAIC FORMULATION

▶ **Formulation**
A formal, algebraic statement of a constrained optimization problem.

The text description of the problem is fine for communicating a general understanding, but in order to address the problem using optimization we need to develop a formal algebraic description, called a **formulation**. A formulation contains explicit definitions of the decision variables, an algebraic expression of the objective function, and algebraic statements of the constraints. This is not as hard as it sounds, but it is a crucial step if you hope to develop a high-quality spreadsheet model that Solver can successfully solve.

A diagram of the situation can help to structure the problem so the algebraic formulation can be written down more easily. A well-constructed diagram is also a valuable communication tool. In this situation, three resources are combined to produce two products. Specific amounts of each resource are needed, and each product generates a known profit. See Figure B-2 for one way to show this. One of the important points is that the total amount of each resource available must be shared between the two products (assuming both are to be produced).

A diagram, coupled with a text-based formulation, often serves as a valuable stepping stone to a formal algebraic formulation. In this situation, a text-based formulation can be stated as follows:

Text-Based Formulation

- Decision Variables: Number of camshafts to make, number of gears to make
- Objective Function: Maximize profit
- Constraints: Must not exceed our resource availability in steel, labor, and machine hours

The text-based formulation helps to structure the problem. The next step is to write the algebraic formulation. The algebraic formulation states the decision variables, objective function, and constraints in algebraic terms. The decision variables correspond to the quantities of camshafts and gears to make. We do not know ahead of time the best values for these (that's what we're trying to determine), so we create a variable to represent each quantity, as follows:

FIGURE B-2

DJJ Enterprises diagram
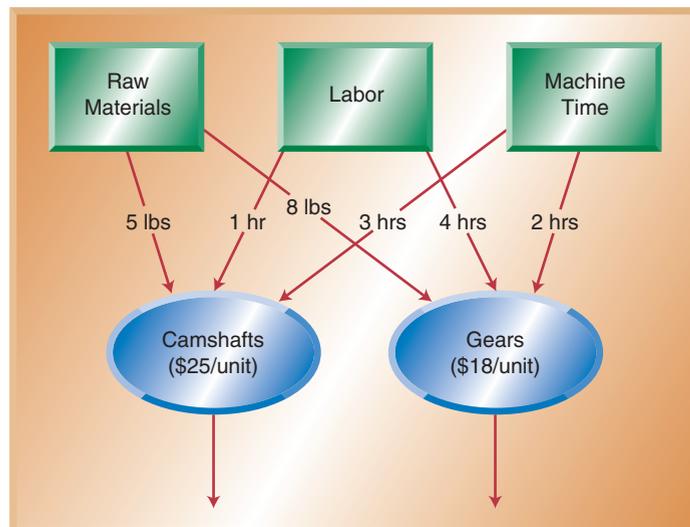
Decision Variables

C = number of camshafts to make
G = number of gears to make

The actual names of the decision variables are not crucial (some people prefer X_1 and X_2, for example); however, it is helpful to name the variables so the names remind you of the quantities they represent.

The next step is to write the objective function in terms of the decision variables just defined. The objective function is simply an expression that determines how much profit will be earned if C camshafts are made and G gears are made. We also indicate whether we want to maximize ("Max") or minimize ("Min") the objective function. Since we know the unit profits of each, the objective function is:

Objective Function

$$\text{Max } 25C + 18G \quad (\text{profit, \$})$$

It is good practice to include the units and the quantity represented by the objective function, as shown in parentheses.

Finally, we need to write expressions for the constraints. In this example, we are limited by the amounts of the resources—steel, labor, and machine time—available. For steel, we know that 5000 lbs are available. We also know that each camshaft uses 5 lbs, and each gear uses 8 lbs. Therefore, if we make C and G camshafts and gears, respectively, it will require $5C + 8G$ lbs of steel. In order for this product mix to be possible, this combined total must not exceed 5000 lbs. We write this in mathematical terms as follows:

$$5C + 8G \leq 5000 \quad (\text{steel, lbs})$$

Using similar logic, we write the constraints for labor and machine time. Convince yourself that these constraints should be written as:

$$1C + 4G \leq 1500 \quad (\text{labor, hrs})$$
$$3C + 2G \leq 1000 \quad (\text{machine time, hrs})$$

Each of these constraints is a **less-than-or-equal-to constraint**. A constraint can also be a **greater-than-or-equal-to constraint** or an **equality constraint**. A problem can have a mix of these constraint types. For example, it could be that we are required to produce at least a certain number of camshafts and gears combined ($\geq$ constraint). Alternately, perhaps we want to force our solution to use exactly 1500 labor hours ($=$ constraint). Strict inequalities ($<$, $>$) are not used in optimization problems; the three constraint relationship types are $\leq$, $=$, and $\geq$.

There are two additional constraints that initially may not seem required. They force each of the decision variables to remain nonnegative; that is, greater than or equal to zero. We refer to these as the **nonnegativity constraints**. From a business sense we know that we cannot produce negative quantities of camshafts and gears. However, neither a mathematical model nor a spreadsheet model "knows" these seemingly obvious things. These constraints are written:

$$C \geq 0$$
$$G \geq 0$$

Most of the time, these would be written together as

$$C, G \geq 0 \quad (\text{nonnegativity})$$

▶ **Less-than-or-equal-to constraint**
A constraint such as $3x_1 + 5x_2 \leq 22$, often used to model a limitation on the amount of a resource that can be used.

▶ **Greater-than-or-equal-to constraint**
A constraint such as $4x_1 + 7x_2 \geq 50$, often used to model a requirement that must be satisfied.

▶ **Equality constraint**
A constraint such as $6x_1 + 3x_2 = 30$, used to specify that a requirement must be met exactly.

▶ **Nonnegativity constraints**
Constraints of the form $x_1 \geq 0$, which are nearly universal in linear programming problems. They are used to represent the fact that negative quantities of products cannot be made, shipped, etc.

Putting all the constraints together, we have:

Constraints

$$5C + 8G \leq 5000 \quad \text{(steel, lbs)}$$
$$1C + 4G \leq 1500 \quad \text{(labor, hrs)}$$
$$3C + 2G \leq 1000 \quad \text{(machine time, hrs)}$$
$$C, G \geq 0 \quad \text{(nonnegativity)}$$

Normally, we consolidate the entire formulation together into a standard format, as follows:

Complete Formulation

Decision Variables

$$C = \text{number of camshafts to make}$$
$$G = \text{number of gears to make}$$

Max $25C + 18G$ (profit, $)

Subject to (or, "s.t.")

$$5C + 8G \leq 5000 \quad \text{(steel, lbs)}$$
$$1C + 4G \leq 1500 \quad \text{(labor, hrs)}$$
$$3C + 2G \leq 1000 \quad \text{(machine time, hrs)}$$
$$C, G \geq 0 \quad \text{(nonnegativity)}$$

Examining the Formulation

The formulation is a concise mathematical description of the problem. It indicates that we want to find the values of C and G that produce the largest value of the objective function while satisfying all the constraints. In this formulation all of the relationships among the variables are linear. That is, all expressions involving the variables C and G consist of a constant multiplied by the variable itself. Combinations of variables can be added (or subtracted) to one another, but there are no nonlinear expressions involving variables, such as C^2, G/C, or $\sqrt{C}$. Thus, this formulation is referred to as a Linear Program (LP), or an LP Formulation. A **Linear Program (LP)** is an optimization problem in which all of the relationships among the decision variables are linear. LPs are much easier to solve, in general, than problems involving nonlinear expressions. The solver built into Excel has the capability to solve both linear and nonlinear problems, but if your problem can be formulated as an LP, it is best to do so because its solution algorithm is faster and more reliable.

A **feasible solution** to an optimization problem in general, or an LP as in this case, is a particular combination of C and G that satisfies *all* of the constraints. An **infeasible solution** violates at least one of the constraints. The **optimal solution** is the feasible solution with the largest (for a "max" problem) or smallest (for a "min" problem) objective function value.

Consider the solution $C = 75$, $G = 200$. Is this solution feasible? To determine if it is, evaluate each of the constraints. The amount of steel required would be $5(75) + 8(200) = 1975$ lbs, which is less than the 5000 lbs available. Similarly, the solution requires $1(75) + 4(200) = 875$ hrs labor (less than the 1500 hours available), and

▶ **Linear program (LP)**
A constrained optimization problem in which all the functions involving decision variables are linear.

▶ **Feasible solution**
A specific combination of values of the decision variables such that all of the constraints are satisfied.

▶ **Infeasible solution**
A specific combination of values of the decision variables such that at least one of the constraints is violated.

▶ **Optimal solution**
The feasible solution with the largest (for a maximization problem) or smallest (for a minimization) objective value.

