Surface to Surface Intersections

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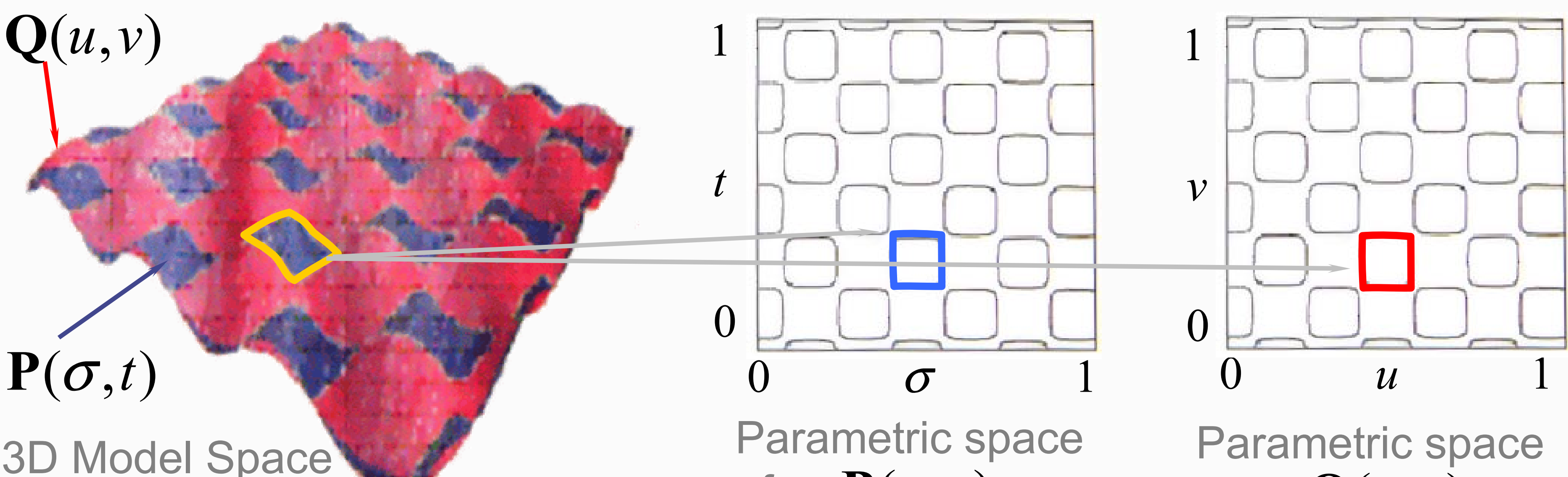
Introduction
Motivation

- Surface to surface intersection (SSI) is needed in:
  - Solid modeling (B-rep)
  - Contouring
  - Numerically controlled machining (Milling)
  - Collision avoidance
  - Feature recognition
  - Manufacturing simulation
  - Computer animation

Introduction

Background

- Intersection of two *parametric surfaces*, $\mathbf{P}(\sigma,t)=\mathbf{Q}(u,v)$ defined in *parametric spaces* $0 \le \sigma,t \le 1$ and $0 \le u,v \le 1$ can have *multiple components* [4].



Parametric space of $\mathbf{P}(\sigma,t)$

Parametric space of $\mathbf{Q}(u,v)$

- An *intersection curve segment* is represented by a continuous trajectory in *parametric space.*

Introduction
Possible Approaches

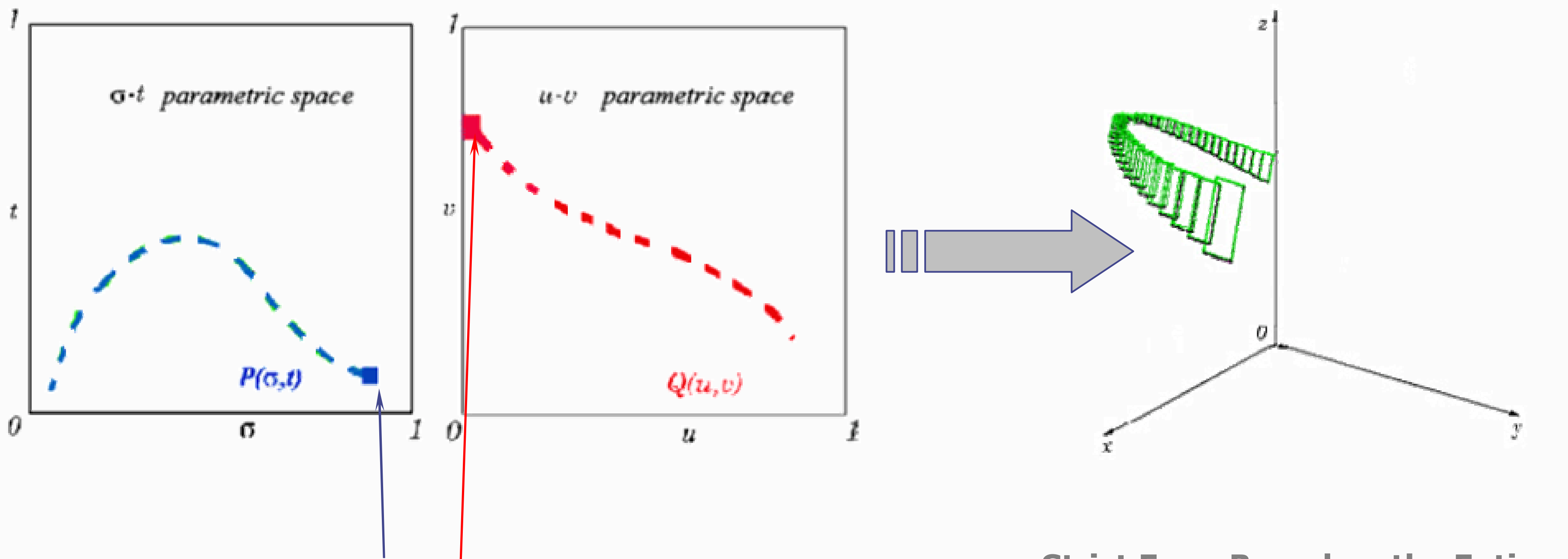
- Three popular methods
  - Lattice methods
    - Issues related to topology, missing roots.
  - Subdivision based methods
    - Issues related to topology, extraneous roots.
  - **Marching scheme (Our Choice)**
    - Intersection curve segment is computed through an IVP.

