Synchronization and Flow in Networks

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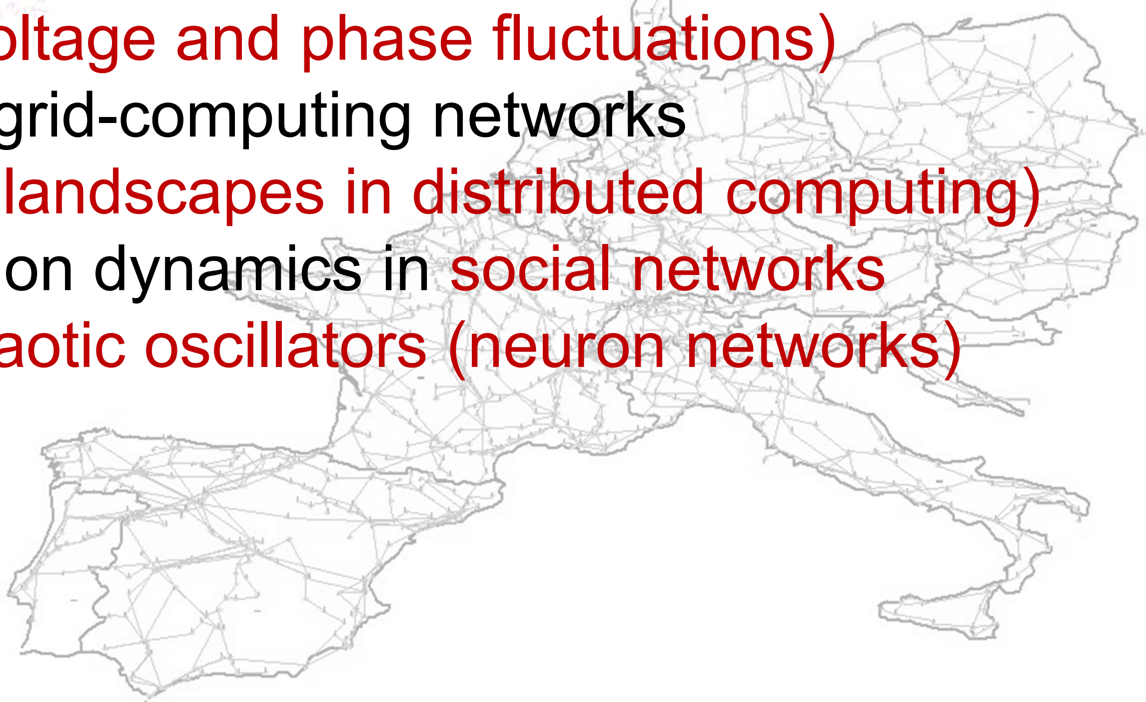
Collective dynamics networks

Examples:

- Internet (packet traffic/flux in search or routing)
- Load-balancing schemes (job allocation among processors)
- Electric power grid (voltage and phase fluctuations)
- High-performance or grid-computing networks
  (task-completion landscapes in distributed computing)
- Information flow, opinion dynamics in social networks
- Coupled nonlinear chaotic oscillators (neuron networks)



Overview

- **Resistor networks and random walks**

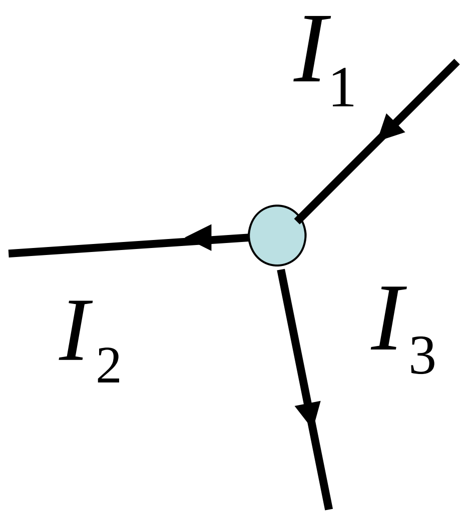
  Doyle & Snell (1984); Chandra *et al.* (1989);
  Tetali (1990); Klein and Randic (1993);
  Wu (2004); Lopez *et al.* (2005); Gallos *et al.* (2007);
  GK *et al.* (2006); GK (2007); Ellens *et al*. (2011);
  Asztalos *et al.* (2012).

- **Synchronization and consensus problems in a noisy environment** in networks
  (the Edwards-Wilkinson process on networks)
  GK *et al.* (2003, 2004, 2006); Guclu *et al.* (2006, 2007)

- **Optimizing synchronization (and flow) in weighted networks**

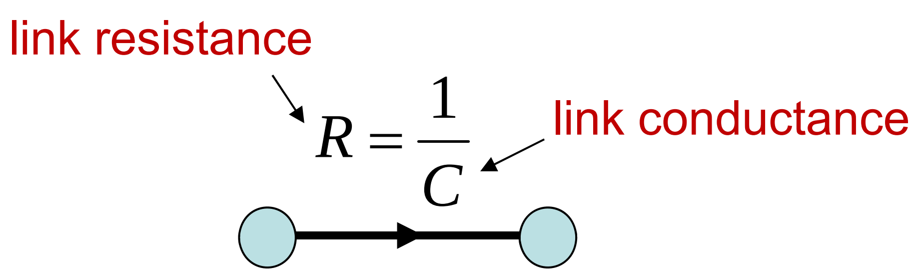
  Zhou *et al.* (2006); Motter *et al.* (2004); GK (2007)

Resistor networks



G. Kirchhoff (1847)

$$I_1 + I_2 + I_3 = 0$$

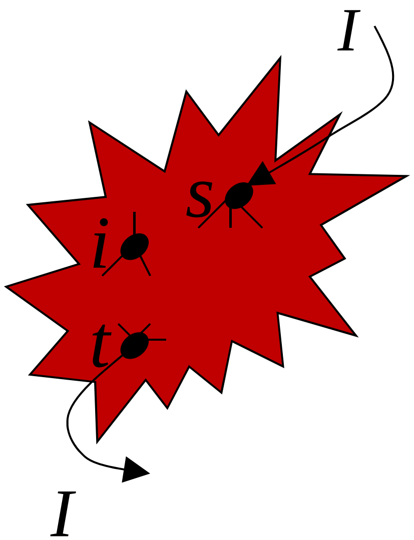


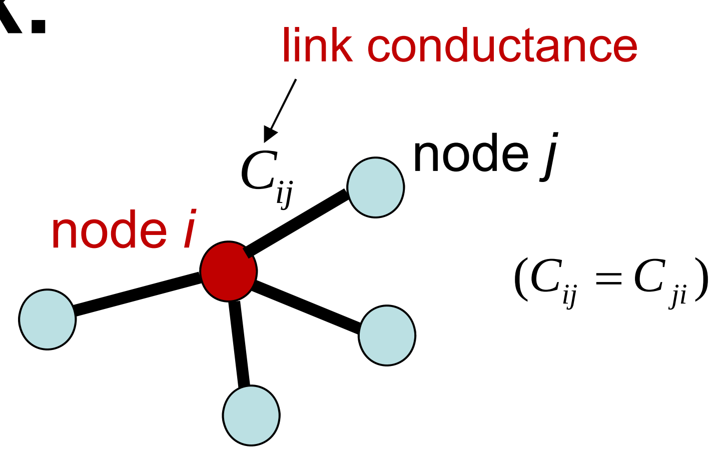
G. Ohm (1827)

