System of Linear Equations

Slide for MA1203 Business Mathematics II

Week 2 & 3

2.1 Systems of Linear Equations

System of Linear Equations

In many economics problems, there are two or more linear equations that must be satisfied *simultaneously*. Then we have a system of linear equations in two or more variables.

Consider a system of two linear equations in two variables.

$$ax + by = h$$

$$cx + dy = k$$

where a, b, c, d, h, and k are real constants and neither a and b nor c and d are both zero.

Graphing Solutions

The solution of a system of two linear equations could be found by finding the point of intersection of two straight lines.

Suppose we are given two straight lines L_1 and L_2 with equations

$$y = m_1x + b_1 \text{ and } y = m_2x + b_2$$

(where m_1, b_1, m_2, and b_2 are constants) that intersect at the point $P(x_0,y_0)$.
The point $P(x_0,y_0)$ lies on the line L_1, so it satisfies the equation $y = m_1x + b_1$. It also lies on the line L_2, so it satisfies the equation $y = m_2x + b_2$.

Example

Find a solution to the two linear equations

$$x + y = 10 \text{ and } x - y = 0.$$

Solutions by Substitution and Elimination

Example

Solve the following system of equations

$$5x + 2y = 3 \text{ and } - x - 4y = 3$$

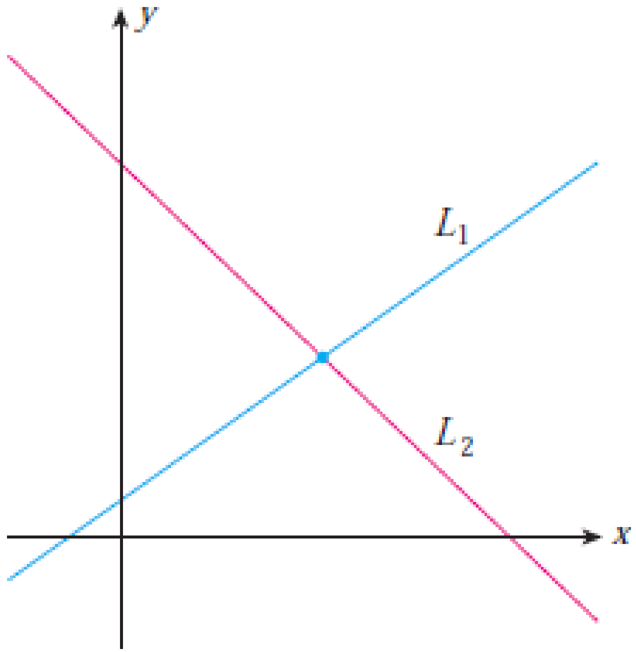
(a) by substitution and

(b) by elimination.

Solutions of System of Linear Equations in Two Variables

Given two lines L_1 and L_2, *one and only one* of the following may occur:

$a.$ L_1 and L_2 intersect at exactly one point.