3(75) + 2(200) = 625 hrs machine time. Both C and G are nonnegative, so this solution is feasible. Now consider $C = 300$, $G = 200$. Verify that this solution requires 3100 lbs steel, 700 hours labor, and 1300 hours machine time. Is the solution feasible? It satisfies two of the constraints, but violates the machine time constraint. Therefore, this solution is infeasible. The point is that if a solution violates even a single constraint, the solution is infeasible.

Is either one of these solutions the optimal solution? The second solution, $C = 300$ and $G = 200$, cannot be optimal because it is not even feasible. The first solution, $C = 75$ and $G = 200$, is at least feasible. Actually, it is one of an infinite number of feasible solutions to this problem. However, it is not the optimal solution. You can try to find a better feasible solution. In fact, for this small problem, you could probably, using pencil, paper, and a calculator, find the optimal solution. If the problem were larger, perhaps with 15 products and 8 constraints, finding the optimal solution by trial-and-error would be nearly impossible.

An ingenious algorithm has been developed to solve LPs. It is called the **Simplex Method** and was developed by George Dantzig in 1947. Improvements have been made in solving large LPs since then, but the Simplex Method is still the most common method for solving them. Learning the intricacies of the Simplex Method is a significant study in itself. Fortunately, Solver, part of Microsoft Excel, implements the Simplex Method. The user does not need to learn the details of the mathematical manipulations, but can focus on developing a high-quality spreadsheet model from an algebraic formulation before setting up the problem in Solver. Once Solver is able to find the optimal solution, the user can perform additional analysis to gain insight into the business problem.

▶ **Simplex Method**
A mathematical algorithm developed by George Dantzig that, when implemented in software, can solve LPs very quickly.

SPREADSHEET MODEL DEVELOPMENT

Now that an algebraic formulation has been developed, we will implement the formulation in the spreadsheet. There is no single way to set up an LP in a spreadsheet, but there are some guidelines that can be established:

- Develop a correct, flexible, and documented model, as described in Supplement A. The user should be able to use the model in a "what-if" manner, changing any decision variable or coefficient in the model without having to change any formulas.

- Divide the worksheet into three sections: one for the decision variables, one for the objective function, and one for the constraints.

- Use the algebraic formulation and the natural structure of the problem to guide the structure of the spreadsheet.

- Use one cell for each decision variable.

- Store the coefficients of the objective function in separate cells, and use another cell to store a formula that calculates the value of the objective function (by referring to the decision variable cells and the coefficient cells).

- Likewise, store the coefficients of the constraints in cells, and write formulas to compute the "left hand side" (LHS) value of each constraint. The **LHS value** is the value of the constraint expression to the left of the ≤, ≥, or = sign. Then store the "right-hand-side" value (**RHS value**) of each constraint next to this formula for easy comparison.

▶ **LHS value**
The value of the constraint expression to the left of the ≤, ≥, or = sign.

▶ **RHS value**
The value of the constraint expression to the right of the ≤, ≥, or = sign.

A spreadsheet for this problem is shown in Figure B-3. It uses $C = 75$ and $G = 200$ as trial values. Note the structure of the model; column A is used for text labels, columns B and C are used to represent camshafts and gears, respectively, and columns D-F are used to compute formulas and show the resource limitations. There is a row (row 5) reserved to contain the decision variables, a row (row 8) to store the coefficients and compute the value of the objective function, and one row for each of the constraints (rows 11–13). As in Supplement A, values highlighted in yellow represent data provided with the problem, and the light blue cell represents the primary output measure, the objective function. In addition, green cells represent the decision variables.

Cells B5:C5 contain values of the decision variables. The user can change these to see the effects of different solutions, visually checking to see if the constraints are satisfied. Cells B8:C8 contain the objective function coefficients, in this case, the unit profits. Cells B11:C13 contain the coefficients of the constraints. Refer to the algebraic formulation given previously. The first constraint is $5C + 8G \leq 5000$. The coefficients are 5 and 8, meaning that 5 lbs of steel are required for each camshaft, and 8 lbs of steel are required for each gear. In cell F11, the value 5000 is stored.

The formula in cell D8 is =B8*B$5+C8*C$5. This references the objective coefficients (B8:C8) and the decision variables (B5:C5) to compute the total profit. If either a decision variable or a coefficient changes, the total profit will still calculate properly. Copy this formula to cells D11:D13 to compute the LHS values of the constraints, that is, the actual resources used by the solution. You could also write three additional formulas in cells D11:D13, but using absolute and relative references allows you to write one formula, and copy it to other cells needing formulas. Cell D11 computes the amount of steel used by the current solution. A $\leq$ sign is entered into cell E11 for readability, and the RHS value is entered in F11.

Testing the Model

After the base case model is developed, you should enter some trial values into the cells as decision variables in order to check the model's calculations. The trial solution shown in Figure B-3 is the same solution we discussed when defining feasible and infeasible solutions. Note that the spreadsheet calculates $8800 profit, 3300 lbs steel used, 1500 hours labor used, and 1000 hours machine time used. These values match those noted previously. We can quickly see from the model that this is a feasible solution by observing that the values in cells D11:D13 are all $\leq$ the values in cells F11:F13.

FIGURE B-3

Spreadsheet for DJJ problem

	A	B	C	D	E	F	G
1	**Example B.1**						
2	**DJJ Enterprises Production Planning**						
3							
4	**Decision Variables**	**Camshafts**	**Gears**				
5	Units to Make	75	200				
6							
7	**Objective**			**Total**			
8	Profit	$25	$18	**$5,475**			
9							
10	**Constraints**			**Used**		**Available**	
11	Steel (lbs)	5	8	1975	<=	5000	
12	Labor (hrs)	1	4	875	<=	1500	
13	Machine Time (hrs)	3	2	625	<=	1000	

D8: =B8*B$5+C8*C$5 (copied to D11:D13)

If the spreadsheet model is larger, the model testing phase is more involved. Finding errors in spreadsheet models is generally not easy, which makes this step especially important. No matter how much a spreadsheet has been used, you should always think about the outputs generated by a model, to see if they make sense given your knowledge of the situation. Excel has tools available to help diagnose problems in models. These are Error Checking (Tools/Error Checking) and Formula Auditing (Tools/Formula Auditing). These are not demonstrated in this supplement, but every spreadsheet modeler should be familiar with them.

SOLVER BASICS

Given the spreadsheet model, you could use it to try out different solutions. As we saw previously, it is easy to tell whether a solution is feasible. Therefore, through some trial and error, you could most likely find the optimal solution to this problem. Since there are only two decision variables, this probably will not take you long. However, for a larger problem, finding the optimal solution using this approach would at best take much longer, and most likely you would never find the optimal solution.

Excel's Solver add-in solves optimization problems. An add-in is an additional piece of software that enhances Excel's core capabilities. By default Solver does not load automatically when Excel starts. To load Solver, go to Tools/Add Ins, and select Solver from the list that appears. If this is the first time you have used Solver on your machine, you may be prompted to insert your Microsoft Office CD so that the Solver software can be copied onto your hard disk. Once you have selected (and possibly installed) Solver from the Add Ins list, it will automatically be loaded every time Excel starts, until you de-select it from the Add Ins list.

Problem Solving Tip: Solver Is not in the Add Ins List   If you are using Excel 2000 or later, Solver should appear on the Add Ins list when you select Tools/Add Ins. The first time you run Solver, you may be prompted to insert your Microsoft Office CD. You will only need to do this once. If you are using Excel 97 or earlier, Solver does not necessarily appear in the Add Ins list. If it does not, you will need to first quit Excel and run the Microsoft Office Setup program from the Microsoft Office CD. Follow the on-screen prompts to install Solver.

SETTING UP AND RUNNING SOLVER

To set up Solver for the DJJ problem, choose Tools/Solver from the menu. A dialog box similar to the one shown in Figure B-4 appears. Setting up an optimization problem with Solver consists of four steps:

1. Specify the **Target Cell** (corresponds to the objective function in the algebraic model).
2. Specify the **Changing Cells** (corresponds to the decision variables in the algebraic model).
3. Specify the **Constraints.**
4. Specify **Solver Options.**

These steps are easy if you have previously completed the algebraic formulation and spreadsheet model. They are very difficult, and perhaps impossible, if you haven't. Most difficulties students have with Solver are not really with Solver, but arise because

▶ **Target Cell**
In Solver, the cell to be maximized or minimized. This cell corresponds to the objective value.
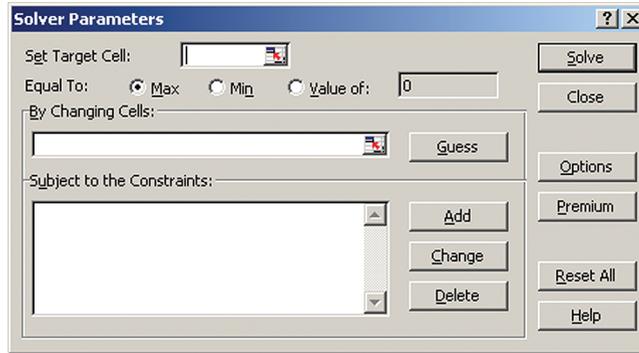
▶ **Changing Cells**
In Solver, the cells that can be changed during the optimization process. These cells correspond to the decision variables.

