Symplectic rigidity and quantum mechanics

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Abstract. I present new links between Symplectic Topology and Quantum Mechanics which have been discovered in the framework of function theory on symplectic manifolds. Recent advances in this emerging theory highlight some rigidity features of the Poisson bracket, a fundamental operation governing the mathematical model of Classical Mechanics. Unexpectedly, the intuition behind this rigidity comes from Quantum Mechanics.

2010 Mathematics Subject Classification. Primary 53D-XX ; Secondary 81S-XX
Keywords. Poisson bracket, symplectic quasi-state, quantization, quantum measurement

> Suddenly the result turned out completely different from what he had expected: again it was $1 + 1 = 2$. “Wait a minute!” he cried out, “Something's wrong here”. And at that very moment, the entire class began whispering the solution to him in unison: “Planck's constant! Planck's constant!”
>
> after M. Pavic, *Landscape Painted with Tea*, 1988

1. Introduction

In the present lecture we discuss an interaction between symplectic topology and quantum mechanics. The interaction goes in both directions. On the one hand, some ideas from quantum mechanics give rise to new notions and structures on the symplectic side and, furthermore, quantum mechanical insights lead to useful symplectic predictions when the topological intuition fails. On the other hand, some phenomena discovered within symplectic topology admit a meaningful translation into the language of quantum mechanics, thus revealing quantum footprints of symplectic rigidity. This subject brings together three disciplines: “hard” symplectic topology, quantum mechanics, and quantization which provides a bridge between classical and quantum worlds.

Let us present this picture in more detail. Symplectic topology originated as a geometric tool for problems of classical mechanics. It studies symplectic manifolds, i.e., even dimensional manifolds M^{2n} equipped with a closed differential 2-form ω which in appropriate local coordinates (p, q) can be written as $\sum_{i=1}^{n} dp_i \wedge dq_i$. To have some interesting examples in mind, think of surfaces with an area form

*Partially supported by the Israel Science Foundation grant 178/13 and the European Research Council Advanced grant 338809.

and their products, as well as of complex projective spaces equipped with the Fubini-Study form, and their complex submanifolds.

Symplectic manifolds model the phase spaces of systems of classical mechanics. Observables (i.e., physical quantities such as energy, momentum, etc.) are represented by functions on M. The states of the system are encoded by Borel probability measures μ on M. Every observable $f : M \to \mathbb{R}$ is considered as a random variable with respect to a state μ. The simplest states are given by the Dirac measure δ_z concentrated at a point $z \in M$. In such a state every observable f has unique deterministic value $f(z)$.

The laws of motion are governed by the *Poisson bracket*, a canonical operation on smooth functions on M given by $\{f, g\} = \sum_j \left(\frac{\partial f}{\partial q_j} \frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q_j} \right)$. The evolution of the system is determined by its energy, a time-dependent Hamiltonian function $f_t : M \to \mathbb{R}$. Hamilton's famous equation describing the motion of the system is given, in the Heisenberg picture, by $\dot{g}_t = \{f_t, g_t\}$, where $g_t = g \circ \phi_t^{-1}$ stands for the time evolution of an observable function g on M under the Hamiltonian flow ϕ_t. The maps ϕ_t are called *Hamiltonian diffeomorphisms.* They preserve the symplectic form ω and constitute a group with respect to composition.

The mathematical model of quantum mechanics starts with a complex Hilbert space H. In what follows we shall focus on finite-dimensional Hilbert spaces only as they are quantum counterparts of compact symplectic manifolds. Observables are represented by Hermitian operators whose space is denoted by $\mathcal{L}(H)$. The states are provided by *density operators*, i.e., positive trace 1 operators $\rho \in \mathcal{L}(H)$. Given an observable A, let $A = \sum_{i=1}^{k} \lambda_i P_i$ be its spectral decomposition, where P_i's are pair-wise orthogonal projectors with $\sum P_i = \mathbb{1}$. In a state ρ the observable A takes the values λ_i with the probability trace$(P_i \rho)$. The *pure states* are provided by rank 1 orthogonal projectors, which we usually identify (ignoring the phase factor) with unit vectors $\xi \in H$. At such a state A takes the value λ_i with the probability $\langle A\xi, \xi \rangle$.

The space $\mathcal{L}(H)$ can be equipped with the structure of a Lie algebra whose bracket is given by $-(i/\hbar)[A, B]$, where $[A, B]$ stands for the commutator $AB - BA$ and $\hbar$ is the Planck constant. While $\hbar$ is a fundamental physical constant, it will play the role of a small parameter of the theory. Exactly as in classical mechanics, the bracket governs the unitary evolution U_t of the system, giving rise to the Schrödinger equation $\dot{G}_t = -(i/\hbar)[F_t, G_t]$, where F_t is the Hamiltonian (i.e., the energy) and $G_t = U_t G U_t^{-1}$ describes the evolution of an observable G.

Quantization is a formalism behind the quantum-classical correspondence, a fundamental principle stating that quantum mechanics contains the classical one as a limiting case when $\hbar \to 0$. Mathematically, the correspondence in question is a linear map $f \mapsto T_\hbar(f)$ between smooth functions on a symplectic manifold and Hermitian linear operators on a complex Hilbert space $H_\hbar$ depending on the Planck constant $\hbar$. The map is assumed to satisfy a number of axioms which are summarized in Table 1.

Let us emphasize that the quantum-classical correspondence is not precise. It holds true up to an error which is small with $\hbar$. In Section 4 we will review some recent joint results with Charles [20] on the sharp bounds for this error in the context

Table 1. Quantum-Classical Correspondence

	CLASSICAL	QUANTUM
	Symplectic manifold (M, ω)	$\mathbb{C}$-Hilbert space H
OBSERVABLES	$f \in C^\infty(M)$	$T_\hbar(f) \in \mathcal{L}(H)$
NORM	Uniform norm $\|f\|$	Operator norm $\|T_\hbar(f)\|_{\mathrm{op}}$
BRACKET	$\{f, g\}$	$-(i/\hbar)[T_\hbar(f), T_\hbar(g)]$
PRODUCT	fg	$T_\hbar(f)T_\hbar(g)$
STATES	Probability measures on M	$\rho \in \mathcal{L}(H), \rho \geq 0, \mathrm{trace}(\rho) = 1$

of *the Berezin-Toeplitz quantization*. An extra bonus provided by this quantization scheme is positivity: $T_\hbar$ sends non-negative functions to positive operators, which is important for the applications to quantum measurements discussed below.

"Hard" symplectic topology, whose birth goes back to the 1980ies (Conley, Zehnder, Gromov, Floer), lead to the discovery of surprising rigidity phenomena involving symplectic manifolds, their subsets, and their diffeomorphisms. These phenomena have been detected with the help of a variety of novel powerful methods, including Floer theory, a version of Morse theory on the loop spaces of symplectic manifolds, which in turns brings together complex analysis and elliptic PDEs (see Section 6.3 below). Achievements of symplectic topology include a wealth of non-trivial symplectic invariants beyond the symplectic volume, surprising features of symplectic maps which distinguish them from general volume-preserving maps (Arnold's fixed points conjectures, Hofer's geometry), and topological constraints on Lagrangian submanifolds, to mention a few of them.

At first glance there is a certain incompatibility between the "output" of hard symplectic topology (symplectic invariants of diffeomorphisms and subsets) and the "input" of quantization (functions). The key to reducing this discrepancy is provided by *function theory on symplectic manifolds*, a recently emerged area which studies manifestations of symplectic rigidity taking place in function spaces associated to a symplectic manifold. On the one hand, these spaces exhibit unexpected properties and interesting structures, giving rise to an alternative intuition and new tools in symplectic topology. On the other hand, they fit well with quantization.

In the present lecture we will discuss two examples of interaction between symplectic topology and quantum mechanics.

Poisson bracket invariants: Even though these symplectic invariants are defined through elementary looking variational problems involving the functional $(f, g) \to \|\{f, g\}\|$, the proof of their non-triviality involves a variety of "hard" methods. These invariants have applications to topology and dynamics. Their quantum footprints lead to quantum measurements theory and the noise operator (see Sections 2 and 5).