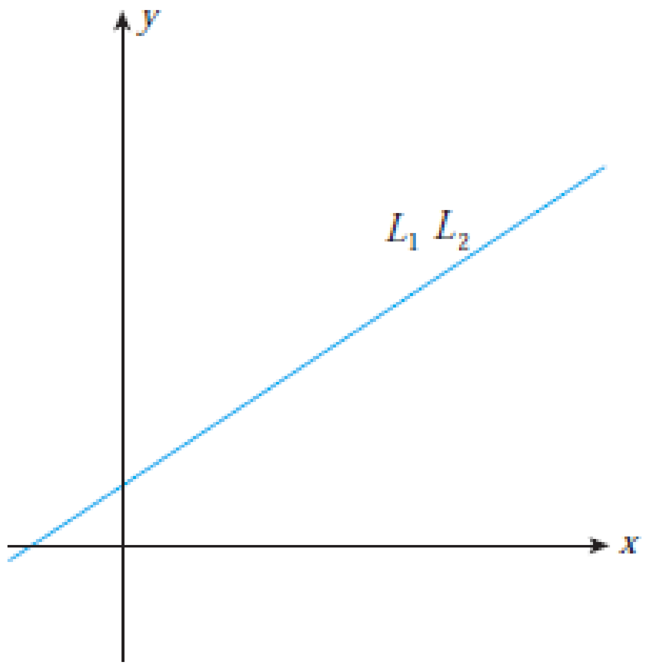
(a) Unique solution

A system of equations with exactly one solution

$$2x - y = 1$$
$$3x + 2y = 12$$

Solutions of System of Linear Equations in Two Variables (2)

b. L_1 and L_2 are parallel and coincident.



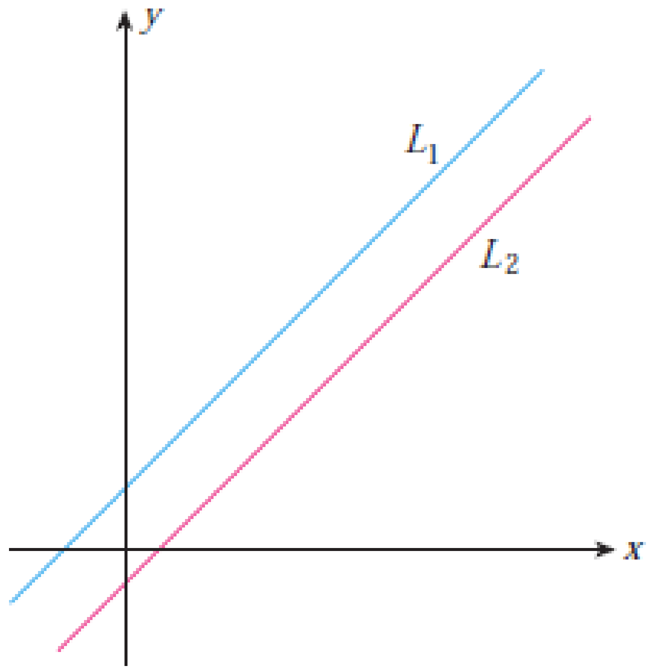
(b) Infinitely many solutions

A system of equations with infinitely many solutions

$$2x - y = 1$$
$$6x - 3y = 3$$

Solutions of System of Linear Equations in Two Variables (3)

c. L_1 and L_2 are parallel and distinct.



(c) No solution

A system of equations that has no solution

$$2x - y = 1$$
$$6x - 3y = 12$$

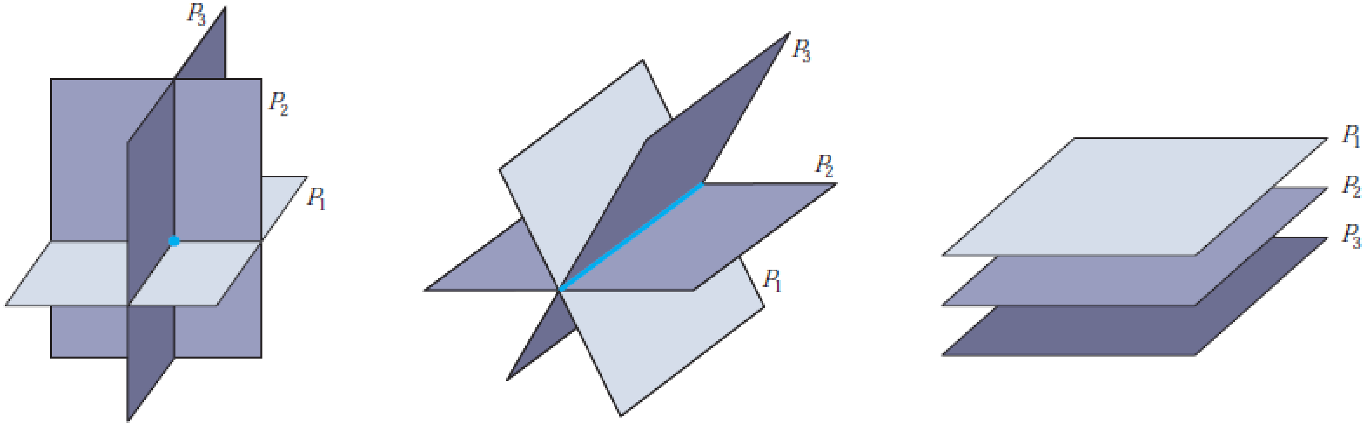
Systems of Linear Equations in Three Variables