▶ **Constraints**
In Solver, the limitations and requirements that must be met.

▶ **Solver Options**
Solver provides many options for the solution process. For LPs, the two most commonly used are Assume Linear Model and Assume Non-Negative.

they either have not fully understood the problem or developed an algebraic formulation, or developed a correct and flexible spreadsheet model.

The Target Cell is the cell that you want to maximize or minimize. That is, it represents the objective value of the model. In this example, Cell D8 calculates the profit obtained from some combination of camshafts and gears. The Target Cell must always contain a formula; otherwise Solver generates an error message. To specify D8 as the Target Cell, either click on Cell D8 directly or type "D8" into the Target Cell field. Then make sure the radio button on the "Equal to" line is set to "Max." There are three choices on this line, but we will only deal with maximizing and minimizing problems. Click in the "By Changing Cells" field. The Changing Cells represent the decision variables of the model, in our case the numbers of camshafts and gears. These values are stored in cells B5 and C5. The easiest way to enter these into the By Changing Cells field is to simply select them directly in the spreadsheet using either the mouse to select both cells or by clicking on one cell, holding down the Shift key, and then using the keyboard arrow keys to select all of the pertinent cells. Alternately, the range "B5:C5" can be typed directly in the dialog box. The Solver Parameters box to this point is shown in Figure B-5. Note that if you type in the cell references directly for the Target Cell and the Changing Cells, you do not have to type in the "$" characters. This is one feature where it doesn't matter if you use absolute or relative cell referencing (see Supplement A for more information on absolute and relative references).

Entering the constraints is slightly more complicated, but not difficult if the spreadsheet model is designed well. To enter a constraint, click the "Add" button. A box appears, as shown in Figure B-6. A constraint in Solver consists of three pieces of
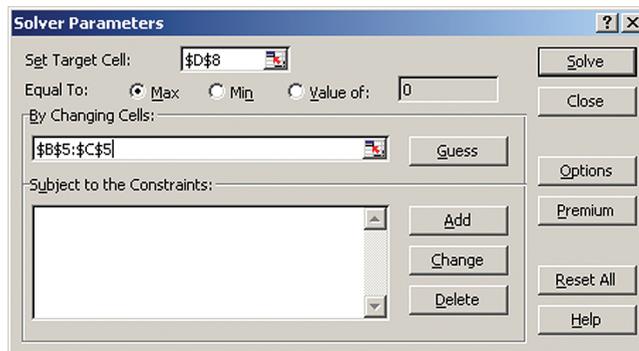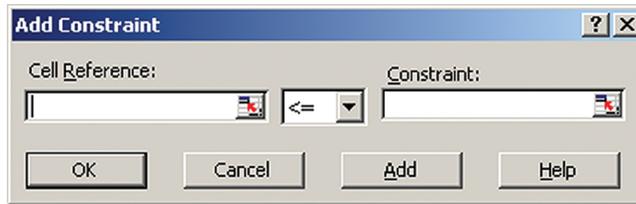
Add Constraint box (empty)

information: A "Cell Reference," which should refer to a spreadsheet cell containing a formula; a relationship operator, which can be ≤, =, or ≥; and a "Constraint," which should refer to a spreadsheet cell containing a value. More specifically, a constraint is of the form:

$$\text{LHS Cell} \quad \{\leq, =, \text{or} \geq\} \quad \text{RHS Cell}$$

Consider our first constraint, dealing with steel. The algebraic form of this constraint is

$$5C + 8G \leq 5000 \quad (\text{steel, lbs})$$

and is implemented in row 11 of the spreadsheet. Cells B11 and C11 store the coefficients, cell D11 contains the formula "=B11*B$5+C11*C$5," cell E11 contains the ≤ relationship, and cell F11 contains the value 5000. How do we fill in the Add Constraint box in Figure B-6? Since cell D11 computes the amount of steel actually used by a given combination of camshafts and gears, D11 is the "LHS Cell" for the constraint. Similarly, since F11 contains the amount of steel available, it is the "RHS Cell" for the constraint. The completed Add Constraint box is shown in Figure B-7.

We have two other constraints to enter, so click "Add" on the Add Constraint box. If you have already clicked "OK," then simply click "Add" on the Solver Parameters box. The other two constraints are entered in exactly the same way. They are both ≤ constraints. Using the previous logic, these constraints should be entered as

$$\text{D12} \leq \text{F12}$$
$$\text{D13} \leq \text{F13}$$

Once you have entered the last constraint, click "OK" on the Add Constraint box. The completed Solver Parameters box is shown in Figure B-8.

Problem Solving Tip: What if some of the constraints are ≥ or = constraints? Most problems contain more than one type of constraint. For these, simply pull down the arrow at the center of the Add Constraint box, and change the relationship to either ≥ or =, as appropriate. There are two additional choices, "int" and "bin" (not shown in the figures). These choices are for problems with integer and binary variables, which are beyond the scope of this example. They represent a special type of restriction on the decision variables.
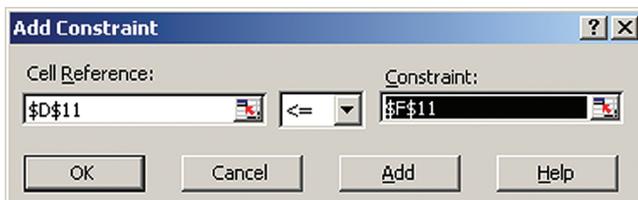

Add Constraint box (complete)
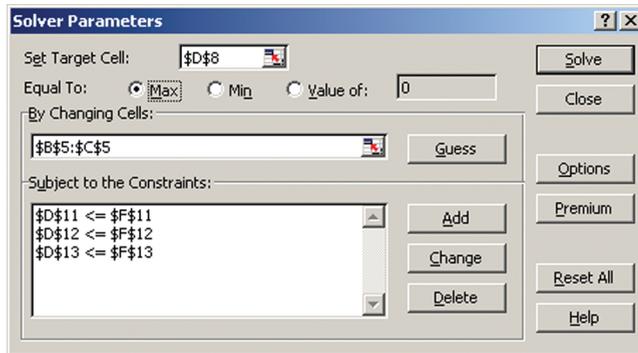
We have completed the first three steps of specifying the problem to Solver. To complete the fourth step, click the "Options" button on the Solver Parameters box. On the resulting Solver Options dialog box, select "Assuming Linear Model" and "Assume Non-Negative," as shown in Figure B-9. Don't worry about or change any of the other options. Once you have done this, click "OK" to return to the Solver Parameters box.

The Assume Non-Negative option adds the requirement that each of the Changing Cells be ≥ 0. This specifies our requirement in the algebraic formulation that $C \geq 0$ and $G \geq 0$. The Assume Linear Model option tells Solver to use the Simplex Method, mentioned previously. Solver actually comprises two solution algorithms, the Simplex Method (for Linear Programming problems) and a more general solution algorithm that can solve both linear and nonlinear problems. For Linear Programming problems, the Simplex Method is much faster and more reliable than the more general algorithm. It also provides more sensitivity analysis information.

Solving the Problem

The problem has now been completely specified. To actually solve it, click "OK" on the completed Solver Parameters box. Solver then applies the algorithm to the problem,
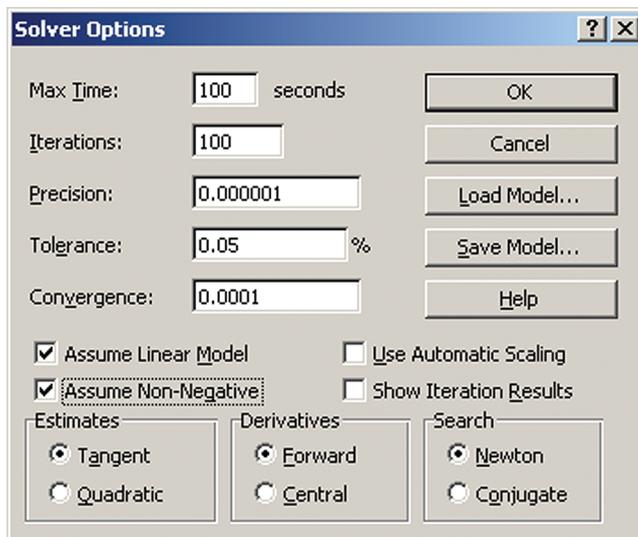
and produces a completion message, as shown in Figure B-10. The values of the Changing Cells (and any other cells dependent on them) also will change in the model. It is very important to read the Solver Results box. This one says "Solver found a solution. All constraints and optimality conditions are satisfied." If you did not obtain the same message for this problem, you should go back and work through the problem again, making sure each step is followed carefully. This message essentially means that Solver was successful in finding the optimal solution, and the solution shown in the spreadsheet is guaranteed to be the optimal solution to the model. There are many possible messages for the Solver Results box. Later we discuss some of the more commonly encountered ones and their meanings. You may notice that you can generate reports (Answer, Sensitivity, and Limits) from this dialog box as well. These reports can help you understand more about the solution and generate additional insights about the business problem. These are discussed later.

