T.4 Applications of Right Angle Trigonometry

Solving Right Triangles

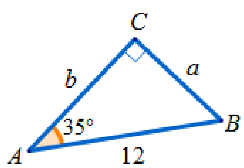
Geometry of right triangles has many applications in the real world. It is often used by carpenters, surveyors, engineers, navigators, scientists, astronomers, etc. Since many application problems can be modelled by a right triangle and trigonometric ratios allow us to find different parts of a right triangle, it is essential that we learn how to apply trigonometry to solve such triangles first.

Definition 4.1 ► To **solve a triangle** means to find the measures of all the unknown **sides** and **angles** of the triangle.

Example 1 ► **Solving a Right Triangle Given an Angle and a Side**

Given the information, solve triangle ABC, assuming that $\angle C = 90°$.

a.



b. $\angle B = 11.4°,\ b = 6\ cm$

Solution ► **a.** To find the length a, we want to relate it to the given length of 12 and the angle of 35°. Since a is opposite angle 35° and 12 is the length of the hypotenuse, we can use the ratio of sine:

$$\frac{a}{12} = \sin 35°$$

Then, after multiplying by 12, we have

$$a = 12 \sin 35° \simeq \mathbf{6.9}$$

round **lengths** to **one decimal place**

Attentin: To be more accurate, if possible, **use the given data** rather than the previously calculated ones, which are most likely already rounded.

Since we already have the value of a, the length b can be determined in two ways: by applying the Pythagorean Theorem, or by using the cosine ratio. For better accuracy, we will apply the cosine ratio:

$$\frac{b}{12} = \cos 35°$$

which gives

$$b = 12 \cos 35° \simeq \mathbf{9.8}$$

Finally, since the two acute angles are complementary, $\angle B = 90° - 35° = \mathbf{55}°$.

We have found the three missing measurements, $a \simeq 6.9$, $b \simeq 9.8$, and $\angle B = 55°$, so the triangle is solved.

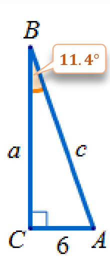


Figure 1

b. To visualize the situation, let's sketch a right triangle with $\angle B = 11.4°$ and $b = 6$ (see *Figure 1*). To find side a, we we would like to set up an equation that relates 6, a, and

$11.4°$. Since $b = 6$ is the opposite and a is the adjacent with respect to $\angle B = 11.4°$, we will use the ratio of tangent:

$$\tan 11.4° = \frac{6}{a}$$

To solve for a, we may want to multiply both sides of the equation by a and divide by $\tan 11.4°$. Observe that this will cause a and $\tan 11.4°$ to interchange (swap) their positions. So, we obtain

$$a = \frac{6}{\tan 11.4°} \simeq \mathbf{29.8}$$

To find side c, we will set up an equation that relates 6, c, and $11.4°$. Since $b = 6$ is the opposite to $\angle B = 11.4°$ and c is the hypothenuse, the ratio of sine applies. So, we have

$$\sin 11.4° = \frac{6}{c}$$

Similarly as before, to solve for c, we can simply interchange the position of $\sin 11.4°$ and c to obtain

$$c = \frac{6}{\sin 11.4°} \simeq \mathbf{30.4}$$

Finally, $\angle A = 90° - 11.4° = \mathbf{78.6}°$, which completes the solution.

In summary, $\angle A = 78.6°$, $a \simeq 29.8$, and $c \simeq 30.4$.

Observation: Notice that after approximated length a was found, we could have used the Pythagorean Theorem to find length c. However, this could decrease the accuracy of the result. For this reason, it is advised that we use the given rather than approximated data, if possible.

Finding an Angle Given a Trigonometric Function Value

So far we have been evaluating trigonometric functions for a given angle. Now, what if we wish to reverse this process and try to recover an angle that corresponds to a given trigonometric function value?