Symplectic quasi-states: These are monotone functionals on the space $C^\infty(M)$ which are linear on every Poisson-commutative subalgebra, but not necessarily on the whole space (see Section 6). The origins of this notion go back to Gleason's theorem, which plays an important role in the discussion of quantum indeterminism. In dimension ≥ 4, symplectic quasi-states come from Floer theory, the cornerstone of "hard" symplectic topology.

Even though the quantum mechanical ingredients in these two examples are quite different, the themes are closely related: symplectic quasi-states provide an efficient tool for studying the Poisson bracket invariants.

In the last two sections, we touch upon other facets of interaction between symplectic topology and quantum mechanics and outline some future research directions.

2. Poisson bracket invariants

2.1. Prologue. One of the first discoveries of function theory on symplectic manifolds is the *C^0-rigidity of the Poisson bracket* [29]. In the quantum world, the bracket $(F,G) \mapsto -(i/\hbar)[F,G]$ is continuous with respect to the operator norm. At first glance, this miserably fails in the classical limit, as the Poisson bracket $\{f,g\}$ depends on the derivatives of functions f and g and hence it may blow up under small perturbations in the uniform norm $\|f\| = \max_M |f(x)|$. Surprisingly, the following feature survives: *The functional $(f,g) \mapsto \|\{f,g\}\|$ is lower semi-continuous in the uniform norm.* We refer to [67, 18] for earlier results in this direction, and [10] for a different proof and generalizations. All known approaches in dimension ≥ 4 involve methods of "hard" symplectic topology.

As a consequence, the functional $\|\{f,g\}\|$ can be canonically extended to all *continuous* functions on M. This is a Cheshire Cat effect: for a pair of continuous functions f and g, the Poisson bracket is in general not defined, albeit its uniform norm is!

The functional $\|\{f,g\}\|$ gives rise to a number of interesting symplectic invariants. One of them plays a central role in our exposition. Let $\vec{f}$ be a finite collection $f_1,...,f_N$ of smooth functions on M. Denote $f^x := \sum_i x_i f_i$, $x \in \mathbb{R}^N$. For a finite open cover $\mathcal{U} = \{U_1,...,U_N\}$ of M introduce the Poisson bracket invariant [57]

$$\mathrm{pb}(\mathcal{U}) = \inf_{\vec{f}} \max_{x,y \in [-1,1]^N} \|\{f^x, f^y\}\| , \tag{1}$$

where the infimum is taken over all partitions of unity subordinated to $\mathcal{U}$. It measures the minimal possible magnitude of non-commutativity of a partition of unity subordinated to $\mathcal{U}$. The Poisson bracket invariant increases under refinements of the covers. As we shall see below, $\mathrm{pb}(\mathcal{U}) > 0$ *provided the sets U_i are "symplectically small"*.

Example 2.1. Assume the sets U_i are metrically small, that is, their diameters with respect to an auxiliary Riemannian metric on M are $\leq \epsilon$. We claim that $\mathrm{pb}(\mathcal{U}) \to +\infty$ as $\epsilon \to 0$. Indeed, for any $K > 0$ fix a pair of functions $u, v : M \to [-1,1]$ with $\|\{u,v\}\| \geq K$. Choose L greater than the Lipschitz constants of u and v. Take any partition of unity $\vec{f} = (f_1,...,f_N)$ subordinated to $\mathcal{U}$. Pick points $z_i \in U_i$, and put $x = (u(z_1),...,u(z_N)) \in [-1,1]^N$, $y = (v(z_1),...,v(z_N)) \in [-1,1]^N$. Note that $f_i(z) = 0$ if $z \notin U_i$ and $|u(z) - u(z_i)| \leq L\epsilon$ if $z \in U_i$. Since $u = \sum_i u f_i$ we get that for every $z \in M$, $|u(z) - f^x(z)| = |\sum_i (u(z) - u(z_i)) f_i(z)| \leq$

$L\epsilon \sum f_i(z) = L\epsilon$, and the same holds for v and f^y. It follows that $f^x \to u$ and $f^y \to v$ in the uniform norm as $\epsilon \to 0$, and hence by C^0-rigidity of the Poisson bracket, $\mathrm{pb}(\mathcal{U}) \geq K/2$ for all ϵ small enough. The claim follows.

2.2. Small scale in symplectic topology. In 1990 Hofer [42] introduced an intrinsic "small scale" on a symplectic manifold: A subset $X \subset M$ is called *displaceable* if there exists a Hamiltonian diffeomorphism ϕ such that $\phi(X) \cap X = \emptyset$.

Example 2.2. Let us illustrate this notion in the case when $M = S^2$ is the two-dimensional sphere equipped with the standard area form. Any disc lying in the upper hemisphere is displaceable: one can send it to the lower hemisphere by a rotation. However the equator (a simple closed curve splitting the sphere into two discs of equal area) is non-displaceable by any area-preserving transformation. This example demonstrates the contrast between symplectic "smallness" and measure-theoretic "smallness": the equator has measure 0, yet it is large from the viewpoint of symplectic topology.

Theorem 2.3 (Rigidity of partitions of unity, [31])**.** $\mathrm{pb}(\mathcal{U}) > 0$ *for every finite open cover of a closed symplectic manifold by displaceable sets.*

2.3. Topological applications. The next result, which readily follows from the rigidity of partitions of unity, provides an application of function theory on symplectic manifolds to topology.

Theorem 2.4 (Non-displaceable fiber theorem, [25])**.** *Let* $\vec{f} = (f_1, ..., f_N) : M \to \mathbb{R}^N$ *be a smooth map of a closed symplectic manifold* M *whose coordinate functions* f_i *pair-wise Poisson commute. Then* $\vec{f}$ *possesses a non-displaceable fiber: for some* $w \in \mathbb{R}^N$*, the set* $\vec{f}^{-1}(w)$ *is non-empty and non-displaceable.*

It is tempting to consider this result as a symplectic counterpart of Gromov's *waist inequality* stating that for any continuous map from the unit n-sphere to $\mathbb{R}^q$, at least one of the fibers has "large" $(n-q)$-dimensional volume (here "large" means at least that of an $(n-q)$-dimensional equator), see [39]. It would be interesting to explore this analogy further.

Detecting non-displaceability of subsets of symplectic manifolds is a classical problem going back to Arnold's seminal Lagrangian intersections conjecture. Theorem 2.4 provides a useful tool in the following situation. Assume that we know *a priori* that all but possibly one fiber of a map $\vec{f} : M \to \mathbb{R}^N$ with Poisson-commuting components are displaceable. Then that particular fiber is necessarily non-displaceable.

Example 2.5 ([4])**.** Consider the standard complex projective space $\mathbb{C}P^n$ equipped with the Fubini-Study symplectic form. Let $[z_0 : \cdots : z_n]$ be the homogeneous coordinates. Consider the map $\vec{f} : \mathbb{C}P^n \to \mathbb{R}^n$ with the components $f_i(z) = |z_i|^2/g(z)$, $i = 1, ..., n$, where $g(z) := \sum_{j=0}^n |z_j|^2$. The coordinate functions f_i, considered as Hamiltonians, generate the standard torus action on the projective space and hence Poisson commute. The image of $\vec{f}$ is the standard n-dimensional simplex $\Delta \subset \mathbb{R}^n$.

Denote its barycenter by b. One readily checks that for every $w \neq b$ the fiber $\vec{f}^{-1}(w)$ is displaceable by a unitary transformation of $\mathbb{C}P^n$ (use a permutation of the coordinates). It follows that the *Clifford torus* $\vec{f}^{-1}(b)$ is non-displaceable in $\mathbb{C}P^n$.

We refer to [28, 22, 7, 34, 66] for further application of quasi-states to non-displaceability of (possibly, singular) Lagrangian submanifolds, fibers of moment maps of Hamiltonian torus actions, and invariant tori of integrable systems.