INTERPRETING THE SOLUTION

As shown in Figure B-11, the optimal solution is to make 100 camshafts and 350 gears. Doing so will yield a profit of $8800 and will require 3300 lbs of steel, 1000 hrs. of labor, and 1000 hrs. of machine time. Notice that we don't use all the steel that is available. In fact, there are 1700 lbs of steel unused. However, all of the labor and machine time are used. Having steel left over might seem nonintuitive, but making more of either camshafts or gears (to use up more steel) would also require more labor and machine time, of which we have no extra. Therefore, we are at a limit of sorts in how far we can "push" this solution. In general, if we have more constraints than decision variables, there will some constraints that are not at their limits. Constraints at their
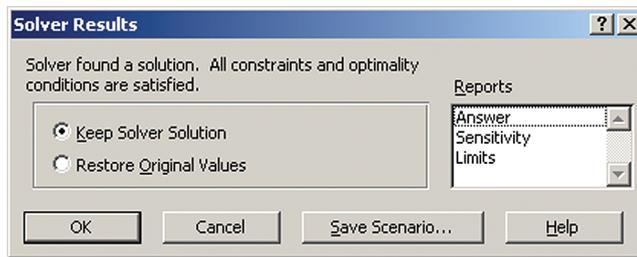
FIGURE B-10

Solver Results box

FIGURE B-11

DJJ optimal solution

	A	B	C	D	E	F
1	**Example B.1**					
2	**DJJ Enterprises Production Planning**					
3						
4	**Decision Variables**	**Camshafts**	**Gears**			
5	Units to Make	100	350			
6						
7	**Objective**			**Total**		
8	Profit	$25	$18	**$8,800**		
9						
10	**Constraints**			**Used**		**Available**
11	Steel (lbs)	5	8	3300	<=	5000
12	Labor (hrs)	1	4	1500	<=	1500
13	Machine Time (hrs)	3	2	1000	<=	1000

▶ **Binding constraints**
Constraints that are satisfied exactly at the optimal solution.

▶ **Nonbinding constraints**
Constraints that are satisfied at the optimal solution, but as strict inequalities ($<$ or $>$).

▶ **Slack**
The difference between the left hand side and the right hand side of a constraint. Loosely speaking, the amount of a resource left over (for a $\leq$ constraint) or the amount by which a requirement is satisfied (for a $\geq$ constraint).

limits when at the optimal solution, that is, with the LHS Value equal to the RHS Value, are called **binding constraints**. Those that are not at their limits are called **nonbinding constraints.** Here the binding constraints are labor and machine time, and the nonbinding constraint is steel. As noted, we have 1700 lbs of steel remaining at this solution. This value is called the **slack** of the constraint, and is simply the difference between the RHS Value and the LHS Value (slack values for binding constraints are equal to 0). This is a fundamental insight gained from the solution. Before solving the problem, we didn't know that labor and machine time would run out, in a sense, before steel. Putting this situation into practical business terms, if we want to do even better than this optimal solution, we will need to address the resource limitations of labor and/or machine time, or find another use for the excess steel.

In specifying the model to Solver, we must be careful to retain the flexibility of the model. This is done in Solver by using only cell references, not specific values. Now we have a powerful tool for analysis into the problem. We can change any parameter value in the model (any value highlighted in yellow), go to Tools/Solver, and click OK. Solver remembers all the problem settings, so we don't need to re-enter them. For example, if we have more labor and/or machine time available, we can enter the new value(s) in cell F12 and/or F13, run Solver again, and immediately see the impact of the modified resource amount. More importantly, we should do this before we actually commit to paying overtime for labor or committing more machine time; this way we can determine if, indeed, it is worthwhile. This ability to immediately reoptimize the problem is extremely powerful, and can lead to additional insights to assist in decision making.

SOLVER SOLUTION REPORTS

Three reports are available when Solver has successfully found an optimal solution. These are the Answer Report, the Sensitivity Report, and the Limits Report. They are shown as choices when Solver completes its work and displays the Solver Results box (Figure B-10). To generate these reports, select them and then click "OK." Solver inserts three new worksheets in the workbook. A detailed discussion of these reports is beyond the scope of this chapter, but an overview is provided of the Answer and Sensitivity Reports. Refer to one of the Management Science texts listed at the end of this supplement for more detailed information.

The Answer Report for this problem is shown in Figure B-12. It contains three sections, one each for the Target Cell, Adjustable (Changing) Cells, and the Constraints. Shown in this report are the "original" and "final" values. The original values are the values that were in the cells when Solver started its calculations, and the final values are the optimal values. By looking at the Target Cell and Adjustable Cells sections, you can identify that the optimal solution contains 100 camshafts and 350 gears, and yields $8800 in profit. The Constraint section contains the cell value (the LHS Value) of the constraint, the actual constraint itself as specified to Solver (the "Formula" column in the report), whether the constraint is binding or nonbinding, and the slack value. Two constraints are shown as binding, as we have already discovered by analyzing the solution in the spreadsheet.

The Answer Report doesn't provide any new information compared to what is available from an analysis of the final spreadsheet. However, it puts the information into a standard format. If two people develop spreadsheets for the same problem, they will likely look different. However, if the models are set up and solved correctly, the

Target Cell (Max)

Cell	Name	Original Value	Final Value
$D$8	Profit Total	$0	$8,800

Adjustable Cells

Cell	Name	Original Value	Final Value
$B$5	Units to Make Camshafts	0	100
$C$5	Units to Make Gears	0	350

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$D$11	Steel (lbs) Used	3300	$D$11<=$F$11	Not Binding	1700
$D$12	Labor (hrs) Used	1500	$D$12<=$F$12	Binding	0
$D$13	Machine Time (hrs) Used	1000	$D$13<=$F$13	Binding	0

FIGURE B-12

Answer Report for DJJ
Enterprises

Answer Reports will look very similar. Another use of the Answer Report is as a debugging tool. Looking in all three sections, the explicit cell references are given for the Target Cell, the Adjustable (Changing) Cells, and the Constraints. The Answer Report provides the settings that were actually set up in Solver, which sometimes differ from what a modeler *thinks* they told Solver!

The Sensitivity Report, shown in Figure B-13, contains information that is not directly available in the spreadsheet model itself. There is one section for the Adjustable (Changing) Cells and one for the Constraints. This report tells what will happen when certain changes are made to the model. We will not discuss all the different aspects of this report. However, two pieces of information from the Sensitivity Report can quickly provide insight into the solution.

In the Adjustable Cells section, the Allowable Increase and Allowable Decrease columns provide an indication as to how sensitive the optimal solution is to changes in the objective coefficients. In this example, the objective coefficients are $25 per camshaft and $18 per gear. From the report, the allowable increase of 2 for camshafts indicates that if the unit profit on camshafts were to increase by *more* than $2 per camshaft (i.e., to more than $27 per camshaft), holding all other data of the problem constant, the optimal solution (100 camshafts, 350 gears) would change. Conversely, the allowable decrease of 20.5 means that the unit profit of camshafts can decline by $20.50 per camshaft (to $4.50), again holding all other data constant, before the optimal solution will change. Therefore, the optimal solution appears quite sensitive to

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$5	Units to Make Camshafts	100	0	25	2	20.5
$C$5	Units to Make Gears	350	0	18	82	1.333333333

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$D$11	Steel (lbs) Used	3300	0	5000	1E+30	1700
$D$12	Labor (hrs) Used	1500	0.4	1500	500	1166.666667
$D$13	Machine Time (hrs) Used	1000	8.2	1000	1416.666667	250

FIGURE B-13

Sensitivity Report for DJJ
Problem

increases in the unit profit for camshafts, but insensitive to decreases in the unit profit. Similar (but opposite in direction) observations can be made for the unit profit of gears. In reality, objective coefficients are rarely known with certainty, so the allowable increase and decrease ranges provide a level of confidence in the optimal solution found by Solver.