Example 2 ► **Finding an Angle Given a Trigonometric Function Value**

Find an angle θ, satisfying the given equation. *Round to one decimal place, if needed.*

a. $\sin\theta = 0.7508$ **b.** $\cos\theta = -0.5$

Solution ► **a.** Since 0.7508 is not a special value, we will not be able to find θ by relating the equation to a special triangle as we did in *section T3, example 8*. This time, we will need to rely on a calculator. To find θ, we want to "undo" the sine. The function that can "undo" the sine is called **arcsine**, or **inverse sine**, and it is often abbreviated by $\mathbf{sin^{-1}}$. By applying the $\sin^{-1}$ to both sides of the equation

$$\sin\theta = 0.7508,$$

we have

$$\cancel{\sin}^{-1}(\cancel{\sin}\,\theta) = \sin^{-1}(0.7508)$$

Since $\sin^{-1}$ "undoes" the sine function, we obtain

round **angles** to **one decimal place**

$$\theta = \sin^{-1} 0.7508 \simeq \mathbf{48.7°}$$

On most calculators, to find this value, we follow the sequence of keys:

2nd or INV or Shift , SIN , 0.7508, ENTER or =

b. In this example, the absolute value of cosine is a special value. This means that θ can be found by referring to the **golden triangle** properties and the **CAST** rule of signs as in *section T3, example 8b*. The other way of finding θ is via a calculator

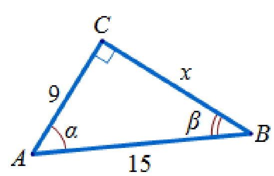
$$\theta = \cos^{-1}(-0.5) = \mathbf{120°}$$

Note: *Calculators are programed to return*
$\mathbf{sin^{-1}}$ *and* $\mathbf{tan^{-1}}$ *as angles from the interval* $[\mathbf{-90°}, \mathbf{90°}]$ *and*
$\mathbf{cos^{-1}}$ *as angles from the interval* $[\mathbf{0°}, \mathbf{180°}]$.
That implies that when looking for an obtuse angle, it is easier to work with $\mathbf{cos^{-1}}$*, if possible, as our calculator will return the actual angle. When using* $\mathbf{sin^{-1}}$ *or* $\mathbf{tan^{-1}}$*, we might need to search for a corresponding angle in the second quadrant on our own.*

More on Solving Right Triangles

Example 3 ► **Solving a Right Triangle Given Two Sides**

Solve the triangle.



Solution ► Since $\triangle ABC$ is a right triangle, to find the length x, we can use the Pythagorean Theorem.

$$x^2 + 9^2 = 15^2$$

so

$$\boldsymbol{x} = \sqrt{225 - 81} = \sqrt{144} = \mathbf{12}$$

To find the angle α, we can relate either $x = 12$, 9, and α, or 12, 15, and α. We will use the second triple and the ratio of sine. Thus, we have

$$\sin\alpha = \frac{12}{15},$$

therefore

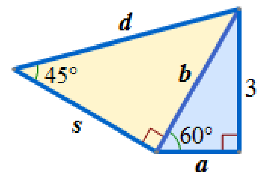
$$\boldsymbol{\alpha} = \sin^{-1}\frac{12}{15} \simeq \mathbf{53.1°}$$

Finally, $\boldsymbol{\beta} = 90° - \alpha \simeq 90° - 53.1° = \mathbf{36.9°}$.

In summary, $\alpha = 53.1°$, $\beta \simeq 36.9°$, and $x = 12$.

Example 4 ► **Using Relationships Between Sides of Special Triangles**

Find the **exact** value of each unknown in the figure.



Solution ► First, consider the blue right triangle. Since one of the acute angles is 60°, the other must be 30°. Thus the blue triangle represents half of an equilateral triangle with the side $\boldsymbol{b}$ and the height of 3 units. Using the relation $h = a\sqrt{3}$ between the height h and half a side a of an equilateral triangle, we obtain

$$\boldsymbol{a}\sqrt{3} = 3,$$

which gives us $\boldsymbol{a} = \frac{3}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\cancel{3}\sqrt{3}}{\cancel{3}} = \boldsymbol{\sqrt{3}}$. Consequently, $\boldsymbol{b} = 2\boldsymbol{a} = \mathbf{2}\boldsymbol{\sqrt{3}}$.

Now, considering the yellow right triangle, we observe that both acute angles are equal to 45° and therefore the triangle represents half of a square with the side $\boldsymbol{s} = \boldsymbol{b} = \mathbf{2}\boldsymbol{\sqrt{3}}$.