2.4. Symplectic size. Symplectic topology provides various ways to measure the *size* of a finite open cover. With an appropriate notion of size at hand, the rigidity of partitions of unity phenomenon admits the following quantitative version:

$$\mathrm{pb}(\mathcal{U}) \cdot \mathrm{Size}(\mathcal{U}) \geq C(\mathcal{U}) , \tag{2}$$

where the positive constant C depends, roughly speaking, on combinatorics of the cover $\mathcal{U}$. Let us mention that inequality (2) was initially guessed on the quantum side, where it admits a transparent interpretation as an uncertainty relation, see Section 5.1 below. Let us describe two versions of size for which (2) holds.

We start with some basic combinatorial invariants of an open cover $\mathcal{U} = \{U_1, \dots, U_N\}$ of M. Consider the graph with vertices $\{1, \dots, N\}$, where two vertices i and j are connected by an edge provided U_i and U_j intersect. By definition, the cover has degree d if the degree of each vertex is at most d. For a natural number p, put $U_i^{(p)} = \bigcup_j U_j$, where j runs over the set of all vertices whose graph distance to i is $\leq p$.

Displacement energy: Let $U \subset V \subset M$ be a pair of open subsets of a symplectic manifold M. We say that U *is displaceable inside* V if there exists a time-dependent Hamiltonian function h_t on M which is supported in V and such that the time-one map ϕ of the Hamiltonian flow generated by h_t displaces U, i.e., $\phi(U) \cap U = \emptyset$. The infimum of the total energy $\int_0^1 \|h_t\| dt$ over all such displacements is called the *displacement energy* of U inside V and is denoted by $e(U, V)$ (see [42]).

Assume now that each set U_i of the cover is displaceable in $U_i^{(p)}$ and define $\mathrm{Size}(\mathcal{U}) := \max_i e(U_i, U_i^{(p)})$. Note that this definition depends on the constant p. It was shown in [57] that (2) holds with $C = C(p, d)$.

Example 2.6 (Greedy covers)**.** Fix an auxiliary Riemannian metric on M, and for $r > 0$ small enough choose a maximal r-net, i.e., a maximal collection of points such that the distance between any two of them is $> r$. Let $\mathcal{U}^{(r)}$ be the collection of metric balls of radius r with the centers at the points of the net. By the maximality of the net, $\mathcal{U}^{(r)}$ is a cover of M. It is not hard to check that for r small enough, the degree d admits an upper bound independent of r. Furthermore, for some p independent of r, $\mathrm{Size}(\mathcal{U}^{(r)}) \sim r^2$. It follows that $\mathrm{pb}(\mathcal{U}^{(r)}) \gtrsim r^{-2}$ as $r \to 0$. One can show that this asymptotic behavior is sharp: $\mathrm{pb}(\mathcal{U}^{(r)}) \sim r^{-2}$.

Covers by balls: Consider the open ball $B^{2n}(R) := \{|p|^2 + |q|^2 < R^2\} \subset \mathbb{R}^{2n}$ equipped with the symplectic form $dp \wedge dq$. A *symplectic ball* of radius R

in a symplectic manifold (M^{2n},ω) is the image of $B^{2n}(R)$ under a symplectic embedding. Consider a finite cover $\mathcal{U}$ of M by symplectic balls. Define Size$(\mathcal{U})$ of such a cover as $\max_i \pi R_i^2$, where R_i are the radii of the balls. Inequality (2) with this definition of size holds true for certain symplectic manifolds (e.g., when $\pi_2(M)=0$), sometimes under an extra assumption that Size$(\mathcal{U})$ is sufficiently small (e.g., for $\mathbb{C}P^n$ with the Fubini-Study form). This was recently proved in increasing generality by the author [57], Seyfaddini [61], and Ishikawa [44] with $C=C(d)$.

It is unclear whether the constant C in (2) can be chosen independent of the degree d of the cover.

2.5. Dynamical applications [12]. Let X_0, X_1, Y_0, Y_1 be a quadruple of compact subsets of a symplectic manifold (M,ω) with $X_0\cap X_1 = Y_0\cap Y_1=\emptyset$. Put $\mathrm{pb}_4(X_0,X_1,Y_0,Y_1) = \inf \|\{f,g\}\|$, where the infimum is taken over all pairs of compactly supported smooth functions $f,g: M\to\mathbb{R}$ such that $f=0$ near X_0, $f=1$ near X_1, $g=0$ near Y_0 and $g=1$ near Y_1. Observe that the $(f,1-f)$ and $(g,1-g)$ are partitions of unity subordinated to the open covers $(M\setminus X_0, M\setminus X_1)$ and $(M\setminus Y_0, M\setminus Y_1)$, respectively. Thus pb_4 can be considered as a version of pb for pairs of open covers. Even though pb_4 is defined through an elementary-looking variational problem involving Poisson brackets, the proof of its non-triviality involves a variety of methods of "hard" symplectic topology. Interestingly enough, the pb_4-invariant enables one to detect Hamiltonian chords, i.e., trajectories of Hamiltonian systems connecting two given disjoint subsets of the phase space.

Theorem 2.7. *Let $X_0,X_1,Y_0,Y_1\subset M$ be a quadruple of compact subsets with $X_0\cap X_1 = Y_0\cap Y_1=\emptyset$ and $\mathrm{pb}_4(X_0,X_1,Y_0,Y_1)=p>0$. Let $G\in C^\infty(M)$ be a Hamiltonian function with $G|_{Y_0}\le 0$ and $G|_{Y_1}\ge 1$, which generates a complete Hamiltonian flow g_t. Then $g_T x\in X_1$ for some point $x\in X_0$ and some time moment $T\in[-1/p,1/p]$.*

This result generalizes to time-dependent Hamiltonian flows. Furthermore, by using a slight modification of pb_4, one can control the sign of T, i.e., one can decide whether the trajectory goes from X_0 to X_1 or vice versa. We refer to [30] for applications of pb_4-based techniques to instabilities in Hamiltonian dynamics.

3. Quantum measurements and noise

3.1. Positive operator valued measures. Let H be a complex Hilbert space. Recall that $\mathcal{L}(H)$ denotes the space of all bounded Hermitian operators on H.

Consider a set Ω equipped with a σ-algebra $\mathcal{C}$ of its subsets. An $\mathcal{L}(H)$-valued *positive operator valued measure* (POVM) F on $(\Omega,\mathcal{C})$ is a countably additive map $F\colon\mathcal{C}\to\mathcal{L}(H)$ which takes each subset $X\in\mathcal{C}$ to a positive operator $F(X)\in\mathcal{L}(H)$ and which is normalized by $F(\Omega)=\mathbb{1}$.

Example 3.1 (POVMs on finite sets)**.** When $\Omega=\Omega_N:=\{1,\dots,N\}$, is a finite set, any POVM F on Ω is fully determined by the N positive Hermitian operators $F_i:=F(\{i\})$ which sum up to $\mathbb{1}$.

POVMs appear in quantum measurement theory [13]. For the purposes of this paper, a POVM F on Ω represents a measuring device coupled with the system, while Ω is interpreted as the space of device readings. According to the basic statistical postulate of POVMs, *when the system is in a pure state ξ, the probability of finding the device F in a subset $X \in \mathcal{C}$ equals $\langle F(X)\xi, \xi\rangle$*. Given an $\mathcal{L}(H)$-valued POVM F on $(\Omega, \mathcal{C})$ and a bounded measurable function $x \colon \Omega \to \mathbb{R}$, one can define the integral $F(x) := \int_\Omega x \, dF \in \mathcal{L}(H)$ as follows. Introduce a measure $\mu_{F,\xi}(X) = \langle F(X)\xi, \xi\rangle$ on Ω and put $\langle F(x)\xi, \xi\rangle = \int_\Omega x \, d\mu_{F,\xi}$, for every state $\xi \in H$. In a state ξ, the function x becomes a random variable on Ω with respect to the measure $\mu_{F,\xi}$ with the expectation $\langle F(x)\xi, \xi\rangle$.

Example 3.2 (Projector valued measures)**.** An important class of POVMs is formed by the *projector valued measures* P, for which all the operators $P(X)$, $X \in \mathcal{C}$ are orthogonal projectors. For instance, every von Neumann observable $A \in \mathcal{L}(H)$ with N pair-wise distinct eigenvalues corresponds to the projector valued measure $\{P_i\}$ on the set $\Omega_N = \{1, \ldots, N\}$ and a random variable $\lambda : \Omega_N \to \mathbb{R}$ such that $A = \sum_{i=1}^N \lambda_i P_i$ is the spectral decomposition of A. In this case the statistical postulate for POVMs agrees with the one of von Neumann's quantum mechanics.