$$V = RI$$

$$(I = CV)$$

For an arbitrary network:





$$\sum_j C_{ij}(V_i - V_j) = I(\delta_{is} - \delta_{it})$$

weighted degree: $C_i = \sum_l C_{il}$

Laplacian: $\Gamma_{ij} \equiv \delta_{ij} C_i - C_{ij}$

$$\sum_j \Gamma_{ij} V_j = I(\delta_{is} - \delta_{it})$$

Formally inverting Γ:

$$\sum_j \Gamma_{ij} V_j = I(\delta_{is} - \delta_{it})$$

$$\Gamma \boldsymbol{\psi}_k = \boldsymbol{\psi}_k \lambda_k$$

$$(k = 0, 1, 2, \ldots, N-1)$$

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \ldots \lambda_{N-1} = \lambda_{\max}$$

$$\boldsymbol{\psi}_0 = N^{-1/2}(1, 1, \ldots, 1)$$

pseudoinverse or Green's function:

$$G_{ij} \equiv \hat{\Gamma}^{-1}_{ij} = \sum_{k=1}^{N-1} \frac{1}{\lambda_k} \psi_{ki} \psi_{kj}$$

$$\boxed{\hat{V}_i \equiv V_i - \overline{V} = I(G_{is} - G_{it})}$$

$$\overline{V} = \sum_l V_l$$

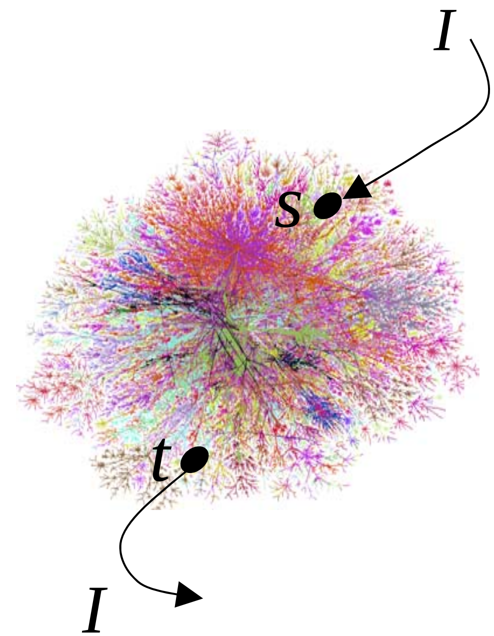
Inversion or exact numerical diag: $O(N^3)$ routines.
Conceptually useful, and can also be employed following exact numerical diagonalization [up to $O(10^4)$ nodes on sparse networks]

$$V_i - V_j = \hat{V}_i - \hat{V}_j = I(G_{is} - G_{it} - G_{js} + G_{jt})$$

Effective two-point resistance:

specifically, for $i=s$, $j=t$:

$$V_s - V_t = I(\underbrace{G_{ss} - G_{st} - G_{ts} + G_{tt}}_{R_{st}})$$



$$\boldsymbol{R_{st} = G_{ss} + G_{tt} - 2G_{ts}}$$

$$R_{st} = \sum_{k=1}^{N-1} \frac{1}{\lambda_k}(\psi_{ks}^2 + \psi_{kt}^2 + 2\psi_{ks}\psi_{kt}) = \sum_{k=1}^{N-1} \frac{1}{\lambda_k}(\psi_{ks} - \psi_{kt})^2$$

$$\frac{2}{\lambda_{N-1}} \leq R_{st} \leq \frac{2}{\lambda_1}$$

Klein and Randic (1993);
GK *et al.* (2006); GK (2007);
Ghosh *et al.* (2008);
Ellens *et al.* (2011).

A Global Observable: the average system resistance

$$\overline{R} = \frac{1}{N(N-1)} \sum_{s \neq t} R_{st} = \frac{2}{N-1} \sum_{k=1}^{N-1} \frac{1}{\lambda_k}$$

$$\max\left(\frac{2}{(N-1)\lambda_1}, \frac{2}{\lambda_{N-1}}\right) < \overline{R} \leq R_{\max} \leq \frac{2}{\lambda_1}$$

$$\overline{R} = \frac{2}{N-1} \sum_{k=1}^{N-1} \frac{1}{\lambda_k} \xrightarrow[N\to\infty]{} 2\int \frac{\rho(\lambda)d\lambda}{\lambda}$$

$\rho(\lambda)$: density of eigenvalues

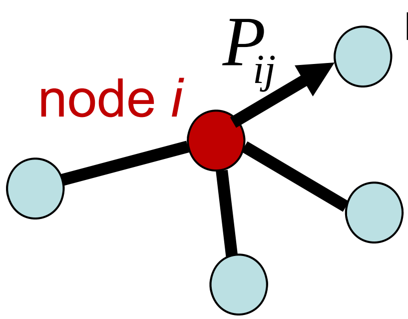
From the theory of disordered systems:

- replica method, Bray & Rodgers (1988); Kim and Kahng (2006)
- effective medium approximation, Monasson (1999); Dorogovtsev et al. (2003)

Random Walks on Networks

Doyle & Snell (1984)
Tetali (1990)

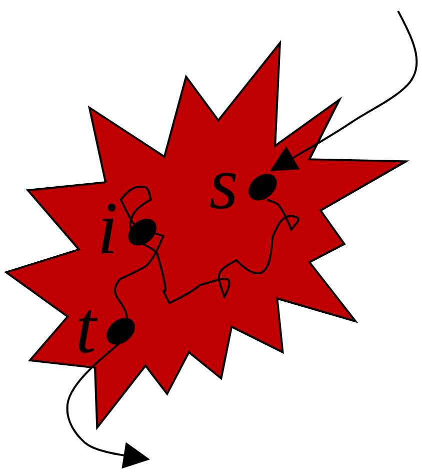
symmetric weighted edges: $C_{ij} = C_{ji}$



Prob$\{i \to j\}$:

$\sum_j P_{ij} = 1$

$$P_{ij} \equiv \frac{C_{ij}}{\sum_l C_{il}} = \frac{C_{ij}}{C_i}$$



expected # of visits to node ***i***, starting at node ***s***, before reaching node ***t*** $(i \neq s, t)$:

$$E_i^{st} = \sum_j E_j^{st} P_{ji} \qquad (E_t^{st} \equiv 0)$$

$$P_{ij} C_i = P_{ji} C_j$$

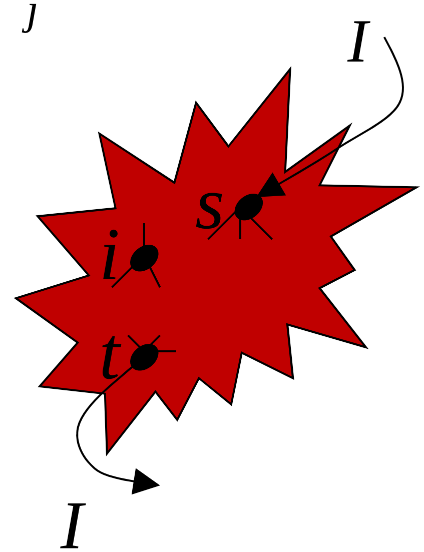
$$E_i^{st} = \sum_j E_j^{st} P_{ij} \frac{C_i}{C_j} \quad \Rightarrow \quad \frac{E_i^{st}}{C_i} = \sum_j P_{ij} \frac{E_j^{st}}{C_j}$$