Finally, using the relation between the diagonal and a side of a square, we have

$$\boldsymbol{d} = \boldsymbol{s}\sqrt{2} = 2\sqrt{3}\sqrt{2} = \mathbf{2}\boldsymbol{\sqrt{6}}.$$

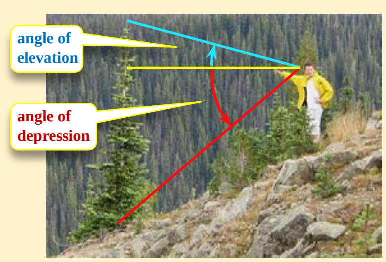
Angles of Elevation or Depression in Applications

The method of solving right triangles is widely adopted in solving many applied problems. One of the critical steps in the solution process is sketching a triangle that models the situation, and labeling the parts of this triangle correctly.

In trigonometry, many applied problems refer to angles of **elevation** or **depression**, or include some navigation terminology, such as **direction** or **bearing**.

Definition 4.2 ► **Angle of elevation** (or **inclination**) is the acute angle formed by a **horizontal** line and the line of sight to an object **above** the horizontal line.

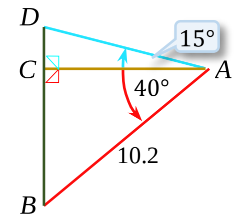
Angle of depression (or **declination**) is the acute angle formed by a **horizontal** line and the line of sight to an object **below** the horizontal line.



Example 5 ► **Applying Angles of Elevation or Depression**

Find the height of the tree in the picture given next to *Definition 4.2*, assuming that the observer sees the top of the tree at an angle of elevation of 15°, the base of the tree at an angle of depression of 40°, and the distance from the base of the tree to the observer's eyes is 10.2 meters.

Solution ► First, let's draw a diagram to model the situation, label the vertices, and place the given data. Then, observe that the height of the tree BD can be obtained as the sum of distances BC and CD.



BC can be found from $\triangle ABC$, by using the ratio of sine of 40°.
From the equation

$$\frac{BC}{10.2} = \sin 40°,$$

we have

$$BC = 10.2 \sin 40° \simeq \mathbf{6.56}$$

To calculate the length DC, we would need to have another piece of information about $\triangle ADC$ first. Notice that the side AC is common for the two triangles. This means that we can find it from $\triangle ABC$, and use it for $\triangle ADC$ in subsequent calculations.
From the equation

$$\frac{CA}{10.2} = \cos 40°,$$

we have

$$CA = 10.2 \cos 40° \simeq 7.8137$$

since we use this result in further calculations, four decimals of accuracy is advised

Now, employing tangent of 15° in $\triangle ADC$, we have

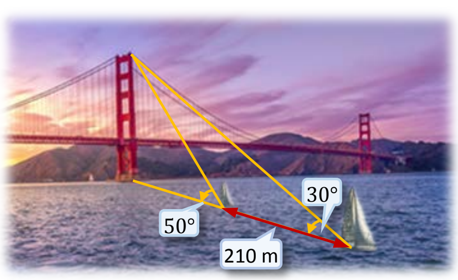
$$\frac{CD}{7.8137} = \tan 15°$$

which gives us

$$CD = 7.8137 \cdot \tan 15° \simeq \mathbf{2.09}$$

Hence the height of the tree is $BC \simeq 6.56 + 2.09 = 8.65 \simeq \mathbf{8.7}$ meters.

Example 6 ► **Using Two Angles of Elevation at a Given Distance to Determine the Height**



The Golden Gate Bridge has two main towers of equal height that support the two main cables. A visitor on a sailboat passing through San Francisco Bay views the top of one of the towers and estimates the angle of elevation to be 30°. After sailing 210 meters closer, the visitor estimates the angle of elevation to the same tower to be 50°. Approximate the height of the tower to the nearest meter.

Solution ► Let's draw the diagram to model the situation and adopt the notation as in *Figure 2*. We look for height $\boldsymbol{h}$, which is a part of the two right triangles ΔABC and ΔDBC.

