SYMMETRIC AUCTIONS

RAHUL DEB[†] AND MALLESH M. PAI[‡]

ABSTRACT. Symmetric (sealed bid) auctions such as first, second, and all-pay auctions are commonly used in practice. One reason for their popularity is that the rules of these auctions are "fair" in that they are anonymous and nondiscriminatory. In an independent private value setting with heterogenous buyers, we characterize the outcomes that can be implemented when the auction designer is restricted to using a symmetric auction format. We show that symmetric auctions can yield a wide variety of discriminatory outcomes, such as revenue maximization and affirmative action. We also characterize the set of implementable outcomes when other desiderata are imposed in addition to symmetry. These additional requirements may prevent the seller from maximizing revenue.

[†]DEPARTMENT OF ECONOMICS, UNIVERSITY OF TORONTO
[‡]DEPARTMENT OF ECONOMICS, UNIVERSITY OF PENNSYLVANIA
E-mail addresses: `rahul.deb@utoronto.ca, mallesh@econ.upenn.edu`.
Date: November 6, 2014.
We would like to thank Dirk Bergemann, Heski Bar-Isaac, Amos Fiat, Johannes Hörner, René Kirkegaard, Bernard Lebrun, Kevin Lleyton-Brown, Steve Matthews, Maher Said, Larry Samuelson, Xianwen Shi, Andrzej Skrzypacz, Jeroen Swinkels, Juuso Toikka, Gabor Virag, and numerous seminar audiences for their insightful comments. We also thank Yunan Li for providing excellent research assistance. Mallesh Pai gratefully acknowledges the support of NSF Grant CCF-1101389.

An optimal auction extends the asymmetry of the buyer roles to the allocation rule itself. The assignment of the good and the appropriate buyer payment will depend not only on the list of offers, but also on the identities of the buyers who submit the bids. In short, an optimal auction under asymmetric conditions violates the principle of buyer anonymity.

J. Riley and W. Samuelson (1981). "Optimal Auctions," *American Economic Review.*

1. Introduction

Symmetric sealed bid auctions or, simply, symmetric auctions are widely used in practice. In these auctions, buyers submit sealed bids, the highest bidder over the reservation bid wins and the transfers are determined via an anonymous function which maps bids to payments. Standard examples are first, second, and all-pay auctions. In each of these auctions, the winner is determined in the same way; instead, the auctions differ in terms of the payment rules for the winning and losing bidders that in turn affect the outcomes through equilibrium bidding. More complex examples are first or second price auctions with an all pay component (usually an entry fee, as in Levin and Smith (1994)), k-price auctions (Güth and Van Damme, 1986), k-price all-pay auctions (Goeree, Maasland, Onderstal, and Turner, 2005), and auctions in which winners pay some combination of the highest and second highest bid (Lebrun, 2013), to mention but a few.

Symmetric auctions have the advantage of having rules that are anonymous and nondiscriminatory. This is one of the reasons that they remain popular in the real world despite the fact that they may not achieve the seller's goals such as revenue maximization (when buyers are ex-ante heterogeneous) or affirmative action.[1] The particular symmetric auction format chosen by the seller depends on his objectives and the environment he faces. For instance, a revenue maximizing seller's preference ranking of the first and second price auction is determined by the buyers' value distributions (Maskin and Riley, 2000; Kirkegaard, 2012).

Motivated by the ubiquity of symmetric auctions, the aim of this paper is to understand the degree of flexibility this format offers for auction design. Specifically, we examine the set of outcomes that a seller can achieve when restricted to using such a mechanism. As a consequence, our analysis uncovers the extent to which a seller can discriminate amongst buyers (via their equilibrium bidding) using a format that appears "fair" on the surface.

Mechanism design subject to symmetry constraints has a long history. In matching theory, fairness is a concern in university and public housing allocations, parking space assignment and student placement in public schools. Hence, two of the most prominent allocation mechanisms (the random priority mechanism (Abdulkadiroğlu and Sönmez, 1998) and the probabilistic serial mechanism (Bogomolnaia and Moulin, 2001)) are anonymous. More generally, anonymity as a condition has a rich background in the axiomatic social choice theory literature.[2] Our analysis departs from this literature in imposing the symmetry requirement on the indirect implementation as opposed to on the direct mechanism. Additionally, we differ from the majority of social

[1]As we discuss below, fairness is often legally mandated.

[2]Often, other notions of fairness such as the "equal treatment of equals" (two agents making the same reports receive the same allocations) and "envy-freeness" (each agent prefers her allocation to that of any other agent) are imposed instead.

choice theory in considering the weaker Bayesian (as opposed to dominant strategy) incentive compatibility criterion commonly employed in auction theory. [3] It should also be pointed out that symmetry axioms play an important role in cooperative game theory (to characterize the Shapley value) and in both the Nash and Kalai-Smorodinsky bargaining solutions.

We consider a general independent private value setting with ex-ante heterogenous buyers. We say that a direct mechanism has a symmetric implementation if there is a symmetric auction that has an equilibrium (in undominated strategies) for which the outcome yields the same (ex-post) allocation rule and the same expected (interim) payments.[4] Our main result (Theorems 1 and 2) is a complete characterization of the set of implementable direct mechanisms. Additionally, we provide a simple qualitative description of the set of direct mechanisms that are not implementable (Corollary 1).

We first argue that a direct mechanism is implementable only if it is a hierarchical mechanism (Border, 1991). In a hierarchical mechanism, there is a (potentially different) nondecreasing index associated with each buyer's valuation, and the allocation rule awards the good to the highest index at each value profile. This is an extremely general class of mechanisms that includes, in particular, both the efficient (the index is the value itself) and the Myerson (1981) optimal auction (the index is the "ironed" virtual value), among many others. We then show that (in a sense we make precise) almost all hierarchical mechanisms are implementable (Corollary 2).[5] In other words, symmetric implementability is a generic property of hierarchical mechanisms. A strength of our analysis is that it requires very mild assumptions on the distributions of buyer valuations. In particular, we do not need to impose any hazard rate assumptions that are commonly employed in the auction theory literature.

In this sense, we view our main result on the versatility of symmetric auctions as akin to (though, of course, not as strong as) the revelation principle for direct mechanisms. The revelation principle states that restricting the seller to direct mechanisms is without loss of generality. Analogously, our characterization shows that restricting the auctioneer to symmetric auction formats does not prevent him from achieving a wide variety of different (and discriminatory) goals. In this regard, our results are also similar in spirit to the recent work of Manelli and Vincent (2010) and Gershkov, Goeree, Kushnir, Moldovanu, and Shi (2013). These authors show that, in the independent private values model, any incentive compatible and individually rational outcome that can be achieved in Bayes-Nash equilibrium can also be achieved (in expectation) in dominant strategies. Thus, as with the case of symmetry, the requirement of dominant strategy implementation is not restrictive in and of itself.

A surprising implication of our main result is that the revenue-optimal outcome can always be achieved via a symmetric auction (Corollary 3). This result counters what appears to be common intuition and received wisdom. Because its direct implementation is asymmetric, the optimal

[3] There is a strand of literature that uses anonymity in combination with Bayesian incentive compatibility (an early paper is d'Aspremont and Peleg (1988)).

[4] Of course, changing the direct mechanism or the value distributions will lead to different symmetric auction implementations.

[5] A consequence of this is that there is an implementable hierarchical mechanism that is arbitrarily close to any nonimplementable one.

auction was believed to be nonanonymous in the earliest seminal work (see epigraph), and since then, there have been numerous instances in the auction theory literature where similar beliefs are stated. Some argue that this observation justifies the removal of legal hurdles that prevent discrimination. In the context of international trade, McAfee and McMillan (1989) used the theory of optimal auctions to show that explicitly discriminating amongst suppliers can reduce the costs of procurement. Their aim was to provide an argument against the 1981 Agreement on Government Procurement (in the General Agreement on Tariffs and Trade), which set out rules to ensure that domestic and international suppliers were treated equally.[6] Similarly, Cramton and Ayres (1996) suggest that, in government license auctions, subsidizing minority owned or local businesses may actually result in more revenue to the government.[7] We show that, at least from a theoretical perspective, such goals can be achieved without explicit discrimination by the auctioneer.

That a symmetric auction can be used to achieve a broad class of different objectives has implications for government procurement auctions, which often have distributional goals in addition to generating revenue (Athey, Coey, and Levin, 2013). Governments often desire to favor certain bidders (small businesses, women, minorities, etc.) who are economically disadvantaged and hence may be unable to compete with stronger bidders unless the auction rules are skewed in their favor. However, such a preferential policy is often viewed as unfair. This policy was successfully challenged in the U.S. Supreme Court case 199 (1995), and states such as California and Michigan have explicitly changed their laws (Proposition 209 and Proposal 2 respectively) to prohibit favored treatment on the basis of race, sex, or ethnicity. In Europe, Article 87(1) of the European Commission Treaty prohibits "aid granted by a Member State or through State resources in any form whatsoever which distorts or threatens to distort competition by favoring certain undertakings..." Our results suggest that, in principle, it is potentially possible to achieve outcomes where particular classes of bidders are favored without having to resort to explicitly biasing the auction.

This implication can also be interpreted another way—symmetry of the auction does not imply fairness of the outcome. In a sense, this intuition is already well known, as ex-ante heterogenous buyers may have different equilibrium strategies even in a symmetric auction. For instance, Maskin and Riley (2000) show that stronger bidders often favor second-price to first-price auctions and that the latter format can yield higher revenues for the seller. The observation that a "fair" and transparent auction can be constructed in a way to implement discriminatory outcomes is important in formulating policy that prevents favoritism.

This latter observation has also been made in the context of affirmative action in college admissions. Opponents of affirmative action often assert that a ban would lead to higher-quality students being admitted. However, it has been argued that it is theoretically possible for universities to alter the criteria for admissions in response to such a ban in a way that can still achieve diversity goals without explicit discrimination (Chan and Eyster, 2003; Fryer, Loury, and Yuret, 2008). Essentially, this can be achieved by shifting weight from academic traits that predict performance to social traits that proxy for race. However, unlike our setting, where bidders are strategic, students cannot choose what to report on their applications.

[6]Such an agreement is also currently present in the World Trade Oraganization, which has replaced the GATT.

[7]Corns and Schotter (1999) test these arguments empirically by conducting a laboratory experiment.

While the main result of the paper is primarily theoretical, we feel that the design of symmetric auctions for real-world applications is an important auction design problem, and our theoretical analysis characterizing implementable outcomes is a necessary step towards this end. With this in mind, we consider a handful of additional desiderata that a seller might want in an auction. We isolate a number of "attractive" properties of first- and second-price auctions and then impose them as additional theoretical requirements on the symmetric implementation. For brevity and tractability, we focus on the case of two bidders and characterize the set of hierarchical mechanisms that are implementable with each of these additional requirements imposed separately. The key takeaway from these characterizations is that the optimal auction is no longer always implementable when these conditions are required in addition to symmetry.

The first property we consider is that of inactive losers; that is, the losers in the auction neither make payments nor receive subsidies. All-pay auctions do not satisfy this property (since losers must also pay their bids), which is perhaps one of the reasons that they are rarely used in practice. We show that hierarchical mechanisms generically have an inactive losers implementation (Proposition 1). The second property we examine is continuity of the payment rule. We provide a necessary and sufficient condition for continuous implementation (Proposition 2). While this condition is not generically satisfied, it is fairly unrestrictive. The third property we consider is monotonicity of the payment rule in the bids. We separately consider both monotonicity of the payment in the opponent's bid (as in the second-price auction) and, monotonicity of the payment in one's own bid (as in the first-price auction). Each of these monotonicity requirements makes the set of implementable mechanisms nongeneric (Propositions 3 and 4, respectively). The last property we consider is ex-post (as opposed to interim) individual rationality of the equilibrium of the symmetric auction implementation. This property implies that, in equilibrium, losers never have to make payments (but can receive subsidies) and that winners do not pay more than their value for the object. Imposing ex-post individual rationality can useful to avoid situations where bidders may not be able to make payments for certain realized bids due to budget constraints. The set of ex-post individually rational implementable hierarchical mechanisms is nongeneric (Proposition 5).

The paper is organized as follows. In Section 2, we describe the model and set up the notation. Section 3 presents an example that highlights our approach by demonstrating the implementation of a "regular" optimal auction with two bidders. The main results are presented in Section 4. We present the characterizations subject to the additional desiderata in Section 5. Finally, concluding remarks are provided in Section 6. The appendix contains proofs and some results that are not in the text.

2. THE MODEL

We consider an independent private value auction setting. A set $N = \{1, 2, \dots, n\}$ of risk-neutral buyers or bidders (used interchangeably) compete for a single indivisible object.[8] Buyer $i \in N$ draws a value $v_i \in V_i \equiv [\underline{v}_i, \overline{v}_i]$ independently from a distribution F_i. We assume that F_i is

[8]Equivalently, our model could be considered as a procurement setting where a firm or government wants a single project to be completed and solicits quotes from contractors, each of whom has an independent private cost.

twice continuously differentiable with corresponding density f_i which is strictly positive throughout the support $[\underline{v}_i, \overline{v}_i]$. Note that both V_i and F_i can be different across i, so we allow for ex-ante heterogenous bidders. We denote $\mathbf{V} \equiv \prod_{j\in N} V_j$ and $\mathbf{V}_{-i} \equiv \prod_{j\neq i} V_j$, with $\mathbf{v} \in \mathbf{V}$ and $\mathbf{v}_{-i} \in \mathbf{V}_{-i}$ denoting typical elements of these sets. As with values, we use the notation $F \equiv \prod_{j\in N} F_j$ and $F_{-i} \equiv \prod_{j\neq i} F_j$. We will use similar notation for other vectors and vector-valued functions throughout the paper.

A *direct mechanism* asks bidders to report their values, and uses these reports to determine allocations and payments. Allocations are determined via an ordered list of functions:

$$a^d = \left(a_1^d, \ldots, a_n^d\right) \quad \text{where} \quad a_i^d : \mathbf{V} \to [0,1] \text{ and } \sum_{i=1}^n a_i^d(\mathbf{v}) \leq 1. \tag{Direct Allocation}$$

Here, $a_i^d(\mathbf{v})$ is the probability that bidder i wins the auction when the profile of reported types is $\mathbf{v}$. The inequality above reflects the fact that the seller has a single unit to sell, so the probability of allocating it cannot exceed 1 at any profile $\mathbf{v}$. Additionally, this allows for the possibility that the seller may choose to withhold the good. Similarly, payments are determined via an ordered list of functions:

$$p^d = \left(p_1^d, \ldots, p_n^d\right) \quad \text{where} \quad p_i^d : \mathbf{V} \to \mathbb{R}. \tag{Direct Payment}$$

Here, $p_i^d(\mathbf{v})$ is the payment made by bidder i when the profile of reported types is $\mathbf{v}$. Note that, when it is positive, this is a transfer to the seller, and, when it is negative, it is a subsidy from the seller. In addition, the bidder may be required to make payments even when she does not receive the object.

Values are private; that is, buyers do not know the realized valuations of other bidders. Hence, each bidder's expected utility from participating in this mechanism is determined by his/her expected allocation and payment. For a given direct mechanism $\left(a^d, p^d\right)$, we define *interim* allocations and payments to be the expected allocations and payments conditioned on truthful reporting by all the bidders. Formally, these are given by

$$a_i^d(v_i) \equiv \int_{\mathbf{V}_{-i}} a_i^d(v_i, \mathbf{v}_{-i}) dF_{-i}(\mathbf{v}_{-i}), \tag{Interim Allocation}$$

$$p_i^d(v_i) \equiv \int_{\mathbf{V}_{-i}} p_i^d(v_i, \mathbf{v}_{-i}) dF_{-i}(\mathbf{v}_{-i}). \tag{Interim Payment}$$

For simplicity, we deliberately abuse notation by denoting interim allocations using the same symbol; the difference is determined by whether the argument is a single value or a value profile.

We make the additional standard assumption that the bidders are risk neutral and that their utilities are quasilinear in the transfers. Conditional on truthful reporting by the other bidders, the interim expected utility for bidder i with value v_i who announces a value v_i' is simply

$$v_i\, a_i^d(v_i') - p_i^d(v_i'). \tag{Bidder Utility}$$

A mechanism $\left(a^d, p^d\right)$ is said to be (Bayesian) *incentive compatible* or simply IC if reporting truthfully is a Bayes-Nash equilibrium, i.e.,

$$v_i\, a_i^d(v_i) - p_i^d(v_i) \geq v_i\, a_i^d(v_i') - p_i^d(v_i') \qquad \forall i \in N,\, \forall v_i, v_i' \in V_i. \tag{IC}$$

Myerson (1981) showed that incentive compatibility implies that the allocation rule a^d pins down the payments p^d up to constants $c_i \in \mathbb{R}$; that is,

$$p_i^d(v_i) = v_i a_i^d(v_i) - \int_{\underline{v}_i}^{v_i} a_i^d(w)dw + c_i. \qquad \text{(Payoff Equivalence)}$$

Additionally, a mechanism is said to be *individually rational* or simply IR if truthful reporting leads to a nonnegative payoff, or

$$v_i\, a_i^d(v_i) - p_i^d(v_i) \geq 0 \qquad \forall v_i \in V_i. \qquad \text{(IR)}$$

2.1. *Symmetric Auctions*

We define a *symmetric auction* as a game with three properties: (i) buyers simultaneously submit real numbers called bids; (ii) the winner is the highest bidder over a given reservation bid (ties are broken uniformly); and (iii) payments are determined via an anonymous payment function. This is an indirect sealed bid auction mechanism with the additional restriction that allocations and payments depend only on the profile of bids and not the identity of the bidders. Formally, in a symmetric auction, each bidder i chooses a bid $b_i \in \mathbb{R}$, and allocations and payments are determined by functions $a^s : \mathbb{R}^n \to [0,1]$ and $p^s : \mathbb{R}^n \to \mathbb{R}$, respectively. Bidder i's allocation or simply her probability of winning the item is given by

$$a^s\left(b_i, \mathbf{b}_{-i}\right) = \begin{cases} \frac{1}{\#\{j \in N : b_j = b_i\}} & \text{when } b_i \geq \max\{\mathbf{b}_{-i}, r\}, \\ 0 & \text{otherwise.} \end{cases} \qquad \text{(Symmetric Auction Allocation)}$$

where r is the reservation bid. As with the values, we use $\mathbf{b}$ and $\mathbf{b}_{-i}$ to denote the vector of all bids and the vector of all bids except that of bidder i, respectively.

Bidder i's payment is given by

$$p^s\left(b_i, \mathbf{b}_{-i}\right), \qquad \text{(Symmetric Auction Payment)}$$

where p^s is invariant to permutations of $\mathbf{b}_{-i}$ but can depend on the underlying distribution of values $(F_1, \ldots, F_n)$. Notice that, since the allocation and payment rules do not depend on the identity of the bidders, we only need a single function, as opposed to lists of functions, to define these mechanisms. Most commonly used auction formats, such as first-price, second-price and all-pay auctions are symmetric in this sense.

In a symmetric auction, a pure strategy (henceforth referred to simply as a strategy) for a bidder i is a mapping

$$\sigma_i : V_i \to \mathbb{R}, \qquad \text{(Buyer Strategy)}$$

that specifies the bid corresponding to each possible value. A profile of strategies $\sigma = (\sigma_1, \ldots, \sigma_n)$ constitutes a (Bayesian Nash) equilibrium of the symmetric auction (a^s, p^s) if each buyer's strategy is a best response to the strategies of other buyers. Formally, this requires that, for all $i \in N$ and $v_i \in V_i$, we have

$$\sigma_i(v_i) \in \underset{b \in \mathbb{R}}{\operatorname{argmax}} \int_{\mathbf{V}_{-i}} \left[v_i\, a^s\left(b, \sigma_{-i}(\mathbf{v}_{-i})\right) - p^s\left(b, \sigma_{-i}(\mathbf{v}_{-i})\right)\right] dF_{-i}(\mathbf{v}_{-i}).$$

Symmetric auctions are useful in situations where the seller knows the underlying value distributions (perhaps from having conducted similar auctions in the past) but cannot condition the mechanism on bidder identity. As we argued in the introduction, one reason for this is that discrimination may be explicitly prohibited by law. Alternatively, the seller could be conducting the auction in an environment where it is easy for bidders to conceal their identities (such as auctions conducted over the Internet). An advantage of a symmetric auction format is that it maintains buyer privacy by ensuring that they are not forced to reveal their identities via their bids. However, we require the buyers to know the underlying value distributions so that they can compute their equilibrium bid. Admittedly, this might be an unrealistic assumption in certain settings. That said, this requirement is imposed in almost all auction theory and, in particular, is necessary for buyers to calculate equilibrium bids even in standard first-price auctions.

We say that an IC and IR direct mechanism $\left(a^d, p^d\right)$ is *implemented* by a symmetric auction (a^s, p^s) if there is a pure strategy equilibrium in undominated strategies of the latter mechanism that yields the same allocation and expected payment as the former. Specifically, we say that a direct mechanism is implementable if there exists an undominated equilibrium strategy profile σ such that, for all $\mathbf{v} \in \mathbf{V}$,[9]

$$a_i^d(\mathbf{v}) = a^s\left(\sigma_i(v_i), \sigma_{-i}(\mathbf{v}_{-i})\right), \tag{1a}$$

$$p_i^d(v_i) = \int_{\mathbf{V}_{-i}} p^s\left(\sigma_i(v_i), \sigma_{-i}(\mathbf{v}_{-i})\right) dF_{-i}(\mathbf{v}_{-i}). \tag{1b}$$

In this notion of implementability, we require the equilibrium allocation of the symmetric auction to be identical to the direct mechanism for each profile of values but the payments to be equal in expectation. This is a partial implementation criterion as we do not require the symmetric auction to have a unique equilibrium.[10]

More generally, we say that an IC and IR direct mechanism $\left(a^d, p^d\right)$ is *implementable* if there exists a symmetric auction (a^s, p^s) that implements it (almost sure and interim implementability are defined analogously). The main goal of this paper is to characterize the set of IC and IR direct mechanisms that are implementable.[11] To make the exposition cleaner, we have deliberately defined implementation only in terms of pure strategies for the bidders. This restriction does not affect any of the results in the paper. We show in the appendix that allowing for mixed strategies does not expand the set of implementable mechanisms (or the set of implementable mechanisms subject to the additional conditions in Section 5).

We will also refer to two additional weaker implementation criteria. The first is *almost sure implementation*, which requires (1) to hold almost surely (over the distribution of buyer values). In other words, according to this criterion, the allocations and interim payments are the same

---

[9]We use the additional restriction of undominated equilibrium strategies to ensure that our symmetric implementation is not based on "implausible" buyer behavior.