Introduction
Marching Scheme

- A *marching scheme* involves:
  - Identifying all components
  - Obtaining an accurate starting point in each component
  - **Tracing the given intersection correctly**
- Assumption:
  - The given surfaces are *Rational Polynomial Parametric (RPP).*
  - We are given an intersection curve segment.
  - No singularities exist in the intersection curve segment.

Introduction
Objective

- Given an error bound on the *starting point* in both *parametric spaces,* obtain a bound for the entire *intersection curve segment* in *3D model space.*



Strict Error Bound on Starting Point (Given)

Strict Error Bound on the Entire Intersection Curve Segment (Goal)

Outline

- **Problem Formulation**
- Error Bounds in Parametric Space
- Error Bounds in 3D Model Space
- Results and Examples
- Conclusions

Problem Formulation
Transversal Intersection

- Transversal intersection formulated as a system of *ordinary differential equations* (ODEs) in *parametric space* [4].

From $\mathbf{P}(\sigma,t)=\mathbf{Q}(u,v)$, we obtain :

$$\frac{d\sigma}{ds}=\frac{Det[\mathbf{c},\mathbf{P}_t,\mathbf{N}^P(\sigma,t)]}{\mathbf{N}^P(\sigma,t)\bullet\mathbf{N}^P(\sigma,t)},$$

$$\frac{dt}{ds}=\frac{Det[\mathbf{P}_\sigma,\mathbf{c},\mathbf{N}^P(\sigma,t)]}{\mathbf{N}^P(\sigma,t)\bullet\mathbf{N}^P(\sigma,t)},$$

$$\frac{du}{ds}=\frac{Det[\mathbf{c},\mathbf{Q}_v,\mathbf{N}^Q(u,v)]}{\mathbf{N}^Q(u,v)\bullet\mathbf{N}^Q(u,v)},$$

$$\frac{dv}{ds}=\frac{Det[\mathbf{Q}_u,\mathbf{c},\mathbf{N}^Q(u,v)]}{\mathbf{N}^Q(u,v)\bullet\mathbf{N}^Q(u,v)},$$

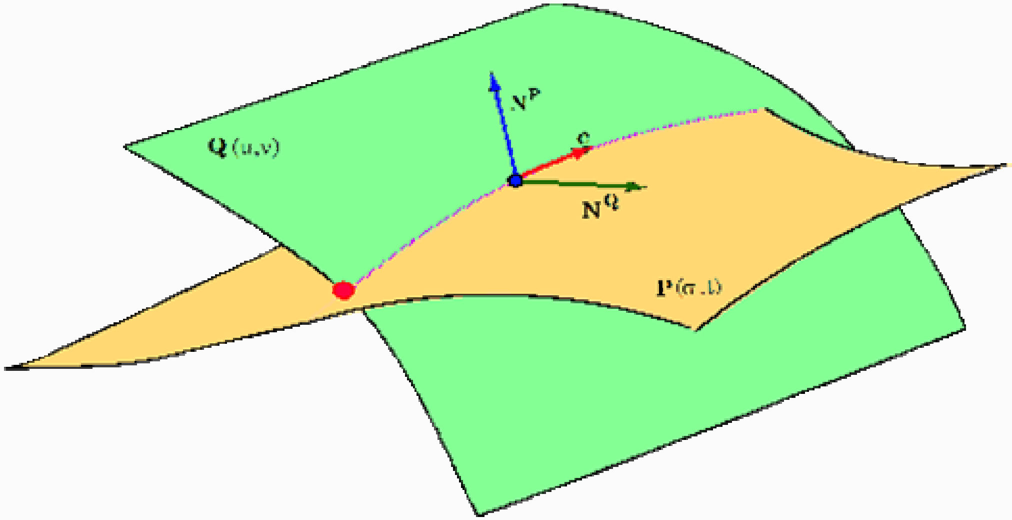
where s is the arc length parameter and

$\mathbf{P}_\sigma,\mathbf{P}_t,\mathbf{Q}_u,\mathbf{Q}_v$ *are partial derivatives*

$\mathbf{N}^P=\mathbf{P}_\sigma\times\mathbf{P}_t$ *Normal to* $\mathbf{P}(\sigma,t)$

$\mathbf{N}^Q=\mathbf{Q}_u\times\mathbf{Q}_v$ *Normal to* $\mathbf{Q}(u,v)$

$$\mathbf{c}=\frac{\mathbf{N}^P\times\mathbf{N}^Q}{\left|\mathbf{N}^P\times\mathbf{N}^Q\right|}$$