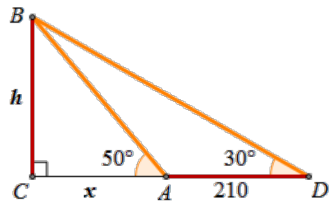


Figure 2

Since trigonometric ratios involve two sides of a triangle, and we already have length AD, a part of the side CD, it is reasonable to introduce another unknown, call it $\boldsymbol{x}$, to represent the remaining part CA. Then, applying the ratio of tangent to each of the right triangles, we produce the following system of equations:

$$\begin{cases} \frac{h}{x} = \tan 50° \\ \frac{h}{x+210} = \tan 30° \end{cases}$$

To solve the above system, we first solve each equation for h

$$\begin{cases} h \simeq 1.1918x \\ h \simeq 0.5774(x+210) \end{cases},$$

and then by equating the right sides, we obtain

substitute to the top equation

$$1.1918x = 0.5774(x+210)$$

$$1.1918x - 0.5774x = 121.254$$

$$0.6144x = 121.254$$

$$x = \frac{121.254}{0.6144} \simeq 197.4$$

Therefore, $\boldsymbol{h} \simeq 1.1918 \cdot 197.4 \simeq \mathbf{235}\ \boldsymbol{m}$.

The height of the tower is approximately 235 meters.

Direction or Bearing in Applications

A large group of applied problems in trigonometry refer to **direction** or **bearing** to describe the location of an object, usually a plane or a ship. The idea comes from following the behaviour of a compass. The magnetic needle in a compass points North. Therefore, the location of an object is described as a clockwise deviation from the SOUTH-NORTH line.

There are two main ways of describing directions:

- One way is by stating the angle θ that starts from the North and opens clockwise until the line of sight of an object. For example, we can say that the point $\boldsymbol{B}$ is seen in the **direction** of 108° from the point $\boldsymbol{A}$, as in *Figure 2a*.
- Another way is by stating the acute angle formed by the South-North line and the line of sight. Such an angle starts either from the North (**N**) or the South (**S**) and opens either towards the East (**E**) or the West (**W**). For instance, the position of the point $\boldsymbol{B}$ in *Figure 2b* would be described as being at a **bearing** of **S**72°**E** (*read:* South 72° towards the East) from the point $\boldsymbol{A}$.

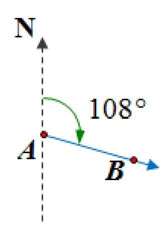


Figure 2a

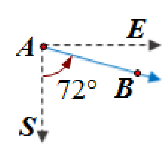


Figure 2b

This, for example, means that:

the direction of 195° can be seen as the bearing **S**15°**W**
and the direction of 290° means the same as **N**70°**W**.

Example 7 ► **Using Direction in Applications Involving Navigation**

An airplane flying at a speed of 400 mi/hr flies from a point A in the direction of 153° for one hour and then flies in the direction of 63° for another hour.

a. How long will it take the plane to get back to the point A?

b. What is the direction that the plane needs to fly in order to get back to the point A?

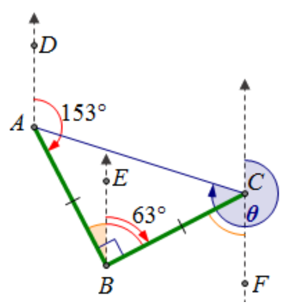


Figure 3

a. First, let's draw a diagram modeling the situation. Assume the notation as in *Figure 3*. Since the plane flies at 153° and the South-North lines $\overleftrightarrow{AD}$ and $\overleftrightarrow{BE}$ are parallel, by the property of interior angles, we have $\angle ABD = 180° - 153° = 27°$. This in turn gives us $\angle ABC = \angle ABE + \angle EBC = 27° + 63° = 90°$. So the $\triangle ABC$ is right angled with $\angle B = 90°$ and the two legs of length $AB = BC = 400\ mi$. This means that the $\triangle ABC$ is in fact a special triangle of the type $45° - 45° - 90°$.

Therefore $AC = AB\sqrt{2} = 400\sqrt{2} \simeq 565.7\ mi$.

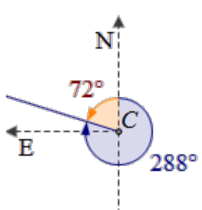
Now, solving the well-known motion formula $R \cdot T = D$ for the time T, we have

$$T = \frac{D}{R} \simeq \frac{\cancel{400}\sqrt{2}}{\cancel{400}} = \sqrt{2} \simeq 1.4142\ hr \simeq \mathbf{1\ hr\ 25\ min}$$

Thus, it will take the plane approximately 1 hour and 25 minutes to return to the starting point A.

b. To direct the plane back to the starting point, we need to find angle θ, marked in blue, rotating clockwise from the North to the ray $\overrightarrow{CA}$. By the property of alternating angles, we know that $\angle FCB = 63°$. We also know that $\angle BCA = 45°$, as $\angle ABD$ is the "half of a square" special triangle. Therefore,

$$\theta = 180° + 63° + 45° = \mathbf{288°}.$$



Thus, to get back to the point A, the plane should fly in the direction of 288°. Notice that this direction can also be stated as **N**72°**W**.

