Mikala Ørsnes Jansen

The stratified homotopy type of the reductive Borel–Serre compactification and applications to algebraic K-theory

PhD Thesis

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Mikala Ørsnes Jansen
Department of Mathematical Sciences
University of Copenhagen
Universitetsparken 5
DK-2100 København Ø
Denmark

mikala@math.ku.dk

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Adviser: Søren Galatius
University of Copenhagen, Denmark

Assessment committee: Lars Hesselholt (chairman)
University of Copenhagen, Denmark

Dan Petersen
Stockholm University, Sweden

Leslie Saper
Duke University, USA

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Abstract

In this thesis, we study the stratified homotopy type of the reductive Borel–Serre compactification, and we investigate how the stratified homotopy type can be used to model unstable algebraic K-theory. The thesis consists of an introduction and two papers, the second of which is joint with Dustin Clausen.

In the first paper, Paper I, we determine the exit path ∞-category of the reductive Borel–Serre compactification of a locally symmetric space associated with a neat arithmetic group. We show that the exit path ∞-category is equivalent to the nerve of a 1-category defined in purely algebraic terms. We derive some immediate corollaries of our result: the homotopy type and, in particular, the fundamental group of the reductive Borel–Serre compactification is determined, and we obtain an identification of the constructible derived category as a derived functor category. To make this identification, we develop several calculational tools applicable to a larger class of stratified spaces.

In the second paper, Paper II, we generalise the exit path 1-category of the reductive Borel–Serre compactification to general linear groups over arbitrary rings, and we show that for a certain class of rings, these categories provide models for unstable algebraic K-theory. For finite fields, the model is in a certain sense better than that given by the plus-construction. We reprove the main result of Paper I using entirely different techniques, and crucially use this proof strategy when dealing with the generalisations. We also define a further generalisation of the exit path 1-category, associating a strict monoidal category to any exact category. We show that these monoidal categories define models for the algebraic K-theory space of exact categories.

Resumé

I denne afhandling studererer vi den stratificerede homotopitype af den reduktive Borel–Serre-kompaktificering, og vi undersøger hvordan den stratificerede homotopitype kan bruges til at modellere ustabil algebraisk K-teori. Afhandlingen består af en introduktion og to artikler, hvoraf den sidste er et samarbejde med Dustin Clausen.

I den første artikel, Paper I, bestemmer vi udgangssti-∞-kategorien af den reduktive Borel–Serre-kompaktificering af et lokalt-symmetrisk rum associeret til en net aritmetisk gruppe. Den er ækvivalent med nerven af en 1-kategori, der kan defineres helt algebraisk. Dette har en række umiddelbare korollarer: vi bestemmer homotopitypen og specielt fundamentalgruppen af den reduktive Borel–Serre-kompaktificering, og vi identificerer den konstruerbare deriverede kategori som en deriveret funktorkategori. Undervejs udvikler vi nogle beregningsværktøjer, som bør kunne bruges på andre interessante stratificerede rum.

I den anden artikel, Paper II, generaliserer vi udgangsti-1-kategorien for den reduktive Borel–Serre-kompaktificering til generelle lineære grupper over associative ringe. Vi viser at for en bestemt klasse af ringe giver disse kategorier modeller for ustabil algebraisk K-teori. For endelige legemer er modellen i en vis forstand bedre end den sædvanlige plus-konstruktion. Vi giver et nyt bevis for hovedresultatet i Paper I og udnytter denne bevisstrategi i arbejdet med generaliseringerne. Vi generaliserer udgangssti-1-kategorien yderligere og associerer dermed en streng monoidal kategori til enhver eksakt kategori. Vi viser at disse monoidale kategorier definerer modeller for det algebraiske K-teorirum for eksakte kategorier.

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Contents

Part I

Introduction

In the first part of this thesis, we describe the approach, ideas and main results of the two papers contained in Part II, focusing on the storyline and on motivating the results. Before turning our attention to the results of the thesis, however, we review the historical and technical background of the work. Finally, we address some ideas for future research.

1. Algebraic K-theory, the reductive Borel–Serre compactification and stratified homotopy theory

The project resulting in the work contained in this thesis was initialised by the following question, put to the author by Dustin Clausen:

What is the stratified homotopy type of the reductive Borel–Serre compactification?

The three topics combined in this work are algebraic K-theory, the reductive Borel–Serre compactification, and stratified homotopy theory. To put it roughly, algebraic K-theory is the motivation, the reductive Borel–Serre compactification is the object of study, and stratified homotopy theory is our toolbox. In order to put our work into context we provide some historical background and perspective on these three topics. We especially want to emphasise the mathematical significance of algebraic K-theory and of the reductive Borel–Serre compactification. To do this, we include a varied selection of applications and references. The author hopes that the scope will serve to illustrate the expanse of these subjects and that the different applications will appeal to different readers.

1.1. The motivation: algebraic K-theory. Algebraic K-theory is a vast subject touching upon many different fields of mathematics, including number theory, algebraic topology, algebraic geometry and homological algebra. The story of algebraic K-theory is long and technical and its applications plentiful — we do our very best to present here a brief historical summary hinting at just some of the many applications within different branches of mathematics. There is a certain lack of linearity in the storyline as we branch off to touch upon different applications as we move forward, but the author hopes that this reflects the historical development and the complexity of the subject.

The beginnings.

In 1957, Grothendieck introduced a group $K(X)$ associated to an algebraic variety X as a crucial ingredient in his formulation of the Riemann–Roch Theorem. More generally, the *Grothendieck group* associated with a commutative monoid M is the universal way to turn the monoid into an abelian group. Grothendieck applied this construction to the monoid of isomorphism classes of objects in an abelian category, particularly the abelian category $M(X)$ of coherent sheaves on X and the subcategory $P(X)$ of locally free sheaves ([Gro71]). The group $K(X)$ is the Grothendieck group associated to $P(X)$. This marked the beginnings of K-theory, and $K(X)$ is now known as $K_0(X)$.

For a ring R, the K-group $K_0(R)$ is the Grothendieck group of isomorphism classes of finitely generated projective R-modules. If R is a Dedekind domain, then $K_0(R) \cong \mathrm{Pic}(R) \oplus \mathbb{Z}$, where $\mathrm{Pic}(R)$ is the Picard group of R. An important application of K_0 in topology was discovered by Wall: if X is a CW complex dominated by a finite complex, then X is itself finite if and only if a certain element $\chi(X)$ in the reduced group $\tilde{K}_0(\mathbb{Z}[\pi_1(X)])$ vanishes ([Wal65]). The element $\chi(X)$ generalises the Euler characteristic and is known as *Wall's finiteness obstruction.*

Inspired by Grothendieck's definition, Atiyah and Hirzebruch defined *topological K-theory* by considering topological vector bundles over compact Hausdorff spaces. They defined higher topological K-groups and showed that these define an extraordinary cohomology theory [AH59, AH61]). It would be several years before a definition of higher *algebraic* K-theory appeared, however. The two theories agree on K_0, but other than that they are vastly different; for one, topological K-theory is 2-periodic due to Bott periodicity.

The lower K-groups.

In 1962, Bass and Schanuel defined an algebraic analogue of the topologists' $K^{-1}(X)$ by setting $K_1(R) := \mathrm{GL}(R)/\mathrm{E}(R)$ for any ring R, where $\mathrm{GL}(R)$ is the stable general linear group and $\mathrm{E}(R)$ is the union of the subgroups $\mathrm{E}_n(R)$ of elementary matrices ([BS62]). This was inspired by Whitehead's work on simple homotopy type. In 1950, Whitehead showed that for a homotopy equivalence $f\colon X \to Y$ between cellular complexes, there is an element $\tau(f)$, the *torsion* of f, belonging to a group $\mathrm{Wh}(\pi_1(X))$, and this element is an obstruction to f being made up of simple moves (expansions and contractions) ([Whi50]). In 1961, Milnor used Reidemeister torsion (a coarser invariant than Whitehead's torsion) to disprove the Hauptvermutung, which conjectured that any two triangulations of a space have a common refinement. Milnor observed that for any group π, the *Whitehead group* $\mathrm{Wh}(\pi)$ is the quotient of $\mathrm{GL}(\mathbb{Z}[\pi])/\mathrm{E}(\mathbb{Z}[\pi]))$ by the subgroup $\pm\pi$ ([Mil61]).

The Whitehead group has important applications in the theory of cobordisms. An h-cobordism (W, M, M') is a smooth closed manifold W with boundary $M \amalg M'$ such that both M and M' are deformation retracts of W. If $\dim M \geq 5$, then an h-cobordism (W, M, M') is diffeomorphic to the cylinder $M \times I$ if and only if the Whitehead torsion $\tau(W, M) \in \mathrm{Wh}(\pi_1(M))$ vanishes. In fact, diffeomorphism classes of h-cobordisms (W, M, M') of M are in bijection with elements of the Whitehead group $\mathrm{Wh}(\pi_1(M))$. This is the s-cobordism Theorem ([Bar63, Maz63, Sta65]).

Bass and Schanuel also defined a relative group $K_0(f)$ for a ring homomorphism $f\colon R \to S$ and constructed a 5-term exact sequence

$$K_1(R) \to K_1(S) \to K_0(f) \to K_0(R) \to K_0(S).$$

Bass defined relative K-groups $K_1(R, I)$ for any ideal I in R, extending the exact sequence above by one for the morphism $f\colon R \to R/I$ ([Bas64]). These groups have implications for the congruence subgroup problem, namely the question: does any finite index subgroup $\Gamma \leq \mathrm{SL}_n(R)$ contain a subgroup $\mathrm{SL}_n(I)$ for some ideal I in R? For $n \geq 3$, the answer is yes if and only if a certain subgroup $SK_1(R, I) \leq K_1(R, I)$ is trivial for all I. Bass–Milnor–Serre proved the congruence subgroup problem for all global fields in [BMS67], also revealing a connection between explicit power reciprocity laws in a global field F and the groups $K_1(\mathscr{O}_F, I)$ where $\mathscr{O}_F$ is the ring of integers in F.

In 1967, Milnor defined $K_2(R)$ as the kernel of the homomorphism $\mathrm{St}(R) \to \mathrm{E}(R)$, where $\mathrm{St}(R)$ is the stable Steinberg group ([Mil71]). In 1968, Matsumoto gave a presentation of K_2

of a field in terms of Steinberg symbols ([Mat69]):

$$K_2(F) = F^\times \otimes F^\times / \langle (a, 1-a) \mid a \neq 0, 1 \rangle.$$

Tate discovered norm residue symbols on $K_2(F)$ (certain functions from $K_2(F)$ to the Brauer group of F), one for each n which is invertible in F ([Tat71]). These symbols give rise to maps $K_2(F)/n \to H^2_{\text{ét}}(F, \mu_n^{\otimes 2})$, where the right hand side is étale cohomology and μ_n is the group of n^{th} roots of unity. The Merkujev–Suslin Theorem states that these maps are isomorphisms for all fields ([Tat76, Mer81, MS82]). The relationship between K-theory and étale cohomology turned out to be even more intricate in view of the motivic Bloch–Kato conjecture, which we touch upon briefly at the end of this section.

Higher algebraic K-theory.

Several definitions of higher algebraic K-theory were proposed during the late 1960s and early 1970s ([Ger71, KV69, Keu71, Mil70, NV68, Swa70, Vol71, Wag73]). The most important one is Quillen's *plus-construction* of 1969 ([Qui71]). Through his work on the Adams conjecture, Quillen was inspired to define the higher K-groups of rings in terms of general linear groups and an intermediary homotopical construction, the plus-construction $B\,\mathrm{GL}(R)^+$ of $B\,\mathrm{GL}(R)$ for any ring R. In rough terms, the plus-construction "corrects" the fundamental group of $B\,\mathrm{GL}(R)$ while preserving the homology. Quillen then defined $K_n(R) := \pi_n(B\,\mathrm{GL}(R)^+)$ for all $n \geq 1$. Quillen's calculations of the homology of general linear groups over finite fields enabled him to completely calculate the K-groups of finite fields ([Qui72]). This construction lacks the important K_0, however, as $B\,\mathrm{GL}(R)$ is connected. On the other hand, studying the unstable spaces $B\,\mathrm{GL}_n(R)^+$ is a very useful tool as demonstrated by Quillen, and they can be interpreted as models for unstable algebraic K-theory.

Quillen provided another definition in 1972, applicable to exact categories; that is, subcategories of abelian categories which are closed under extension. Given an exact category $\mathscr{E}$, Quillen defined a category $Q(\mathscr{E})$, the so-called *Q-construction*, and set $K_n(\mathscr{E}) := \pi_{n-1}|Q(\mathscr{E})|$, where $|Q(\mathscr{E})|$ is the geometric realisation of $Q(\mathscr{E})$. Using the Q-construction, he was able to prove fundamental theorems about algebraic K-theory, like the Additivity, Resolution, Dévissage and Localisation Theorems ([Qui73b]). When $\mathscr{E} = \mathcal{P}_R$ is the exact category of finitely generated projective modules over a ring R, then $\Omega|Q(\mathcal{P}_R)| \simeq K_0(R) \times B\,\mathrm{GL}(R)^+$, recovering the plus construction definition ([Qui73b, Gra76]).

Quillen observed in a 1971 preprint that the infinite loop space machine developed by Segal ([Seg74]) also recovers this homotopy type: $\Omega B|i\mathcal{P}_R| \simeq K_0(R) \times B\,\mathrm{GL}(R)^+$, where $i\mathcal{P}_R$ is the monoidal category of finitely generated projective R-modules and isomorphisms between them ([FM94, Appendix Q]). This is somehow the homotopical analogue of Grothendieck's construction, viewing the K-theory space as the group completion of the topological monoid $|i\mathcal{P}_R|$.

We return briefly to the theory of cobordisms. Having established that h-cobordisms are determined by the Whitehead group, topologists began to study pseudo-isotopies of a manifold M, i.e. diffeomorphisms of $M \times I$ which are the identity on $M \times \{0\}$. This turned out to also have connections with algebraic K-theory. For example, if $\mathcal{P}(M)$ denotes the topological group of pseudo-isotopies of M, then $\pi_0(\mathcal{P}(M))$ is isomorphic to the product of the Whitehead group with certain coefficients and a quotient of $K_2(\mathbb{Z}[\pi_1(M)])$ ([HW73]).

The work of Segal and the study of pseudo-isotopies inspired Waldhausen to make another important definition of higher algebraic K-theory, announced in the mid 1970's, the details appearing in [Wal85]. Waldhausen's definition is applicable to categories with suitable notions of cofibrations and weak equivalences, now often known as *Waldhausen categories*. These include exact categories with isomorphisms as weak equivalences, but also more flexible settings such as the category $R_f(X)$ of finite split retractions $Y \to X$ with homotopy equivalences over X as the weak equivalences — this gives rise to what Waldhausen called *algebraic K-theory of spaces*. For a Waldhausen category $\mathcal{C}$ with weak equivalences w, Waldhausen defined a simplicial category $wS_\bullet\mathcal{C}$, now known as the $S_\bullet$*-construction*, the S standing for Segal. He then studied the space $K(\mathscr{C}, w) := \Omega|wS_\bullet\mathscr{C}|$. He observed that for an exact category $\mathscr{E}$, the edgewise subdivision of $wS_\bullet\mathscr{E}$ is homotopy equivalent to the nerve of $Q(\mathscr{E})$, recovering Quillen's definition. Waldhausen made significant advances on the foundations of algebraic K-theory through his work.

Other applications and more recent advances.

In 1972, Lichtenbaum conjectured a connection between the K-groups $K_i(\mathscr{O}_F)$ of the rings of integers in a totally real number field F and values of the ζ-function ζ_F. More precisely, he conjectured that $|\zeta_F(1-2i)| = |K_{4i-2}(\mathscr{O}_F)|/|K_{4i-1}(\mathscr{O}_F)|$ for all $i \geq 1$ ([Lic73]). This was later corrected slightly, after Lee–Szczarba's calculation of $K_3(\mathbb{Z}) \cong \mathbb{Z}/48\mathbb{Z}$ ([LS76]), and Quillen reformulated the conjecture in terms of étale cohomology, conjecturing also the existence of an Atiyah–Hirzebruch type spectral sequence from étale cohomology abutting to a certain completion of the K-groups ([Qui75]). Such a spectral sequence can be found in Thomason's work, where he studies the higher algebraic K-theory of algebraic varieties ([Tho85]). In 2009, Voevodsky proved the motivic Bloch–Kato conjecture relating Milnor K-theory and Galois cohomology via the norm residue map ([Voe11]). This implies the Quillen–Lichtenbaum conjecture and with this result, the K-groups of the integers are almost completely determined. The only piece missing is the question of whether the groups $K_{4k}(\mathbb{Z})$ vanish; this is equivalent to the Kummer–Vandiver conjecture about the class group of cyclotomic integers made independently by Kummer and Vandiver in relation with their work on Fermat's Last Theorem ([Van46, Kur92]).

In the ∞-categorical setting, it is possible to define algebraic K-theory via universal properties. One of the main results of [BGT13] is that connective algebraic K-theory is a universal additive invariant. This brings us nicely back to the beginning of this story, identifying K-theory as a group completion in much the same way that the Grothendieck group turns a monoid into a group. We will end our summary here, well aware of the fact that there is a lot more to be told of this story, especially of the more modern approaches to and advances within algebraic K-theory. For more details of the history, we refer to Weibel's excellent survey covering the development of algebraic K-theory before 1980 ([Wei99]).

1.2. The object of study: the reductive Borel–Serre compactification. The reductive Borel–Serre compactification is a certain compactification of the locally symmetric space associated with a neat arithmetic group. We present some background on compactifications of locally symmetric spaces, provide more details on the Borel–Serre and reductive Borel–Serre compactifications, and finally stress the mathematical importance of the reductive Borel–Serre compactification by naming some key applications and properties.

An arithmetic group is a group arising as the integer points of an algebraic group, e.g. $\mathrm{SL}_n(\mathbb{Z})$, $\mathrm{GL}_n(\mathbb{Z})$, $\mathrm{Sp}_{2n}(\mathbb{Z})$. Cohomology of arithmetic groups is an important subject, as it has influences in a number of different mathematical fields through its connections with algebraic K-theory, reduction theory, automorphic forms, Galois representations and Hecke operators, among other things.

An arithmetic group Γ acts on a symmetric space X. If Γ is an arithmetic group in a sufficiently nice reductive linear algebraic group $\mathbf{G}$ over $\mathbb{Q}$, then X is the space of maximal compact subgroups of $\mathbf{G}(\mathbb{R})$ on which Γ acts by conjugation. In particular, $X \cong \mathbf{G}(\mathbb{R})/K$ for a choice of maximal compact subgroup $K \leq \mathbf{G}(\mathbb{R})$, with Γ acting by left multiplication on the homogeneous space. When Γ is torsion free, then the quotient $\Gamma\backslash X$ is a locally symmetric space, in particular a smooth manifold, and moreover a model for the classifying space of Γ. This allows us to study them using geometric tools and illustrates the fact that arithmetic groups are much more than just discrete groups.

The locally symmetric space $\Gamma\backslash X$ is, however, very rarely compact (cf. the Godement compactness criterion). The problem of compactifying it has given rise to a vast number of compactifications well-suited for different purposes; the classical compactifications are the Satake compactifications ([Sat60]), the Baily–Borel compactification ([BB66]), the Borel–Serre compactification ([BS73]), the reductive Borel–Serre compactification ([Zuc83]), and toroidal compactifications ([AMRT75]). In this thesis, we study the Borel–Serre and reductive Borel–Serre compactifications.

The *Borel–Serre compactification* $\Gamma\backslash\overline{X}^{BS}$ was introduced in 1973 by Borel and Serre ([BS73]). When Γ is torsion free, it is a smooth manifold with corners with the same homotopy type as $\Gamma\backslash X$, and thus provides a compact geometric model for the classifying space of Γ. Borel exploited this construction to calculate the stable real cohomology of certain classical groups, and this in turn enabled him to calculate the ranks of the K-groups of the ring of integers $\mathscr{O}_F$ in a number field F ([Bor74]). It was also used by Quillen to show that these same K-groups $K_i(\mathscr{O}_F)$ are finitely generated ([Qui73a]).

If one is interested in L^2-cohomology of the group Γ, then the Borel–Serre compactification is not a suitable space to study — it has too much boundary in some sense. The problem is that $\Gamma\backslash\overline{X}^{BS}$ does not admit L^2-partitions of unity; the obstruction is a nilmanifold factor in each boundary stratum corresponding to the unipotent radical of a parabolic subgroup associated with the given stratum. In 1982, Zucker defined the *reductive Borel–Serre compactification* $\overline{\Gamma\backslash X}^{RBS}$ as a quotient of the Borel–Serre compactification by collapsing these nilmanifold factors ([Zuc83]).

We want to comment on the structure of these two compactifications to illustrate what happens when we take the quotient. This will be slightly technical, but we focus on the case where Γ is a finite index neat subgroup in $\mathrm{GL}_n(\mathbb{Z})$ to make it more accessible. The rational parabolic subgroups of the reductive algebraic group $\mathbf{GL}_n$ are the $\mathrm{GL}_n(\mathbb{Z})$-conjugates of block upper triangular subgroups. The Borel–Serre compactification $\Gamma\backslash\overline{X}^{BS}$ is naturally stratified as a manifold with corners over the poset $\mathscr{P}_\Gamma$ of Γ-conjugacy classes of rational parabolic subgroups of $\mathbf{GL}_n$. This stratification descends along the quotient map $\Gamma\backslash\overline{X}^{BS} \to \overline{\Gamma\backslash X}^{RBS}$, equipping the reductive Borel–Serre compactification with a natural stratification over $\mathscr{P}_\Gamma$. The stratum in $\Gamma\backslash\overline{X}^{BS}$ corresponding to the class of a rational parabolic subgroup $\mathbf{P}$ is a model for the classifying space of the group $\Gamma_{\mathbf{P}}$ given by intersecting with $\mathbf{P}$. The corresponding stratum in $\overline{\Gamma\backslash X}^{RBS}$ is a model for classifying space of the quotient $\Gamma_{\mathbf{L_P}} = \Gamma_{\mathbf{P}}/\Gamma_{\mathbf{N_P}}$, where $\Gamma_{\mathbf{N_P}}$ is the subgroup given by intersecting with the unipotent radical $\mathbf{N_P} \leq \mathbf{P}$. If $\mathbf{P}$ is block upper triangular, then the unipotent radical $\mathbf{N_P}$ is the subgroup all of whose diagonal blocks are the identity and the Levi quotient $\mathbf{L_P} = \mathbf{P}/\mathbf{N_P}$ can be identified with the diagonal blocks.