A linear system composed of three linear equations in three variables x, y, and z has the general form

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

Each equation represents a *plane* in three-dimensional space, and the *solution(s) of the system* is precisely the point(s) of intersection of the three planes defined by the three linear equations that make up the system.

The system has one and only one solution, infinitely many solutions, or no solution, depending on whether and how the planes intersect one another.



(a) A unique solution (b) Infinitely many solutions (c) No solution

Linear Equations in *n* Variables

A linear equation in n variables, $x_1, x_2, \dots, x_n$ is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = c$$

where $a_1, a_2, \dots, a_n$ (not all zero) and c are constants.

As in the case of systems involving two or three variables, it can be shown that only three possibilities exist regarding the nature of the solution of such a system:

(1) a unique solution,

(2) infinitely many solutions, or

(3) no solution.

2.2 Systems of Linear Equations: Unique Solutions

The Gauss–Jordan Method

The **Gauss–Jordan elimination method** is a technique for solving systems of linear equations of any size.

This method involves a sequence of operations on a system of linear equations to obtain at each stage an equivalent system—that is, a system having the same solution as the original system. The reduction is complete when the original system has been transformed so that it is in a certain standard form from which the solution can be easily read.

The operations of the Gauss–Jordan elimination method are

1. Interchange any two equations.
2. Replace an equation by a nonzero constant multiple of itself.
3. Replace an equation by the sum of that equation and a constant multiple of any other equation.

Example

Solve the following system of equations:

$$\begin{gathered} 2x + 4y + 6z = 22 \\ 3x + 8y + 5z = 27 \\ -x + y + 2z = 2 \end{gathered}$$

Augmented Matrix

With the aid of matrices, which are rectangular arrays of numbers, we can eliminate writing the variables at each step of the reduction and thus save ourselves a great deal of work.

For example, the system

$$\begin{aligned} 2x + 4y + 6z &= 22 \\ 3x + 8y + 5z &= 27 \\ -x + y + 2z &= 2 \end{aligned}$$

may be represented by the matrix

$$\left[\begin{array}{ccc|c} 2 & 4 & 6 & 22 \\ 3 & 8 & 5 & 27 \\ -1 & 1 & 2 & 2 \end{array}\right]$$

The submatrix consisting of the first three columns of the matrix is called the coefficient matrix of the system. The matrix itself is referred to as the augmented matrix of the system since it is obtained by joining the matrix of coefficients to the column (matrix) of constants. The vertical line separates the column of constants from the matrix of coefficients.

Row-reduced form of a matrix

1. Each row consisting entirely of zeros lies below all rows having nonzero entries.
2. The first nonzero entry in each (nonzero) row is 1 (called a leading 1).
3. In any two successive (nonzero) rows, the leading 1 in the lower row lies to the right of the leading 1 in the upper row.
4. If a column in the coefficient matrix contains a leading 1, then the other entries in that column are zeros.

Example

Determine which of the following matrices are in row-reduced form.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 2 & -2 \\ 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 1 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \end{array}\right]$$

Row Operations

We need an adaptation of the Gauss–Jordan elimination method in solving systems of linear equations using matrices.

The three operations on the equations of a system translate into the following row operations on the corresponding augmented matrices.

1. Interchange any two equations.

   Interchange any two rows.

