Surface Area and Volume Weeks 14-16

CK-12 Foundation

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Chapter 1

Surface Area and Volume

1.1 The Polyhedron

Learning Objectives

- Identify polyhedra.
- Understand the properties of polyhedra.
- Use Euler's formula solve problems.
- Identify regular (Platonic) polyhedra.

Introduction

In earlier chapters you learned that a **polygon** is a two-dimensional (planar) figure that is made of three or more points joined together by line segments. Examples of polygons include triangles, quadrilaterals, pentagons, or octagons. In general, an n - gon is a polygon with n sides. So a triangle is a 3 - gon, or 3-sided polygon, a pentagon is a 5 - gon, or 5-sided polygon.



You can use polygons to construct a 3-dimensional figure called a **polyhedron** (plural: **polyhedra**). A polyhedron is a 3-dimensional figure that is made up of polygon faces. A cube is an example of a polyhedron and its faces are squares (quadrilaterals).

Polyhedron or Not

A polyhedron has the following properties:

- It is a 3-dimensional figure.
- It is made of polygons and only polygons. Each polygon is called a **face** of the polyhedron.
- Polygon faces join together along segments called **edges.**
- Each edge joins exactly two faces.
- Edges meet in points called **vertices.**
- There are no gaps between edges or vertices.



a polyhedron (prism)

Example 1

Is the figure a polyhedron?



Yes. A figure is a polyhedron if it has all of the properties of a polyhedron. This figure:

- Is 3-dimensional.
- Is constructed entirely of flat polygons (triangles and rectangles).
- Has faces that meet in edges and edges that meet in vertices.
- Has no gaps between edges.

- Has no non-polygon faces (e.g., curves).
- Has no concave faces.

Since the figure has all of the properties of a polyhedron, it is a polyhedron.

Example 2

Is the figure a polyhedron?



No. This figure has faces, edges, and vertices, but all of its surfaces are *not* flat polygons. Look at the end surface marked A. It is flat, but it has a curved edge so it is not a polygon. Surface B is not flat (planar).

Example 3

Is the figure a polyhedron?



No. The figure is made up of polygons and it has faces, edges, and vertices. But the faces do not fit together—the figure has gaps. The figure also has an overlap that creates a concave surface. For these reasons, the figure is not a polyhedron.

Face, Vertex, Edge, Base

As indicated above, a polyhedron joins faces together along edges, and edges together at vertices. The following statements are true of any polyhedron:

• Each edge joins exactly two faces.

• Each edge joins exactly two vertices.

To see why this is true, take a look at this prism. Each of its edges joins two faces along a single line segment. Each of its edges includes exactly two vertices.



Let's count the number of faces, edges, and vertices in a few typical polyhedra. The square pyramid gets its name from its base, which is a square. It has 5 faces, 8 edges, and 5 vertices.



Other figures have a different number of faces, edges, and vertices.

rectangular prism	octahedron	pentagonal prism
base: rectangle	base: triangle	base: pentagon
faces: 6	faces: 8	faces: 7
edges: 12	edges: 12	edges: 15
vertices: 8	vertices: 6	vertices: 10

If we make a table that summarizes the data from each of the figures we get:

Figure	Vertices	Faces	Edges
Square pyramid	5	5	8
Rectangular prism	8	6	12
Octahedron	6	8	12
Pentagonal prism	10	7	15

Table 1.1:

Do you see a pattern? Calculate the sum of the number of vertices and edges. Then compare that sum to the number of edges:

Table 1.2:

Figure	V	F	E	V + F
square pyramid	5	5	8	10
rectangular	8	6	12	14
prism				
octahedron	6	8	12	14
pentagonal	10	7	15	17
prism				

Do you see the pattern? The formula that summarizes this relationship is named after mathematician Leonhard Euler. Euler's formula says, for any polyhedron:

Euler's Formula for Polyhedra

or

$$v + f = e + 2$$

You can use Euler's formula to find the number of edges, faces, or vertices in a polyhedron.

Example 4

Count the number of faces, edges, and vertices in the figure. Does it conform to Euler's formula?



There are 6 faces, 12 edges, and 8 vertices. Using the formula:

$$v + f = e + 2$$

 $8 + 6 = 12 + 2$

So the figure conforms to Euler's formula.

Example 5

In a 6-faced polyhedron, there are 10 edges. How many vertices does the polyhedron have? Use Euler's formula.

v + f = e + 2	Euler's formula
v + 6 = 10 + 2	Substitute values for f and e
v = 6	Solve

There are 6 vertices in the figure.

Example 6

A 3-dimensional figure has 10 vertices, 5 faces, and 12 edges. It is a polyhedron? How do you know?

Use Euler's formula.

v + f = e + 2	Euler's formula
$10 + 5 \neq 12 + 2$	Substitute values for v, f , and e
$15 \neq 14$	Evaluate

The equation does not hold so Euler's formula does not apply to this figure. Since all polyhedra conform to Euler's formula, this figure must not be a polyhedron.

Regular Polyhedra

Polyhedra can be named and classified in a number of ways—by side, by angle, by base, by number of faces, and so on. Perhaps the most important classification is whether or not a polyhedron is **regular** or not. You will recall that a **regular polygon** is a polygon whose sides and angles are all congruent.

A polyhedron is regular if it has the following characteristics:

- All faces are the same.
- All faces are congruent regular polygons.
- The same number of faces meet at every vertex.
- The figure has no gaps or holes.
- The figure is convex—it has no indentations.



regular polygons



convex

non-convex

Example 7

Is a cube a regular polyhedron?



All faces of a cube are regular polygons—squares. The cube is convex because it has no indented surfaces. The cube is simple because it has no gaps. Therefore, a cube is a regular polyhedron.

A polyhedron is **semi-regular** if all of its faces are regular polygons and the same number of faces meet at every vertex.

- Semi-regular polyhedra often have two different kinds of faces, both of which are regular polygons.
- **Prisms** with a regular polygon base are one kind of semi-regular polyhedron.
- Not all semi-regular polyhedra are prisms. An example of a non-prism is shown below.



semi-regular: prism equilateral triangles and squares

Completely irregular polyhedra also exist. They are made of different kinds of regular and irregular polygons.





semi-regular: squares and equilateral triangles

So now a question arises. Given that a polyhedron is regular if all of its faces are congruent regular polygons, it is convex and contains no gaps or holes. How many *regular* polyhedra actually exist?

In fact, you may be surprised to learn that only five regular polyhedra can be made. They are known as the **Platonic** (or noble) solids.



Note that no matter how you try, you can't construct any other regular polyhedra besides the ones above.

Example 8

How many faces, edges, and vertices does a tetrahedron (see above) have?

Faces : 4, edges : 6, vertices : 4

Example 9

Which regular polygon does an icosahedron feature?

An equilateral triangle

Review Questions

Identify each of the following three-dimensional figures:





- 6. Below is a list of the properties of a polyhedron. Two of the properties are not correct. Find the incorrect ones and correct them.
 - It is a 3 dimensional figure.
 - Some of its faces are polygons.
 - Polygon faces join together along segments called edges.
 - Each edge joins three faces.
 - There are no gaps between edges and vertices.

Complete the table and verify Euler's formula for each of the figures in the problem.

Table 1.3:

Figure	# vertices	# edges	# faces		
Pentagonal prism					
Rectangular pyra-					
mid					
Triangular prism					
Trapezoidal prism					

7.

Review Answers

Identify each of the following three dimensional figures:

- 1. pentagonal prism
- 2. rectangular pyramid
- 3. triangular prism
- 4. triangular pyramid
- 5. trapezoidal prism
- 6. Below is a list of the properties of a polyhedron. Two of the properties are not correct. Find the incorrect ones and correct them.
 - It is a 3 dimensional figure.
 - Some of its faces are polygons. All of its faces are polygons.
 - Polygon faces join together along segments called edges.
 - Each edge joins three faces. *Each edge joins two faces*.
 - There are no gaps between edges and vertices.

Complete the table and verify Euler's formula for each of the figures in the problem.

Figure	# vertices	# edges	# faces
Pentagonal prism	10	15	7
Rectangular pyra-	5	8	5
mid			
Triangular prism	6	9	5
Trapezoidal prism	8	12	6

Table 1.4:

7. In all cases vertices + faces = edges + 2

Representing Solids 1.2

Learning Objectives

- Identify isometric, orthographic, cross-sectional views of solids.
- Draw isometric, orthographic, cross-sectional views of solids.
- Identify, draw, and construct nets for solids.