Let us emphasise the two important things to take note of in the above. First of all, the poset of rational parabolic subgroups underlies these constructions in a very important way. Secondly, the topological quotient $\Gamma\backslash\overline{X}^{BS} \to \overline{\Gamma\backslash X}^{RBS}$ is detected by the homotopy types of the individual strata as certain group quotients given by the internal structure of the parabolic subgroups. As a small remark, the name reductive Borel–Serre compactification stems from the fact that the Levi quotient $\mathbf{L_P} = \mathbf{P}/\mathbf{N_P}$ is a reductive algebraic group. The parabolic subgroups are not reductive in general.

The reductive Borel–Serre compactification has come to play a central role in the theory of compactifications. We here present a brief overview of some of its important applications and properties. By construction, it is well-suited for the study of L^p-cohomology of Γ ([Zuc83, Zuc86, Zuc01]). It dominates all Satake compactifications ([BJ06, III.15.2]) and plays an important role in parametrising the continuous spectrum of the Laplacian on $\Gamma\backslash X$ ([JM02]). When $\Gamma\backslash X$ is Hermitian, the Baily–Borel compactification is a projective variety ([BB66]), but this is not necessarily the case for $\overline{\Gamma\backslash X}^{RBS}$, as it may have odd dimensional real boundary strata. Nevertheless, the cohomology of $\overline{\Gamma\backslash X}^{RBS}$ carries a natural mixed Hodge structure ([Zuc04]) and is motivic ([AZ12]).

The reductive Borel–Serre compactification is used to define weighted cohomology of Γ. This is defined as the hypercohomology of a constructible complex of sheaves on $\overline{\Gamma\backslash X}^{RBS}$ depending on an auxiliary parameter, the so-called weight profiles ([GHM94]). Intersection cohomology, an analogue of singular cohomology rectifying the failure of Poincaré duality for singular spaces, can likewise be defined as the hypercohomology of a constructible complex of sheaves ([GM83]). For different weight profiles, weighted cohomology recovers the ordinary cohomology of Γ, intersection cohomology of the Baily–Borel compactification (when $\Gamma\backslash X$ is Hermitian), L^2-cohomology of Γ (when this is finite dimensional), compactly supported cohomology of $\Gamma\backslash X$, and Franke's weighted L^2 cohomology ([GHM94, Fra98, Nai99]). Weighted cohomology is the main ingredient in the topological trace formula calculating the Lefschetz number of Hecke correspondences on weighted cohomology ([GM92, GM03]). This exploits the fact that Hecke operators extend to the reductive Borel–Serre compactification (they do not extend to the Borel–Serre compactification), and moreover that the singularities of $\overline{\Gamma\backslash X}^{RBS}$ are not too complicated, making calculations more tractable.

Finally, the reductive Borel–Serre compactification motivated the theory of $\mathscr{L}$-modules introduced by Saper ([Sap05a, Sap05b]). These are combinatorial analogues of constructible complexes of sheaves on $\overline{\Gamma\backslash X}^{RBS}$. Saper used the theory to settle a conjecture made independently by Rapoport and Goresky–MacPherson relating the intersection cohomology of certain Satake compactifications with that of the reductive Borel–Serre compactification ([Rap86, GM88]). This allows us to transfer cohomological calculations from the more singular spaces, Satake compactifications, to the reductive Borel–Serre compactification. As a special case, this recovers and generalises the main result on weighted cohomology, relating weighted cohomology of Γ with intersection cohomology of the Baily–Borel compactification of $\Gamma\backslash X$ ([GHM94]).

Let us remark here that the link between algebraic K-theory and the reductive Borel–Serre compactification may not be altogether clear at this point. Indeed, the link is in some sense the main result of this thesis. We will, however, motivate why such a connection may not be entirely surprising when we turn our attention to the results of the thesis. In this project, we study the reductive Borel–Serre compactification $\overline{\Gamma\backslash X}^{RBS}$ as a topological space from a purely homotopical viewpoint. We incorporate, however, the important natural stratification over the poset $\mathscr{P}_\Gamma$. This leads us on to the next section, where we address the tools which we use to do this.

1.3. The toolbox: stratified homotopy theory. The study of stratified spaces arose from a desire to understand topological spaces that are not quite manifolds, but can be partitioned into pieces which are. Examples include algebraic varieties, mapping cylinders of maps between manifolds, orbit spaces by certain group actions, and compactifications of locally symmetric spaces. In addition to the partition of the space, one needs to specify the rigidity with which these pieces fit together, that is, the type of neighbourhood each piece has in the total space. Different conditions on these neighbourhoods lead to different notions of stratified spaces. These include Whitney stratified spaces ([Whi65]), Thom–Mather stratified spaces ([Mat12]), homotopically stratified sets ([Qui88]), and more recently conically smooth stratified spaces ([AFT17]). The theory of homotopically stratified sets moved away from the geometric origins by considering filtered spaces subject to certain homotopical conditions on the neighbourhoods, whereas the other three examples given are subject to some sort of geometric conditions.

It is a classical result that there is an equivalence between locally constant sheaves on a sufficiently nice topological space and representations of its fundamental groupoid: the so-called monodromy equivalence. In unpublished work, MacPherson made the observation that analogously to how locally constant sheaves are classified by paths in the topological space, the constructible sheaves on a stratified space are classified by so-called exit paths. Constructible sheaves are sheaves which are locally constant along each stratum, and exit paths are stratum preserving paths, that is, if a stratum X_i is in the closure of another stratum X_j, then the path can move from X_i into X_j, but it cannot return. Treumann gave a 2-categorical version of this result ([Tre09]), and Lurie developed the ∞-categorical setting that we will be working with.

We consider conically stratified (poset-stratified) spaces as defined by Lurie ([Lur17, Appendix A]). This is a rather weak notion of stratified spaces which includes the four examples mentioned above. A conically stratified space is, roughly put, the data of a topological space equipped with a well-behaved partition, subject to certain local topological conditions. To a conically stratified space X, we can associate its exit path ∞-category $\Pi_\infty^{\mathrm{exit}}(X)$ ([Lur17, Appendix A]). This is a refinement of the fundamental ∞-groupoid incorporating the additional structure of a stratification — the exit path ∞-category should really be interpreted as the analogue or generalisation of the fundamental ∞-groupoid for stratified spaces. Lurie proved that for sufficiently nice conically stratified spaces, the ∞-category of constructible space-valued sheaves is equivalent to the ∞-category of representations of the exit path ∞-category:

$$\mathrm{Shv}_{\mathrm{cbl}}(X, \mathcal{S}) \simeq \mathrm{Fun}(\Pi_\infty^{\mathrm{exit}}(X), \mathcal{S}).$$

This generalises MacPherson's observation and cements the analogy with the fundamental ∞-groupoid.

The work of Lurie has been used and developed in several directions. We briefly review some of these developments to put our tools into a broader context. Recall that the homotopy hypothesis states that up to weak homotopy equivalence, ∞-groupoids are equivalent to topological spaces. In other words, any homotopy type can be realised as an ∞-groupoid. This hypothesis was put forward by Grothendieck in a letter to Quillen — in fact, he made the stronger conjecture that an n-groupoid should be equivalent to a homotopy-n-type and that this equivalence should be achieved by the fundamental n-groupoid ([Gro83]). Whether the homotopy hypothesis has been confirmed depends on which definition of n-groupoids one has in mind, but it has been formalised and verified in many cases. A formal statement, as proved by Quillen ([Qui67]), is that there is a Quillen equivalence

$$(|\cdot| \dashv \mathrm{Sing})\colon\ \mathrm{Top} \to \mathrm{sSet}$$

where Top and sSet are equipped with the classical model structures, $|\cdot|$ is geometric realisation and Sing is the singular simplicial set (i.e. the fundamental ∞-groupoid in this setting).

Several people have worked on establishing a *stratified homotopy hypothesis*. Ayala, Francis, Rozenblyum and Tanaka have developed an ∞-category of conically smooth stratified spaces $\mathcal{S}\mathrm{trat}$, and they show that the exit path ∞-category provides a fully faithful embedding $\mathcal{S}\mathrm{trat} \to \mathrm{Cat}_\infty$, thus providing a candidate for a stratified homotopy hypothesis ([AFT17], [AFR17]). A slightly different form of a stratified homotopy hypothesis is obtained by Haine: for a fixed poset P, the ∞-category of quasicategories with a conservative functor to P can be obtained from the category of P-stratified topological spaces by inverting a class of weak equivalences ([Hai19]).

In a different direction, Barwick, Glasman and Haine have generalised the exit path ∞-category to schemes, also providing a classification of constructible sheaves on the scheme as representations of its exit path ∞-category — they call this the exodromy equivalence ([BGH20]). In the same paper, they extend the exodromy equivalence to their newly developed pyknotic setting ([BH19]). From a different angle, this extends the monodromy equivalence of Bhatt–Scholze which considers continuous representations out of a refined fundamental group using the pro-étale topology on a scheme ([BS15]). The theory of pyknotic sets and the closely related, independently developed theory of condensed mathematics of Clausen–Scholze ([Sch19]) aim to provide foundations for studying algebraic objects carrying topologies, but this is rather off-topic for this thesis.

2. The story of this thesis

In this thesis, we approach the reductive Borel–Serre compactification from an algebro-topological angle, studying $\overline{\Gamma\backslash X}^{RBS}$ homotopically and forgetting the otherwise important geometric structure. We crucially take into account the natural stratification of $\overline{\Gamma\backslash X}^{RBS}$ by determining its stratified homotopy type. We then detach ourselves further from $\overline{\Gamma\backslash X}^{RBS}$ by, in some sense, forgetting that there was even a space there to study. More precisely, we study the stratified homotopy type independently, generalising it significantly and extrapolating from this a model for the algebraic K-theory space. More precisely, we find a candidate model for unstable algebraic K-theory.

The project was initialised by a suggestion of Dustin Clausen, coauthor of the second paper, and we wish to remark here that he anticipated the existence of such a model for the algebraic K-theory space.

2.1. Flags and parabolic subgroups. We begin by drawing some parallels between the algebraic K-theory space and the reductive Borel–Serre compactification. These are not at all formal statements, but serve to motivate why the connection established in this thesis may not come as a complete surprise. Recall that a filtration in a vector space V is a sequence of subspaces $0 = V_0 \subset V_1 \subset \cdots \subset V_k = V$. The associated graded of such a filtration is the graded object given by the successive quotients $\bigoplus_{i=1}^{k} V_i/V_{i-1}$. A flag is a sequence of proper subspaces, i.e. a filtration given by strict inclusions. Flags, filtrations and associated gradeds are built into the structure of both the reductive Borel–Serre compactification and the algebraic K-theory spaces as we will give examples of below.

Let $\mathscr{E}$ be an exact category. Recall that the K-theory space $K(\mathscr{E})$ can be defined as the loop space of the geometric realisation of Quillen's Q-construction, $\Omega|Q(\mathscr{E})|$, or as the loop space of the total realisation of Waldhausen's $S_\bullet$-construction, $\Omega|wS_\bullet\mathscr{E}|$.

Quillen's Q-construction $Q(\mathscr{E})$ is a category whose objects are those of $\mathscr{E}$ and whose morphisms $x \to y$ are equivalence classes of diagrams $x \twoheadleftarrow z \rightarrowtail y$, where two such diagrams are equivalent if the middle terms are isomorphic via an isomorphism which commutes with the maps to x and y. It follows that a morphism $x \to y$ is equivalent to identifying x as a subquotient of y, which in turn is equivalent to providing a three step filtration of y and an identification of x with the middle term in the associated graded.

Filtrations and associated gradeds also appear in Waldhausen's $S_\bullet$-construction. An object in the n-simplices of Waldhausen's $S_\bullet$-construction $wS_\bullet\mathscr{E}$ is a sequence $V_1 \rightarrowtail \cdots \rightarrowtail V_n$ of admissible monomorphisms together with a choice of subquotients V_j/V_i for all $i < j$. The morphisms are given by isomorphisms of the resulting diagrams. An isomorphism class of such a diagram encodes the data of a filtration together with all possible choices of associated gradeds.

We turn our attention to the reductive Borel–Serre compactification. Consider the reductive algebraic group $\mathbf{GL}_n$ over $\mathbb{Q}$. Parabolic subgroups of $\mathbf{GL}_n$ are conjugates of block upper triangular subgroups, and they correspond to stabilisers of flags in $\mathbb{Q}^n$. The standard Borel subgroup for example, is the subgroup of upper triangular matrices and it stabilises the complete flag $0 \subset \mathbb{Q} \subset \mathbb{Q}^2 \subset \cdots \subset \mathbb{Q}^n$. The unipotent radical of a block upper triangular subgroup is given by the matrices all of whose diagonal blocks are the identity. Taking the quotient of a parabolic subgroup by its unipotent radical results in the Levi quotient, which in this case is a product of general linear groups, since only the diagonal blocks remain. In terms of flags, modding out by the unipotent radical corresponds to taking the associated graded.

Recall that the reductive Borel–Serre compactification is constructed as a quotient of the Borel–Serre compactification by collapsing nilmanifold factors in the boundary strata corresponding to unipotent radicals of parabolic subgroups. A boundary stratum of the Borel–Serre compactification $\Gamma\backslash\overline{X}^{BS}$ is a model for the classifying space of $\Gamma_{\mathbf{P}}$ for some rational parabolic subgroup $\mathbf{P}$, whereas the corresponding stratum in $\overline{\Gamma\backslash X}^{RBS}$ is a model for the classifying space of the arithmetic group $\Gamma_{\mathbf{L_P}}$ in the Levi quotient $\mathbf{L_P} = \mathbf{P}/\mathbf{N_P}$.

In other words, the stratification of the Borel–Serre compactification encodes the poset of (Γ-conjugacy classes of) rational parabolic subgroups and remembers these groups, and the reductive Borel–Serre compactification also encodes the poset of (Γ-conjugacy classes of) rational parabolic subgroups but remembers the Levi quotients instead. Translating this into flags, we can think of the Borel–Serre compactification as encoding the poset of flags and automorphisms of these, whereas the reductive Borel–Serre compactification encodes the poset of flags but automorphisms of the associated gradeds.

We wish to highlight one more construction of the K-theory space, this time through Quillen's plus-construction. One concrete model for the plus-construction $B\,\mathrm{GL}(R)^+$ is given by the Volodin space. The Volodin space $X(R)$ is the subspace $X(R) = \bigcup_{n,\sigma} B(\mathrm{U}_n(R)^\sigma) \subset B\,\mathrm{GL}(R)$, where $\mathrm{U}_n(R) \leq \mathrm{GL}_n(R)$ is the subgroup of upper triangular matrices, n runs through the natural numbers, and σ runs through the permutation matrices and acts by conjugation on $\mathrm{U}_n(R)$. The topological quotient $B\,\mathrm{GL}(R)/X(R)$ is a model for the plus-construction $B\,\mathrm{GL}(R)^+$ ([Sus81]).

For fixed n and varying σ, the groups $\mathrm{U}_n(R)^\sigma$ generate the subgroup of elementary matrices $\mathrm{E}_n(R) \leq \mathrm{GL}_n(R)$. For $R = \mathbb{Z}$, consider a rational parabolic subgroup $\mathbf{P} \leq \mathbf{GL}_n$ and the subgroup $\mathrm{GL}_n(\mathbb{Z}) \cap \mathbf{N_P}(\mathbb{Q}) \leq \mathrm{GL}_n(\mathbb{Z})$ given by the unipotent radical $\mathbf{N_P} \leq \mathbf{P}$. For varying $\mathbf{P}$, these subgroups also generate $\mathrm{E}_n(\mathbb{Z})$. One can think of the reductive Borel–Serre compactification as collapsing these subgroups systematically instead of just taking a topological quotient as in Volodin's construction.

2.2. The reductive Borel–Serre category and algebraic K-theory. The main result of [I] is that the exit path ∞-category of the reductive Borel–Serre compactification $\overline{\Gamma\backslash X}^{RBS}$ is equivalent to the nerve of a 1-category $\mathscr{C}_\Gamma^{RBS}$ defined purely in terms of rational parabolic subgroups, their unipotent radicals and the conjugation action of Γ on these groups. The two main things to underline about this result, are that (1) what was *a priori* an ∞-category is essentially a 1-category and (2) the definition of $\mathscr{C}_\Gamma^{RBS}$ makes no reference to the space $\overline{\Gamma\backslash X}^{RBS}$. This result has some direct consequences: it determines the homotopy type, classifies constructible sheaves and provides a combinatorial incarnation of the constructible derived category. In this introduction, we will instead focus on telling the story of how we naturally move from this result to defining a model for the algebraic K-theory space.

To keep things simple, we consider a local commutative ring R. Let $i\mathcal{P}_R$ denote the symmetric monoidal category of finitely generated projective R-modules and isomorphisms. This is equivalent to the disjoint union $M(R) = \coprod_n \mathrm{GL}_n(R)$, viewing each group as a one object category and with the monoidal product given by direct sum. The group completion of the topological monoid $|M(R)| = \coprod_n B\,\mathrm{GL}_n(R)$ is a model for the algebraic K-theory space, $K(R) \simeq \Omega B|M(R)|$.

Now, the category $\mathscr{C}_\Gamma^{RBS}$ makes perfect sense for $\Gamma = \mathrm{GL}_n(\mathbb{Z})$ — the torsion that is a complication when considering the space $\overline{\Gamma\backslash X}^{RBS}$ is inconsequential on the "spaceless" side of things. In fact, the category $\mathscr{C}_\Gamma^{RBS}$ can be defined verbatim for $\Gamma = \mathrm{GL}_n(R)$ and any commutative ring R. We call this the *reductive Borel–Serre category* associated to $\mathrm{GL}_n(R)$ and denote it by $\mathscr{C}_n^{RBS}(R)$.

Disregarding issues with torsion for the sake of illustration, we have models for the classifying space of $\Gamma = \mathrm{GL}_n(\mathbb{Z})$, namely the locally symmetric space $\Gamma\backslash X$ and even better the compact model given by the Borel–Serre compactification $\Gamma\backslash\overline{X}^{BS}$. If we think of the topological monoid $|M(\mathbb{Z})|$ as a disjoint union of the homotopy types of the Borel–Serre compactifications associated with $\mathrm{GL}_n(\mathbb{Z})$ for varying n, then it is a small step from there to wondering what would happen if we consider instead the disjoint union of the *stratified* homotopy types of the *reductive* Borel–Serre compactifications associated with $\mathrm{GL}_n(\mathbb{Z})$ or even $\mathrm{GL}_n(R)$.

For a local commutative ring R, we define a monoidal category $M^{RBS}(R) = \coprod_n \mathscr{C}_n^{RBS}(R)$ with monoidal product given by direct sum. This is no longer symmetric monoidal. We can strictify this monoidal category, defining a strict monoidal category $M_{\mathcal{P}_R}$ associated to the exact category $\mathcal{P}_R$ of finitely generated projective R-modules, by using the correspondence between parabolic subgroups and flags, Levi quotients and associated gradeds.

In fact, for any exact category $\mathscr{E}$, we can define a strict monoidal category $M_{\mathscr{E}}$ by translating the definition of $M_{\mathcal{P}_R}$ word for word to the more general setting. One should think of the objects of $M_{\mathscr{E}}$ as associated gradeds and the morphisms as given by flags and refinements of flags. In [II] we prove that the classifying space $B|M_{\mathscr{E}}|$ of the topological monoid $|M_{\mathscr{E}}|$ is homotopy equivalent to the geometric realisation $|Q(\mathscr{E})|$ of Quillen's Q-construction. In particular, the loop space $\Omega B|M_{\mathscr{E}}|$ is a model for the algebraic K-theory space $K(\mathscr{E})$. The proof crucially uses the fact that a morphism $x \to y$ in $Q(\mathscr{E})$ is a three-step filtration of y with middle graded piece x.

The justification for considering and exploring this model for the algebraic K-theory space lies to a great extent in its geometric origins: it comes from a geometric object which is of great interest in its own right. One might even hope to be able to transfer tools from the geometric side to the algebraic side via this connection. Ultimately, however, we are interested in investigating whether the categories $\mathscr{C}_n^{RBS}(R)$ provide models for unstable algebraic K-theory. More precisely, does the colimit of these categories $\mathscr{C}^{RBS}(R)$ identify with the 0-component of the algebraic K-theory space, $B\,\mathrm{GL}(R)^+$? In [II], we make some homology calculations which unveil promising behaviour in this direction. We provide another identification of the exit path ∞-category of the reductive Borel–Serre compactification in [II] — the proof uses entirely different techniques to the one given in [I], and the two proofs provide very different insights into the structure of the reductive Borel–Serre compactification. This proof strategy allows us to make some homology calculations for finite fields and rings with many units in the sense of [NS89], e.g. an infinite field or a local ring with infinite residue field. In these cases, we do find that $\mathscr{C}^{RBS}(R) \simeq B\,\mathrm{GL}(R)^+$. In addition to this, we find that for finite fields our model exhibits much better homological stability properties than the general linear groups, and thus in a certain sense provides a better model for unstable algebraic K-theory than the plus-constructions.

