Systems of Linear Equations in two variables (4.1)

1. Solve by graphing
2. Solve using substitution
3. Solve by elimination by addition
4. Applications

Opening example

- A restaurant serves two types of fish dinners- small for $5.99 each and a large order for $8.99. One day, there were 134 total orders of fish and the total receipts for these 134 orders was $1024.66. How many small orders and how many large fish plates were ordered?

Systems of Two Equations in Two variables

- Given the linear system
- ax + by = c
- dx + ey = f
- A solution is an ordered pair $\left(x_0, y_0\right)$
- that will satisfy each equation (make a true equation when substituted into that equation). The solution set is the set of all ordered pairs that satisfy both equations. In this section, we wish to find the solution set of a system of linear equations.

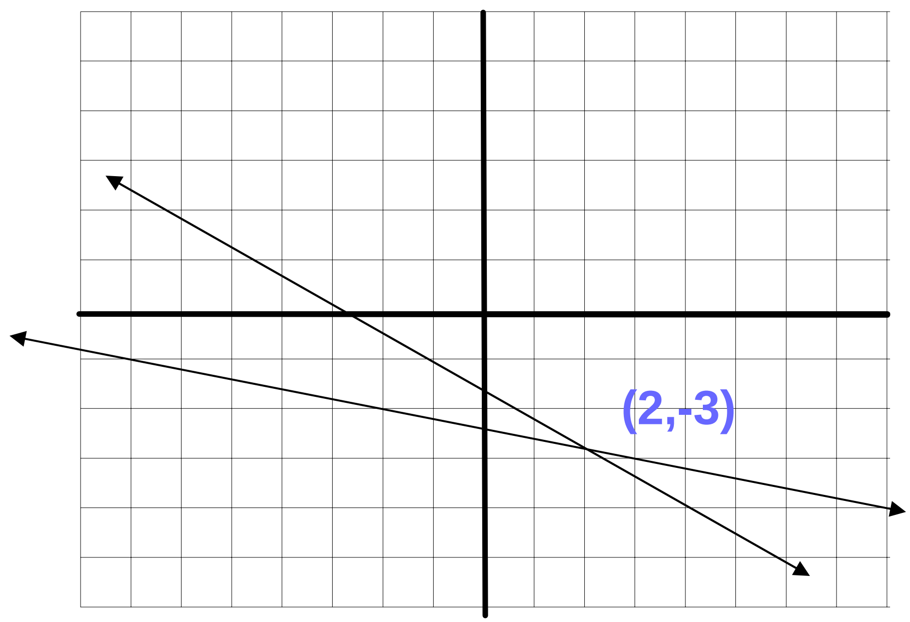
Solve by graphing

- One method to find the solution of a system of linear equations is to graph each equation on a coordinate plane and to determine the point of intersection (if it exists). The drawback of this method is that it is not very accurate in most cases, but does give a general location of the point of intersection. Lets take a look at an example:

- Solve the system by graphing:
- $3x + 5y = -9$
- $x + 4y = -10$

Solve by graphing

- 3x + 5y = -9
- x+ 4y = -10
- Intercept method:
- If x = 0, y= -9/5
- If y = 0 , x = - 3
- Plot points and draw line



- Second line:
- (0, -5/2) , (-10,0)
- From the graph we see that the point of intersection is (2,-3).
- Check: 3(2)+5(-3)= -9 and 2+4(-3)= -10 both check.

Another example:

- Now, you try one:
- Solve the system by graphing:
- $2x+3 = y$
- $X+2y = -4$
- The solution is (-2,-1)

Method of Substitution

- Although the method of graphing is intuitive, it is not very accurate in most cases. There is another method that is 100% accurate- it is called the method of substitution. This method is an algebraic one. This method works well when the coefficients of x or y are either 1 or -1. For example, let's solve the previous system
- $2x + 3 = y$
- $X + 2y = -4$

using the method of substitution. This steps for this method are as follows:

1) Solve one of the equations for either x or y.
2) Substitute that result into the other equation to obtain an equation in a single variable (either x or y).
3) Solve the equation for that variable.
4. Substitute this value into any convenient equation to obtain the value of the remaining variable.

Method of substitution

- $2x+3 = y$
- $x + 2y = -4$
- $x + 2(2x+3) = -4$
- $x + 4x + 6 = -4$
- $5x + 6 = -4$
- $5x = -10$
- $x = -2$

- If $x = -2$ then from the first equation, we have $2(-2)+3 = y$ or
- $-1 = y$. Our solution is
- $(-2, -1)$

Another example:

- Solve the system using substitution:
- 3x-2y=-7
- y = 2x -3

- Solution:

$$3x-2y=7$$

$$y=2x-3$$

$$3x-2(2x-3)=7$$

$$3x-4x+6=7$$

$$-x=1$$

$$x=-1$$

$$y=2(-1)-3 \rightarrow y=-5$$

Terminology:

- 1. A consistent linear system is one that has one or more solutions.
  - A) If a consistent system has exactly one solution then it is said to be independent. An independent system will occur when two lines have different slopes.
  - B) if a consistent system has more than one solution, then it is said to be dependent. A dependent system will occur when two lines have the same slope and the same y intercept. In other words, the two equations are identical. The graphs of the lines will coincide with one another and there will be an infinite number of points of intersection.
- 2. An inconsistent linear system is one that has no solutions. This will occur when two lines have the same slope but different y intercepts. In this case, the lines will be parallel and will never intersect.

An example:

- Determine if the system is consistent, independent, dependent or inconsistent:
- 1) 2x – 5y = 6
- -4x + 10y = -1
- Solve each equation for y to obtain the slope intercept form of the equation:

$$2x - 5y = 6$$
$$2x - 6 = 5y$$
$$\frac{2x-6}{5} = y$$
$$\frac{2x}{5} - \frac{6}{5} = y$$

$$-4x + 10y = -1$$
$$10y = 4x - 1$$
$$y = \frac{4x}{10} - \frac{1}{10} = \frac{2}{5}x - \frac{1}{10}$$

- Since each equation has the same slope but different y intercepts, they will not intersect. This is an inconsistent system

Elimination by Addition

- The method of substitution is not preferable if none of the coefficients of x and y are 1 or -1. For example, substitution is not the preferred method for the system below: $2x - 7y = 3$
  - $-5x + 3y = 7$
  - A better method is elimination by addition: First of all, we need to know what operations can be used to produce equivalent systems. They are as follows:
  - 1) Two equations can be interchanged.
  - 2\. An equation is multiplied by a non-zero constant.
  - 3\. An equation is multiplied by a non-zero constant and then added to another equation.