Looking to the Constraint section, we focus on the Shadow Price column, as well as the Allowable Increase and Allowable Decrease. The **shadow price** for a constraint is the amount the optimal objective value will change, if the RHS of the constraint is increased by one unit. For example, the shadow price of the labor constraint is shown as 0.4. This means that if the number of labor hours were increased by one (to 1501), the optimal level of profit would increase by $0.40. The correct units would be $0.40 per labor hour. This is commonly referred to as the **marginal value** of the resource. This rate of change in the objective function does not continue indefinitely; the Allowable Increase column tells us that if we increase the labor available by up to 500 hours (to 2000 total hours), each one-hour increase will result in $0.40 of additional profit. Similar logic applies to the allowable decrease. By comparing shadow prices of constraints having the same units, it is easy to see which constraints will affect the objective value the most. Here, additional machine time is much more valuable than labor, since each additional machine hour can be used to generate an additional $8.20 in profit. Finally, the shadow price for the steel constraint is shown as 0, or $0 per pound of steel. Note that steel is a nonbinding constraint. Therefore, since only 3300 pounds of steel are required at the optimal solution, an additional pound of steel is worthless in terms of generating additional profit at this time. In general, shadow prices of nonbinding constraints are always zero.

▶ **Shadow price**
The amount by which the optimal objective value will change if the right hand side of a constraint is increased by one unit, assuming all other coefficients of the problem remain the same.

▶ **Marginal value**
Another name for shadow price.

RECAP

To address the DJJ Enterprises problem, we first developed a better understanding through a picture and a structured text formulation. Then we developed the formal algebraic formulation, and created a spreadsheet model based on it. We provided the target cell, changing cells, and constraints to Solver and set the appropriate options. After solving the problem, we generated solution reports and interpreted the results. Although the example in this chapter is simplified, this is the basic process you can use for addressing all linear programming problems.

OUTCOMES OF LINEAR PROGRAMMING PROBLEMS

When an LP is solved, there are several possible outcomes. One outcome is an optimal solution, as we have seen for the example. Solver shows a Solver Results box (see Figure B-10) saying that "Solver found a solution. All constraints and optimality conditions are satisfied." This is what we usually hope for when setting up and solving an LP. However, there are two additional possible outcomes when solving an LP. One occurs when Solver cannot find any feasible solution to the problem. Suppose we had these two constraints:

▶ **Infeasible problem**
An LP in which it is impossible for all the constraints to be satisfied simultaneously.

$$x_1 + x_2 \geq 10$$
$$x_1 + x_2 \leq 3$$

There is no combination of values for x_1 and x_2 that satisfies both constraints. When an LP does not have a feasible solution, we say the problem is **infeasible.** When this is

discovered by Solver, Solver displays the Solver Results box shown in Figure B-14. If you get this message, the values of the decision variables are not meaningful. Be careful to always read the Solver Results message.

Some problems indeed are infeasible, but many times this box is obtained when the modeler has made an error in the model itself, or in how the constraints were submitted to Solver. Even a small, simple error, such as entering a $\leq$ constraint incorrectly as a $\geq$ constraint, can cause the model to be infeasible. Careful scrutiny of the model is needed in order to debug it.

Essentially the opposite of a problem being infeasible is when the problem is **unbounded.** Being unbounded means that the objective value can increase (for a maximization problem) or decrease (for a minimization problem) without bound, that is, without ever running into a constraint. Consider the following problem:

$$\text{Max} \quad x_1$$
$$\text{subject to}$$
$$x_1 + x_2 \geq 6$$
$$x_1, x_2 \geq 0$$

► **Unbounded problem**
An LP in which the objective value can increase (for a maximization problem) or decrease (for a minimization problem) without bound.

You can see from the constraints that as long as the sum of x_1 and x_2 is at least six, and both x_1 and x_2 are nonnegative, the solution is feasible. Further, since the objective is to maximize x_1, we simply set x_2 to be some value (say, zero), and make x_1 infinitely large. The objective value can grow without bound; this is the source of the term "unbounded." The Solver Results box that appears when Solver encounters an unbounded model is shown in Figure B-15. Even though the language of this box is not as clear as it could be, it does make sense. The Set Cell (Target Cell) values do not converge on a specific, finite value. Therefore, the problem is unbounded.

Another Solver Results box you may encounter is shown in Figure B-16. This says, "The conditions for Assume Linear Model are not satisfied." This box appears when the user checks "Assume Linear Model" in Solver Options, but Solver does not recognize the model as being linear. Solver does its best to determine if the model is linear.
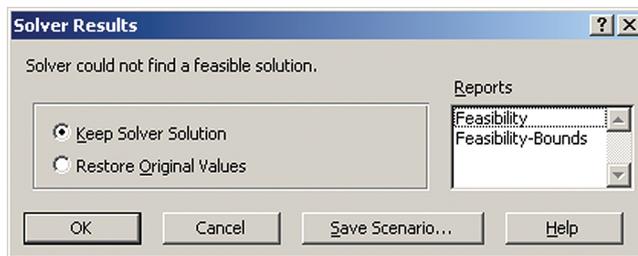

FIGURE B-14

Solver Results box indicating infeasible problem
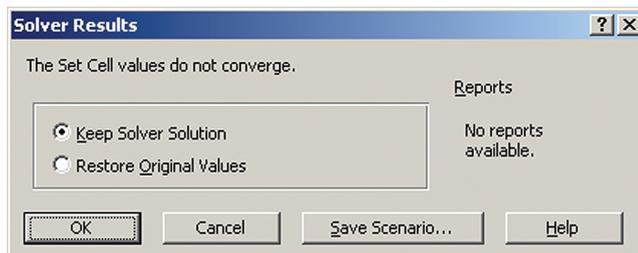
FIGURE B-15

Solver Results box indicating unbounded problem

Solver Results

The linearity conditions required by this LP Solver are
not satisfied.

Reports

Linearity

◉ Keep Solver Solution
○ Restore Original Values

OK   Cancel   Save Scenario...   Help
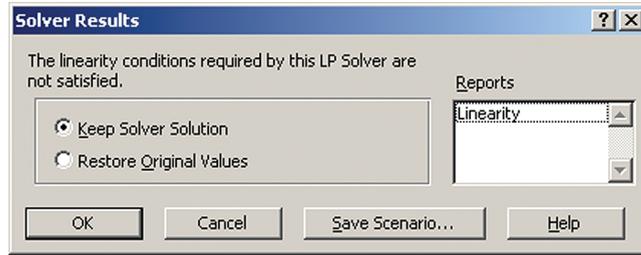
If you are sure your problem is linear, be sure all spreadsheet formulas involving variables are linear. Solver's check for a model being linear is not perfect, however, and sometimes simply solving the model again will cause a more normal completion message. Particular culprits for true nonlinearity are constraints involving percentages or ratios (usually these can be put into a linear form), or the use of functions such as IF, MAX, MIN, and other functions in the model. These functions are nonlinear and cannot be used to manipulate the decision variables in an LP.

INSIDE OM

As we discussed in the opening paragraphs of this chapter, everyone uses informal optimization thinking to make many routine decisions. However, formal optimization is used in all types of decisions in real organizations. On a large scale, Continental Airlines recently developed an optimization-based system to better recover from unexpected events such as inclement weather, aircraft mechanical problems, and crew (pilots and flight attendants) availability. Disruptions in one part of the flight network cause ripple effects throughout, because planes and crews do not get where they need to go. CALEB Technologies developed a system called CrewSaver that has already helped Continental recover from major disruptions ranging from weather delays to the September 11, 2001 terrorist attacks. Since 2000, CALEB estimates it has saved Continental $40 million due to more effective recovery procedures.

On a smaller scale, but no less significant to the organization, Welch's has developed a spreadsheet-based optimization tool to help them develop the optimal recipes for fruit juices. Since raw ingredient supplies and prices vary and demand also changes, it is necessary that management have a tool that will provide the optimal allocation of the raw materials to the products. The tool is used each month to aid management in the logistics plan. The savings during the first year (1994) was between $130,000 and $170,000.

Supplement Highlights

1 People use informal "optimization" to make decisions almost every day.

2 Organizations use formal optimization methods to address problems across the organization, from optimal pricing to locating a new facility.

3 The algebraic formulation of an LP comprises the definitions of the decision variables, an algebraic statement of the objective function, and algebraic statements of the constraints.

4 The spreadsheet model for an optimization problem should be guided by the algebraic formulation.

5 Solver, an Excel Add In, is able to solve both linear and nonlinear problems. This supplement focuses on solving linear problems.

6 After solving an LP, you must interpret the results to see if they make sense, fix problems with the model, and find the insights useful for management.

7 Solver can generate the Answer and Sensitivity Reports. The Sensitivity Report provides additional information about what happens to the solution when certain coefficients of the problem are changed.