3. Questions for the future

There are many unanswered questions in the wake of this work that would be interesting to pursue. We present here some suggestions for future research.

3.1. Constructible complexes of sheaves and $\mathscr{L}$-modules. Both the weighted cohomology of an arithmetic group Γ and intersection cohomology of the reductive Borel–Serre compactification $\overline{\Gamma\backslash X}^{RBS}$ can be defined as the hypercohomology of certain constructible complexes of sheaves on $\overline{\Gamma\backslash X}^{RBS}$. The theory of $\mathscr{L}$-modules as defined by Saper provides a combinatorial analogue of constructible complexes of sheaves on $\overline{\Gamma\backslash X}^{RBS}$ ([Sap05a, Sap05b]). For an associative ring R, our identification of the exit path ∞-category of the reductive Borel–Serre compactification yields an equivalence between the constructible derived category of sheaves of R-modules on $\overline{\Gamma\backslash X}^{RBS}$ and a derived functor category:

$$D_{\mathrm{cbl}}(\mathrm{Shv}(\overline{\Gamma\backslash X}^{RBS}, R)) \simeq D(\mathrm{Fun}(\mathscr{C}_\Gamma^{RBS}, \mathrm{Mod}_R)).$$

An $\mathscr{L}$-module, however, contains strictly more data than a constructible complex of sheaves. So if one thinks of $\mathscr{L}$-modules as a combinatorial analogue of constructible complexes of sheaves, then the equivalence above can be interpreted as providing an actual combinatorial incarnation. It is not, however, completely straightforward to analyse this equivalence on a concrete complex of sheaves, and it is even less straightforward how one should translate an $\mathscr{L}$-module into an object in the derived functor category.

There are two directions in which one could exploit a concrete description. Geometrically, one can study the cohomology of $\overline{\Gamma\backslash X}^{RBS}$ with coefficients given by a constructible complex of sheaves in a "point-free" way by studying functor cohomology instead. Algebraically, given a complex of functors $\mathscr{C}_\Gamma^{RBS} \to \mathrm{Mod}_R$ corresponding to an interesting cohomology theory on $\overline{\Gamma\backslash X}^{RBS}$, one could try to generalise it to a complex of functors $\mathscr{C}_n^{RBS}(R) \to \mathrm{Mod}_R$ out of the reductive Borel–Serre category associated to $\mathrm{GL}_n(R)$ or even to a complex of functors $M_{\mathscr{E}} \to \mathrm{Mod}_R$, where $M_{\mathscr{E}}$ is the monoidal category associated with an exact category $\mathscr{E}$ defined in [II]. Such complexes of functors could turn out to be an interesting tool for studying algebraic K-theory, stably or unstably. The first complexes to consider should be the intersection and weighted cohomology complexes.

3.2. Exploring the model for the algebraic K-theory space. There are many unanswered questions about the model $K(\mathscr{E}) \simeq \Omega B|M_{\mathscr{E}}|$ of the algebraic K-theory space. For a commutative ring R, the reductive Borel–Serre categories $\mathscr{C}_n^{RBS}(R)$ generalising $\mathscr{C}_\Gamma^{RBS}$ come equipped with canonical embeddings $\mathrm{GL}_n(R) \to \mathscr{C}_n^{RBS}(R)$, and we can also consider the stable categories $\mathrm{GL}(R) \to \mathscr{C}^{RBS}(R)$.

Suppose R is a ring with many units in the sense of [NS89], e.g. an infinite field. Our homology calculations imply that for $n \geq 1$, the functor $\mathrm{GL}_n(R) \to \mathscr{C}_n^{RBS}(R)$ exhibits the geometric realisation $|\mathscr{C}_n^{RBS}(R)|$ as the plus-construction $B\,\mathrm{GL}_n(R)^+$ with respect to the maximal perfect subgroup. If k is a finite field, then this is true stably, i.e. $|\mathscr{C}^{RBS}(k)| \simeq B\,\mathrm{GL}(k)^+$. As mentioned, our model has better homological stability properties for finite fields than the general linear groups; more precisely, if k is a finite field with $\mathrm{char}(k) = p$, then $H^*(\mathscr{C}_n^{RBS}(k), \mathbb{F}_p) = 0$ for all $* > 0$. In contrast, $H^*(\mathrm{GL}_n(k), \mathbb{F}_p)$ is known to contain unstable non-trivial classes in positive degree for all n ([MP87, LS18]) and is still largely unknown.

It would be interesting to explore these properties and this model further. Some questions to address are the following:

* In what generality does the functor $\mathrm{GL}(R) \to \mathscr{C}^{RBS}(R)$ exhibit $|\mathscr{C}^{RBS}(R)|$ as the plus-construction $B\,\mathrm{GL}(R)^+$? This holds for all fields and all rings with many units as remarked above, but we do not know whether it is true for rings like $\mathbb{Z}$ or $\mathbb{Z}/(p^2)$ for example.
* Do we find nice homological stability properties for a more general class of rings? One could try to establish homotopy fibre sequences of the form
$$|S(n)| \to |\mathscr{C}^{RBS}_{n-1}(R)| \to |\mathscr{C}^{RBS}_{n}(R)|,$$
for a given ring R. A candidate for $S(n)$ is the complex of special unimodular frames studied by Nesterenko–Suslin which is $(n - \mathrm{sr}(R) - 1)$-acyclic, where $\mathrm{sr}(R)$ is the stable rank of R ([NS89]).
* The monoidal category $M_{\mathscr{E}}$ is not symmetric monoidal, so *a priori* its geometric realisation $|M_{\mathscr{E}}|$ is just an $\mathbb{E}_1$-space. Does it satisfy higher homotopy-commutativity?

3.3. The Deligne–Mumford compactification and other stratified homotopy types. We believe that one should be able to apply the machinery developed in this thesis in order to determine the stratified homotopy type of other stratified spaces. We also believe that they can be used to determine the stratified homotopy type of stratified stacks. This requires a suitable definition of exit path ∞-categories for stacks, for example the EI-∞-category that classifies constructible sheaves.

For a concrete example, consider the moduli stack $\overline{\mathscr{M}_{g,n}}$ of stable genus g curves with n marked points and let $\overline{\mathscr{M}^{an}_{g,n}}$ denote its underlying analytic stack (for $n = 0$, we omit the n). A candidate for the exit path ∞-category of $\overline{\mathscr{M}^{an}_{g,n}}$ is already in the literature. In 1984, Charney and Lee defined a category $\mathcal{CL}_g$ and showed that there is a rational homology equivalence between the geometric realisation $|\mathcal{CL}_g|$ and the coarse moduli space of $\overline{\mathscr{M}^{an}_{g}}$ ([CL84]). Generalising the category $\mathcal{CL}_g$ to include marked points, Ebert and Giansiracusa both generalised and strengthened the result of Charney–Lee by showing that the geometric realisation of the resulting category $\mathcal{CL}_{g,n}$ is in fact homotopy equivalent to the homotopy type of the stack $\overline{\mathscr{M}^{an}_{g,n}}$. Moreover, both $\mathcal{CL}_{g,n}$ and $\overline{\mathscr{M}^{an}_{g,n}}$ are naturally stratified, and the established homotopy equivalence is functorial with respect to inclusions of strata and closures of strata ([EG08]).

We believe that $\mathcal{CL}_{g,n}$ is in fact the exit path ∞-category of the stack $\overline{\mathscr{M}^{an}_{g,n}}$ for any appropriate definition of exit path ∞-categories for stacks. Establishing this would recover and strengthen the results of Charney–Lee and Ebert–Giansiracusa. A classification of the constructible sheaves in terms of a 1-category would provide a tool for studying cohomology of $\overline{\mathscr{M}_{g,n}}$ with coefficients which are not necessarily constant — there does not yet seem to be much in the literature along these lines. By applying the tools to Satake compactifications of locally symmetric spaces, one could also strengthen a similar result of Charney–Lee on the cohomology of Satake compactifications ([CL83]).

3.4. A model for the Hermitian K-theory space and other generalisations. In order to define the monoidal category $M_{\mathscr{E}}$ associated to an exact category $\mathscr{E}$, we generalised the exit path category $\mathscr{C}_\Gamma^{RBS}$ to general linear groups and exploited the correspondence between parabolic subgroups of $\mathbf{GL}_n$ and flags. One could easily imagine doing this procedure for other classical groups, for example the symplectic groups and orthogonal groups. We expect that one would obtain similar models for the symplectic and Hermitian K-theory spaces by considering isotropic flags and comparing a certain intermediary Q-construction in our proof with the appropriate analogue of Quillen's Q-construction.

References

[I] Mikala Ørsnes Jansen. The stratified homotopy type of the reductive Borel–Serre compactification. Preprint: arXiv:2012.10777, 2020.

[II] Dustin Clausen and Mikala Ørsnes Jansen. The reductive Borel–Serre compactification as a model for unstable algebraic K-theory. Preprint, 2020.

[AFR17] David Ayala, John Francis, and Nick Rozenblyum. A stratified homotopy hypothesis. Preprint: arXiv:1502.01713, 2017.

[AFT17] David Ayala, John Francis, and Hiro Lee Tanaka. Local structures on stratified spaces. *Adv. Math.*, 307:903–1028, 2017.

[AH59] M. F. Atiyah and F. Hirzebruch. Riemann-Roch theorems for differentiable manifolds. *Bull. Amer. Math. Soc.*, 65:276–281, 1959.

[AH61] M. F. Atiyah and F. Hirzebruch. Vector bundles and homogeneous spaces. In *Proc. Sympos. Pure Math., Vol. III*, pages 7–38. American Mathematical Society, Providence, R.I., 1961.

[AMRT75] A. Ash, D. Mumford, M. Rapoport, and Y. Tai. *Smooth compactification of locally symmetric varieties.* Math. Sci. Press, Brookline, Mass., 1975. Lie Groups: History, Frontiers and Applications, Vol. IV.

[AZ12] Joseph Ayoub and Steven Zucker. Relative Artin motives and the reductive Borel-Serre compactification of a locally symmetric variety. *Invent. Math.*, 188(2):277–427, 2012.

[Bar63] Dennis Barden. The structure of manifolds. PhD thesis, Cambridge University, 1963.

[Bas64] H. Bass. K-theory and stable algebra. *Inst. Hautes Études Sci. Publ. Math.*, (22):5–60, 1964.

[BB66] W. L. Baily, Jr. and A. Borel. Compactification of arithmetic quotients of bounded symmetric domains. *Ann. of Math. (2)*, 84:442–528, 1966.

[BGH20] Clark Barwick, Saul Glasman, and Peter Haine. Exodromy. Preprint: arXiv:1807.03281, 2020.

[BGT13] Andrew J. Blumberg, David Gepner, and Gonçalo Tabuada. A universal characterization of higher algebraic K-theory. *Geom. Topol.*, 17(2):733–838, 2013.

[BH19] Clark Barwick and Peter Haine. Pyknotic objects, i. basic notions. Preprint: arXiv:1904.09966, 2019.

[BJ06] Armand Borel and Lizhen Ji. *Compactifications of symmetric and locally symmetric spaces.* Mathematics: Theory & Applications. Birkhäuser Boston, Inc., Boston, MA, 2006.

[BMS67] H. Bass, J. Milnor, and J.-P. Serre. Solution of the congruence subgroup problem for SL_n $(n \geq 3)$ and Sp_{2n} $(n \geq 2)$. *Inst. Hautes Études Sci. Publ. Math.*, (33):59–137, 1967.

[Bor74] Armand Borel. Stable real cohomology of arithmetic groups. *Ann. Sci. École Norm. Sup. (4)*, 7:235–272 (1975), 1974.

[BS62] H. Bass and S. Schanuel. The homotopy theory of projective modules. *Bull. Amer. Math. Soc.*, 68:425–428, 1962.

[BS73] A. Borel and J.-P. Serre. Corners and arithmetic groups. *Comment. Math. Helv.*, 48:436–491, 1973. Avec un appendice: Arrondissement des variétés à coins, par A. Douady et L. Hérault.

[BS15] Bhargav Bhatt and Peter Scholze. The pro-étale topology for schemes. *Astérisque*, (369):99–201, 2015.

[CL83] Ruth Charney and Ronnie Lee. Cohomology of the Satake compactification. *Topology*, 22(4):389–423, 1983.

[CL84] Ruth Charney and Ronnie Lee. Moduli space of stable curves from a homotopy viewpoint. *J. Differential Geom.*, 20(1):185–235, 1984.

[EG08] Johannes Ebert and Jeffrey Giansiracusa. On the homotopy type of the Deligne-Mumford compactification. *Algebr. Geom. Topol.*, 8(4):2049–2062, 2008.

[FM94] Eric M. Friedlander and Barry Mazur. Filtrations on the homology of algebraic varieties. *Mem. Amer. Math. Soc.*, 110(529):x+110, 1994. With an appendix by Daniel Quillen.

[Fra98] Jens Franke. Harmonic analysis in weighted L_2-spaces. *Ann. Sci. École Norm. Sup. (4)*, 31(2):181–279, 1998.

[Ger71] S. M. Gersten. On the functor K_2. I. *J. Algebra*, 17:212–237, 1971.

[GHM94] M. Goresky, G. Harder, and R. MacPherson. Weighted cohomology. *Invent. Math.*, 116(1-3):139–213, 1994.

[GM83] Mark Goresky and Robert MacPherson. Intersection homology. II. *Invent. Math.*, 72(1):77–129, 1983.

[GM88] Mark Goresky and Robert MacPherson. Weighted cohomology of Satake compactifications. Centre de recherches mathématiques, preprint # 1593, 1988.

[GM92] Mark Goresky and Robert MacPherson. Lefschetz numbers of Hecke correspondences. In *The zeta functions of Picard modular surfaces*, pages 465–478. Univ. Montréal, Montreal, QC, 1992.

[GM03] Mark Goresky and Robert MacPherson. The topological trace formula. *J. Reine Angew. Math.*, 560:77–150, 2003.

[Gra76] Daniel Grayson. Higher algebraic K-theory. II (after Daniel Quillen). In *Algebraic K-theory (Proc. Conf., Northwestern Univ., Evanston, Ill., 1976)*, pages 217–240. Lecture Notes in Math., Vol. 551, 1976.

[Gro71] Alexander Grothendieck. Classes de faisceaux et théoréme de Riemann-Roch. mimeographed notes, princeton 1957. In *Théorie des intersections et théorème de Riemann-Roch (SGA 6)*, Lecture Notes in Mathematics, Vol. 225, pages 20–77. Springer-Verlag, Berlin-New York, 1971.

[Gro83] Alexander Grothendieck. Pursuing stacks. Unpublished manuscript, 1983.

[Hai19] Peter Haine. On the homotopy theory of stratified spaces, 2019. Preprint: arXiv:1811.01119.

[HW73] Allen Hatcher and John Wagoner. *Pseudo-isotopies of compact manifolds.* Société Mathématique de France, Paris, 1973. Astérisque, No. 6.

[JM02] L. Ji and R. MacPherson. Geometry of compactifications of locally symmetric spaces. *Ann. Inst. Fourier (Grenoble)*, 52(2):457–559, 2002.

[Keu71] Frans Keune. Algèbre homotopique et K-théorie algébrique. *C. R. Acad. Sci. Paris Sér. A-B*, 273:A592–A595, 1971.

[Kur92] Masato Kurihara. Some remarks on conjectures about cyclotomic fields and K-groups of $\mathbf{Z}$. *Compositio Math.*, 81(2):223–236, 1992.

[KV69] Max Karoubi and Orlando Villamayor. Foncteurs K^n en algèbre et en topologie. *C. R. Acad. Sci. Paris Sér. A-B*, 269:A416–A419, 1969.

[Lic73] Stephen Lichtenbaum. Values of zeta-functions, étale cohomology, and algebraic K-theory. In *Algebraic K-theory, II: "Classical" algebraic K-theory and connections with arithmetic (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 489–501. Lecture Notes in Math., Vol. 342, 1973.

[LS76] Ronnie Lee and R. H. Szczarba. The group $K_3(Z)$ is cyclic of order forty-eight. *Ann. of Math. (2)*, 104(1):31–60, 1976.

[LS18] Anssi Lahtinen and David Sprehn. Modular characteristic classes for representations over finite fields. *Adv. Math.*, 323:1–37, 2018.

[Lur17] Jacob Lurie. Higher algebra. www.math.ias.edu/ lurie/papers/HA.pdf, 2017.

[Mat69] Hideya Matsumoto. Sur les sous-groupes arithmétiques des groupes semi-simples déployés. *Ann. Sci. École Norm. Sup. (4)*, 2:1–62, 1969.

[Mat12] John Mather. Notes on topological stability. *Bull. Amer. Math. Soc. (N.S.)*, 49(4):475–506, 2012.

[Maz63] Barry Mazur. Relative neighborhoods and the theorems of Smale. *Ann. of Math. (2)*, 77:232–249, 1963.

[Mer81] A. S. Merkurjev. On the norm residue symbol of degree 2. *Dokl. Akad. Nauk SSSR*, 261(3):542–547, 1981.

[Mil61] John Milnor. Two complexes which are homeomorphic but combinatorially distinct. *Ann. of Math. (2)*, 74:575–590, 1961.

[Mil71] John Milnor. *Introduction to algebraic K-theory*. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1971. Annals of Mathematics Studies, No. 72.

[Mil70] John Milnor. Algebraic K-theory and quadratic forms. *Invent. Math.*, 9:318–344, 1969/70.

[MP87] R. James Milgram and Stewart B. Priddy. Invariant theory and $H^*(\mathrm{GL}_n(\mathbf{F}_p); \mathbf{F}_p)$. In *Proceedings of the Northwestern conference on cohomology of groups (Evanston, Ill., 1985)*, volume 44, pages 291–302, 1987.

[MS82] A. S. Merkurjev and A. A. Suslin. K-cohomology of Severi-Brauer varieties and the norm residue homomorphism. *Izv. Akad. Nauk SSSR Ser. Mat.*, 46(5):1011–1046, 1135–1136, 1982.

[Nai99] Arvind Nair. Weighted cohomology of arithmetic groups. *Ann. of Math. (2)*, 150(1):1–31, 1999.

[NS89] Yu. P. Nesterenko and A. A. Suslin. Homology of the general linear group over a local ring, and Milnor's K-theory. *Izv. Akad. Nauk SSSR Ser. Mat.*, 53(1):121–146, 1989.

[NV68] A. Nobile and O. E. Villamayor. Sur la K-théorie algébrique. In *Comptes Rendus du K-Colloque (Montpellier, 1968), Vol. II*, pages 44–94. Univ. of Montpellier, Montpellier, 1968.

[Qui67] Daniel G. Quillen. *Homotopical algebra*. Lecture Notes in Mathematics, No. 43. Springer-Verlag, Berlin-New York, 1967.

[Qui71] Daniel Quillen. Cohomology of groups. In *Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2*, pages 47–51. 1971.

[Qui72] Daniel Quillen. On the cohomology and K-theory of the general linear groups over a finite field. *Ann. of Math. (2)*, 96:552–586, 1972.

[Qui73a] Daniel Quillen. Finite generation of the groups K_i of rings of algebraic integers. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 179–198. Lecture Notes in Math., Vol. 341, 1973.

[Qui73b] Daniel Quillen. Higher algebraic K-theory. I. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 85–147. Lecture Notes in Math., Vol. 341, 1973.

[Qui75] Daniel Quillen. Higher algebraic K-theory. In *Proceedings of the International Congress of Mathematicians (Vancouver, B. C., 1974), Vol. 1*, pages 171–176, 1975.

[Qui88] Frank Quinn. Homotopically stratified sets. *J. Amer. Math. Soc.*, 1(2):441–499, 1988.

[Rap86] Michael Rapoport. Letter to Borel. 1986.

[Sap05a] Leslie Saper. $\mathcal{L}$-modules and micro-support. Preprint: arXiv:0112251, 2005. To appear in Annals of Mathematics.

[Sap05b] Leslie Saper. $\mathcal{L}$-modules and the conjecture of Rapoport and Goresky-MacPherson. Number 298, pages 319–334. 2005. Automorphic forms. I.

[Sat60] Ichir Satake. On compactifications of the quotient spaces for arithmetically defined discontinuous groups. *Ann. of Math. (2)*, 72:555–580, 1960.

[Sch19] Peter Scholze. Lectures on condensed mathematics. www.math.uni-bonn.de/people/scholze/Condensed.pdf, 2019.

[Seg74] Graeme Segal. Categories and cohomology theories. *Topology*, 13:293–312, 1974.

[Sta65] John R. Stallings. On infinite processes leading to differentiability in the complement of a point. In *Differential and Combinatorial Topology (A Symposium in Honor of Marston Morse)*, pages 245–254. Princeton Univ. Press, Princeton, N.J., 1965.

[Sus81] A. A. Suslin. On the equivalence of K-theories. *Comm. Algebra*, 9(15):1559–1566, 1981.

[Swa70] Richard G. Swan. Nonabelian homological algebra and K-theory. In *Applications of Categorical Algebra (Proc. Sympos. Pure Math., Vol. XVII, New York, 1968)*, pages 88–123. Amer. Math. Soc., Providence, R.I., 1970.

[Tat71] John Tate. Symbols in arithmetic. In *Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1*, pages 201–211. 1971.

[Tat76] John Tate. Relations between K_2 and Galois cohomology. *Invent. Math.*, 36:257–274, 1976.

[Tho85] R. W. Thomason. Algebraic K-theory and étale cohomology. *Ann. Sci. École Norm. Sup. (4)*, 18(3):437–552, 1985.