[10]Given the fact that we allow for very general value distributions, it is perhaps unrealistic to expect a symmetric auction implementation to have a unique equilibrium. Note that even standard auction formats like the first or second price auction can have multiple equilibria in our model. This is because our setting is more general than even the fairly unrestrictive conditions required for uniqueness in first price auctions (Lebrun, 2006).

[11]Since the additional requirement of IR only involves changing the payment rules by a constant, our characterization results can also be viewed as simply characterizing the set of IC direct mechanisms which are implementable.

except at a measure zero set of values. The second is *interim implementation*, which requires the allocation rule (as with the payment) to be implemented in an expected sense or that $a_i^d(v_i) = \int_{\mathbf{V}_{-i}} a^s\left(\sigma_i(v_i), \sigma_{-i}(\mathbf{v}_{-i})\right) dF_{-i}(\mathbf{v}_{-i})$. The recent work on the equivalence of Bayesian and dominant strategy implementability (Manelli and Vincent, 2010; Gershkov, Goeree, Kushnir, Moldovanu, and Shi, 2013) uses an even weaker notion that instead requires the expected utilities (as opposed to interim allocations and payments separately) of the agents to be the same.

3. Example: Implementing the Optimal Auction with Two Buyers

In this section, we explain our approach by describing a symmetric implementation of the optimal auction when there are two buyers. For simplicity, we additionally assume that the distributions of both buyers satisfy the increasing virtual value property. Formally, this condition requires that, for each buyer $i \in N$, the virtual value

$$\phi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} \qquad \text{(Virtual Value)}$$

is increasing in v_i. An implication is that ϕ_i^{-1} is a single valued function.

We denote the allocation and payment rule of the optimal auction by (a^*, p^*). Recall that in the optimal auction, bidders announce their values and the mechanism awards the good to the bidder who has the highest positive virtual value (it is without loss to assume that ties are broken equally). Hence, when bidders draw their values from different distributions, this direct mechanism is not symmetric, as the allocation rule depends on the bidder-specific value distribution.

A natural way to attempt a symmetric implementation of the optimal auction is to construct a payment rule such that it is an equilibrium for both bidders to bid their virtual values. The auction could then allocate the good to the higher bid and have a reservation bid of 0. We denote the set of virtual values of bidder i by

$$B_i \equiv \left[\phi_i(\underline{v}_i), \phi_i(\overline{v}_i)\right].$$

The distribution F_i over V_i induces a distribution G_i over the set B_i of virtual values.

We claim that the optimal auction can be implemented if we can construct a payment rule p^s that satisfies

$$p_i^*(v_i) = \int_{B_j} p^s\left(\phi_i(v_i), b_j\right) dG_j(b_j) \qquad \text{for } i \neq j \text{ and all } v_i \in V_i.$$

This is simply a restatement of the implementability requirement where equilibrium strategies of bidding the virtual value have been substituted in. This claim is easy to see:

(1) Suppose that buyer i with value v_i bids $b_i \in B_i$ but $b_i \neq \phi_i(v_i)$. This is equivalent to her reporting a value $\phi_i^{-1}(b_i) \neq v_i$ in the direct mechanism (a^*, p^*), which yields a lower payoff because the optimal auction is IC.

(2) Suppose that buyer i with value v_i bids $b_i \notin B_i$. This can be detected with positive probability by the auctioneer when the other bidder is bidding truthfully. This is because there will be a positive measure of bids b_j such that $(b_i, b_j) \notin (B_1 \times B_2) \cup (B_2 \times B_1)$. Such off-equilibrium bids can be discouraged by making the payments high enough at these bids.

We now construct such a symmetric payment rule. Since it is easy to discourage bids that lie outside the support of the virtual values, the payment rule is deliberately defined only for equilibrium bid profiles $(b_i, b_j) \in (B_1 \times B_2) \cup (B_2 \times B_1)$. We separately construct the payment for bids that lie in the supports of only one and both virtual value distributions respectively. In equilibrium, bids $b_i \in B_i \backslash B_j$ are made only by buyer i. Hence, for such bids, we can simply define the payment rule to be the interim payment from the optimal auction or

$$p^s(b_i, b_j) = p_i^*(\phi_i^{-1}(b_i)) \text{ when } b_i \in B_i \backslash B_j \text{ and } b_j \in B_j.$$

To construct the payments for bids $b_i \in B_1 \cap B_2$ that lie in the support of both virtual value distributions, we first observe that, for asymmetric buyers ($F_1 \neq F_2$), there exists a $\hat{b} \in \mathbb{R}$ such that $G_1(\hat{b}) \neq G_2(\hat{b})$. In other words, different value distributions yield different virtual values distributions. Consider the payment rule

$$p^s(b_i, b_j) = \begin{cases} p^u(b_i) & \text{if } b_j \geq \hat{b} \text{ and } b_j \in B_1 \cup B_2, \\ p^l(b_i) & \text{if } b_j < \hat{b} \text{ and } b_j \in B_1 \cup B_2, \end{cases} \tag{2}$$

where

$$\begin{aligned} p^u(b_i) &= \frac{p_1^*(\phi_1^{-1}(b_i))G_1(\hat{b}) - p_2^*(\phi_2^{-1}(b_i))G_2(\hat{b})}{G_1(\hat{b}) - G_2(\hat{b})}, \\ p^l(b_i) &= \frac{p_2^*(\phi_2^{-1}(b_i))[1 - G_2(\hat{b})] - p_1^*(\phi_1^{-1}(b_i))[1 - G_1(\hat{b})]}{G_1(\hat{b}) - G_2(\hat{b})}. \end{aligned} \tag{3}$$

According to this payment rule, a bidder i who bids b_i pays an amount $p^u(b_i)$ when her opponent bids higher than $\hat{b}$ and an amount $p^l(b_i)$ when her opponent's bid is lower than $\hat{b}$. Hence, the expected payment of a bidder i who bids $b_i \in B_1 \cap B_2$ when bidder j bids $\phi_j(v_j)$ for all $v_j \in V_j$ is

$$p^u(b_i)[1 - G_j(\hat{b})] + p^l(b_i)G_j(\hat{b}) = p_i^*(\phi_i^{-1}(b_i)), \tag{4}$$

which is precisely the required payment for implementation.

Notice also that the above equation (4) can be used to derive the expressions for p^u and p^l. An equivalent matrix representation is the following system for $b_i \in B_1 \cap B_2$

$$\mathcal{M} \begin{bmatrix} p^u(b_i) \\ p^l(b_i) \end{bmatrix} = \begin{bmatrix} p_1^*\left(\phi_1^{-1}(b_i)\right) \\ p_2^*\left(\phi_2^{-1}(b_i)\right) \end{bmatrix} \quad \text{where} \quad \mathcal{M} = \begin{bmatrix} 1 - G_2(\hat{b}) & G_2(\hat{b}) \\ 1 - G_1(\hat{b}) & G_1(\hat{b}) \end{bmatrix}. \tag{5}$$

By definition, $G_1(\hat{b}) \neq G_2(\hat{b})$ implies that $\mathcal{M}$ is a full rank matrix. Therefore (5) has a solution for all $b_i \in B_1 \cap B_2$, and p^u, p^l can be obtained by inverting $\mathcal{M}$. Note that this logic bears some resemblance to the intuition in Crémer and McLean (1988) and McAfee and Reny (1992), who study the possibility of full surplus extraction in auctions when different buyers' values are correlated. In their setting, in addition to bidding for the object, buyers are forced to make a "side bet" on their opponents' reported types (in our construction, the analogous "side bet" is whether your opponent's bid is above or below $\hat{b}$). When values are correlated, these side bets have different expected values for different types of the same buyer, which allows the payment rule to effectively discriminate between them. They show that a full rank condition (analogous to requiring $\mathcal{M}$ to be full rank) on the value distributions is sufficient for full surplus extraction.

In summary, the symmetric payment rule that implements the optimal auction in this example is

$$p^s(b_i, b_j) = \begin{cases} p^u(b_i) & \text{if } b_i \in B_1 \cap B_2,\ b_j \geq \hat{b} \text{ and } b_j \in B_1 \cup B_2, \\ p^l(b_i) & \text{if } b_i \in B_1 \cap B_2,\ b_j < \hat{b} \text{ and } b_j \in B_1 \cup B_2, \\ p_1^*(\phi_1^{-1}(b_i)) & \text{if } b_i \in B_1 \backslash B_2 \text{ and } b_j \in B_2, \\ p_2^*(\phi_2^{-1}(b_i)) & \text{if } b_i \in B_2 \backslash B_1 \text{ and } b_j \in B_1. \end{cases}$$

The following numerical example illustrates this construction.

Example 1. Consider a setting with two buyers. Buyer 1 has a value that is uniformly distributed over $[2, 4]$, while buyer 2's value is uniformly distributed over $[1, 2]$. The seller wants to conduct a symmetric implementation of the optimal auction. In this setting, the virtual value of buyer 1 is $\phi_1(v_1) = 2v_1 - 4$, and the virtual value of buyer 2 is $\phi_2(v_2) = 2v_2 - 2$. Therefore, buyer 1's virtual value (bid) is uniformly distributed over $B_1 \equiv [0, 4]$, while buyer 2's is uniformly distributed over $B_2 \equiv [0, 2]$.

We begin by deriving the interim payments. These can be determined using (Payoff Equivalence) as follows:

$$\begin{aligned} p_1^*(v_1) &= v_1 a_1^*(v_1) - \int_2^{v_1} a_1^*(w)dw = v_1 \min\{v_1 - 2, 1\} - \int_2^{v_1} \min\{w - 2, 1\}dw \\ &= \begin{cases} \frac{v_1^2}{2} - 2 & \text{for } v_1 \in [2, 3] \\ \frac{5}{2} & \text{for } v_1 \in (3, 4] \end{cases} \end{aligned}$$

and

$$\begin{aligned} p_2^*(v_2) &= v_2 a_2^*(v_2) - \int_1^{v_2} a_2^h(w)dw \\ &= v_2 \left[\frac{v_2 - 1}{2}\right] - \int_1^{v_2} \left[\frac{w - 1}{2}\right] dw = \frac{v_2^2 - 1}{4} \qquad \text{for } v_2 \in [1, 2]. \end{aligned}$$

Interim payments expressed in terms of bids are

$$p_1^*\left(\phi_1^{-1}(b_1)\right) = \begin{cases} \frac{b_1^2}{8} + b_1 & \text{for } b_1 \in [0, 2], \\ \frac{5}{2} & \text{for } b_1 \in (2, 4], \end{cases} \tag{6}$$

$$p_2^*\left(\phi_1^{-1}(b_2)\right) = \frac{b_2^2}{16} + \frac{b_2}{4} \qquad \text{for } b_2 \in [0, 2]. \tag{7}$$

Consider now $\hat{b} = 1$, for which we have that $G_1(\hat{b}) = \frac{1}{4}$ and $G_2(\hat{b}) = \frac{1}{2}$. This choice of $\hat{b}$ yields

$$p^u(b_i) = -\frac{b_i}{2} \qquad \text{and} \qquad p^l(b_i) = \frac{5b_i}{2} + \frac{b_i^2}{4},$$

from which we can define the symmetric payment rule for equilibrium bids:

$$p^s(b_i, b_j) = \begin{cases} -\frac{b_i}{2} & \text{if } b_i \in [0, 2] \text{ and } b_j \in [1, 4], \\ \frac{5b_i}{2} + \frac{b_i^2}{4} & \text{if } b_i \in [0, 2] \text{ and } b_j \in [0, 1), \\ \frac{5}{2} & \text{if } b_i \in (2, 4] \text{ and } b_j \in [0, 2]. \end{cases}$$

Our main result in the next section builds on the intuition in this example. The key difficulty in a symmetric implementation is that the same bid, when made by different bidders, must lead to the appropriate, potentially different interim payments. For this to be the case, the payment rule needs to be designed in a way that utilizes the difference in the distribution of the equilibrium bids of each bidder. In this example, we simply had to charge different amounts depending on whether the opponent's bid was above or below $\hat{b}$. The proof of the main result contains the substantially harder generalization of this construction to n bidders.

4. CHARACTERIZATION OF MECHANISMS WITH SYMMETRIC IMPLEMENTATIONS

In this section, we present and discuss the main result— a characterization of implementable IC and IR direct mechanisms. A constructive approach to determining whether a particular direct mechanism is implementable would require first the design of a symmetric auction and then a derivation of its equilibrium. However, deriving equilibria for a given symmetric auction can be a hard task. For instance, it is well known that it is difficult to obtain closed-form solutions for equilibrium bids in the first-price auction for arbitrary distributions. We show that the set of implementable mechanisms is a subset of the set of hierarchical mechanisms. This simplifies our task.

We begin by defining *hierarchical allocation rules*.[12] These are generated by an ordered list $I = (I_1, \ldots, I_n)$ of *index functions* that are nondecreasing mappings $I_i : V_i \to \mathbb{R}$ for $i \in N$. A hierarchical allocation rule is generated from a given list of index functions I as follows

$$a_i^h(\mathbf{v}) = \begin{cases} \frac{1}{\#\{j \in N : I_j(v_j) = I_i(v_i)\}} & \text{when } I_i(v_i) \geq \max\{I_{-i}(\mathbf{v}_{-i}), 0\}, \\ 0 & \text{otherwise.} \end{cases} \qquad \text{(Hierarchical Allocation)}$$

Each buyer's value is transformed into an index via the index function. The good is then allocated to the buyer with the highest positive index, and ties are broken equally. Restricting allocations to buyers with positive indices is essentially equivalent to setting a reservation bid. Choosing a reserve of 0 for the index functions is without loss of generality, as all bids can always be moved up or down by a constant. In addition, note that index functions can be chosen so that allocations occur above different reservation values across the buyers.

A *hierarchical mechanism* (I, p^h) is an IC and IR mechanism that consists of index functions I and payment functions p^h. The allocation a^h is determined as shown above from the index functions. For the results that follow, we find it convenient to denote a hierarchical mechanism in terms of the index functions I as opposed to the allocation rule a^h. If two lists of index functions I and I' generate the same allocation rule a^h, then it must be that one is a monotone transformation of the other. Formally, if I and I' generate the same allocation a^h, then there exists a monotone function $\Gamma : \mathbb{R} \to \mathbb{R}$ such that $I_i(v_i) = \Gamma(I_i'(v_i))$ for all i and v_i. The particular choice of index functions that correspond to a given allocation a^h does not matter for the statement of any of our results.

Since the index functions are nondecreasing, having a higher value implies a weakly higher probability of winning. This implies that every hierarchical allocation rule a^h has associated IC transfers p^h (pinned down to constants) that yield a hierarchical mechanism. All mechanisms in applied mechanism design that we are aware of fall within the class of hierarchical mechanisms (we provide examples of nonhierarchical mechanisms below). In the efficient Vickrey auction,

[12]This term was introduced by Border (1991).

values serve as indices or $I_i(v_i) = v_i$, and in the optimal auction (with increasing virtual values) the indices are given by the virtual values or $I_i(v_i) = \phi_i(v_i)$. When the virtual values are not increasing, the index functions are simply the "ironed" virtual value functions (Myerson, 1981). Alternatively, suppose an auctioneer with affirmative action concerns wants to "subsidize" a historically disadvantaged bidder i over a bidder j where the latter has index $I_j(v_j) = v_j$. The index for bidder i could reflect either a flat subsidy $I_i(v_i) = v_i + s$ (where $s > 0$) or a percentage subsidy $I_i(v_i) = s\, v_i$ (where $s > 1$).

We show below that any implementable mechanism must effectively be a hierarchical mechanism. This allows us to focus on this smaller class of mechanisms, which in turn simplifies the implementation task, as in the previous section. Since the allocation rule of a symmetric auction that implements a hierarchical mechanism must allocate the good to the bidder with the highest index, a natural assumption is to make equilibrium bids correspond to the index values. Then, constructing the symmetric implementation essentially boils down to finding a symmetric payment rule that yields the same interim payments. Given a hierarchical mechanism (I, p^h), the distribution F_i on the set of values V_i induces a distribution G_i on the set of indices or bids

$$B_i \equiv \{I_i(v_i) \mid v_i \in V_i\}. \tag{Bid Space}$$

At times, we will slightly abuse notation and use G_i as both a distribution and a measure. The meaning will be clear depending on whether the argument of G_i is a real number or a set. The notation G_i deliberately suppresses the dependence on the index function I_i; the meaning will always be clear from the context. Since index functions I are not necessarily strictly increasing, the induced distributions G_i may have atoms. Additionally, notice that the set B_i need not be an interval because the index functions I may be discontinuous.

A hierarchical allocation mechanism (I, p^h) can be implemented if we can find a symmetric payment function p^s such that

$$p_i^h(v_i) = \int_{\mathbf{B}_{-i}} p^s\left(I_i(v_i), \mathbf{b}_{-i}\right) dG_{-i}(\mathbf{b}_{-i}) \qquad \text{for all } i \in N \text{ and } v_i \in V_i. \tag{$\star$}$$

If such a symmetric payment function exists, it follows that an equilibrium of the symmetric auction with this payment rule will involve each buyer i with value v_i bidding their index $I_i(v_i)$. By construction, such bids generate the required allocation.

The intuition is straightforward and identical to that of the example. Suppose that a bidder with value v_i makes a bid $b_i' \in B_i$ other than her index so $b_i' \neq I_i(v_i)$. Her corresponding allocation and payment would be identical to what she would get by reporting a value $v_i' \in I_i^{-1}(b_i')$, resulting in lower utility as the direct mechanism (I, p^h) is IC.[13] Off-equilibrium bids $b_i' \notin B_i$, which lie outside the bid space, can be punished by requiring high expected payments at these bids.

We now present our main result as two separate theorems.

Theorem 1. *Suppose that a direct revelation mechanism (a^d, p^d) is implementable. Then, there exists an implementable hierarchical mechanism (I, p^h) such that its implementation is an almost sure implementation of (a^d, p^d).*

[13]Here, $I_i^{-1}(\cdot)$ is the correspondence defined by $I_i^{-1}(b_i) = \{v_i \in V_i \mid I_i(v_i) = b_i\}$.

Theorem 1 says that it is essentially without loss to restrict attention to hierarchical mechanisms. It states that, for any implementable direct mechanism, there is an implementable hierarchical mechanism that almost surely has exactly the same allocation and payments. An implication is that, for any implementation of any nontrivial objective, a principal can restrict attention to hierarchical mechanisms. This result is intuitive. Clearly, a nonhierarchical mechanism cannot have an implementation in pure strategies because, if so, the allocation rule could have been generated by an index rule with indices equal to the equilibrium bids in the symmetric auction. The appendix contains the argument for mixed strategies.

Theorem 2 provides conditions that characterize the set of implementable hierarchical mechanisms.[14]

Theorem 2. *A hierarchical mechanism (I, p^h) is implementable if and only if, for any pair of distinct buyers $i, j \in N$ who have the same distribution of bids ($G_i = G_j$), and any pair of values for these two buyers $v_i \in V_i$, $v_j \in V_j$ satisfying $I_i(v_i) = I_j(v_j)$, we have that $p^h_i(v_i) = p^h_j(v_j)$.*

We can decompose this into two parts:

(1) Whenever bid distributions G_i differ across the buyers, the condition of the theorem is vacuously satisfied, and therefore, it is possible to construct a payment rule so that ($\star$) is satisfied. When there are two bidders, a payment rule like the one in the previous section can be used to construct the implementation. The construction for more that two bidders is considerably more complicated and can be found in the appendix.
(2) When two buyers are such that the two induced bid distributions are the same, that is $G_i = G_j$, then the interim payments must be the same for any two values (one for each buyer) that correspond to the same index. This is because it is no longer possible to generate different equilibrium expected payments for distinct buyers who make the same bid.

We now present two examples of hierarchical mechanisms that cannot be implemented, thus showing that the theorem is not vacuously true. In the first example, the good is allocated randomly and in the second, the seller would like to subsidize one of the buyers.

Example 2. There are two buyers. Buyer 1 has a value uniformly distributed on $[0, 1]$. Buyer 2 has a value uniformly distributed on $[0.5, 1]$. The seller assigns the good at random (with equal probability) to each of the two buyers irrespective of their value. Buyer 1 is never asked to pay anything, whereas buyer 2 is always asked to pay 0.25.

Notice that this mechanism is a hierarchical mechanism where each bidder's index function is a constant nonnegative function or $I_1(v_1) = I_2(v_2) \geq 0$ for all $v_1 \in [0, 1]$ and $v_2 \in [.5, 1]$. Here, the bid space just consists of a single point, and distributions G_1, G_2 are degenerate and therefore satisfy $G_1 = G_2$. However, the payments differ. Therefore this mechanism violates the conditions of Theorem 2. It follows that there is no symmetric implementation of this direct revelation mechanism.

[14]The theorem is actually slightly stronger. The conditions are also necessary and sufficient for (the weaker criterion of) interim implementability of a hierarchical mechanism.

Example 3. Consider an environment where there are two buyers. Buyer 1 has a value v_1 that is uniformly distributed on $[0,1]$. Buyer 2 has a value v_2 which is uniformly distributed on $[1,2]$.

Suppose that the seller would like to "subsidize" the bid of buyer 1 by a dollar. Put differently, buyer 2 wins the good if and only if his value exceeds that of buyer 1 by 1. Therefore, for any $v_1 \in [0,1]$, the interim allocation probabilities are given by

$$a_1^h(v_1) = a_2^h(1+v_1).$$

The IC and IR payments are chosen to be such that the lowest type of both buyers for whom there is no probability of winning neither make payments nor are paid. This is clearly a hierarchical mechanism with index functions $I_1(v_1) = I_2(v_1+1)$, where $I_1(\cdot)$ is strictly increasing on the interval $[0,1]$.

Observe that this implies that the distributions over the bid spaces are identical, since G_1 and G_2 are both $\mathbb{U}[0,1]$. Moreover, incentive compatibility pins down payments, and therefore we have

$$p_2^h(v_1+1) = p_1^h(v_1) + a_1^h(v_1).$$

For all values $v_1 \in (0,1]$, therefore, the above equation implies that

$$p_2^h(v_1+1) \neq p_1^h(v_1).$$

Since the interim payments differ for values that have the same index and the bid spaces have identical distributions, symmetric payments cannot be constructed to implement this mechanism.

However, note that this mechanism could have been implemented if buyer 2's value distribution was anything even slightly different that $\mathbb{U}[1,2]$, as this would imply that $G_1 \neq G_2$.

The conditions in Theorem 2 were on the distributions of the bid space. The following corollary qualitatively describes the types of hierarchical allocation rules that cannot be implemented.

Corollary 1. *Suppose that a hierarchical mechanism (I, p^h) is not implementable. Then there must exist two distinct buyers j and j' such that their index functions can be written as*

$$\text{for } i = j, j' : \ I_i(v_i) = \Gamma(F_i(v_i)) \text{ for almost every } v_i \in V_i,$$

for some non-decreasing function $\Gamma(\cdot)$.