Key Terms

objective B2
constraints B2
decision variables B2
constrained optimization problem B2
formulation B4
less-than-or-equal-to constraint B5
greater-than-or-equal-to constraint B5
equality constraint B5
nonnegativity constraints B5

Linear Program (LP) B6
feasible solution B6
infeasible solution B6
optimal solution B6
Simplex Method B7
LHS value B7
RHS value B7
Target Cell B9
Changing Cells B9

Constraints (in Solver) B9
Solver Options B9
binding constraints B14
nonbinding constraints B14
slack B14
shadow price B16
marginal value B16
infeasible problem B16
unbounded problem B17

Solved Problems

• Problem 1

TowAlong makes trailers at plants in Kansas City, Denver, and Raleigh, and ships these units to distribution centers in Birmingham, Milwaukee, Los Angeles, and Seattle. In planning production for the next year, TowAlong estimates unit shipping cost between any plant and distribution center, plant capacities, and distribution center demands. These numbers are given in the table. TowAlong faces the problem of determining how much to ship between each plant and distribution center to minimize the total transportation cost, while not exceeding capacity and while meeting demand.

(a) Formulate an LP to minimize the total shipping cost.
(b) Set up and solve the problem on a spreadsheet. What is the optimal solution?

| | **Distribution Center** | | | | |
| | | | **Los** | | |
Plant	**Birmingham**	**Milwaukee**	**Angeles**	**Seattle**	**Capacity**
Kansas City	$35	$40	$60	$120	12,000
Denver	$30	$30	$45	$130	8,000
Raleigh	$60	$65	$50	$100	5,000
Demand	9,000	3,000	9,500	1,500	

• Solution

(a) There are essentially 12 decisions to make when determining how much to ship from each plant to each distribution center. These decisions can be represented by x_{ij}, where i represents the plant, and j the distribution center. Therefore, we have the variables $x_{11}, x_{12}, x_{13}, x_{14}, \ldots, x_{34}$. Demand must be satisfied, and capacity constraints must not be violated. The

algebraic formulation can be stated as follows:

$$\text{Minimize Total Cost} = 35x_{11} + 40x_{12} + 65x_{13} + 120x_{14}$$

$$+ 30x_{21} + 30x_{22} + 45x_{23} + 130x_{24}$$

$$+ 60x_{31} + 65x_{32} + 50x_{33} + 100x_{34}$$

subject to

$$x_{11} + x_{12} + x_{13} + x_{14} \leq 12{,}000 \quad \text{(Kansas City capacity)}$$

$$x_{21} + x_{22} + x_{23} + x_{24} \leq 8000 \quad \text{(Denver capacity)}$$

$$x_{31} + x_{32} + x_{33} + x_{34} \leq 5000 \quad \text{(Raleigh capacity)}$$

$$x_{11} + x_{21} + x_{31} \geq 9000 \quad \text{(Birmingham demand)}$$

$$x_{12} + x_{22} + x_{32} \geq 3000 \quad \text{(Milwaukee demand)}$$

$$x_{13} + x_{23} + x_{33} \geq 9500 \quad \text{(Los Angeles demand)}$$

$$x_{14} + x_{24} + x_{34} \geq 1500 \quad \text{(Seattle demand)}$$

$$x_{ij} \geq 0 \text{ for all combinations} \quad \text{(nonnegativity)}$$

(b) The spreadsheet model can be structured in different ways, but it makes sense to arrange it in a way that will communicate the information clearly to management. The solved spreadsheet is shown. The minimum total cost is $1,010,000. To achieve this, ship 9000 from Kansas City to Birmingham, 1000 from Kansas City to Milwaukee, 2000 from Denver to Milwaukee, 6000 from Denver to Los Angeles, 3500 from Raleigh to Los Angeles, and 1500 from Raleigh to Seattle.

The spreadsheet has three sections: unit costs, transportation plan, and transportation costs. The 12 decision variables are the 12 cells in the transportation plan section. Based on this plan and the unit costs, the cost for each plant/distribution center combination is computed, and then totaled to get the objective function. Each row of the transportation plan section represents the amounts shipped from each plant summed in cells F15:F17, and each column represents the amounts received at each distribution center, summed in cells B18:E18. The Solver Settings are as follows for this model:

	A	B	C	D	E	F	G
1	**TowAway Transportation Problem**						
2							
3	**Unit Transportation Costs and Capacity/Demand Estimates**						
4			**Distribution Center**				
5	**Plant**	*Birmingham*	*Milwaukee*	*Los Angeles*	*Seattle*	**Capacity**	
6	Kansas Cty	$35	$40	$60	$120	12000	
7	Denver	$30	$30	$45	$130	8000	
8	Raleigh	$60	$65	$50	$100	5000	
9	**Demand**	**9000**	**3000**	**9500**	**1500**		
10						$F$15: =SUM(B15:E15)	
11						(copy down)	
12	**Transportation Plan**						
13			**Distribution Center**				
14	**Plant**	*Birmingham*	*Milwaukee*	*Los Angeles*	*Seattle*	**Shipped**	
15	Kansas Cty	9000	1000	0	0	10000	
16	Denver	0	2000	6000	0	8000	
17	Raleigh	0	0	3500	1500	5000	
18	**Received**	**9000**	**3000**	**9500**	**1500**		
19							
20			$B$18: =SUM(B15:B17) (copy right)				
21	**Transportation Costs**						
22			**Distribution Center**			$B$24: =B15*B6	
23	**Plant**	*Birmingham*	*Milwaukee*	*Los Angeles*	*Seattle*	(copy right & down) Outbound	
24	Kansas Cty	$315,000	$40,000	$0	$0	**$355,000**	
25	Denver	$0	$60,000	$270,000	$0	**$330,000**	
26	Raleigh	$0	$0	$175,000	$150,000	**$325,000**	
27	**Inbound**	**$315,000**	**$100,000**	**$445,000**	**$150,000**		
28						$F$24: =SUM(B24:E24)	
29			$B$27: =SUM(B24:B26) (copy right)			(copy down)	
30	**Total Transportation Costs:**				**$1,010,000**		
31				$E$30: =SUM(F24:F26)			
32							

Target Cell	E30 (minimize)		B18:E18 ≥ B9:E9 (demand constraints)
Changing Cells	B15:E17	Options	Assume Linear Model
Constraints	F15:F17 ≤ F6:F8 (capacity constraints)		Assume Non-Negative

• Problem 2

Chris is a student at the local college. Being on the go through the day, eating habits and nutrition sometimes do not receive as much attention as they should. In an effort to improve nutrition, Chris wonders if there is a way to eat a reasonably nutritious lunch and dinner diet at the local fast-food chain. Chris has collected some nutritional information about the menu items most often consumed. This isn't necessarily a diet that Chris will eat every day. However, since Chris often studies for hours at a time at the restaurant, it is likely that both lunch and dinner will sometimes be eaten there. Chris has been advised by a doctor to be careful about high sodium intake.

Nutritional and cost data are provided in the table. Chris has decided to use linear programming to try to find a reasonably nutritious diet that minimizes sodium intake. Chris has determined that the nutrition requirements indicate that the diet must contain 2000–2500 calories, have no more than 85 grams of fat, and contain at least 100% of the "daily value" (DV) of Vitamin C, Vitamin A, Calcium, and Iron. Chris has $15 budgeted for the two meals combined. Chris is willing to eat no more than two of each item from the menu.

	Large Burger	Small Burger	Chicken Strips	Chili	Deluxe Salad	Side Salad	Baked Potato	French Fries	Milk	Shake
Cost	$2.20	$0.99	$2.30	$0.99	$1.95	$0.99	$1.75	$1.50	$0.79	$1.50
Calories	410	270	410	200	410	140	390	400	120	330
Total Fat (g)	19	9	18	5	32	8	12	17	4.5	8
Vitamin C %DV	10	0	0	4	60	30	60	6	0	0
Vitamin A %DV	6	0	0	15	170	140	15	2	10	15
Calcium %DV	6	4	0	8	35	8	8	0	25	30
Iron %DV	25	20	6	10	20	6	15	8	0	15
Sodium (mg)	910	630	1470	870	840	570	160	340	120	150

• Solution

This problem is actually quite similar to the example in the chapter, only larger. As such, the spreadsheet will be arranged in a similar way. A text-based formulation is given. Note the objective function here is not monetary, but rather to minimize sodium intake, subject to a set of constraints, one of which is a budget constraint.

Text-Based Formulation
- Decision Variables: Number of each menu item to purchase and consume.
- Objective Function: Minimize sodium intake.
- Constraints: Calories, fat, Vitamin A, Vitamin C, Calcium, Iron, total cost, and the maximum of two of each menu item.