[Tre09] David Treumann. Exit paths and constructible stacks. *Compos. Math.*, 145(6):1504–1532, 2009.

[Van46] H. S. Vandiver. Fermat's last theorem. Its history and the nature of the known results concerning it. *Amer. Math. Monthly*, 53:555–578, 1946.

[Voe11] Vladimir Voevodsky. On motivic cohomology with $\mathbb{Z}/l$-coefficients. *Ann. of Math. (2)*, 174(1):401–438, 2011.

[Vol71] I. A. Volodin. Algebraic K-theory as an extraordinary homology theory on the category of associative rings with a unit. *Izv. Akad. Nauk SSSR Ser. Mat.*, 35:844–873, 1971.

[Wag73] J. Wagoner. Buildings, stratifications, and higher K-theory. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 148–165. Lecture Notes in Math., Vol. 341, 1973.

[Wal65] C. T. C. Wall. Finiteness conditions for CW-complexes. *Ann. of Math. (2)*, 81:56–69, 1965.

[Wal85] Friedhelm Waldhausen. Algebraic K-theory of spaces. In *Algebraic and geometric topology (New Brunswick, N.J., 1983)*, volume 1126 of *Lecture Notes in Math.*, pages 318–419. Springer, Berlin, 1985.

[Wei99] Charles A. Weibel. The development of algebraic K-theory before 1980. In *Algebra, K-theory, groups, and education (New York, 1997)*, volume 243 of *Contemp. Math.*, pages 211–238. Amer. Math. Soc., Providence, RI, 1999.

[Whi50] J. H. C. Whitehead. Simple homotopy types. *Amer. J. Math.*, 72:1–57, 1950.

[Whi65] Hassler Whitney. Tangents to an analytic variety. *Ann. of Math. (2)*, 81:496–549, 1965.

[Zuc86] Steven Zucker. L_2-cohomology and intersection homology of locally symmetric varieties. II. *Compositio Math.*, 59(3):339–398, 1986.

[Zuc01] Steven Zucker. On the reductive Borel-Serre compactification: L^p-cohomology of arithmetic groups (for large p). *Amer. J. Math.*, 123(5):951–984, 2001.

[Zuc04] Steven Zucker. On the reductive Borel-Serre compactification. III. Mixed Hodge structures. *Asian J. Math.*, 8(4):881–911, 2004.

[Zuc83] Steven Zucker. L_2-cohomology of warped products and arithmetic groups. *Invent. Math.*, 70(2):169–218, 1982/83.

Part II

Papers

PAPER I

The stratified homotopy type of the reductive Borel–Serre compactification

This chapter contains the preprint version of the following paper:

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THE STRATIFIED HOMOTOPY TYPE OF THE REDUCTIVE BOREL–SERRE COMPACTIFICATION

MIKALA ØRSNES JANSEN

Abstract. We identify the exit path ∞-category of the reductive Borel–Serre compactification as the nerve of a 1-category defined purely in terms of rational parabolic subgroups and their unipotent radicals. As an immediate consequence, we identify the fundamental group of the reductive Borel–Serre compactification, recovering a result of Ji–Murty–Saper–Scherk, and we obtain a combinatorial incarnation of constructible complexes of sheaves on the reductive Borel–Serre compactification as elements in a derived functor category.

Contents

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1. Introduction

Background. Let $\mathbf{G}$ be a connected reductive linear algebraic group defined over $\mathbb{Q}$ whose centre is anisotropic over $\mathbb{Q}$. Let $\Gamma \leq \mathbf{G}(\mathbb{Q})$ be a neat arithmetic group, and consider the symmetric space X of maximal compact subgroups of $\mathbf{G}(\mathbb{R})$ on which Γ acts by conjugation. The locally symmetric space $\Gamma\backslash X$ is a model for the classifying space of Γ and a smooth manifold; it is compact if and only if the $\mathbb{Q}$-rank of $\mathbf{G}$ is zero. The problem of compactifying such locally symmetric spaces has given rise to a number of different compactifications well suited for different purposes. In this paper we study the Borel–Serre and reductive Borel–Serre compactifications.

The *Borel–Serre compactification* $\Gamma\backslash\overline{X}^{BS}$, introduced in 1973 by Borel and Serre, is a compactification of $\Gamma\backslash X$ with the same homotopy type ([BS73]). This construction enabled Borel to calculate the rank of the K-groups $K_i(O_K)$, where O_K is the ring of intergers in a number field K ([Bor74]). Quillen also used the Borel–Serre compactification to show that these same K-groups, $K_i(O_K)$, are finitely generated ([Qui73a]).

The *reductive Borel–Serre compactification* $\Gamma\backslash\overline{X}^{RBS}$ was introduced by Zucker in 1982 to facilitate the study of L^2-cohomology of $\Gamma\backslash X$ ([Zuc83], see also [GHM94]). It is a quotient of the Borel–Serre compactification, $\Gamma\backslash\overline{X}^{BS} \to \Gamma\backslash\overline{X}^{RBS}$, and it remedies the failure of $\Gamma\backslash\overline{X}^{BS}$ to support L^2-partitions of unity. The reductive Borel–Serre compactification has been studied extensively and has come to play a central role in the theory of compactifications. It is well-suited for studying the L^p-cohomology of $\Gamma\backslash X$ ([Zuc01]), it dominates all Satake compactifications ([BJ06, III.15.2]), and it plays an important role in parametrising the continuous spectrum of $\Gamma\backslash X$ ([JM02]). It is used to define weighted cohomology ([GHM94]) which is the main ingredient in the topological trace formula ([GM92, GM03]) exploiting the fact that Hecke operators extend to the reductive Borel–Serre compactification. It has moreover motivated the theory of $\mathscr{L}$-modules ([Sap05a, Sap05b]) which is used to prove a conjecture of Rapoport ([Rap86]) and Goresky–MacPherson ([GM88]) relating the intersection cohomology of the reductive Borel–Serre compactification with that of certain Satake compactifications.

We will study these spaces as stratified topological spaces and determine their stratified homotopy type, or more precisely their exit path ∞-categories. The Borel–Serre compactification $\Gamma\backslash\overline{X}^{BS}$ is naturally stratified as a manifold with corners over the poset of Γ-conjugacy classes of rational parabolic subgroups of $\mathbf{G}$. This stratification descends along the quotient map $\Gamma\backslash\overline{X}^{BS} \to \Gamma\backslash\overline{X}^{RBS}$, equipping the reductive Borel–Serre compactification with a natural stratification.

For a sufficiently nice stratified space X, one can define its *exit path ∞-category* $\Pi_\infty^{\mathrm{exit}}(X)$, an analogue of the fundamental ∞-groupoid for topological spaces. Intuitively, the exit path ∞-category has as objects the points of the stratified space and as morphisms the paths which can only move "upwards" in the stratification, i.e. if $X_i \subset \overline{X_j}$ for two distinct strata $X_i, X_j \subset X$, then a path can move from X_i to X_j, but not the other way. The higher simplices are stratum preserving homotopies between such paths.

The most important feature of the exit path ∞-category is that it classifies constructible sheaves, that is, sheaves which are locally constant on each stratum. This generalises the classical result that for a sufficiently nice topological space X, the monodromy functor gives an equivalence between representations of the fundamental groupoid and locally constant sheaves on X. It was observed by MacPherson that for stratified spaces, one can define an exit path category which in the same way classifies constructible sheaves. Treumann gave a 2-categorical version of this result ([Tre09]), and Lurie developed the ∞-categorical setting, defining the exit path ∞-category and generalising MacPherson's observation ([Lur17, Theorem A.9.3]).

Main results. Let $\mathbf{G}$, Γ and X be as above, and assume that $\mathbf{G}$ has positive $\mathbb{Q}$-rank, so that $\Gamma\backslash X$ is non-compact. Let $\mathscr{P}$ denote the poset of rational parabolic subgroups of $\mathbf{G}$. For all $\mathbf{P} \in \mathscr{P}$, let $\mathbf{N_P} \leq \mathbf{P}$ denote the unipotent radical of $\mathbf{P}$ and write $\Gamma_{\mathbf{N_P}} = \Gamma \cap \mathbf{N_P}(\mathbb{Q})$.

Our main theorem is the following.

Theorem 4.3. *The exit path ∞-category of the reductive Borel–Serre compactification $\Gamma\backslash\overline{X}^{RBS}$ is canonically equivalent to the nerve of its homotopy category. This in turn is equivalent to the category $\mathscr{C}_\Gamma^{RBS}$ with objects the rational parabolic subgroups of $\mathbf{G}$ and hom-sets*

$$\mathscr{C}_\Gamma^{RBS}(\mathbf{P},\mathbf{Q}) = \{\gamma \in \Gamma \mid \gamma\mathbf{P}\gamma^{-1} \leq \mathbf{Q}\}/\Gamma_{\mathbf{N_P}}, \quad \textit{for all} \quad \mathbf{P},\mathbf{Q} \in \mathscr{P},$$

where $\Gamma_{\mathbf{N_P}}$ acts by right multiplication, and composition is given by multiplication of representatives.

The two important things to note here, is that the exit path ∞-category is equivalent to a 1-category, and that the definition of this 1-category makes no reference to the space $\Gamma\backslash\overline{X}^{RBS}$, but is defined purely in terms of the poset of rational parabolic subgroups, their unipotent radicals and the conjugation action of Γ on this poset. As an intermediate step towards this identification, we identify the exit path ∞-categories of the partial Borel–Serre compactification $\overline{X}^{BS}$ of X and the Borel–Serre compactification $\Gamma\backslash\overline{X}^{BS}$ of $\Gamma\backslash X$. We also show that the equivalences can be chosen to be compatible with the quotient maps $\overline{X}^{BS} \to \Gamma\backslash\overline{X}^{BS} \to \Gamma\backslash\overline{X}^{RBS}$.

We have the following corollaries of the main theorem.

Corollary 5.1. *The reductive Borel–Serre compactification $\Gamma\backslash\overline{X}^{RBS}$ is weakly homotopy equivalent to the geometric realisation of $\mathscr{C}_\Gamma^{RBS}$.*

This immediately recovers the following result of Ji–Murty–Saper–Scherk ([JMSS15, Corollary 5.3]).

Corollary 5.2. *The fundamental group of the reductive Borel–Serre compactification* $\Gamma\backslash\overline{X}^{RBS}$ *is isomorphic to the group* Γ/E_Γ*, where* $E_\Gamma \lhd \Gamma$ *is the normal subgroup generated by the subgroups* $\Gamma_{\mathbf{N_P}} \leq \Gamma$ *as* $\mathbf{P}$ *runs through all rational parabolic subgroups of* $\mathbf{G}$*.*

For a stratified space X and an associative ring R, let $D(\mathrm{Shv}_1(X,R))$ denote the classical derived category of sheaves on X with values in left R-modules, and let LMod_R^1 denote the category of left R-modules. Let $D_{\mathrm{cbl}}(\mathrm{Shv}_1(X,R))$ denote the full subcategory spanned by the complexes whose homology is constructible, and let $D_{\mathrm{cbl,cpt}}(\mathrm{Shv}_1(X,R))$ denote the full subcategory spanned by the complexes whose homology is constructible *and* whose stalk complexes are perfect chain complexes. As another corollary, we get the following expression of these derived categories of sheaves as derived functor categories.

Corollary 5.6. *Let* R *be an associative ring. There is an equivalence of categories*

$$D_{\mathrm{cbl}}(\mathrm{Shv}_1(\Gamma\backslash\overline{X}^{RBS},R)) \simeq D(\mathrm{Fun}(\mathscr{C}_\Gamma^{RBS},\mathrm{LMod}_R^1))$$

which restricts to an equivalence

$$D_{\mathrm{cbl,cpt}}(\mathrm{Shv}_1(\Gamma\backslash\overline{X}^{RBS},R)) \simeq D_{\mathrm{cpt}}(\mathrm{Fun}(\mathscr{C}_\Gamma^{RBS},\mathrm{LMod}_R^1)),$$

where $D_{\mathrm{cpt}}(\mathrm{Fun}(\mathscr{C}_\Gamma^{RBS},\mathrm{LMod}_R^1)) \subset D(\mathrm{Fun}(\mathscr{C}_\Gamma^{RBS},\mathrm{LMod}_R^1))$ *is the full subcategory spanned by the complexes of functors* $F_\bullet$ *such that the complex* $F_\bullet(x)$ *is perfect for all* $x \in X$*.*

This can be interpreted as a combinatorial incarnation of constructible complexes of sheaves on the reductive Borel–Serre compactification. In fact, we get an ∞-categorical result (see Proposition 5.5), but we state the 1-categorical consequence here as this speaks of the more classical constructible derived category.

The definition of the category $\mathscr{C}_\Gamma^{RBS}$ generalises to a purely algebraic setting of a group acting on a poset. We make this precise in Section 6. For finite groups with a split BN-pair of characteristic p, we recover (the opposite of) the orbit category on the collection of p-radical subgroups, an object that has been studied extensively in finite group theory ([Alp87, AF90, Bou84, JMO92a, JMO92b, Gro02, Gro18], see also Section 6.2).

Calculational tools. In order to identify the exit path ∞-category of the reductive Borel–Serre compactification, we develop some calculation tools. First of all we follow ideas of Woolf ([Woo09]) and identify the mapping spaces in the exit path ∞-category as the fibres of certain fibrations, namely the end point evaluation fibrations out of the so-called *homotopy link*, which comes from the theory of homotopically stratified sets developed by Quinn ([Qui88]). With a little extra data, the resulting long exact sequences of homotopy groups enable us to identify the homotopy types of the mapping spaces. In particular, we can use this to determine whether the mapping spaces have contractible components, implying that the exit path ∞-category is canonically equivalent to the nerve of its homotopy category.

We go on to study group actions on stratified spaces and to identify the exit path ∞-categories of the quotient stratified spaces in particularly nice cases. This is done in Theorems 3.9 and 3.14 and we believe that these results should be applicable to a larger class of interesting stratified spaces. These results should be compared with a result of Chen–Looijenga ([CL15, Theorem 1.7]): we rephrase and slightly strengthen their result in certain situations, and the conditions that need to be satisfied for our result to apply are a local version of their conditions (see also Remark 3.20). It should be stressed, however, that the settings differ a great deal and that, apart from allowing local conditions and local data, ours is the more restrictive setting since we do not deal with the cases where the quotient stratified space is an orbispace.

Further work. In joint work with Dustin Clausen, we investigate how generalisations of the categories $\mathscr{C}_\Gamma^{RBS}$ model unstable algebraic K-theory ([CØJ20]). For an associative ring A and any finitely generated projective A-module M, we introduce a category $\mathrm{RBS}(M)$ which is defined purely in terms of linear algebra internal to M and which naturally generalises $\mathscr{C}_\Gamma^{RBS}$. We show that if A is a finite field or a ring with many units, e.g. a local ring with infinite residue field, then these categories model unstable algebraic K-theory. For rings with many units, $|\mathrm{RBS}(M)|$ recovers the plus-construction of $B\mathrm{GL}(M)$. In the case of finite fields, however, this model is in a certain sense better than the one given by the plus-construction as it gets rid of the complicated unstable $\mathbb{F}_p$-homology of $\mathrm{GL}_n(k)$ for k a finite field of characteristic p. We also show that these categories in a natural way stabilise to provide a model for the algebraic K-theory space $K(A)$.

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Notation and conventions. By *∞-category* we mean quasicategory, that is, a simplicial set satisfying the extension property for all inner horn inclusions. We refer to [Lur09] for details.

By the *homotopy category* of an ∞-category $\mathscr{C}$, we mean the 1-category $h\mathscr{C}$ with objects the 0-simplices of $\mathscr{C}$ and morphisms the 1-simplices subject to certain relations given by the 2-simplices (see [Lur09, §1.2.3]). We say that an ∞-category $\mathscr{C}$ is *canonically equivalent* to (the nerve of) its homotopy category if the canonical map $\mathscr{C} \to N(h\mathscr{C})$ is an equivalence.

Given a (1)-category $\mathscr{C}$, we define its *geometric realisation* $|\mathscr{C}|$ as the geometric realisation of its nerve $N_\bullet\mathscr{C}$. We reserve the term *classifying space* for groups and monoids, i.e. the classifying space BG of a group G is the geometric realisation of the one object category with morphisms the elements of G.

For a group G, an element $g \in G$ and a subgroup $H \leq G$, we write ${}^gH = gHg^{-1}$ and $H^g = g^{-1}Hg$ for the conjugated subgroups. And similarly for algebraic groups and subgroups.

For a set A and a subset $B \subset A$, we denote the set difference by $A - B$.

We write $[n] = \{0 < 1 < \cdots < n\}$ for the linearly ordered poset with $n+1$ elements.

2. Stratified homotopy theory

We recall the definitions of conically stratified (poset-stratified) spaces and exit path ∞-categories following Lurie ([Lur17, Appendix A]). We also recall the homotopy link from the theory of homotopically stratified sets introduced by Quinn ([Qui88]) and two important properties of this object, as this will be essential for developing our calculational tools. We state the main property of the exit path ∞-category, namely the fact that constructible sheaves are equivalent to representations of it, and we show that this result can be extended to the constructible derived category.

2.1. **Poset-stratified spaces.**

In the following, all posets are topologised by the Alexandroff topology, i.e. the open sets are the upwards closed sets.

Definition 2.1. A *poset-stratified space* (or simply *stratified space*) is a continuous map $s\colon X \to I$, where X is a topological space and I a poset. The poset I is called the *poset of strata* and the subspace $X_i = s^{-1}(i)$ is called the i'th *stratum*. A *stratum preserving* (or *stratified*) map from $s\colon X \to I$ to $r\colon Y \to J$ is a pair of continuous maps $(f\colon X \to Y, \theta\colon I \to J)$ such that $r \circ f = \theta \circ s$. ◁

Remark 2.2. When no confusion can occur, we omit the poset of strata and refer to a stratified space $s\colon X \to I$ simply by X. If we want to stress that X is stratified over the poset I, then we say that X is an I-stratified space. Similarly, when considering a stratum preserving map (f, θ), we may omit the order-preserving map θ.

The strata X_i, $i \in I$, define a partition of X, and continuity of s is equivalent to requiring the upward unions of strata to be open in X, i.e. $\bigcup_{j \geq i} X_j \subset X$ is open for all $i \in I$. The closure relations in the partition translate to poset relations in I: if $X_i \subseteq \overline{X_j}$, then $i \leq j$. It will often be the case in naturally occurring examples that $X_i \subseteq \overline{X_j}$ if and only if $i \leq j$. ○

Definition 2.3. Let $s\colon Y \to I$ be a stratified space. The *(open) cone* on Y is the stratified space $s^{\triangleleft}\colon C(Y) \to I^{\triangleleft}$ defined as follows: the poset of strata is $I^{\triangleleft} := I \cup \{-\infty\}$ with $-\infty \leq i$ for all $i \in I$; as a set $C(Y) = (Y \times (0,1)) \coprod *$, and $U \subseteq C(Y)$ is open, if and only if

(i) $U \cap (Y \times (0,1))$ is open,
(ii) $* \in U$ implies $Y \times (0, \varepsilon) \subseteq U$ for some $\varepsilon > 0$.

The stratification map $s^{\triangleleft}$ is given by $s^{\triangleleft}(x,t) = s(x)$ and $s^{\triangleleft}(*) = -\infty$. ◁

Remark 2.4. The topology of $C(Y)$ above coincides with the teardrop topology on the open cone of Y (see for example [HTWW00]). If Y is compact Hausdorff, then $C(Y)$ is homeomorphic to the pushout $(Y \times [0,1)) \coprod_{Y \times \{0\}} *$.

Note that if Y is metrisable, then by [HTWW00, Lemma 3.15], so is the stratified cone $C(Y)$. ○

Definition 2.5. An I-stratified space X is *conically stratified at* $x \in X_i \subseteq X$, if there exists

(i) a topological space V,
(ii) an $I_{>i}$-stratified space L,

(iii) and a stratified homeomorphism $(\varphi, \theta)\colon V \times C(L) \xrightarrow{\cong} U$ onto a neighbourhood U of x in X, where $\theta\colon (I_{>i})^{\triangleleft} \to I$ is the canonical identification of $(I_{>i})^{\triangleleft}$ with $I_{\geq i} \subseteq I$.

We say that X is *metrisably conically stratified* at x, if there exists a conical neighbourhood as above such that the union $V \cup X_j$ is metrisable for all $j > i$.

A stratified space X is *conically stratified* if it is conically stratified at all points, and it is *metrisably conically stratified* if it is metrisably conically stratified at all points. ◁

Remark 2.6. We will in most situations say conically stratified space and only write conically stratified I-stratified space if we want to stress the poset I.

We will often write that a point $x \in X_i$ admits a conical neighbourhood $\varphi\colon V \times C(L) \xrightarrow{\cong} U$, in which case we implicitly assume that L is an $I_{>i}$-stratified space, $V = U \cap X_i$ and φ is a stratified homeomorphism identifying $I^{\triangleleft}_{>i}$ with $I_{\geq i}$.

We call L a *link space* of x in X. We cannot in general speak of *the* link space of x, although in many cases it will be well-defined up to some sort of equivalence. If a stratified space X is equipped with link bundles (in the sense of [GHM94]), then X is conically stratified and the link spaces of any two points $x, x' \in X_i$ are stratified homeomorphic. If X is homotopically stratified (in the sense of [Qui88]), then X is conically stratified and the link spaces of any two points $x, x' \in X_i$ are homotopy equivalent. ○

Example 2.7. Let M be a smooth manifold with corners of dimension n, i.e. a space modelled smoothly upon open subsets of a quadrant in $\mathbb{R}^n$. A point $x \in M$ has *index* j, if there is a chart (U, φ) on M, such that $\varphi(x)$ has exactly j coordinates equal to zero. Let $M_j \subseteq M$ denote the subspace consisting of points of index j; it is a smooth manifold of dimension $n - j$. The standard stratification of M as a manifold with corners is by the path components of the M_j, $j = 0, \ldots, n$. Let $N \subseteq M_j \subseteq M$ be a stratum, that is a path component, and let $x \in N$. N is of codimension j in M, and there is a conical neighbourhood $x \in U \cong V \times C(\Delta^{j-1})$, where Δ^{j-1} is the standard $(j-1)$-simplex stratified as a manifold with corners.