RWs and Resistor Networks

Doyle & Snell (1984)
Tetali (1990)

recall for resistor networks: $(i \neq s,t)$

$$\sum_j C_{ij}(V_i - V_j) = 0 \quad \Rightarrow \quad V_i = \sum_j P_{ij} V_j$$



E_i^{st}/C_i and V_i obey the same harmonic equation

$$(E_t^{st} \equiv 0)$$

$$\boxed{E_i^{st} = C_i(V_i - V_t)} \quad \text{with } I = 1 \text{ (unit current)}$$

$$I_{ij} = C_{ij}(V_i - V_j) = E_i^{st} P_{ij} - E_j^{st} P_{ji}$$

RW node betweenness ("load")

expected # of visits to node *i*, starting at node *s*, before reaching node *t* :

$$E_i^{st} = C_i(V_i - V_t) \qquad (I = 1)$$

$$E_i^{st} = C_i(V_i - V_t) = C_i(G_{is} - G_{it} - G_{ts} + G_{tt})$$

RW node betwenness:

$$b_i \equiv \frac{1}{N-1}\sum_{s \neq t} E_i^{st} = \frac{1}{2(N-1)}\sum_{s \neq t}(E_i^{st} + E_i^{ts}) =$$

$$\frac{C_i}{2(N-1)}\sum_{s \neq t}(G_{tt} + G_{ss} - 2G_{ts}) = N\frac{C_i}{2}\overline{R}$$

local load:

$$b_i = N\frac{C_i}{2}\overline{R}$$

global average load:

$$\overline{b} = \frac{1}{2}\left(\sum_i C_i\right)\overline{R}$$

First-Passage and Commute Times in RWs

$$E_i^{st} = C_i(V_i - V_t) = C_i(G_{is} - G_{it} - G_{ts} + G_{tt})$$

Chandra et al. (1989);
Tetali (1990);
GK (2007); Ellens *et al.* (2011).

expected **first-passage time**:
(expected # of steps of RW)

$$\tau^{st} = \sum_i E_i^{st} = \sum_i C_i(G_{is} - G_{it} - G_{ts} + G_{tt})$$

expected **commute time**:

$$\tau^{st} + \tau^{ts} = \sum_i C_i(G_{ss} + G_{tt} - 2G_{st}) = (\sum_i C_i)R_{st}$$

average expected first-passage time:
(averaged over all pairs of nodes
in the graph)

$$\overline{\tau} = \frac{\sum_i C_i}{2}\overline{R}$$

(a specific realization of ***Little's law***:)

$$\overline{\tau} = \overline{b}$$

Synchronization/Coordination/Consensus in Coupled Systems

- individual units or agents (represented by static or mobile nodes) attempt to adjust their local state variables (e.g., pace, load, alignment, coordination) in a decentralized fashion. Craig Reynolds (1987); Vicsek *et al*. (1995); Cavagna *et al*. (2010).
- nodes interact or communicate only with their local neighbors in the network, possibly to improve global performance or coordination.
- nodes react (perform corrective actions) to the information or signal received from their neighbors
- Applications: autonomous coordination, unmanned aerial vehicles, microsatellite clusters, traffic flow, sensor and communication networks, load balancing, flocking, distributed decision making in social networks

Olfati-Saber (2005); Olfati-Saber, Fax & Murray (2007); Hunt (2010).

flocking birds

spontaneous brain activity (fMRI)
(Justin Vincent)

IP activity
(Zeus load balancer)

Synchronization in Networks in a Noisy Environment: the Edwards-Wilkinson Process on a Network

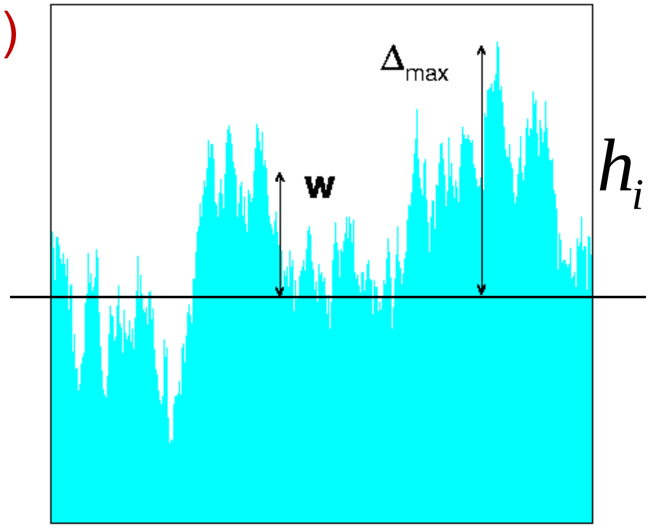
$$\partial_t h_i(t) = -\sum_j C_{ij}(h_i - h_j) + \eta_i(t) \qquad \langle \eta_i(t)\eta_j(t')\rangle = 2\delta_{ij}\delta(t-t')$$

(stochastic consensus problem)

global observable:
(spread or **width of the synchronization landscape**)

$$\left\langle w^2(t) \right\rangle = \left\langle \frac{1}{N}\sum_{i=1}^{N}(h_i - \bar{h})^2 \right\rangle$$

$$\bar{h} = \sum_l h_l$$



i [index of node]

D. Hunt *et al*., *PRL* (2010);
B. Kozma *et al.*, *PRL* (2005);
B. Kozma *et al*., *PRL* (2004).

Synchronization in Networks in a Noisy Environment: the Edwards-Wilkinson Process on a Network

$$\partial_t h_i(t) = -\sum_j \Gamma_{ij} h_j + \eta_i(t)$$

Laplacian: $\Gamma_{ij} \equiv \delta_{ij} C_i - C_{ij}$

$$\partial_t \tilde{h}_k(t) = -\lambda_k \tilde{h}_k(t) + \tilde{\eta}_k(t)$$

$$\langle \tilde{\eta}_k(t) \tilde{\eta}_{k'}(t') \rangle = 2\delta_{kk'} \delta(t-t')$$

$$0 = \lambda_0 \le \lambda_1 \le \lambda_2 \le \ldots \le \lambda_{N-1} = \lambda_{\max}$$

$$\left\langle w^2(t) \right\rangle = \left\langle \frac{1}{N} \sum_{i=1}^{N} (h_i - \bar{h})^2 \right\rangle = \frac{1}{N} \sum_{k=1}^{N-1} \left\langle \tilde{h}_k^2(t) \right\rangle$$

$$\langle w^2(t) \rangle = \frac{1}{N} \sum_{k=1}^{N-1} \frac{1}{\lambda_k} (1 - e^{-2\lambda_k t})$$