The other difference with this problem is that we cannot realistically order a fractional quantity of an item. Therefore, we will use Solver's capability to restrict the decision variables to be integers. You can consult one of the Management Science books in the bibliography for more information about integer variables.

A solved spreadsheet is shown. While the spreadsheet is somewhat large, there is only one unique formula written, which is for the objective value (total sodium) in cell L8. This formula is then copied to cells L10:L17 to compute the LHS Values of the constraints. The formula in cell L8 is =SUMPRODUCT (B8:K8,B$4:K$4). The SUMPRODUCT function was covered in Supplement A, but briefly, it multiplies each corresponding pair of entries (e.g., B8*B$4, C8*C$4, etc.), and then adds up these individual products.

Somewhat surprisingly, the optimal solution is to eat 1 Small Burger, 1 Deluxe Salad, 2 Baked Potatoes, and 2 Shakes. The total Sodium is 2090 mg. Considering the nutritional and cost constraints, we see that they are all satisfied, with Iron being the only constraint that is actually binding. The cost of this diet is $9.44, significantly less than our budget. The Solver Settings are as follows for this model:

Target Cell	L8 (minimize)
Changing Cells	B4:K4
Constraints	L10 ≥ N10 (calorie lower limit)

L11:L12 ≤ N11:12 (calorie & fat upper limits)

L13:L16 > N13:16 (Vitamins A & C, Calcium, & Iron lower limits)

L17 ≤ N17 (cost upper limit)

B4:K4 ≥ B5:K5 (item lower limit)

B4:K4 ≤ B6:K6 (item upper limit)

B4:K4 integer (to restrict to only integer values)

Options   Assume Linear Model

Assume Non-Negative

The B4:K4 integer constraint is entered in the same way as the other constraints, using the Add Constraint dialog box. When pulling down the relationship selection menu, choose "int" rather than ≤, ≥, or = . Integer restrictions should not automatically be added to a problem, even if the variables need to be integer in the real situation. The reason is that integer restrictions can dramatically increase the time required to solve a problem. It is usually best to first solve the problem without integer restrictions, and add them only if necessary. See one of the references at the end of the chapter for more information.

	A	B X1: Large Burger	C X2: Small Burger	D X3: Chicken Strips	E X4: Chili	F X5: Deluxe Salad	G X6: Side Salad	H X7: Baked Potato	I X8: French Fries	J X9: Milk	K X10: Shake	L	M	N
3														
4	**Number to Eat**	0	1	0	0	1	0	2	0	0	2			
5	**Minimum Required**	0	0	0	0	0	0	0	0	0	0			
6	**Maximum Allowed**	2	2	2	2	2	2	2	2	2	2			
7												**Total**		
8	**Sodium (mg)**	910	630	1470	870	840	570	160	340	120	150	**2090**		
9												**Total**		**Allowed or Required**
10	**Calories (min required)**	410	270	410	200	410	140	390	400	120	330	2120	>=	2000
11	**Calories (max allowed)**	410	270	410	200	410	140	390	400	120	330	2120	<=	2500
12	**Total Fat (g)**	19	9	18	5	32	8	12	17	4.5	8	81	<=	85
13	**Vitamin C %DV**	10	0	0	4	60	30	60	6	0	0	180	>=	100
14	**Vitamin A %DV**	6	0	0	15	170	140	15	2	10	15	230	>=	100
15	**Calcium %DV**	6	4	0	8	35	8	8	0	25	30	115	>=	100
16	**Iron %DV**	25	20	6	10	20	6	15	8	0	15	100	>=	100
17	**Cost**	$2.20	$0.99	$2.30	$0.99	$1.95	$0.99	$1.75	$1.50	$0.79	$1.50	$9.44	<=	$15.00

• Problem 3

IMC, Inc. needs to schedule production of a certain item for the next 4 months. The unit cost is estimated to be $12 for the first 2 months and $14 for the last 2 months. From the company's forecasting models, monthly demands are estimated to be 400, 750, 950, and 900 units, respectively. IMC can produce a maximum of 850 units each month on regular time. Overtime can be scheduled during the second and third months, which increases monthly capacity by 200 units. However, units produced on overtime cost $4 more to produce. Excess production can be stored at a cost of $3 per unit per month, but a maximum of 50 units can be stored during any month. Assuming that beginning and ending inventory levels are zero, how should the production be scheduled so as to minimize total costs?

• *Solution*

Similar to Solved Problem 1, we will let the structure of this problem drive the way the spreadsheet model is arranged. Essentially we must allow the user (or Solver) to change the quantities associated with regular and overtime production, and the model must compute the inventories (if any) as well as all costs.

A completed and solved spreadsheet is shown. Input data is listed in the Costs and Capacities section. The production plan contains the primary logic of the model. Each month the same pattern occurs. We start with some beginning inventory, add to that through regular and overtime production, experience demand, and result with an ending inventory. From this production plan and the resulting inventories, the costs can be calculated. The solution to this problem is to have regular production

	A	B	C	D	E	F	G	H
1	**IMC Production/Inventory Scheduling Problem**							
2	**Multiple-Period Planning**							
3								
4		**Month1**	**Month2**	**Month3**	**Month4**			
5	**Costs**							
6	Regular Production Cost ($/unit)	$12	$12	$14	$14			
7	Overtime Production Cost ($/unit)		$16	$18				
8	Inventory Holding Cost ($/unit-month)	$3	$3	$3	$3			
9								
10	**Capacities**							
11	Regular Production Capacity (units)	850	850	850	850			
12	Overtime Production Capacity (units)	0	200	200	0			
13	Inventory Capacity (units)	50	50	50	50			
14								
15						$C$17: =B21 copy to D17:E17		
16	**Production Plan**							
17	Beginning Inventory	0	0.00	50.00	50.00			
18	Regular Production	*400*	*800*	*850*	*850*	$B$21: =B17+B18+B19-B20 copy to C21:E21		
19	Overtime Production	*0*	*0*	*100*	*0*			
20	Demand (Sales)	400	750	950	900			
21	Ending Inventory	0.00	50.00	50.00	0.00	$F$24: =SUM(B24:E24) copy to F25:F26		
22			$B$24: =B18*B6 copy to B24:E25					
23	**Costs**						Total	
24	Regular Production	$ 4,800	$ 9,600	$ 11,900	$ 11,900	$ 38,200		
25	Overtime Production	$ -	$ -	$ 1,800	$ -	$ 1,800		
26	Inventory	$ 0	$ 150	$ 150	$ (0)	$ 300		
27	Total	$ 4,800	$ 9,750	$ 13,850	$ 11,900	$ 40,300	<----minimize this	
28		$B$26: =B21*B8 copy to C26:E26						
29		$B$27: =SUM(B24:B26) copy to C27:F27						
30								

be 400 in Month 1, 800 in Month 2, and 850 in Months 3 and 4. During Month 3, we will also plan 100 units of overtime production. This plan results in the lowest possible total cost, $40,300.

The Solver Settings are listed. The third constraint, requiring ending inventories to be ≥0, forces all demand to be met during the month in which it occurs (a negative inventory would indicate that some demand was not met).

Target Cell   F27 (minimize)
Changing Cells   B18:E19

Constraints   B18:E19 ≤ B11:E12 (production capacity constraints)
B21:E21 ≤ B13:E13 (inventory capacity constraints)
B21:E21 ≥ 0 (inventory cannot fall below zero)

Options   Assume Linear Model
Assume Non-Negative

Discussion Questions

1. What are the three main elements of an optimization problem? Give an example of each.

2. Describe a recent decision you made that could be viewed from an optimization perspective. In a few words, what were the "decision variables?" What was the objective? What were the constraints? Did you have more than one objective?

3. Think of a company you are familiar with. Describe at least three problems the company faces that could be addressed, at least in part, through optimization. The problems can be either strategic or operational in nature.

4. When specifying the Target Cell, Changing Cells, and Constraints to Solver, why is it important to use only cell references rather than fixed numbers?

5. Explain the meaning of binding and nonbinding constraints.

6. Why is it important to conduct sensitivity analysis after solving an optimization problem?

Problems

1. DynaTune is trying to determine how to allocate its $145,000 advertising budget for a new product. Radio and television are the two media outlets selected for advertising. The table below shows the costs of advertising in these two media and the estimated number of new customers reached by increasing amounts of advertising.

Media Type & Number of Ads	Number New Customers Reached	Cost per Ad
Radio 1–10	900	$1,000
Radio 11–20	700	$900
Radio 21–30	400	$800
Television 1–5	10,000	$12,000
Television 6–10	7,500	$10,000
Television 11–15	5,000	$8,000

For example, each of the first 10 ads the company places on the radio will cost $1000 and is expected to reach 900 new customers. Each of the next 10 will cost $900 and is expected to reach 700 new customers. Assume the company will purchase no more than 30 radio ads and no more than 15 television ads.