Problem Formulation
Tangential Intersection

- ODEs have the same form as in transversal intersection case

$$\sigma' = \frac{Det[\mathbf{c}, \mathbf{P}_t, \mathbf{N}^P(\sigma,t)]}{\mathbf{N}^P(\sigma,t) \bullet \mathbf{N}^P(\sigma,t)}, \quad t' = \frac{Det[\mathbf{P}_\sigma, \mathbf{c}, \mathbf{N}^P(\sigma,t)]}{\mathbf{N}^P(\sigma,t) \bullet \mathbf{N}^P(\sigma,t)}, \quad u' = \frac{Det[\mathbf{c}, \mathbf{Q}_v, \mathbf{N}^Q(u,v)]}{\mathbf{N}^Q(u,v) \bullet \mathbf{N}^Q(u,v)}, \quad v' = \frac{Det[\mathbf{Q}_u, \mathbf{c}, \mathbf{N}^Q(u,v)]}{\mathbf{N}^Q(u,v) \bullet \mathbf{N}^Q(u,v)},$$

$$\mathbf{N}^P = \mathbf{P}_\sigma \times \mathbf{P}_t \quad \textit{Normal to } \mathbf{P}(\sigma,t) \textit{ and } \mathbf{N}^Q = \mathbf{Q}_u \times \mathbf{Q}_v \quad \textit{Normal to } \mathbf{Q}(u,v)$$

$\mathbf{c}$ is determined using the second derivatives of the surfaces.

- From the condition of *equal normal curvatures* we obtain the equation

$$b_{11}(\sigma')^2 + 2b_{12}(\sigma')(t') + b_{22}(t')^2 = 0$$

where b_{11}, b_{12}, b_{22} are functions of the *first* and *second fundamental form coefficients* of the surfaces.

- For a unique marching direction, $(b_{12}^2 - b_{11}b_{22}) = 0$ and $(b_{12}^2 + b_{11}^2 + b_{22}^2) \neq 0$
  - **Thus if:** $b_{11} \neq 0$, **or if:** $b_{11} = 0,\ b_{22} \neq 0$

$$\mathbf{c} = \frac{\nu \mathbf{P}_\sigma + \mathbf{P}_t}{|\nu \mathbf{P}_\sigma + \mathbf{P}_t|}, \text{ where } \nu = -\frac{b_{12}}{b_{11}}$$

$$\mathbf{c} = \frac{\mathbf{P}_\sigma + \mu \mathbf{P}_t}{|\mathbf{P}_\sigma + \mu \mathbf{P}_t|}, \textit{ where } \mu = -\frac{b_{12}}{b_{22}}$$

Problem Formulation
Vector IVP for ODE

- Given a *starting point (initial condition)* belonging to an *intersection curve segment,* we can integrate the system of ODEs.

- The system of ODEs with the *starting point* represents an initial value problem (IVP).
  - Written in vector notation as:

$$\begin{bmatrix} \frac{d\sigma}{ds} \\ \frac{dt}{ds} \\ \frac{du}{ds} \\ \frac{dv}{ds} \end{bmatrix} = \begin{bmatrix} f_1(\sigma,t,u,v) \\ f_2(\sigma,t,u,v) \\ f_3(\sigma,t,u,v) \\ f_4(\sigma,t,u,v) \end{bmatrix}$$

$$\frac{d\mathbf{y}}{ds} = \mathbf{f}(\mathbf{y}), \quad \mathbf{y}(\mathbf{0}) = \mathbf{y}_0$$

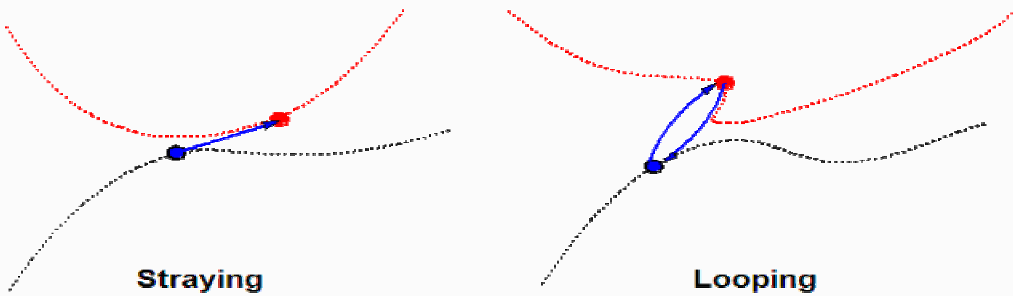
Outline

- Problem Formulation
- **Error Bounds in Parametric Space**
  - **Review of Standard Schemes**
  - **Interval Arithmetic**
  - **Validated Interval Scheme**
- Error Bounds in 3D Model Space
- Results and Examples
- Conclusions

Error Bounds in Parametric Space
Review of Standard Schemes

- Well-known Standard Schemes:
  - Runge-Kutta Method
  - Adams-Bashforth Method
  - Taylor Series Method

- Properties of Standard Schemes:
  - They are approximation schemes and introduce a *truncation error*
  - They do not consider *uncertainty in initial conditions*
  - They are prone to *rounding errors*
  - They suffer from *straying* or *looping* near closely spaced features



Error Bounds in Parametric Space
Interval Arithmetic (Introduction)

- Intervals are defined by [2]:

$$[a] \equiv [\underline{a}, \overline{a}] \equiv \{x | \underline{a} \leq x \leq \overline{a}\}, \ \underline{a}, \overline{a}, x \in \mathbf{R}, \ \underline{a} \leq \overline{a}$$