A somewhat simplistic description of quantum measurement is as follows: an experimentalist after setting a quantum measuring device (i.e., a POVM F) chooses an arbitrary collection of functions x_α on Ω and performs a measurement whose outcome is the collection of operators $F(x_\alpha) \in \mathcal{L}(H)$. Such a procedure is called *a joint unbiased approximate measurement* of the observables $A_\alpha := F(x_\alpha) \in \mathcal{L}(H)$. The expectation of A_α in every state ξ coincides with the one of the measurement procedure (hence *unbiased*), in spite of the fact that actual probability distributions determined by the observable A_α and the pair (F, x_α) could be quite different (hence *approximate*). Let us mention that every finite collection of observables admits a joint unbiased approximate measurement.

3.2. Uncertainty and noise. In quantum mechanics, "all measurements are uncertain, but some of them are less uncertain than others[1]." By uncertainty we mean appearance of positive variances. For instance, the variance of an observable A in a state ξ equals $\mathbb{V}(A, \xi) = \langle A^2\xi, \xi\rangle - \langle A\xi, \xi\rangle^2$. Heisenberg's famous uncertainty principle states that

$$\mathbb{V}(A, \xi) \cdot \mathbb{V}(B, \xi) \geq \frac{1}{4} \cdot |\langle [A, B]\xi, \xi\rangle|^2 \,. \tag{3}$$

This inequality can be interpreted as follows (see [56], p. 93): consider an ensemble of quantum particles prepared in the state ξ. Let us measure for half of the particles the observable A and for the other half B. The variances of the corresponding statistical procedures will necessarily satisfy (3).

In general, the variance increases under an unbiased approximate measurement. Assume that the latter is provided by a POVM F on Ω together with a random variable $x : \Omega \to \mathbb{R}$ with $F(x) = A$. Then $\mathbb{V}(F, x, \xi) = \mathbb{V}(A, \xi) + \langle \Delta_F(x)\xi, \xi\rangle$, where

[1] cf. G. Orwell, *Animal farm*, 1945.

$\Delta_F(x) := F(x^2) - F(x)^2$ is *the noise operator*, see [14]. The noise operator, which is known to be positive, measures the increment of the variance. Furthermore, $\Delta_F(x) = 0$ provided F is a projector valued measure. From the viewpoint of quantum mechanics, the projective measurements are as good (or sharp) as it gets, i.e., they carry the least uncertainty.

The following property of the noise operator is crucial for our purposes ([41], Theorem 7.5). For any POVM F, any pair of random variables x, y, and any state $\xi \in H$,

$$\langle \Delta_F(x)\xi, \xi\rangle \cdot \langle \Delta_F(y)\xi, \xi\rangle \geq \frac{1}{4} \cdot |\langle [F(x), F(y)]\xi, \xi\rangle|^2 . \tag{4}$$

An interesting consequence of this inequality is the following *uncertainty jump* phenomenon for joint unbiased approximate measurements [43] which manifests the increase of uncertainty due to a measurement. Assume that F provides a joint measurement for a pair of observables A and B: $F(x) = A$, $F(y) = B$. Then $\mathbb{V}(F, x, \xi) \cdot \mathbb{V}(F, y, \xi) \geq |\langle [A, B]\xi, \xi\rangle|^2$ for every quantum state ξ, and this inequality is sharp. Comparing this with the Heisenberg uncertainty principle, we see that the coefficients at $|\langle [A, B]\xi, \xi\rangle|^2$ jump from 1/4 to 1.

Observe that if A and B commute, they admit a simultaneous diagonalization. The corresponding projector valued measure provides a noiseless joint unbiased measurement of A and B.

Let us conclude this discussion with a remark on joint *biased* measurements (we learned this concept from P. Busch). Recently Kachkovskiy and Safarov [45] settled a long-standing problem in operator theory by proving that "almost-commutativity yields near commutativity". More precisely, every pair of observables A, B can be approximated by a commuting pair with the error $\leq C\|[A, B]\|_{\text{op}}^{1/2}$, where the constant C does not depend on A, B and the dimension. Thus, by allowing such an error (or bias), we reduce the noise to 0. Interestingly enough, on the classical side, i.e., for functions on symplectic manifolds, almost-commutativity yields near commutativity in dimension 2 (Zapolsky, [68]), albeit not in higher dimensions. An ingenious counter-example was constructed by Buhovsky in [11].

3.3. Inherent noise of a POVM. Let $F = \{F_1, \ldots, F_N\}$ be a POVM on the finite set Ω_N. Considering F as a measuring device, we address the following question: *Can one refine it so that the new device is able to produce the same unbiased approximate measurements as F which are as noiseless as possible?*

Quantum mechanics provides a suitable notion of refinement: Let G be a POVM on some space Θ, and let $\vec{f} = (f_1, \ldots, f_N)$ be a partition of unity on Θ (i.e., non-negative measurable functions which sum up to 1) such that $F_i = G(f_i)$ for all i. We say that F is *a smearing* (or coarse-graining) of G, and we refer to G as a refinement of F. Observe that the POVM G can reproduce all the measurements performed by F. Indeed, $F(x) = G(f^x)$ with $f^x := \sum_i x_i f_i$ for every random variable $x = (x_1, \ldots, x_N)$ on Ω_N. Smearing can be interpreted as a randomization: fix a quantum state ξ, and imagine that we first perform the G-measurement whose result is a random point $\theta \in \Theta$ distributed according to $\mu_{G,\xi}$, and then θ jumps to a point $i \in \Omega_N$ with probability $f_i(\theta)$. The POVM F provides a correct statistical description of this two-step procedure.

The noise increases under smearings [51]: $\Delta_G(f^x) \leq \Delta_F(x)$. In order to quantify the level of noise produced after smearing, we will restrict to random variables x from the cube $[-1,1]^N$. Define the *inherent noise* of the POVM F as $\mathcal{N}(F) := \inf_{G,\vec{f}} \max_{x\in[-1,1]^N} \|\Delta_G(f^x)\|_{\mathrm{op}}$, where the infimum is taken over all pairs $(G,\vec{f})$ providing a refinement of F. By inequality (4), the inherent noise admits a lower bound in terms of the magnitude of non-commutativity of F. It is given by the following *unsharpness principle*:

$$\mathcal{N}(F) \geq \frac{1}{2} \cdot \max_{x,y\in[-1,1]^N} \|[F(x),F(y)]\|_{\mathrm{op}} . \tag{5}$$

In the opposite direction, if all F_i's commute, then $\mathcal{N}(F) = 0$. This follows immediately from the simultaneous diagonalizability of commuting Hermitian operators. The behavior of the function $\mathcal{N}(F)$ on the space of all POVMs F is still unexplored. We refer the reader to [46] for an intriguing link between quantum noise production and non-commutativity in the context of quantum computing.

4. Berezin-Toeplitz quantization

4.1. Introducing the quantization. POVMs play a crucial role in the context of the Berezin-Toeplitz quantization, a mathematical model of quantum-classical correspondence [3, 6, 40, 9, 48, 60, 19]. A closed symplectic manifold (M^{2n},ω) is called *quantizable* if $[\omega]/(2\pi) \in H^2(M,\mathbb{Z})$. For such a manifold one can construct its *Berezin-Toeplitz* quantization which is given by the following data:

- a subset $\Lambda \subset \mathbb{R}_{>0}$ having 0 as a limit point;
- a family $H_\hbar$ of finite-dimensional complex Hilbert spaces, $\hbar \in \Lambda$;
- a family of $\mathcal{L}(H_\hbar)$-valued positive operator valued measures $G_\hbar$ on M.