Elimination by Addition

- For our system, we will seek to eliminate the x variable. The coefficients of the x variables are 2 and -5. The least common multiple of 2 and 5 is 10. Our goal is to obtain coefficients of x that are additive inverses of each other.
- We can accomplish this by multiplying the first equation by 5

Multiply second equation by 2.

2

- x Next, we can add the two equations to eliminate the x-variable.
- – Solve for y
- Substitute y value into original 7 equation and solve for x
- y Write solution as an ordered pair

=

- S: 2x-7y=3 / -5x+3y=7 $\rightarrow$

$$\left.\begin{aligned} 5(2x-7y) &= 5(3) \\ 2(-5x+3y) &= 2(7) \end{aligned}\right\} \rightarrow$$

$$\left.\begin{aligned} 10x-35y &= 15 \\ -10x+6y &= 14 \end{aligned}\right\} \rightarrow$$

$$0x-29y = 29 \rightarrow$$

$$y = -1$$

$$2x-7(-1) = 3 \rightarrow$$

$$2x+7 = 3 \rightarrow$$

$$2x = -4$$

$$x = -2$$

$$(-2,-1)$$

Solve using elimination by addition

- Solve 2x-5y = 6
- -4x+10y=-1
- 1. Eliminate x by multiplying equation 1 by 2 .
- 2. Add two equations
- 3. Upon adding the equations, both variables are eliminated producing the false equation
- 0 = 11
- 4. Conclusion: If a false equation arises, the system is inconsistent and there is no solution.

- Solution:

$$\left.\begin{aligned} 2x-5y&=6 \\ -4x+10y&=-1 \end{aligned}\right\} \rightarrow$$

$$\left.\begin{aligned} 4x-10y&=12 \\ -4x+10y&=-1 \end{aligned}\right\} \rightarrow$$

$$0=11\}$$

Applications

- A man walks at a rate of 3 miles per hour and jogs at a rate of 5 miles per hour. He walks and jogs a total distance of 3.5 miles in 0.9 hours. How long does the man jog?
- Solution: Let x represent the amount of time spent walking and y represent the amount of time spent jogging. Since the total time spent walking and jogging is 0.9 hours, we have the equation
- $x + y = 0.9$. We are given the total distance traveled as 3.5 miles. Since Distance = Rate x time, we have [distance walking] + [distance jogging] = [total distance] . Distance walking = $3x$ and distance jogging = $5y$. Then [distance walking] plus [distance jogging] $\longrightarrow$ $3x+5y= 3.5$.

Application continued

- We can solve the system using substitution.
- 1. Solve the first equation for y
- 2. Substitute this expression into the second equation.
- 3. Solve 2nd equation for x
- 4. Use this x value to find the y value
- 5. Answer the question.
- 6. Time spent jogging is 0.4 hours.

- Solution:

$$x + y = 0.9$$
$$y = 0.9 - x$$
$$3x + 5y = 3.5$$
$$3x + 5(0.9 - x) = 3.5$$
$$3x + 4.5 - 5x = 3.5$$
$$-2x = -1$$
$$x = 0.5$$
$$0.5 + y = 0.9$$
$$y = 0.4$$

Now, solve the opening example:

- A restaurant serves two types of fish dinners- small for $5.99 each and a large order for $8.99. One day, there were 134 total orders of fish and the total receipts for these 134 orders was $1024.66. How many small orders and how many large fish plates were ordered? (60 small orders and 74 large orders)

4.2 Systems of Linear equations and Augmented Matrices

It is impractical to solve more complicated linear systems by hand. Computers and calculators now have built in routines to solve larger and more complex systems. Matrices, in conjunction with graphing utilities and or computers are used for solving more complex systems. In this section, we will develop certain matrix methods for solving two by two systems.

Matrices

- A matrix is a rectangular array of numbers written within brackets. Here is an example of a matrix which has three rows and three columns: The subscripts give the "address" of each entry of the matrix. For example the entry a_{23}
- Is found in the second row and third column

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Matrix solutions of linear systems

- When solving systems of linear equations, the coefficients of the variables played an important role. We can represent a linear system of equations using what is called an augmented matrix, a matrix which stores the coefficients and constants of the linear system and then manipulate the augmented matrix to obtain the solution of the system. Here is an example:

- $x + 3y = 5$
- $2x - y = 3$

$$\left[\begin{array}{cc|c} 1 & 3 & 5 \\ 2 & -1 & 3 \end{array}\right]$$

Generalization

- Linear system:

$$a_{11}x_1 + a_{12}x_2 = k_1$$
$$a_{21}x_1 + a_{221}x_2 = k_2$$

- Associated augmented matrix:

$$\left[\begin{array}{cc|c} a_{11} & a_{12} & k_1 \\ a_{21} & a_{22} & k_2 \end{array}\right]$$

Operations that Produce Row-Equivalent Matrices:

- 1. Two rows are interchanged:
- $$R_i \leftrightarrow R_j$$
- 2. A row is multiplied by a nonzero constant:

$$kR_i \rightarrow R_i$$

- 3. A constant multiple of one row is added to another row: $$kR_j + R_i \rightarrow R_i$$

Solve using Augmented matrix:

- Solve
- x +3y=5
- 2x – y=3
- 1. Augmented system
- 2. Eliminate 2 in 2nd row by row operation
- 3. Divide row two by -7 to obtain a coefficient of 1.
- 4. Eliminate the 3 in first row, second position.
- 5. Read solution from matrix

- $$\left[\begin{array}{cc|c} 1 & 3 & 5 \\ 2 & -1 & 3 \end{array}\right]$$

$$-2R_1 + R_2 \rightarrow$$

$$\left[\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & -7 & -7 \end{array}\right]$$

$$R_2 / -7 \rightarrow R_2 \rightarrow$$

$$\left[\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & 1 & 1 \end{array}\right]$$

$$-3R_2 + R_1 \rightarrow R_1 \rightarrow$$

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right] \rightarrow x = 2, y = 1; (2,1)$$