- Example:

$$\pi = 3.14159265\,3589793238\,46\ldots$$
$$\pi \in [3.141\,,\ 3.142] = [\pi]$$

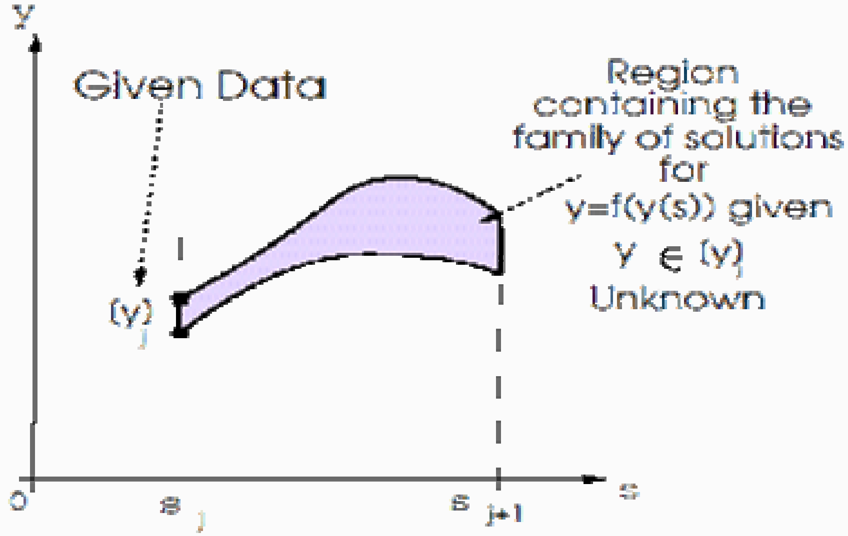
- Basic *interval arithmetic* operations are defined by:

$$\begin{aligned}
[a,b] + [c,d] &= [a+c, b+d] \\
[a,b] - [c,d] &= [a-d, b-c] \\
[a,b] \cdot [c,d] &= [\min(a\cdot c, a\cdot d, b\cdot c, b\cdot d), \max(a\cdot c, a\cdot d, b\cdot c, b\cdot d)] \\
[a,b] \,/\, [c,d] &= [\min(a/c, a/d, b/c, b/d), \max(a/c, a/d, b/c, b/d)], \quad \text{for } 0 \notin [c,d]
\end{aligned}$$

Error Bounds in Parametric Space

Interval Arithmetic (Solution of IVPs)

- For strict bounds for IVPs in *parametric space,* we employ a *validated interval scheme* for ODEs [3].
- The error in *starting point* is bounded by an *initial interval*.
- *Interval solution* represents a family of solutions passing through the *initial interval* satisfying the governing ODEs.



Error Bounds in Parametric Space
Validated Interval Scheme (Introduction)

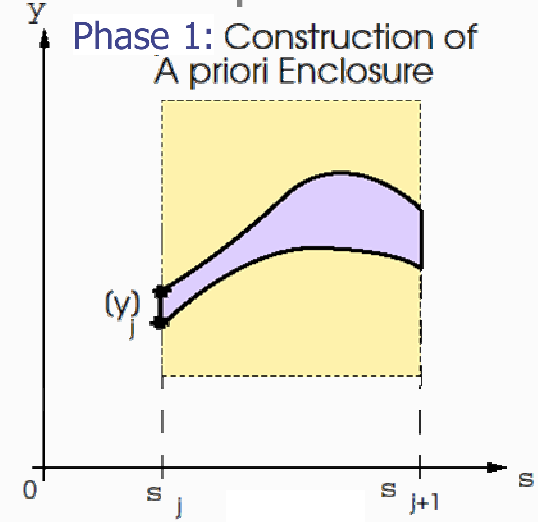
- Every *step* of a *validated interval scheme* involves [3]:
  - Computing an interval valued function $[\mathbf{y}](s)$ such that:
    - $\mathbf{y}(s) \in [\mathbf{y}](s)$, and
    - The width of the $[\mathbf{y}](s)$ is below a given tolerance

    $\forall\ s \in [s_j, s_{j+1}]$
  - Verifying the *existence* and *uniqueness* of the solution in $[s_j, s_{j+1}]$.

Error Bounds in Parametric Space

Validated Interval Scheme (Overview)

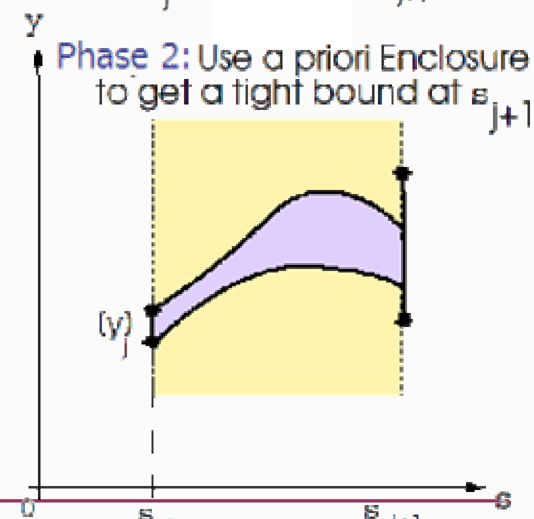
- One *step* of a *validated interval scheme* is done in *two* phases:



- **Phase I Algorithm**
  - A *step size* $h_j = s_{j+1} - s_j$
  - An *a priori enclosure* $[\tilde{y}_j]$ such that:

$$y(s) \in [\tilde{y}_j], \quad \forall \ s \in [s_j, s_{j+1}]$$

- **Phase II Algorithm**
  - Using $[\tilde{y}_j]$ compute a *tighter bound* $[y_{j+1}]$ at s_{j+1} .