To each function $f \in C^\infty(M)$ corresponds the *Toeplitz operator* $T_\hbar(f) := \int_M f dG_\hbar$. We assume that the ($\mathbb{R}$-linear) map $T_\hbar : C^\infty(M) \to \mathcal{L}(H_\hbar)$ is surjective for all $\hbar$, and that additionally it satisfies the following properties:

(P1) **(norm correspondence)** $\|f\| - O(\hbar) \leq \|T_\hbar(f)\|_{\mathrm{op}} \leq \|f\|$;

(P2) **(the correspondence principle)**

$$\| -(i/\hbar)[T_\hbar(f),T_\hbar(g)] - T_\hbar(\{f,g\})\|_{\mathrm{op}} = O(\hbar) ;$$

(P3) **(quasi-multiplicativity)** $\|T_\hbar(fg) - T_\hbar(f)T_\hbar(g)\|_{\mathrm{op}} = O(\hbar)$;

(P4) **(trace correspondence)**

$$\left|\mathrm{trace}(T_\hbar(f)) - (2\pi\hbar)^{-n}\int_M f\frac{\omega^n}{n!}\right| = O(\hbar^{-(n-1)}) ,$$

for all $f, g \in C^\infty(M)$.

While the quantization sends classical observables to quantum ones, on the states it acts in the opposite direction. To every quantum state $\xi \in H_\hbar$, $|\xi| = 1$, corresponds a probability measure given by

$$\mu_{\hbar,\xi}(X) = \langle G_\hbar(X)\xi, \xi\rangle \tag{6}$$

for every Borel subset $X \subset M$. One can interpret this in the spirit of the wave-particle duality as follows: for a fixed value of $\hbar$, every quantum state has a classical footprint, a particle distributed over the classical phase space according to the measure $\mu_{G_\hbar,\xi}$. The geometry of these measures for meaningful sequences of quantum states in the classical limit $\hbar \to 0$ is still far from being understood.

4.2. Sharp remainder bounds. The remainders $O(\hbar)$ in (P1)–(P4) above depend on functions f, g, uniformly on compact sets in C^∞-topology. In a recent paper with L. Charles [20] we proposed the following structure of remainders. Denote by $|f|_m$ the C^m-norm of a function f. Then the remainders $O(\hbar)$ in (P1)–(P3) have the following form:

(P1) $\le \alpha |f|_2 \hbar$;

(P2) $\le \beta(|f|_1 \cdot |g|_3 + |f|_2 \cdot |g|_2 + |f|_3 \cdot |g|_1)\hbar$.

(P3) $\le \gamma(|f|_0 \cdot |g|_2 + |f|_1 \cdot |g|_1 + |f|_2 \cdot |g|_0)\hbar$.

These remainder bounds are essentially sharp. Assuming (P4), one can show that *for any Berezin-Toeplitz quantization scheme* $\alpha \ge \alpha_0$, $\beta \ge c\alpha^{-2}$ and $\gamma \ge \gamma_0$, where the positive constants α_0, c, γ_0 depend only on (M, ω) and the auxiliary Riemannian metric entering the definition of the C^m-norms.

A quantization with such remainder bounds exists for every quantizable manifold (M, ω) [20]. An interesting question that we learned from S. Gelfand is whether the integrality of the class $[\omega]/2\pi$ is a necessary condition for the existence of a Berezin-Toeplitz quantization.

Example 4.1. In the case of closed Kähler manifolds the construction of quantization is very transparent and goes as follows (see e.g. [60] for a survey). Pick a holomorphic Hermitian line bundle L over M whose Chern connection has curvature $i\omega$. Define the Planck constant $\hbar$ by $1/k$, where $k \in \mathbb{N}$ is large enough. Write L^k for the k-th tensor power of L. The space $H_\hbar$ lies in the Hilbert space $V_\hbar$ of all L_2-sections of L^k equipped with the canonical Hermitian product. Let $\Pi_\hbar : V_\hbar \to H_\hbar$ be the orthogonal projection. In this language the Toeplitz operators $T_\hbar(f)$ act by composition of the multiplication by f and projection: $s \mapsto \Pi_\hbar(fs)$ for every $s \in H_\hbar$. The Berezin-Toeplitz POVMs $G_\hbar$ come from the Kodaira embedding theorem. Recall that the latter provides a map $M \to \mathbb{P}(H_\hbar^*)$ which sends each point $z \in M$ to the hyperplane $\{s \in H_\hbar \;:\; s(z) = 0\}$. Denote by $P_{\hbar,z}$ the orthogonal projector of $H_\hbar$ to the line orthogonal to this hyperplane. One can show that there exists a smooth function $R_\hbar$ (called the *Rawnsley function*) such that $dG_\hbar(z) = R_\hbar(z)P_{\hbar,z}d\mathrm{Vol}(z)$.

5. Quantum footprints of symplectic rigidity

5.1. The noise-localization uncertainty relation. Let $\mathcal{U} = \{U_1, \dots, U_N\}$ be a finite open cover of M. Given a particle z on M, we wish to localize it in the phase space, i.e., provide an answer to the following question: to which of the sets U_i does z belong? Of course, the question is ambiguous even if the particle is completely deterministic (i.e., a point $z \in M$) due to overlaps between the sets U_i. In order to resolve the ambiguity, let us make the required assignment $z \mapsto U_j$ at random: fix a partition of unity $\vec{f} = (f_1, \dots, f_N)$ subordinated to $\mathcal{U}$ and register z in U_j with probability $f_j(z)$. Since f_j is supported in U_j, this procedure provides "the truth, but not the whole truth".

In case the particle is distributed over M according to a probability measure μ, the probability of registration in U_j equals $\int f_j d\mu$. With this remark at hand, let us describe the quantum version of our registration procedure. We assume that the manifold (M, ω) is quantizable and fix a scheme of the Berezin-Toeplitz quantization. In a state $\xi \in H$, the quantum particle is distributed over M according to the measure $\mu_{\hbar,\xi}$ and hence the probability of registration in U_j equals $\int f_j d\mu_{\hbar,\xi} = \langle T_\hbar(f_j)\xi, \xi\rangle$. In other words, the quantum registration procedure is governed by the POVM $F_\hbar := \{T_\hbar(f_j)\}$. The next result provides an estimate for the inherent noise $\mathcal{N}(F_\hbar)$ of this POVM. Recall that $\mathrm{pb}(\mathcal{U})$ stands for the Poisson bracket invariant defined in (1).

Theorem 5.1 ([57]). *Assume that* $\mathrm{pb}(\mathcal{U}) > 0$. *There exist constants* $C_+ > 0$ *and* $\hbar_0 > 0$ *depending on* $\vec{f}$ *such that* $C_+\hbar \geq \mathcal{N}(F_\hbar) \geq \frac{1}{2}\,\mathrm{pb}(\mathcal{U}) \cdot \hbar + O(\hbar^2)$ *for* $\hbar \leq \hbar_0$.

The upper bound follows from the fact that $F_\hbar$ is a smearing of the Berezin-Toeplitz POVM $G_\hbar$ on M associated to the partition of unity $\vec{f}$. Thus, writing $f^x := \sum x_i f_i$, we get that $\mathcal{N}(A) \leq \sup_{x \in [-1,1]^N} \|T_\hbar((f^x)^2) - T_\hbar(f^x)^2\|_{\mathrm{op}} = O(\hbar)$, where the last equality follows from quasi-multiplicativity property (P3) of the Berezin-Toeplitz quantization. The lower bound is an immediate consequence of the unsharpness principle (5) and the correspondence principle. Thus the assumption $\mathrm{pb}(\mathcal{U}) > 0$ (which, for instance, holds true if all the sets of the cover are displaceable, see Theorem 2.3) guarantees that the quantum registration procedure produces positive noise of the order $\sim \hbar$.

Applying inequality (2) with an appropriate notion of the size, we conclude that $\mathcal{N}(F_\hbar) \cdot \mathrm{Size}(\mathcal{U}) \geq C'\hbar$, where the positive constant C' depends only on combinatorics of the cover. This is *a noise-localization uncertainty relation* which can be considered as a quantum counterpart of the rigidity of partitions of unity phenomenon in symplectic topology. It reflects the trade-off between the precision of the phase space localization of a quantum particle and the magnitude of the inherent noise of the corresponding measurement.