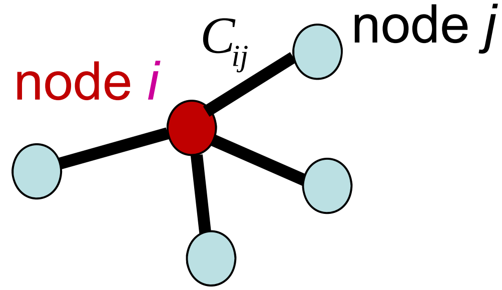
(starting from a flat landscape, $h_i(0) = 0$)

$$\langle w^2 \rangle = \langle w(\infty)^2 \rangle = \frac{1}{N} \sum_{k=1}^{N-1} \frac{1}{\lambda_k} \xrightarrow[N \to \infty]{} \int \frac{\rho(\lambda) d\lambda}{\lambda}$$

steady-state width

Connection between the two-point resistance and the steady-state fluctuations

for *any* graph:



$$\partial_t h_i(t) = -\sum_j C_{ij}(h_i - h_j) + \eta_i(t) \qquad \sum_j C_{ij}(V_i - V_j) = I(\delta_{is} - \delta_{it})$$

$$\Gamma_{ij} = \delta_{ij} C_i - C_{ij} \qquad G_{ij} \equiv \hat{\Gamma}^{-1}_{ij}$$

$$\langle (h_s - \bar{h})(h_t - \bar{h}) \rangle = G_{st}$$

$$\langle (h_s - h_t)^2 \rangle = G_{ss} + G_{tt} - 2G_{st}$$

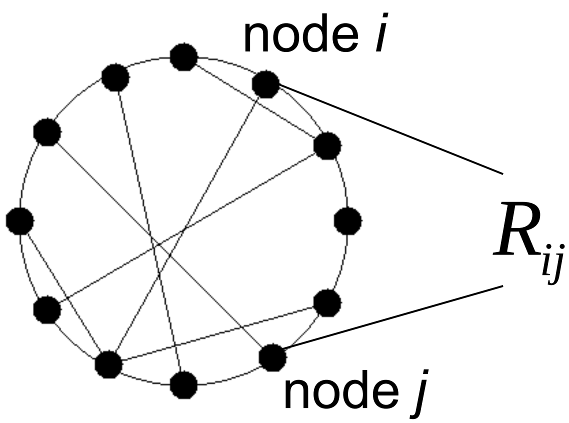
$$\hat{V}_i = I(G_{is} - G_{it})$$

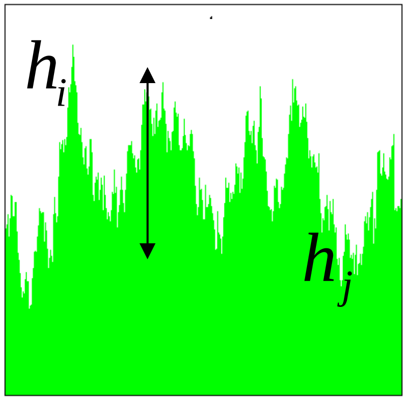
$$R_{st} = G_{ss} + G_{tt} - 2G_{st}$$

Connection between the two-point resistance and the steady-state fluctuations

$$R_{ij} = G_{ii} + G_{jj} - 2G_{ij} = \langle (h_i - h_j)^2 \rangle$$

(for *any* graph)





$$\bar{R} = \frac{2}{N(N-1)} \sum_{i<j} R_{ij} = \frac{N}{N-1} 2\langle w^2 \rangle \approx 2\langle w^2 \rangle$$

GK *et al.*, (2006)
GK (2007).

Application to Weighted Complex Networks

$$\text{minimize} \quad \bar{R}/2 = \frac{\bar{\tau}}{\sum_{i,j} C_{ij}} = \langle w^2 \rangle$$

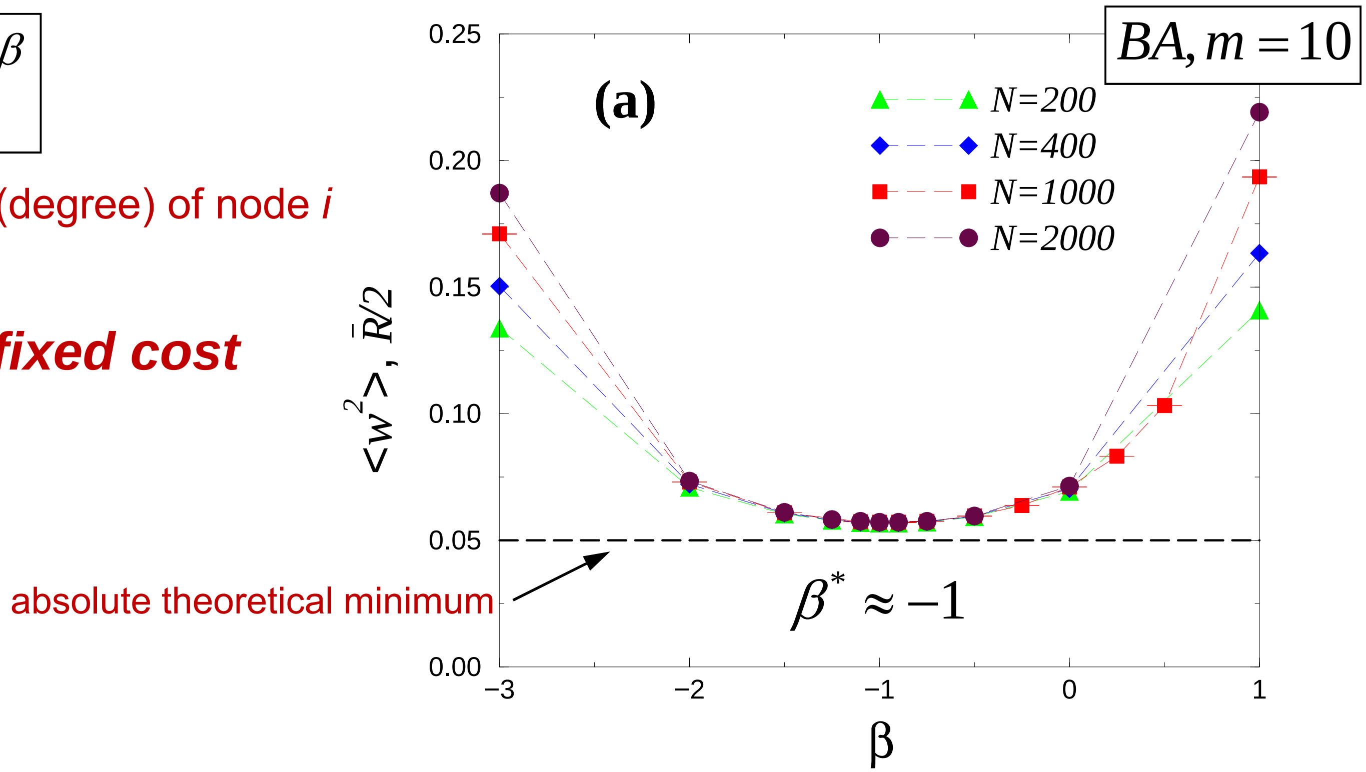
- heterogeneous degrees
- weighted links (Barrat *et al*. (2004)