Error Bounds in Parametric Space

Validated Interval Scheme (Phase I : Validation)

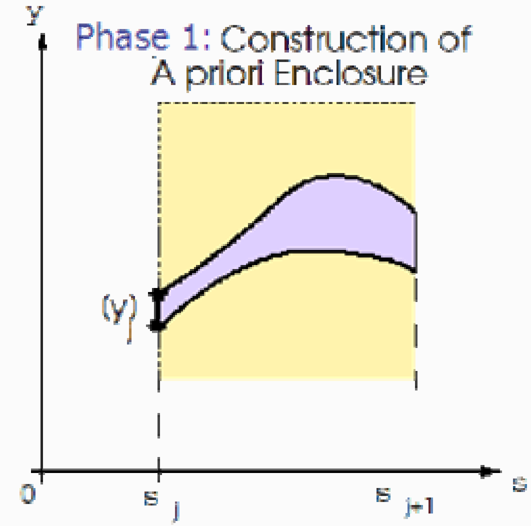
- A pair of $[\tilde{\mathbf{y}}_j]$ and h_j satisfying the relation:

$$[\tilde{\mathbf{y}}_j] \supseteq [\mathbf{y}_j] + \mathbf{f}([\tilde{\mathbf{y}}_j])h_j$$

  - This assures *existence* and *uniqueness* of the solution.
  - This method is called a *constant enclosure method* [3].

- The *a priori enclosure* $[\tilde{\mathbf{y}}_j]$ bounds the true solution in the *parametric space* $\forall\ s \in [s_j, s_{j+1}]$.

- Numerical implementation
  - Choosing a $[\tilde{\mathbf{y}}_j]$ and,
  - Iterating to find a corresponding h_j .



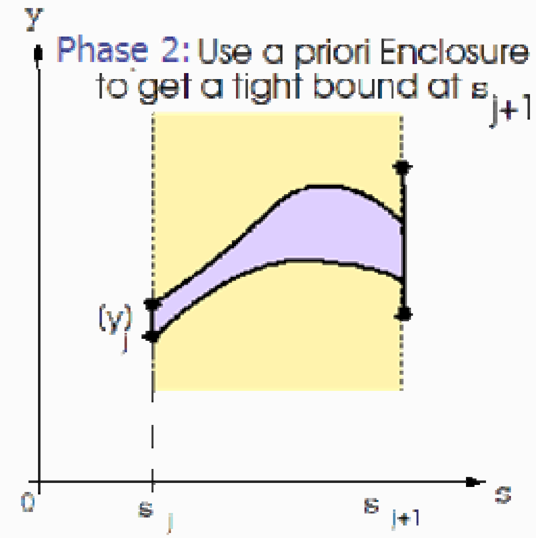
Error Bounds in Parametric Space

Validated Interval Scheme (Phase II : Tighter Bound)

- Using the *a priori enclosure* we
  - find a tighter bound $[\mathbf{y}_{j+1}]$ at s_{j+1} [3].
- This phase helps in the propagation of the solution by providing an *initial interval* for the successive step.
- The key idea is to use:
  - Interval version of *Taylor's formula* [3].

$$[\mathrm{y}_{j+1}] = [\mathrm{y}_j] + \sum_{i=1}^{k-1} h_j^{\,i} \mathbf{f}^{[i]}([\mathrm{y}_j]) + h_j^{\,k} \mathbf{f}^{[k]}([\tilde{\mathrm{y}}_j])$$

where $\mathbf{f}^{[i]}([\mathrm{y}_j])$ represents the i^{th} *Taylor coefficient* obtained using a technique called Automatic Differentiation [3].



Error Bounds in Parametric Space

Validated Interval Scheme (Application to SSI)

- We represent the surfaces as *interval surfaces.*
  - *Interval surfaces* have *interval coefficients* and are written as:

$$[\mathbf{P}](\sigma, t) \text{ and } [\mathbf{Q}](u, v)$$

- We obtain a *vector interval ODE system* :

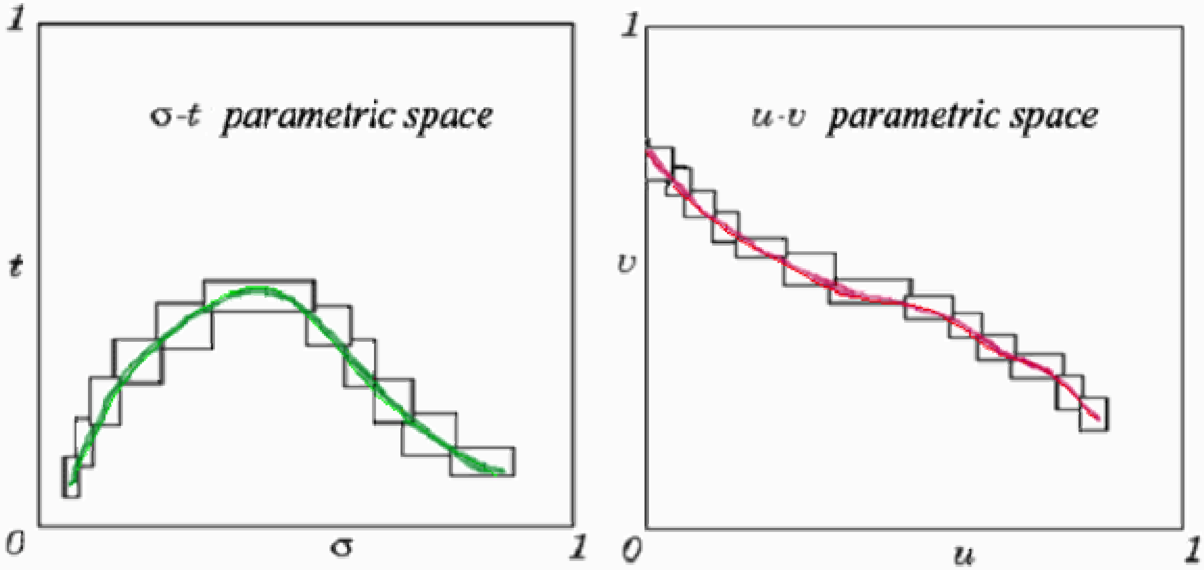
$$\left[\frac{d\sigma}{ds} \ \frac{dt}{ds} \ \frac{du}{ds} \ \frac{dv}{ds} \right]^T = \frac{d\mathbf{y}}{ds} = \mathbf{f}([\mathbf{y}(s)])$$

- With an *interval initial condition* :

$$[\mathbf{y}_0] = [[\sigma_0] \ \ [t_0] \ \ [u_0] \ \ [v_0]]^T$$

Error Bounds in Parametric Space

- Validated ODE solver produces *a priori enclosures* in *parametric space* of each surface, guaranteed to contain the *true intersection curve segment*.