Solving a system using augmented matrix methods

- X+2y=4
- X+(1/2)y=4

1. Eliminate fraction in second equation.
2. Write system as augmented matrix.
3. Multiply row 1 by -2 and add to row 2
4. Divide row 2 by -3
5. Multiply row 2 by -2 and add to row 1.
6. Read solution : x = 4, y = 0
7. (4,0)

$$x + 2y = 4$$

$$x + \frac{1}{2}y = 4 \rightarrow 2x + y = 8$$

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 2 & 1 & 8 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -3 & 0 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 0 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 0 \end{array}\right] \rightarrow$$

Solving a system using augmented matrix methods

- $10x - 2y = 6$
- $-5x + y = -3$
- 1. Represent as augmented matrix.
- 2. Divide row 1 by 2
- 3. Add row 1 to row 2 and replace row 2 by sum
- 4. Since 0 = 0 is always true, we have a dependent system. The two equations are identical and there are an infinite number of solutions.

$$\left[\begin{array}{cc|c} 10 & -2 & 6 \\ -5 & 1 & -3 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 5 & -1 & 3 \\ -5 & 1 & -3 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 5 & -1 & 3 \\ 0 & 0 & 0 \end{array}\right]$$

Another example

- Solve $$5x-2y=-7$$ $$y=\frac{5}{2}x+1$$
- Rewrite second equation : $$2y=5x+2\rightarrow$$ $$-5x+2y=2$$
- Since we have an impossible equation, there is no solution. The two lines are parallel and do not intersect.

$$\left.\begin{aligned}5x-2y&=-7\\-5x+2y&=2\end{aligned}\right\}\rightarrow$$

$$\left[\begin{array}{cc|c}5 & -2 & -7\\-5 & 2 & 2\end{array}\right]\rightarrow$$

$$\left[\begin{array}{cc|c}5 & -2 & -7\\0 & 0 & -5\end{array}\right]$$

4.3 Gauss Jordan Elimination

- Any linear system must have exactly one solution, no solution, or an infinite number of solutions. Just as in the 2X2 case, the term **consistent** is used to describe a system with a **unique solution**, **inconsistent** is used to describe a system with no solution, and **dependent** is used for a system with an infinite number of solutions.

Matrix representations of consistent, inconsistent and dependent systems

- The following matrix representations of three linear equations in three unknowns illustrate the three different cases:

- Case I : consistent

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{array}\right)$$

- From this matrix representation, you can determine that
- x = 3, y = 4 and z= 5

Matrix representations of consistent, inconsistent and dependent systems

- Case 2:
- Inconsistent case:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

- From the second row of the matrix, we find that
- 0x + 0y +0z =6 or 0 = 6, an impossible equation. From this, we conclude that there are no solutions to the linear system.

Matrix representations of consistent, inconsistent and dependent systems

- Case 3:
- Dependent system

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

- When two rows of a matrix representation consist entirely of zeros, we conclude that two the linear equations were identical and therefore, the system is dependent.

Reduced row echelon form

- A matrix is said to be in **reduced row echelon form** or, more simply, in **reduced form**, if :
- 1. Each row consisting entirely of zeros is below any row having at least one non-zero element.
- 2. The leftmost nonzero element in each row is 1.
- 3. All other elements in the column containing the leftmost 1 of a given row are zeros.
- 4. The leftmost 1 in any row is to the right of the leftmost 1 in the row above.

Examples of reduced row echelon form:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 3 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & 8 \end{array}\right)$$

Solving a system using Gauss-Jordan Elimination

- Problem:

Example. Solve:

$x + y - z = -2$

$2x - y + z = 5$

$-x + 2y + 2z = 1$

Karl Frederick Gauss:

- At the age of seven, **Carl Friedrich Gauss** started elementary school, and his potential was noticed almost immediately. His teacher, Büttner, and his assistant, Martin Bartels, were amazed when Gauss summed the integers from 1 to 100 instantly by spotting that the sum was 50 pairs of numbers each pair summing to 101.

Example

- solution

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 2 & -1 & 1 & 5 \\ -1 & 2 & 2 & 1 \end{array}\right]$$

We begin by writing the system as an augmented matrix

Example continued:

- We already have a 1 in the diagonal position of first column. Now we want 0's below the 1. The first 0 can be obtained by multiplying row 1 by -2 and adding the results to row 2 :

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & -3 & 3 & 9 \\ -1 & 2 & 2 & 1 \end{array}\right]$$

Row 1 is unchanged:
-2 times row 1 is added to row 2
Row 3 is unchanged

Example continued:

- The second 0 can be obtained by adding row 1 to row 3:

Row 1 is unchanged
Row 2 is unchanged
Row 1 is added to Row 3

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & -3 & 3 & 9 \\ 0 & 3 & 1 & -1 \end{array}\right]$$

Example continued:

- Moving to the second column, we want a 1 in the diagonal position (where there is now
- –3). We get this by dividing every element in row 2 by -3:

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & 1 & -1 & -3 \\ 0 & 3 & 1 & -1 \end{array}\right]$$

- Row 1 is unchanged
- Row 2 is divided by –3
- Row 3 is unchanged

Example continued:

- To obtain a 0 below the 1 , we multiply row 2 by -3 and add it to the third row:

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 4 & 8 \end{array}\right]$$

Row 1 is unchanged
Row 2 is unchanged
-3 times row 2 is added to row 3

Example continued:

- To obtain a 1 in the third position of the third row, we divide that row by 4. Rows 1 and 2 do not change.

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Example continued:

- We can now work upwards to get zeros in the third column above the 1 in the third row. We will add R3 to R2 and replace R2 with that sum **and** add R3 to R1 and replace R1 with the sum . Row 3 will not be changed. All that remains to obtain reduced row echelon form is to eliminate the 1 in the first row, 2nd position.

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

Example continued:

- To get a zero in the first row and second position, we multiply row 2 by -1 and add the result to row 1 and replace row 1 by that result. Rows 2 and 3 remain unaffected.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

Final result:

- We can now “read” our solution from this last matrix. We have $x = 1$, $y = -1$ and $z = 2$. Written as an ordered triple, we have
- (1, -1, 2). This is a **consistent** system with a unique solution.