Manifolds with corners have more rigourous structure, namely mapping cylinder neighbourhoods of each stratum, not just conical neighbourhoods of points — this is the case for many naturally occurring stratified spaces, but we only need to local data, so we refrain from going into this. ○

Definition 2.8. The *standard stratified n-simplex* is the standard n-simplex

$$\Delta^n = \{(t_0, \ldots, t_n) \in [0,1]^n \mid \textstyle\sum t_i = 1\}$$

stratified by the map $s_n\colon \Delta^n \to [n]$ defined by

$$s_n(t_0, \ldots, t_k, 0, \ldots, 0) = k \quad \text{if } t_k \neq 0.$$

In other words, s_n maps the subspace $\Delta^{0,\ldots,k} - \Delta^{0,\ldots,k-1} \subset \Delta^n$ to k, where $\Delta^{0,\ldots,j}$ denotes the face spanned by the vertices $0, 1, \ldots, j$. ◁

Remark 2.9. Note that the standard stratified simplex is not stratified as a manifold with corners, but rather in a way that retains the combinatorial information. It can be identified with the $(n+1)$-fold stratified closed mapping cone of a point, where the closed mapping cone is the stratified space obtained by replacing $(0,1)$ by $(0,1]$ in Definition 2.3. ○

2.2. **Homotopy links.** It will be convenient for us to use a homotopical version of link spaces, namely the homotopy link defined by Quinn in his study of homotopically stratified sets ([Qui88]).

Definition 2.10. Let X be a topological space and $Y \subseteq X$ a subspace. The *homotopy link* of the pair (X,Y) is the subspace

$$H(X,Y) = \{\gamma\colon [0,1] \to X \mid \gamma(0) \in Y,\ \gamma((0,1]) \subseteq X - Y\} \subseteq C([0,1],X)$$

of the path space of X equipped with the compact-open topology. ◁

Let X and $Y \subseteq X$ a closed subspace. If we stratify X over $\{0 < 1\}$ by sending Y to 0 and $X - Y$ to 1, then the points of $H(X,Y)$ can be identified with the stratum preserving maps $\sigma\colon \Delta^1 \to X$ starting in Y and ending in $X - Y$.

We need two fundamental facts about the homotopy link which hold when the pair (X,Y) is sufficiently nice:

(i) the end point evaluation map
$$e\colon H(X,Y) \to Y \times X, \quad \gamma \mapsto (\gamma(0), \gamma(1)),$$
is a fibration (see Corollary A.3 and Proposition 2.11),

(ii) the homotopy link serves as a homotopical replacement for link spaces or more generally for neighbourhoods admitting a nearly strict deformation retraction (see Proposition A.8 and Proposition 2.12).

These results are well-known, but we have been unable to locate a source which does not work in a much more general or slightly different setting so for the sake of self-containment, we have chosen to include the proofs in this fairly elementary point-set topological setting in Appendix A. These results explain our need to work with *metrisably* conically stratified spaces.

The following is a direct consequence of Corollary A.3, since the evaluation at zero map $e_0\colon H(Y \times C(Z), Y \times \{*\}) \to Y$ is a fibration for any topological spaces Y and Z.

Proposition 2.11. *Let X be an I-stratified space. Let $i \in I$, $x \in X_i$ and $j > i$. Suppose x has a conical neighbourhood U and set $V = U \cap X_i$. If the union $V \cup X_j$ is metrisable, then the end point evaluation map $e\colon H(X_j \cup V, V) \to V \times X_j$ is a fibration.*

Let X be an I-stratified space and suppose $x \in X_i$ has a conical neighbourhood

$$\varphi\colon V \times C(L) \xrightarrow{\cong} U.$$

Then the map

$$U \times [0,1] \to U, \quad (\varphi(v,[l,s]),t) \mapsto \varphi(v,[l,st])$$

is a (stratum preserving) nearly strict deformation retraction into $V = U \cap X_i$ (Definition A.4). For all $i < j$ and any fixed $\varepsilon \in (0,1)$, we have a map

$$\Psi^j_{\varphi,\varepsilon}\colon V \times L_j \to H(X_j \cup V, V), \quad (v,l) \mapsto \gamma_{v,l,\varepsilon},$$
$$\text{where} \quad \gamma_{v,l,\varepsilon}\colon [0,1] \to X_j \cup V, \quad t \mapsto \varphi(v,[l,t\varepsilon]).$$

That is, $\gamma_{v,l,\varepsilon}$ is the path tracing the cone coordinate from the apex to ε for fixed $v \in V$ and $l \in L$ in the other coordinates. The following is a direct application of Proposition A.8, since the map $\Psi^j_{\varphi,\varepsilon}$ is the composition of the following three maps

(i) the inclusion at ε, $V \times L_j \to V \times L_j \times (0,1)$, $(v,l) \mapsto (v,l,\varepsilon)$,
(ii) the homeomorphism $V \times L_j \times (0,1) \xrightarrow{\cong} U \cap X_j$ given by φ,
(iii) and the map $U \cap X_j \to H(X_j \cup V, V)$ of Proposition A.8.

Proposition 2.12. *Let X be an I-stratified space. Let $i \in I$, $x \in X_i$ and $j > i$. Suppose x has a conical neighbourhood $\varphi\colon V \times C(L) \xrightarrow{\cong} U$. If the union $V \cup X_j$ is metrisable, then for any choice of $\varepsilon \in (0,1)$, the map $\Psi^j_{\varphi,\varepsilon}\colon V \times L_j \to H(X_j \cup V, V)$ defined above is a homotopy equivalence. In particular, if V is weakly contractible, then the map $L_j \to H(X_j \cup V, V)$, $l \mapsto \gamma_{x,l,\varepsilon}$, is a weak homotopy equivalence.*

2.3. **Exit path ∞-categories and constructible sheaves.** Following the work of Lurie, we recall the definitions of the exit path ∞-category and of constructible sheaves, and we state the classification of constructible sheaves as representations of the exit path ∞-category ([Lur17, Appendix A]).

In the following $s_n\colon \Delta^n \to [n]$ will denote the standard stratified n-simplex as defined in Definition 2.8, and $\mathrm{Sing}(X)$ denotes the singular set of a topological space.

Definition 2.13. Let $s\colon X \to I$ be a conically stratified space. The *exit path ∞-category* of $X \to I$ is the subsimplicial set $\Pi^{\mathrm{exit}}_\infty(X,I) \subset \mathrm{Sing}(X)$ whose n-simplices are the maps $\sigma\colon \Delta^n \to X$ for which there is an order preserving map $\theta\colon [n] \to I$ such that $s \circ \sigma = \theta \circ s_n$.

Remark 2.14. If θ and θ' satisfy $\theta \circ s_n = \theta' \circ s_n$, then $\theta = \theta'$, so we can also define the exit path ∞-category as the simplicial set with n-simplices the stratum preserving maps $\sigma\colon \Delta^n \to X$.

From now on, we write $\Pi^{\mathrm{exit}}_\infty(X) = \Pi^{\mathrm{exit}}_\infty(X,I)$, letting the poset I be implicit in the notation.∘

The following theorem justifies the name and notation.

Theorem 2.15 ([Lur17, Theorem A.6.4])**.** *For a conically stratified space X, the simplicial set $\Pi^{\mathrm{exit}}_\infty(X)$ is an ∞-category.*

Thus we have a functor $\Pi^{\mathrm{exit}}_\infty\colon \mathrm{Strat} \to \mathrm{Cat}_\infty$ (of 1-categories).

Definition 2.16. We define the *exit path 1-category* of a conically stratified space X as the homotopy category of the exit path ∞-category of X and we denote it by $\Pi^{\mathrm{exit}}_1(X)$. ◁

Remark 2.17. Note that we are not taking the enriched homotopy category, but just the underlying 1-category; we deal with the higher homotopy in the mapping spaces of $\Pi_\infty^{\mathrm{exit}}(X)$ separately. ○

The following remark should provide some intuition for the exit path ∞-category.

Remark 2.18. Let X be a conically I-stratified space.

* The 0-simplices of $\Pi_\infty^{\mathrm{exit}}(X)$ are the points of X.
* Identifying $\Delta^1 \cong [0,1]$, the 1-simplices of $\Pi_\infty^{\mathrm{exit}}(X)$ are the paths $\sigma\colon [0,1] \to X$ which satisfy $\sigma(0) \in X_i$ and $\sigma((0,1]) \subseteq X_j$ for some $i \leq j$ in I. In other words, the exit paths either stay within one stratum or leave the deeper stratum instantaneously entering the stratum containing the end point. We see that the homotopy link $H(X_i \cup X_j, X_i)$ of X_i in $X_i \cup X_j$ is a subset of the 1-simplices of $\Pi_\infty^{\mathrm{exit}}(X)$.
* The morphisms in $\Pi_1^{\mathrm{exit}}(X)$ are represented by 1-simplices of $\Pi_\infty^{\mathrm{exit}}(X)$ as described above, but composition is hard to describe concretely. Intuitively, however, we can think of the composite of two such paths in $\Pi_1^{\mathrm{exit}}(X)$ as the concatenation.
* For all $i \in I$, $\Pi_\infty^{\mathrm{exit}}(X)$ contains the fundamental ∞-groupoid $\mathrm{Sing}(X_i)$ as the full subcategory spanned by the points of X_i. ○

The most important feature of the exit path ∞-category is that for sufficiently well-behaved stratified spaces, it classifies constructible sheaves. We state this for sheaves with values in any compactly generated ∞-category. Lurie proves it for space-valued sheaves, but the generalisation is well-known and quite elementary to prove. However, since we have been unable to locate a proof in the literature, we have included a detailed proof in the appendix, also in the hope that it makes these results more accessible to a reader without a background in ∞-categories. We refer to [Lur17, Section A.5] and Appendix B for proofs and details.

For a topological space X and a compactly generated ∞-category $\mathscr{C}$, we denote by $\mathrm{Shv}(X, \mathscr{C})$ the ∞-category of $\mathscr{C}$-valued sheaves on X (see Appendix B.1).

Definition 2.19. Let X be an I-stratified space and let $\mathscr{C}$ be a compactly generated ∞-category. A sheaf $\mathcal{F} \in \mathrm{Shv}(X, \mathscr{C})$ is *constructible* if for every $i \in I$, the restriction $\mathcal{F}|_{X_i}$ is a locally constant sheaf in $\mathrm{Shv}(X_i, \mathscr{C})$. We denote by $\mathrm{Shv}_{\mathrm{cbl}}(X, \mathscr{C})$ the full subcategory spanned by the constructible sheaves. ◁

Definition 2.20. A poset I is said to satisfy the *ascending chain condition* if every non-empty subset of I has a maximal element. ◁

The following theorem generalises the monodromy equivalence which classifies locally constant sheaves as representations of the fundamental ∞-groupoid.

Theorem 2.21 ([Lur17, Theorem A.9.3] and Theorem B.9)**.** *Let $\mathscr{C}$ be a compactly generated ∞-category. Suppose X is a conically I-stratified space which is paracompact and locally contractible, and that I satisfies the ascending chain condition. Then there is an equivalence of ∞-categories*

$$\Psi_X\colon \mathrm{Fun}(\Pi_\infty^{\mathrm{exit}}(X), \mathscr{C}) \to \mathrm{Shv}_{\mathrm{cbl}}(X, \mathscr{C}).$$

Remark 2.22. The result is stated in [Lur17] for spaces which are locally of singular shape, but we wish to avoid going into the technicalities involved in defining this notion. ◦

Corollary 2.23 ([Lur17, Corollary A.9.4])**.** *Suppose X is a conically I-stratified space which is paracompact and locally contractible and where I satisfies the ascending chain condition. The inclusion $\Pi_\infty^{\mathrm{exit}}(X) \hookrightarrow \mathrm{Sing}(X)$ is a weak homotopy equivalence of simplicial sets, i.e. the induced map of geometric realisations $|\Pi_\infty^{\mathrm{exit}}(X)| \to |\,\mathrm{Sing}(X)|$ is a homotopy equivalence.*

2.4. **The constructible derived category of sheaves.** If the exit path ∞-category is equivalent to the nerve of its homotopy category, then the classification of constructible sheaves as representations of the exit path ∞-category can be extended to give an expression of the constructible derived category of sheaves (of R-modules) in terms of the exit path 1-category. We stress that the observations made in this section are quite elementary for anyone with a background in ∞-categories. We have chosen to be quite detailed for the sake of other potential readers.

For a Grothendieck abelian category $\mathcal{A}$ we denote by $\mathscr{D}(\mathcal{A})$ the (unbounded) derived ∞-category of $\mathcal{A}$ (see [Lur17, §1.3]). The homotopy category of $\mathscr{D}(\mathcal{A})$ is the classical (unbounded) derived category $D(\mathcal{A})$ of $\mathcal{A}$.

Let R be an associative ring and let LMod_R^1 denote the 1-category of left R-modules. Viewing R as a discrete $\mathbb{E}_1$-ring, let LMod_R denote the ∞-category of left R-module spectra. Then

$$\mathscr{D}(\mathrm{LMod}_R^1) \xrightarrow{\simeq} \mathrm{LMod}_R$$

by [Lur17, Proposition 7.1.1.16]. In particular, $D(R) := D(\mathrm{LMod}_R^1)$ is equivalent to the homotopy category of LMod_R. By [Lur17, Proposition 7.2.4.2], the ∞-category LMod_R is compactly generated and the subcategory of compact objects is the ∞-category $\mathrm{Perf}_\infty(R)$ of perfect modules ([Lur17, §7.2.4]). Under the equivalence $\mathscr{D}(\mathrm{LMod}_R^1) \xrightarrow{\simeq} \mathrm{LMod}_R$, perfect modules correspond to perfect chain complexes, i.e. complexes which are quasi-isomorphic to bounded chain complexes whose terms are finitely generated projective modules (Corollary 7.2.4.5 and Example 7.2.4.25 of [Lur17]). Let $\mathrm{Perf}_1(R) \subseteq D(R)$ denote the full subcategory spanned by the perfect chain complexes.

Let $\mathrm{Shv}_1(X,R)$ denote the 1-category of sheaves on X with values in LMod_R^1. This is a Grothendieck abelian category, and we consider the derived ∞-category $\mathscr{D}(\mathrm{Shv}_1(X,R))$.

Remark 2.24. The canonical functor

$$\mathscr{D}(\mathrm{Shv}_1(X,R)) \to \mathrm{Shv}(X, \mathscr{D}(R)) \simeq \mathrm{Shv}(X, \mathrm{LMod}_R)$$

is fully faithful with essential image the full subcategory $\mathrm{Shv}^{\mathrm{hyp}}(X, \mathrm{LMod}_R)$ of hypercomplete sheaves, that is, sheaves which satisfy descent with respect to any hypercovering not just covering sieves ([Lur09, §6.5.2], see also the discussion at [mat]). Constructible sheaves are hypercomplete by [Lur17, Proposition A.5.9], and we note that the subcategory

$$\mathrm{Shv}_{\mathrm{cbl}}(X, \mathrm{LMod}_R) \subseteq \mathrm{Shv}^{\mathrm{hyp}}(X, \mathrm{LMod}_R)$$

corresponds to the full subcategory $\mathscr{D}_{\mathrm{cbl}}(\mathrm{Shv}_1(X,R)) \subseteq \mathscr{D}(\mathrm{Shv}_1(X,R))$ spanned by the complexes whose homology sheaves are constructible. Similarly, we see that the subcategory of constructible compact-valued sheaves (i.e. whose stalk complexes are compact objects in LMod_R, see Definition B.8)

$$\mathrm{Shv}_{\mathrm{cbl,cpt}}(X, \mathrm{LMod}_R) \subseteq \mathrm{Shv}_{\mathrm{cbl}}(X, \mathrm{LMod}_R)$$

corresponds to the subcategory $\mathscr{D}_{\mathrm{cbl,cpt}}(\mathrm{Shv}_1(X,R)) \subseteq \mathscr{D}_{\mathrm{cbl}}(\mathrm{Shv}_1(X,R))$ spanned by the complexes whose homology sheaves are constructible and whose stalk complex is a perfect chain complex. ◦

Definition 2.25. The *constructible derived category of sheaves* on X with values in left R-modules is the full subcategory

$$D_{\mathrm{cbl}}(\mathrm{Shv}_1(X,R)) \subseteq D(\mathrm{Shv}_1(X,R))$$

spanned by the complexes of sheaves with constructible homology. The *constructible compact-valued derived category of sheaves* on X with values in left R-modules is the full subcategory

$$D_{\mathrm{cbl,cpt}}(\mathrm{Shv}_1(X,R)) \subseteq D(\mathrm{Shv}_1(X,R))$$

spanned by the complexes of sheaves whose homology sheaves are constructible and whose stalk complex is a perfect chain complex. ◁

We give two examples which are of interest in the study of the reductive Borel-Serre compactification. First we make the following observation.

Remark 2.26. Suppose R is a regular Noetherian ring of finite Krull dimension. Then it has finite global dimension, and thus any bounded below chain complex whose terms are finitely generated is quasi-isomorphic to a bounded complex whose terms are finitely generated projective. Therefore a constructible complex of sheaves in the sense of [GM83, §1.4] is a constructible compact-valued sheaf in the sense of Definition B.8. ◦

Example 2.27.

(i) Let X be a topological pseudomanifold with a fixed stratification and let k be a field. Intersection homology of X can be defined as the cohomology of a complex of sheaves $\mathbf{IC}_p(X)$ on X taking values in k-vector spaces ([GM80], [GM83]). The complexes $\mathbf{IC}_p(X)$ are constructible and compact-valued [GM83, §3].

(ii) The weighted cohomology of the arithmetic group Γ is defined as the cohomology of a complex of sheaves $\mathbf{W}^p\mathbf{C}^\bullet(\mathbf{E})$ on the reductive Borel–Serre compactification associated with Γ taking values in complex vector spaces ([GHM94]). The complexes $\mathbf{W}^p\mathbf{C}^\bullet(\mathbf{E})$ are constructible and compact-valued [GHM94, Theorem 17.6]. ◦

We have the following theorem.

Theorem 2.28. *Let X be a paracompact, locally contractible conically I-stratified space with I satisfying the ascending chain condition, and let R be an associative ring. Suppose the exit* *path ∞-category $\Pi_\infty^{\mathrm{exit}}(X)$ is equivalent to the nerve of its homotopy category $\Pi_1^{\mathrm{exit}}(X)$. Then there is an equivalence of ∞-categories*

$$\mathrm{Shv}_{\mathrm{cbl}}(X, \mathrm{LMod}_R) \simeq \mathscr{D}(\mathrm{Fun}\left(\Pi_1^{\mathrm{exit}}(X), \mathrm{LMod}_R^1\right)),$$

which restricts to an equivalence

$$\mathrm{Shv}_{\mathrm{cbl,cpt}}(X, \mathrm{LMod}_R) \simeq \mathscr{D}_{\mathrm{cpt}}(\mathrm{Fun}\left(\Pi_1^{\mathrm{exit}}(X), \mathrm{LMod}_R^1\right)),$$

where $\mathscr{D}_{\mathrm{cpt}}(\mathrm{Fun}\left(\Pi_1^{\mathrm{exit}}(X), \mathrm{LMod}_R^1\right))$ is the full subcategory spanned by the complexes of functors $F_\bullet$ such that the complex $F_\bullet(x)$ is perfect for all $x \in X$.

Proof. Propositions 1.3.4.25 and 1.3.5.15 of [Lur17] give us the first of the following two equivalences, and the second is the one of Theorem 2.21.

$$\mathscr{D}(\mathrm{Fun}(\Pi_1^{\mathrm{exit}}(X), \mathrm{LMod}_R^1)) \xrightarrow{\simeq} \mathrm{Fun}(\Pi_\infty^{\mathrm{exit}}(X), \mathrm{LMod}_R) \xrightarrow{\simeq} \mathrm{Shv}_{\mathrm{cbl}}(X, \mathrm{LMod}_R).$$

The restriction to perfect complexes objects is a consequence of Corollary B.12. □

Taking homotopy categories, we get the following corollary.

Corollary 2.29. *In the situation of Theorem 2.28 there is an equivalence of (1-)categories*

$$D_{\mathrm{cbl}}(\mathrm{Shv}_1(X, R)) \simeq D(\mathrm{Fun}(\Pi_1^{\mathrm{exit}}(X), \mathrm{LMod}_R^1))$$

which restricts to an equivalence

$$D_{\mathrm{cbl,cpt}}(\mathrm{Shv}_1(X, R)) \simeq D_{\mathrm{cpt}}(\mathrm{Fun}(\Pi_1^{\mathrm{exit}}(X), \mathrm{LMod}_R^1))$$

where $D_{cpt}(\mathrm{Fun}(\Pi_1^{\mathrm{exit}}(X), \mathrm{LMod}_R^1))$ is the full subcategory spanned by the complexes of functors $F_\bullet$ such that the complex $F_\bullet(x)$ is perfect for all $x \in X$.

3. Calculational tools

If X is a metrisably conically stratified space, then the end point evaluation maps from appropriately chosen homotopy links are fibrations. We identify the mapping spaces of $\Pi_\infty^{\mathrm{exit}}(X)$ with the fibres of these fibrations, and we analyse the resulting long exact sequences of homotopy groups in order to determine the homotopy type of the mapping spaces. This follows ideas of Woolf ([Woo09]). We apply these tools to determine the exit path ∞-category of quotients of sufficiently contractible stratified spaces under well-behaved group actions. This strengthens a result of Chen-Looijenga ([CL15]).