- The union of *a priori enclosures* bounds the *true intersection curve segment* in *parametric space*.

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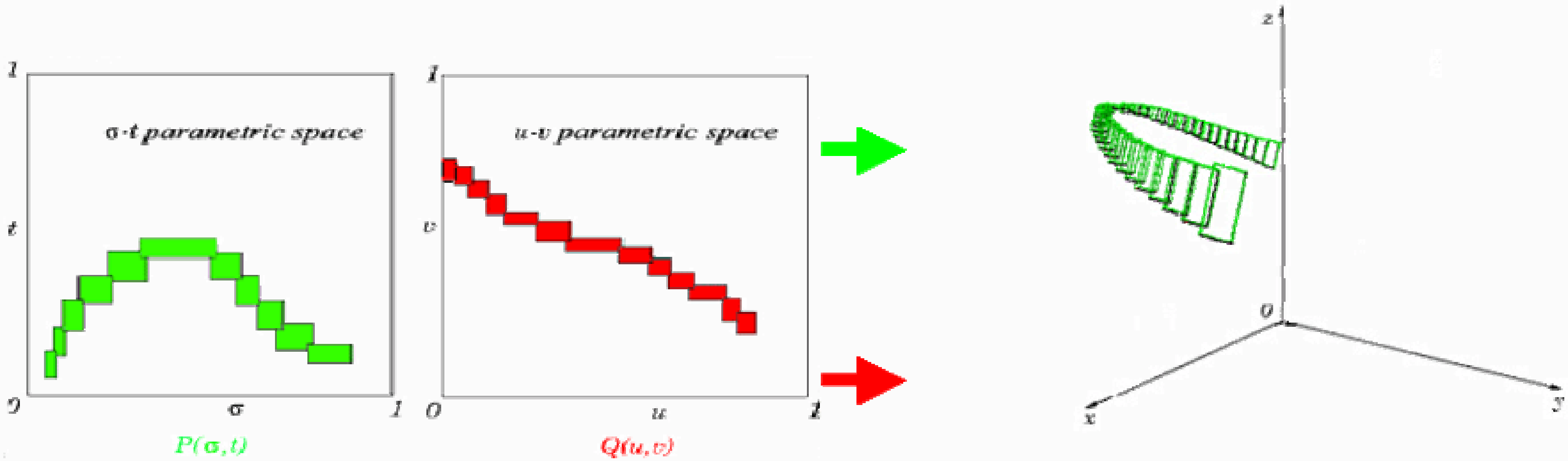
Error Bounds in 3D Model Space

Mapping into 3D Model Space

- Mapping from *parametric space* to *3D model space*
  - using corresponding surfaces $[\mathbf{P}](\sigma,t)$ or $[\mathbf{Q}](u,v)$
  - coupled with *rounded interval arithmetic* evaluation



- Ensures continuous error bounds in *3D model space* [1] guaranteed to contain the *true curve of intersection*.

Outline

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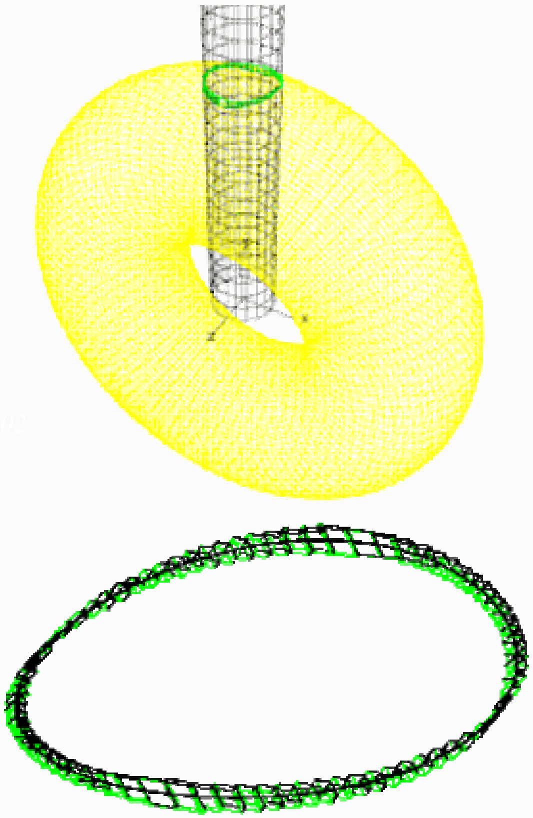
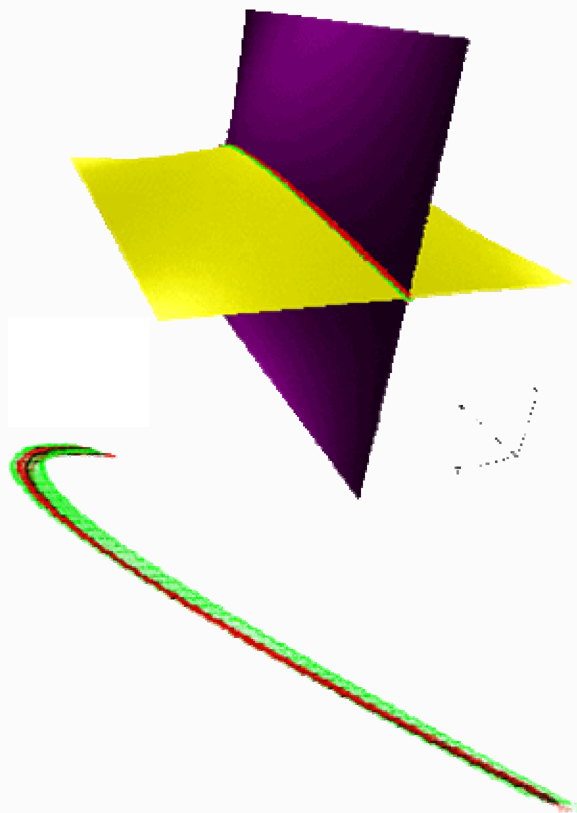
Results & Examples