The above corollary demonstrates that the only nonimplementable hierarchical mechanisms are ones where there are two buyers whose indices corresponding to a value depend solely on the "statistical rank" of that value in the distribution of that buyer's values. This is a very specific and small subset of hierarchical mechanisms; in fact the set of implementable mechanisms is generic in a topological sense, formalized below.

For each buyer i, the distribution F_i defines a measure space on V_i. Consider the space of index functions for buyer i, as an L_p space where $1 \leq p \leq \infty$. The space of index functions $I = (I_1, \ldots, I_n)$ is topologized with the product topology and denoted $\mathcal{I}$. Since a finite product of complete normed vector spaces is a Baire space, standard topological notions of genericity are well defined. Recall that a property is said to be *generically satisfied* on a topological space if the set that does not satisfy it is a meager set (or conversely, the set that does satisfy it is a residual set). Further, recall that a set in a topological space is meagre if it can be expressed as the union of countably many nowhere dense subsets in that space.

Corollary 2. *Generically, on $\mathcal{I}$, $G_i \neq G_j$ for every pair of buyers i and j.*[15]

The intuition and proof for this result are straightforward: two index functions that result in the same distribution over bids can be made different by slightly perturbing them.

The fact that a large number of disparate objectives can be achieved either exactly or arbitrarily closely via a symmetric implementation is one of the main insights of this paper, and it is worth repeating its two main implications. The first is that symmetry need not imply fairness—just because an auction treats the bids of different buyers similarly, this does not imply that the resulting outcomes are equal from an ex-ante perspective. The second is that careful auction design can allow the mechanism designer to achieve a wide variety of goals in environments where explicit favoritism is impractical or prohibited. For instance, the auction designer can choose formats which favor weaker bidders without explicitly biasing the mechanism. This can be helpful for governments striving to reach distributional goals (favoring small businesses, minorities, etc.) without facing legal challenges over favoritism policies. Alternatively, this can be useful to encourage competition (and thereby enhance revenue) amongst asymmetric bidders in settings such as online auctions where the seller may have good knowledge about value distributions (from previous auctions conducted) but bids are placed anonymously. In fact, the following corollary points out that the revenue-maximizing auction can always be implemented. Since this seems to counter prevailing intuition, we feel that this is perhaps one of the most surprising results of the paper.[16]

Corollary 3. *The optimal auction can be implemented symmetrically.*

It is worth reiterating that the above corollary requires no hazard rate assumptions on the value distributions. When the distributions satisfy the increasing virtual value property, it is easy to show that if the bidders are asymmetric, the distribution over virtual values must also be different. When the virtual values are not increasing then the proof of the Corollary shows that if the distributions over the 'ironed' virtual values are the same then the condition of the Theorem must be satisfied.

We end this section by pointing out that, if the implementation criterion is weakened, the principal can achieve the outcomes corresponding to certain nonhierarchical mechanisms using randomization. The principal can randomize by choosing amongst a set of mechanisms via a lottery. After choosing one such mechanism from the set, the principal can announce it to the buyer. For instance, the principal could toss a coin and choose between a first- and second-price auction.

[15] It is possible to restate this corollary to say instead that implementability is a generic property in the space of hierarchical mechanisms (instead of in the space of index functions that do not include payments). We have deliberately chosen not to do so to avoid the distracting technicalities inherent in defining the appropriate topology on the space of hierarchical mechanisms. The complications arise from the fact that the index functions restrict the payments (up to constants) via incentive compatibility, so we cannot simply employ a product topology over index functions and payments.

[16] For instance, in an influential paper, Cantillon (2008) conjectured that bidder asymmetries hurt the auctioneer in any anonymous mechanism after showing that this is not the case in the optimal auction. Corollary 3 answers this conjecture in the negative by showing that the optimal auction can be implemented by an anonymous mechanism.

Having chosen, the buyer is informed of the auction format and the game proceeds. Such randomization is appropriate for a principal concerned about expected outcomes (as in Manelli and Vincent, 2010; Gershkov, Goeree, Kushnir, Moldovanu, and Shi, 2013).

Randomization can be useful in achieving the outcomes of both unimplementable hierarchical allocation mechanisms and nonhierarchical mechanisms. A simple two-buyer example of a nonhierarchical mechanism is one where irrespective of the values, buyer 1 gets the good 25% of the time and buyer 2 gets it 75%. Clearly, this is not a hierarchical allocation since our definition of the latter requires the equal breaking of ties. Another example is a mechanism in which the seller randomly allocates the good 50% of the time and runs a second price auction the remaining 50%.

A mechanism (a^d, p^d) is defined to be a *randomization* over a set of mechanisms $\mathscr{M}$, if there is a measure ζ defined on $\mathscr{M}$ such that

$$a_i^d(v_i) = \int_{\mathscr{M}} a_i(v_i) d\zeta((a,p)) \qquad \text{and} \qquad p_i^d(v_i) = \int_{\mathscr{M}} p_i(v_i) d\zeta((a,p)).$$

The lemma below shows that all IC and IR direct mechanisms can be obtained as a randomization over hierarchical mechanisms. This lemma follows from results in Border (1991) and Mierendorff (2011).

Lemma 1. *Every IC and IR direct mechanism is a randomization over the set of hierarchical mechanisms.*

Clearly, the outcome from any mechanism that is a randomization over implementable hierarchical mechanisms can be achieved in such an ex-ante sense. The auctioneer can just randomly choose (using measure ζ) from the symmetric auctions that correspond to the implementable hierarchical mechanisms. Note that, strictly speaking, this is not interim implementation as we defined it. However, for practical applications, it serves the same purpose, as randomization is done before the chosen symmetric auction is announced to the buyers. The next corollary summarizes this discussion, and in it, we use the terminology *outcomes are achievable* to clarify the distinction from interim implementation.

Corollary 4. *The outcomes from an IC and IR direct mechanism are achievable if it is a randomization over implementable hierarchical mechanisms.*

Finally, we discuss the two examples of the unimplementable mechanisms and examine whether their outcomes can be achieved via randomization.

Example 2. (Continued) Recall that, in this example, the seller assigns the good at random (with equal probability), buyer 1 is never asked to pay anything, while buyer 2 is always asked to pay 0.25. The outcome from the mechanism can be achieved by randomizing with equal probability over two implementable hierarchical mechanisms. In the first hierarchical mechanism, buyer 1 is always awarded the good irrespective of value and is not asked to pay anything. In the second hierarchical mechanism, buyer 2 is always given the good irrespective of value and is asked to pay 0.5.

Example 3. (Continued) Recall that, in this example, buyer 2 wins the good if and only if her value exceeds that of buyer 1 by 1. The outcome of this mechanism cannot be achieved using randomization.

Consider the index function $I_1(v) = I_2(v+1) = v$. By observation, the allocation rule a^h corresponding to these index functions is the unique (almost everywhere) maximizer of

$$\int_{\mathbf{V}} \left(\sum_{j \in \{1,2\}} a^d_j(v_j, \mathbf{v}_{-j}) I_j(v_j) f_j(v_j) \right) d\mathbf{v},$$

amongst all IC direct allocations a^d.

Therefore, for any hierarchical allocation rule $\tilde{a}^h \neq a^h$ that differs from a^h at a positive measure subset of values, it must be that

$$\int_{\mathbf{V}} \left(\sum_{j \in \{1,2\}} \tilde{a}^h_j(v_j, \mathbf{v}_{-j}) I_j(v_j) f_j(v_j) \right) d\mathbf{v} < \int_{\mathbf{V}} \left(\sum_{j \in \{1,2\}} a^h_j(v_j, \mathbf{v}_{-j}) I_j(v_j) f_j(v_j) \right) d\mathbf{v}.$$

Moreover, any allocation rule that is equal to a^h almost everywhere is not implementable. Therefore, a^h is not a randomization over implementable hierarchical allocations, so its outcome is not achievable.

5. Additional Desiderata

Our main result from the previous section showed that a large class of mechanisms can be implemented symmetrically. However, symmetry is just one desideratum of a practical implementation. In this section, we consider a number of additional properties that one might want in an auction implementation and discuss how these properties, along with symmetry, restrict the set of implementable outcomes. Essentially, the goal is to identify some properties in first- and second-price auctions which can be construed to be desirable and impose them as additional restrictions on a symmetric auction implementation. Throughout the section, we consider the case of two bidders ($n = 2$), primarily for the sake of brevity and tractability.[17] A key takeaway from this section is that the optimal auction is no longer always implementable under these additional requirements.

5.1. *Inactive Losers*

An important property of first- and second-price auctions is that losers neither make nor receive payments. With the notable exception of charity auctions (see, for instance Goeree, Maasland, Onderstal, and Turner, 2005), most auctions conducted in the real world have this feature. It is often argued that requiring the loser to pay reduces participation, which is one of the reasons that all-pay auctions are seldom used in practice. Hence, this might be construed as a shortcoming of Theorem 2: the symmetric implementation that we construct there may require both the winner and the losers to make payments.

A hierarchical mechanism has a symmetric, *inactive losers implementation* (a^s, p^s) if $p^s(b_i, b_j) = 0$ whenever $b_i < b_j$. Note that such an implementation may sometimes require the winner to make payments that are greater than his value and thus may not be ex-post IR (we consider the ex-post

[17] Barring Propositions 4 and 5, we can extend the results in this section to more than two bidders.

IR requirement later in Section 5.4). We now state a condition that is necessary and sufficient for there to exist such an implementation.

Inactive Losers Condition: Consider a hierarchical mechanism (I, p^h) that induces distributions G_1 and G_2 on the set of bids. This mechanism satisfies the inactive losers condition if, for all $\tilde{b}$ such that there is a constant $\alpha > 0$ for which $G_1(b) = \alpha G_2(b)$ for all $b \leq \tilde{b}$, we have $\alpha p_1^h(v_1) = p_2^h(v_2)$ for all $v_i \in I_i^{-1}(\tilde{b})$.

The necessity of this condition for an inactive losers implementation is intuitive. Consider a bid $\tilde{b}$ for which $G_1(b) = \alpha G_2(b)$ for all $b \leq \tilde{b}$. For any $v_i \in I_i^{-1}(\tilde{b})$, the interim payments for any inactive losers implementation must satisfy

$$p_2^h(v_2) = \int_{b_1 \leq \tilde{b}} p^s(\tilde{b}, b_1) dG_1(b_1) = \int_{b_2 \leq \tilde{b}} p^s(\tilde{b}, b_2) \alpha dG_2(b_2) = \alpha p_1^h(v_1).$$

The next proposition argues that this condition is also sufficient. The sufficiency follows from a similar construction to that utilized in Theorem 2.

Proposition 1. *Suppose that $n = 2$. An implementable hierarchical mechanism (I, p^h) has an inactive losers implementation if and only if the induced bid distributions G_1 and G_2 satisfy the inactive losers condition.*

We should point out that the inactive losers condition is generically satisfied in the sense of Corollary 2. Intuitively, this is because any mechanism that does not satisfy it can be converted to one that does by slightly perturbing it at the lower bound of the support of the bid distribution. Put differently, almost all hierarchical mechanisms have an inactive losers implementations. That said, we now revisit Example 1 to show that the inactive losers condition is not vacuous and can be violated even in the optimal auction.

Example 1. (Continued) Recall that buyer 1 has a value that is uniformly distributed over $[2, 4]$, while buyer 2's value is uniformly distributed over $[1, 2]$ and the seller wants to maximize revenue. The distributions of virtual values are $G_1 \sim \mathbb{U}[0, 4]$ and $G_2 \sim \mathbb{U}[0, 2]$ respectively. Therefore, for any $\tilde{b} \in [0, 2]$, $G_1(b) = .5G_2(b)$ for all $b \leq \tilde{b}$.

Now consider $\tilde{b} = 2$. From (6) and (7), the interim payments at this bid are $p_1^*\left(\phi_1^{-1}(2)\right) = \frac{5}{2}$ and $p_2^*\left(\phi_2^{-1}(2)\right) = \frac{3}{4}$. Note that the payment of buyer 1 is not twice that of buyer 2, so the inactive losers condition is not satisfied.

5.2. *Continuity*

The basic construction we used in the example of Section 3 consisted of discontinuous payment rules where a buyer i's payment discontinuously changed depending on whether their opponent bid above or below the cutoff bid $\hat{b}$. Of course, conditional on winning, the payments in first- and second-price auctions are continuous in the profile of bids (since losers do not pay, payments in these auctions are not continuous unconditionally). We now examine the effect that the additional requirement of continuity has on the set of implementable mechanisms.

A hierarchical mechanism has a symmetric, *continuous implementation* (a^s, p^s) if $p^s(b_i, b_j)$ is continuous in both b_i, b_j.

We show that the existence of a continuous implementation is equivalent to there being no *non-trivial atoms* in the hierarchical mechanism. A non-trivial atom is one in which there is a positive measure of values of buyer i which have the same index b, and this index also lies in the support of the bid space of buyer j. Formally, a nontrivial atom exists if, for a buyer i, there are two distinct values $v_i, v_i' \in V_i$ such that $I_i(v_i) = I_i(v_i') = b$ and $b \in B_j$. Such atoms can occur in natural applications such as the optimal auction when the buyers' value distributions do not satisfy the monotone hazard rate condition.

The absence of nontrivial atoms is a necessary condition for a continuous implementation. To see this, note that, in any symmetric implementation, at such an atom, it must be the case that $\sigma_i(v_i) = \sigma_i(v_i') = \sigma_j(v_j) = b$ for all $v_j \in I_j^{-1}(b)$. In other words, this says that, in any implementation, it must be that all types at the nontrivial atom make the same bid. However, if the payment p^s is continuous, buyer j has an incentive to bid slightly higher than b. Bidding slightly higher would lead to a continuous increase in payment but a discontinuous increase in the probability of winning, so $\sigma_j(v_j)$ is not a best response for v_j. The result below shows that this is the only additional condition required for the existence of a continuous implementation.

Proposition 2. *Suppose that $n = 2$. An implementable hierarchical mechanism (I, p^h) has a continuous implementation if and only if it has no non-trivial atoms.*

The intuition for this result can be easily seen by revisiting the example of the regular optimal auction of Section 3. When the virtual values are increasing, the resulting optimal auction does not have non-trivial atoms. The payment rule we constructed was discontinuous in the opponent's bid b_j but can easily be smoothed around the point of the discontinuity while ensuring that the interim payments remain the same. The simplest way to do this is linearly, which is illustrated below in Figure 1. Additionally, when the hierarchical mechanism has no non-trivial atoms, it is

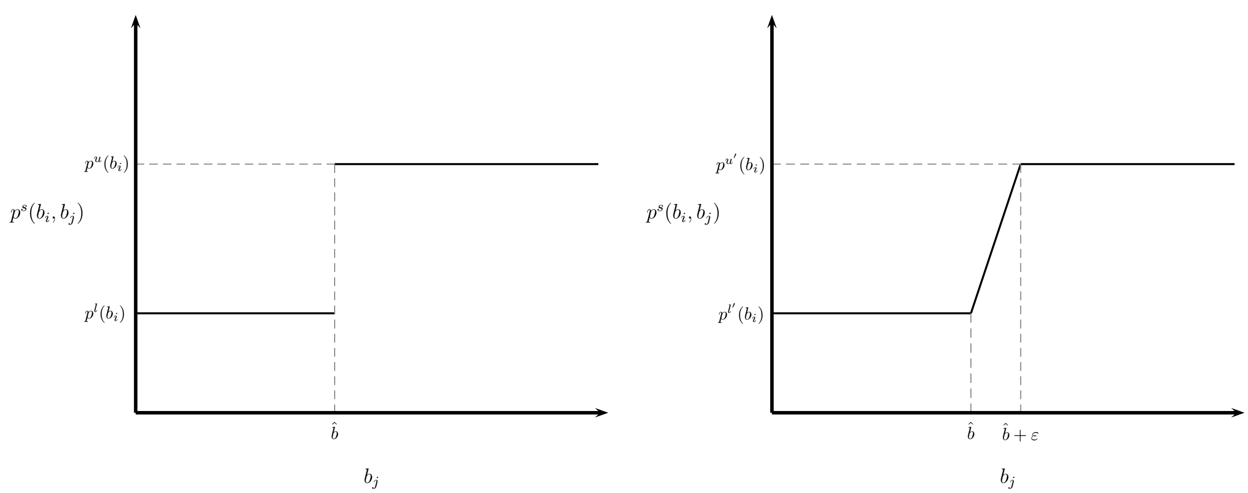

FIGURE 1. Continuous Symmetric Implementation

also possible to achieve continuity in b_i. The constructive proof in the appendix demonstrates this.

5.3. *Monotonicity*

Incentive compatibility implies that a buyer's interim payments in any symmetric auction must be nondecreasing in his value. However, the payment rule need not be monotone in an ex-post sense. For instance, in the payment (2) we constructed for the example in Section 3, we make neither the restriction that $p^u(b_i) \geq p^l(b_i)$ nor that p^u and p^l are increasing in b_i. In other words, we so far have not restricted ex-post payments from our symmetric implementations to be monotone in either in a buyer's bid or their opponent's bid. Of course, conditional on winning, the payment rules for both first- and second- price auctions are monotone in such an ex-post sense.

We first examine the effect of imposing monotonicity in the opponent's bid, a property of second-price auctions. We say a hierarchical mechanism has a symmetric, *monotone in opponent's bid implementation* (a^s, p^s) if $p^s(b_i, b_j)$ is nondecreasing in b_j. The next result provides necessary and sufficient conditions for such an implementation to exist.

Proposition 3. *Suppose that $n = 2$. An implementable hierarchical allocation mechanism (I, p^h) has a monotone in opponent's bid implementation if and only if whenever G_i first order stochastically dominates G_j, it is the case that $p_i(I_i^{-1}(b)) \leq p_j(I_j^{-1}(b))$ for all $b \in B_1 \cap B_2$.*

Intuition for the sufficiency of the above conditions can be understood by examining payments p^u and p^l in the example of Section 3. For this particular construction, monotonicity in the opponent's bid requires that $p^u(b_i) \geq p^l(b_i)$ for all b_i. From (3), this happens if and only if

$$\frac{1}{G_1(\hat{b}) - G_2(\hat{b})}\left(p_1^*\left(I_1^{-1}(b_i)\right) - p_2^*\left(I_2^{-1}(b_i)\right)\right) \geq 0.$$

The sufficiency of our condition is immediate and its necessity is easily established in the proof in the appendix. Observe that if neither distribution first order stochastically dominates the other, there must exist pivot bids $\hat{b}$ and $\hat{b}'$ such that $G_1(\hat{b}) > G_2(\hat{b})$ and $G_1(\hat{b}') < G_2(\hat{b}')$. For bids at which $p_1^*(I_1^{-1}(b_i)) > p_2^*(I_2^{-1}(b_i))$, we can construct the payments p^u and p^l in (3) by pivoting around $\hat{b}$ and similarly we can pivot around $\hat{b}'$ when $p_1^*(I_1^{-1}(b_i)) < p_2^*(I_2^{-1}(b_i))$. This payment rule would satisfy the above inequality and would hence be monotone in the opponent's bid. Note that the condition is not satisfied generically. Intuitively, this is because it is possible to perturb a hierarchical mechanism that violates it to get another mechanism that continues to violate this condition.

Below, we show that Example 1 violates fails this condition. Additionally, we modify Example 3 slightly to show that there are cases where one distribution first order stochastically dominates the other, but the condition is satisfied.

Example 1. (Continued) In this example, $G_1 \sim \mathbb{U}[0, 4]$ first order stochastically dominates $G_2 \sim \mathbb{U}[0, 2]$. However, for bid $b = 1$ the payments (given by equations 6, 7) are

$$p_1^*(\phi_1^{-1}(1)) = \frac{9}{8} > \frac{5}{16} = p_2^*(\phi_2^{-1}(1)),$$

violating the condition of the proposition.

Example 3. (Continued) Recall that, in this example, buyer 1's value distribution $F_1 \sim \mathbb{U}[0, 1]$ and buyer 2's value distribution $F_2 \sim \mathbb{U}[1, 2]$. Now, unlike previously, suppose that the seller subsidizes the bid of buyer 1 by one and a half dollars (instead of one dollar). In this case, $G_1 \sim \mathbb{U}[1.5, 2.5]$ strictly first order stochastically dominates $G_2 \sim \mathbb{U}[1, 2]$, so this mechanism is implementable. Additionally, it is easy to show that $p_1(I_1^{-1}(b)) \leq p_2(I_2^{-1}(b))$ for all bids b. Therefore, there is a monotone in opponent's bid implementation for this mechanism.

We now examine the effect of requiring monotonicity in a buyer's own bid. We say a hierarchical mechanism has a symmetric, *monotone in own bid implementation* (a^s, p^s) if $p^s(b_i, b_j)$ is nondecreasing in b_i. This requirement restricts the relative rates at which the interim payments of both buyers can increase in their bids for any implementable mechanism. Put differently, if one buyer's payment increases very rapidly, then this monotonicity requirement will place a lower bound on the rate at which the other buyer's payment must increase.

For simplicity, the characterization restricts attention to hierarchical mechanisms with strictly increasing and differentiable index functions—this ensures that the implied distribution over bids for any buyer has a density. Moreover, the characterization involves slightly different necessary and sufficient conditions, as we have been unable to derive a single characterizing condition. The sufficient condition involves the slopes $\frac{dp_i^h(I_i^{-1}(b_i))}{db_i}$ of the payments on the common part of the supports of the bid spaces $B_1 \cap B_2$. Since $B_1 \cap B_2$ is a closed interval, these derivatives refer to the left (right) derivative at the upper (lower) bound of the support.

Proposition 4. *Suppose that $n = 2$. Consider an implementable hierarchical mechanism (I, p^h) with differentiable and strictly increasing index functions. (I, p^h) has a monotone in own bid implementation if, for all distinct $i, j \in \{1, 2\}$ and for all $b \in B_1 \cap B_2$ for which $\frac{dp_i^h(I_i^{-1}(b))}{db} > 0$, we have*

$$\inf\left\{\frac{g_i(\tilde{b})}{g_j(\tilde{b})} : \begin{array}{l}\tilde{b} \in B_1 \cup B_2, \\ g_i(\tilde{b}) + g_j(\tilde{b}) > 0\end{array}\right\} \leq \frac{\frac{dp_j^h(I_j^{-1}(b))}{db}}{\frac{dp_i^h(I_i^{-1}(b))}{db}} \leq \sup\left\{\frac{g_i(\tilde{b})}{g_j(\tilde{b})} : \begin{array}{l}\tilde{b} \in B_1 \cup B_2, \\ g_i(\tilde{b}) + g_j(\tilde{b}) > 0\end{array}\right\}, \tag{8}$$

with a strict lower (upper) inequality unless the corresponding infimum (supremum) is reached on a set of bids with positive G_j (G_i) mass.