Introduction

The best way to represent a three-dimensional figure is to use a solid model. Unfortunately, models are sometimes not available. There are four primary ways to represent solids in two dimensions on paper. These are:

- An isometric (or perspective) view.
- An orthographic or blow-up view.
- A cross-sectional view.
- A net.

Isometric View

The typical three-dimensional view of a solid is the **isometric** view. Strictly speaking, an isometric view of a solid does not include **perspective**. Perspective is the illusion used by artists to make things in the distance look smaller than things nearby by using a vanishing point where parallel lines converge.

The figures below show the difference between an isometric and perspective view of a solid.





isometric view

As you can see, the perspective view looks more "real" to the eye, but in geometry, isometric representations are useful for measuring and comparing distances.

The isometric view is often shown in a transparent "see-through" form.



Color and shading can also be added to help the eye visualize the solid.

Example 1

Show isometric views of a prism with an equilateral triangle for its base.



Example 2

Show a see-through isometric view of a prism with a hexagon for a base.



Orthographic View

An **orthographic** projection is a blow-up view of a solid that shows a flat representation of each of the figure's sides. A good way to see how an orthographic projection works is to construct one. The (non-convex) polyhedron shown has a different projection on every side.



To show the figure in an orthographic view, place it in an imaginary box.



Now project out to each of the walls of the box. Three of the views are shown below.



A more complete orthographic blow-up shows the image of the side on each of the six walls of the box.



The same image looks like this in fold out view.



Example 3



 $Show \ an \ orthographic \ view \ of \ the \ figure.$

First, place the figure in a box.



Now project each of the sides of the figure out to the walls of the box. Three projections are shown.



You can use this image to make a fold-out representation of the same figure.



Cross Section View

Imagine slicing a three-dimensional figure into a series of thin slices. Each slice shows a **cross-section** view.



The cross section you get depends on the angle at which you slice the figure.

Example 4

What kind of cross section will result from cutting the figure at the angle shown?



Example 5

What kind of cross section will result from cutting the figure at the angle shown?





Example 6

What kind of cross section will result from cutting the figure at the angle shown?



Nets

One final way to represent a solid is to use a net. If you cut out a net you can fold it into a model of a figure. Nets can also be used to analyze a single solid. Here is an example of a net for a cube.



There is more than one way to make a net for a single figure.



However, not all arrangements will create a cube.



Example 7

What kind of figure does the net create? Draw the figure.



The net creates a box-shaped rectangular prism as shown below.



Example 8

What kind of net can you draw to represent the figure shown? Draw the net.



A net for the prism is shown. Other nets are possible.



Review Questions

- 1. Name four different ways to represent solids in two dimensions on paper.
- 2. Show an isometric view of a prism with a square base.

Given the following pyramid:



3. If the pyramid is cut with a plane parallel to the base, what is the cross section?

- 4. If the pyramid is cut with a plane passing through the top vertex and perpendicular to the base, what is the cross section?
- 5. If the pyramid is cut with a plane perpendicular to the base but not through the top vertex, what is the cross section?

Sketch the shape of the plane surface at the cut of this solid figure.



- 6. Cut AB
- 7. Cut CD
- 8. For this figure, what is the cross section?



Draw a net for each of the following:



Review Answers

1. Name four different ways to represent solids in two dimensions on paper.

Isometric, orthographic, cross sectional, net

2. Show an isometric view of a prism with a square base.



Given the following pyramid:

3. If the pyramid is cut with a plane parallel to the base, what is the cross section? square

- 4. If the pyramid is cut with a plane passing through the top vertex and perpendicular to the base, what is the cross section? *triangle*
- 5. If the pyramid is cut with a plane perpendicular to the base but not through the top vertex, what is the cross section? *trapezoid*

Sketch the shape of the plane surface at the cut of this solid figure.





1.3 Prisms

Learning Objectives

- Use nets to represent prisms.
- Find the surface area of a prism.
- Find the volume of a prism.

Introduction

A prism is a three-dimensional figure with a pair of parallel and congruent ends, or bases. The sides of a prism are parallelograms. Prisms are identified by their bases.



Surface Area of a Prism Using Nets

The prisms above are **right prisms**. In a right prism, the lateral sides are perpendicular to the bases of prism. Compare a right prism to an **oblique prism**, in which sides and bases are not perpendicular.



right prism oblique prism

Two postulates that apply to area are the Area Congruence Postulate and the Area Addition Postulate.

Area Congruence Postulate:

If two polygons (or plane figures) are congruent, then their areas are congruent.

Area Addition Postulate:

The surface area of a three-dimensional figure is the sum of the areas of all of its non-overlapping parts.

You can use a net and the Area Addition Postulate to find the surface area of a right prism.



From the net, you can see that that the surface area of the entire prism equals the sum of the figures that make up the net:

Total surface area = area A + area B + area C + area D + area E + area F

Using the formula for the area of a rectangle, you can see that the area of rectangle A is:

 $\begin{aligned} A &= l \cdot w \\ A &= 10 \cdot 5 = 50 \text{ square units} \end{aligned}$

Similarly, the areas of the other rectangles are inserted back into the equation above.

Total surface area = area A + area B + area C + area D + area E + area FTotal surface area = $(10 \cdot 5) + (10 \cdot 3) + (10 \cdot 5) + (10 \cdot 3) + (5 \cdot 3) + (5 \cdot 3)$ Total surface area = 50 + 30 + 50 + 30 + 15 + 15Total surface area = 190 square units

Example 9

Use a net to find the surface area of the prism.



The area of the net is equal to the surface area of the figure. To find the area of the triangle, we use the formula:

A = 1/2 hb where h is the height of the triangle and b is its base.

Note that triangles A and E are congruent so we can multiply the area of triangle A by 2.

area = area A + area B + area C + area D + area E
= 2(area A) + area B + area C + area D
=
$$2\left[\frac{1}{2}(9 \cdot 12)\right] + (6 \cdot 9) + (6 \cdot 12) + (6 \cdot 15)$$

= 108 + 54 + 72 + 90
= 324

Thus, the surface area is 324 square units .

Surface Area of a Prism Using Perimeter

This hexagonal prism has two regular hexagons for bases and six sides. Since all sides of the hexagon are congruent, all of the rectangles that make up the lateral sides of the three-dimensional figure are also congruent. You can break down the figure like this.



The surface area of the rectangular sides of the figure is called the **lateral area** of the figure. To find the lateral area, you could add up all of the areas of the rectangles.

lateral area = 6 (area of one rectangle)
= 6
$$(s \cdot h)$$

= 6sh

Notice that 6s is the perimeter of the base. So another way to find the lateral area of the figure is to multiply the perimeter of the base by h, the height of the figure.

lateral area
$$= 6sh$$

 $= (6s) \cdot h$
 $= (perimeter)h$
 $= Ph$

Substituting P, the perimeter, for 6s, we get the formula for any lateral area of a right prism:

lateral area of a prism
$$= Ph$$

Now we can use the formula to calculate the total surface area of the prism. Using P for the perimeter and B for the area of a base:

Total surface area = lateral area + area of 2 bases = (perimeter of base \cdot height) + 2 (area of base) = Ph + 2 B

To find the surface area of the figure above, first find the area of the bases. The regular hexagon is made of six congruent small triangles. The altitude of each triangle is the **apothem** of the polygon. Note: be careful here—we are talking about the altitude of the triangles, not the height of the prism. We find the length of the altitude of the triangle using the Pythagorean Theorem, $a = \sqrt{4^2 - 2^2} \approx 3.46$



So the area of each small triangle is:

$$A \text{ (triangle)} = \frac{1}{2}ab$$
$$= \frac{1}{2}(3.46)(4)$$
$$= 6.92$$

The area of the entire hexagon is therefore:

$$A \text{ (base)} = 6 \text{ (area of triangle)}$$
$$= 6 \cdot 6.92$$
$$= 41.52$$

You can also use the formula for the area of a regular polygon to find the area of each base:

$$A \text{ (polygon)} = \frac{1}{2}aP$$
$$= \frac{1}{2}(3.46)(24)$$
$$= 41.52$$
Now just substitute values to find the surface area of the entire figure above.

$$(\text{total surface area}) = Ph + 2B$$

= $[(6 \cdot 4) \cdot 10] + 2(41.52)$
= $(24 \cdot 10) + 83.04$
= $240 + 83.04$
= 323.04 square units

You can use the formula A = Ph + 2B to find the surface area of any right prism.

Example 10

Use the formula to find the total surface area of the trapezoidal prism.



The dimensions of the trapezoidal base are shown. Set up the formula. We'll call the height of the entire prism H to avoid confusion with h, the height of each trapezoidal base.