Example 2

- Solve the system:

- $3x - 4y + 4z = 7$
- $x - y - 2z = 2$
- $2x - 3y + 6z = 5$

Example 2 continued

- Begin by representing the system as an augmented matrix:

- Augmented matrix:

$$\left(\begin{array}{ccc|c} 3 & -4 & 4 & 7 \\ 1 & -1 & -2 & 2 \\ 2 & -3 & 6 & 5 \end{array}\right)$$

Continuation of example 2

- Since the first number in the second row is a 1, we interchange rows 1 and 2 and leave row 3 unchanged:
- New matrix:

$$\left(\begin{array}{ccc|c} 1 & -1 & -2 & 2 \\ 3 & -4 & 4 & 7 \\ 2 & -3 & 6 & 5 \end{array}\right)$$

Continuation of example 2:

- In this step, we will get zeros in the entries beneath the 1 in the first column: Multiply row 1 by -3 , add to row 2 and replace row 2; and -2*R1+R3 and replace R3:
- New matrix:

$$\left(\begin{array}{ccc|c} 1 & -1 & -2 & 2 \\ 0 & -1 & 10 & 1 \\ 0 & -1 & 10 & 1 \end{array}\right)$$

Final result:

- To get a zero in the third row, second entry we multiply row 2 by -1 and add the result to R3 and replace R3 by that sum: Notice this operations “wipes out” row 3 so row consists entirely of zeros.

- New matrix: This matrix corresponds to a dependent system with an infinite number of solutions:

$$\left(\begin{array}{ccc|c} 1 & -1 & -2 & 2 \\ 0 & -1 & 10 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Representation of a solution of a dependent system

$$\left(\begin{array}{ccc|c} 1 & -1 & -2 & 2 \\ 0 & -1 & 10 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

- We can interpret the second row of this matrix as **–y + 10z=1**
- Or **10z – 1 = y**
- So, if we let **z = t** (arbitrary real number, then in terms of t,
- **y = 10t-1.**
- Next we can express the variable x n terms of t as follows: From the first row of the matrix, we have
- **x – y -2z = 2**. If **z = t** and
- **y = 10t – 1**, we have
- **x – (10t-1)-2t = 2 or x -12t +1=2 ; x -12t = 1 or x = 12t+1**
- Our general solution can now be expressed in terms of t:
- **(12t+1,10t-1,t),** where t is an arbitrary real number

4.4 Matrices: Basic Operations

Addition and Subtraction of matrices

- To add or subtract matrices, they must be of the same order, mxn. To add matrices of the same order, add their corresponding entries. To subtract matrices of the same order, subtract their corresponding entries. The general rule is as follows using mathematical notation:

$$A + B = \left[a_{ij} + b_{ij} \right]$$

$$A - B = \left[a_{ij} - b_{ij} \right]$$

An example:

- 1. Add the matrices

$$\begin{bmatrix} 4 & -3 & 1 \\ 0 & 5 & -2 \\ 5 & -6 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 2 & 3 \\ 6 & -7 & 9 \\ 0 & -4 & 8 \end{bmatrix}$$

- First, note that each matrix has dimensions of 3X3, so we are able to perform the addition. The result is shown at right:

- Solution: Adding corresponding entries we have

$$\begin{bmatrix} 3 & -1 & 4 \\ 6 & -2 & 7 \\ 5 & -10 & 8 \end{bmatrix}$$

Subtraction of matrices

- Now, we will subtract the same two matrices

$$\begin{bmatrix} 4 & -3 & 1 \\ 0 & 5 & -2 \\ 5 & -6 & 0 \end{bmatrix} - \begin{bmatrix} -1 & 2 & 3 \\ 6 & -7 & 9 \\ 0 & -4 & 8 \end{bmatrix}$$

- Subtract corresponding entries as follows:

$$\begin{bmatrix} 4-(-1) & -3-2 & 1-3 \\ 0-6 & 5-(-7) & -2-9 \\ 5-0 & -6-(-4) & 0-8 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -5 & -2 \\ -6 & 12 & -11 \\ 5 & -2 & -8 \end{bmatrix}$$

Scalar Multiplication

- The **scalar product** of a number *k* and a matrix **A** is the matrix denoted by *k***A,** obtained by multiplying each entry of **A** by the number *k* . The number *k* is called a **scalar. In mathematical notation,**

$$kA = \left[ka_{ij} \right]$$

Example of scalar multiplication

- Find (-1)A where
- A = $\begin{bmatrix} -1 & 2 & 3 \\ 6 & -7 & 9 \\ 0 & -4 & 8 \end{bmatrix}$

- Solution:
- (-1)A=
- -1 $\begin{bmatrix} -1 & 2 & 3 \\ 6 & -7 & 9 \\ 0 & -4 & 8 \end{bmatrix}$

$$= (-1)\begin{bmatrix} -1 & 2 & 3 \\ 6 & -7 & 9 \\ 0 & -4 & 8 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -3 \\ -6 & 7 & -9 \\ 0 & 4 & -8 \end{bmatrix}$$

Alternate definition of subtraction of matrices:

- The definition of subtract of two real numbers a and b is
- a – b = a + (-1)b or a plus the opposite of b. We can define subtraction of matrices similarly:
- If A and B are two matrices of the same dimensions, then
- **A – B** = **A + (-1)B,** where -1 is a scalar.

An example

- The example at right illustrates this procedure for 2 2X2 matrices.
- Solution:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$+(-1)\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} -5 & -6 \\ -7 & -8 \end{bmatrix}$$

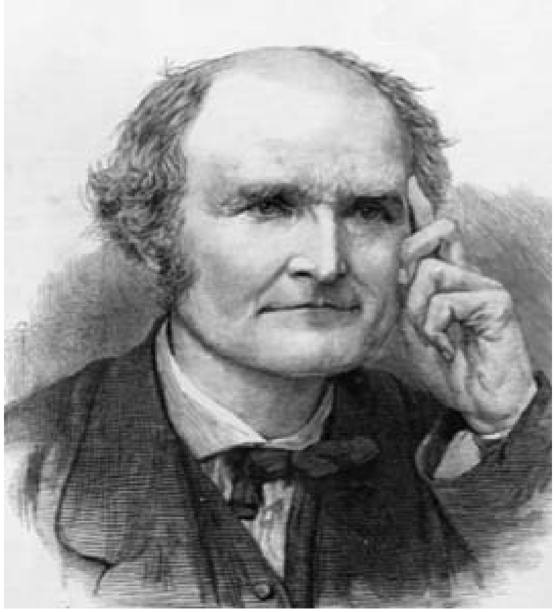
$$\begin{bmatrix} -4 & -4 \\ -4 & -4 \end{bmatrix}$$

Matrix product

- The method of multiplication of matrices is not as intuitive and may seem strange, although this method is extremely useful in many mathematical applications.
- Matrix multiplication was introduced by an English mathematician named Arthur Cayley
- (1821-1895) . We will see shortly how matrix multiplication can be used to solve systems of linear equations.