Conversely, (I, p^h) has a monotone in own bid implementation only if, for all distinct $b, b' \in B_1 \cap B_2$, we have

$$\inf\left\{\frac{g_i(\tilde{b})}{g_j(\tilde{b})} : \begin{array}{l}\tilde{b} \in B_1 \cup B_2, \\ g_i(\tilde{b}) + g_j(\tilde{b}) > 0\end{array}\right\} \leq \frac{p_j^h(I_j^{-1}(b')) - p_j^h(I_j^{-1}(b))}{p_i^h(I_i^{-1}(b')) - p_i^h(I_i^{-1}(b))} \leq \sup\left\{\frac{g_i(\tilde{b})}{g_j(\tilde{b})} : \begin{array}{l}\tilde{b} \in B_1 \cup B_2, \\ g_i(\tilde{b}) + g_j(\tilde{b}) > 0\end{array}\right\}, \tag{9}$$

with a strict lower (upper) inequality unless the corresponding infimum (supremum) is reached on a set of bids with positive G_j (G_i) mass.

The astute reader might observe that, on the surface, it seems that the necessary condition is stronger than the sufficient condition in the above proposition (divide the numerator and denominator of the central term of (9) by $b' - b$ and take the limit $b' \to b$ to get the same central term of (8)). However, consider a case where neither the infinimum nor supremum is achieved on a set of positive mass. Further, suppose both inequalities in (9) are satisfied strictly for every pair b', b. It may still be the case that for some b, one of the inequalities in (8) may be satisfied only as an equality. In this case, the sufficient condition will be violated, while the necessary condition is satisfied. This discussion also demonstrates that the gap between these conditions is (loosely speaking) quite small.

Note that the above conditions are always satisfied whenever neither bid space B_1 or B_2 is a subset of the other. In that case, for distinct $i, j \in \{1, 2\}$, we get

$$\inf \left\{ \frac{g_i(\tilde{b})}{g_j(\tilde{b})} : \begin{array}{l} \tilde{b} \in B_1 \cup B_2, \\ g_i(\tilde{b}) + g_j(\tilde{b}) > 0 \end{array} \right\} = 0 \text{ and } \sup \left\{ \frac{g_i(\tilde{b})}{g_j(\tilde{b})} : \begin{array}{l} \tilde{b} \in B_1 \cup B_2, \\ g_i(\tilde{b}) + g_j(\tilde{b}) > 0 \end{array} \right\} = \infty,$$

because there are bids $b_i \in B_i$, $b_j \in B_j$ that do not lie in the intersection $b_i, b_j \notin B_i \cap B_j$. Note that, once again, the necessary condition is not generically satisfied (the intuition is identical to the monotone in opponent's bid case) and below, we show that it is, in particular, not satisfied by Example 1.

Example 1. (Continued) Since $G_1 \sim \mathbb{U}[0, 4]$ and $G_2 \sim \mathbb{U}[0, 2]$, we have

$$\inf \left\{ \frac{g_2(\tilde{b})}{g_1(\tilde{b})} : \begin{array}{l} \tilde{b} \in B_1 \cup B_2, \\ g_1(\tilde{b}) + g_2(\tilde{b}) > 0 \end{array} \right\} = 0 \text{ and } \sup \left\{ \frac{g_2(\tilde{b})}{g_1(\tilde{b})} : \begin{array}{l} \tilde{b} \in B_1 \cup B_2, \\ g_1(\tilde{b}) + g_2(\tilde{b}) > 0 \end{array} \right\} = 2.$$

For bids $b' = 1$ and $b = 0$, the necessary condition is violated because

$$\frac{p_1^*(\phi_1^{-1}(1)) - p_1^*(\phi_1^{-1}(0))}{p_2^*(\phi_2^{-1}(1)) - p_2^*(\phi_2^{-1}(0))} = \frac{9/8}{5/16} > 2.$$

5.4. *Ex-Post IR*

While the symmetric implementations we construct for Theorem 2 are by definition IR, they are IR in an interim sense. As we have argued above, the equilibrium however need not be IR in an ex-post sense: certain bid profiles may result in losing bidders having to make payments or winners having to pay more than their valuation. This is unappealing and may result in certain bidders choosing not to participate. Perhaps more importantly, this may result in non-payment by budget-constrained bidders. This is because a bidder's valuation may reflect her ability to pay for the good. Additionally, certain bidders who plan to pay by obtaining a loan may be unable to obtain credit upon losing the auction.[18]

Formally, we say that a hierarchical mechanism (I, p^h) has a symmetric, *ex-post IR implementation* (a^s, p^s) with associated equilibrium strategies σ, if for all $\mathbf{v} \in \mathbf{V}$ and $i \in N$, we have $p^s(\sigma_i(v_i), \sigma_{-i}(\mathbf{v}_{-i})) \le v_i a^s(\sigma_i(v_i), \sigma_{-i}(\mathbf{v}_{-i}))$.

This states that at any bid profile that occurs in equilibrium, winning buyers are never charged more than their value and losers do not have to make payments, although they may receive subsidies (which implies that losers may not be inactive). Notice that, when there are ties, the above inequality implies that buyers only have to pay in the event that they win.

The ex-post IR requirement places a bound on the payments that the symmetric auction can require buyers to make at both winning and losing bids. As in the case with inactive losers implementation, the optimal auction may not have an ex-post IR implementation. In fact, this can be demonstrated by once again revisiting Example 1.

[18]If we were to take the procurement interpretation of our model, the ex-post IR requirement would ensure that firms can cover their costs and complete the project.

Example 1. (Continued) Recall that buyer 1 with value $v_1 = 3$ has a virtual value of $\phi_1(3) = 2$, always wins the good, and pays $p_1^*(3) = \frac{5}{2}$. For there to be a symmetric ex-post IR implementation, there must exist a symmetric payment p^s such that

$$\int_0^2 p^s(2, b_2)dG_2(b_2) = \frac{5}{2}$$

which in turn implies that there must exist at least one $b \in [0, 2]$ such that

$$p^s(2, b) \geq \frac{5}{2}.$$

However, note that a buyer 2 with value $v_2 = 2$ also has the virtual value $\phi_2(2) = 2$. Since there is a $b \in [0, 2]$ such that $p^s(2, b) \geq \frac{5}{2}$, there will be a bid profile in the support of the equilibrium bids at which buyer 2 is paying more than her value. This violates the ex-post IR requirement.

We derive necessary and sufficient conditions for a hierarchical mechanism to admit a symmetric ex-post IR implementation. Due to the complexity of the characterization, we need to make two additional assumptions. As in Proposition 4, we first restrict attention to hierarchical mechanisms (I, p^h) in which the index functions I are differentiable and strictly increasing. Second, we further restrict attention to the case where the lower bounds of the supports of the bid space do not coincide, or $I_1(\underline{v}_1) \neq I_2(\underline{v}_2)$. The characterization for this case is easier to state. In the appendix, we present the characterization for allocation rules in which $I_1(\underline{v}_1) = I_2(\underline{v}_2)$.

Without loss of generality, we assume that bidder 1's bid space has the lower support, or

$$I_1(\underline{v}_1) = \underline{b}_1 < \underline{b}_2 = I_2(\underline{v}_2).$$

Additionally, we define $I_i(\overline{v}_i) = \overline{b}_i$ for $i \in \{1, 2\}$ and

$$v(b) \equiv \min\{I_1^{-1}(b), I_2^{-1}(b)\} \qquad \text{for } b \in B_1 \cap B_2,$$

as the lower of the values of the two buyers corresponding to a bid b that lies in both bid spaces. Recall that, since we have restricted attention to strictly increasing index functions, this inverse is well defined.

We can now state a simple first necessary condition that a hierarchical mechanism (I, p^h) must satisfy to have an ex-post IR implementation.

Condition C1: The distribution of values F_1 and F_2 induce distributions G_1 and G_2 such that

$$\forall b \in B_1 \cap B_2: \quad v(b)G_2(b) \geq p_1^h\left(I_1^{-1}(b)\right). \tag{C1}$$

This is an intuitive necessary condition. $v(b)$ is the maximum amount that can be charged to a winning buyer who bids $b \in B_1 \cap B_2$ and whose opponent bids $b' \in B_1 \cap B_2$, $b' \leq b$. Since the auction is symmetric, such a profile of bids will not reveal the identity of the winning bidder, so the ex-post IR requirement restricts the payment to be lower than both possible values of the winning bidder. Hence, bidder 1's interim payment $p_1^h\left(I_1^{-1}(b)\right)$ cannot be higher than $v(b)G_2(b)$ for any bid $b \in B_1 \cap B_2$. Note that the necessity of this condition does not hinge on the lower bounds of the supports of the bid spaces being different, and C1 will continue to remain necessary when $\underline{b}_1 = \underline{b}_2$. We revisit Example 1 yet again and show that it violates this condition.

Example 1. (Continued) Once again, consider buyer 1 with value $v_1 = 3$, at which the interim payment is $p_1^*(3) = \frac{5}{2}$. At the bid $\phi_1(3) = 2$, Condition C1 is violated because

$$v(2) = \min\{\phi_1^{-1}(2), \phi_2^{-1}(2)\} = \min\{3, 2\} = 2,$$

and hence,

$$v(2)G_2(2) = 2 < p_1^*\left(\phi_1^{-1}(2)\right) = \frac{5}{2}.$$

It remains to derive a similar condition for the interim payment of buyer 2, which accounts for the fact that the lower bounds of the supports of the bid distributions differ ($\underline{b}_1 < \underline{b}_2$). Suppose that one buyer bids $b \in B_1 \cap B_2$ while the other bid is in $[\underline{b}_1, \underline{b}_2)$. Then, it is clear that the buyer bidding b is buyer 2, so payments in this range of bids can be chosen to be up to her value $I_2^{-1}(b)$ which may be higher than $v(b)$. By contrast, when buyer 1 bids b, she can never be charged more than $v(b)$ even if her value $I_1^{-1}(b)$ is strictly greater. This argument yields an analogous necessary condition for buyer 2.

Condition C1': The distribution of values F_1 and F_2 induce distributions G_1 and G_2 such that

$$\forall b \in B_1 \cap B_2 : \quad v(b)\left(G_1(b) - G_1(\underline{b}_2)\right) + I_2^{-1}(b)G_1(\underline{b}_2) \geq p_2^h\left(I_2^{-1}(b)\right). \tag{C1'}$$

However, conditions C1 and C1′ together need not be sufficient. This is because ensuring the appropriate interim payment for buyer 1 places a bound on the amount that can be extracted from buyer 2 from bids that lie in the common support $B_1 \cap B_2$. Suppose that, at a bid b, the interim payment $p_1^h\left(I_1^{-1}(b)\right)$ of buyer 1 is substantially lower than that of buyer 2, which is $p_2^h\left(I_2^{-1}(b)\right)$. This may prevent the seller from extracting the entire expected payment $v(b)[G_1(b) - G_1(\underline{b}_2)]$ from buyer 2 when buyer 1's bids lie in the range $[\underline{b}_2, b]$.

Hence, we need to derive the maximum payment $\eta(b) \leq v(b)[G_1(b) - G_1(\underline{b}_2)]$ that can be extracted symmetrically from buyer 2 when (i) she bids $b \in B_1 \cap B_2$, (ii) positive payments are only taken when b is the winning bid (i.e., the other buyer's bids are in the range $[\underline{b}_2, b]$) and (iii) buyer 1's expected payment from bid b is $p_1^h\left(I_1^{-1}(b)\right)$.

In words, we need to define payments for bids $b \in B_1 \cap B_2$ in a way that maximizes the amount extracted from buyer 2 while ensuring that buyer 1's expected payment remains $p_1^h\left(I_1^{-1}(b)\right)$. If this amount extracted is greater than the required payment $p_2^h\left(I_2^{-1}(b)\right)$ for buyer 2, subsidies can always be provided when buyer 1's bids lie in the range $[\underline{b}_1, \underline{b}_2)$ because, in equilibrium, such bids can only come from buyer 1.

We now need some additional notation. First, we define the following function for $b \in B_2$, which depends on the ratios of the densities:

$$L(b) = \begin{cases} \infty & \text{if } g_1(b) = g_2(b) = 0, \\ \frac{g_1(b)}{g_2(b)} & \text{otherwise.} \end{cases}$$

That is, $L(\cdot)$ is the likelihood ratio of a buyer bidding b being buyer 1 versus buyer 2. Further, we define

$$\underline{\ell} \equiv \min_{b \in B_2} \{ L(b) \}.$$

This is the lowest value of the likelihood ratio for bids in B_2. Since index functions are assumed to be differentiable and strictly increasing, densities g_1 and g_2 are well defined and continuous on B_1 and B_2 respectively. As a result, $\underline{\ell}$ is well defined and positive when $\overline{b}_2 \leq \overline{b}_1$ and 0 when $\overline{b}_2 > \overline{b}_1$.

Additionally, we define the sets

$$\gamma(\ell) \equiv \left\{ b \in B_2 \,\middle|\, L(b) \leq \ell \right\}$$

as the set of bids less than b where the likelihood ratio is at most ℓ, and

$$\bar{\bar{\gamma}}(\ell) \equiv \left\{ b \in B_2 \,\middle|\, L(b) = \ell \right\}$$

similarly as the set of bids less than b where the likelihood ratio is exactly ℓ. These sets will be useful to describe payment rules that derive $\eta(b)$. To obtain $\eta(b)$, we concentrate the maximum payment $v(b)$ on bids that are more likely to lie in the bid space of buyer 1 relative to that of buyer 2, and buyer 1's interim payment is then guaranteed by providing a subsidy at bids that are least likely.

When Condition C1 holds, that is, when $v(b)G_2(b) \geq p_1^h \left(I_1^{-1}(b) \right)$, the following two cases are mutually exclusive and exhaustive for any $b \in B_1 \cap B_2$.[19]

$$G_2(\bar{\bar{\gamma}}(\underline{\ell})) > 0 \qquad \text{OR} \qquad v(b)G_2(b) = p_1^h \left(I_1^{-1}(b) \right). \tag{Case 1}$$

$$G_2(\bar{\bar{\gamma}}(\underline{\ell})) = 0 \qquad \text{AND} \qquad v(b)G_2(b) > p_1^h \left(I_1^{-1}(b) \right). \tag{Case 2}$$

$\eta(b)$ needs to be derived separately for each of these two cases, and hence, we analyze them separately below.

Case 1. Let $\hat{B}$ be a subset of $\bar{\bar{\gamma}}(\underline{\ell})$ such that

$$v(b)G_2 \left([\underline{b}_2, b] \backslash \hat{B} \right) \geq p_1^h \left(I_1^{-1}(b) \right).$$

If $v(b)G_2(b) = p_1^h \left(I_1^{-1}(b) \right)$, then $\hat{B}$ must be a G_2-null setl else consider any set $\hat{B}$ that satisfies the above inequality and has a strictly positive measure.

We now define a payment rule,

$$\hat{p}(b, b') = \begin{cases} v(b) & \text{for } b' \in [\underline{b}_2, b] \backslash \hat{B}, \\ s & \text{for } b' \in \hat{B}, \\ 0 & \text{for } b' \in B_2 \text{ and } b' \notin \left([\underline{b}_2, b] \cup \hat{B} \right). \end{cases} \tag{C2,P1}$$

where s is chosen to solve

$$v(b)G_2 \left([\underline{b}_2, b] \backslash \hat{B} \right) + sG_2 \left(\hat{B} \right) = p_1^h \left(I_1^{-1}(b) \right).$$

[19] Recall that we use G_i to represent both a measure and a CDF.

Note that s here is a subsidy. We set

$$\eta(b) = \int_{\underline{b}_2}^{\overline{b}_2} \hat{p}(b,b')dG_1(b'). \tag{10}$$

Observe that $\eta(b)$ does not depend on the choice of $\hat{B}$. In addition, observe that, when $\overline{b}_2 > \overline{b}_1$, then $\hat{B} \subset (\overline{b}_1, \overline{b}_2]$ and $\eta(b) = v(b)$.

Case 2. Since $G_2(\bar{\bar{\gamma}}(\underline{\ell})) = 0$, it must be that $\overline{b}_2 \leq \overline{b}_1$. Here, we define the payment rule $\hat{p}_\ell$ for $\ell > \underline{\ell}$ as follows:

$$\hat{p}_\ell(b,b') = \begin{cases} v(b) & \text{for } b' \in [\underline{b}_2, b] \backslash \gamma(\ell), \\ s & \text{for } b' \in \gamma(\ell), \\ 0 & \text{for } b' \in B_2 \text{ and } b' \notin \left([\underline{b}_2, b] \cup \hat{B}\right). \end{cases} \tag{C2,P2}$$

where s is chosen to solve

$$v(b)G_2\left([\underline{b}_2, b]\backslash\gamma(\ell)\right) + sG_2(\gamma(\ell)) = p_1^h\left(I_1^{-1}(b)\right).$$

Note that, for ℓ close to $\underline{\ell}$, s is negative, so the payment rule $\hat{p}_\ell$ is ex-post IR. Define:

$$\eta_\ell(b) = \int_{\underline{b}_2}^{\overline{b}_2} \hat{p}_\ell(b,b')dG_1(b'), \tag{11}$$

and let

$$\eta(b) = \lim_{\ell \downarrow \underline{\ell}} \left[\eta_\ell(b)\right].$$

We can now define the second condition.

Definition 5.1 (Condition C2)**.** The distribution of values F_1 and F_2 induce distributions G_1 and G_2 such that

$$\forall b \in B_1 \cap B_2: \quad \eta(b) + I_2^{-1}(b)G_1(\underline{b}_2) \geq p_2^h\left(I_2^{-1}(b)\right), \tag{C2}$$

with the inequality holding strictly for any b such that

$$G_2(\bar{\bar{\gamma}}(\underline{\ell})) = 0 \text{ and } v(b)G_2(b) > p_1^h\left(I_1^{-1}(b)\right).$$

The following proposition states that the two conditions C1 and C2 are necessary and sufficient for a symmetric ex-post IR implementation.

Proposition 5. *Suppose that $n = 2$. Consider an implementable hierarchical allocation mechanism (I, p^h) with differentiable and strictly increasing index functions such that the lower bounds of the supports of the bid distributions differ; that is, $\underline{b}_1 < \underline{b}_2$. Then, Conditions C1 and C2 are necessary and sufficient for there to exist a symmetric, ex-post IR implementation of (I, p^h).*

We end this section by observing that Proposition 5 can be adapted to accommodate entry fees. In many practical situations, auctions are often conducted in two steps: buyers first pay to participate, following which the auction is conducted. Such entry fees can relax ex-post IR constraints of the auction itself, as buyers are making a part of the payment before participating. In particular, if the seller could charge a high enough entry fee, he would not need the buyers to make payments in the auction and could offer rebates instead. Having sunk the entry cost, ex-post IR would then be obtained automatically. Conditions C1 and C2 can be appropriately weakened to accommodate a given entry fee; the construction in this section can simply be altered so that winning bidder never pays more than her value plus the fee and the loser never has to pay more than the fee.

6. Concluding Remarks

Symmetric sealed bid auction formats are commonly used in the real world. An advantage of these formats is that they are anonymous and nondiscriminatory, which are requirements that are often necessary for legal and practical reasons. We characterized the set of outcomes that are theoretically achievable via symmetry auctions. We have thus shown that symmetry, in and of itself, does not prevent the auctioneer from achieving a wide variety of goals. In particular, the optimal auction can be implemented, as can auctions in which certain bidders are subsidized over others. In this sense, we view our main result as similar in spirit to the "revelation principle" for direct mechanisms. We have also shown that imposing additional properties (inactive losers, continuity, monotonicity and ex-post IR) results in stronger restrictions on the set of implementable outcomes. In particular, the optimal auction is not always implementable with these additional properties.

First- and second-price auctions are the most commonly used symmetric auction formats. An interesting avenue for future research is to isolate further properties of these formats that make them popular in practical applications and design auctions that have these properties. Apart from the desiderata considered in this paper, two other properties that we think are important are "simplicity" and distribution independence of description of the mechanism. For the behavior of the buyers to be predictable, the mechanism employed should have simple, transparent rules, and buyers should be able to compute their equilibrium strategies easily. While the implementations we construct have comparatively complex payment rules, the equilibrium bids can easily be derived by buyers. By contrast, first price auctions have simple rules, but equilibrium bids may be hard to compute. Needless to say, one of the challenges in designing simple mechanisms is the definition of simplicity itself.

Finally, the symmetric auctions we construct depend critically on the fact that the seller knows the underlying distribution of values. Moreover, equilibrium bidding requires the buyers to possess this knowledge as well. This is a widespread assumption in mechanism design and, more generally, in Bayesian games. In fact, knowledge about value distributions is necessary even for revenue ranking first- and second-price auctions. Just computing equilibrium bids in the former format requires bidders to posses this knowledge. However, this assumption may be unsuitable for some practical applications. Brooks (2013a,b) studies an environment where buyers are informed about the value distributions but the seller is not and attempts to elicit this information. Such a situation may arise when a novice seller faces seasoned bidders. By contrast, even a seasoned auctioneer may not have exact knowledge about distributions, but may know certain summary statistics such as moments of the value distributions (see Ollár and Penta (2013) for a study on full implementation in such environments). An important topic for future research is a further development of auction theory in the case where sellers have no or limited information (such as statistics of the distribution that can be estimated from previous auction data).

APPENDIX A. PROOFS FROM SECTION 4

A.1. *Proof of Theorem 1*

Fix a direct revelation mechanism (a^d, p^d), and suppose it has a symmetric implementation (a^s, p^s) where buyers use strategies σ. We are left to construct a symmetrically implementable hierarchical mechanism with the desired property.

Pure Strategies. First suppose the mechanism (a^d, s^d) has an implementation in pure strategies. In this case, note that the hierarchical mechanism defined as

$$I_i(v_i) = \sigma_i(v_i), \; p_i^h(v_i) = p_i^d(v_i),$$

has a symmetric implementation which (exactly) implements the direct revelation mechanism.

Mixed Strategies. So now suppose the implementation of the direct revelation mechanism is in mixed strategies. Fix the symmetric auction game. A mixed strategy equilibrium in this setting is a mapping for each buyer i, $\sigma_i : V_i \to \Delta\mathbb{R}$, that is, buyer i with value v_i randomizes over bids with probability measure $\sigma_i(v_i)$.

A little notation is useful. Given that buyer i's values are distributed according to F_i, and that when he has value v_i, he randomizes over bids with measure $\sigma_i(v_i)$, let G_i^r denote the implied distribution over bids.