T.4 Exercises

Vocabulary Check *Complete each blank with the most appropriate term or number from the given list:* ***depression, East, elevation, inverse, ratio, right, sides, sight, solve, South, three, 25.***

1. To _________ a triangle means to find the measure of all _______ angles and all three ________.

2. The value of a trigonometric function of an acute angle of a _____________ triangle represents the _____________ of lengths of appropriate sides of this triangle.

3. The value of an _____________ trigonometric function of a given number represents the angle.

4. The acute angle formed by a line of sight that falls *below* a horizontal line is called an angle of ________________. The acute angle formed by a line of __________ that rises *above* a horizontal line is called an angle of ________________.

5. The bearing **S**$25°$**E** indicates a line of sight that forms an angle of ____° to the __________ of a line heading ___________.

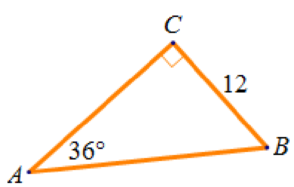
Concept check *Using a calculator, find an angle θ satisfying the given equation. Leave your answer in decimal degrees rounded to the nearest tenth of a degree if needed.*

6. $\sin\theta = 0.7906$ **7.** $\cos\theta = 0.7906$ **8.** $\tan\theta = 2.5302$

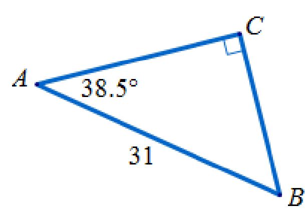
9. $\cos\theta = -0.75$ **10.** $\tan\theta = \sqrt{3}$ **11.** $\sin\theta = \frac{3}{4}$

Concept check *Given the data, solve each triangle ABC with $\angle C = 90°$.*

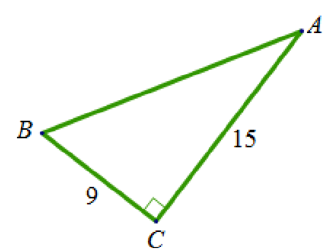
12.



13.



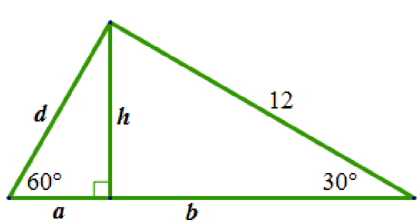
14.



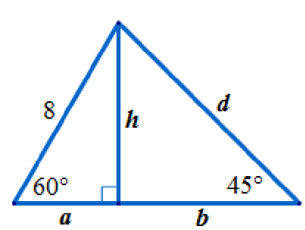
15. $\angle A = 42°,\ b = 17$ **16.** $a = 9.45,\ c = 9.81$ **17.** $\angle B = 63°12',\ b = 19.1$

*Find the **exact** value of each unknown in the figure.*

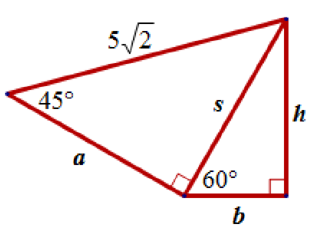
18.



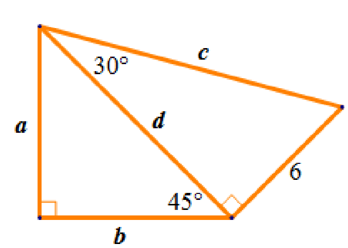
19.



20.



21.



22. A circle of radius 6 inches is inscribed in a regular hexagon. Find the exact length of one of its sides.

23. Find the perimeter of a regular hexagon that is inscribed in a circle of radius 8 meters.

24. A guy wire 77.4 meters long is attached to the top of an antenna mast that is 71.3 meters high. Find the angle that the wire makes with the ground.

25. A 100-foot guy wire is attached to the top of an antenna. The angle between the guy wire and the ground is 62°. How tall is the antenna to the nearest foot?