3.1. **Mapping spaces, fibrations and long exact sequences.** We have already observed that the homotopy link is a subset of the 1-simplices in the exit path ∞-category. It turns out that the mapping spaces in the exit path ∞-category can be identified with subspaces of the homotopy link.

Proposition 3.1. *Let X be a conically I-stratified space, let $i < j$ in I and choose $x \in X_i$, $y \in X_j$. Let V be a neighbourhood of x in X_i. The mapping space $M(x,y)$ of the exit path ∞-category $\Pi_\infty^{\mathrm{exit}}(X)$ can be identified with the fibre $F(x,y) = e^{-1}(x,y)$ of the end point evaluation map $e\colon H(V \cup X_j, V) \to V \times X_j$, $\gamma \mapsto (\gamma(0), \gamma(1))$.*

Proof. Write $S := \Pi_\infty^{\mathrm{exit}}(X)$. We use the following model for the mapping space:

$$M(x,y) = \{x\} \times_S S^{\Delta^1_\bullet} \times_S \{y\},$$

that is, an n-simplex of $M(x,y)$ is a simplicial map $\sigma\colon (\Delta^n \times \Delta^1)_\bullet \to S$ which satisfies $\sigma(\Delta^n_\bullet \times \{0\}) = \{x\}$ and $\sigma(\Delta^n_\bullet \times \{1\}) = \{y\}$ (see [Lur09, §1.2.2]).

The simplicial sets $(\mathrm{Sing}\, X)^{\Delta^1_\bullet}$ and $\mathrm{Sing}(X^{|\Delta^1_\bullet|})$ are isomorphic via the adjunction $|-| \dashv \mathrm{Sing}$ and the exponential law for topological spaces. By translating the conditions on the subsimplicial sets $M(x,y) \subseteq (\mathrm{Sing}\, X)^{\Delta^1_\bullet}$ and $\mathrm{Sing}(F(x,y)) \subseteq \mathrm{Sing}(X^{|\Delta^1_\bullet|})$ across this isomorphism, we see that it restricts to an isomorphism $M(x,y) \cong \mathrm{Sing}(F(x,y))$. □

Remark 3.2. Let X be a conically I-stratified space, let $i,j \in I$, $x \in X_i$, $y \in X_j$ and let V be a neighbourhood of x in X_i. The proposition above implies that if $i \neq j$, then $M(x,y) \subset H(V \cup X_j, V)$. If $i = j$, then $M(x,y) \cap H(V \cup X_i, V) = \emptyset$, as V is a neighbourhood of x in X_i. In this case, however, $M(x,y)$ is the mapping space in the ∞-category $\mathrm{Sing}(X_i)$ which can be identified with the fibre of the path space fibration of X_i with respect to the basepoint x. Hence, $M(x,y)$ is either empty or homotopy equivalent to the loop space $\Omega(X_i, x)$. ○

The end point evaluation map from the homotopy link is a fibration in certain situations, for example when the stratified space is metrisably conically stratified (Proposition 2.11). The following proposition simply rewrites the long exact sequence of homotopy groups arising from this fibration. To state the proposition, we need to fix some notation and various basepoints and maps — this is done below in what we for future reference will call a preamble.

Preamble 3.3. Let X be a conically I-stratified space and let $i < j$ in I. Fix points $x_i \in X_i$, $x_j \in X_j$ and suppose there is a conical neighbourhood U of x_i in X with a stratified homeomorphism $\varphi_i\colon V_i \times C(L_i) \to U$, where V_i is a weakly contractible neighbourhood of x_i in X_i and the union $V_i \cup X_j$ is metrisable. Suppose $M(x_i, x_j) \neq \emptyset$ and fix a path $\gamma_{ij} \in M(x_i, x_j)$ and an $\varepsilon \in (0,1)$.

The end point evaluation map

$$e_{ij}\colon H(V_i \cup X_j, V_i) \to V_i \times X_j, \qquad \gamma \mapsto \big(\gamma(0), \gamma(1)\big)$$

is a fibration (Proposition 2.11) and in view of Proposition 3.1, we may identify $M(x_i, x_j)$ with the fibre $e_{ij}^{-1}(x_i, x_j)$. The map

$$\Psi_{ij}\colon L_{ij} \to H(V_i \cup X_j, V_i), \qquad l \mapsto \big(\gamma_{x_i,l,\varepsilon}\colon t \mapsto \varphi_i(x_i, [l, t\varepsilon])\big)$$

is a homotopy equivalence (Proposition 2.12). Fix a homotopy inverse

$$\Psi_{ij}^h\colon H(V_i \cup X_j, V_i) \to L_{ij}$$

and a homotopy

$$h\colon H(V_i \cup X_j, V_i) \times [0,1] \to H(V_i \cup X_j, V_i), \quad h\colon \mathrm{id} \sim \Psi_{ij} \circ \Psi_{ij}^h.$$

Consider the embedding of the j'th link space stratum L_{ij} into X_j

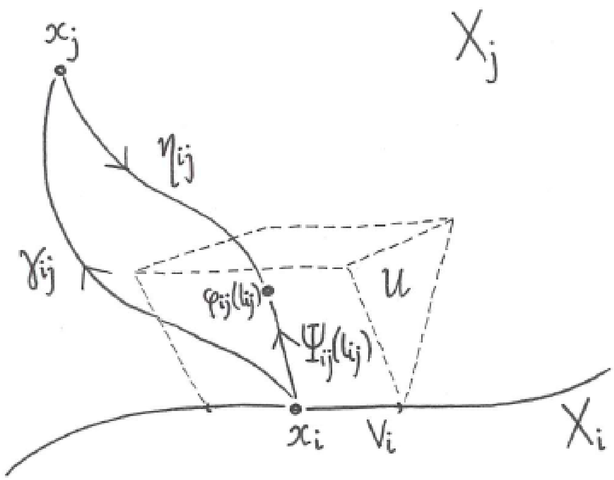
$$\varphi_{ij}\colon L_{ij} \to X_j, \quad l \mapsto \varphi_i(x_i, [l, \varepsilon]).$$

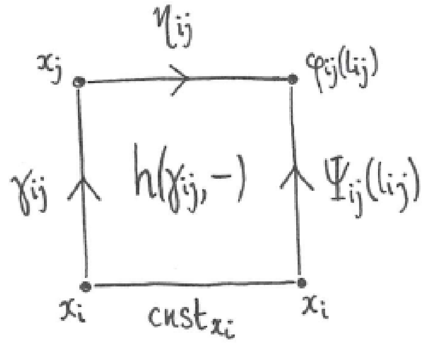
Finally, set $l_{ij} := \Psi^h_{ij}(\gamma_{ij}) \in L_{ij}$ and define a path

$$\eta_{ij} := h(\gamma_{ij}, -)(1)\colon [0,1] \to X_j$$

from x_j to $\varphi_{ij}(l_{ij})$.

The situation can be pictured as follows.





○

Proposition 3.4. *In the situation of Preamble 3.3, there is a long exact sequence of homotopy groups*

$$\cdots \longrightarrow \pi_n(L_{ij}, l_{ij}) \longrightarrow \pi_n(X_j, x_j) \longrightarrow \pi_{n-1}(M(x_i, x_j), \gamma_{ij}) \longrightarrow \cdots$$

$$\cdots \longrightarrow \pi_1(L_{ij}, l_{ij}) \xrightarrow{\varphi} \pi_1(X_j, x_j) \xrightarrow{\partial} \pi_0(M(x_i, x_j), \gamma_{ij}) \longrightarrow \cdots$$

The map φ is given by conjugation by η_{ij}:

$$\varphi\colon \pi_1(L_{ij}, l_{ij}) \longrightarrow \pi_1(X_j, x_j), \qquad [\alpha] \mapsto [\eta_{ij}^{-1} * ((\varphi_{ij})_* \alpha) * \eta_{ij}],$$

and the boundary map ∂ is given by concatenation with γ_{ij}:

$$\partial\colon \pi_1(X_j, x_j) \longrightarrow \pi_0(M(x_i, x_j), \gamma_{ij}), \qquad [\alpha] \mapsto [\alpha * \gamma_{ij}].$$

Proof. We have a long exact sequence of homotopy groups arising from the fibration e_{ij} in which we may replace $\pi_n(V_i, x_i)$ by 0:

$$\cdots \longrightarrow \pi_n(H(V_i \cup X_j, V_i), \gamma_{ij}) \longrightarrow \pi_n(X_j, x_j) \longrightarrow \pi_{n-1}(M(x_i, x_j), \gamma_{ij}) \longrightarrow \cdots$$

We replace $\pi_n(H(V_i \cup X_j, V_i), \gamma_{ij})$ by $\pi_n(L_{ij}, l_{ij})$ via the homotopy equivalence Ψ_{ij} and a basepoint change from $\Psi_{ij}(l_{ij})$ to γ_{ij}:

$$C_{h(\gamma_{ij},-)} \circ (\Psi_{ij})_* \colon \pi_n(L_{ij}, l_{ij}) \xrightarrow{\cong} \pi_n(H(V_i \cup X_j, V_i), \gamma_{ij})$$

where $C_{h(\gamma_{ij},-)}$ denotes conjugation by the path $t \mapsto h(\gamma_{ij}, t)$.

To see that the maps are as claimed, let pr_j denote the projection to X_j and $C_{\eta_{ij}}$ conjugation by η_{ij}. Then

$$\varphi = (\mathrm{pr}_j \circ e_{ij})_* \circ C_{h(\gamma_{ij},-)} \circ (\Psi_{ij})_* = C_{\eta_{ij}} \circ (\varphi_{ij})_*.$$

For the boundary map ∂, note that it is equal to the following composite

$$\pi_1(X_j, x_j) \xleftarrow[\cong]{(\mathrm{pr}_j \circ e_{ij})_*} \pi_1\Big(H(V_i \cup X_j, V_i), M(x_i, x_j), \gamma_{ij}\Big) \xrightarrow{\delta} \pi_0(M(x_i, x_j), \gamma_{ij}),$$

where the middle term is the relative homotopy group and δ is the boundary map in the long exact sequence of homotopy groups of the pair $(H(V_i \cup X_j, V_i), M(x_i, x_j))$. This is given by sending a map $f \colon [0,1] \to H(V_i \cup X_j, V_i)$ representing an element in the relative π_1 to the starting point $f(0) \in M(x_i, x_j)$. The inverse to $(\mathrm{pr}_j \circ e_{ij})_*$ is given by lifting a loop $[0,1] \to X_j$ to a path $[0,1] \to H(X_j \cup V_i, V_i)$ with end point γ_{ij} ([Hat02, proof of Theorem 4.41]). This is independent of the choice of lift, so for any $\alpha \colon [0,1] \to X_j$ with $\alpha(0) = \alpha(1) = x_j$, we may choose the lift $\widetilde{\alpha} \colon t \mapsto \alpha|_{[0,1-t]} * \gamma_{ij}$, and we see that ∂ is given by $[\alpha] \mapsto [\widetilde{\alpha}(0)] = [\alpha * \gamma_{ij}]$ as claimed. □

For mapping spaces within a stratum, Remark 3.2 gives the following identification.

Proposition 3.5. *Let X be a conically I-stratified space. Let $i \in I$ and fix $x_i, x_i' \in X_i$. If $M(x_i, x_i') \neq \emptyset$, then $M(x_i, x_i')$ has the homotopy type of the loop space $\Omega(X_i, x_i)$. In particular, $\pi_n(M(x_i, x_i'), \gamma) \cong \pi_{n+1}(X_i, x_i)$ for all $n \geq 0$ and any choice of basepoint $\gamma \in M(x_i, x_i')$.*

We have the following corollary.

Corollary 3.6. *Let X be a metrisably conically I-stratified space whose strata are locally weakly contractible. Let $i < j$, $x_i \in X_i$ and $x_j \in X_j$, and assume $M(x_i, x_j) \neq \emptyset$. Then the assumptions of Preamble 3.3 are satisfied. Assume additionally that for any choice of $\gamma_{ij} \in M(x_i, x_j)$ in the situation of Preamble 3.3, the following holds:*

(i) the map $\varphi_{ij} \colon L_{ij} \to X_j$ is injective on π_1,
(ii) $\pi_n(X_j, x_j) = 0$ for all $n > 1$,
(iii) $\pi_n(L_{ij}, l_{ij}) = 0$ for all $n > 1$.

Then the mapping space $M(x_i, x_j)$ has contractible path components and the set of path components fits into a 5-term exact sequence

$$0 \to \pi_1(L_{ij}, l_{ij}) \xrightarrow{\varphi} \pi_1(X_j, x_j) \xrightarrow{\partial} \pi_0(M(x_i, x_j)) \to \pi_0(L_{ij}) \to \pi_0(X_j) \to 0,$$

where φ and ∂ are as described in Proposition 3.4. In particular, if (i)-(iii) hold for all $i < j$, $x_i \in X_i$ and $x_j \in X_j$ with $M(x_i, x_j) \neq \emptyset$, then the exit path ∞-category is canonically equivalent to the nerve of its homotopy category $\Pi_1^{\mathrm{exit}}(X)$ and the hom-sets in $\Pi_1^{\mathrm{exit}}(X)$ can *be identified using the exact sequences above and the isomorphisms* $\pi_0(M(x_i, x_i')) \cong \pi_1(X_i, x_i)$ *for* $x_i, x_i' \in X_i$ *in the same path component.*

This result identifies (the homotopy type of) the mapping spaces in the exit path ∞-category, but does not tell us anything about composition. If, however, the stratified space is sufficiently contractible, then we can use Corollary 3.6 to identify the exit path ∞-category as in the following corollary.

Corollary 3.7. *Let* X *be a metrisably conically* I*-stratified space with path connected, weakly contractible strata* X_i*, and suppose* X *admits conical neighbourhoods with weakly contractible strata. Then the exit path* ∞*-category* $\Pi_\infty^{\mathrm{exit}}(X)$ *is canonically equivalent to the nerve of its homotopy category* $\Pi_1^{\mathrm{exit}}(X)$ *which in turn is equivalent to the poset of strata* I*.*

3.2. **Group actions and exit path ∞-categories.** In this section we determine the exit path ∞-category of stratified spaces obtained via suitably well-behaved group actions. The results should be compared with [CL15, Theorem 1.7] (see Remark 3.20).

A (left) action of a discrete group G on a stratified space $s\colon X \to I$ consists of compatible continuous (left) actions of G on X and I, i.e. such that the stratification map s is equivariant. Recall that an action of G on X is *properly discontinuous* if each point $x \in X$ has a neighbourhood U such that the set $\{g \in G \mid g.U \cap U \neq \emptyset\}$ is finite.

Remark 3.8. A word of warning: the cones in the following theorem are the stratified cones of Definition 2.3. If L_i is compact Hausdorff, then it coincides with the usual topological cone, but generally they are different. In Corollaries 3.14 and 3.15 below we present a weaker version of this theorem in which we allow neighbourhoods in X which locally look like (stratified) topological cones. ○

Theorem 3.9. *Let* $X \to I$ *be a stratified space with path connected, weakly contractible strata, with* I *satisfying the ascending chain condition, and with surjective stratification map. Suppose* G *is a discrete group acting on* $X \to I$ *and let* $\pi\colon X \to G\backslash X$ *denote the quotient map. For any* $i \in I$*, denote by* G_i *the stabiliser of* i *and let* $G_i^\ell \leq G_i$ *denote the subgroup which fixes* X_i *pointwise. Suppose that for all* $i \in I$ *and all* $x \in X_i$ *there is:*

(i) a G_i^ℓ*-invariant neighbourhood* U *of* x *in* X *satisfying*

$$\{g \in G \mid g.U \cap U \neq \emptyset\} = G_i^\ell,$$

and such that $V = U \cap X_i$ *is weakly contractible.*

(ii) a stratified space $L \to I_{>i}$ *with weakly contractible strata and equipped with an action of* G_i^ℓ *(where the action on* $I_{>i}$ *is the restriction of the one of* G_i*), and whose stratification map is surjective.*

(iii) a G_i^ℓ*-equivariant stratified homeomorphism*

$$\varphi\colon V \times C(L) \xrightarrow{\cong} U,$$

where G_i^ℓ *acts on the* L*-coordinate of the left hand side,* $g.(x, [l, t]) = (x, [g.l, t])$*, and such that* φ *restricts to the identity on* $V \times \{*\}$*.*

(iv) assume additionally that for all $j > i$, the union $V \cup X_j$ and its image $\pi(V \cup X_j)$ are metrisable.

Then $X \to I$ and $G\backslash X \to G\backslash I$ are metrisably conically stratified spaces whose exit path ∞-categories are canonically equivalent to the nerves of their homotopy categories. The exit path 1-category of X is equivalent to the poset I, and the exit path 1-category of $G\backslash X$ is equivalent to the category $\mathscr{C}_{G,X}$ with objects the elements of I and hom-sets

$$\mathscr{C}_{G,X}(i,j) = \{g \in G \mid g.i \leq j\}/G_i^{\ell},$$

where G_i^{ℓ} acts by right multiplication and with composition given by the product in G. Moreover, the equivalences can be chosen such that the following diagram commutes, where π_* is the functor induced by the quotient map $\pi\colon X \to G\backslash X$ and the top vertical map sends $i \leq j$ to the morphism $i \to j$ represented by the identity element of G:

$$\begin{array}{ccc} I & \longrightarrow & \mathscr{C}_{G,X} \\ {\scriptstyle\sim}\downarrow & & \downarrow{\scriptstyle\sim} \\ \Pi_1^{\mathrm{exit}}(X) & \xrightarrow[\pi_*]{} & \Pi_1^{\mathrm{exit}}(G\backslash X) \end{array}$$

Before tackling the proof, we make some preliminary observations.

Observation 3.10. Note first of all that $G\backslash I$ is indeed a poset. The action is order preserving and we have assumed I to satisfy the ascending chain condition, so we cannot have $g.i < i$ for any $g \in G$, $i \in I$. Hence, setting $[i] \leq [j]$, if $i \leq g.j$ for some $g \in G$ defines a partial order on $G\backslash I$.

For any $i \leq j$, we have $G_j^{\ell} \leq G_i^{\ell}$ by assumption (i) and assumption (iii), since the neighbourhood U is stratified over $I_{\geq i}$ with $U_j \neq \emptyset$, and $g \in G_j^{\ell}$ fixes U_j. This together with the fact that ${}^gG_i^{\ell} = G_{g.i}^{\ell}$ for all $i \in I$, $g \in G$, implies that the category $\mathscr{C}_{G,X}$ is well-defined. ∘

Proof. By assumption (i), the equivalence relations on U_i induced by G and G_i^{ℓ} agree, so the conical neighbourhoods in X descend to $G\backslash X$:

$$\hat{\varphi}\colon V \times C(G_i^{\ell}\backslash L) \xrightarrow{\cong} G_i^{\ell}\backslash U.$$

It follows that the quotient $G\backslash X \to G\backslash I$ is indeed a conically stratified space. By (iv), both X and $G\backslash X$ are metrisably conically stratified.

We now analyse the link spaces and show that the mapping spaces of $\Pi_\infty^{\mathrm{exit}}(G\backslash X)$ have contractible components. Let $i, j \in I$ with images $\hat{\imath}, \hat{\jmath} \in G\backslash I$ and suppose $\hat{\imath} \leq \hat{\jmath}$. Write $G_{ij} = \{g \in G \mid g.i \leq j\}$. Let $x_i \in X_i$, $x_j \in X_j$ with images $\hat{x}_i$, $\hat{x}_j$ in $G\backslash X$ and let $\varphi_i\colon V_i \times C(L_i) \xrightarrow{\cong} U_i$ be a conical neighbourhood of x_i as in (i)-(iv). Consider the corresponding conical neighbourhood $\hat{\varphi}_i\colon V_i \times C(G_i^{\ell}\backslash L_i) \xrightarrow{\cong} G_i^{\ell}\backslash U_i$ of $\hat{x}_i$. The $\hat{\jmath}$'th stratum of the link space $\mathscr{L}_{\hat{\imath}} := G_i^\ell \backslash L_i$ is

$$\mathscr{L}_{\hat{\imath}\hat{\jmath}} = G_i^\ell \Big\backslash \Big(\coprod_{\overline{g} \in G_j \backslash G_{ij}} L_{i(g^{-1}.j)} \Big)$$

where the action of G_j on G_{ij} in the indexing set is given by left multiplication. For $x \in L_{i(g^{-1}.j)}$, and $u \in G_i^\ell$, we have $u.x \in L_{i(ug^{-1}.j)} = L_{i((gu^{-1})^{-1}.j)}$. Hence, the quotient map $G_j \backslash G_{ij} \to G_j \backslash G_{ij} / G_i^\ell$ induces an isomorphism

$$\pi_0(\mathscr{L}_{\hat{\imath}\hat{\jmath}}) \cong G_j \backslash G_{ij} / G_i^\ell.$$

Moreover, for a given $[g] \in G_j \backslash G_{ij} / G_i^\ell$, the corresponding component is homeomorphic to the quotient of $L_{i(g^{-1}.j)}$ by the action of $G_i^\ell \cap G_{g^{-1}.j}$.