Total surface area = PH + 2B

Now find the area of each trapezoidal base. You can do this by using the formula for the area of a trapezoid. (Note that the height of the trapezoid, 2.46 is small h.)

$$A = \frac{1}{2}h(b_1 + b_2)$$

= $\frac{1}{2}(2.64)[10 + 4]$
= 18.48 square units

Now find the perimeter of the base.

$$P = 10 + 4 + 4 + 4$$

= 22

Now find the total surface area of the solid.

(total surface area) = Ph + 2B= (22)(21) + 2(18.48) = 462 + 36.96 = 498.96 square units

Volume of a Right Rectangular Prism

Volume is a measure of how much space a three-dimensional figure occupies. In everyday language, the volume tells you how much a three-dimensional figure can hold. The basic unit of volume is the cubic unit—cubic centimeter, cubic inch, cubic meter, cubic foot, and so on. Each basic cubic unit has a measure of 1 for its length, width, and height.



Two postulates that apply to volume are the Volume Congruence Postulate and the Volume Addition Postulate.

Volume Congruence Postulate

If two polyhedrons (or solids) are congruent, then their volumes are congruent.

Volume Addition Postulate

The volume of a solid is the sum of the volumes of all of its non-overlapping parts.

A right rectangular prism is a prism with rectangular bases and the angle between each base and its rectangular lateral sides is also a right angle. You can recognize a right rectangular prism by its "box" shape.

You can use the Volume Addition Postulate to find the volume of a right rectangular prism by counting boxes. The box below measures 2 units in height, 4 units in width, and 3 units in depth. Each layer has 2×4 cubes or 8 cubes.

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Together, you get three groups of $2 \cdot 4$ so the total volume is:

$$V = 2 \cdot 4 \cdot 3$$
$$= 24$$

The volume is 24 cubic units.

This same pattern holds for any right rectangular prism. Volume is giving by the formula:

$$Volume = l \cdot w \cdot h$$

Example 11

Find the volume of this box.



Use the formula for volume of a right rectangular prism.

$$V = l \cdot w \cdot h$$
$$V = 8 \cdot 10 \cdot 7$$
$$V = 560$$

So the volume of this rectangular prism is 560 cubic units.

Volume of a Right Prism

Looking at the volume of right prisms with the same height and different bases you can see a pattern. The computed area of each base is given below. The height of all three solids is the same, 10.



Putting the data for each solid into a table, we get:

Table	1.5:
-------	------

Solid	Height	Area of base	Volume
Box	10	300	3000
Trapezoid	10	140	1400
Triangle	10	170	1700

The relationship in each case is clear. This relationship can be proved to establish the following formula for any right prism:

Volume of a Right Prism

The volume of a right prism is V = Bh.

where B is the area of the base of the three-dimensional figure, and h is the prism's height (also called altitude).

Example 12

Find the volume of the prism with a triangular equilateral base and the dimensions shown in centimeters.



To find the volume, first find the area of the base. It is given by:

$$A = \frac{1}{2}bh$$

The height (altitude) of the triangle is 10.38 cm. Each side of the triangle measures 12 cm. So the triangle has the following area.

$$A = \frac{1}{2}bh$$

= $\frac{1}{2}(10.38)(12)$
= 62.28

Now use the formula for the volume of the prism, V = Bh, where B is the area of the base (i.e., the area of the triangle) and h is the height of the prism. Recall that the "height" of the prism is the distance between the bases, so in this case the height of the prism is 15 cm. You can imagine that the prism is lying on its side.

$$V = Bh$$

= (62.28)(15)
= 934.2

Thus, the volume of the prism is 934.2 cm^3 .

Example 13

Find the volume of the prism with a regular hexagon for a base and 9 - inch sides.



You don't know the apothem of the figure's base. However, you do know that a regular hexagon is divided into six congruent equilateral triangles.



You can use the Pythagorean Theorem to find the apothem. The right triangle measures 9 by 4.5 by a, the apothem.

$$9^2 = 4.5^2 + n^2$$

 $81 = 20.25 + n^2$
 $60.75 = n^2$
 $7.785 = n$

$$A = \frac{1}{2} (\text{apothem}) \text{ (perimeter)}$$
$$= \frac{1}{2} (7.785)(6 \cdot 9)$$
$$= 210.195 \text{ sq in}$$

Thus, the volume of the prism is given by:

$$V = Bh$$

= 210.195 \cdot 24
= 5044.7 cu in

Review Questions

For each of the following find the surface area using

- a. the method of nets and
- b. the perimeter.



- 3. The base of a prism is a right triangle whose legs are 3 and 4 and show height is 20. What is the total area of the prism?
- 4. A right hexagonal prism is 24 inches tall and has bases that are regular hexagons measuring 8 inches on a side. What is the total surface area?
- 5. What is the volume of the prism in problem #4?

For problems 6 and 7:

A barn is shaped like a pentagonal prism with dimensions shown in feet:



- 6. How many square feet (excluding the roof) are there on the surface of the barn to be painted?
- 7. If a gallon of paint covers 250 square feet , how many gallons of paint are needed to paint the barn?
- 8. A cardboard box is a perfect cube with an edge measuring 17 inches. How many cubic feet can it hold?
- 9. A swimming pool is 16 feet wide, 32 feet long and is uniformly 4 feet deep. How many cubic feet of water can it hold?
- 10. A cereal box has length 25 cm, width 9 cm and height 30 cm. How much cereal can it hold?

Review Answers

- 1. 40.5 in^2
- $2.~838~\mathrm{cm}^2$
- 3. 252 square units
- 4. 1484.6 square units
- 5. 7981.3 cubic units
- 6. 2450 square feet
- 7. 10 gallons of paint
- 8. 2.85 cubic feet (be careful here. The units in the problem are given in inches but the question asks for feet.)
- 9. 2048 cubic feet $% \left({{\left({{{\left({{{\left({{{\left({{{\left({{{}}}} \right)}} \right.}$
- 10. 6750 $\rm cm^3$

1.4 Cylinders

Learning Objectives

- Find the surface area of cylinders.
- Find the volume of cylinders.

• Find the volume of composite three-dimensional figures.

Introduction

A **cylinder** is a three-dimensional figure with a pair of parallel and congruent circular ends, or **bases**. A cylinder has a single curved side that forms a rectangle when laid out flat.

As with prisms, cylinders can be *right* or *oblique*. The side of a right cylinder is perpendicular to its circular bases. The side of an oblique cylinder is not perpendicular to its bases.



right cylinder oblique cylinder

Surface Area of a Cylinder Using Nets

You can deconstruct a cylinder into a net.



The area of each base is given by the area of a circle:

$$A = \pi r^{2}$$

= $\pi (5)^{2}$
= 25π
 $\approx (25)(3.14) = 78.5$

The area of the rectangular lateral area L is given by the product of a width and height. The height is given as 24. You can see that the width of the area is equal to the circumference of the circular base.



To find the width, imagine taking a can-like cylinder apart with a scissors. When you cut the lateral area, you see that it is equal to the circumference of the can's top. The circumference of a circle is given by $C = 2\pi r$,

the lateral area, L, is

$$L = 2\pi rh$$

= 2\pi(5)(24)
= 240\pi
\approx (240)(3.14) = 753.6

Now we can find the area of the entire cylinder using A = (area of two bases) + (area of lateral side).

$$A = 2(75.36) + 753.6$$

= 904.32

You can see that the formula we used to find the total surface area can be used for any right cylinder.

Area of a Right Cylinder

The surface area of a right cylinder, with radius r and height h is given by A = 2B + L, where B is the area of each base of the cylinder and L is the lateral area of the cylinder.

Example 1

Use a net to find the surface area of the cylinder.



First draw and label a net for the figure.



Calculate the area of each base.

$$A = \pi r^{2}$$

= $\pi (8)^{2}$
= 64π
 $\approx (64)(3.14) = 200.96$

Calculate L.

$$L = 2\pi rh$$

= 2\pi (8)(9)
= 144\pi
\approx (240)(3.14) = 452.16

Find the area of the entire cylinder.

$$A = 2(200.96) + 452.16$$
$$= 854.08$$

Thus, the total surface area is approximately 854.08 square units

Surface Area of a Cylinder Using a Formula

You have seen how to use nets to find the total surface area of a cylinder. The postulate can be broken down to create a general formula for all right cylinders.

$$A = 2B + L$$



Notice that the base, B, of any cylinder is:

$$B = \pi r^2$$

The lateral area, L, for any cylinder is:

 $L = \text{width of lateral area} \cdot \text{height of cylinder}$ $= \text{circumference of base} \cdot \text{height of cylinder}$ $= 2\pi r \cdot h$

Putting the two equations together we get:

$$A = 2B + L$$

= 2(\pi r^2) + 2\pi r \cdot h

Factoring out a $2\pi r$ from the equation gives:

 $A = 2\pi r(r+h)$

The Surface Area of a Right Cylinder

A right cylinder with radius r and height h can be expressed as:

$$A = 2\pi r^2 + 2\pi rh$$

or:

$$A = 2\pi r(r+h)$$

You can use the formulas to find the area of any right cylinder.