Let us denote by $\breve{a}_i(b_i)$ the interim winning probability of buyer i when he bids b_i, with associated interim payment $\breve{p}_i(b_i)$. Note that $\breve{a}_i(b_i)$ is non-decreasing in b_i for all buyers i.

The following observation shows that the bids over which different values of a given buyer randomize are disjoint and ordered.

Observation 1. *For any buyer i and values $v_i < v_i'$, the support of the distributions of bids by these two values is completely ordered, i.e.*

$$\max\{\breve{a}_i(b_i) : b_i \in supp(\sigma_i(v_i))\} \leq \min\{\breve{a}_i(b_i) : b_i \in supp(\sigma_i(v_i'))\}.$$

Proof. Firstly, note that if buyer i with value v_i mixes over bids $b_i < b_i'$ with $\breve{a}_i(b_i) < \breve{a}_i(b_i')$, then he must be indifferent between these bids. Therefore $\breve{p}_i(b_i') - \breve{p}_i(b_i) = v_i(\breve{a}_i(b_i') - \breve{a}_i(b_i))$, implying that v_i' cannot be indifferent between both these bids.

So now suppose buyer i with value $v_i' > v_i$ has an equilibrium bid b_i'' with $\breve{a}_i(b_i'') < \breve{a}_i(b_i')$. Combining the equilibrium constraints that v_i prefers to bid b_i' than b_i'' and that v_i' prefers to bid b_i'' than b_i', we have a contradiction. The observation follows. □

Observation 2. *For any buyer i, the set of values $v_i \in V_i$ such that*

$$\exists b_i, b_i' \in supp(\sigma_i(v_i)) : \breve{a}_i(b_i) \neq \breve{a}_i(b_i')$$

has F_i measure 0*.*

Proof. By Observation 1 we have that the support of distribution of bids for a given buyer is effectively disjoint. Therefore, at most a countable number of values for any buyer can have two bids with different interim probabilities of winning in their support: since the range of $\breve{a}_i(\cdot)$ is $[0, 1]$ and the reals can have an at most countable set of positive length intervals. Since F_i is differentiable, the measure of a countable set of values is 0. $\square$

Finally consider the hierarchical mechanism (I, p^h) constructed as follows. Fix a buyer i. Some buyer types may be following properly mixed strategies. These can be separated into two parts.

(1) By Observation 2 there are an at most countable set of buyer types who randomize over different values which result in different probabilities of getting the good. For each of these values define $I_i(v_i) = b_i$ for some $b_i \in \text{supp}(\sigma_i(v_i))$, with $p_i^h(v_i) = \int_{b_{-i}} p^s(b_i, b_{-i}) dG^r_{-i}(b_{-i})$.

(2) Finally the remaining types for any buyer i can be partitioned into intervals, such that all types in each interval receive a constant probability of winning. Any randomization by any value in this interval must be such that for any two distinct bids $b_i < b_i'$, such that $b_i, b_i' \in \text{supp}(\sigma_i(v_i))$, $\breve{a}_i(b_i) = \breve{a}_i(b_i')$. It follows that $\breve{a}_i(\cdot)$ is constant on $[b_i, b_i']$. This implies that there cannot be a positive measure of values of other buyers $-i$ that submit bids in the interval $[b_i, b_i']$. For any value in each such interval, $[\underline{v}, \bar{v}]$,[20] define $I_i(v) = b$ for some b within the smallest interval within $\cup_{v \in [\underline{v}, \bar{v}]} \text{supp}(\sigma_i(v_i))$ which carries all the mass. Further, define $p_i^h(v) = p_i^d(v)$.

For each remaining value v_i, $\sigma_i(v_i)$ is a single point, and we define $I_i(v_i) = \sigma_i(v_i)$, $p_i^h(v_i) = p_i^d(v_i)$.

The hierarchical mechanism (I, p^h) has a symmetric implementation by construction. Further, by construction, it achieves the same ex-post allocation and interim payment almost surely (that is, for all buyer values other than the countable set of buyers in point 1 above). Therefore this implementation is an almost sure implementation of the original direct revelation mechanism.

A.2. *Proof of Theorem 2*

Sufficiency. At a high level, we generalize the ideas in our construction of the two bidder example in Section 3. Recall that our goal is to construct a symmetric auction game which implements a hierarchical mechanism (I, p^h) with corresponding allocation rule a^h.

We construct a symmetric auction game which has a pure strategy Bayes-Nash equilibrium in which buyer i with value v_i reports $I_i(v_i)$. By construction therefore, the allocation of this mechanism equals a^h. We are left to show that:

i. As constructed, this auction game implements the desired payments p^h.
ii. For each buyer i, bidding according to $I_i(\cdot)$ constitutes a Bayes-Nash equilibrium of the construction auction.

Step 1: Preliminaries. Our goal is to show that we can construct a symmetric $p^s : \mathbb{R}^n \to \mathbb{R}$, such that

$$\forall i, \forall v_i \in V_i : \quad p_i^h(v_i) = \int_{\mathbf{B}_{-i}} p^s\left(I_i(v_i), \mathbf{b}_{-i}\right)\ \mathrm{d}G_{-i}(\mathbf{b}_{-i}). \tag{12}$$

In other words, we need to show that we can construct a p^s such that each buyer i's expected payment, in expectation over candidate equilibrium bids of other buyers, equals $p_i^h(\cdot)$.

[20] Note that while this is written as a closed interval, the interval may be closed, open, or half open, half closed.

Step 2: Full Rank Events. We say that an event $E \subseteq \mathbb{R}^{n-1}$ is *symmetric* if

$$\text{for every permutation } \rho : \{1,2,\dots,n-1\} \to \{1,2,\dots,n-1\},$$
$$(b_1, b_2, \dots b_{n-1}) \in E \implies (b_{\rho(1)}, b_{\rho(2)}, \dots b_{\rho(n-1)}) \in E.$$

We start with a simple observation.

Observation 3. *Consider $k \leq n$ symmetric events $E_1, E_2 \dots E_k \subseteq \mathbb{R}^{n-1}$ and define the $k \times k$ matrix*

$$\mathcal{M} \equiv [G_{-i}(E_j)]_{i,j=1}^{k}.$$

If matrix $\mathcal{M}$ is full rank, then, there exists a symmetric payment rule p^s such that

$$\forall i = 1, \dots k, \; \forall v_i \in V_i : \quad p_i^h(v_i) = \int_{\mathbf{B}_{-i}} p^s(I_i(v_i), \mathbf{b}_{-i}) \, dG_{-i}(\mathbf{b}_{-i}). \tag{13}$$

In particular, if $k = n$, then there exists a payment rule p^s that satisfies (12).

Proof. Define the payment rule this way: there are k numbers associated with each bid $b \in \mathbb{R}$, denote the j^{th} number $\pi_j(b)$. Suppose a bid b is made by a buyer, and other buyers make the profile of bids $\mathbf{b}_-$. For each event E_j that occurs among other buyers' bids, i.e. $\mathbf{b}_- \in E_j$, the buyer is asked to pay $\pi_j(b)$. Formally, the payment function is defined as

$$p^s(b, \mathbf{b}_-) = \sum_{j=1}^{k} \pi_j(b) \chi_{\{\mathbf{b}_- \in E_j\}}, \tag{14}$$

where χ is the characteristic function. Note that since each of the E_j's are symmetric (by assumption) the payment rule defined thus is symmetric as well.

Given this definition of p^s, the expected payment made by buyer i bidding $b_i \in B_i$ when all other buyers are bidding according to their candidate equilibrium strategies is

$$\begin{aligned}
&\int_{\mathbf{B}_{-i}} p^s(b_i, \mathbf{b}_{-i}) \, \mathrm{d}G_{-i}(\mathbf{b}_{-i}), \\
&= \int_{\mathbf{B}_{-i}} \left(\sum_{j=1}^{k} \pi_j(b_i) \chi_{\{\mathbf{b}_{-i} \in E_j\}} \right) \mathrm{d}G_{-i}(\mathbf{b}_{-i}), \\
&= \sum_{j=1}^{k} \pi_j(b_i) G_{-i}(E_j).
\end{aligned}$$

By the full rank assumption, for any $b \in \mathbb{R}$, there exists a solution $\pi(b) \in \mathbb{R}^k$ to the system of equations:

$$\mathcal{M}\pi(b) = \tilde{p}(b), \tag{15}$$

$$\text{where } \tilde{p}(b) = [\tilde{p}_1(b), \dots, \tilde{p}_k(b)]^T,$$

$$\tilde{p}_i(b) = \begin{cases} p_i^h(I_i^{-1}(b)) & \text{if } b \in B_i, \\ \max_{i \in N}\{\overline{v}_i\} & \text{otherwise.} \end{cases} \tag{16}$$

Therefore, the payment rule defined using $\pi(\cdot)$ that satisfies this system of equations satisfies (13). When $k = n$, the constructed system satisfies (12). □

Theorem 3 shows that there always exist such events.

Theorem 3. *For any $n > 1$ and any $k \leq n$ such that $G_1, G_2, \ldots G_k$ are all pairwise distinct, there exist symmetric events $E_1, \ldots, E_k \subseteq \mathbb{R}^{n-1}$ such that the $(k \times k)$ matrix $\mathcal{M} = [G_{-i}(E_j)]_{i,j=1}^k$ has full rank.*

A proof of the theorem is deferred to Appendix C.

Step 3: Matching Payments. First consider the case where $G_i \neq G_{i'}$ for all $i \neq i'$. Then by Theorem 3 there exist symmetric events $E_1, E_2 \ldots E_n \subseteq \mathbb{R}^{n-1}$ such that the $n \times n$ matrix $\mathcal{M} = [G_{-i}(E_j)]$ is full rank. Therefore, by Observation 3, we can construct a symmetric payment rule p^s that matches the desired interim payment rule p^h when all buyers make their candidate equilibrium bids, i.e., satisfies (12).

Now to consider the other case, i.e. there exist i, i' such that $G_i = G_{i'}$. Note that if $G_i = G_{i'}$ for some $i \neq i'$, then $G_{-i} = G_{-i'}$.

We define N_U as the set of "distributionally unique buyers." Formally for any induced distribution over bids, G define $N_G = \{i \in N : G_i = G\}$. Now we can define $N_U = \cup_{i \in N} \{\min\{N_{G_i}\}\}$. In other words, N_U is the largest subset of N s.t. for any distinct $i, i' \in N_U$, $G_i \neq G_{i'}$. Renumber the buyers so that the first $|N_U|$ buyers are distributionally unique. By Theorem 3, we can construct full row rank events for these buyers. We are then done, because by assumption, if $G_i = G_{i'}$ we have that $p_i^h(I_i^{-1}(b)) = p_{i'}^h(I_{i'}^{-1}(b))$.

Step 4: Equilibrium. We have already shown that if each buyer followed the candidate equilibrium strategy, the desired payment rule p^h is implemented. We are left to show that following the candidate strategy (i.e. that buyer i with value v_i bids $I_i(v_i)$) is a Bayes-Nash equilibrium of the game.

Consider buyer i, with value v_i. His candidate equilibrium bid is $b_i = I_i(v_i)$. Let us divide possible deviations into two types:

(1) Buyer i bids $b_i' \in B_i$.
(2) Buyer i bids $b_i' \notin B_i$.

Since the original mechanism (a^h, p^h) is Bayes Incentive Compatible, it should be clear that deviations of type 1 cannot be profitable. If player i with value v_i deviates to some other $b_i' = I_i(v_i')$. Assuming all other players are playing their equilibrium strategies player i will win the good with probability $a_i^h(v_i')$ and make an expected payment of $p_i^h(v_i')$. Incentive compatibility of the original direct revelation mechanism guarantees that:

$$v_i a_i^h(v_i) - p_i^h(v_i) \geq v_i a_i^h(v_i') - p_i^h(v_i').$$

By construction (15,16), deviations of type 2 will require the buyer to make an expected payment of $\max_{i \in N}\{\overline{v}_i\}$ and hence, such deviations cannot be profitable.

Therefore our candidate equilibrium strategies constitute a Bayes-Nash equilibrium of the symmetric auction game we constructed, concluding our proof of sufficiency.

Necessity. We now show that our condition is necessary for there to exist a symmetric implementation. Let us consider a hierarchical allocation rule with index functions $I_1, \ldots, I_n$ such that for buyers 1 and 2, $G_1 = G_2$.

Firstly, note that any other index function I' that implements the same allocation rule must be a strictly monotone transform of I. Therefore the resulting distributions will be such that $G_1' = G_2'$. It is therefore without loss to only check whether there exists an implementation corresponding to the 'original' index rule I.

Pick v_1, v_2 such that $I_1(v_1) = I_2(v_2)$, and $p_1^h(v_1) \neq p_2^h(v_2)$. Note that $a_1^h(v_1) = a_2^h(v_2)$ since $G_1 = G_2 \implies G_{-1} = G_{-2}$.

Recall that a symmetric implementation in pure strategies is a symmetric payment rule p^s, such that for all buyers i and all valuations v_i in V_i,

$$p_i^h(v_i) = \int_{\mathbf{B}_{-i}} p^s(I_i(v_i), \mathbf{b}_{-i}) dG_{-i}(\mathbf{b}_{-i}).$$

Since $G_1 = G_2$, the product distributions G_{-1} and G_{-2} are also the same. Therefore for any b,

$$\int_{\mathbf{B}_{-1}} p^s(b, \mathbf{b}_{-1}) dG_{-1}(\mathbf{b}_{-1}) = \int_{\mathbf{B}_{-2}} p^s(b, \mathbf{b}_{-2}) dG_{-2}(\mathbf{b}_{-2}).$$

For $b = I_1^{-1}(v_1) \ \left(= I_2^{-1}(v_2)\right)$, we have the required contradiction.

Mixed Strategies. We now argue that allowing for mixed strategies does not expand the set of implementable mechanisms.

Consider a mixed implementation such that the resulting distribution over bids of buyer i is G_i^r. It follows from Observation 2 that $G_1 = G_2$ implies $G_1^r = G_2^r$. As a result, $G_{-1}^r = G_{-2}^r$.

Suppose a hierarchical mechanism (I, p^h) cannot be implemented in pure strategies. Then without loss of generality, $G_1 = G_2$ and there are values v_1 and v_2 such that condition (ii) of Theorem 2 is violated. For interim implementation in mixed strategies we must have that

$$a_i^h(v_i) = \int_{B_i} \breve{a}_i(b_i) d\sigma_i(v_i)(b_i),$$
$$p_i^h(v_i) = \int_{B_i} \breve{p}_i(b_i) d\sigma_i(v_i)(b_i).$$

Now suppose (without loss of generality) that there is a mixed strategy symmetric implementation of the case where $G_1 = G_2$, $p_1^h(v_1) > p_2^h(v_2)$ and $I_1(v_1) = I_2(v_2)$. Then, buyer 1 with value v_1, strictly prefers the strategy $\sigma_2(v_2)$ over $\sigma_1(v_1)$ (since $a_1^h(v_1) = a_2^h(v_2)$ by assumption), contradicting the assumption that these strategies constitute an equilibrium.

A.3. *Proof of Corollary 1*

Without loss of generality, consider only buyers 1 and 2. Since the auction does not have a symmetric implementation, it must be the case that $G_1 = G_2$. First consider the case that index functions I_1 and I_2 are continuous.

Suppose v_1, v_2 are such that $F_1(v_1) = F_2(v_2)$ but $I_1(v_1) > I_2(v_2)$— if no such v_1, v_2 exists we are done. Define

$$v_1' = \max\{v \in V_1 : I_1(v) = I_2(v_2)\}.$$

By continuity of I_1, v_1' exists. By monotonicity of I_1, $v_1' < v_1$. By assumption, $G_1(I_1(v_1')) = G_2(I_2(v_2))$. Combining, we have that

$$F_1(v_1) > F_1(v_1') = G_1(I_1(v_1')) = G_2(I_2(v_2)) \geq F_2(v_2),$$

implying that $F_1(v_1) > F_2(v_2)$. This contradicts our assumption that $F_1(v_1) = F_2(v_2)$.

Now suppose I_1 and I_2 are not necessarily continuous. The common support must lie on an at most countable collection of intervals and at most countable atoms. For any point in the interior of any interval in the support of G_1, and any atom, the above argument shows that

$$\text{for } i = 1, 2: \ \ I_i(v_i) = \Gamma(F_i(v_i)),$$

for any v such that $I_1(v_1)$ is in the interior of an interval in the support of G_1 or an atom on G_1. This leaves only measure 0 end points of the intervals, of which there are an at most countable set. These correspond to discontinuities in the index rules, which are also at most countable. □

A.4. *Proof of Corollary 2*

For each buyer i, consider the index function I_i as a point in the L_1 space (the same argument works with any L_p norm) with respect to the measure space defined by measure F_i on V_i. The space I is topologized by the product topology. Since this is finite product of complete normed vector spaces, it is a Baire space, and therefore standard topological notions of genericity are well defined.

To see the desired result, first note that by Corollary 1, the condition of Theorem 2 is violated only if there exists a nondecreasing function Γ and bidders $j, j' \in N$ such that for any $i = j, j'$, $I_i(v_i) = \Gamma(F_i(v_i))$ for almost all $v_i \in V_i$. Consider the set E_S defined on sets $S \subseteq N$ which is a subset of the set of index rules, defined as:

$$E_S = \{I \, : \, \text{for all } i \in S, \, I_i(v_i) = \Gamma(F_i(v_i)) \ \text{ almost everywhere}\}.$$

We show that $E := \cup_{S \subset N} E_S$ is a meagre set.

First note that E is closed since it is the finite union of closed sets. Each set E_S is closed since the limit of any sequence of I's that violates condition (i) of Corollary 1 for the subset S will also violate this condition for S.

Thus, if we can show that E has a nonempty interior it will be nowhere dense and we are done. The following lemma delivers this result.

Lemma 2. *Consider any hierarchical mechanism $I \in E$. Then for any $\epsilon > 0$, there is an index rule I' which satisfies condition (i) such that for each buyer i,*

$$\int_{V_i} |I_i(v) - I_i'(v)| dF_i(v) \leq \epsilon.$$

Proof. For any ϵ small, consider an increasing function $X_j^\epsilon : V_j \to [0, \epsilon]$, such that $F_j(\{v_j | X_j^\epsilon(v_j) \neq 0\}) \neq 0$. Define $I_j' = I_j + X_j^\epsilon$. Clearly one can select X_j^ϵ for each j such that $I' \notin E$. □

A.5. *Proof of Corollary 3*

Recall from Myerson (1981) that if the function $\phi_i(v_i)$, defined as

$$\phi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)},$$

is non-decreasing in v_i, then the allocation rule for the optimal auction is defined by the hierarchical allocation rule with the index rule ϕ_i for buyer i. If ϕ_i is not non-decreasing, then the optimal allocation rule is given by the "ironed" virtual value $\overline{\phi}_i$. Let G_i be the distribution over bids of buyer i induced by $\overline{\phi}_i$.

The following simple lemma shows if two buyers (without loss of generality 1 and 2) have the same distribution of (possibly ironed) virtual values, then the two buyers also have the same function mapping value into virtual value. Therefore, the hierarchical allocation rule implementing the optimal auction either induces different distributions of virtual values, or if not, then this Lemma shows it satisfies the condition of Theorem 2. The Corollary follows.

Lemma 3. *Suppose two buyers are such that $G_1 = G_2$. Then $V_1 = V_2$ and $\overline{\phi}_1 = \overline{\phi}_2$.*

Proof. Define $v_i(b) = \overline{\phi}_i^{-1}(b)$ for $b \in B_1$. Since $\overline{\phi}_i(\cdot)$ need not be strictly increasing, it follows that $\overline{\phi}_i^{-1}(\cdot)$ is a correspondence. Define $\underline{v}_i(b) = \inf \overline{\phi}_i^{-1}(b)$ and $\overline{v}_i(b) = \sup \overline{\phi}_i^{-1}(b)$.
Since $\overline{\phi}_i$ non-decreasing, it follows that

$$G_i(b) = F_i(\overline{v}_i(b)).$$

There can be at most a countable number of pooling intervals in $\overline{\phi}_i$ (see Myerson, 1981, Section 6). Each of these pooling intervals correspond to an atom in G_i. We denote the set of atomic bids by $\mathcal{B}_i \subseteq B_i$, denote by β_{in} the bid that corresponds to the n^{th} atom in G_i, the size of the atom is denoted by

$$\varsigma_{in} = F_i(\overline{v}_i(\beta_{in})) - F_i(\underline{v}_i(\beta_{in})).$$

$\overline{\phi}_i$ is differentiable everywhere else, therefore so is $v_i(\cdot)$ whenever it is a singleton. For any $b \in B_i \backslash \mathcal{B}_i$, differentiating we have that

$$g_i(b) = f_i(v_i(b))v_i'(b).$$

For any $b \in B_i \backslash \mathcal{B}_i$, we know that $\phi_i(v_i(b)) = \overline{\phi}_i(v_i(b))$, and therefore by the definition of ϕ_i

$$v_i(b) - \frac{1 - F_i(v_i(b))}{f_i(v_i(b))} = b.$$

$$\implies v_i(b) - \frac{1 - G_i(b)}{g_i(b)} v_i'(b) = b. \tag{17}$$

Observation 4. *Consider any interval $[\underline{b}, \overline{b}]$ in the support of G such that there are no atoms in this interval. Further, suppose $v_1(\overline{b}) = v_2(\overline{b})$. Then $v_1(b) = v_2(b)$ for every $b \in [\underline{b}, \overline{b}]$.*

Proof. From (17), we know that for each $b \in [\underline{b}, \overline{b}]$, and $i = 1, 2$

$$\begin{aligned} v_i(b) - \frac{1 - G_i(b)}{g_i(b)} v_i'(b) &= b, \\ \implies v_1(b) \lesseqqgtr v_2(b) \iff v_1'(b) &\lesseqqgtr v_2'(b), \end{aligned}$$

where the implication follows from the fact that $G_1 = G_2$. Therefore if $v_1(b) \neq v_2(b)$ for some $b \in [\underline{b}, \overline{b}]$, it cannot be that $v_1(\overline{b}) = v_2(\overline{b})$. □

For any $\beta_{in} \in \mathcal{B}_i$, the 'ironed' virtual value pools all buyers in $[\underline{v}_i(b), \overline{v}_i(b)]$. Therefore

$$\begin{aligned} \beta_{in} &= \frac{\int_{\underline{v}_i(\beta_{in})}^{\overline{v}_i(\beta_{in})} \phi_i(v) f_i(v) \, \mathrm{d}v}{F_i(\overline{v}_i(\beta_{in})) - F_i(\underline{v}_i(\beta_{in}))}, \\ &= \underline{v}_i(\beta_{in}) - (\overline{v}_i(\beta_{in}) - \underline{v}_i(\beta_{in})) \frac{1 - F_i(\overline{v}_i(\beta_{in}))}{F_i(\overline{v}_i(\beta_{in})) - F_i(\underline{v}_i(\beta_{in}))}, \\ &= \underline{v}_i(\beta_{in}) - (\overline{v}_i(\beta_{in}) - \underline{v}_i(\beta_{in})) \frac{1 - G_i(\beta_{in})}{\varsigma_{in}}. \end{aligned} \tag{18}$$

Since $G_1 = G_2$, both have the same (at most countable set of) atoms— we denote the set of atoms $\mathcal{B}$ with generic element β_n of 'size' ς_n.