2. Replace an equation by a nonzero constant multiple of itself.

   Replace any row by a nonzero constant multiple of itself.

3. Replace an equation by the sum of that equation and a constant multiple of any other equation.

   Replace any row by the sum of that row and a constant multiple of any other row.

Notations for Gauss-Jordan Elimination using Matrices

A column in a coefficient matrix is called a unit column if one of the entries in the column is a 1 and the other entries are zeros.

Notation for Row Operations

Letting R_i denote the ith row of a matrix, we write:

Operation 1 $R_i \leftrightarrow R_j$ to mean: Interchange row i with row j.

Operation 2 cR_i to mean: Replace row i with c times row i.

Operation 3 $R_i + aR_j$ to mean: Replace row i with the sum of row i and a times row j.

The Gauss-Jordan Elimination Method

1. Write the augmented matrix corresponding to the linear system.
2. Interchange rows (operation 1), if necessary, to obtain an augmented matrix in which the first entry in the first row is nonzero. Then pivot the matrix about this entry.
3. Interchange the second row with any row below it, if necessary, to obtain an augmented matrix in which the second entry in the second row is nonzero. Pivot the matrix about this entry.
4. Continue until the final matrix is in row-reduced form.

Example Solve the system of linear equations given by

$$3x - 2y + 8z = 9$$
$$-2x + 2y + z = 3$$
$$x + 2y - 3z = 8$$

Applied Example:
Manufacturing: Production Scheduling

Ace Novelty wishes to produce three types of souvenirs: Types *A*, *B*, and *C*.

To manufacture a Type *A* souvenir requires 2 minutes on Machine I, 1 minute on Machine II, and 2 minutes on Machine III. A Type *B* souvenir requires 1 minute on Machine I, 3 minutes on Machine II, and 1 minute on Machine III. A Type *C* souvenir requires 1 minute on Machine I and 2 minutes each on Machines II and III.

There are 3 hours available on Machine I, 5 hours available on Machine II, and 4 hours available on Machine III for processing the order.

How many souvenirs of each type should Ace Novelty make in order to use all of the available time?

2.3 Systems of Linear Equations: Underdetermined and Overdetermined Systems

A System an Infinite Number of Solutions

Solve the system of linear equations given by

1. $x + 2y = 4$
   $3x + 6y = 12$

2. $x + 2y - 3z = -2$

   $3x - y - 2z = 1$

   $2x + 3y - 5z = -3$

A System of Equations That Has No Solution

Solve the system of linear equations given by

$$x + y + z = 1$$

$$3x - y - z = 4$$

$$x + 5y + 5z = -1$$

A System with More Equations Than Variables

Solve the system of linear equations given by

$$\begin{gathered} x + 2y = 4 \\ x - 2y = 0 \\ 4x + 3y = 12 \end{gathered}$$

A System with More Variables Than Equations

Solve the system of linear equations given by

$$x + 2y - 3z + w = -2$$

$$3x - y - 2z - 4w = 1$$

$$2x + 3y - 5z + w = -3$$

Applied Example

Traffic Control

The Figure 7 on the right shows the flow of downtown traffic in a certain city during the rush hours on a typical weekday. The arrows indicate the direction of traffic flow on each one-way road, and the average number of vehicles per hour entering and leaving each intersection appears beside each road.

5th Avenue and 6th Avenue can each handle up to 2000 vehicles per hour without causing congestion, whereas the maximum capacity of both 4th Street and 5th Street is 1000 vehicles per hour. The flow of traffic is controlled by traffic lights installed at each of the four intersections.

a. Write a general expression involving the rates of flow—*x*1, *x*2, *x*3, *x*4—and suggest two possible traffic-flow patterns that will ensure no traffic congestion.

b. Suppose that the part of 4th Street between 5th Avenue and 6th Avenue is to be resurfaced and that traffic flow between the two junctions must therefore be reduced to at most 300 vehicles per hour. Find two possible traffic-flow patterns that will result in a smooth flow of traffic.