Arthur Cayley (1821-1895)

- Introduced matrix multiplication



Product of a Row Matrix and a Column Matrix

- In order to understand the general procedure of matrix multiplication, we will introduce the concept of the product of a row matrix by a column matrix. A row matrix consists of a single row of numbers while a column matrix consists of a single column of numbers. If the number of columns of a row matrix equals the number of rows of a column matrix, the product of a row matrix and column matrix is defined. Otherwise, the product is not defined. For example, a row matrix consists of 1 row of 4 numbers so this matrix has four columns. It has dimensions
- 1 X 4. This matrix can be multiplied by a column matrix consisting of 4 numbers in a single column (this matrix has dimensions 4X1.

Row by column multiplication

- 1X4 row matrix multiplied by a 4X1 column matrix: Notice the manner in which corresponding entries of each matrix are multiplied:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ 6 \\ 7 \\ 8 \end{pmatrix} = 1 \cdot 5 + 2 \cdot 6 + 3 \cdot 7 + 4 \cdot 8 = 70$$

Revenue of a car dealer

- A car dealer sells four model types: A,B,C,D. On a given week, this dealer sold 10 cars of model A, 5 of model B, 8 of model C and 3 of model D. The selling prices of each automobile are respectively $12,500, $11,800, $15,900 and $25,300. Represent the data using matrices and use matrix multiplication to find the total revenue.

Solution using matrix multiplication

- We represent the number of each model sold using a row matrix (4X1) and we use a 1X4 column matrix to represent the sales price of each model. When a 4X1 matrix is multiplied by a 1X4 matrix, the result is a 1X1 matrix of a single number.

$$\begin{bmatrix} 10 & 5 & 8 & 3 \end{bmatrix} \begin{bmatrix} 12,500 \\ 11,800 \\ 15,900 \\ 25,300 \end{bmatrix} = \begin{bmatrix} 10(12,500) + 5(11,800) + 8(15,900) + 3(25,300) \end{bmatrix} = \begin{bmatrix} 387,100 \end{bmatrix}$$

Matrix Product

- If **A** is an *m x p* matrix and **B** is a *p x n* matrix, the **matrix product** of *A* and *B* denoted by ***AB*** is an *m x n* matrix whose element in the *ith* row and *jth* column is the real number obtained from the product of the *Ith* row of **A** and the *jth* column of **B.** If the number of columns of **A** does **not equal** the number of rows of **B**, the matrix product **AB** is **not defined.**

Multiplying a 2X4 matrix by a 4X3 matrix to obtain a 4X2

- The following is an illustration of the product of a 2 x 4 matrix with a 4 x 3 . First, the number of columns of the matrix on the left equals the number of rows of the matrix on the right so matrix multiplication is defined. A row by column multiplication is performed three times to obtain the first row of the product:
- 70 80 90.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{pmatrix} = \begin{pmatrix} 70 & 80 & 90 \\ 158 & 184 & 210 \end{pmatrix}$$

Final result

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{pmatrix} = \begin{pmatrix} 70 & 80 & 90 \\ 158 & 184 & 210 \end{pmatrix}$$

Undefined matrix multiplication

Why is this matrix multiplication not defined? The answer is that the left matrix has three columns but the matrix on the right has only two rows. To multiply the second row [4 5 6] by the third column, $\begin{bmatrix} 3 \\ 7 \end{bmatrix}$ there is no number to pair with 6 to multiply.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix} \quad \text{is not defined}$$

More examples:

Given A = $\begin{bmatrix} 3 & 1 & -1 \\ 2 & 0 & 3 \end{bmatrix}$ B= $\begin{bmatrix} 1 & 6 \\ 3 & -5 \\ -2 & 4 \end{bmatrix}$

Find AB if it is defined: $\begin{bmatrix} 3 & 1 & -1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 3 & -5 \\ -2 & 4 \end{bmatrix}$

Solution:

$$\begin{bmatrix} 3 & 1 & -1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 3 & -5 \\ -2 & 4 \end{bmatrix}$$

- Since A is a 2 x 3 matrix and B is a 3 x 2 matrix, **AB** will be a 2 x 2 matrix.
- 1. Multiply first row of A by first column of B : $\longrightarrow$
- 3(1) + 1(3) +(-1)(-2)=8
- 2. First row of A times **second** column of B: $\longrightarrow$
- 3(6)+1(-5)+ (-1)(4)= 9
- 3. Proceeding as above the final result is

$$\begin{bmatrix} 8 & 9 \\ -4 & 24 \end{bmatrix}$$

Is Matrix Multiplication Commutative?

- Now we will attempt to multiply the matrices in reverse order:
- BA =

$$\begin{bmatrix} 1 & 6 \\ 3 & -5 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 3 & 1 & -1 \\ 2 & 0 & 3 \end{bmatrix}$$

- Now we are multiplying a 3 x 2 matrix by a 2 x 3 matrix. This matrix multiplication is defined but the result will be a 3 x 3 matrix. Since AB does not equal BA, matrix multiplication is **not commutative.**
- **BA=**

$$\begin{bmatrix} 15 & 1 & 17 \\ -1 & 3 & -18 \\ 2 & -2 & 14 \end{bmatrix}$$

Practical application

- Suppose you a business owner and sell clothing. The following represents the number of items sold and the cost for each item: Use matrix operations to determine the total revenue over the two days:
- Monday: 3 T-shirts at $10 each, 4 hats at $15 each, and 1 pair of shorts at $20.
  Tuesday: 4 T-shirts at $10 each, 2 hats at $15 each, and 3 pairs of shorts at $20.

Solution of practical application

- Represent the information using two matrices: The product of the two matrices give the total revenue:

Unit price of each item:

Qty sold of each item on Monday

Qty sold of each item on Tuesday

$$\begin{bmatrix} 10 & 15 & 20 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 4 & 2 \\ 1 & 3 \end{bmatrix}$$

- Then your total revenue for the two days is =[110 130]
  Price Quantity=Revenue

4.5 Inverse of a Square Matrix

In this section, we will learn how to find an inverse of a square matrix (if it exists) and learn the definition of the identity matrix.