Let us mention also that the pb_4-invariant defined in Section 2.5 appears in the study of quantum noise for joint measurements [57].

5.2. Zooming into the wave length scale. Let us emphasize that in the noise-localization uncertainty relation above the cover $\mathcal{U}$ is fixed as $\hbar \to \infty$, that is, we localize our particle to a symplectically small, but fixed scale. What happens on an $\hbar$-dependent scale? Let us focus on the case of greedy covers (see Example 2.6 above), where the sets U_i are metric balls of radii $r \ll 1$, while the combinatorial parameters d and p are fixed. For the sake of concreteness, let us assume that $r = R\sqrt{\hbar}$, where R is fixed and $\hbar \to 0$, i.e., we work on the physically meaningful wave length scale. One can show [20] that if R is large and the functions entering the partition of unity have controlled derivatives, the noise is of the order $\sim R^{-2}$, and in particular the noise-localization uncertainty holds. The main difficulty here is that the functions of the partitions of unity depend on $\hbar$, and thus in order to run the argument used in the proof of Theorem 5.1 above one has to deal with the Toeplitz operators of the form $T_\hbar(f_\hbar)$. At this point the sharp remainder bounds presented in Section 4.2 enter the play.

6. From quantum indeterminism to quasi-states

6.1. Gleason's theorem. In his foundational book [64] von Neumann defined quantum states as real valued functionals $\rho : \mathcal{L}(H) \to \mathbb{R}$ satisfying three simple axioms: $\rho(\mathbb{1}) = 1$ (*normalization*), $\rho(A) \geq 0$ if $A \geq 0$ (*positivity*) and *linearity*. Next, he showed that each such functional can be written as $\rho(A) := \text{trace}(\rho A)$, where ρ is a density operator. Interpreting $\rho(A)$ as the expectation of the observable A at the state ρ, von Neumann concluded that for any quantum state ρ there exists an observable A such that the variance $\rho(A^2) - \rho(A)^2$ is strictly positive. In other words, in sharp contrast with Dirac δ-measures in classical mechanics, there are no quantum states in which the values of all observables are deterministic.

This conclusion, known as the impossibility to introduce hidden variables into quantum mechanics, caused a passionate discussion among physicists: it was criticized first by Hermann [35] and later on by Bohm and Bell (see e.g. [2]). They argued that the linearity axiom only makes sense for observables A, B that can be measured simultaneously, that is commute: $[A, B] = 0$. This led to the following definition: A *quantum quasi-state* is a functional $\rho\colon \mathcal{L}(H) \to \mathbb{R}$ which satisfies the positivity and normalization axioms, while the linearity is relaxed as follows: ρ is linear on every commutative subspace of $\mathcal{L}(H)$ (*quasi-linearity*).

However, in 1957 Gleason proved the following remarkable theorem: *If H has complex dimension* 3 *or greater, any quantum quasi-state is linear, that is, it is a quantum state.* This confirms Neumann's conclusion. Citing Peres [56, p. 196], "Gleason's theorem is a powerful argument against the hypothesis that the stochastic behavior of quantum tests can be explained by the existence of a subquantum world, endowed with hidden variables whose values unambiguously determine the outcome of each test."

6.2. Symplectic quasi-states [25]. Let us now mimic the definition of a quantum quasi-state in classical mechanics, using the quantum-classical correspondence and having in mind that commuting Hermitian operators correspond to Poisson-commuting functions. Let (M, ω) be a closed symplectic manifold. A *symplectic* *quasi-state* on M is a functional $\zeta\colon C(M) \to \mathbb{R}$ such that $\zeta(1) = 1$ (*normalization*), $\zeta(f) \geq 0$ for $f \geq 0$ (*positivity*), and ζ is linear on every Poisson-commutative subspace (*quasi-linearity*). Recall that the C^0-rigidity of the Poisson bracket provides a natural notion of Poisson-commuting *continuous functions*, see Section 2.1 above.

In contrast to quantum mechanics, *certain symplectic manifolds admit non-linear symplectic quasi-states.* This "anti-Gleason phenomenon" in classical mechanics has been established for various complex manifolds including complex projective spaces and their products, toric manifolds, blow ups and coadjoint orbits [25, 55, 62, 34, 17].

In terms of the existence mechanism for symplectic quasi-states there is a mysterious dichotomy (vaguely resembling the rank two versus higher rank dichotomy in Lie theory). In dimension two (i.e., for surfaces), symplectic quasi-states exist in abundance. Their construction is provided by Aarnes' theory of topological quasi-states [1], whose motivation was to explore validity of Gleason theorem for algebras of functions on topological spaces, where the quasi-linearity is understood as linearity on all singly-generated subalgebras. In fact, in dimension two topological and symplectic quasi-states coincide. Interestingly enough, all known non-linear symplectic quasi-states in higher dimensions come from Floer theory. We refer to Section 6.3 below for a discussion, and to [58, 23, 54] for more details. In general, Floer-homological quasi-states do not admit a simple description. However, there is one exception.

Example 6.1 (Median quasi-state)**.** First, we define a quasi-state $\zeta : C(S^2) \to \mathbb{R}$ on smooth Morse functions $f \in C^\infty(S^2)$, where the sphere S^2 is equipped with the area form ω of total area 1. Recall that the *Reeb graph* Γ of f is obtained from S^2 by collapsing connected components of the level sets of f to points, see Figure 1. In the case of S^2, the Reeb graph is necessarily a tree. Denote by

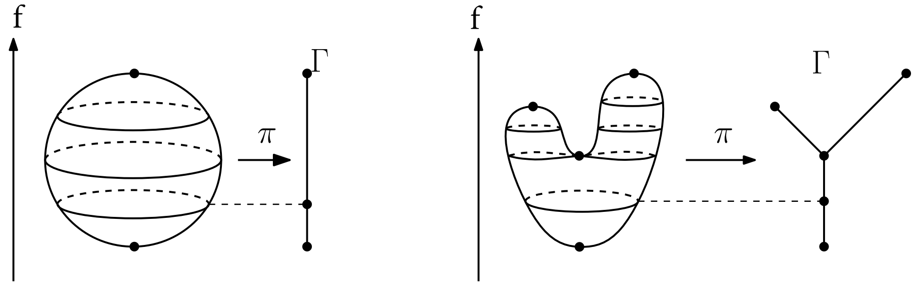

Figure 1. The Reeb graph

$\pi : S^2 \to \Gamma$ the natural projection. The push-forward of the area on the sphere is a probability measure on Γ. It is not hard to show (and in fact, this is well known in combinatorial optimization) that there exists a unique point $m \in \Gamma$, called the *median* of Γ, such that each connected component of $\Gamma \setminus \{m\}$ has measure $\leq \frac{1}{2}$, see [24, Section 5.3]. Define $\zeta(f)$ as the value of f on the level $\pi^{-1}(m)$. It turns out that ζ is Lipshitz in the uniform norm and its extension to $C(M)$ is a non-linear quasi-state, the one which comes from Floer theory on S^2.

One can show that the median quasi-state on S^2 is dispersion free: $\zeta(f^2) = \zeta(f)^2$ for every $f \in C^\infty(S^2)$. It is unknown whether this holds true for Floer homological quasi-states in higher dimensions.

Interestingly enough, Floer-homological quasi-states come with a package of additional features which make them useful for various applications in symplectic topology. In particular, $\zeta(f) = 0$ for every function f with displaceable support (*vanishing property*). This immediately yields that a finite open cover of M by displaceable subsets does not admit a subordinated Poisson-commuting partition of unity, which in turn is equivalent to the non-displaceable fiber theorem. Indeed, assume that $f_1, \dots, f_N$ are pair-wise commuting functions with displaceable supports which sum up to 1. By the vanishing property, $\zeta(f_i) = 0$. By normalization and quasi-linearity, $1 = \zeta\big(\sum f_i\big) = \sum \zeta(f_i) = 0$, and we get a contradiction.