   (a) Formulate an LP to maximize the number of new customers reached.
   (b) Set up and solve the problem on a spreadsheet.
   (c) What is the optimal solution? Explain the rationale for the solution.
   (d) Suppose the number of new customers reached by 11–15 television ads is 7500 and the number reached by 6–10 ads is 5000. Resolve the problem. What is wrong with this solution? Is Solver incorrect, or is there some missing logic in the model?

2. Montana Pulp (MP) produces recycled paperboard by combining 4 grades of recycled paper stock. Each grade of stock has a different strength, color, and texture. The strength, color, and texture of the finished paperboard are a weighted average of those characteristics of the paper stock inputs. The table provides the characteristics of the paper inputs and their costs per ton. An order for 500 tons of paperboard with a strength of at least 7, a color of at least 5, and a texture of at least 6 has been received. MP needs to find the minimal-cost mix of inputs required to satisfy this order.

Paper Stock	Strength	Color	Texture	Cost/Ton
1	8	9	8	$150
2	6	7	5	$110
3	5	5	6	$90
4	3	4	5	$50

   (a) Formulate an LP to minimize the cost of the order.
   (b) Set up and solve the problem on a spreadsheet.

   (c) What is the optimal solution? Explain the rationale for the solution.

3. A company has 3 manufacturing plants (in Atlanta, Tulsa, and Springfield) that produce a product that is then shipped to 1 of 4 distribution centers. The 3 plants can produce 13, 18, and 12 truckloads of product each week, respectively. Each distribution center needs 10 truckloads of product each week. The shipping costs per truckload between the plants and distribution centers are given in the table. The company needs to determine how much to ship from each plant to each distribution center, and would like to minimize total shipping costs.

Plant	Distribution Center			
	A	B	C	D
Atlanta	$800	$1300	$400	$700
Tulsa	$1100	$1400	$600	$1000
Springfield	$600	$1200	$800	$900

   (a) Formulate an LP to minimize the total shipping costs.
   (b) Set up and solve the problem on a spreadsheet.
   (c) What is the optimal solution? Explain the rationale for the solution.

4. FiberTech makes newsprint for newspapers at 3 mills, A, B, and C. The cost of producing newsprint is estimated to be $210 at mill A, $225 at B, and $220 at C. Five primary geographical markets are served from these mills. The monthly demand at each market, the shipping cost (per ton) between each mill and market, and the monthly production capacity of each mill are given in the table.

	Mill	Market					Capacity (tons/month)
		1	2	3	4	5	
Shipping	A	$20	$25	$30	$15	$35	1200
Cost/Ton	B	$30	$20	$32	$28	$19	1500
	C	$25	$18	$28	$23	$31	900
Monthly Demand (tons)		600	100	500	800	500	

FiberTech would like to assign production responsibilities to the mills, and also specify how much should be shipped from each plant to each market, so as to minimize total production and distribution costs.
   (a) Formulate an LP to minimize total production and distribution costs.
   (b) Set up and solve the problem on a spreadsheet.
   (c) What is the optimal solution? Explain the rationale for the solution.

5. Northwest Pipe (NP) makes water pipe. NP is planning production for the next 7 months, March through September. The forecast demands (in thousands of feet) are, respectively, 40, 60, 70, 80, 90, 100, and 80. NP can make 75,000 feet of pipe per month using regular time production, at a cost of $1.25 per foot. They can make up to an additional 15,000 feet using overtime production at a cost of $1.50 per foot. Any pipe made in one month and sold in a later month incurs an inventory holding cost of $0.15 per foot, per month. NP expects to end February with 5000 feet of pipe, and would like to plan to end September with 10,000 feet in inventory. NP would like to plan their production schedule to minimize total cost during the next 7 months.

(a) Formulate an LP to minimize total costs.
(b) Set up and solve the problem on a spreadsheet.
(c) What is the optimal solution? Explain the rationale for the solution.

6. A1 Credit provides credit information to its customers throughout the country, 24 hours per day. Credit representatives answer customer calls and provide information. Based on demand patterns, the estimated number of representatives needed during 4-hour time periods of each day are shown in the table.

Time Period	Number Representatives Needed
midnight–4 AM	3
4 AM–8 AM	6
8 AM–noon	13
noon–4 PM	15
4 PM–8 PM	12
8 PM–midnight	9

Employees work shifts of 8 consecutive hours, and shifts can start at the beginning of any of the 6 periods shown in the table. Assume this schedule will be repeated day after day, so that someone starting at 8 PM will contribute to the need for representatives in the Midnight-4 AM time period. The supervisor has been instructed to schedule employees so that all demands can be met with a minimum possible number of employees.

(a) Formulate an LP to minimize the number of representatives needed.
(b) Set up and solve the problem on a spreadsheet.
(c) What is the optimal solution? Explain the rationale for the solution.

Case: Exeter Enterprises

Exeter Enterprises produces premium golf clubs. They are in need of a planning tool that will help them make decisions over the next 8 quarters. The demand forecast for the upcoming 8-quarter period is developed every quarter, and the production manager must determine how many people to hire and lay off, and also how much overtime and subcontracting to schedule.

Information relevant to the problem is shown. The demand for the upcoming 8 quarters is for the following number of sets of clubs, respectively: 1920, 2160, 1440, 1200, 2040, 2400, 1740, and 1500. Clubs can be inventoried from one period to the next, incurring the holding cost. Backorders are also allowed, although the company does not prefer to use them.

Cost Data

Regular time labor cost per hour	$12.50
Overtime labor cost per hour	$18.75
Subcontracting cost per unit (labor only)	$125.00
Backorder cost per unit per quarter	$25.00
Inventory holding cost per unit per quarter	$10.00
Hiring cost per employee	$800.00
Firing cost per employee	$500.00

Capacity Data

Beginning workforce	90	employees
Beginning inventory	0	units
Production standard per unit	8	hours of labor per unit
Regular time available per period	160	hours per period per employee
Overtime available per period	40	hours per period per employee

The production manager would ideally like a tool in which one could enter the decision values (number of sets to assign to regular time, overtime, and subcontracting; number of sets to backorder; and number of employees to hire and lay off each quarter). The manager could then compute a cost breakdown, to compare different plans. Ultimately, the production manager would like the tool to be able to find the plan minimizing the total costs over the next 8 quarters.

Questions

1. Develop a spreadsheet decision tool that will allow the production manager to experiment with different plans.

2. Develop an optimization model using Solver to allow the manager to find the minimal-cost plan. Make what you think are reasonable assumptions about the maximum number of backorders allowed, as well as the maximum number of employees that could be hired and laid off in any one quarter. After solving the problem, you may decide to make additional assumptions.

3. Write a short summary of your findings, comparing several plans entered manually, as well as the optimal plan.

Interactive Learning

1. Spreadsheets for Example B.1 and Solved Problems 1, 2, and 3 are available on the CD.

2. Frontline Systems, Inc. developed Solver for Microsoft and continues to develop enhanced versions of Solver. Go to http://www.solver.com and gather information about Frontline's products and services. Prepare a one-page summary.

3. Lindo Systems, Inc. is a developer of several different optimization tools, including What's Best, which is a competitor to Solver. Go to http://www.lindo.com/table/wbt.html and gather information about this product. How does it compare to Solver?

4. Go to http://www.solver.com/solutions.htm. Optimization examples from Finance, Investment, Production, Distribu-

tion, Purchasing, and Scheduling are available. Choose two examples from two different areas. Study the examples, and prepare a 10-minute presentation to explain the examples.

5. On Wendy's Web site (http://www.wendys.com), you can build a meal and obtain all the nutritional information. Click on the "Nutritional Guide" to access this part of the Web site. Build a meal plan for a one-day diet, and assess the nutritional value. Compare and contrast this diet-planning tool with the optimization-based approach presented in Solved Problem 2.

Bibliography

Lawrence, John A., and Barry A. Pasternack. *Applied Management Science: A Computer-Integrated Approach for Decision Making.* 2nd ed. New York: Wiley, 2002.

Ragsdale, Cliff T., *Spreadsheet Modeling & Decision Analysis,* 4th ed. Stamford, CT: Thomson, 2004.

Schuster, Edmund W., and Stuart J. Allen. "Raw Material Management at Welch's, Inc." *Interfaces,* Vol. 28, No. 5, 1998, 13–28.

Winston, Wayne L., and S. Christian Albright. *Practical Management Science* (with CD-ROM Update), 2nd ed. Duxbury, 2003.

Yu, Gang, Michael Arguello, Gao Song, Sandra McCowan, and Anna White. "A New Era for Crew Recovery at Continental Airlines." *Interfaces,* Vol. 33, No. 1, 2003, 5–22.