$$C_{ij} \propto A_{ij} (k_i k_j)^{\beta}$$

k_i: number of neighbors (degree) of node *i*

- ***limited resources/fixed cost***

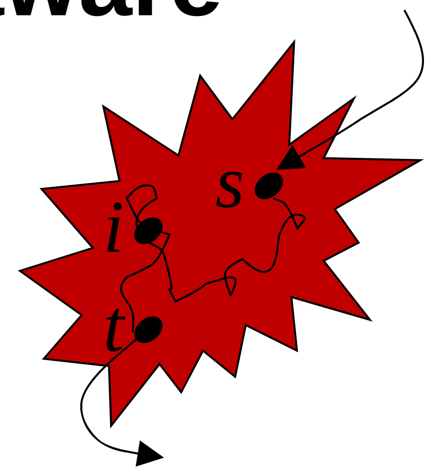
$$\sum_{i,j} C_{ij} = \text{fixed}$$



G. Korniss, *Phys. Rev. E* **75**, 051121 (2007).

Connection with congestion-aware local RW "routing" schemes

Guimerà et al. '02, Zhao et al., '05;
Danila et al., '06; Sreenivasan et al. '06

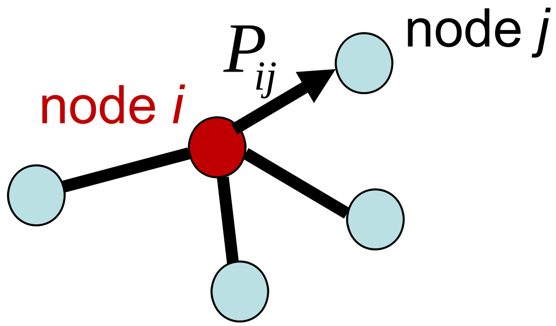


- packets are generated with ***identical rate*** φ at each node
- identical processing capabilities for each node
  (e.g., can forward one packet per unit time)
- **throughput is limited by the most congested node**

$$b_i = N \frac{C_i}{2} \overline{R}$$

$$\varphi b_i < 1, \quad \forall i \quad \Rightarrow \quad \varphi_c = \frac{1}{b_{\max}}$$

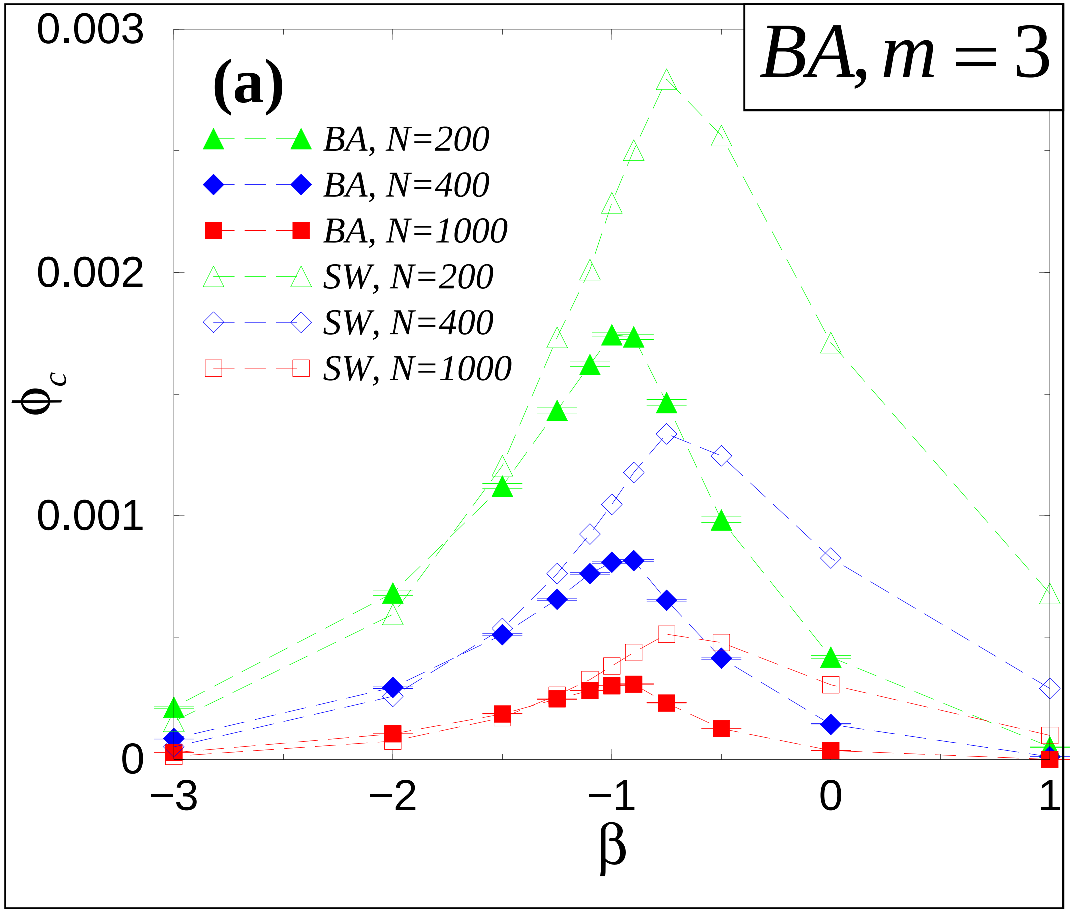
Special case of weighted RWs:



$RW:$ $\text{Prob}\{i \to j\}:$

$$P_{ij} = \frac{C_{ij}}{C_i} = \frac{A_{ij}(k_i k_j)^{\beta}}{\sum_l A_{il}(k_i k_l)^{\beta}} = \frac{A_{ij} k_j^{\beta}}{\sum_l A_{il} k_l^{\beta}} \propto A_{ij} k_j^{\beta}$$

network throughput:



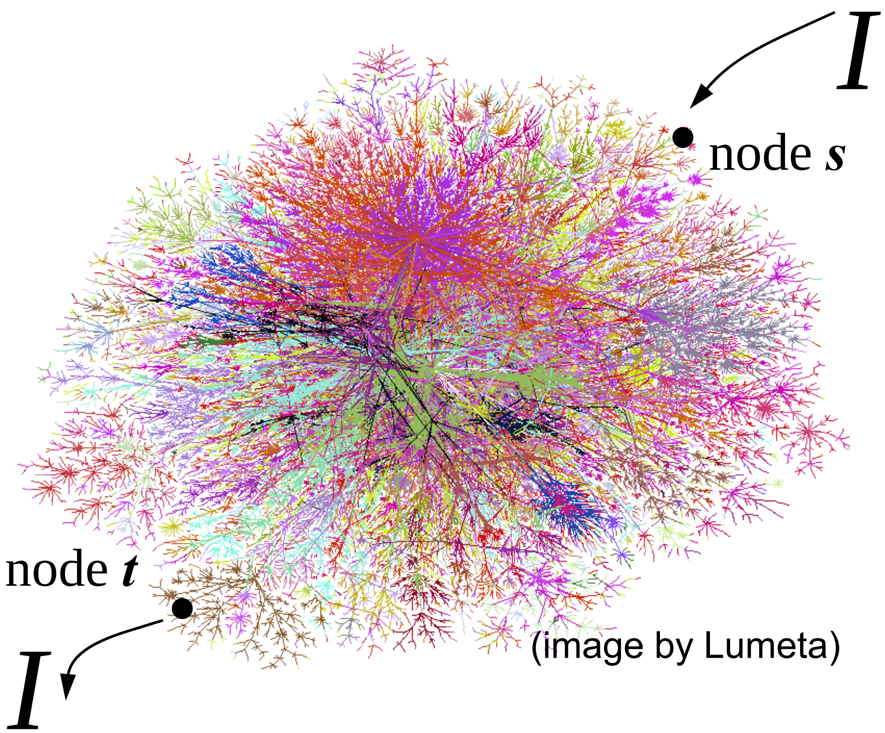
there exists a β^* where φ_c is maximum

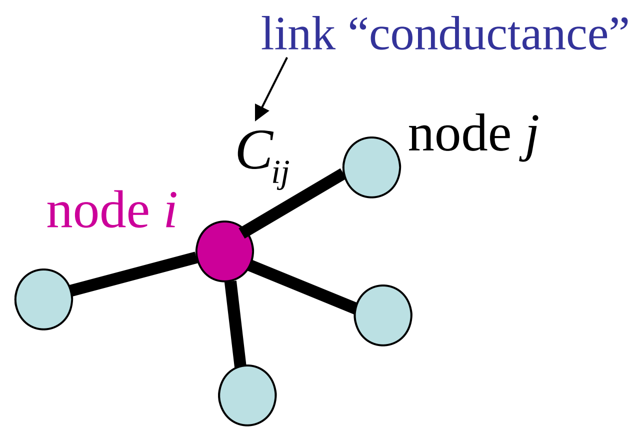
$$\beta^* \approx -1, \quad 1 << m << N$$

$$\overline{b} = \frac{1}{2}\sum_i C_i \overline{R} = \min$$

$$\overline{\tau} = \overline{b} = \min$$

Transport and Flow in Complex Networks: Cascading Overload Failures in Networks with Distributed Flows





$$\sum_j C_{ij}(V_i - V_j) = I(\delta_{is} - \delta_{it})$$

López et al., *PRL* (2005);
GK et al., *PLA* (2005); GK, *PRE* (2007);
Asztalos *et al.*, *EPJB* (2012).

- fundamental model for ***distributed flows*** in networks
- local flow and ***extreme flow distributions***: $P(|I_i|)$, $P(|I_i|^{max})$
- identify ***critical*** nodes and links (with large load) and ***vulnerability*** spots
- ***cascading failures***

Flow Optimization Using Current Flows in Weighted Resistor Networks

$$C_{ij} \propto A_{ij}(k_i k_j)^{\beta}$$

(pseudo) inverse of the network Laplacian

$$|I_{ij}^{st}| = C_{ij}\,|G_{is} - G_{it} - G_{js} + G_{jt}| \xrightarrow{\frac{1}{N-1}\sum_{s,t}} l_{ij} = \frac{1}{N-1}\sum_{s,t}|I_{ij}^{st}|$$

current-flow edge betwenness

$$I_i^{st} = (1/2)\sum_j |I_{ij}^{st}| \xrightarrow{\frac{1}{N-1}\sum_{s,t}} l_i = \frac{1}{N-1}\sum_{s,t}|I_i^{st}|$$

current-flow node betwenness

Brandes and Fleischer (2005);
GK *et al*. (2012);
Asztalos *et al.*, *EPJB* (2012).

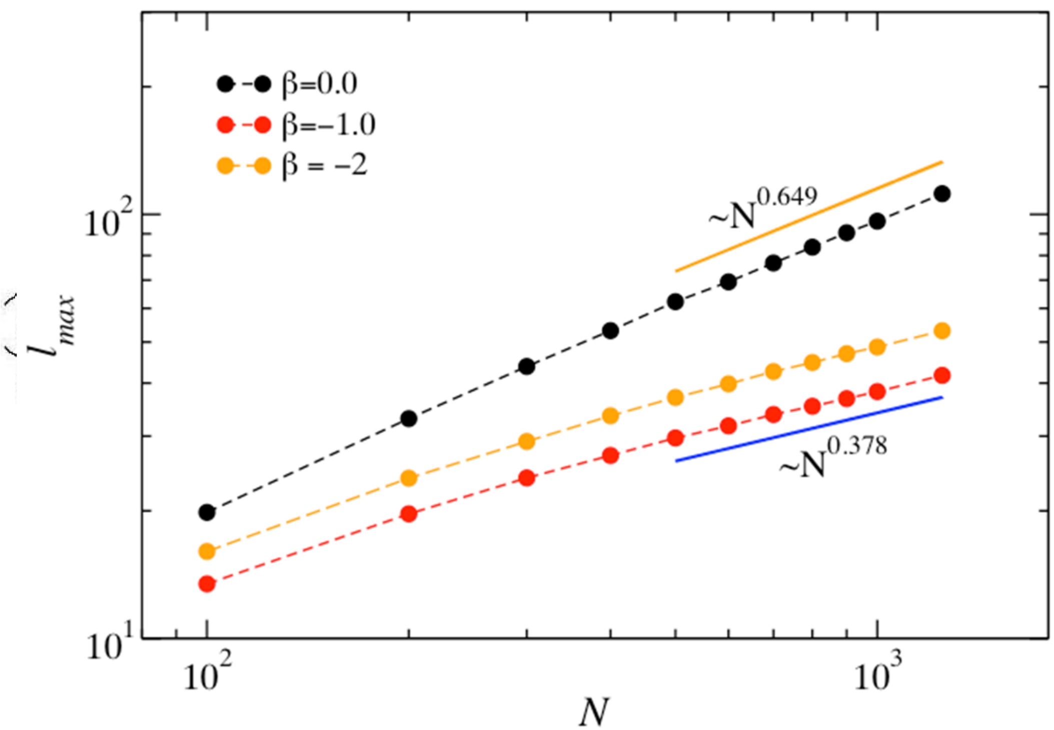
Maximizing Transport Capacity (general ucSF graph, $P(k) \sim k^{-\gamma}$)

- Assume Φ units of current entering the network
    - Each node has unit processing capability
- Congestion-free network if : $\Phi l_i \leq 1$ for all *i* nodes