Example 2

Use the formula to find the surface area of the cylinder.



Write the formula and substitute in the values and solve.

$$A = 2(\pi r^2) + 2\pi rh$$

= 2(3.14)(15)(15) + 2(3.14)(15)(48)
= 1413 + 4521.6
= 5934.6 square inches

Example 3

Find the surface area of the cylinder.



Write the formula and substitute in the values and solve.

$$A = 2\pi r(r+h)$$

= 2(3.14)(0.75)[0.75+6]
= 31.7925 square inches

Example 4

Find the height of a cylinder that has radius 4 cm and surface area of 226.08 sq cm. Write the formula with the given information and solve for h.

$$A = 2\pi r(r+h)$$

$$226.08 = 2(3.14)(4)[4+h]$$

$$226.08 = 25.12[4+h]$$

$$226.08 = 100.48 + 25.12h$$

$$5 = h$$

Volume of a Right Cylinder

You have seen how to find the volume of any right prism.

$$V = Bh$$

where B is the area of the prism's base and h is the height of the prism.



As you might guess, right prisms and right cylinders are very similar with respect to volume. In a sense, a cylinder is just a "prism with round bases." One way to develop a formula for the volume of a cylinder is to compare it to a prism. Suppose you divided the prism above into slices that were 1 unit thick.



The volume of each individual slice would be given by the product of the area of the base and the height. Since the height for each slice is 1, the volume of a single slice would be:

 $V \text{ (single slice)} = \text{area of base} \cdot \text{height}$ $= B \cdot 1$ = B

Now it follows that the volume of the entire prism is equal to the area of the base multiplied by the number of slices. If there are h slices, then:

$$V \text{ (whole prism)} = B \cdot \text{number of slices}$$
$$= Bh$$

Of course, you already know this formula from prisms. But now you can use the same idea to obtain a formula for the volume of a cylinder.



Since the height of each unit slice of the cylinder is 1, each slice has a volume of $B \cdot (1)$, or B. Since the base has an area of πr^2 , each slice has a volume of πr^2 and:

$$V \text{ (whole cylinder)} = B \cdot \text{number of slices}$$
$$= Bh$$
$$= \pi r^2 h$$

This leads to a postulate for the volume of any right cylinder.

Volume of a Right Cylinder

The volume of a right cylinder with radius r and height h can be expressed as:

Volume =
$$\pi r^2 h$$

Example 5

Use the postulate to find the volume of the cylinder.



Write the formula from the postulate. Then substitute in the values and solve.

$$V = \pi r^2 h$$

= (3.14)(6.5)(6.5)(14)
= 1857.31 cubic inches

Example 6

What is the radius of a cylinder with height 10 cm and a volume of 250π ? Write the formula. Solve for r.

$$V = \pi r^2 h$$

$$250\pi = \pi r^2 (10)$$

$$250\pi/10\pi = r^2$$

$$25 = r^2$$

$$5 = r$$

Composite Solids

Suppose this pipe is made of metal. How can you find the volume of metal that the pipe is made of?



The basic process takes three steps.

Step 1: Find the volume of the entire cylinder as if it had no hole.

Step 2: Find the volume of the hole.

Step 3: Subtract the volume of the hole from the volume of the entire cylinder.

Here are the steps carried out. First, use the formula to find the volume of the entire cylinder. Note that since d, the diameter of the pipe, is 6 cm, the radius is half of the diameter, or 3 cm.

 $V = \pi r^{2} h$ = (3.14)(3)(3)(5) = 141.3 cubic inches

Now find the volume of the inner empty "hole" in the pipe. Since the pipe is 1 inch thick, the diameter of the hole is 2 inches less than the diameter of the outer part of the pipe.

$$d$$
 (inner pipe) = d (outer pipe) - 2
= $6 - 2$
= 4

The radius of the hole is half of 4 or 2.

$$V = \pi r^{2}h$$

= (3.14)(2)(2)(5)
= 62.8 cubic inches

Now subtract the hole from the entire cylinder.

$$V(\text{pipe}) = V \text{ (cylinder)} - V \text{ (hole)}$$
$$= 141.3 - 62.8$$
$$= 78.5 \text{ cubic inches}$$

Example 7

Find the solid volume of this cinder block. Its edges are 3 cm thick all around. The two square holes are identical in size.



Find the volume of the entire solid block figure. Subtract the volume of the two holes. To find the volume of the three-dimensional figure:

$$V = l \cdot w \cdot h$$

= 21 \cdot 21 \cdot 26
= 11,466 cubic cm

Now find the length of the sides of the two holes. The width of the entire block is 21 cm. This is equal to:

width of block = 3 edges + 2 holes 21 = 3(3 cm) + 2n 21 = 9 + 2n 12 = 2n6 = n

So the sides of the square holes are 6 cm by 6 cm. Now the volume of each square hole is:

> $V = l \cdot w \cdot h$ = 6 \cdot 6 \cdot 26 = 936 cubic cm

Finally, subtract the volume of the two holes from the volume of the entire brick.

V(block) = V(solid) - V(holes)= 11,466 - 2(936)= 9,594 cubic cm

Review Questions



Complete the following sentences. They refer to the figure above.

1. The figure above is a _____

- 2. The shape of the lateral face of the figure is _____

- 5. Draw the net for this cylinder and use the net to find the surface area of the cylinder.



6. Use the formula to find the volume of this cylinder.



- 7. Matthew's favorite mug is a cylinder that has a base area of 9 square inches and a height of 5 inches. How much coffee can he put in his mug?
- 8. Given the following two cylinders which of the following statements is true:





- (a) Volume of A < Volume of B
- (b) Volume of A > Volume of B
- (c) Volume of A = Volume of B
- 9. Suppose you work for a company that makes cylindrical water tanks. A customer wants a tank that measures 9 meters in height and 2 meters in diameter. How much metal should you order to make this tank?
- 10. If the radius of a cylinder is doubled what effect does the doubling have on the volume of this cylinder? Explain your answer.

Review Answers

- 1. Cylinder
- 2. Rectangle
- 3. Circle
- 4. Height
- 5. Surface area = 266π in²



- 6. 250 π cm²
- 7. Volume = 45 in^3
- 8. Volume of A <volume of B
- 9. $18\pi \text{ m}^2$
- 10. The volume will be quadrupled

1.5 Pyramids

Learning Objectives

- Identify pyramids.
- Find the surface area of a pyramid using a net or a formula.
- Find the volume of a pyramid.

Introduction

A pyramid is a three-dimensional figure with a single base and a three or more non-parallel sides that meet at a single point above the base. The sides of a pyramid are triangles.



A **regular pyramid** is a pyramid that has a regular polygon for its base and whose sides are all congruent triangles.

Surface Area of a Pyramid Using Nets

You can deconstruct a pyramid into a net.



To find the surface area of the figure using the net, first find the area of the base:

$$A = s^{2}$$

= (12)(12)
= 144 square units

Now find the area of each isosceles triangle. Use the Pythagorean Theorem to find the height of the triangles. This height of each triangle is called the **slant height** of the pyramid. The slant height of the pyramid is the altitude of one of the triangles. Notice that the slant height is larger than the altitude of the triangle.



We'll call the slant height n for this problem. Using the Pythagorean Theorem:

$$(11.66)^{2} = 6^{2} + n^{2}$$
$$136 = 36 + n^{2}$$
$$100 = n^{2}$$
$$10 = n$$

Now find the area of each triangle:

$$A = \frac{1}{2}hb$$

= $\frac{1}{2}(10)(12)$
= 60 square units

As there are 4 triangles:

$$A(\text{triangles}) = 4(60)$$
$$= 240 \text{ square units}$$

Finally, add the total area of the triangles to the area of the base.

$$A(\text{total}) = A(\text{triangles}) + A(\text{base})$$
$$= 240 + 144$$
$$= 384 \text{ square units}$$

Example 1

Use the net to find the total area of the regular hexagonal pyramid with an apothem of 5.19. The dimensions are given in centimeters.