Identity Matrix for Multiplication:

- 1 is called the multiplication identity for real numbers since a(1) = a
- For example 5(1)=5

If a matrix is a square matrix (has same number of rows and columns), then all such matrices have an identity element for multiplication .

Identity matrices

- **2 x 2 identity matrix:**

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- **3 x 3 identity matrix**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Identity Matrix Multiplication

- **AI = A** (Verify the multiplication)
- We can also show that **IA = A** and in general **AI = IA = A** for all square matrices **A**.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \bullet \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Inverse of a Matrix

- All real numbers (excluding 0) have an inverse.

$$a \cdot \frac{1}{a} = 1$$

- For example

$$5 \cdot \frac{1}{5} = 1$$

Matrix Inverses

- Some (not all) square matrices also have matrix inverses

If the inverse of a matrix A, exists, we shall call it A^{-1}

- Then,

$$A \cdot A^{-1} = A^{-1} \cdot A = I_n$$

Inverse of a 2 x 2 matrix

- There is a simple procedure to find the inverse of a two by two matrix. This procedure **only works for the 2 x 2 case.**

An example will be used to illustrate the procedure:

Inverse of a 2 x 2 matrix

- Find the inverse of

$$\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

- Δ = delta= difference of product of diagonal elements

- Step 1: Determine whether or not the inverse actually exists. We will define

  as

  Δ **(2)2- 1(3);**

  Δ is the difference of the product of the diagonal elements of the matrix.

- **In order for the inverse of a 2 x 2 matrix to exist,** Δ **cannot equal to zero.**
- **If** happens Δ to be zero, then we conclude the inverse does not exist and we stop all calculations.
- In our case Δ = 1, so we can proceed.

Inverse of a two by two matrix

$$\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

- Step 2. Reverse the entries of the main diagonal consisting of the
- two 2's. In this case, no apparent change is noticed.

Step 3. Reverse the signs of the other diagonal entries 3 and 1 so they become -3 and -1

Step 4. Divide each element of the matrix by Δ

which in this case is 1, so no apparent change will be noticed.

Solution

- The inverse of the matrix is then

$$\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

- To verify that this is the inverse, we will multiply the original matrix by its inverse and hopefully obtain the 2 x 2 identity matrix:
- $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} =$

$$\begin{bmatrix} 4-3 & -6+6 \\ 2-2 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

General procedure to find the inverse matrix

- We use a more general procedure to find the inverse of a 3 x 3 matrix.

Problem: Find the inverse of the matrix

$$\begin{bmatrix} 1 & -1 & 3 \\ 2 & 1 & 2 \\ -2 & -2 & 1 \end{bmatrix}$$

Steps to find the inverse of any matrix

- 1. Augment this matrix with the 3 x 3 identity matrix.

1. 2. Use elementary row operations to transform the matrix on the left side of the vertical line to the 3 x 3 identity matrix. The row operation is used for the **entire row** so that the matrix on the right hand side of the vertical line will also change.

2. 3. When the matrix on the left is transformed to the 3 x 3 identity matrix, the matrix **on the right of the vertical line is the inverse.**

Procedure

- Here are the necessary row operations:
- Step 1: Get zeros below the 1 in the first column by multiplying row 1 by -2 and adding the result to R2. Row 2 is replaced by this sum.
- Step2. Multiply R1 by 2, add result to R3 and replace R3 by that result.
- Step 3. Multiply row 2 by (1/3) to get a 1 in the second row first position.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ -2 & -2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[r_3+2r_1=R_3]{r_2-2r_1=R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 0 & 3 & -4 & -2 & 1 & 0 \\ 0 & -4 & 7 & 2 & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{3}r_2=R_2}$$

TO NEXT LINE

Continuation of procedure:

- Step 4. Add R1 to R2 and replace R1 by that sum.
- Step 5. Multiply R2 by 4, add result to R3 and replace R3 by that sum.
- Step 6. Multiply R3 by 3/5 to get a 1 in the third row, third position.

FROM ABOVE LINE

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -\frac{4}{3} & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & -4 & 7 & 2 & 0 & 1 \end{array}\right] \xrightarrow[r_3+4r_2=R_3]{r_1+r_2=R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{5}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & -\frac{4}{3} & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{5}{3} & -\frac{2}{3} & \frac{4}{3} & 1 \end{array}\right] \quad \frac{3}{5}r_3 = R_3$$

TO NEXT LINE

Final result

- Step 7. Eliminate the 5/3 in the first row third position by multiplying row 3 by -5/3 and adding result to Row 1.
- Step 8. Eliminate the -4/3 in the second row, third position by multiplying R3 by 4/3 and adding result to R2.
- Step 9. You now have the identity matrix on the left, which is our goal.

FROM ABOVE LINE

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{5}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & -\frac{4}{3} & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 & -\frac{2}{5} & \frac{4}{5} & \frac{3}{5} \end{array}\right] \xrightarrow[r_2 + \frac{4}{3} r_3 = R_2]{r_1 - \frac{5}{3} r_3 = R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & -\frac{6}{5} & \frac{7}{5} & \frac{4}{5} \\ 0 & 0 & 1 & -\frac{2}{5} & \frac{4}{5} & \frac{3}{5} \end{array}\right]$$

The inverse matrix

- The inverse matrix appears on the right hand side of the vertical line and is displayed below. Many calculators as well as computers have software programs that can calculate the inverse of a matrix quite easily. If you have access to a TI 83, consult the manual to determine how to find the inverse using a calculator.

$$\begin{bmatrix} 1 & -1 & -1 \\ -\frac{6}{5} & \frac{7}{5} & \frac{4}{5} \\ -\frac{2}{5} & \frac{4}{5} & \frac{3}{5} \end{bmatrix}$$

4.6 Matrix Equations and Systems of Linear Equations

In this section, you will study matrix equations and how to use them to solve systems of linear equations as well other applications.

Matrix equations

- Let’s review one property of solving equations involving real numbers. Recall
- If ax = b then x = $\frac{1}{a}b$ or $\frac{b}{a}$
- A similar property of matrices will be used to solve systems of linear equations.
- Many of the basic properties of matrices are similar to the properties of real numbers with the exception that matrix multiplication is not commutative.