Observation 5. *Consider any atom $\beta_n \in \mathcal{B}$ of size ς_n, and suppose that $\overline{v}_1(\beta_n) = \overline{v}_2(\beta_n)$. Then we have that $\underline{v}_1(\beta_n) = \underline{v}_2(\beta_n)$, i.e. $v_1(\beta_n) = v_2(\beta_n)$.*

Proof. By (18), we have for $i = 1, 2$

$$\begin{aligned} \beta_n &= \underline{v}_i(\beta_n) - (\overline{v}_i(\beta_n) - \underline{v}_i(\beta_n)) \frac{1 - G_i(\beta_n)}{\varsigma_n}, \\ &= \underline{v}_i(\beta_n) \left(1 + \frac{1 - G_i(\beta_n)}{\varsigma_n}\right) - \overline{v}_i(\beta_n). \end{aligned}$$

Therefore if $\overline{v}_1(\beta_n) = \overline{v}_2(\beta_n)$, then $\underline{v}_1(\beta_n) = \underline{v}_2(\beta_n)$. □

Finally, letting $\overline{b}$ be the upper bound of the support of $G_1 (= G_2)$, note that by definition:

$$v_1(\overline{b}) = v_2(\overline{b}) = \overline{b}.$$

The fact that $v_1(\cdot) = v_2(\cdot)$ now follows from this initial condition and Observations 4 and 5. Therefore $G_1 = G_2 \implies \overline{\phi}_1 = \overline{\phi}_2$. □

A.6. *Proof of Lemma 1*

Define the set of non-decreasing interim allocation rules achieved by some index rule as $\mathscr{H}_M$ the set of all feasible, non-decreasing interim allocation rules by $\mathscr{F}_M$ and the set of all feasible interim allocation rules by $\mathscr{F}$. By feasible, we mean that this interim allocation rule can result from some feasible ex-post allocation rule. The proof follows from two observations.

Observation 6. *$\mathscr{F}_M$ is a compact subset of L_2^n in the weak/weak* topology $\sigma(L_2^n, L_2^n)$.*

Proof. Lemma 8 of Mierendorff (2011) shows that the set of feasible interim allocation rules $\mathscr{F}$ is a compact convex subset of L_2^n in this topology.

By observation, $\mathscr{F}_M$ is convex. We now argue that $\mathscr{F}_M$ is also compact in this topology. By the Eberlein-Smulian theorem (Theorem 6.34, Aliprantis and Border, 2006), sequential compactness and compactness coincide in this topology. It is therefore enough to show that if for some sequence $\{a^n\}_{n=1}^{\infty} \subset \mathscr{F}_M$, $a^n \rightharpoonup a$, then $a \in \mathscr{F}_M$. Since each a^n is monotone, it is a function of bounded variation and therefore by Helly's selection theorem, there exists a subsequence which converges pointwise. Therefore a is also non-decreasing, and $a \in \mathscr{F}_M$, concluding our argument. □

Therefore, we have that the closure of the convex hull of $\mathscr{H}_M$ is a subset of $\mathscr{F}_M$ or

$$\overline{\text{conv}(\mathscr{H}_M)} \subseteq \mathscr{F}_M.$$

Observation 7. *For any index function $I : V \to \mathbb{R}^n$, there exists a hierarchical allocation rule $a^h \in \mathscr{H}_M$ which solves*

$$\max_{a \in \mathscr{F}_M} \int_V \left(\sum_j a_j(v) I_j(v) f_j(v) \right) dv. \tag{I-OPT-M}$$

Proof. If I is non-decreasing, i.e. $I_j(v)$ is non-decreasing in v for each $j \in N$, then the solution to (I-OPT-M) is in $\mathscr{H}_M$. This follows easily from the definition of hierarchical allocation rule. Since at every profile of values, the good is allotted to the buyer with the higher index, the rule maximizes the 'index revenue' profile-by-profile. Therefore it solves the maximization problem (I-OPT-M).

So let us consider the solution to (I-OPT-M) for other index functions. We can re-write the problem as

$$\max_{a \in \mathscr{F}} \int_V \left(\sum_j a_j(v) I_j(v) f_j(v) \right) dv,$$
$$a \text{ is non-decreasing.}$$

In this case, we can 'relax' the non-decreasing constraint into the objective function. By the ironing procedure of Myerson (1981), there exists an 'ironed' non-decreasing index rule $\hat{I}$ such that the solution to the above problem is the same as

$$\max_{a \in \mathscr{F}} \int_V \left(\sum_j a_j(v) \hat{I}_j(v) f_j(v) \right) dv.$$

Note that the corresponding hierarchical rule for index rule $\hat{I}$ lies in $\mathscr{H}_M$. □

To conclude the proof, suppose by way of contradiction that

$$\overline{\text{conv}(\mathscr{H}_M)} \subsetneq \mathscr{F}_M.$$

Then there exists $a \in \mathscr{F}_M$ such that $a \notin \overline{\text{conv}(\mathscr{H}_M)}$. By Corollary 7.47 of Aliprantis and Border (2006) there exists an $I \in L_2^n$ such that

$$\langle a, I \rangle > \max_{a' \in \overline{\text{conv}(\mathscr{H}_M)}} \langle a', I \rangle,$$

where $\langle a, I \rangle$ is the standard inner product $\int_V \left(\sum_j a_j(v) I_j(v) f_j(v) \right) dv$.

By Observation 7, the hierarchical allocation rule corresponding to I solves (I-OPT-M), implying that

$$\langle a, I\rangle > \max_{a' \in \mathscr{F}_M} \langle a', I\rangle.$$

Since $a \in \mathscr{F}_M$, this is a contradiction. It follows that

$$\overline{\text{conv}(\mathscr{H}_M)} = \mathscr{F}_M.$$

The Lemma follows. □

Appendix B. Proofs from Section 5

We should note in advance that all the proofs that follow are written discussing the possibility or impossibility of pure strategy implementations satsifying the additional desiderata. One may wonder about mixed strategy implementations. Observation 2 above showed that in any mixed strategy implementation, an at most probability 0 set of values for any buyer can be mixing over bids that achieve different probabilities of winning. It follows that to induce the same interim allocation rule, any mixed strategy implementation must induce the same distribution over bids as some pure strategy index rule implementing the allocation rule. Therefore our results extend to implementation in mixed strategies as well. We omit the details in the interests of brevity.

B.1. *Proof of Proposition 1*

We begin by showing necessity. Suppose the inactive losers condition does not hold. This implies that there is $\tilde{b}$ such that $G_1(b) = \alpha G_2(b)$ for all $b \leq \tilde{b}$ but $\alpha p_1^h(\tilde{b}) \neq p_2^h(\tilde{b})$. By observation it must be the case that $\underline{b}_1 = \underline{b}_2 = \underline{b}$.

If there is an inactive losers implementation $p(b_i, b_j)$, we have the following:

$$p_2^h(\tilde{b}) = \int_{\underline{b}}^{\tilde{b}} p(\tilde{b}, b')dG_1(b') = \alpha \int_{\underline{b}}^{\tilde{b}} p(\tilde{b}, b')dG_2(b') = \alpha p_1^h(\tilde{b})$$

which is a contradiction. Notice that the above equation follows from the fact that a inactive loser implementation by definition requires that the losing bidder make or receive no payments.

We now show sufficiency. We construct the payment rule for an arbitrary $\tilde{b}$, for each of the two cases of the inactive losers condition.

Case 1. There is a $\min\{\underline{b}_1, \underline{b}_2\} < b^* < \tilde{b}$ such that $\frac{G_1(\tilde{b})}{G_2(\tilde{b})} \neq \frac{G_1(b^*)}{G_2(b^*)}$. This implies that the matrix

$$\begin{bmatrix} G_1(\tilde{b}) - G_1(b^*) & G_1(b^*) \\ G_2(\tilde{b}) - G_2(b^*) & G_2(b^*) \end{bmatrix}$$

has full rank. This in turn means that following system of equations has a solution for x, y

$$p_1^h(\tilde{b}) = x[G_2(\tilde{b}) - G_2(b^*)] + yG_2(b^*),$$

$$p_2^h(\tilde{b}) = x[G_1(\tilde{b}) - G_1(b^*)] + yG_1(b^*).$$

Finally, setting

$$p(\tilde{b}, b') = \begin{cases} x & \text{for } b^* \leq b' < \tilde{b} \\ y & \text{for } b' < b^* \\ 0 & \text{otherwise} \end{cases}$$

results in the desired interim payments.

Case 2. There is a constant $\alpha > 0$ such $G_1(b) = \alpha G_2(b)$ for all $\min\{\underline{b}_1, \underline{b}_2\} \leq b \leq \tilde{b}$.

In this case, the condition implies that $\alpha p_1(\tilde{b}) = p_2(\tilde{b})$. We can simply set

$$p(\tilde{b}, b') = \begin{cases} \frac{p_2(\tilde{b})}{G_1(\tilde{b})} & \text{for } b' < \tilde{b} \\ 0 & \text{otherwise} \end{cases}$$

Clearly, in both cases, the corresponding construction results in the desired interim payments and is an inactive losers implementation. □

B.2. *Proof of Proposition 2*

Sufficiency. We argue sufficiency first. Suppose the hierarchical mechanism (I, p^h) has no non-trivial atoms.

To begin, note that there exists a hierarchical mechanism (I', p^h) which implements exactly the same ex-post allocation rule and interim payment rule, such that I'_i is strictly increasing for each $i = 1, 2$. To see this, consider any "atom" of buyer i over bid b. By assumption, $I_j^{-1}(b) = \phi$. Define $I'_{i'}(v'') = I'_{i'}(v'') + \epsilon$ for $i' = 1, 2$ and v'' s.t. $I_{i'}(v'') > b$ and some $\epsilon > 0$. Further, "continuously stretch" the $I'_i(v)$ for $v \in I_i^{-1}(b)$ over $[b, b + \epsilon']$ for some $\epsilon' < \epsilon$. Proceed inductively for each atom in I. Note that by construction the ex-post allocation rule implemented by I' is the same as that by I.

Therefore, we now may now suppose I is strictly increasing wlog. Observe that we do not require the index rules to be continuous in values, and therefore there may be jump discontinuities. To ensure continuity in the bid space, we therefore need to be careful about the outcomes at these discontinuities.

Formally, note that since I_1 and I_2 are strictly increasing, G_1 and G_2 do not have any atoms. Further let $\underline{b}_i = I_i(\underline{v}_i)$, $\overline{b}_i = I_i(\overline{v}_i)$. Define $\overline{B}_i = [\underline{b}_i, \overline{b}_i]$, the smallest interval that contains the set of equilibrium bids B_i of buyer i.

Observe that for any $b \in I_i(V_i)$, we require the interim expected payment to be exactly $p_i^h(I_i^{-1}(b))$. We will now extend the desired interim payments for any $b \in \overline{B}_i \backslash B_i$—previously when continuity was not a concern, we set it to be some large payment to deter a deviation. Define it as:

$$p_i^*(b) = \sup_{\beta \in B_i, \beta < b} p_i^h(I_i^{-1}(\beta)) + \overline{v}(b) \left(G_{-i}(b) - \sup_{\beta \in B_i : \beta < b} G_{-i}(b) \right),$$

where $\overline{v}(b) = \inf\{I_i^{-1}(\beta) : \beta \geq b, \beta \in B_i\}$. Note that by construction, $p_i^*(b)$ is continuous in b—the strict increasing-ness of I_1 and I_2 guarantees that G_1 and G_2 are continuous in b. Further note that by construction for $b \in B_i$, $p_i^*(b) = p_i^h(I_i^{-1}(b))$.

We now use this continuous $p_i^*(\cdot)$ to construct an ex-post payment rule that is continuous in both own bid and the opponent's bid. Consider the construction of Theorem 2, with the proviso that the interim expected payment for buyer i bidding $b_i \in \overline{B}_i \backslash B_i$ is $p_i^*(b_i)$.

First consider the case that $b_i \in \overline{B}_i, b_i \notin \overline{B_j}$. In this case the constant payment rule $p^s(b_i, b_j) = p_i^*(b_i)$ for any $b_j \in B_j$ is continuous in both arguments.

Next consider the following payment rule for bids $b_i \in \overline{B}_1 \cap \overline{B}_2$. Let $\hat{b}$ be a bid such that $G_1(\hat{b}) \neq G_2(\hat{b})$. For a given $\varepsilon > 0$, we define $e_i(\varepsilon)$ as

$$e_i(\varepsilon) = \mathbb{E}_i[b_i - \hat{b} \mid \hat{b} \leq b_i \leq \hat{b} + \varepsilon].$$

Consider the payment rule defined as:

$$p^s(b_i, b_j) = \begin{cases} p^u(b_i) & \text{if } b_j \geq \hat{b} + \varepsilon, \\ p^l(b_i) + \frac{b_j - \hat{b}}{\varepsilon}\left(p^u(b_i) - p^l(b_i)\right) & \text{if } \hat{b} \leq b_j < \hat{b} + \varepsilon, \\ p^l(b_i) & \text{if } b_j < \hat{b}. \end{cases}$$

Note that this payment rule is continuous in b_j.

The expected payment for a bid b_i by bidder i is then

$$p^u(b_i)[1 - G_j(\hat{b} + \varepsilon)] + \frac{e_j(\varepsilon)}{\varepsilon}\left(p^u(b_i) - p^l(b_i)\right) + G_j(\hat{b})p^l(b_i),$$

which then yields the following system of equations

$$\begin{bmatrix} 1 - G_2(\hat{b} + \varepsilon) + \frac{e_2(\varepsilon)}{\varepsilon} & G_2(\hat{b}) - \frac{e_2(\varepsilon)}{\varepsilon} \\ 1 - G_1(\hat{b} + \varepsilon) + \frac{e_1(\varepsilon)}{\varepsilon} & G_1(\hat{b}) - \frac{e_1(\varepsilon)}{\varepsilon} \end{bmatrix} \begin{bmatrix} p^u(b_i) \\ p^l(b_i) \end{bmatrix} = \begin{bmatrix} p_1^*(b_i) \\ p_2^*(b_i) \end{bmatrix}.$$

Note that since $\frac{1-G_2(\hat{b})}{1-G_1(\hat{b})} \neq \frac{G_2(\hat{b})}{G_1(\hat{b})}$ and $e_i(\varepsilon)$ is continuous, there exists a small enough $\varepsilon > 0$ such that

$$\frac{1 - G_2(\hat{b} + \varepsilon) + \frac{e_2(\varepsilon)}{\varepsilon}}{1 - G_1(\hat{b} + \varepsilon) + \frac{e_1(\varepsilon)}{\varepsilon}} \neq \frac{G_2(\hat{b}) - \frac{e_2(\varepsilon)}{\varepsilon}}{G_1(\hat{b}) - \frac{e_1(\varepsilon)}{\varepsilon}}.$$

Finally, note by construction $p_i^*(b_i)$ is continuous in b_i. This in turn implies that p^u and p^l are continuous in b_i which completes the proof.

Necessity. Consider any hierarchical mechanism (I, p^h) such that I has a non-trival atom. Suppose that buyer 1 has a positive mass on bid b, and some buyer 2 of value $v_2 \in V_2$ also bids b. Bidding $b + \epsilon$ for $\epsilon > 0$, ϵ small will result in a jump in buyer 2's probability of winning the good. However, by continuity of payments requires that buyer 2's payment must increase continuously. This results in a contradiction. □

B.3. *Proof of Proposition 3*

Consider the payment rule we constructed which pivots around a point $\hat{b}$:

$$\begin{bmatrix} 1 - G_2(\hat{b}) & G_2(\hat{b}) \\ 1 - G_1(\hat{b}) & G_1(\hat{b}) \end{bmatrix} \begin{bmatrix} p^u(b_i) \\ p^l(b_i) \end{bmatrix} = \begin{bmatrix} p_1^h\left(I_1^{-1}(b_i)\right) \\ p_2^h\left(I_2^{-1}(b_i)\right) \end{bmatrix}.$$

Inverting, we get

$$\begin{bmatrix} p^u(b_i) \\ p^l(b_i) \end{bmatrix} = \frac{1}{G_1(\hat{b}) - G_2(\hat{b})} \begin{bmatrix} G_1(\hat{b}) & -G_2(\hat{b}) \\ -(1-G_1(\hat{b})) & 1-G_2(\hat{b}) \end{bmatrix} \begin{bmatrix} p_1^h\left(I_1^{-1}(b_i)\right) \\ p_2^h\left(I_2^{-1}(b_i)\right) \end{bmatrix}$$
$$= \frac{1}{G_1(\hat{b}) - G_2(\hat{b})} \begin{bmatrix} G_1(\hat{b})p_1^h\left(I_1^{-1}(b_i)\right) - G_2(\hat{b})p_2^h\left(I_2^{-1}(b_i)\right) \\ (1-G_2(\hat{b}))p_2^h\left(I_2^{-1}(b_i)\right) - (1-G_1(\hat{b}))p_1^h\left(I_1^{-1}(b_i)\right) \end{bmatrix}.$$

For monotonicity in the opponents bid, we require that $p^u(b_i) - p^l(b_i) \geq 0$ or, equivalently, that

$$\frac{1}{G_1(\hat{b}) - G_2(\hat{b})}\left(p_1^h\left(I_1^{-1}(b_i)\right) - p_2^h\left(I_2^{-1}(b_i)\right)\right) \geq 0. \tag{19}$$

We can now consider two cases.

Case (1): Neither distribution first order stochastically dominates the other.

This implies that there exist $\hat{b}$ and $\hat{b}'$ such that $G_1(\hat{b}) > G_2(\hat{b})$ and $G_1(\hat{b}') < G_2(\hat{b}')$. This immediately implies that there exists a symmetric implementation.

We can now use these to construct a monotone payment rule. For all b_i where $p_1^h\left(I_1^{-1}(b_i)\right) - p_2^h\left(I_1^{-1}(b_i)\right) > 0$, we pivot the payment around $\hat{b}$. Similarly, for all b_i where $p_1^h\left(I_1^{-1}(b_i)\right) - p_2^h\left(I_1^{-1}(b_i)\right) < 0$, we pivot the payment around $\hat{b}'$. This will ensure that (19) is satisfied and hence that payments are monotone.

Case (2): One of the distributions first order stochastically dominates the other, wlog G_1 first order stochastically dominates G_2.

We first show that in this case there exists a symmetric monotone implementation. If $G_1 = G_2$, then we are done. If not then we can take any $\hat{b}$ such that $G_1(\hat{b}) < G_2(\hat{b})$ and construct the usual payment rule. Clearly, this condition implies that (19) will be satisfied.

We now show the converse. Assume without loss of generality that G_1 strictly first order stochastically dominates G_2. Now suppose, in contradiction, that there is a $b \in B_1 \cap B_2$ such that $p_1^h(I_1^{-1}(b)) > p_2^h(I_2^{-1}(b))$ and that there is a symmetric and monotone implementation p^s. First order stochastic dominance would then imply that

$$p_1^h(I_1^{-1}(b)) = \int_{B_2} p^s(b,b_2)dG_2(b_2) \leq \int_{B_1} p^s(b,b_1)dG_1(b_1) = p_2^h(I_2^{-1}(b)),$$

which isn't possible. This completes the proof. $\square$

B.4. *Proof of Proposition 4*

First note that:

$$\begin{aligned}\frac{\mathrm{d}p_i^h(I_i^{-1}(b))}{\mathrm{d}b} &= \frac{\mathrm{d}p_i^h(v_i)}{\mathrm{d}v_i}\Big|_{v_i=I_i^{-1}(b)}\frac{\mathrm{d}I_i^{-1}(b)}{\mathrm{d}b}\\ &= v_i\frac{\mathrm{d}a_i(v_i)}{\mathrm{d}v_i}\frac{1}{I_i'(v_i)}\Big|_{v_i=I_i^{-1}(b)}\\ &= v_i\frac{\mathrm{d}G_j(I_i(v_i))}{\mathrm{d}v_i}\frac{1}{I_i'(v_i)}\Big|_{v_i=I_i^{-1}(b)}\\ &= v_i g_j(I_i(v_i))\Big|_{v_i=I_i^{-1}(b)}\\ &= I_i^{-1}(b)g_j(b).\end{aligned}$$

Since I_j is strictly increasing and differentiable by assumption, $g_j(b)$ exists and therefore, so does $\frac{\mathrm{d}p_i^h(I_i^{-1}(b))}{\mathrm{d}b}$.

Sufficiency. To see that the conditions are sufficient, recall that any symmetric implementation $p^s(b,b')$ must be such that for any $b \in B_1 \cap B_2$

$$\int_{B_2} p^s(b,b')g_2(b')db' = p_1^h(I_1^{-1}(b)),$$
$$\int_{B_1} p^s(b,b')g_1(b')db' = p_2^h(I_2^{-1}(b)).$$

Note that $p_i^h(I_i^{-1}(b))$ is non-decreasing in b by assumption for $i = 1,2$. Therfore for an implementation that is monotone in own bid, it is sufficient to find p^s such that:

$$\int_{B_2} \frac{\mathrm{d}p^s(b,b')}{\mathrm{d}b}g_2(b')db' = \frac{\mathrm{d}p_1^h(I_1^{-1}(b))}{\mathrm{d}b}, \tag{20a}$$
$$\int_{B_1} \frac{\mathrm{d}p^s(b,b')}{\mathrm{d}b}g_1(b')db' = \frac{\mathrm{d}p_2^h(I_2^{-1}(b))}{\mathrm{d}b}, \tag{20b}$$

where $\frac{\mathrm{d}p^s(b,b')}{\mathrm{d}b} \geq 0$.

If both $\frac{\mathrm{d}p_1^h(I_1^{-1}(b))}{\mathrm{d}b}$ and $\frac{\mathrm{d}p_2^h(I_2^{-1}(b))}{\mathrm{d}b}$ equal 0, setting $\frac{\mathrm{d}p^s(b,b')}{\mathrm{d}b} = 0$ solves (20). So suppose not. Without loss suppose $\frac{\mathrm{d}p_2(I_2^{-1}(b)}{\mathrm{d}b} \neq 0$, the other case follows symmetrically.