The area of the hexagonal base is given by the formula for the area of a regular polygon. Since each side of the hexagon measures 6 cm, the perimeter is $6 \cdot 6$ or 36 cm. The apothem, or perpendicular distance to the center of the hexagon is 5.19 cm.

$$A = \frac{1}{2} (\text{apothem}) (\text{perimeter})$$
$$= \frac{1}{2} (5.19)(36)$$
$$= 93.42 \text{ square cm}$$

Using the Pythagorean Theorem to find the slant height of each lateral triangle.

$$(14)^2 = 3^2 + n^2$$

 $196 = 9 + n^2$
 $187 = n^2$
 $13.67 = n$

Now find the area of each triangle:

$$A = \frac{1}{2}hb$$
$$= \frac{1}{2}(13.67)(6)$$
$$= 41 \text{ square cm}$$

Together, the area of all six triangles that make up the lateral sides of the pyramid are

A = 6(area of each triangle) = $6 \cdot 41$ = 246 square cm

Add the area of the lateral sides to the area of the hexagonal base.

$$A(\text{total}) = A(\text{triangles}) + A(\text{base})$$
$$= 246 + 93.42$$
$$= 339.42 \text{ square cm}$$

Surface Area of a Regular Pyramid

To get a general formula for the area of a regular pyramid, look at the net for this square pyramid.



The slant height of each lateral triangle is labeled l (the lowercase letter L), and the side of the regular polygon is labeled s. For each lateral triangle, the area is:

$$A = \frac{1}{2}ls$$

There are *n* triangles in a regular polygon—e.g., n = 3 for a triangular pyramid, n = 4 for a square pyramid, n = 5 for a pentagonal pyramid. So the total area, *L*, of the lateral triangles is:

$$L = n \cdot \text{(area of each lateral triangle)}$$
$$= n \left(\frac{1}{2}ls\right)$$

If we rearrange the above equation we get:

$$L = \left(\frac{1}{2}ln \cdot s\right)$$

Notice that $n \cdot s$ is just the perimeter, P, of the regular polygon. So we can substitute P into the equation to get the following postulate.

$$L = \left(\frac{1}{2}lP\right)$$

To get the total area of the pyramid, add the area of the base, B, to the equation above.

$$A = \frac{1}{2}lP + B$$

Area of a Regular Pyramid

The surface area of a regular pyramid is

$$A = \frac{1}{2}lP + B$$

where l is the slant height of the pyramid and P is the perimeter of the regular polygon that forms the pyramid's base, and B is the area of the base.

Example 2

A tent without a bottom has the shape of a hexagonal pyramid with a slant height l of 30 feet. The sides of the hexagonal perimeter of the figure each measure 8 feet. Find the surface area of the tent that exists above ground.

For this problem, B is zero because the tent has no bottom. So simply calculate the lateral area of the figure.

$$A = \frac{1}{2}lP + B$$

= $\frac{1}{2}lP + 0$
= $\frac{1}{2}lP$
= $\frac{1}{2}(30)(6 \cdot 8)$
= 720 square feet

Example 3

A pentagonal pyramid has a slant height l of 12 cm. The sides of the pentagonal perimeter of the figure each measure 9 cm. The apothem of the figure is 6.19 cm. Find the surface area of the figure.



First find the lateral area of the figure.

$$L = \frac{1}{2}lP$$

= $\frac{1}{2}(12)(5 \cdot 9)$
= 270 square cm

Now use the formula for the area of a regular polygon to find the area of the base.

$$A = \frac{1}{2} (\text{apothem}) (\text{perimeter})$$
$$= \frac{1}{2} (6.19) (5 \cdot 9)$$
$$= 139.3605 \text{ square cm}$$

Finally, add these together to find the total surface area.

 $139.3605 + 270 \approx 409.36$ square centimeters

Estimate the Volume of a Pyramid and Prism

Which has a greater volume, a prism or a pyramid, if the two have the same base and height? To find out, compare prisms and pyramids that have congruent bases and the same height.

Here is a base for a triangular prism and a triangular pyramid. Both figures have the same height. Compare the two figures. Which one appears to have a greater volume?



The prism may appear to be greater in volume. But how can you prove that the volume of the prism is greater than the volume of the pyramid? Put one figure inside of the other. The figure that is smaller will fit inside of the other figure.



This is shown in the diagram on the above. Both figures have congruent bases and the same height. The pyramid clearly fits inside of the prism. So the volume of the pyramid must be smaller.

Example 4

Show that the volume of a square prism is greater than the volume of a square pyramid.

Draw or make a square prism and a square pyramid that have congruent bases and the same height.



square pyramid

Now place the one figure inside of the other. The pyramid fits inside of the prism. So when two figures have the same height and the same base, the prism's volume is greater.



In general, when you compare two figures that have congruent bases and are equal in height, the prism will have a greater volume than the pyramid.

The reason should be obvious. At the "bottom," both figures start out the same—with a square base. But the pyramid quickly slants inward, "cutting away" large amounts of material while the prism does not slant.

Find the Volume of a Pyramid and Prism

Given the figure above, in which a square pyramid is placed inside of a square prism, we now ask: how many of these pyramids would fit inside of the prism?



To find out, obtain a square prism and square pyramid that are both hollow, both have no bottom, and both have the same height and congruent bases.



Now turn the figures upside down. Fill the pyramid with liquid. How many full pyramids of liquid will fill the prism up to the top?

In fact, it takes exactly three full pyramids to fill the prism. Since the volume of the prism is:

$$V = Bh$$

where B stands for the area of the base and h is the height of the prism, we can write:

 $3 \cdot (\text{volume of square pyramid}) = (\text{volume of square prism})$

or:

(volume of square pyramid) =
$$\frac{1}{3}$$
(volume of square prism)

And, since the volume of a square prism is Bh, we can write:

$$V = \frac{1}{3}Bh$$

This can be written as the Volume Postulate for pyramids.

Volume of a Pyramid

Given a right pyramid with a base that has area B and height h:

$$V = \frac{1}{3}Bh$$

Example 5

Find the volume of a pyramid with a right triangle base with sides that measure 5 cm, 8 cm, and 9.43 cm. The height of the pyramid is 15 cm.

First find the area of the base. The longest of the three sides that measure 5 cm, 8 cm, and 9.43 cm must be the hypotenuse, so the two shorter sides are the legs of the right triangle.

$$A = \frac{1}{2}hb$$
$$= \frac{1}{2}(5)(8)$$
$$= 20 \text{ square cm}$$

Now use the postulate for the volume of a pyramid.

$$V(\text{pyramid}) = \frac{1}{3}Bh$$
$$= \frac{1}{3}(20)(15)$$
$$= 100 \text{ cubic cm}$$

Example 6

Find the altitude of a pyramid with a regular pentagonal base. The figure has an apothem of 10.38 cm, 12 cm sides, and a volume of 2802.6 cu cm.

First find the area of the base.

$$A(\text{base}) = \frac{1}{2}aP$$
$$= \frac{1}{2}(10.38)(5 \cdot 12)$$
$$= 311.4 \text{ square cm}$$

Now use the value for the area of the base and the postulate to solve for h.

$$V(\text{pyramid}) = \frac{1}{3}Bh$$
$$2802.6 = \frac{1}{3}(311.4)h$$
$$27 = h$$

Review Questions

Consider the following figure in answering questions 1 - 4.



- 1. What type of pyramid is this?
- 2. Triangle ABE is what part of the figure?
- 3. Segment AE is a(n) _____ of the figure.
- 4. Point E is the _____
- 5. How many faces are there on a pyramid whose base has 16 sides?

A right pyramid has a regular hexagon for a base. Each edge measures $2\sqrt{22}$. Find

- 6. The lateral surface area of the pyramid
- 7. The total surface area of the pyramid
- 8. The volume of the pyramid

- 9. The Transamerica Building in San Francisco is a pyramid. The length of each edge of the square base is 149 feet and the slant height of the pyramid is 800 feet. What is the lateral area of the pyramid? How tall is the building?
- 10. Given the following pyramid:



With c = 22 mm, b = 17 mm and $volume = 1433.67 \text{ mm}^3$ what is the value of a?

Review Answers

- 1. Rectangular pyramid
- 2. Lateral face
- 3. Edge
- 4. Apex
- $5.\ 16$
- 6. 135.6 square units
- 7. 200.55 square units
- 8. 171.84 cubic units
- 9. Lateral surface area = 238,400 square feet Height = 796.5 feet
- 10. A = 11.5 mm

1.6 Cones

Learning Objectives

• Find the surface area of a cone using a net or a formula.

• Find the volume of a cone.

Introduction

A cone is a three-dimensional figure with a single curved base that tapers to a single point called an *apex*. The base of a cone can be a circle or an oval of some type. In this chapter, we will limit the discussion to circular cones. The apex of a right cone lies above the center of the cone's circle. In an oblique cone, the apex is not in the center.