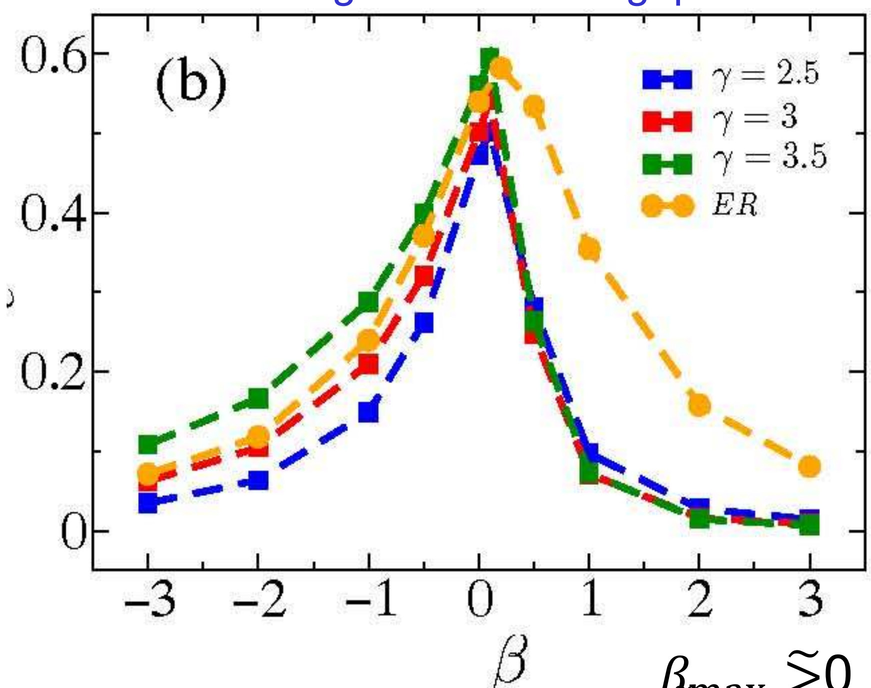
$$C_{ij} \propto A_{ij} (k_i k_j)^{\beta}$$

Transport capacity (network throughput) – the maximum Φ unit of inserted current for which the network is congestion free:

$$\Phi_c = \frac{1}{l_{max}}$$



edge-limited throughput



$\beta_{max} \gtrsim 0$

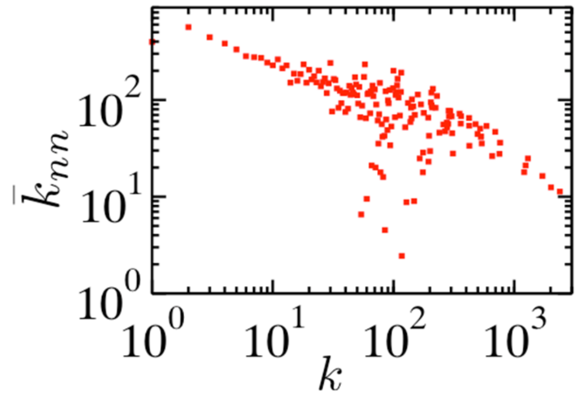
Asztalos *et al.*,*EPJB* (2012).

Maximizing Transport Capacity

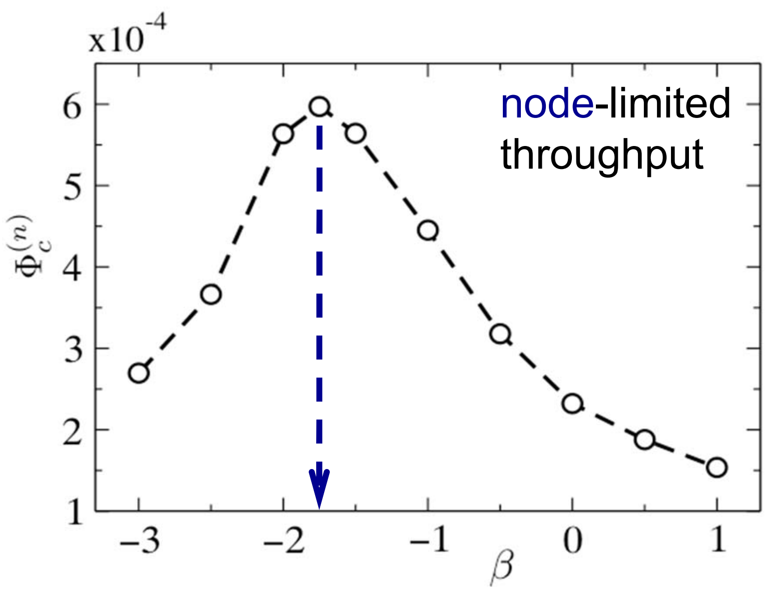
Real Data: **Internet** – infrastructural backbone for communication and information

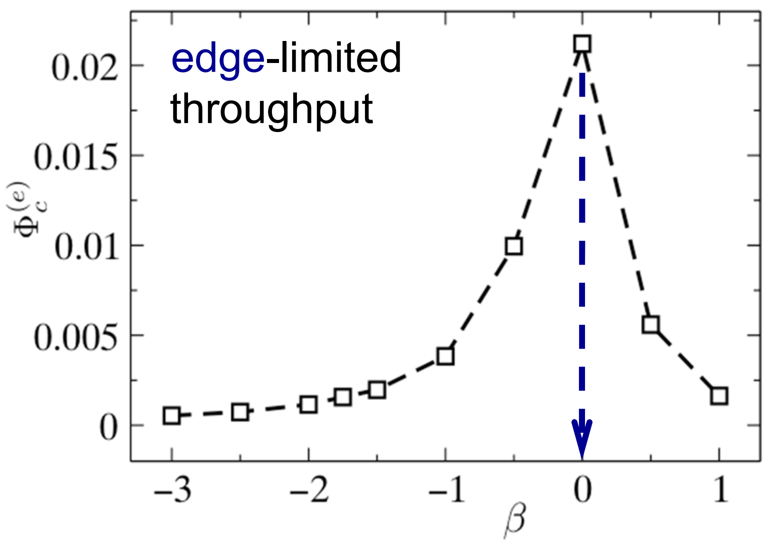
- N=22963, <k>=4.22
- scale-free network
- exhibits disassortative mixing by degree