Solving a matrix equation

- Given an n x n matrix A and an n x 1 column matrix B and a third matrix denoted by X, we will solve the matrix equation AX = B for X.

$$AX = B$$

$$A^{-1}(AX) = A^{-1}B$$

$$\left(A^{-1}A\right)X = A^{-1}B$$

$$\left(I_n\right)X = A^{-1}B$$

$$X = A^{-1}B$$

- Reasons for each step:
- 1. Given (Note: since A is n x n , X must by n x p , where p is a natural number)
- 2. Multiply on the left by A inverse.
- 3. Associative property of matrices
- 4. Property of matrix inverses.
- 5. Property of the identity matrix
- (I is the n x n identity matrix since X is n x p).
- 6. Solution. Note A inverse is on the left of B. The order cannot be reversed because matrix multiplication is not commutative.

An example:

- Use matrix inverses to solve the system below:

$$
\begin{aligned}
x \quad &+ y \quad + 2z = 1 \\
2x \quad &+ y \qquad\quad\; = 2 \\
x \quad &+ 2y \; + 2z = 3
\end{aligned}
$$

- 1. Determine the matrix of coefficients, A, the matrix X, containing the variables x, y, and z. and the column matrix B, containing the numbers on the right hand side of the equal sign.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 0 \\ 1 & 2 & 2 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Continuation:

$$\begin{aligned} x &+ y &+ 2z &= 1 \\ 2x &+ y & &= 2 \\ x &+ 2y &+ 2z &= 3 \end{aligned}$$

- 2. Form the matrix equation AX=B . Multiply the 3 x 3 matrix A by the 3 x 1 matrix X to verify that this multiplication produces the 3 x 3 system on the left:

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 0 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Problem continued:

- If the matrix A inverse exists, then the solution is determined by multiplying A inverse by the column matrix B. Since A inverse is 3 x 3 and B is 3 x 1, the resulting product will have dimensions 3 x1 and will store the values of x , y and z.

$$X = A^{-1}B$$

- The inverse matrix A can be determined by the methods of a previous section or by using a computer or calculator. The display is shown below:

$$X = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} \\ -1 & 0 & 1 \\ \frac{3}{4} & \frac{-1}{4} & \frac{-1}{4} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Solution

When the product of A inverse and matrix B is found the result is as follows:

$$X = A^{-1}B$$

$$X = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} \\ -1 & 0 & 1 \\ \frac{3}{4} & \frac{-1}{4} & \frac{-1}{4} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$X = \begin{bmatrix} 0 \\ 2 \\ \frac{-1}{2} \end{bmatrix}$$

- The solution can be interpreted from the X matrix: x = 0, y = 2 and z = -1/2 . Written as an ordered triple of numbers, the solution is
- (0 , 2 , -1/2)

Another example: Using matrix techniques to solve a linear system

- Solve the system below using the inverse of a matrix

$$x + 2y + z = 1$$
$$2x - y + 2z = 2$$
$$3x + y + 3z = 4$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 2 \\ 3 & 1 & 3 \end{bmatrix}$$

- The coefficient matrix A is displayed to the left:: The inverse of A does not exist. We cannot use the technique of multiplying A inverse by matrix B to find the variables x, y and z. Whenever, the inverse of a matrix does not exist, we say that the matrix is singular.
- There are two cases were inverse methods will not work:
- **1. if the coefficient matrix is singular**
- **2. If the number of variables is not the same as the number of equations.**

Application

- Production scheduling: Labor and material costs for manufacturing two guitar models are given in the table below: Suppose that in a given week $1800 is used for labor and $1200 used for materials. How many of each model should be produced to use exactly each of these allocations?

Guitar model	Labor cost	Material cost
A	$30	$20
B	$40	$30

Solution

- Let A be the number of model A guitars to produce and B represent the number of model B guitars. Then, multiplying the labor costs for each guitar by the number of guitars produced, we have
- 30x + 40y = 1800
- Since the material costs are \$20 and \$30 for models A and B respectively, we have 20A + 30B = 1200.
- This gives us the system of linear equations:
- 30A+ 40B = 1800
- 20A+30B=1200
- We can write this as a matrix equation:

$$\begin{bmatrix} 30 & 40 \\ 20 & 30 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1800 \\ 1200 \end{bmatrix}$$

solution

- Using the result

$$X = A^{-1}B$$

$$A = \begin{bmatrix} 30 & 40 \\ 20 & 30 \end{bmatrix}$$

- The inverse of matrix A is

$$\begin{bmatrix} 0.3 & -0.4 \\ -0.2 & 0.3 \end{bmatrix}$$

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0.3 & -0.4 \\ -0.2 & 0.3 \end{bmatrix} \begin{bmatrix} 1800 \\ 1200 \end{bmatrix} = \begin{bmatrix} 60 \\ 0 \end{bmatrix}$$

- Produce 60 model A guitars and no model B guitars.

4.7 Leontief Input-Output Analysis

- In this section, we will study an important economic application of matrix inverses and matrix multiplication.
- This branch of applied mathematics is called **input-output analysis** and was first proposed by Wassily Leontief, who won the Nobel Prize in economics in 1973 for his work in this area.