The height of a cone, h, is the perpendicular distance from the center of the cone's base to its apex.

Surface Area of a Cone Using Nets

Most three-dimensional figures are easy to deconstruct into a net. The cone is different in this regard. Can you predict what the net for a cone looks like? In fact, the net for a cone looks like a small circle and a sector, or part of a larger circle.



The diagrams below show how the half-circle sector folds to become a cone.



Note that the circle that the sector is cut from is much larger than the base of the cone.

Example 1

Which sector will give you a taller cone—a half circle or a sector that covers three-quarters of a circle? Assume that both sectors are cut from congruent circles.

Make a model of each sector.

The half circle makes a cone that has a height that is about equal to the radius of the semi-circle.



The three-quarters sector gives a cone that has a wider base (greater diameter) but its height as not as great as the half-circle cone.



Example 2

Predict which will be greater in height—a cone made from a half-circle sector or a cone made from a one-third-circle sector. Assume that both sectors are cut from congruent circles.

The relationship in the example above holds true—the greater (in degrees) the sector, the smaller in height of the cone. In other words, the fraction 1/3 is less than 1/2, so a one-third sector will create a cone with greater height than a half sector.

Example 3

Predict which will be greater in diameter—a cone made from a half-circle sector or a cone made from a one-third-circle sector. Assume that the sectors are cut from congruent circles

Here you have the opposite relationship—the larger (in degrees) the sector, the *greater* the diameter of the cone. In other words, 1/2 is greater than 1/3, so a one-half sector will create a cone with greater diameter than a one-third sector.
Surface Area of a Regular Cone

The surface area of a regular pyramid is given by:



$$A = \left(\frac{1}{2}lP\right) + B$$

where l is the slant height of the figure, P is the perimeter of the base, and B is the area of the base.

Imagine a series of pyramids in which n, the number of sides of each figure's base, increases.



As you can see, as n increases, the figure more and more resembles a circle. So in a sense, a circle approaches a polygon with an infinite number of sides that are infinitely small.

In the same way, a cone is like a pyramid that has an infinite number of sides that are infinitely small in length.

Given this idea, it should come as no surprise that the formula for finding the total surface area of a cone is similar to the pyramid formula. The only difference between the two is that the pyramid uses P, the perimeter of the base, while a cone uses C, the circumference of the base.

$$A(\text{pyramid}) = \frac{1}{2}lP + B$$
$$A(\text{cone}) = \frac{1}{2}lC + B$$

Surface Area of a Right Cone

The surface area of a right cone is given by:

$$A = \frac{1}{2}lC + B$$

Since the circumference of a circle is $2\pi r$:

$$A(\text{cone}) = \frac{1}{2}lC + B$$
$$= \frac{1}{2}l(2\pi r) + B$$
$$= \pi rl + B$$

You can also express B as πr^2 to get:

$$A(\text{cone}) = \pi r l + B$$
$$= \pi r l + \pi r^{2}$$
$$= \pi r (l + r)$$

Any of these forms of the equation can be used to find the surface area of a right cone.

Example 4

Find the total surface area of a right cone with a radius of 8 cm and a slant height of 10 cm. Use the formula:

$$A(\text{cone}) = \pi r(l+r)$$

= (3.14)(8)[10+8]
= 452.16 square cm

Example 5

Find the total surface area of a right cone with a radius of 3 feet and an altitude (not slant height) of 6 feet.



Use the Pythagorean Theorem to find the slant height:



$$l^{2} = r^{2} + h^{2}$$

= (3)² + (6)²
= 45
$$l = \sqrt{45}$$

= 3\sqrt{5}

Now use the area formula.

$$A(\text{cone}) = \pi r(l+r)$$

= (3.14)(3)[3 $\sqrt{5}$ + 3]
 \approx 91.5 square cm

Volume of a Cone

Which has a greater volume, a pyramid, cone, or cylinder if the figures have bases with the same "diameter" (i.e., distance across the base) and the same altitude? To find out, compare pyramids, cylinders, and cones that have bases with equal diameters and the same altitude.

Here are three figures that have the same dimensions—cylinder, a right regular hexagonal pyramid, and a right circular cone. Which figure appears to have a greater volume?



It seems obvious that the volume of the cylinder is greater than the other two figures. That's because the pyramid and cone taper off to a single point, while the cylinder's sides stay the same width.

Determining whether the pyramid or the cone has a greater volume is not so obvious. If you look at the bases of each figure you see that the apothem of the hexagon is congruent to the radius of the circle. You can see the relative size of the two bases by superimposing one onto the other.



From the diagram you can see that the hexagon is slightly larger in area than the circle. So it follows that the volume of the right hexagonal regular pyramid would be greater than the volume of a right circular cone. And indeed it is, but only because the area of the base of the hexagon is slightly greater than the area of the base of the circular cone.

The formula for finding the volume of each figure is virtually identical. Both formulas follow the same basic form:

$$V = \frac{1}{3}Bh$$

Since the base of a circular cone is, by definition, a circle, you can substitute the area of a

circle, πr^2 for the base of the figure. This is expressed as a volume postulate for cones.

Volume of a Right Circular Cone

Given a right circular cone with height h and a base that has radius r:

$$V = \frac{1}{3}Bh$$
$$= \frac{1}{3}\pi r^2 h$$

Example 6

Find the volume of a right cone with a radius of 9 cm and a height of 16 cm. Use the formula:

$$V = \frac{1}{3}Bh$$

= $\frac{1}{3}\pi r^2 h$
= $\frac{1}{3}(3.14)(9)(9)(16)$
= 1356.48 cubic cm

Example 7

Find the volume of a right cone with a radius of 10 feet and a slant height of 13 feet. Use the Pythagorean theorem to find the height:



$$r^{2} + h^{2} = l^{2}$$

$$(10)^{2} + h^{2} = (13)^{2}$$

$$h^{2} = (13)^{2} - (10)^{2}$$

$$h^{2} = 69$$

$$h = 8.31$$

Now use the volume formula.

$$V = \frac{1}{3}\pi r^2 h$$

= $\frac{1}{3}(3.14)(10)(10)(8.31)$
= 869.78 cubic feet

Review Questions

1. Find the surface area of



2. Find the surface area of



3. Find the surface area of a cone with a height of 4 m and a base area of 281.2 m^2

In problems 4 and 5 find the missing dimension. Round to the nearest tenth of a unit.

- 4. Cone: volume = 424 cubic meters Diameter = 18 meters Height = _____
- 5. Cone: surface area = 153.5 in² Radius = 4 inches Slant height = _____
- 6. A cone shaped paper cup is 8 cm high with a diameter of 5 cm. If a plant needs 240 ml of water, about how many paper cups of water are needed to water it? (1 mL = 1 cubic cm)

In problems 7 and 8 refer to this diagram. It is a diagram of a yogurt container. The yogurt container is a truncated cone.



- 7. What is the surface area of the container?
- 8. What is the volume of the container?

- 9. Find the height of a cone that has a radius of 2 cm and a volume of 23 cm^3
- 10. A cylinder has a volume of 2120.6 cm^3 and a base radius of 5 cm. What is the volume of a cone with the same height but a base radius of 3 cm?

Review Answers

- 1. 483.8 square units
- 2. 312.6 square units
- 3. Surface area = 75.4 m^2
- 4. Height = 5 meters
- 5. Slant height = 8 inches
- 6. 1.2 cups (approximately)
- 7. Surface area of the container = 152.62 cm^2
- 8. The volume of the container = 212.58 cm^3
- 9. Height = 5.49 cm
- 10. Volume of the cone = 254.45 cm^3

1.7 Spheres

Learning Objectives

- Find the surface area of a sphere.
- Find the volume of a sphere.

Introduction

A sphere is a three-dimensional figure that has the shape of a ball.



Spheres can be characterized in three ways.

- A sphere is the set of all points that lie a fixed distance r from a single center point O.
- A sphere is the surface that results when a circle is rotated about any of its diameters.



• A sphere results when you construct a polyhedron with an infinite number of faces that are infinitely small. To see why this is true, recall regular polyhedra.



As the number of faces on the figure increases, each face gets smaller in area and the figure comes more to resemble a sphere. When you imagine figure with an *infinite* number of faces, it would look like (and be!) a sphere.

Parts of a Sphere

As described above, a sphere is the surface that is the set of all points a fixed distance from a center point O. Terminology for spheres is similar for that of circles.

The distance from O to the surface of the sphere is r, the radius.



The **diameter**, d, of a sphere is the length of the segment connecting any two points on the sphere's surface and passing through O. Note that you can find a diameter (actually an infinite number of diameters) on any plane within the sphere. Two diameters are shown in each sphere below.