Average degree of nearest neighbors of nodes with degree k



Vertex load – net number of data packets passing through an AS



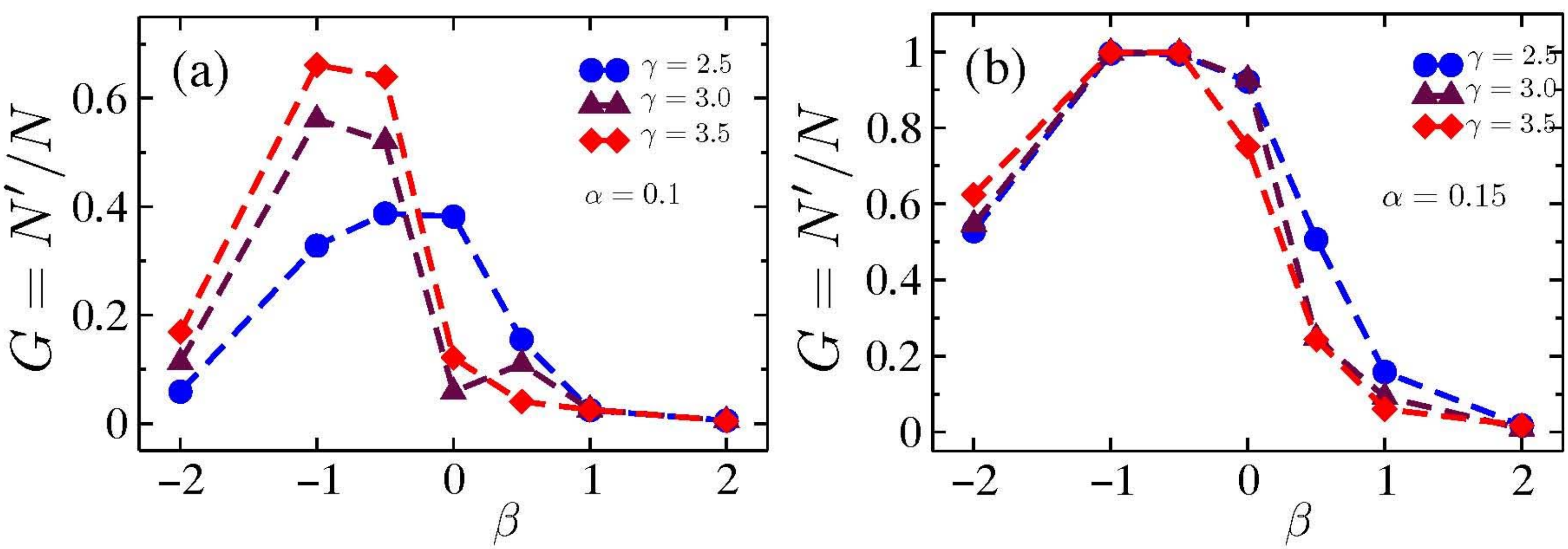


Asztalos *et al.*, *EPJB* (2012).

Improving Network Robustness to Cascading Failures

- Node forwarding capabilities proportional to the initial load: $C_i = (1+\alpha) l_i^0$
- Node fails when $l_i > C_i$ α - tolerance parameter Motter & Lai, *PRE (*2002).

Cascade of failing nodes – subsequent removal of failed nodes, triggered by a **single** node removal, till none of the loads exceeds their capacities

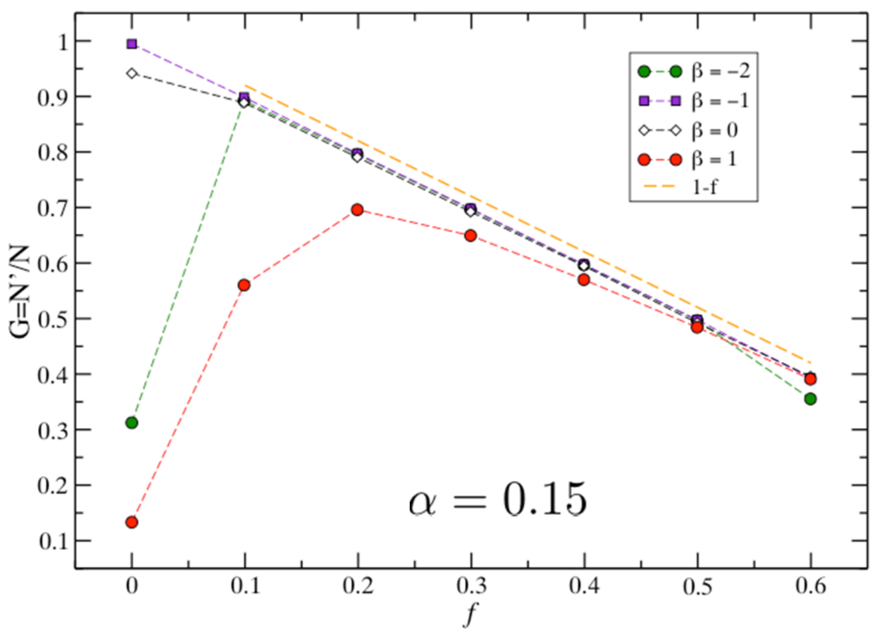


Robustness and network efficiency is optimal for $\beta \approx -1$

Yang *et al.*, *PRE* (2009).
Asztalos *et al.*, *EPJB* (2012).

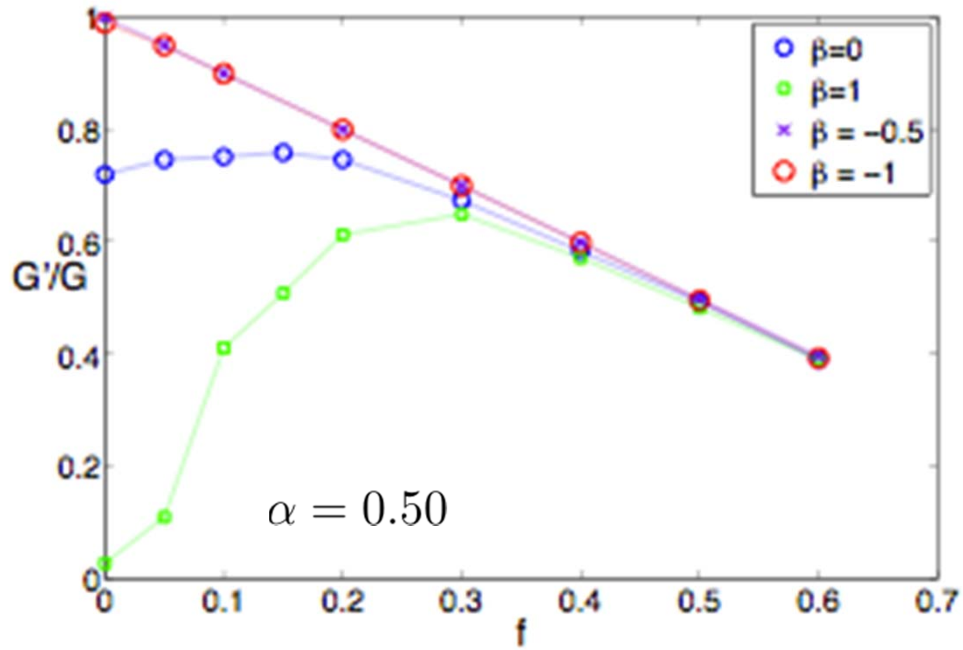
Preemptive node removal on uncorrelated **scale-free networks** in the presence of distributed flow and shortest-path flow

$N = 1000, \langle k \rangle = 5, \gamma = 2.5$



distributed flow

$N = 1024, \langle k \rangle = 5, \gamma = 2.5$



shortest-path flow

Motter, *PRL* (2004).
GK *et al*. (2012).

Some Recommended Reading

P.G. Doyle and J.L. Snell (1984), http://arxiv.org/abs/math/0001057

A.K. Chandra et al., *Proc. 21st ACM STOC* 575-586 (1989); http://dx.doi.org/10.1145/73007.73062

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