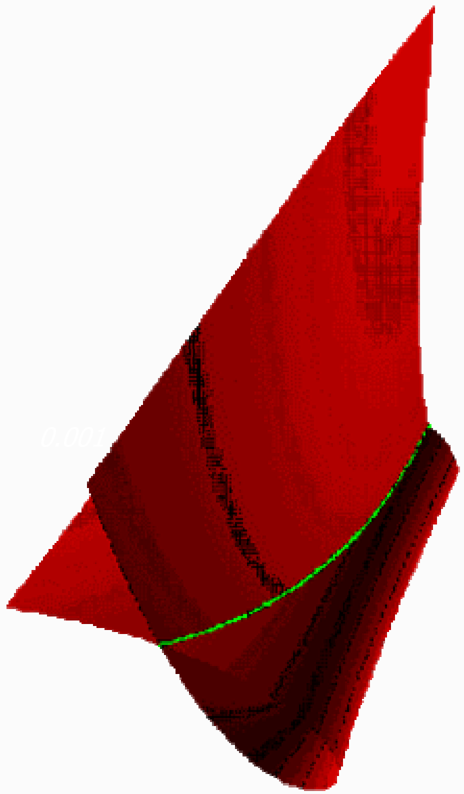
Error Bounds in 3D Model Space (Transversal)