A chord for a sphere is similar to the chord of a circle except that it exists in three dimensions. Keep in mind that a diameter is a kind of chord—a special chord that intersects the center of the circle or sphere.



A secant is a line, ray, or line segment that intersects a circle or sphere in two places and extends outside of the circle or sphere.



A tangent intersects the circle or sphere at only one point.



In a circle, a tangent is perpendicular to the radius that meets the point where the tangent intersects with the circle. The same thing is true for the sphere. All tangents are

perpendicular to the radii that intersect with them.

Finally, a sphere can be sliced by an infinite number of different planes. Some planes include point O, the center of the sphere. Other points do not include the center.



Surface Area of a Sphere

You can infer the formula for the surface area of a sphere by taking measurements of spheres and cylinders. Here we show a sphere with a radius of 3 and a right cylinder with both a radius and a height of 3 and express the area in terms of π .



Now try a larger pair, expressing the surface area in decimal form.



r = 42 mm h = 42 mm

surface area = 22,155.84 sq mm surface area = 22,155.84 sq mm

Is it a coincidence that a sphere and a cylinder whose radius and height are equal to the radius of the sphere have the exact same surface area? Not at all! In fact, the ancient Greeks used a method that showed that the following formula can be used to find the surface area of any sphere (or any cylinder in which r = h).

The Surface Area of a Sphere is given by:

 $A = 4\pi r^2$

Example 1

Find the surface area of a sphere with a radius of 14 feet. Use the formula.

$$A = 4\pi r^2$$

= $4\pi (14)^2$
= $4\pi (196)$
= 784π
= 2461.76 square feet

Example 2

Find the surface area of the following figure in terms of π .



The figure is made of one half sphere or hemisphere, and one cylinder without its top.

$$A(\text{half sphere}) = \frac{1}{2}A(\text{sphere})$$
$$= \frac{1}{2} \cdot 4\pi r^2$$
$$= 2\pi (576)$$
$$= 1152\pi \text{ square cm}$$

Now find the area of the cylinder without its top.

$$A(\text{topless cylinder}) = A(\text{cylinder}) - A(\text{top})$$
$$= 2(\pi r^2) + 2\pi rh - \pi r^2$$
$$= \pi r^2 + 2\pi rh$$
$$= \pi (576) + 2\pi (24)(50)$$
$$= 2976\pi \text{ square cm}$$

Thus, the total surface area is $1152\pi+2976\pi=4128\pi$ square cm

Volume of a Sphere

A sphere can be thought of as a regular polyhedron with an infinite number of congruent polygon faces. A series polyhedra with an increasing number of faces is shown.



As n, the number of faces increases to an infinite number, the figure approaches becoming a sphere.

So a sphere can be thought of as a polyhedron with an infinite number faces. Each of those faces is the base of a pyramid whose vertex is located at O, the center of the sphere. This is shown below.

Each of the pyramids that make up the sphere would be congruent to the pyramid shown. The volume of this pyramid is given by:

$$V(\text{each pyramid}) = \frac{1}{3}Bh$$



To find the volume of the sphere, you simply need to add up the volumes of an infinite number of infinitely small pyramids.

$$V(\text{all pyramids}) = V_1 + V_2 + V_3 + \dots + V_n$$

= $\frac{1}{3}B_1h + \frac{1}{3}B_2h + \frac{1}{3}B_3h + \dots + \frac{1}{3}B_nh$
= $\frac{1}{3}h(B_1 + B_2 + B_3 + \dots + B_n)$

Now, it should be obvious that the sum of all of the bases of the pyramids is simply the surface area of the sphere. Since you know that the surface area of the sphere is $4\pi r^2$, you can substitute this quantity into the equation above.

$$V(\text{all pyramids}) = \frac{1}{3}h (B_1 + B_2 + B_3 + \ldots + B_n)$$
$$= \frac{1}{3}h(4\pi r^2)$$

Finally, as n increases and the surface of the figure becomes more "rounded," h, the height of each pyramid becomes equal to r, the radius of the sphere. So we can substitute r for h. This gives:

$$V(\text{all pyramids}) = \frac{1}{3}r(4\pi r^2)$$
$$= \frac{4}{3}\pi r^3$$

We can write this as a formula.

Volume of a Sphere

Given a sphere that has radius r

$$V = \frac{4}{3}\pi r^3$$

Example 3

Find the volume of a sphere with radius 6.25 m.

Use the postulate above.

$$V(\text{sphere}) = \frac{4}{3}\pi r^{3}$$

= $\frac{4}{3}(3.14)(6.25)(6.25)(6.25)$
= 1022.14 cubic m

Example 4

A sphere has a volume of $(85\pi)1/3$. Find its diameter.

Use the postulate above. Convert (85)1/3 to an improper fraction, 256/3.

$$V(\text{sphere}) = \frac{4}{3}\pi r^3$$
$$\frac{256}{3}\pi = \frac{4}{3}\pi r^3$$
$$\left(\frac{3}{4}\right)\left(\frac{256}{3}\right)\pi = \pi r^3$$
$$\frac{192}{3} = \pi r^3$$
$$64 = r^3$$
$$4 = r$$

Since r = 4, the diameter is 8 units.

Review Questions

1. Find the radius of the sphere that has a volume of 335 cm^3 .

Determine the surface area and volume of the following shapes:





- 4. The radius of a sphere is 4. Find its volume and total surface area.
- 5. A sphere has a radius of 5. A right cylinder, having the same radius has the same volume. Find the height and total surface area of the cylinder.

In problems 6 and 7 find the missing dimension.

- 6. Sphere : volume = 296 cm^3 Diameter = _____
- 7. Sphere : surface area is 179 in². Radius = $_$
- 8. Tennis balls with a diameter of 3.5 inches are sold in cans of three. The can is a cylinder. Assume the balls touch the can on the sides, top and bottom. What is the volume of the space not occupied by the tennis balls?
- 9. A sphere has surface area of 36π in². Find its volume.
- 10. A giant scoop, operated by a crane, is in the shape of a hemisphere of radius = 21 inches. The scoop is filled with melted hot steel. When the steel is poured into a cylindrical storage tank that has a radius of 28 inches, the melted steel will rise to a height of how many inches?

Review Answers

1. Radius = 4.39 cm

- 2. Surface area = 706.86 cm^2 Volume = 1767.15 cm^3
- 3. Surface area = surface area of hemisphere + surface area of cone = 678.58 in² Volume = 2544.69 in³
- 4. Volume = 268.08 units³ Surface area = 201.06 units²
- 5. Height = 20/3 units total surface area = 366.52 units²
- 6. Diameter = 8.27 inches
- 7. Radius = 3.77 inches
- 8. Volume of cylinder = 32.16π in³ volume of tennis balls = 21.44π in³ Volume of space not occupied by tennis balls = 33.67 in³
- 9. Volume = 113.10 in^3
- 10. Height of molten steel in cylinder will be 7.88 inches

1.8 Similar Solids

Learning Objectives

• Find the volumes of solids with bases of equal areas.

Introduction

You've learned formulas for calculating the volume of different types of solids—prisms, pyramids, cylinders, and spheres. In most cases, the formulas provided had special conditions. For example, the formula for the volume of a cylinder was specific for a *right* cylinder.



Now the question arises: What happens when you consider the volume of two cylinders that have an equal base but one cylinder is non-right—that is, oblique. Does an oblique cylinder have the same volume as a right cylinder if the two share bases of the same area?

Parts of a Solid

Given, two cylinders with the same height and radius. One cylinder is a right cylinder, the other is oblique. To see if the volume of the oblique cylinder is equivalent to the volume of the right cylinder, first observe the two solids.



Since they both have the same circular radius, they both have congruent bases with area:

$$A = \pi r^2.$$

Now cut the right cylinder into a series of n cross-section disks each with height 1 and radius r.



It should be clear from the diagram that the total volume of the n disks is equal to the volume of the original cylinder.

Now start with the same set of disks. Shift each disk over to the right. The volume of the shifted disks must be exactly the same as the unshifted disks, since both figures are made out of the same disks.



It follows that the volume of the oblique figure is equal to the volume of the original right cylinder.



In other words, if the radius and height of each figure are congruent:

$$V(\text{right cylinder}) = V(\text{oblique cylinder})$$
$$V(\text{any cylinder}) = \frac{1}{3}Bh$$
$$V(\text{any cylinder}) = \frac{1}{3}\pi r^2h$$

The principle shown above was developed in the seventeenth century by Italian mathematician Francisco Cavalieri. It is known as **Cavalieri's Principle.** (Liu Hui also discovered the same principle in third-century China, but was not given credit for it until recently.) The principle is valid for any solid studied in this chapter.