26. From the top of a lighthouse 52 m high, the angle of depression to a boat is $4°15'$. How far is the boat from the base of the lighthouse?

27. A security camera in a bank is mounted on a wall 9 feet above the floor. What angle of depression should be used if the camera is to be directed to a spot 6 feet above the floor and 12 feet from the wall?

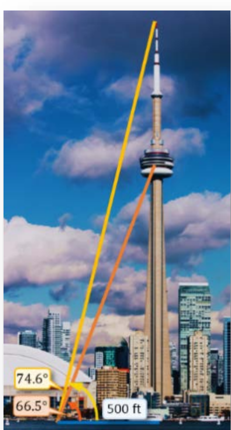
28. For a person standing 100 meters from the center of the base of the Eiffel Tower, the angle of elevation to the top of the tower is 71.6°. How tall is the Eiffel Tower?

29. Find the altitude of an isosceles triangle having a base of 184.2 cm if the angle opposite the base is $68°44'$.

Analytic Skills

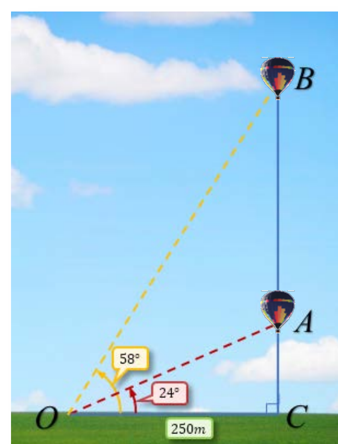
30. From city A to city B, a plane flies 650 miles at a bearing of **N**48°**E**. Then the plane flies 810 miles from city B to city C at a bearing of **S**42°**E**. Find the distance AC and the bearing directly from A to C.

31. A plane flies at 360 km/h for 30 minutes in the direction of 137°. Then, it changes its direction to 227° and flies for 45 minutes. How far and in what direction is the plane at that time from the starting point?

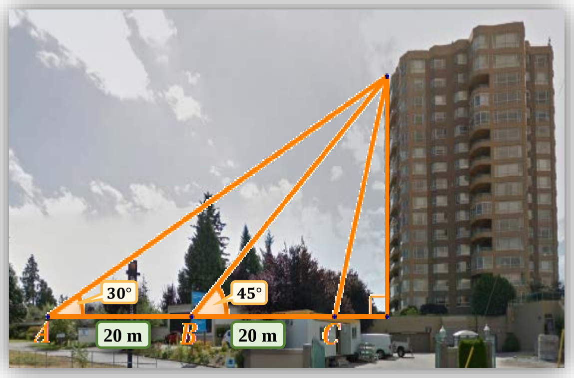


32. The tallest free-standing tower in the world is the CNN Tower in Toronto, Canada. The tower includes a rotating restaurant high above the ground. From a distance of 500 ft the angle of elevation to the pinnacle of the tower is 74.6°. The angle of elevation to the restaurant from the same vantage point is 66.5°. How tall is the CNN Tower including its pinnacle? How far below the tower is the restaurant located?

33. A hot air balloon is rising upward from the earth at a constant rate, as shown in the accompanying figure. An observer 250 meters away spots the balloon at an angle of elevation of 24°. Two minutes later, the angle of elevation of the balloon is 58°. At what rate is the balloon ascending? Answer to the nearest tenth of a meter per second.



34. A hot air balloon is between two spotters who are 1.2 mi apart. One spotter reports that the angle of elevation of the balloon is 76°, and the other reports that it is 68°. What is the altitude of the balloon in miles?



35. From point A the angle of elevation to the top of the building is 30°, as shown in the accompanying figure. From point B, 20 meters closer to the building, the angle of elevation is 45°. Find the angle of elevation of the building from point C, which is another 20 meters closer to the building.

36. For years the Woolworth skyscraper in New York held the record for the world's tallest office building. If the length of the shadow of the Woolworth building increases by 17.4 m as the angle of elevation of the sun changes from 44° to 42°, then how tall is the building?

37. A policeman has positioned himself 150 meters from the intersection of two roads. He has carefully measured the angles of the lines of sight to points A and B as shown in the drawing. If a car passes from A to B in 1.75 sec and the speed limit is 90 km/h, is the car speeding?