The quantitative versions given by inequality (2) are more subtle. Roughly speaking, they involve the following inequality relating Floer-homological symplectic quasi-states and the Poisson brackets [31]. There exists a constant C, depending on ζ, such that

$$|\zeta(f+g) - \zeta(f) - \zeta(g)| \le C\sqrt{\|\{f,g\}\|} \;\; \forall f,g \in C^\infty(M) \; . \tag{7}$$

The simplest manifold for which existence of a non-linear symplectic quasi-state is still unknown is the standard symplectic torus $\mathbb{T}^4$. Fortunately, every closed symplectic manifold admits a weaker structure given by so-called *partial symplectic quasi-states* [25], which are powerful enough for proving the rigidity of partitions of unity. These are normalized, positive, $\mathbb{R}_+$-homogeneous functionals ζ on $C(M)$ which satisfy $\zeta(f+g) = \zeta(g)$ provided $\{f,g\} = 0$ and g has displaceable support.

6.3. Floer theory and persistence modules. Let us briefly sketch a construction of partial symplectic quasi-states. For simplicity, we shall deal with closed symplectic manifolds with $\pi_2(M) = 0$. The symplectic structure ω induces a functional $\mathcal{A} : LM \to \mathbb{R}$ on the space LM of all contractible loops $z : S^1 \to M$ in M. Given such a loop z, take any disc $D \subset M$ spanning z and put $\mathcal{A}(z) = -\int_D \omega$. Since ω is a closed form and $\pi_2(M) = 0$, this functional is well defined. Its critical points are degenerate: they form the submanifold of all constant loops. In order to resolve this degeneracy, fix a time-dependent Hamiltonian $f_t : M \to \mathbb{R}$, $t \in S^1$, and define a perturbation $\mathcal{A}_f : LM \to \mathbb{R}$ of $\mathcal{A}$ by $\mathcal{A}_f(z) = \mathcal{A}(z) + \int_0^1 f_t(z(t))dt$. This is the *classical action functional.* Roughly speaking, Floer theory is the Morse theory for $\mathcal{A}_f$. Ironically, the perturbations become the main object of interest.

According to the least action principle, the critical points of $\mathcal{A}_f$ are precisely the contractible 1-periodic orbits of the Hamiltonian flow ϕ_t generated by f_t. Denote by P the set of such orbits (generically, there is a finite number of them). For $a \in \mathbb{R}$ put $P_a := \{z \in P \;:\; \mathcal{A}_f(z) < a\}$.

The space LM carries a special class of Riemannian metrics associated to loops of ω-compatible almost complex structures on M. Pick such a metric and look at the space $T(x,y)$ of the gradient trajectories of $\mathcal{A}_f$ connecting two critical points $x, y \in P$. Note that in M such a trajectory is a path of loops, i.e., a cylinder. A great insight of Floer was that these cylinders satisfy a version of the Cauchy-Riemann equation with asymptotic boundary conditions. Even though the gradient flow of $\mathcal{A}_f$ is ill defined, this boundary problem is well posed and Fredholm. Generically, $T(x,y)$ is a manifold. If its dimension vanishes, it consists of a finite number of points. Put $n(x,y) \in \mathbb{Z}_2$ to be the parity of $T(x,y)$ if $\dim T(x,y) = 0$, and declare $n(x,y) = 0$ otherwise.

For $a \in \mathbb{R}$, consider the linear map d of the vector space $\mathrm{Span}_{\mathbb{Z}_2}(P_a)$ given by $dx = \sum_y n(x,y)y$. It turns out that d is a differential, i.e., $d^2 = 0$. Define the Floer homology $HF_a(f) := \mathrm{Ker}(d)/\mathrm{Im}(d)$.

The inclusion $P_a \subset P_b$ for $a < b$ induces a canonical morphism on homologies $\pi_{ab} : HF_a(f) \to HF_b(f)$. These morphisms satisfy some natural axioms which enables one to consider the collection $V(f) := (HF_a(f), \pi_{ab})_{a<b}$ as a *persistence module*, an algebraic object which incidentally plays a crucial role in modern topological data analysis. According to the structure theorem, for each such module there exists a unique *barcode*, i.e., a finite collection $\mathcal{B}$ of (possibly infinite) intervals $I = (\alpha, \beta]$, $\alpha < \beta \leq +\infty$ with multiplicities such that $V(f) = \bigoplus_{I \in \mathcal{B}} \mathbb{Z}_2(I)$. The building block $\mathbb{Z}_2(I)$ is a persistence module $(W_a, \pi_{ab})_{a<b}$ such that $W_a = \mathbb{Z}_2$ for $a \in I$ and 0 otherwise, and $\pi_{ab} = \mathbb{1}$ for $a, b \in I$ and 0 otherwise.

Order the vertices of infinite rays in $\mathcal{B}$ and denote by $c(f)$ the maximal one. It is called *a spectral invariant* of f. It turns out that the functional $f \mapsto c(f)$ is continuous in the C^0-topology and hence it can be extended to all continuous Hamiltonians, including time-independent functions $f \in C(M)$. One can show that the limit $\zeta(f) := \lim_{s \to +\infty} c(sf)/s$ is a partial symplectic quasi-state.

Interestingly enough, the persistence module $V(f)$, which is called *filtered Floer homology*, in fact depends only on the time-one map $\phi = \phi_1$ of the Hamiltonian flow generated by f. Accordingly we will denote it by $V(\phi)$. This is a fundamental algebraic invariant of a Hamiltonian diffeomorphism. Its barcode $\mathcal{B} = \mathcal{B}(\phi)$ provides various interesting invariants of Hamiltonian diffeomorphisms which are robust with respect to C^0-perturbations of Hamiltonians. Furthermore, the correspondence which sends a Hamiltonian diffeomorphism ϕ to its barcode $\mathcal{B}(\phi)$ is Lipschitz with respects to the Hofer metric on diffeomorphisms and the so-called bottleneck distance on barcodes.

The construction sketched above generalizes to manifolds with non-trivial π_2. In this case the action functional $\mathcal{A}_f$ on LM becomes multivalued and one has to develop a version of Morse-Novikov theory on covering spaces of LM. In certain situations, one can show that partial symplectic quasi-states obtained in this way are actually genuine symplectic quasi-states. At this point a new character enters the play, namely the multiplicative structure of Floer homology with respect to the pair-of-pants product, or, equivalently, the structure of the quantum homology algebra QH of (M, ω). The latter is a deformation of the intersection product on the homology of M which takes into account pseudo-holomorphic spheres in M. If QH is semi-simple [24] or, more generally contains a field as a direct summand (McDuff), then (M, ω) carries a genuine symplectic quasi-state.

Let us conclude this section with some historical remarks and references. Floer homology was invented by Floer and spectral invariants by Viterbo, Schwartz, and Oh [53, 58, 54]. The theory of persistence modules was pioneered by Edelsbrunner, Harer, and Carlsson, among others [16, 65]. Applications of persistence modules to symplectic topology appeared in recent works by Shelukhin and the author [59], as well as by Usher and Zhang [63].

6.4. Algebraic aspects of quasi-states. The quasi-linearity axiom of quasi-states makes perfect sense in the context of finite dimensional Lie algebras $\mathfrak{g}$ over $\mathbb{R}$. A function $\zeta\colon \mathfrak{g} \to \mathbb{R}$ is called a *Lie quasi-state* [27] if it is linear on every abelian subalgebra. Under mild regularity assumptions, Lie quasi-states exhibit rigid behavior. For instance, Gleason's theorem readily yields that every Lie quasi-state on the unitary algebra $\mathfrak{u}(n)$ with $n \geq 3$ which is bounded in a neighborhood of 0 is necessarily linear.

In the case of the symplectic algebra $\mathfrak{g} = \mathfrak{sp}\,(2n, \mathbb{R})$ (see [27]) the space of Lie quasi-states which are bounded near 0 is infinite-dimensional. Nevertheless, continuous Lie quasi-states exhibit rigidity. Denote by $\mathcal{Q}(\mathfrak{g})$ the quotient of the space of all continuous Lie quasi-states on $\mathfrak{g}$ by the Lie coalgebra $\mathfrak{g}^*$. It turns out that for $n \geq 3$, $\dim \mathcal{Q}(\mathfrak{sp}\,(2n, \mathbb{R})) = 1$. This statement is false for $n = 1$ and the classification is still open for $n = 2$. We refer to [5] for recent study of Lie quasi-states on other Lie algebras, including examples of non-linear Lie quasi-states in the solvable case. A general theory of Lie quasi-states is still missing.