Volume of a Solid Postulate (Cavalieri's Principle):

The volumes of two objects are equal if the areas of their **corresponding** cross-sections are in all cases equal. Two cross-sections **correspond** if they are intersections of the figure with planes equidistant from a chosen base plane.

Example 1

Prove (informally) that the two circular cones with the same radius and height are equal in volume.



As before, we can break down the right circular cone into disks.



Now shift the disks over.



You can see that the shifted-over figure, since it uses the very same disks as the straight figure, must have the same volume. In fact, you can shift the disks any way you like. Since you are always using the same set of disks, the volume is the same. \blacklozenge



Keep in mind that Cavalieri's Principle will work for any two solids as long as their bases are equal in area (not necessarily congruent) and their cross sections change in the same way.

Example 2

A rectangular pyramid and a circular cone have the same height, and base area. Are their volumes congruent?

Yes. Even though the two figures are different, both can be computed by using the following formula:

$$V(\text{cone}) = \frac{1}{3}B_{circle}h \text{ and } V(\text{pyramid}) = \frac{1}{3}B_{rectangle}h$$

Since

$$B_{circle} = B_{rectangle}$$

Then

$$V(\text{cone}) = V(\text{rectangle})$$

Similar or Not Similar

Two solids of the same type with **equal ratios** of corresponding linear measures (such as heights or radii) are called similar solids.



To be similar, figures need to have corresponding linear measures that are in proportion to one another. If these linear measures are not in proportion, the figures are not similar.

Example 1

Are these two figures similar?



If the figures are similar, all ratios for corresponding measures must be the same. The ratios are:

width
$$=$$
 $\frac{6}{9} = \frac{2}{3}$
height $=$ $\frac{14}{21} = \frac{2}{3}$
depth $=$ $\frac{8}{12} = \frac{2}{3}$

Since the three ratios are equal, you can conclude that the figures are similar.

Example 2

Cone A has height 20 and radius 5. Cone B has height 18 and radius 6. Are the two cones similar?

If the figures are similar all ratios for corresponding measures must be the same.

The ratios are:

height
$$= \frac{20}{16} = \frac{5}{4}$$

radius $= \frac{18}{6} = \frac{3}{1}$

Since the ratios are different, the two figures are not similar.

Compare Surface Areas and Volumes of Similar Figures

When you compare similar two-dimensional figures, area changes as a function of the square of the ratio of

For example, take a look at the areas of these two similar figures.



The ratio between corresponding sides is:

$$\frac{\text{length } (A)}{\text{length } (B)} = \frac{12}{6} = \frac{2}{1}$$

The ratio between the areas of the two figures is the square of the ratio of the linear measurement:

$$\frac{\operatorname{area}(A)}{\operatorname{area}(B)} = \frac{12 \cdot 8}{6 \cdot 4} = \frac{96}{24} = \frac{4}{1} = \frac{2^2}{1}$$

This relationship holds for solid figures as well. The ratio of the areas of two similar figures is equal to the **square** of the ratio between the corresponding linear sides.

Example 3

Find the ratio of the surface area between the two similar figures C and D.



Since the two figures are similar, you can use the ratio between any two corresponding measurements to find the answer. Here, only the radius has been supplied, so:

$$\frac{\text{radius } (C)}{\text{radius } (D)} = \frac{6}{4} = \frac{3}{2}$$

The ratio between the areas of the two figures is the square of the ratio of the linear measurements:

$$\frac{\operatorname{area}(C)}{\operatorname{area}(D)} = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$$

Example 4

If the surface area of the small cylinder in the problem above is 80, what is the surface area of the larger cylinder?

From above we, know that:

$$\frac{\operatorname{area}(C)}{\operatorname{area}(D)} = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$$

So the surface area can be found by setting up equal ratios

$$\frac{9}{4} = \frac{n}{80\pi}$$

Solve for n.

$$n = 180\pi$$

The ratio of the volumes of two similar figures is equal to the **cube** of the ratio between the corresponding linear sides.

Example 5

Find the ratio of the volume between the two similar figures C and D.



As with surface area, since the two figures are similar you can use the height, depth, or width of the figures to find the linear ratio. In this example we will use the widths of the two figures.

$$\frac{w(\text{small})}{w(\text{large})} = \frac{15}{20} = \frac{3}{4}$$

The ratio between the volumes of the two figures is the **cube** of the ratio of the linear measurements:

$$\frac{\text{volume }(C)}{\text{volume }(D)} = \left(\frac{3}{4}\right)^3 = \frac{27}{64}$$

Does this cube relationship agree with the actual measurements? Compute the volume of each figure.

$$\frac{\text{volume (small)}}{\text{volume (large)}} = \frac{5 \times 9 \times 15}{62/3 \times 12 \times 20} = \frac{535}{1600} = \frac{27}{64}$$

As you can see, the ratio holds. We can summarize the information in this lesson in the following postulate.

Similar Solids Postulate:

If two solid figures, A and B are similar and the ratio of their linear measurements is $\frac{a}{b}$, then the ratio of their surface areas is:

$$\frac{\text{surface area A}}{\text{surface area B}} = \left(\frac{a}{b}\right)^2$$

The ratio of their volumes is:

$$\frac{\text{volume A}}{\text{volume B}} = \left(\frac{a}{b}\right)^3$$

Scale Factors and Models

The ratio of the linear measurements between two similar figures is called the scaling factor. For example, we can find the scaling factor for cylinders E and F by finding the ratio of any two corresponding measurements.



Using the heights, we find a scaling factor of:

$$\frac{h(small)}{h(large)} = \frac{8}{16} = \frac{1}{2}.$$

You can use a scaling factor to make a model.

Example 6

Doug is making a model of the Statue of Liberty. The real statue has a height of 111 feet and a nose that is 4.5 feet in length. Doug's model statue has a height of 3 feet. How long should the nose on Doug's model be?

First find the scaling factor.

$$\frac{height(model)}{height(statue)} = \frac{3}{111} = \frac{1}{37} = 0.027$$

To find the length of the nose, simply multiply the height of the model's nose by the scaling factor.

nose(model) = nose(statue)
$$\cdot$$
 (scaling factor)
= 4.5 \cdot 0.027
= 0.122 feet

In inches, the quantity would be:

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nose(model) =
$$0.122$$
 feet $\cdot 12$ inches/feet
= 1.46 inches

Example 7

An architect makes a scale model of a building shaped like a rectangular prism. The model measures 1.4 ft in height, 0.6 inches in width, and 0.2 inches in depth. The real building will be 420 feet tall. How wide will the real building be?

First find the scaling factor.

 $\frac{height(real)}{height(model)} = \frac{420}{1.4} = \frac{300}{1} = 300$

To find the width, simply multiply the width of the model by the scaling factor.

width(real) = width(model)
$$\cdot$$
 (scaling factor)
= 0.6 \cdot 300
= 180 feet

Review Questions

- 1. How does the volume of a cube change if the sides of a cube are multiplied by 4? Explain.
- 2. In a cone if the radius and height are doubled what happens to the volume? Explain.
- 3. In a rectangular solid, is the sides are doubled what happens to the volume? Explain.
- 4. Two spheres have radii of 5 and 9. What is the ratio of their volumes?
- 5. The ratio of the volumes of two similar pyramids is 8 : 27. What is the ratio of their total surface areas?
- 6. (a) Are all spheres similar?
 - (b) Are all cylinders similar?
 - (c) Are all cubes similar? Explain your answers to each of these.
- 7. The ratio of the volumes of two tetrahedron is 1000 : 1. The smaller tetrahedron has a side of length 6 centimeters. What is the side length of the larger tetrahedron?

Refer to these two similar cylinders in problems 8 - 10:

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- 8. What is the similarity ratio of cylinder A to cylinder B?
- 9. What is the ratio of surface area of cylinder A to cylinder B?
- 10. What is the ratio of the volume of cylinder B to cylinder A?

Review Answers

- 1. The volume will be 64 times greater. Volume = s^3 New volume = $(4s)^3$
- 2. Volume will be 8 times greater.
- 3. The volume will be 8 times greater (2w)(2l)(2h) = 8 wlh = 8 (volume of first rectangular solid)
- 4. $5^3/9^3$
- 5. 4/9
- 6. All spheres and all cubes are similar since each has only one linear measure. All cylinders are not similar. They can only be similar if the ratio of the radii = the ratio of the heights.
- $7.\ 60\ \mathrm{cm}$
- 8. 20/5 = 4/1
- 9. 16/1
- 10. $1/4^3$