Pick a set $\underline{B}$ with $G_1(\underline{B}) > 0$ such that $\frac{g_2(b')}{g_1(b')} \leq \frac{\frac{\mathrm{d}p_1^h(I_1^{-1}(b))}{\mathrm{d}b}}{\frac{\mathrm{d}p_2^h(I_2^{-1}(b))}{\mathrm{d}b}}$ for all $b' \in \underline{B}$. Similarly pick a set $\overline{B}$ with $G_2(\overline{B}) > 0$ such that $\frac{\frac{\mathrm{d}p_1^h(I_1^{-1}(b))}{\mathrm{d}b}}{\frac{\mathrm{d}p_2^h(I_2^{-1}(b))}{\mathrm{d}b}} \geq \frac{g_2(b')}{g_1(b')}$ for all $b' \in \underline{B}$.

Finally consider a p^s s.t.:

$$\frac{\mathrm{d}p^s(b,b')}{\mathrm{d}b} = \begin{cases}\bar{x} & b' \in \overline{B}\\ \underline{x} & b' \in \underline{B}\\ 0 & \text{otherwise}\end{cases}$$

Substituting into (20):

$$\underline{x}G_2(\underline{B}) + \overline{x}G_2(\overline{B}) = \frac{\mathrm{d}p_1^h(I_1^{-1}(b))}{\mathrm{d}b},$$
$$\underline{x}G_1(\underline{B}) + \overline{x}G_1(\overline{B}) = \frac{\mathrm{d}p_2^h(I_2^{-1}(b))}{\mathrm{d}b}.$$

By construction, therefore $\underline{x}$ and $\overline{x}$ must be positive. Formally, notice that if this system does not have a non-negative solution, then the Farkas alternative of:

$$y_1 \frac{G_2(\underline{B})}{G_1(\underline{B})} + y_2 \geq 0,$$
$$y_1 \frac{G_2(\overline{B})}{G_1(\overline{B})} + y_2 \geq 0,$$
$$y_1 \frac{\frac{\mathrm{d}p_1(I_1^{-1}(b))}{\mathrm{d}b}}{\frac{\mathrm{d}p_2(I_2^{-1}(b))}{\mathrm{d}b}} + y_2 < 0,$$

must have a solution. Clearly this is impossible by assumption since $\frac{\frac{\mathrm{d}p_1(I_1^{-1}(b))}{\mathrm{d}b}}{\frac{\mathrm{d}p_2(I_2^{-1}(b))}{\mathrm{d}b}} \in \left[\frac{G_2(\underline{B})}{G_1(\underline{B})}, \frac{G_2(\overline{B})}{G_1(\overline{B})}\right]$. Therefore, by construction, we have shown that $\frac{\mathrm{d}p^s(b,b')}{\mathrm{d}b} \geq 0$ for all b'.

Necessity. Once again, recall that any symmetric implementation $p^s(b', b)$ must be such that for any $b' \in B_1 \cap B_2$

$$\int_{B_2} p^s(b', b) g_2(b) db = p_1^h(I_1^{-1}(b')),$$
$$\int_{B_1} p^s(b', b) g_1(b) db = p_2^h(I_2^{-1}(b')).$$

Suppose the condition (9) is violated for some $b'' > b'$. Any symmetric implementation must satisfy:

$$\int_{B_2} (p^s(b'', b) - p^s(b', b)) g_2(b) db = p_1^h(I_1^{-1}(b'') - p_1^h(I_1^{-1}(b')), \tag{21a}$$
$$\int_{B_1} (p^s(b'', b) - p^s(b', b)) g_1(b) db = p_2^h(I_2^{-1}(b'') - p_2^h(I_2^{-1}(b')). \tag{21b}$$

Analogous to the Farkas Lemma argument above, when (9) is violated, there cannot exist a solution to (21) such that $p^s(b'', b) - p^s(b', b) \geq 0$, for all $b \in B_1 \cup B_2$. □

B.5. *Proof of Proposition 5*

We will first prove the theorem as stated for differentiable and strictly increasing index rules (that is, no atoms in G_i). Later, we will extend to the more general case.

Proof. We first demonstrate sufficiency, and then argue necessity.

Sufficiency. For simplicity, we will only define payments for equilibrium bids; off-equilibrium bids can be discouraged in the same way as in the proof of Theorem 2. We consider the two cases of

Condition C2 separately. For (Case 1) we consider the following payment rule:

$$p^s(b,b') = \begin{cases} \frac{1}{G_1(\underline{b}_2)}\left[p_2^h(I_2^{-1}(b)) - \eta(b)\right] & \text{for } b' \in [\underline{b}_1, \underline{b}_2) \\ \hat{p}(b,b') & \text{for } b' \notin [\underline{b}_1, \underline{b}_2) \end{cases}$$

where $\hat{p}$ is given by (C2,P1), and $\eta(b)$ is given by (10).

Similarly, (Case 2) we consider the following payment rule:

$$p^s(b,b') = \begin{cases} \frac{1}{G_1(\underline{b}_2)}\left[p_2^h(I_2^{-1}(b)) - \eta_l(b)\right] & \text{for } b' \in [\underline{b}_1, \underline{b}_2) \\ \hat{p}_l(b,b') & \text{for } b' \notin [\underline{b}_1, \underline{b}_2) \end{cases}$$

where in this case $\hat{p}_\ell$ is given by (C2,P2) for a given $\ell > \underline{\ell}$, and $\eta_l(b)$ is as defined in (11). From Condition C2 and continuity there is a ℓ close enough to $\underline{\ell}$ for which $p^s(b,b') < v(b)$ for $b' \in [\underline{b}_1, \underline{b}_2)$.

By construction, for each buyer, his expected payment will equal his interim payment in the hierarchical mechanism. Condition (C2) guarantees that in the range $b' \in [\underline{b}_1, \underline{b}_2)$, the implementation still satisfies ex-post IR, $p^s(b,b') \leq I_2^{-1}(b)$. Indeed it might be the case that $p^s(b,b') < 0$.

Necessity. We first verify these conditions are necessary for implementation in a symmetric auction where bidding the actual index rule is each buyer's equilibrium strategy. We then show that the same conditions also rule out other implementations as well.

Let us verify that C1 is necessary. So suppose not, i.e. suppose: $v(b)G_2(b) < p_1^h(I_1^{-1}(b))$ for some $b \in B_1 \cap B_2$. Note that if a buyer bids b, and the other bidder bids $b' \in B_1 \cap B_2$, the maximum she can be asked to pay without violating ex-post IR is $v(b)$. But now, for bidder 1, it follows that the maximum expected payment that she can be asked to make is $v(b)G_2(b)$. If her required payment, $p_1^h(I_1^{-1}(b))$, exceeds this, then there cannot be a symmetric, ex-post IR implementation.

With buyer 2, there is a little more 'wriggle room.' When buyer 2 bids $b \in B_1 \cap B_2$, she could be a winner in some 'asymmetric' profiles; i.e. when buyer 1 bids in the range $[\underline{b}_1, \underline{b}_2)$. At these bid profiles, a potentially higher payment (up to $I_2^{-1}(b)$) can be extracted from buyer 2. Condition (C2) then guarantees that the required interim payment, $p_2^h(I_2^{-1}(b))$ can be extracted.

Note that in the construction of either $\hat{p}$ or $\hat{p}_\ell$, either the maximum permissible amount $v(b)$ is being paid by the *winning* buyer, or a rebate of s is being returned to the buyer who bids b. The rebates are being paid when the other buyer's bid b' has the lowest possible value of $L(b')$. This means that the rebates are worth the lowest possible in expectation to a winning buyer 2, because they occur where $L(\cdot)$ is minimized.

We begin by considering the following maximization problem for $b \in B_1 \cap B_2$ and any given $s \leq 0$:

$$m_s(b) = \max_{\varrho(\cdot)} \int_{\underline{b}_2}^{\bar{b}_1} \varrho(b')dG_1(b'), \tag{Max-P}$$

$$\text{s.t.} \int_{\underline{b}_2}^{\bar{b}_2} \varrho(b')dG_2(b') = p_1^h(I_1^{-1}(b)), \tag{λ}$$

$$s \leq \varrho(b') \leq v(b), \qquad \forall b' \in [\underline{b}_2, b], \tag{$\delta(b'),\kappa(b')$}$$

$$s \leq \varrho(b') \leq 0, \qquad \forall b' \in (b, \max\{\bar{b}_1, \bar{b}_2\}]. \tag{$\delta(b'),\kappa(b')$}$$

To understand this optimization program in words, fix a bid b. Think of $\varrho(\cdot)$ as the payment made by the buyer in this case as a function of the other buyer's bid. The program asks what the maximum expected payment that can be extracted from buyer 2 is subject to constraints we describe next. The first constraint requires that the expected payment of buyer 1 under $\varrho(\cdot)$ is his correct interim payment. The latter two constraints require that $\varrho(\cdot)$ is pointwise bounded below by s and bounded above by the maximum possible ex-post IR payment $v(b)$ when winning and 0 when losing. The terms in the parentheses to the right of the constraints denote the corresponding dual (co-state) variables.

We claim that $\lim_{s\downarrow-\infty} m_s(b) = \eta(b)$. When $v(b)G_2(b) = p_1^h(I_1^{-1}(b))$, then $\varrho(b') = v(b)$ for all $b' \in [\underline{b}_2, b]$ is the only feasible function, so this case is trivial. Hence, we focus on the case $v(b)G_2(b) > p_1^h(I_1^{-1}(b))$.

The Hamiltonian in this case is:

$$\begin{aligned} g_1(b') - \lambda g_2(b') + \delta(b') - \kappa(b') = 0, \\ \implies \frac{g_1(b')}{g_2(b')} - \lambda + \frac{1}{g_2(b')}(\delta(b') - \kappa(b')) = 0 \end{aligned}$$

with complementary slackness conditions:

$$\begin{aligned} & \delta(b')(s - \varrho(b')) = 0, \\ \text{for } b' \in [\underline{b}_2, b], \quad & \kappa(b')(v(b) - \varrho(b')) = 0, \\ \text{for } b' \in (b, \max\{\bar{b}_1, \bar{b}_2\}], \quad & \kappa(b')\varrho(b') = 0, \\ \text{and} \quad & \delta(b'), \kappa(b') \geq 0. \end{aligned}$$

By observation, the solution to this for any s is 'bang bang', i.e.

$$\varrho(b') = \begin{cases} s & \text{if } L(b') \leq \lambda^\star, \\ v(b) & \text{if } L(b') > \lambda^\star, b' \in [\underline{b}_2, b], \\ 0 & \text{otherwise.} \end{cases}$$

with $\lambda^\star$ selected such that the corresponding primal equation binds for $\varrho(\cdot)$ selected thus. The corner case that needs care is when $G_2(\bar{\bar{\gamma}}(\underline{\ell})) > 0$. In this case, there is a positive measure of $b' \in [\underline{b}_2, \bar{b}_2]$ such that $L(b') = \underline{\ell}$. Here, the solution is bang bang, but possibly (depending on s), there is $\hat{B} \subseteq \bar{\bar{\gamma}}(\underline{\ell})$ such that

$$\varrho(b') = \begin{cases} s & \text{if } b' \in \hat{B} \subseteq \bar{\bar{\gamma}}(\underline{\ell}), \\ v(b) & \text{if } b' \in [\underline{b}_2, b] \backslash \hat{B}, \\ 0 & \text{otherwise.} \end{cases}$$

It follows by construction, therefore, that $\lim_{s\downarrow-\infty} m_s(b) = \eta(b)$. Therefore, subject to the payment rule extracting the appropriate interim payment $p_1^h(I_1^{-1}(b))$ when buyer 1 bids b, $\eta(b)$ is the maximum expected payment that can be extracted from buyer 2 when she bids b and buyer 1 makes a bid higher than $\underline{b}_2$. It follows therefore that if inequality (C2) is violated, there cannot be an implementation satisfying both symmetry and ex-post individual rationality.

Next, consider any other mechanism (I', p^h) with a differentiable index rule, that implements the same mechanism. Then, it must be that

$$I_i'(v_i) = \Gamma(I_i(v_i)).$$

for some differentiable and strictly increasing function $\Gamma : \mathbb{R} \to \mathbb{R}$. Note that the resulting distribution on bids, which we shall denote by G_i', is

$$G_i'(\Gamma(b)) = G_i(b).$$

Note that this implies that

$$g_i'(\Gamma(b))\Gamma'(b) = g_i(b).$$

Our previous arguments already imply that Conditions C1 and C2, written in terms of G_i''s are necessary for an implementation. By the equations above, we see that for $b \in B_1 \cap B_2$

$$v(b)G_2'(\Gamma(b)) \geq p_1^h(I_1^{-1}(b)) \implies v(b)G_2(b) \geq p_1^h(I_1^{-1}(b)).$$

Also, for any $b \in B_2$,

$$L(b) = \frac{g_1(b)}{g_2(b)} = \frac{g_1'(\Gamma(b))}{g_2'(\Gamma(b))}.$$

Therefore our conditions in terms of the original G_i's are necessary for any pure strategy implementation. $\square$

Weakly increasing index rules. So far we have only considered strictly increasing index rules. If the index rules are not strictly increasing, the corresponding bid distributions will have atoms. Denote by $\mathcal{B}_i$ the atoms in G_i. For $b_i \in \mathcal{B}_i$, the size of the atom is $G_i(\{b_i\})$—recall that this is a measure and not a density. Further, $I_i^{-1}(\cdot)$ may be correspondence— $v(\cdot)$ may not be well defined. Redefine $v(b)$ as

$$v(b) = \inf\{v \in I_1^{-1}(b) \cup I_2^{-1}(b)\}.$$

Note that when $I_1^{-1}(b)$ and $I_2^{-1}(b)$ are singletons, this is the same as the old definition of $v(b)$. Now Condition C1 will be as before with this extended definition of $v(\cdot)$.

Next, note that Condition C2 depends on g_1/g_2, which again may not be well defined. We redefine $L(\cdot)$ as follows

$$L(b) = \begin{cases} \frac{g_1(b)}{g_2(b)} & b \in B_2 \text{ and } b \notin \mathcal{B}_1 \cup \mathcal{B}_2, \\ \frac{G_1(\{b\})}{G_2(\{b\})} & b \in \mathcal{B}_1 \cap \mathcal{B}_2, \\ 0 & b \in \mathcal{B}_2 \backslash \mathcal{B}_1. \end{cases}$$

We can now redefine $\eta(b)$ with this definition $L(b)$. It should be clear that Conditions C1 and C2 thus extended are necessary and sufficient.

B.6. *Symmetric Ex-Post IR Implementation with Common Lower Bound of Bid Space Support*

We now use the previous intuition to derive axioms for the case where $\underline{b}_1 = \underline{b}_2$. This adds a little more complexity to our analysis. To see why, recall that our previous implementation 'heavily' used the fact that $\underline{b}_1 < \underline{b}_2$. In particular, profiles of the sort (b, b') where $b \in B_1 \cap B_2$ and $b' < \underline{b}_2$ were used as a sort of residual claimant. The payment of the winning buyer in profiles could be set as high v_2 to make up for any 'shortfall' in buyer 2's expected payment vis-a-vis her interim payment. Conversely, she can be given a rebate to make up for any surplus.

Since $G_1(\underline{b}_2) = 0$, Condition C2 rewritten in this case reflects the fact that there is no such region to make up for any shortfall:

Definition B.1 (Condition C2′)**.** Condition C2′ requires that for all b in $B_1 \cap B_2$, with $\underline{b}_1 = \underline{b}_2$

$$\eta(b) \geq p_2^h(I_2^{-1}(b)) \tag{22}$$

with the inequality holding strictly for any b such that:

$$G_2(\bar{\bar{\gamma}}(\underline{\ell})) = 0 \text{ and } v(b)G_2(b) > p_1^h(I_1^{-1}(b)).$$

Intuitively, Condition $C2'$ requires that the maximum expected payment $\eta(b)$ that can be extracted from buyer 2 when she bids b, among all payment rules that extract exactly $p_1^h(I_1^{-1}(b))$ from buyer 1 in expectation, is more than $p_2^h(I_2^{-1}(b))$. In the previous section this was enough, because any excess $\eta(b) - p_2^h(I_2^{-1}(b))$ can be rebated to buyer 2 when the other buyer bids in the range $[\underline{b}_1, \underline{b}_2)$. Now, this is no longer enough.

We need an additional condition to account for the fact that there is no lower region to 'rebate' any surplus to. We now write down the exact analog condition, i.e. that the minimum expected payment $\zeta(b)$ that can be extracted from buyer 2 when she bids b, among all payment rules that extract exactly $p_1^h(I_1^{-1}(b))$ from buyer 1 in expectation, is at most $p_2^h(I_2^{-1}(b))$.

If both conditions hold, there clearly exists a payment rule which will achieve the required implementation, since the set of all payment rules that extract exactly $p_1^h(I_1^{-1}(b))$ from buyer 1 in expectation is convex.

We consider two cases depending on the ordering of the upper bound of the possible bids, $\bar{b}_1$ and $\bar{b}_2$.

If $\bar{b}_1 > \bar{b}_2$, we can rebate money to buyer 2 similarly as before—in this case when the other bidder bids in the range $(\bar{b}_2, \bar{b}_1]$. In this case define $\zeta(b) = 0$ for all $b \in B_1 \cap B_2$.

Now let us consider the other case, i.e. that $\bar{b}_1 \leq \bar{b}_2$—in this case $B_1 \subseteq B_2$. We need some additional notation. First, we define

$$\bar{\ell} = \max_{b' \in ([\underline{b}_2, \bar{b}_2] \setminus \{b : g_1(b) = g_2(b) = 0\})} L(b').$$

As before $\bar{\ell}$ is well defined. As before, there are two sub-cases. The first sub-case is when

$$G_2(\bar{\bar{\gamma}}(\bar{\ell})) > 0.$$

Let $\hat{B} \subset \bar{\bar{\gamma}}(\bar{\ell}(b)) > 0$ be a (potentially empty) subset such that:

$$v(b)G_2\left([\underline{b}, b] \setminus \hat{B}\right) \geq p_1^h(I_1^{-1}(b)).$$

We now define a payment rule

$$\hat{p}'(b,b') = \begin{cases} v(b) & \text{for } b' \in [\underline{b}_2, b]\backslash \hat{B}, \\ s & \text{for } b' \in \hat{B}, \\ 0 & \text{o.w.} \end{cases} \tag{C3,P1}$$

where s is chosen to solve

$$v(b)(G_2([\underline{b}, b]\backslash \hat{B})) + sG_2(\hat{B}) = p_1^h(I_1^{-1}(b)).$$

Notice that s here is a subsidy. We set:

$$\zeta(b) = \int_{\underline{b}_2}^{\overline{b}_2} \hat{p}'(b,b')dG_1(b').$$

The second sub-case is when

$$G_2(\bar{\bar{\gamma}}(\bar{\ell})) = 0,$$

we define the payment rule for $\ell < \bar{\ell}$

$$\hat{p}'_\ell(b,b') = \begin{cases} v(b) & \text{for } b' \in [\underline{b}, b],\ L(b') \leq \ell, \\ s & \text{for } L(b') > \ell, \\ 0 & \text{otherwise.} \end{cases} \tag{C3,P2}$$

where s is chosen to solve

$$v(b)G_2(\{b' : b' \in [\underline{b}, b], L(b') \leq \ell\}) + sG_2(\{b' : L(b') > \ell\}) = p_1^h(I_1^{-1}(b)).$$

Here we set

$$\zeta(b) = \lim_{\ell \uparrow \bar{\ell}} \left[\int_{\underline{b}_2}^{b} \hat{p}'_\ell(b,b')dG_1(b') \right].$$

Definition B.2 (Condition C3)**.** Condition C3 requires that

$$\zeta(b) \leq p_2^h(I_2^{-1}(b))$$

with the inequality holding strictly when:

$$G_2(\bar{\bar{\gamma}}(\bar{\ell})) = 0 \text{ and } v(b)G_2(b) > p_1^h(I_1^{-1}(b)).$$

We can now state the proposition

Proposition 6. *Suppose there are 2 buyers. Consider a hierarchical allocation mechanism (I, p^h) with differentiable and strictly increasing index functions such that the lower bounds of the supports of the bid distributions are the same, that is, $\underline{b}_1 = \underline{b}_2$. Then Conditions C1, C2′ and C3 are necessary and sufficient for there to exist a symmetric, ex-post IR implementation of (I, p^h).*

The proof follows from also considering the analogous minimization problem to (Max-P) and is omitted.

APPENDIX C. FULL RANK EVENTS

A little more notation will be useful. We say that an event $E \subseteq \mathbb{R}^{n-1}$ is of type l if there exists a $\beta \in \mathbb{R}$ such that E is the event "l randomly chosen buyers out of the $n-1$ have bids of β or less." For any number k, let $[k] \equiv \{1, 2, \ldots, k\}$. For any set K, $|K| = k$, $l \leq k$, define

$$\binom{K}{l} \equiv \{X : X \subseteq K, |X| = l\},$$

that is, the set of all subsets of K of cardinality exactly l.

By definition, if E is an event of type l with corresponding β, then

$$G_{-i}(E) = \frac{l!(n-1-l)!}{(n-1)!} \sum_{M \in \binom{N \setminus i}{l}} \prod_{j \in M} G_j(\beta). \tag{23}$$

We also allow for an event of type l to have a random cutoff $\tilde{\beta} \in \Delta\mathbb{R}$. This corresponds to the event that there are l randomly chosen buyers out of the $n-1$ and each of them has a bid less than an i.i.d. realization of the random variable $\tilde{\beta}$. Denote by $G_j(\tilde{\beta})$ the probability that a draw according to G_j is less than or equal to the random variable $\tilde{\beta}$.

Note that if we have an event E of type l with corresponding cutoff $\tilde{\beta}$,

$$G_{-i}(E) = \frac{l!(n-1-l)!}{(n-1)!} \sum_{M \in \binom{N \setminus i}{l}} \prod_{j \in M} G_j(\tilde{\beta}). \tag{24}$$

Recall the theorem:

Theorem 2. For any $n > 1$ and any $k \leq n$ such that $G_1, G_2, \ldots G_k$ are all pairwise distinct, there exist symmetric events $E_1, \ldots, E_k \subseteq \mathbb{R}^{n-1}$ such that the $(k \times k)$ matrix $\mathcal{M} = [G_{-i}(E_j)]_{i,j=1}^{k}$ has full rank.