Lie quasi-states have a group theoretic counterpart, quasi-morphisms, which play an important role in various areas of group theory and dynamics (see e.g., [36, 15, 58] for a survey). Recall that a *homogeneous quasi-morphism* on a group G is a function $\mu : G \to \mathbb{R}$ which satisfies $\mu(x^n) = n\mu(x)$ for all $x \in G$ and $n \in \mathbb{Z}$, and which is "a homomorphism up to a bounded error", i.e., there exists *the defect* $C \geq 0$ such that $|\mu(xy) - \mu(x) - \mu(y)| \leq C$ for all $x, y \in G$. One can readily show that homogeneous quasi-morphisms restrict to morphisms on abelian subgroups. Thus, given a homogeneous quasi-morphism on a Lie group, its pull-back under the exponential map is a Lie quasi-state. For instance, the generator of $\mathcal{Q}(\mathfrak{sp}\,(2n, \mathbb{R}))$ is obtained in this way from the Maslov quasi-morphism on the universal cover of the symplectic group $\mathrm{Sp}(2n, \mathbb{R})$.

Interestingly enough, the link between quasi-morphisms and quasi-states persists in the symplectic category. In fact, all Floer homological symplectic quasi-states, as well as some symplectic quasi-states on surfaces, are associated to quasi-morphisms on the universal cover of the group of Hamiltonian diffeomorphisms [24]. Furthemore, the constant C in the Poisson bracket inequality (7) is governed by the defect of the corresponding quasi-morphism [31].

7. Symplectic displacement and quantum speed limit

Fix a Berezin-Toeplitz quantization $T_\hbar$ on a quantizable closed symplectic manifold (M, ω). A fundamental result in symplectic topology states that the symplectic displacement energy of a symplectic ball of radius r in M is $\sim r^2$ provided r

is small enough [42, 47]. Interestingly enough, quantum mechanics provides an intuitive, albeit mathematically flawed, explanation of this result in terms of the *quantum speed limit*, a universal bound on the energy required for moving a quantum state into an orthogonal one. Such a bound was discovered by Mandelstam and Tamm [49] as early as in 1945 and refined by Margolus and Levitin [50] in 1998. The argument, due to Charles and the author, goes as follows. Let f_t be a classical Hamiltonian displacing a ball B of radius $\sim \sqrt{\hbar}$. Choose a state $\xi \in H_\hbar$ such that the corresponding measure (6) is concentrated in B. According to the Egorov theorem, the quantum-classical correspondence takes the Hamiltonian flow $\phi_t : M \to M$ of f_t to the Schrödinger evolution $U_t : H_\hbar \to H_\hbar$ generated by the quantum Hamiltonian $T_\hbar(f_t)$. The measure corresponding to the state $U_1\xi$ is concentrated in $\phi_1(B)$. The quantum-classical correspondence translates the condition $\phi_1(B) \cap B = \emptyset$ into $\langle U_1\xi, \xi\rangle \approx 0$. The quantum speed limit yields $\int_0^1 \|T_\hbar(f_t)\|_{\text{op}} dt \geq (\pi/2)\hbar$, which by the norm correspondence (P1) of the Berezin-Toeplitz quantization means that $\int_0^1 \|f_t\| dt \gtrsim \hbar \sim r^2$, and we are done. Of course, the devil is in the remainders, which at the scale $r \sim \sqrt{\hbar}$ become non-negligible and cannot be ignored. Nevertheless, the above argument can be rigorously applied another way around to the speed limit of semiclassical orthogonalization of semiclassical states. As it is proved in [21], the speed limit becomes more restrictive at the scales exceeding the wave-length. Roughly speaking, one can show that the energy required for the orthogonalization of a semiclassical state occupying a ball of radius $\sim \hbar^\varepsilon$, $\varepsilon \in [0, 1/2)$ is at least $\sim \hbar^{2\varepsilon}$.

8. Epilogue

In spite of a number of advances outlined in the present lecture, quantum footprints of symplectic rigidity remain largely unexplored. We conclude with a brief discussion of open problems and future directions.

Covers vs. packings: Covers by symplectic balls (cf. Section 2.4 above) have a prominent cousin, symplectic packings, i.e., collections of pair-wise disjoint standard symplectic balls in a symplectic manifold. They are subject to various constraints which have been intensively studied since the birth of symplectic topology. For instance, at most 1/2 (resp., 288/289) of the volume of the complex projective plane $\mathbb{C}P^2$ can be filled by a symplectic packing with two (resp., eight) balls of equal radii [38, 52]. *What are the quantum counterparts of symplectic packing obstructions?* Interestingly enough, symplectic packings with derivative bounds were used by Fefferman and Phong [32] in their study of the eigenvalue counting function for pseudo-differential operators with positive symbol. It would be interesting to explore this link in the context of the Berezin-Toeplitz quantization.

A particular class of two-ball packings consists of a ball B and its image $\phi(B)$ under a Hamiltonian diffeomorphism displacing B. In this case, the above-mentioned packing obstruction states that a ball $B \subset \mathbb{C}P^2$ is non-displaceable whenever its volume exceeds $\text{Vol}(\mathbb{C}P^2)/4$. This statement can be quantized along the lines sketched in Section 7 above.

QUASI-STATES REVISITED: Let (M, ω) be a quantizable closed symplectic manifold admitting a non-linear symplectic quasi-state ζ (e.g., the 2-sphere of area 2π equipped with the median quasi-state). Fix a Berezin-Toeplitz quantization $T_\hbar : C^\infty(M) \to \mathcal{L}(H_\hbar)$. *Does there exist a family of continuous functionals* $\zeta_\hbar : \mathcal{L}(H_\hbar) \to \mathbb{R}$ *such that* $\lim_{\hbar\to 0} \zeta_\hbar(T_\hbar(f)) = \zeta(f)$ *for every* $f \in C^\infty(M)$*?* A "natural" quantum analogue of a symplectic quasi-state would be a continuous Lie quasi-state on $\mathcal{L}(H)$. However, this naive guess does not work: such a quasi-state is necessarily linear by Gleason's theorem.

CLOSED ORBITS ON THE QUANTUM SIDE: Another potential connection between symplectic rigidity and quantum mechanics is provided by the Gutzwiller-type trace formula [8] which establishes a relation between the periodic orbits of an autonomous flow generated by a classical Hamiltonian f and the spectrum of the corresponding quantum Hamiltonian $T_\hbar(f)$. Consider *the density of states* $\rho_\hbar(E) := \sum_\lambda \delta((E-\lambda)\hbar^{-1})$, where λ runs over the spectrum of $T_\hbar(f)$. Roughly speaking, non-constant periodic orbits are responsible for rapid oscillations of $\rho_\hbar$ as $\hbar \to 0$. Uribe proposed that these oscillations capture a "hard" symplectic invariant called the Hofer-Zehnder capacity. Furthermore, it seems likely that for a meaningful class of Hamiltonians f, the trace formula contains an information about the spectral invariant $c(f)$ (a project in progress with Charles, Le Floch, and Uribe). *What about the full barcode of the persistence module associated to* f*?* Note that the Floer homology described in Section 6.3 is built on contractible closed orbits of Hamiltonian flows. *Is it possible to distinguish between contractible and non-contractible orbits on the quantum side?* So far, these questions are out of reach.

Acknowledgments. Many results surveyed above are joint with M. Entov. This text has inevitable overlaps with my book [58] with D. Rosen. I thank P. Biran, L. Buhovsky, L. Charles, Y. Eliashberg, M. Entov, D. McDuff, D. Rosen, E. Shelukhin, and F. Zapolsky for fruitful and pleasant collaboration on various aspects of this paper. I am grateful to P. Busch, Y. Le Floch, and A. Uribe for illuminating discussions, as well as to M. Entov and I. Polterovich for very helpful comments on the manuscript. I thank A. Iacob for a superb copyediting. Parts of this paper were written during my stay at the University of Chicago.

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