Condition (i) implies that for all $k \in I$, the action of G_k / G_k^ℓ on the weakly contractible space X_k is free and properly discontinuous. Hence, for any $g \in G_{ij}$, so is the action of $G_i^\ell \cap G_{g^{-1}.j} / G_{g^{-1}.j}^\ell$ on the weakly contractible space $L_{i(g^{-1}.j)}$. In fact, for any $\varepsilon \in (0,1)$, the inclusion

$$L_{i(g^{-1}.j)} \hookrightarrow X_{g^{-1}.j}, \quad l \mapsto \varphi_i(x_i, [l, \varepsilon]),$$

defines a morphism of fibre bundles as below, where $\hat{X}_{\hat{\jmath}} = G \backslash (\coprod_{\overline{g} \in G_j \backslash G} X_{g^{-1}.j})$ is the $\hat{\jmath}$'th stratum of $G \backslash X$.

$$\begin{array}{ccc}
(G_i^\ell \cap G_{g^{-1}.j}) / G_{g^{-1}.j}^\ell & \hookrightarrow & G_{g^{-1}.j} / G_{g^{-1}.j}^\ell \\
\big\downarrow & & \big\downarrow \\
L_{i(g^{-1}.j)} & \hookrightarrow & X_{g^{-1}.j} \\
\big\downarrow & & \big\downarrow \\
\mathscr{L}_{\hat{\imath}\hat{\jmath}} & \hookrightarrow & \hat{X}_{\hat{\jmath}}
\end{array}$$

It follows that for any choice of basepoint in the $[g]$'th component of $\mathscr{L}_{\hat{\imath}\hat{\jmath}}$, the inclusion $\mathscr{L}_{\hat{\imath}\hat{\jmath}} \to \hat{X}_{\hat{\jmath}}$ induces the inclusion

$$(G_i^\ell \cap G_{g^{-1}.j}) / G_{g^{-1}.j}^\ell \hookrightarrow G_{g^{-1}.j} / G_{g^{-1}.j}^\ell$$

on π_1 and the higher homotopy groups of both $\mathscr{L}_{\hat{\imath}\hat{\jmath}}$ and $\hat{X}_{\hat{\jmath}}$ vanish. By Corollary 3.6, the mapping space $M(\hat{x}_i, \hat{x}_j)$ of $\Pi_\infty^{\mathrm{exit}}(G \backslash X)$ has contractible components. As this holds for any choice of points $\hat{x}_i$, $\hat{x}_j$ and the strata are Eilenberg-Maclane spaces, the exit path ∞-category $\Pi_\infty^{\mathrm{exit}}(G \backslash X)$ is canonically equivalent to the nerve of its homotopy category $\Pi_1^{\mathrm{exit}}(G \backslash X)$.

We now define a functor $\mathscr{C}_{G,X} \to \Pi_1^{\mathrm{exit}}(G \backslash X)$ and show that this is an equivalence by examining the exact sequences of Corollary 3.6 in more detail. Denote by $\pi_* \colon \Pi_1^{\mathrm{exit}}(X) \to \Pi_1^{\mathrm{exit}}(G \backslash X)$ the map induced by π. By Corollary 3.7, the exit path ∞-category of X is canonically equivalent to the nerve of its homotopy category $\Pi_1^{\mathrm{exit}}(X)$ which in turn is equivalent to the poset of strata I. In view of this, if $x, x' \in X$ are connected by a morphism in $\Pi_1^{\mathrm{exit}}(X)$, then it is unique and we denote it by $p_{x\to x'}$. Choose basepoints $x_i \in X_i$ for all $i \in I$ and define a functor $F\colon \mathscr{C}_{G,X} \to \Pi_1^{\mathrm{exit}}(G\backslash X)$ as follows

$$F(i) = \pi(x_i), \quad \text{and} \quad F([g]\colon i \to j) = \pi_*(p_{x_i \to g^{-1}.x_j}).$$

This is well-defined, since for $u \in G_i^\ell$, the path $p_{x_i\to u^{-1}.x_i}$ is the trivial loop at x_i.

To see that F is fully faithful, let $i \neq j$ and suppose $g_{ij} \in G_{ij}$ (if $G_{ij} = \emptyset$, then the hom-sets on both sides are empty and there is nothing to prove). Set $\gamma_{ij} = F([g_{ij}]) \in M(\hat{x}_i, \hat{x}_j)$ and fix, according to Preamble 3.3, a compatible basepoint $l_{ij} \in \mathscr{L}_{\hat{\imath}\hat{\jmath}}$. Then we have a commutative diagram of exact sequences as below, and as this holds for any choice of g_{ij}, F is bijective on hom-sets for $i \neq j$ by an extended 5-lemma (see for example [Hat02, §4.1 Exercise 9]).

$$\begin{array}{ccccccccc}
0 \to (G_i^\ell \cap G_{g_{ij}^{-1}.j})/G^\ell_{g_{ij}^{-1}.j} & \longrightarrow & G_j/G_j^\ell & \xrightarrow{-\cdot g_{ij}} & \mathscr{C}_{G,X}(i,j) & \longrightarrow & G_j\backslash G_{ij}/G_i^\ell \to 0 \\
\downarrow\cong & & \downarrow\cong & & \downarrow F & & \downarrow\cong \\
0 \longrightarrow \pi_1(\mathscr{L}_{\hat{\imath}\hat{\jmath}}, l_{ij}) & \xrightarrow[\varphi]{} & \pi_1(\hat{X}_{\hat{\jmath}}, \hat{x}_j) & \xrightarrow[\partial]{} & \pi_0(M(\hat{x}_i,\hat{x}_j),\gamma_{ij}) & \longrightarrow & \pi_0(\mathscr{L}_{\hat{\imath}\hat{\jmath}}, l_{ij}) \to 0
\end{array}$$

For $i = j$, bijectivity on hom-sets follows from the fact that $\hat{X}_{\hat{\imath}}$ is a $K(G_i/G_i^\ell, 1)$. The functor is essentially surjective since the strata are path connected, so we have established the desired equivalence. If, in addition to the functor F, we choose the equivalence $I \xrightarrow{\sim} \Pi_1^{\mathrm{exit}}(X)$ which sends i to x_i and $i < j$ to $p_{x_i\to x_j}$, then the diagram in the statement of the theorem commutes. □

Remark 3.11. Note that the equivalence F defined in the proof of Theorem 3.9 mirrors the identification of the fundamental group of a (nice) topological space with the group of deck transformations of its universal cover: if $\pi\colon \widetilde{X} \to X$ is a universal cover of a space X, where $\widetilde{X}$ (and thus X) is locally path connected, with basepoints $\widetilde{x} \in \widetilde{X}$ and $x = \pi(\widetilde{x}) \in X$, and if G is the group of deck transformations, then we may define a group isomorphism $G \to \pi_1(X, x)$ by sending g to (the homotopy class of) the loop $\pi \circ p_g$, where p_g is the unique (up to homotopy) path in $\widetilde{X}$ from $\widetilde{x}$ to $g^{-1}.\widetilde{x}$.

In this respect, the map $\pi\colon X \to G\backslash X$ can be interpreted as a stratified universal cover — see [Woo09] for more about stratified covers (of homotopically stratified sets). ○

We wish to extend Theorem 3.9 to a larger class of stratified spaces. Corollaries 3.14 and 3.15 below are an application of Theorem 3.9 giving a version of this theorem applicable to the case when X is not necessarily conically stratified, but does admit neighbourhoods which are "conical" with respect to the stratified topological cone defined below. In particular, it will apply to the reductive Borel–Serre compactification as we will see in Section 4.2.

Definition 3.12. Let $s\colon Y \to I$ be a stratified space. The *(open) stratified topological cone* on Y is the stratified space $s^\triangleleft\colon C^t(Y) \to I^\triangleleft$ defined as follows: the poset of strata is

$I^{\triangleleft} := I \cup \{-\infty\}$ with $-\infty \leq i$ for all $i \in I$, and $C^t(Y) = (Y \times [0,1)) \cup_{Y\times\{0\}} *$ is the pushout, and the stratification map is given by $s^{\triangleleft}(x,t) = s(x)$ and $s^{\triangleleft}(*) = -\infty$. ◁

Remark 3.13. As remarked below Definition 2.3, the stratified topological cone $C^t(Y)$ agrees with the stratified cone $C(Y)$, when Y is compact Hausdorff. ○

Note that the corollary below does not provide information about composition in the exit path category. For this we need additional data as described in Corollary 3.15 and Remark 3.16.

Corollary 3.14. *Let $X \to I$ be a stratified space with path connected, weakly contractible strata, with I satisfying the ascending chain condition, and with surjective stratification map. Suppose G is a discrete group acting on $X \to I$ and let $\pi\colon X \to G\backslash X$ denote the quotient map. Let for all $i \in I$, G_i denote the stabiliser of i and let $G_i^\ell \leq G_i$ denote the subgroup which fixes X_i pointwise. Suppose that for all $i \in I$ and all $x \in X_i$ there is:*

(i) a G_i^ℓ-invariant neighbourhood U of x in X satisfying

$$\{g \in G \mid g.U \cap U \neq \emptyset\} = G_i^\ell,$$

and such that $V = U \cap X_i$ is weakly contractible.

(ii) a stratified space $L \to I_{>i}$ with weakly contractible strata which is equipped with an action of G_i^ℓ such that the quotient $G_i^\ell\backslash L$ is compact Hausdorff (where the action on $I_{>i}$ is the restriction of the one of G_i), and whose stratification map is surjective.

(iii) a G_i^ℓ-equivariant stratified homeomorphism

$$\varphi\colon V \times C^t(L) \xrightarrow{\cong} U,$$

where G_i acts on the L-coordinate of the left hand side, $g.(x,[l,t]) = (x,[g.l,t])$, and such that φ restricts to the identity on $V \times \{*\}$.

(iv) assume additionally that the image $\pi(V \cup X_j)$ is metrisable for all $j > i$.

Then $G\backslash X \to G\backslash I$ is a metrisably conically stratified space, and the exit path ∞-category of $G\backslash X$ is equivalent to the nerve of its homotopy category $\Pi_1^{\mathrm{exit}}(G\backslash X)$. The exit path 1-category is equivalent to a category with objects the elements of I and hom-sets

$$\mathrm{Hom}(i,j) = \{g \in G \mid g.i \leq j\}/G_i^\ell,$$

where G_i^ℓ acts by right multiplication.

Proof. As in Theorem 3.9, $G\backslash I$ has a natural partial order. The maps φ descend to the quotient

$$\hat{\varphi}\colon V \times C^t(G_i^\ell\backslash L) \xrightarrow{\cong} G_i^\ell\backslash U,$$

and since $G_i^\ell\backslash L$ is assumed to be compact Hausdorff, the stratified topological cone $C^t(G_i^\ell\backslash L)$ coincides with $C(G_i^\ell\backslash L)$. We thus conclude that $G\backslash X \to G\backslash I$ is a metrisably conically stratified space. The proof of Theorem 3.9 goes through word for word up until defining the functor F, so we conclude that $\Pi_\infty^{\mathrm{exit}}(G\backslash X)$ is canonically equivalent to its homotopy category.

Since the strata of X are path connected, we can fix basepoints $x_i \in X_i$ for all i and any object of $\Pi_1^{\mathrm{exit}}(G\backslash X)$ will be equivalent to $\hat{x}_i = \pi(x_i)$ for some i. To determine the hom-sets in $\Pi_1^{\mathrm{exit}}(G\backslash X)$, we choose a path $\gamma_{ij} \in M(\hat{x}_i, \hat{x}_j)$ for $i \neq j$ (if the mapping space is empty, there is nothing to prove), and the identification of the set of path components as in the proof of Theorem 3.9 follows through word for word. As does the identification for $i = j$, since $\hat{X}_{\hat{\imath}}$ is again a $K(G_i/G_i^\ell, 1)$. □

What is lacking in the above situation is a collection of compatible exit paths that allow us to also analyse the composition in $\Pi_1^{\mathrm{exit}}(G\backslash X)$. If such a collection exists, then we can fully identify the exit path 1-category as in the following corollary (see also Remark 3.16).

Corollary 3.15. *Suppose that in the situation of Corollary 3.14, we can choose basepoints $x_i \in X_i$ for all $i \in I$ and paths $\gamma_{ij}^g\colon [0,1] \to X$ with $\gamma_{ij}^g(0) = x_i$ and $\gamma_{ij}^g(1) = g^{-1}.x_j$ for all $i, j \in I$ and $g \in G$ with $g.i \leq j$. Assume that these paths satisfy the following conditions:*

(i) $\gamma_{ij}^g \in H(X_i \cup X_{g^{-1}.j}, X_i)$, when $g.i < j$,
(ii) $\gamma_{ij}^g \in C([0,1], X_i)$, when $g.i = j$,
(iii) γ_{ii}^u is the constant loop at x_i for all $u \in G_i^\ell$,
(iv) the concatenations are functorial: for all $i, j, k \in I$, and $g, h \in G$ with $g.i \leq j$, $h.j \leq k$, we have equalities in $\Pi_1^{\mathrm{exit}}(G\backslash X)$:

$$(\pi \circ \gamma_{jk}^h) * (\pi \circ \gamma_{ij}^g) = (\pi \circ \gamma_{ik}^{hg}).$$

Then the functor $F\colon \mathscr{C}_{G,X} \to \Pi_1^{\mathrm{exit}}(G\backslash X)$, $F(i) = \pi(x_i)$, $F([g]\colon i \to j) = \pi \circ \gamma_{ij}^g$, is an equivalence, where $\mathscr{C}_{G,X}$ is the category with objects the elements of I and hom-sets

$$\mathscr{C}_{G,X}(i,j) = \{g \in G \mid g.i \leq j\}/G_i^\ell,$$

where G_i^ℓ acts by right multiplication, and with composition given by the product in G.

Proof. As in Theorem 3.9, the category $\mathscr{C}_{G,X}$ is well-defined, and conditions (i)-(iv) imply that F is well-defined. Moreover, it fits into the proof of Corollary 3.14 where it is seen to be an equivalence. □

Remark 3.16. In the situation of Corollary 3.14, one can weaken the topology of X in order to obtain a conically stratified space with the same quotient space $G\backslash X$. This is similar to the trick used by Milnor to construct universal bundles in [Mil56]. More specifically, let X^w denote the space whose underlying set is that of X equipped with the coarsest topology such that

* the stratification map $X^w \to I$ is continuous,
* the map $X^w \to G\backslash X$ is a quotient map,
* the inclusions $X_i \hookrightarrow X^w$, $i \in I$, are embeddings.

Suppose we have a stratified space $L \to I$ equipped with an action of G and let G act on the stratified topological cone $C^t(L) \to I^{\triangleleft}$ by acting on the L-coordinate of the cone and fixing the apex. If $G\backslash L$ is compact Hausdorff, then $C^t(L)^w \cong C(L^w)$. Hence, if X admits neighbourhoods as in Corollary 3.14, then X^w is a conically stratified space with quotient space $G\backslash X$ and we are almost in the situation of Theorem 3.9. However, the metrisability conditions that need to be verified for Theorem 3.9 to apply, mean that we cannot make a reasonable general statement using X^w as an intermediary construction. At least not with the tools at hand.

In concrete cases, one can analyse the topology of X^w and, having verified the metrisability conditions, apply Theorem 3.9 directly. On the other hand, in many cases one can probably avoid considering the weaker topology by finding a collection of paths as in Corollary 3.15. At first sight, finding such a collection of paths looks like a tedious task, but as we will see in the case of the reductive Borel–Serre compactification, there may exists a cover of the stratified space X, from which such a collection naturally descends. In our case, the paths will descend from the partial Borel–Serre compactification. ◦

Observation 3.17. Suppose we are in the situation of either Theorem 3.9 or Corollary 3.15 and consider for some subset $J \subseteq I$, the image of the union $\bigcup_{j\in J} X_j$ under the quotient map $X \to G\backslash X$:

$$\bigcup_{j\in J} \hat{X}_{\hat{\jmath}} \subseteq G\backslash X.$$

Then the exit path ∞-category of $\bigcup_{j\in J} \hat{X}_{\hat{\jmath}}$ is also equivalent to the nerve of its homotopy category and the functor $\mathscr{C}_{G,X} \to \Pi_1^{\mathrm{exit}}(G\backslash X)$ defined in the proof of Theorem 3.9 or Corollary 3.15 restricts to an equivalence

$$\mathscr{C}_{G,X}(J) \longrightarrow \Pi_1^{\mathrm{exit}}(\textstyle\bigcup_{j\in J} \hat{X}_{\hat{\jmath}}).$$

of the full subcategory $\mathscr{C}_{G,X}(J)$ spanned by the elements J and the exit path 1-category of $\bigcup_{j\in J} \hat{X}_{\hat{\jmath}}$. ◦

The following corollary recovers the result of [CL15, Theorem 1.7] in the cases to which their theorem also applies (see also Remark 3.20).

Corollary 3.18. *If in the situation of Theorem 3.9 or Corollary 3.15, the space $G\backslash X$ is paracompact and locally contractible, then it is weakly homotopy equivalent to the geometric realisation of the category $\mathscr{C}_{G,X}$. Moreover, this equivalence is functorial with respect to inclusions of unions of strata in the sense that the restriction functors of Observation 3.17 are also weak homotopy equivalences.*

Proof. We have a zig-zag of functors of ∞-categories,

$$N(\mathscr{C}_{G,X}) \longrightarrow N(\Pi_1^{\mathrm{exit}}(G\backslash X)) \longleftarrow \Pi_\infty^{\mathrm{exit}}(G\backslash X) \longrightarrow \mathrm{Sing}(G\backslash X),$$

where the first is given by the equivalence $F\colon \mathscr{C}_{G,X} \to \Pi_1^{\mathrm{exit}}(G\backslash X)$, the second is the canonical map from $\Pi_\infty^{\mathrm{exit}}(G\backslash X)$ to the nerve of its homotopy category, and the third is the weak homotopy equivalence of Corollary 2.23. Functoriality with respect to inclusions of unions of strata follows directly from Observation 3.17. □

We also have the following corollary, which is the case when all the subgroups G_i^ℓ are trivial.

Corollary 3.19. *Let $X \to I$ be a metrisably conically stratified space with I satisfying the ascending chain condition. Suppose the strata of X are path connected and weakly contractible and that we can choose conical neighbourhoods with weakly contractible strata. Let G be a discrete group acting on $X \to I$ with metrisable quotient space $G\backslash X$. If the action of G on X is free and properly discontinuous, then $G\backslash X \to G\backslash I$ is a metrisably conically stratified space. Moreover, the exit path category of $G\backslash X$ is equivalent to the nerve of its homotopy category which is equivalent to the category $\mathscr{C}_{G,X}$ with objects the elements of I, hom-sets*

$$\mathscr{C}_{G,X}(i,j) = \{g \in G \mid g.i \leq j\},$$

and composition given by the product in G. The equivalence can be chosen such that it is functorial with respect to inclusions of unions of strata. If moreover, $G\backslash X$ is locally contractible, then it is weakly homotopy equivalent to the geometric realisation $|\mathscr{C}_{G,X}|$.

Remark 3.20. The results of this section should be compared with [CL15, Theorem 1.7]. We rephrase and slightly strengthen their result in certain situations. The settings differ a great deal; for one we refrain from talking about stacks, orbifolds and orbispaces in this paper, and the conditions on the space X in Theorem 3.9 are much more restrictive. In particular, we are only able to compare with (a subset of) the situations in which [CL15, Theorem 1.7] gives a homotopy equivalence. In these cases, however, we strenghten their result by determining not just a homotopy type which is functorial with respect to inclusions of unions of strata (Corollary 3.18), but the exit path ∞-category. We also believe that the conditions needed for our result to apply may be somewhat easier to check, since they are local.

The difference between the exit path ∞-category and a homotopy equivalence which is functorial with respect to inclusions of unions of strata can be summed up as the difference between considering a space and its suspension. The mapping spaces of the exit path ∞-category keep track of the glueing data and we may lose this information by only considering the homotopy type. Consider the following example: let X be a finite CW-complex with $\pi_1(X) \neq 0$ and trivial homology, e.g. the 2-skeleton of the Poincaré homology sphere. Then the (unreduced) suspension SX is contractible, since it is simply connected and $SX \to *$ is a homology isomorphism. Stratify SX over $\{0 < 1\}$ by sending $[x, 0]$ to 0 and $[x, t]$ to 1 for $t > 0$ — this is conically stratified, as X is compact Hausdorff. The map $SX \to B[1] \cong [0, 1]$, $[x, t] \mapsto t$, is a homotopy equivalence which is functorial with respect to inclusions of unions of strata. However, X is a link space of the point $[x, 0]$ in SX, so the exit path ∞-category of SX is equivalent to the topological category $\mathscr{C}$ with objects 0 and 1 and morphism spaces $\mathscr{C}(i, i) \simeq *$, $i = 0, 1$, and $\mathscr{C}(0, 1) \simeq X$. ◦

4. The reductive Borel–Serre compactification

We introduce the Borel–Serre and reductive Borel–Serre compactifications of a locally symmetric space $\Gamma\backslash X$ associated with an arithmetic group Γ. Zucker's original definition of the reductive Borel–Serre compactification is as a quotient of the Borel–Serre compactification. Our construction is slightly different and follows the presentation of [JMSS15]: it will be the quotient of a suitable stratified space under an action of Γ allowing us to apply the calculational tools developed in the previous section. We refer the reader to the original papers [BS73] and [Zuc83] and also to [GHM94],[BJ06] and [JMSS15] for more details. In Section 4.2 we interpret these spaces as poset-stratified spaces and determine their exit path ∞-categories, proving the main result of this paper.