Proof. Fix the number of buyers $n > 1$. We will prove the lemma by induction on k.
Base Case: $k = 2$. Since $G_1 \neq G_2$, pick $\beta^\star \in \mathbb{R}$ s.t. $G_1(\beta^\star) \neq G_2(\beta^\star)$. Now pick E_1 to be an event of type 1 with cutoff $\beta^\star$, and $E_2 = \mathbb{R}^{n-1} \setminus E_1$. The corresponding matrix $\mathcal{M}$ is

$$\mathcal{M} = \begin{bmatrix} \frac{1}{n-1}\sum_{i \neq 1} G_i(\beta^\star) & 1 - \frac{1}{n-1}\sum_{i \neq 1} G_i(\beta^\star) \\ \frac{1}{n-1}\sum_{i \neq 2} G_i(\beta^\star) & 1 - \frac{1}{n-1}\sum_{i \neq 2} G_i(\beta^\star) \end{bmatrix}.$$

By observation, this is full rank.
Inductive Hypothesis. Suppose this is true for all $k \leq \hat{k}$ for some $\hat{k} < n$.
Inductive step. We will show this true for $k = \hat{k} + 1$. By the inductive hypothesis we have events $E_1, \ldots, E_{\hat{k}} \subseteq \mathbb{R}^{n-1}$ such that

$$\mathcal{M} = [G_{-i}(E_j)]_{i,j=1}^{\hat{k}} \text{ is full rank.}$$

We need to show that we can find a $E_{\hat{k}+1}$ such that

$$\mathcal{M}' = [G_{-i}(E_j)]_{i,j=1}^{\hat{k}+1} \text{ is full rank.}$$

Note that since $\mathcal{M}$ is full rank, there exists a unique row-vector $\alpha \in \mathbb{R}^{\hat{k}}$ such that:

$$\alpha\mathcal{M} = \left[G_{-(\hat{k}+1)}(E_1), G_{-(\hat{k}+1)}(E_2), \dots G_{-(\hat{k}+1)}(E_{\hat{k}})\right]$$

If it is not the case that

$$\sum_{i=1}^{\hat{k}} \alpha_i = 1,$$

then we are already done. To see this note that we can select $E_{\hat{k}+1} = \mathbb{R}^{n-1}$. With this selection, $\mathcal{M}'$ will be full rank, since $G_{-i}(\mathbb{R}^{n-1}) = 1$ for all i by definition, and therefore

$$\sum_{i=1}^{\hat{k}} \alpha_i G_{-i}(\mathbb{R}^{n-1}) \neq G_{-(\hat{k}+1)}(\mathbb{R}^{n-1}).$$

We will now proceed to prove that there exists an event such that M' is full rank. In particular, we will show that either $\mathbb{R}^{n-1}$ suffices or there must exist an event of type 1 to $\hat{k}$. So suppose that for any event $E_{\hat{k}+1}$ of type 1, the matrix $\mathcal{M}'$ is not full rank. For any event of type 1 with corresponding cutoff β, by (23)

$$G_{-i}(E_{\hat{k}+1}) = \frac{1}{n-1} \sum_{j=1, j\neq i}^{n} G_j(\beta).$$

Since by assumption no such event $E_{\hat{k}+1}$ results in a full rank matrix, we have that for all $E_{\hat{k}+1}$ of type 1 with corresponding β,

$$G_{-(\hat{k}+1)}(E_{\hat{k}+1}) = \sum_{i=1}^{\hat{k}} \alpha_i G_{-i}(E_{\hat{k}+1}),$$

$$\implies \forall \beta \in \mathbb{R},\ G_{\hat{k}+1}(\beta) = \sum_{i=1}^{\hat{k}} \alpha_i G_i(\beta).$$

As notational shorthand, we will write this as

$$G_{\hat{k}+1} = \sum_{i=1}^{\hat{k}} \alpha_i G_i.$$

Claim 1. Suppose $\hat{l} \leq \hat{k}$ is such that for all $l = 1 \dots \hat{l}$, selecting $E_{\hat{k}+1}$ from events of types 1 to $\hat{l}$ cannot make $\mathcal{M}'$ full rank. Then for all $l = 1 \dots \hat{l}$:

$$(G_{\hat{k}+1})^l = \sum_{i=1}^{\hat{k}} \alpha_i (G_i)^l, \tag{25}$$

$$\sum_{M \in \binom{[\hat{k}]}{l}} \left(1 - \sum_{i \in M} \alpha_i\right) \prod_{i \in M} G_i = \left(\sum_{M \in \binom{[\hat{k}]}{l}} \prod_{i \in M} G_i\right) - G_{\hat{k}+1} \sum_{M \in \binom{[\hat{k}]}{l-1}} \left(1 - \sum_{i \in M} \alpha_i\right) \prod_{i \in M} G_i. \tag{26}$$

Recall that (25) is notational shorthand for

$$\forall \tilde{\beta} \in \Delta\mathbb{R}: \; (G_{\hat{k}+1}(\tilde{\beta}))^l = \sum_{i=1}^{\hat{k}} \alpha_i (G_i(\tilde{\beta}))^l,$$

Proof of Claim 1. We prove this claim by induction on $\hat{l}$. While the base case $\hat{l} = 1$ is true by observation, to build intuition we will now prove it for the case of $\hat{l} = 2$. Since by assumption no event of type 2 produces a full rank matrix, it must be that for every event E of type 2,

$$G_{-(\hat{k}+1)}(E) = \sum_{i=1}^{\hat{k}} \alpha_i G_{-i}(E).$$

Substituting in from (23), and canceling terms, we have

$$\begin{aligned}
\sum_{M \in \binom{[k]}{2}} \prod_{i \in M} G_i &= \sum_{q=1}^{\hat{k}} \alpha_q \left(\sum_{M \in \binom{[\hat{k}+1] \setminus q}{2}} \prod_{i \in M} G_i \right), \\
&= \sum_{q=1}^{\hat{k}} \alpha_q \left(\sum_{M \in \binom{[\hat{k}] \setminus q}{2}} \prod_{i \in M} G_i + G_{\hat{k}+1} \sum_{i=1, i \neq q}^{\hat{k}} G_i \right),
\end{aligned}$$

since $\sum_{i=1}^{\hat{k}} \alpha_i = 1$, we have,

$$= \sum_{M \in \binom{[\hat{k}]}{2}} \left(1 - \sum_{i \in M} \alpha_i \right) \prod_{i \in M} G_i + G_{\hat{k}+1} \sum_{i=1}^{\hat{k}} (1 - \alpha_i) G_i.$$

By observation therefore we have (26) for $\hat{l} = 2$. Substituting in that $\sum_i \alpha_i G_i = G_{\hat{k}+1}$, we have

$$\begin{aligned}
\sum_{M \in \binom{[k]}{2}} \prod_{i \in M} G_i &= \sum_{M \in \binom{[\hat{k}]}{2}} \left(1 - \sum_{i \in M} \alpha_i \right) \prod_{i \in M} G_i + \left(\sum_{i=1}^{\hat{k}} \alpha_i G_i \right) \sum_{i=1}^{\hat{k}} G_i - (G_{\hat{k}+1})^2 \\
\implies 0 &= \sum_{M \in \binom{[\hat{k}]}{2}} \left(- \sum_{i \in M} \alpha_i \right) \prod_{i \in M} G_i + \left(\sum_{i=1}^{\hat{k}} \alpha_i G_i \right) \sum_{i=1}^{\hat{k}} G_i - (G_{\hat{k}+1})^2.
\end{aligned}$$

Canceling terms, we have

$$\begin{aligned}
0 &= \sum_{i=1}^{\hat{k}} \alpha_i (G_i)^2 - (G_{\hat{k}+1})^2, \\
\implies (G_{\hat{k}+1})^2 &= \sum_{i=1}^{\hat{k}} \alpha_i (G_i)^2,
\end{aligned}$$

as desired.

For our inductive hypothesis, assume that (25,26) are true for all $l \leq \hat{l} - 1$ and now suppose that no event of type $\hat{l}$ can make matrix $\mathcal{M}'$ full rank. We are therefore left to show (25,26) for $l = \hat{l}$. It therefore must be that for any event E of type $\hat{l}$,

$$G_{-(\hat{k}+1)}(E) = \sum_{i=1}^{\hat{k}} \alpha_i G_{-i}(E).$$

Substituting in from (23), and canceling terms, we have

$$\begin{aligned}\sum_{M \in \binom{[\hat{k}]}{\hat{l}}} \prod_{i \in M} G_i &= \sum_{q=1}^{\hat{k}} \alpha_q \left(\sum_{M \in \binom{[\hat{k}+1] \setminus q}{\hat{l}}} \prod_{i \in M} G_i \right), \\ &= \sum_{q=1}^{\hat{k}} \alpha_q \left(\sum_{M \in \binom{[\hat{k}] \setminus q}{\hat{l}}} \prod_{i \in M} G_i + G_{\hat{k}+1} \sum_{M \in \binom{[\hat{k}] \setminus q}{\hat{l}-1}} \prod_{i \in M} G_i \right),\end{aligned}$$

since $\sum_{i=1}^{\hat{k}} \alpha_i = 1$, we have,

$$= \sum_{M \in \binom{[\hat{k}]}{\hat{l}}} \left(1 - \sum_{i \in M} \alpha_i \right) \prod_{i \in M} G_i + G_{\hat{k}+1} \sum_{M \in \binom{[\hat{k}]}{\hat{l}-1}} (1 - \sum_{i \in M} \alpha_i) \prod_{i \in M} G_i.$$

Therefore, we have (26) as desired for $\hat{l}$. Rearranging, we have

$$\sum_{M \in \binom{[\hat{k}]}{\hat{l}}} \left(\sum_{i \in M} \alpha_i \right) \prod_{i \in M} G_i = G_{\hat{k}+1} \left(\sum_{M \in \binom{[\hat{k}]}{\hat{l}-1}} (1 - \sum_{i \in M} \alpha_i) \prod_{i \in M} G_i \right).$$

Substituting the term in the parentheses on the right hand side from (26) for $\hat{l} - 1$,

$$\sum_{M \in \binom{[\hat{k}]}{\hat{l}}} \left(\sum_{i \in M} \alpha_i \right) \prod_{i \in M} G_i = G_{\hat{k}+1} \left(\left(\sum_{M \in \binom{[\hat{k}]}{l-1}} \prod_{i \in M} G_i \right) - G_{\hat{k}+1} \sum_{M \in \binom{[\hat{k}]}{l-2}} \left(1 - \sum_{i \in M} \alpha_i \right) \prod_{i \in M} G_i. \right).$$

Proceeding inductively and collecting terms, we have

$$\sum_{M\in\binom{[\hat{k}]}{\hat{l}}}\left(\sum_{i\in M}\alpha_i\right)\prod_{i\in M}G_i = G_{\hat{k}+1}\left(\sum_{s=0}^{\hat{l}-2}(-1)^s(G_{\hat{k}+1})^s\sum_{M\in\binom{[\hat{k}]}{\hat{l}-1-s}}\prod_{i\in M}G_i + (-1)^{\hat{l}-1}(G_{\hat{k}+1})^{\hat{l}-1}\right)$$

$$\implies \sum_{M\in\binom{[\hat{k}]}{\hat{l}}}\left(\sum_{i\in M}\alpha_i\right)\prod_{i\in M}G_i = (-1)^{\hat{l}-1}(G_{\hat{k}+1})^{\hat{l}} + G_{\hat{k}+1}\sum_{M\in\binom{[\hat{k}]}{\hat{l}-1}}\prod_{i\in M}G_i$$

$$+ G_{\hat{k}+1}\left(\sum_{s=1}^{\hat{l}-2}(-1)^s(G_{\hat{k}+1})^s\sum_{M\in\binom{[\hat{k}]}{\hat{l}-1-s}}\prod_{i\in M}G_i\right).$$

$$\implies 0 = (-1)^{\hat{l}-1}(G_{\hat{k}+1})^{\hat{l}} + \sum_{M\in\binom{[\hat{k}]}{\hat{l}-1}}\left(\sum_{i\in M}\alpha_i G_i\right)\prod_{i\in M}G_i$$

$$+ G_{\hat{k}+1}\left(\sum_{s=1}^{\hat{l}-2}(-1)^s(G_{\hat{k}+1})^s\sum_{M\in\binom{[\hat{k}]}{\hat{l}-1-s}}\prod_{i\in M}G_i\right).$$

$$\implies 0 = (-1)^{\hat{l}-1}(G_{\hat{k}+1})^{\hat{l}} + \sum_{M\in\binom{[\hat{k}]}{\hat{l}-1}}\left(\sum_{i\in M}\alpha_i G_i\right)\prod_{i\in M}G_i - (G_{\hat{k}+1})^2\sum_{M\in\binom{[\hat{k}]}{\hat{l}-2}}\prod_{i\in M}G_i$$

$$+ G_{\hat{k}+1}\left(\sum_{s=2}^{\hat{l}-2}(-1)^s(G_{\hat{k}+1})^s\sum_{M\in\binom{[\hat{k}]}{\hat{l}-1-s}}\prod_{i\in M}G_i\right).$$

Substituting in from (25) for $l = 2$,

$$\implies 0 = (-1)^{\hat{l}-1}(G_{\hat{k}+1})^{\hat{l}} + \sum_{M\in\binom{[\hat{k}]}{\hat{l}-1}}\left(\sum_{i\in M}\alpha_i G_i\right)\prod_{i\in M}G_i - \left(\sum_{i=1}^{\hat{k}}\alpha_i(G_i)^2\right)\sum_{M\in\binom{[\hat{k}]}{\hat{l}-2}}\prod_{i\in M}G_i$$

$$+ G_{\hat{k}+1}\left(\sum_{s=2}^{\hat{l}-2}(-1)^s(G_{\hat{k}+1})^s\sum_{M\in\binom{[\hat{k}]}{\hat{l}-1-s}}\prod_{i\in M}G_i\right).$$

Canceling terms

$$\implies 0 = (-1)^{\hat{l}-1}(G_{\hat{k}+1})^{\hat{l}} - \sum_{M\in\binom{[\hat{k}]}{\hat{l}-2}}\left(\sum_{i\in M}\alpha_i (G_i)^2\right)\prod_{i\in M}G_i$$

$$+ G_{\hat{k}+1}\left(\sum_{s=2}^{\hat{l}-2}(-1)^s(G_{\hat{k}+1})^s\sum_{M\in\binom{[\hat{k}]}{\hat{l}-1-s}}\prod_{i\in M}G_i\right).$$

Continuing to open out the summation and cancel terms, we have, as desired,

$$(G_{\hat{k}+1})^{\hat{l}} = \sum_{i=1}^{\hat{k}} \alpha_i (G_i)^{\hat{l}}.$$

This concludes the proof of the claim. $\square$

Having shown Claim 1, we now show that there exist an event of type 1 to $\hat{k}$ such that the matrix $\mathcal{M}'$ has full rank. To see this, assume the converse. Then, by (25) we have that

$$\forall l = 1 \dots \hat{k},\ (G_{\hat{k}+1})^l = \sum_{i=1}^{\hat{k}} \alpha_i (G_i)^l,$$

and further we know by our previous arguments that

$$1 = \sum_{i=1}^{\hat{k}} \alpha_i.$$

We can rewrite these together as

$$\forall l = 0 \dots \hat{k},\ (G_{\hat{k}+1})^l = \sum_{i=1}^{\hat{k}} \alpha_i (G_i)^l,$$

We now have $\hat{k}+1$ functional equations, but only $\hat{k}$ variables (α's). Since the distributions are different, it should be intutive that this system of equations cannot have a solution.

Claim 2. Suppose the distributions G_1 to $G_{\hat{k}+1}$ are pairwise different. Then,

$$\exists \tilde{\beta} \in \Delta\mathbb{R} \text{ s.t. } G_1(\tilde{\beta}) \text{ to } G_{\hat{k}+1}(\tilde{\beta}) \text{ are all different.}$$

Proof. Consider the subset of $\mathbb{R}^{\hat{k}+1}$ defined as

$$S \equiv \{(a_1, a_2, \dots, a_{\hat{k}+1}) : \exists \tilde{\beta} \in \Delta\mathbb{R} \text{ s.t. } a_j = G_j(\tilde{\beta}) \text{ for } j = 1, \dots, \hat{k}+1\}.$$

Further, for every j, j', define $X_{j,j'} \subseteq \mathbb{R}^{\hat{k}+1}$

$$X_{j,j'} \equiv \{(a_1, a_2, \dots, a_{\hat{k}+1}) : a_j = a_{j'}\}.$$

Note that each $X_{j,j'}$ is a $\hat{k}$ dimensional subspace of $\mathbb{R}^{\hat{k}+1}$.

By definition S is convex. Since the distributions are pairwise different, for every j, j' there exists $\beta \in \mathbb{R}$ such that $G_j(\beta) \neq G_{j'}(\beta)$. Therefore for each j, j', $S \not\subseteq X_{j,j'}$. Further note that $X \equiv \bigcup_{j \neq j'} X_{j,j'}$ is not convex, so $S \not\subseteq X$, and therefore we have our desired result. $\square$

Note that by Claim 2, possibly by adding a little weight on a low β such that $G_j(\beta) = 0$ for all j, we have that there exists $\tilde{\beta} \in \Delta\mathbb{R}$ such that all $G_1(\tilde{\beta})$ to $G_{\hat{k}+1}(\tilde{\beta})$ are pairwise different, and also different from 1.

Therefore, for this $\tilde{\beta}$, there must exist a solution to:

$$\forall l = 0, \dots, \hat{k}, (G_{\hat{k}+1}(\tilde{\beta}))^l = \sum_{i=1}^{\hat{k}} \alpha_i (G_i(\tilde{\beta}))^l.$$

Taking the appropriate Farkas alternative, therefore, for the previous system to have a solution, here there must exist a non-zero solution $\nu \in \mathbb{R}^{\hat{k}+1}$ to:[21]

$$\forall i = 1, \ldots \hat{k} + 1 : \sum_{l=0}^{\hat{k}} \nu_l (G_i(\tilde{\beta}))^l = 0.$$

But note that this suggests there are $\hat{k} + 1$ distinct roots of the $\hat{k}$ degree polynomial

$$\sum_{l=0}^{\hat{k}} \nu_l x^l,$$

which is impossible. Therefore there is no solution to

$$\forall l = 0, \ldots, \hat{k}, (G_{\hat{k}+1}(\tilde{\beta}))^l = \sum_{i=1}^{\hat{k}} \alpha_i (G_i(\tilde{\beta}))^l,$$

concluding our proof. □

[21]The Farkas Lemma states that either the system $Cx = d$ has a solution or $yC = 0$, $yd > 0$ has a solution but never both. For the latter system to have no solution, it must be that for every non-zero y such that $yC = 0$, it is the case that $yd = 0$. This is the version we stated.

References

(1995): *Adarand Constructors, Inc. v. Pena*, vol. 515 U.S. 200. Supreme Court.

ABDULKADIROĞLU, A., AND T. SÖNMEZ (1998): "Random serial dictatorship and the core from random endowments in house allocation problems," *Econometrica*, pp. 689–701.

ALIPRANTIS, C. D., AND K. C. BORDER (2006): *Infinite Dimensional Analysis*. Springer-Verlag.

ATHEY, S., D. COEY, AND J. LEVIN (2013): "Set-Asides and Subsidies in Auctions," *American Economic Journal: Microeconomics*, 5(1), 1–27.

BOGOMOLNAIA, A., AND H. MOULIN (2001): "A new solution to the random assignment problem," *Journal of Economic Theory*, 100(2), 295–328.

BORDER, K. C. (1991): "Implementation of reduced form auctions: A geometric approach," *Econometrica*, pp. 1175–1187.

BROOKS, B. A. (2013a): "Revenue sharing in second-price auctions," Discussion paper.

——— (2013b): "Surveying and selling: Belief and surplus extraction in auctions," Discussion paper.

CANTILLON, E. (2008): "The effect of Bidders' Asymmetries on Expected Revenue in Auctions," *Games and Economic Behavior*, 62(1), 1–25.

CHAN, J., AND E. EYSTER (2003): "Does banning affirmative action lower college student quality?," *American Economic Review*, pp. 858–872.

CORNS, A., AND A. SCHOTTER (1999): "Can affirmative action be cost effective? An experimental examination of price-preference auctions," *American Economic Review*, 89(1), 291–305.

CRAMTON, P., AND I. AYRES (1996): "Deficit reduction through diversity: How affirmative action at the FCC increased auction competition," *Stanford Law Review*, 48, 761–815.

CRÉMER, J., AND R. P. MCLEAN (1988): "Full extraction of the surplus in Bayesian and dominant strategy auctions," *Econometrica*, 56(6), 1247–1257.

D'ASPREMONT, C., AND B. PELEG (1988): "Ordinal Bayesian incentive compatible representations of committees," *Social Choice and Welfare*, 5(4), 261–279.

FRYER, R. G., G. C. LOURY, AND T. YURET (2008): "An economic analysis of color-blind affirmative action," *Journal of Law, Economics, and Organization*, 24(2), 319–355.

GERSHKOV, A., J. GOEREE, A. KUSHNIR, B. MOLDOVANU, AND X. SHI (2013): "On the equivalence of Bayesian and dominant strategy implementation," *Econometrica*, 81(1), 197–220.

GOEREE, J. K., E. MAASLAND, S. ONDERSTAL, AND J. L. TURNER (2005): "How (not) to raise money," *Journal of Political Economy*, 113(4), 897–918.

GÜTH, W., AND E. VAN DAMME (1986): "A comparison of pricing rules for auctions and fair division games," *Social Choice and Welfare*, 3(3), 177–198.

KIRKEGAARD, R. (2012): "A Mechanism Design Approach to Ranking Asymmetric Auctions," *Econometrica*, 80(5), 2349–2364.

LEBRUN, B. (2006): "Uniqueness of the equilibrium in first-price auctions," *Games and Economic Behavior*, 55(1), 131–151.

——— (2013): "Revenue-Superior Variants of the Second-Price Auction," Discussion paper.

LEVIN, D., AND J. L. SMITH (1994): "Equilibrium in auctions with entry," *American Economic Review*, pp. 585–599.

MANELLI, A., AND D. VINCENT (2010): "Bayesian and Dominant-Strategy Implementation in the Independent Private-Values Model," *Econometrica*, 78(6), 1905–1938.

MASKIN, E., AND J. RILEY (2000): "Asymmetric auctions," *Review of Economic Studies*, 67(3), 413–438.

MCAFEE, R., AND J. MCMILLAN (1989): "Government procurement and international trade," *Journal of International Economics*, 26(3), 291–308.

MCAFEE, R. P., AND P. J. RENY (1992): "Correlated information and mecanism design," *Econometrica*, pp. 395–421.

MIERENDORFF, K. (2011): "Asymmetric reduced form auctions," *Economics Letters*, 110(1), 41–44.

MYERSON, R. (1981): "Optimal auction design," *Mathematics of Operations Research*, 6(1), 58–73.

OLLÁR, M., AND A. PENTA (2013): "Full Implementation and Belief Restrictions," University of Wisconsin.