Wassily Leontief http://www.iioa.org/leontief/Life.html

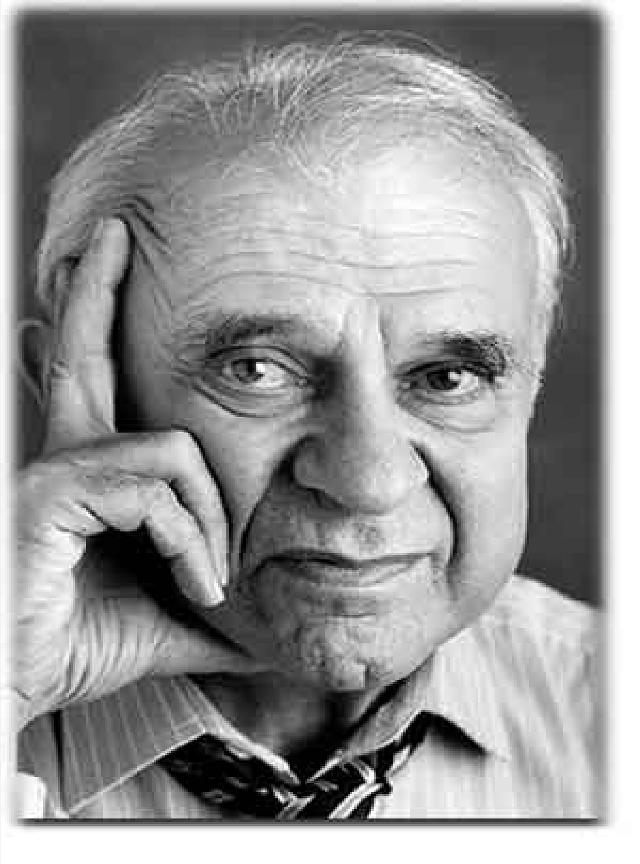
- Born: 1906
  Place of Birth: St. Petersburg, Russia
  Residence: U.S.A.
  Affiliation: Harvard University, Cambridge, Wassily Leontief was born August 5th, 1905 in St. Petersburg, the son of Wassily W. Leontief and his wife Eugenia. A brilliant student, he enrolled in the newly renamed University of Leningrad at only 15 years old.
- He got in trouble by expressing vehement opposition to the lack of intellectual and personal freedom under the country's Communist regime, which had taken power three years earlier. He was arrested several times.

  At Harvard, he developed his theories and methods of Input-Output analysis. This work earned him the Nobel prize in Economics in 1973 for his analysis of America's production machinery. His analytic methods, as the Nobel committee observed, became a permanent part of production planning and forecasting in scores of industrialized nations and in private corporations all over the world.

Wassily Leontief

http://www.iioa.org/leontief/Life.html

- Wassily Leontief in 1983. Photo taken by Gregory Edwards For more information on Professor Leontief, click on the link at the top of this slide.



Two industry model

- We start with an economy that has only two industries : agriculture and energy to illustrate the method and then this method will generalized to three or more industries. These two industries depend upon each other . For example, each dollar’s worth of agriculture produced requires 0.40 dollars of agriculture and 0.20 dollars of energy. Each dollar’s worth of energy produced requires
- 0.20 of agriculture and 0.10 of energy. So, both industries have an internal demand for each others resources. Let us suppose there is an external demand of 12 million dollars of agriculture and 9 million dollars of energy.

Matrix equations

- Let x represent the total output from agriculture and y represent the total output of energy.
- The equations
- $x = 0.4x+0.2y$
- $y = 0.2x + 0.1y$
- can be used to represent the internal demand for agriculture and energy.
- The external demand of 12 and 9 million must also be met so the revised equations are :
- $x = 0.4x+0.2 y + 12$
- $y = 0.2x + 0.1y + 9$
- These equations can be represented by the following matrix equation:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.4 & 0.2 \\ 0.2 & 0.1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 12 \\ 9 \end{bmatrix}$$

Technology matrix (M)

$$\begin{bmatrix} 0.4 & 0.2 \\ 0.2 & 0.1 \end{bmatrix} =$$

A

$\uparrow$

Read left to right, then up

$$\begin{array}{r} A \rightarrow \\[2em] E \rightarrow \end{array} \begin{bmatrix} \left(\begin{array}{l} \textit{input from} \\ \textit{agriculture} \\ \textit{to produce} \\ \$1 \textit{ of agriculture} \end{array}\right) & \left(\begin{array}{l} \textit{input from} \\ \textit{agriculture} \\ \textit{to produce } \$1 \\ \textit{of energy} \end{array}\right) \\ \left(\begin{array}{l} \textit{input of energy} \\ \textit{to produce } \$1 \\ \textit{of agriculture} \end{array}\right) & \left(\begin{array}{l} \textit{input of energy} \\ \textit{to produce } \$1 \textit{ of} \\ \textit{energy} \end{array}\right) \end{bmatrix} = \mathbf{M}$$

Matrix equations

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.4 & 0.2 \\ 0.2 & 0.1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 12 \\ 9 \end{bmatrix}$$

We can represent these matrices symbolically as follows: X = MX+D $\longrightarrow$

X – MX = D $\longrightarrow$

IX – MX = D $\longrightarrow$

(I – M)X = D $\longrightarrow$

$$X = (I - M)^{-1} D$$

if the inverse of (I – M) exists.

Solution

- We will now find

$$X = (I - M)^{-1} D$$

1.First, find (I – M):

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.4 & 0.2 \\ 0.2 & 0.1 \end{bmatrix} = \begin{bmatrix} 0.6 & -0.2 \\ -0.2 & 0.9 \end{bmatrix}$$

The inverse of (I – M) is:

$$\begin{bmatrix} 1.8 & .4 \\ .4 & 1.2 \end{bmatrix}$$

Solution:

- After finding the inverse of (I – M), multiply that result by the external demand matrix D . The answer is to produce a total of 25.2 million dollars of agriculture and 15.6 million dollars of energy to meet both the internal demands of each resource and the external demand.

$$\begin{bmatrix} 1.8 & .4 \\ .4 & 1.2 \end{bmatrix} \begin{bmatrix} 12 \\ 9 \end{bmatrix} = \begin{bmatrix} 25.2 \\ 15.6 \end{bmatrix}$$

Advantages of method

- Suppose consumer demand changes from $12 million dollars of agriculture to $8 million dollars and energy consumption changes from $9 million to $5 million. Find the output for each sector that is needed to satisfy this final demand.

Solution:

- Recall that our general solution of the problem is

$$X = (I - M)^{-1} D$$

The only change in the problem is the external demand matrix. (I – M) did not change. Therefore, our solution is to multiply the inverse of (I –M) by the new external demand matrix, D.

Solution

$$X = (I - M)^{-1} D$$

$$\begin{bmatrix} 1.8 & .4 \\ .4 & 1.2 \end{bmatrix} \begin{bmatrix} 8 \\ 5 \end{bmatrix} = \begin{bmatrix} 16.4 \\ 9.2 \end{bmatrix}$$

More than two sectors of the economy

- This method can also be used if there are more than two sectors of the economy. If there are three sectors, say agriculture, building and energy, the technology matrix M will be a 3 x 3 matrix. The solution to the problem will still be $X = (I - M)^{-1} D$
- although, in this case, it is necessary to determine the inverse of a 3 x 3 matrix.