Torus and *cylinder*



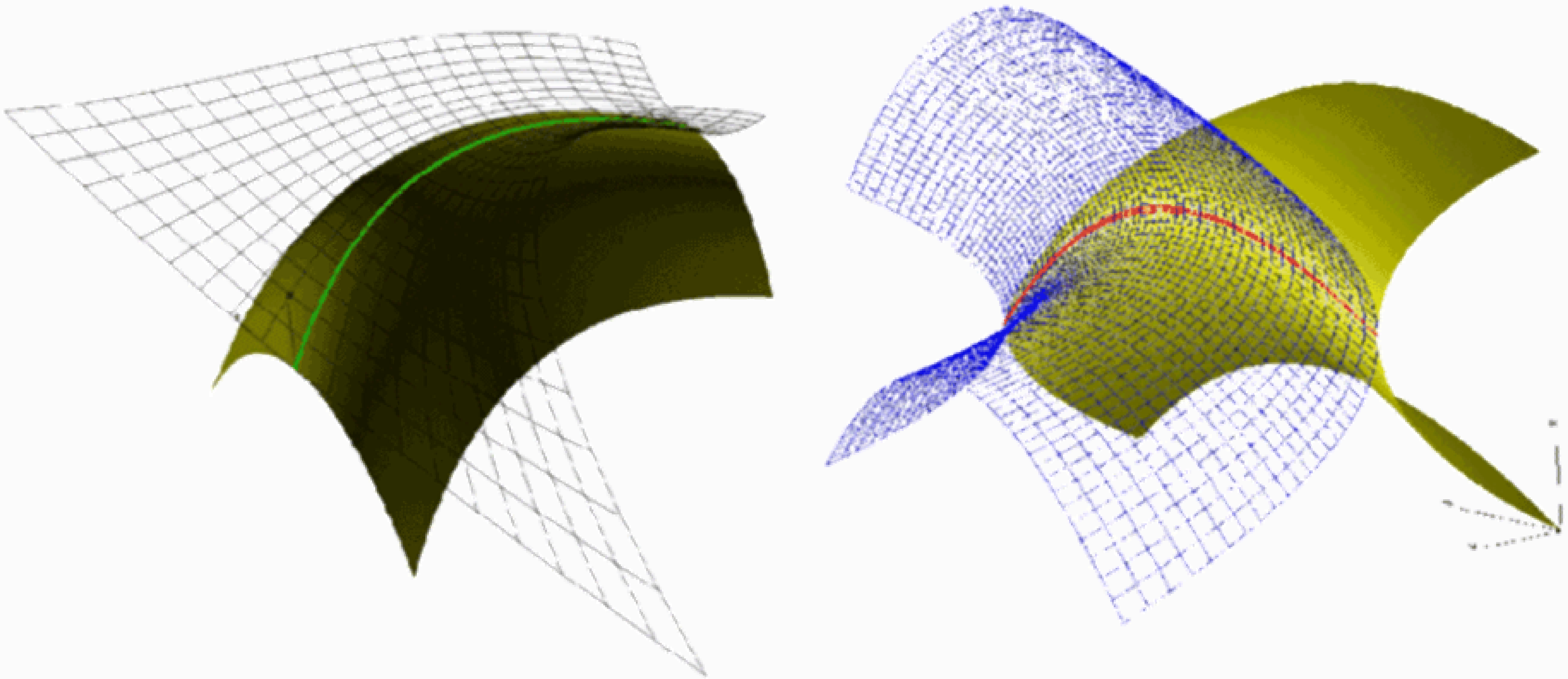
Two *bi-cubic* surfaces



Self intersection of
a *bi-cubic* surface

Results & Examples

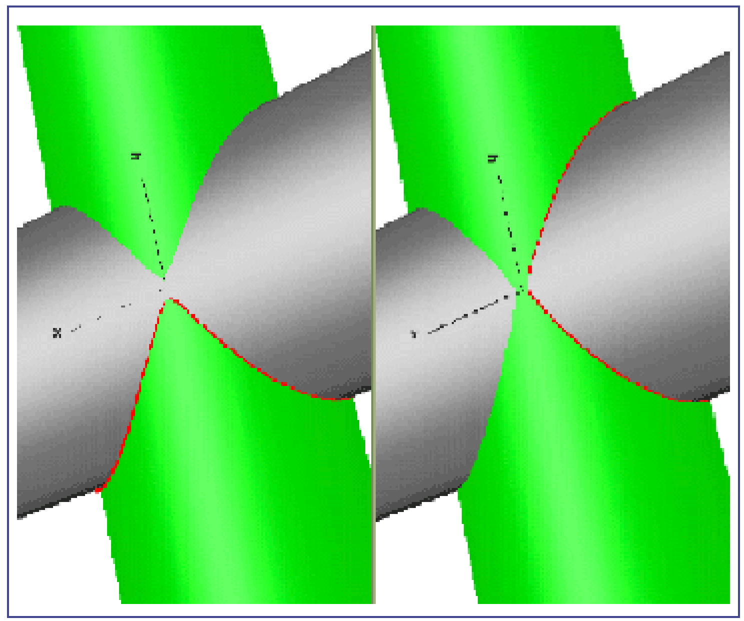
Error Bounds in 3D Model Space (Tangential)

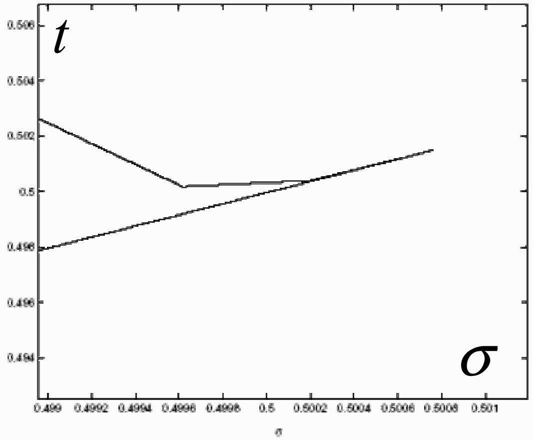


Tangential intersections of parametric surfaces

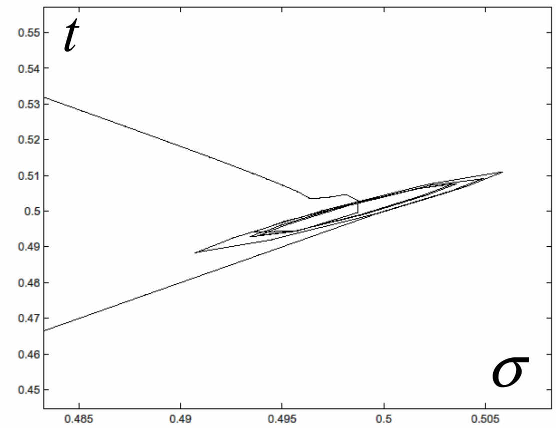
Results & Examples

Preventing Straying and Looping



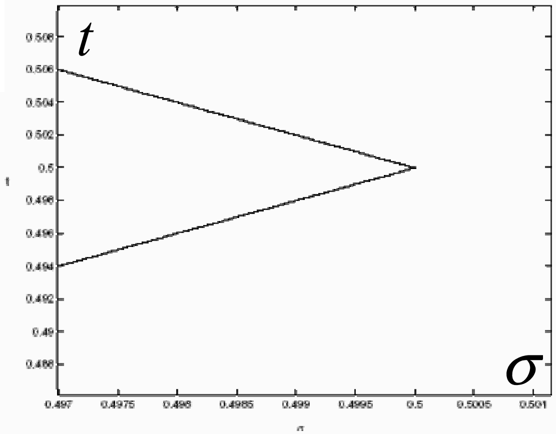


Adams-Bashforth



Runge-Kutta

Perturbation	Steps Required by the Method
+0.000003	1139
0.0	Singularity Reported
-0.000003	1303



Result from a validated interval scheme

Validated ODE solver can correctly trace the *intersection curve segment* even through closely spaced features, where standard methods fail.

Outline

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Conclusions

Merits

- We realize *validated error bounds* in *3D model space* which enclose the *true curve of intersection.*
- The scheme can prevent the phenomenon of *straying or looping*.
- The scheme can accommodate the errors in:
  - initial condition
  - rounding during digital computation
- *Validated error bounds* for surface intersection is essential in *interval boundary representation* for consistent *solid models* [5].

Conclusions
Limitations and Future Work

- Limitations
  - We assume that we have
    - Identified each intersection curve segment
    - *Strict error bound* on the starting point
  - Increasing width of the interval solutions due to
    - Rounding
    - Phenomenon of *wrapping*
- Scope for future work
  - Identification of all components
  - Accurate evaluation of *starting points* in each of the component

Acknowledgements

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