4.1. **Locally symmetric spaces and compactifications.** Let $\mathbf{G}$ be a connected reductive linear algebraic group defined over $\mathbb{Q}$ and let $\Gamma \leq \mathbf{G}(\mathbb{Q})$ be an arithmetic group. We will assume that the centre of $\mathbf{G}$ is anisotropic over $\mathbb{Q}$, i.e. of $\mathbb{Q}$-rank 0 (see [GHM94, §3], [BS73, §5.0] for details on why we can reduce to this case). Choose a maximal compact subgroup $K \leq G = \mathbf{G}(\mathbb{R})$ and consider the symmetric space $X \cong G/K$ of maximal compact subgroups of G and denote by $x_0 \in X$ the basepoint corresponding to K. The space X is homeomorphic to Euclidean space and the action of Γ by left multiplication is properly discontinuous. If Γ is torsion-free then the quotient $\Gamma\backslash X$ is a locally symmetric space which by the Godement compactness criterion is compact if and only if $\mathbf{G}$ has $\mathbb{Q}$-rank 0 ([BJ06, III.2.15]). From now on we assume that $\mathbf{G}$ has positive $\mathbb{Q}$-rank. We will need to assume that Γ is neat later on: recall that a subgroup $H \subset \mathbf{G}(\mathbb{Q})$ is *neat*, if for some (hence any) faithful representation $\rho\colon \mathbf{G} \to \mathbf{GL}_n$ over $\mathbb{Q}$, the subgroup of $\mathbb{C}^\times$ generated by the eigenvalues of $\rho(h)$ is torsion-free for all $h \in H$. If H is neat, then it is torsion-free. Any arithmetic group contains a finite index neat subgroup ([Bor69, §17.6]).

Given a rational parabolic subgroup $\mathbf{P} \leq \mathbf{G}$, denote by $\mathbf{N_P} \leq \mathbf{P}$ the unipotent radical of $\mathbf{P}$ and by $\mathbf{L_P} = \mathbf{P}/\mathbf{N_P}$ the Levi quotient. Let $\mathbf{S_P}$ denote the maximal $\mathbb{Q}$-split torus in the centre of $\mathbf{L_P}$, and let $\mathbf{M_P} = \bigcap_\chi \ker \chi^2$ denote the intersection of the kernels of the squares of all rationally defined characters on $\mathbf{L_P}$. Write $A_\mathbf{P} = \mathbf{S_P}(\mathbb{R})^0$ for the identity component of the real points of $\mathbf{S_P}$, and $M_\mathbf{P} = \mathbf{M_P}(\mathbb{R})$. Then the real points $L_\mathbf{P} = \mathbf{L_P}(\mathbb{R})$ has a direct sum decomposition $L_\mathbf{P} = A_\mathbf{P} \times M_\mathbf{P}$, which induces the *rational Langlands decomposition* of $P = \mathbf{P}(\mathbb{R})$:

$$P \cong N_\mathbf{P} \times A_{\mathbf{P},x_0} \times M_{\mathbf{P},x_0},$$

where $N_\mathbf{P} = \mathbf{N_P}(\mathbb{R})$, and $A_{\mathbf{P},x_0}$ and $M_{\mathbf{P},x_0}$ are the lifts of $A_\mathbf{P}$ and $M_\mathbf{P}$ to the unique Levi subgroup which is stable under the extended Cartan involution of G associated with K. Since $G = PK$, P acts transitively on X, and so the Langlands decomposition of P gives rise to the *horospherical decomposition* of X

$$X \cong N_\mathbf{P} \times A_{\mathbf{P},x_0} \times X_{\mathbf{P},x_0},$$

where $X_{\mathbf{P},x_0}$ is the symmetric space

$$X_{\mathbf{P},x_0} = M_{\mathbf{P},x_0}/(M_{\mathbf{P},x_0} \cap K) \cong L_\mathbf{P}/A_\mathbf{P}K_\mathbf{P},$$

where $K_\mathbf{P} \leq M_\mathbf{P}$ corresponds to $M_{\mathbf{P},x_0} \cap K$. We define the *geodesic action* of $A_\mathbf{P}$ on X by identifying $A_\mathbf{P}$ with the lift $A_{\mathbf{P},x_0}$ and letting it act on X by the translation action on the $A_{\mathbf{P},x_0}$-factor of the horospherical decomposition. This action turns out to be independent of the basepoint x_0 ([BS73, §3.2]). From now on we omit the reference to the basepoint x_0.

For every rational parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ define the *Borel–Serre boundary component* as

$$e(\mathbf{P}) = N_{\mathbf{P}} \times X_{\mathbf{P}} \cong X/A_{\mathbf{P}}$$

according to the decompositions above. As a set, we define the *partial Borel–Serre compactification* as the disjoint union of the Borel–Serre boundary components for all rational parabolic subgroups of $\mathbf{G}$.

$$\overline{X}^{BS} := \coprod_{\mathbf{P}} e(\mathbf{P}),$$

where we also interpret $\mathbf{G}$ as a parabolic subgroup of itself with $e(\mathbf{G}) = X_{\mathbf{G}} = X$.

To define the topology on $\overline{X}^{BS}$, we define corners associated with the rational parabolic subgroups which simultaneously equip $\overline{X}^{BS}$ with the structure of a manifold with corners. Let $\mathbf{P}$ be a rational parabolic subgroup of $\mathbf{G}$ and let $\Delta_{\mathbf{P}}$ denote the simple roots of P with respect to $A_{\mathbf{P}}$ ([BJ06, III.1.13]). We denote the value of a character α on $a \in A_{\mathbf{P}}$ by a^{α} and make the identification

$$A_{\mathbf{P}} \xrightarrow{\cong} (\mathbb{R}_{>0})^{\Delta_{\mathbf{P}}}, \quad a \mapsto (a^{-\alpha})_{\alpha \in \Delta_{\mathbf{P}}}.$$

The closure of $A_{\mathbf{P}}$ in $\mathbb{R}^{\Delta_{\mathbf{P}}}$ is $(\mathbb{R}_{\geq 0})^{\Delta_{\mathbf{P}}}$ and we denote this by $\overline{A_{\mathbf{P}}}$. The rational parabolic subgroups containing $\mathbf{P}$ correspond bijectively to subsets of $\Delta_{\mathbf{P}}$. Denote by $\Delta_{\mathbf{P}}^{\mathbf{Q}} \subseteq \Delta_{\mathbf{P}}$ the subset corresponding to $\mathbf{Q} \geq \mathbf{P}$ and define a point $o_{\mathbf{Q}} \in \overline{A_{\mathbf{P}}}$ with coordinates $o_{\mathbf{Q}}^{-\alpha} = 1$ for all $\alpha \in \Delta_{\mathbf{P}}^{\mathbf{Q}}$ and $o_{\mathbf{Q}}^{-\alpha} = 0$ for $\alpha \notin \Delta_{\mathbf{P}}^{\mathbf{Q}}$. Note that $o_{\mathbf{P}}$ corresponds to the origin in $\mathbb{R}^{\Delta_{\mathbf{P}}}$ and that $o_{\mathbf{G}}$ corresponds to the point all of whose coordinates are 1. For every rational parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ define the *corner associated with* $\mathbf{P}$ as

$$X(\mathbf{P}) = \overline{A_{\mathbf{P}}} \times e(\mathbf{P}) = \overline{A_{\mathbf{P}}} \times N_{\mathbf{P}} \times X_{\mathbf{P}},$$

and for $\mathbf{Q} \geq \mathbf{P}$, we identify $e(\mathbf{Q})$ with $(A_{\mathbf{P}} \cdot o_{\mathbf{Q}}) \times N_{\mathbf{P}} \times X_{\mathbf{P}}$ ([BJ06, III.5.6]). In particular, we identify $e(\mathbf{P})$ with $\{o_{\mathbf{P}}\} \times N_{\mathbf{P}} \times X_{\mathbf{P}}$ and X with $A_{\mathbf{P}} \times N_{\mathbf{P}} \times X_{\mathbf{P}}$. Equip $\overline{X}^{BS}$ with the finest topology such that for all rational parabolic subgroups $\mathbf{P}$ of $\mathbf{G}$, the inclusion

$$X(\mathbf{P}) \cong \coprod_{\mathbf{Q} \geq \mathbf{P}} e(\mathbf{Q}) \longrightarrow \coprod_{\mathbf{Q}} e(\mathbf{Q}) = \overline{X}^{BS}$$

is an open embedding. This equips $\overline{X}^{BS}$ with the structure of a real analytic manifold with corners which is Hausdorff and paracompact ([BS73, Theorem 7.8]).

For each rational parabolic subgroup $\mathbf{P}$, the action of P on X descends to an action on the boundary component $e(\mathbf{P})$. The action of $\mathbf{G}(\mathbb{Q})$ on X can be extended to an action on $\overline{X}^{BS}$ which permutes the boundary components, $g.e(\mathbf{P}) = e({}^{g}\mathbf{P})$, and which restricts to the action of $\mathbf{P}(\mathbb{Q})$ on $e(\mathbf{P})$ ([BS73, Proposition 7.6],[BJ06, III.5.13]).

The action of Γ on $\overline{X}^{BS}$ is properly discontinuous and the quotient $\Gamma\backslash\overline{X}^{BS}$ is compact Hausdorff ([BS73, Theorem 9.3],[BJ06, III.5.14]). If Γ is torsion-free, then the action is free and the quotient map $\overline{X}^{BS} \to \Gamma\backslash\overline{X}^{BS}$ is a local homeomorphism. In particular, $\Gamma\backslash\overline{X}^{BS}$ is also a manifold with corners. In this case, since X can be identified with the interior of $\overline{X}^{BS}$, the spaces $\Gamma\backslash\overline{X}^{BS}$ and $\Gamma\backslash X$ are homotopy equivalent and models for the classifying space of Γ.

We move on to define the reductive Borel–Serre compactification. For every rational parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ define the *reductive Borel–Serre boundary component* as

$$\hat{e}(\mathbf{P}) = X_{\mathbf{P}}$$

so that we have a projection $e(\mathbf{P}) \to \hat{e}(\mathbf{P})$ dropping the factor $N_{\mathbf{P}}$ in the Borel–Serre boundary component. We define the *partial reductive Borel–Serre compactification* as a set

$$\overline{X}^{RBS} = \coprod_{\mathbf{P}} \hat{e}(\mathbf{P}),$$

where we once again interpret $\mathbf{G}$ as a parabolic subgroup with $\hat{e}(\mathbf{G}) = X_G = X$.

The projections $e(\mathbf{P}) \to \hat{e}(\mathbf{P})$ define a surjection $\overline{X}^{BS} \to \overline{X}^{RBS}$ and we equip $\overline{X}^{RBS}$ with the quotient topology. The action of $\mathbf{G}(\mathbb{Q})$ on $\overline{X}^{BS}$ descends to a continuous action on $\overline{X}^{RBS}$ and the quotient $\Gamma\backslash\overline{X}^{RBS}$ is a compact Hausdorff space ([JMSS15, Lemma 2.4]).

Remark 4.1. It should be noted here that the quotient topology on $\overline{X}^{RBS}$ does *not* agree with that of the uniform construction in [BJ06] which is much weaker. We believe that the uniform construction agrees with the weaker topology $(\overline{X}^{RBS})^w$ of Remark 3.16 with respect to the action of Γ. $\circ$

We have a commutative diagram of quotient maps as on the left below, which, if Γ is neat, restricts to a commutative diagram of fibre bundles as on the right for each rational parabolic subgroup $\mathbf{P}$, where $\Gamma_{\mathbf{P}} = \Gamma \cap \mathbf{P}(\mathbb{Q})$ and $\Gamma_{\mathbf{L_P}} = \Gamma_{\mathbf{P}}/\Gamma_{\mathbf{N_P}}$ with $\Gamma_{\mathbf{N_P}} = \Gamma \cap \mathbf{N_P}(\mathbb{Q})$. The fibre of the lower horizontal map is the nilmanifold $\Gamma_{\mathbf{N_P}}\backslash N_{\mathbf{P}}$. Zucker originally defined the reductive Borel–Serre compactification as the quotient of $\Gamma\backslash\overline{X}^{BS}$ given by collapsing these nilmanifold fibres.

$$\begin{array}{ccc} \overline{X}^{BS} & \longrightarrow & \overline{X}^{RBS} \\ \downarrow & & \downarrow \\ \Gamma\backslash\overline{X}^{BS} & \longrightarrow & \Gamma\backslash\overline{X}^{RBS} \end{array} \qquad \begin{array}{ccc} e(\mathbf{P}) & \longrightarrow & \hat{e}(\mathbf{P}) \\ \downarrow & & \downarrow \\ \Gamma_{\mathbf{P}}\backslash e(\mathbf{P}) & \longrightarrow & \Gamma_{\mathbf{L_P}}\backslash\hat{e}(\mathbf{P}) \end{array}$$

We will need the following observation: the spaces $\overline{X}^{BS}$, $\Gamma\backslash\overline{X}^{BS}$ and $\Gamma\backslash\overline{X}^{RBS}$ are metrisable. Indeed, the partial Borel–Serre compactification is a second-countable manifold with corners ([BS73, Theorem 7.8]), the Borel–Serre compactification is a compact manifold with corners, and the reductive Borel–Serre compactification is compact Hausdorff and locally metrisable.

4.2. **Stratifications and exit path ∞-categories.** In this section we use the results of Section 3.2 to determine the exit path ∞-categories of the Borel–Serre and reductive Borel–Serre compactifications.

We stratify the partial compactifications over the poset $\mathscr{P}$ of rational parabolic subgroups in the obvious way, sending the boundary component $e(\mathbf{P})$ respectively $\hat{e}(\mathbf{P})$ to $\mathbf{P}$. For the partial Borel–Serre compactification, this is the natural stratification of $\overline{X}^{BS}$ as a manifold with corners; the boundary component $e(\mathbf{P})$ is of codimension $\dim A_{\mathbf{P}}$. The action of Γ on $\overline{X}^{BS}$ and $\overline{X}^{RBS}$ is a stratum preserving continuous action with Γ acting on $\mathscr{P}$ by conjugation. Since a parabolic subgroup is its own normaliser, the stabiliser of $\mathbf{P}$ is $\Gamma_{\mathbf{P}} = \Gamma \cap \mathbf{P}(\mathbb{Q})$ in both cases.

Assume Γ to be torsion free. The action of Γ on $\overline{X}^{BS}$ is free and properly discontinuous, the strata $e(\mathbf{P})$ are contractible and locally contractible, and, being a manifold with corners, $\overline{X}^{BS}$ is metrisably conically stratified and the link spaces have contractible strata. Therefore Corollary 3.19 applies.

Theorem 4.2. *The exit path ∞-categories of the partial Borel–Serre compactification $\overline{X}^{BS}$ and the Borel–Serre compactification $\Gamma\backslash\overline{X}^{BS}$ are canonically equivalent to the nerves of their homotopy categories. The homotopy category $\Pi_1^{\mathrm{exit}}(\overline{X}^{BS})$ is equivalent to the poset $\mathscr{P}$, and $\Pi_1^{\mathrm{exit}}(\Gamma\backslash\overline{X}^{BS})$ is equivalent to the category $\mathscr{C}_\Gamma^{BS}$ with objects the rational parabolic subgroups of $\mathbf{G}$ and hom-sets*

$$\mathscr{C}_\Gamma^{BS}(\mathbf{P},\mathbf{Q}) = \{\gamma \in \Gamma \mid {}^\gamma\mathbf{P} \leq \mathbf{Q}\}, \quad \textit{for all} \quad \mathbf{P},\mathbf{Q} \in \mathscr{P},$$

and composition given by the product in Γ.

Identifying the exit path ∞-category of the reductive Borel–Serre compactification requires a little more work in order to determine appropriate conical neighbourhoods. Moreover, the partial reductive Borel–Serre compactification is "topologically conically stratified" on non-compact link spaces, so the exit path simplicial set is not necessarily an ∞-category. We exploit that the paths in $\overline{X}^{BS}$ descend to define a compatible collection paths in $\overline{X}^{RBS}$ and apply Corollary 3.15. We now assume Γ to be neat.

Theorem 4.3. *The exit path ∞-category of the reductive Borel–Serre compactification $\Gamma\backslash\overline{X}^{RBS}$ is canonically equivalent to the nerve of its homotopy category. The homotopy category $\Pi_1^{\mathrm{exit}}(\Gamma\backslash\overline{X}^{RBS})$ is equivalent to the category $\mathscr{C}_\Gamma^{RBS}$ with objects the rational parabolic subgroups of $\mathbf{G}$ and hom-sets*

$$\mathscr{C}_\Gamma^{RBS}(\mathbf{P},\mathbf{Q}) = \{\gamma \in \Gamma \mid {}^\gamma\mathbf{P} \leq \mathbf{Q}\}/\Gamma_{\mathbf{N_P}}, \quad \textit{for all} \quad \mathbf{P},\mathbf{Q} \in \mathscr{P},$$

where $\Gamma_{\mathbf{N_P}}$ acts by right multiplication, and composition is given by multiplication of representatives.

Proof. We will show that the stratified space $\overline{X}^{RBS} \to \mathscr{P}$ equipped with the action of Γ satisfies the conditions of Corollary 3.15. Note first of all that the fixing group $\Gamma_{\mathbf{P}}^{\ell}$ of the action of Γ on $\overline{X}^{RBS}$ is $\Gamma_{\mathbf{P}}^{\ell} = \Gamma_{\mathbf{N_P}}$.

For all $\mathbf{P} \in \mathscr{P}$, stratify $\overline{A_{\mathbf{P}}}$ over $\mathscr{P}_{\geq \mathbf{P}}$ by sending $A_{\mathbf{P}} \cdot o_{\mathbf{Q}}$ to $\mathbf{Q}$ for all $\mathbf{Q} \geq \mathbf{P}$. These stratifications are compatible with the stratification of $\overline{X}^{BS}$ as we have identified $e(\mathbf{Q})$ with $(A_{\mathbf{P}} \cdot o_{\mathbf{Q}}) \times N_{\mathbf{P}} \times X_{\mathbf{P}} \subseteq X(\mathbf{P})$. For any $t > 0$ and any rational parabolic subgroup $\mathbf{P}$, set

$$\overline{A_{\mathbf{P}}}(t) := \{a \in \overline{A_{\mathbf{P}}} \mid a^{-\alpha} < t \text{ for all } \alpha \in \Delta_{\mathbf{P}}\} \subseteq \overline{A_{\mathbf{P}}}$$

stratified as a subspace of $\overline{A_{\mathbf{P}}}$. Note that $\overline{A_{\mathbf{P}}}(t) \cong [0,t)^{\Delta_{\mathbf{P}}}$.

Let $\mathbf{P}$ be a rational parabolic subgroup of $\mathbf{G}$. The group $\Gamma_{\mathbf{L_P}}$ is torsion free, as Γ is neat, so it acts freely and properly discontinuously on the stratum $\hat{e}(\mathbf{P})$, Hence, we may choose an open, relatively compact and contractible subset $W \subseteq \hat{e}(\mathbf{P})$ such that

$$\{\gamma \in \Gamma_{\mathbf{P}} \mid \gamma.W \cap W \neq \emptyset\} = \Gamma_{\mathbf{N_P}},$$

and we can view W as a subspace of the quotient $\Gamma_{\mathbf{L_P}} \backslash \hat{e}(\mathbf{P})$. Having compact fibres, the fibre bundle $\Gamma_{\mathbf{P}} \backslash e(\mathbf{P}) \to \Gamma_{\mathbf{L_P}} \backslash \hat{e}(\mathbf{P})$ is proper, and therefore the preimage $V \subseteq \Gamma_{\mathbf{P}} \backslash e(\mathbf{P})$ of W under this map is relatively compact. The preimage of V in $e(\mathbf{P})$ is $N_{\mathbf{P}} \times W$, and it follows that there is a $t > 0$ such that the equivalence relations induced by Γ and $\Gamma_{\mathbf{P}}$ on the subspace

$$\overline{A_{\mathbf{P}}}(t) \times N_{\mathbf{P}} \times W \subseteq \overline{A_{\mathbf{P}}}(t) \times e(\mathbf{P}) \subseteq X(\mathbf{P}) \subseteq \overline{X}^{BS}$$

of the partial Borel–Serre compactification agree ([Zuc86, 1.5]).

Consider the subspace

$$\ell_{\mathbf{P}} = \{a \in \overline{A_{\mathbf{P}}} \mid \textstyle\sum_{\alpha \in \Delta_{\mathbf{P}}} a^{-\alpha} = 1\} \subseteq \overline{A_{\mathbf{P}}}$$

and stratify it accordingly over $\mathscr{P}_{>\mathbf{P}}$. This is just the standard (topological) $(\dim A_{\mathbf{P}} - 1)$-simplex stratified as a manifold with corners; the $\mathbf{Q}$'th stratum of $\ell_{\mathbf{P}}$ is $\ell_{\mathbf{PQ}} = \ell_{\mathbf{P}} \cap (A_{\mathbf{P}} \cdot o_{\mathbf{Q}})$. Let $C(\ell_{\mathbf{P}}) \to \mathscr{P}_{\geq \mathbf{P}}$ denote the stratified cone on $\ell_{\mathbf{P}}$ (as $\ell_{\mathbf{P}}$ is compact Hausdorff, this agrees with the stratified topological cone $C^t(\ell_{\mathbf{P}})$). There is a stratum preserving embedding

$$C(\ell_{\mathbf{P}}) \to \overline{A_{\mathbf{P}}}(t)$$

given by sending $[a,s] \in C(\ell_{\mathbf{P}})$ to the point $b \in \overline{A_{\mathbf{P}}}(t)$ satisfying $b^{-\alpha} = sta^{-\alpha}$.

Define a stratified space $\mathscr{L}_{\mathbf{P}} \to \mathscr{P}_{>\mathbf{P}}$ as the quotient of $\ell_{\mathbf{P}} \times N_{\mathbf{P}} \to \mathscr{P}_{>\mathbf{P}}$ given by applying the quotients

$$\ell_{\mathbf{PQ}} \times N_{\mathbf{P}} \to \ell_{\mathbf{PQ}} \times N_{\mathbf{Q}} \backslash N_{\mathbf{P}}$$

to the strata of $\ell_{\mathbf{P}} \times N_{\mathbf{P}}$. The embedding

$$C(\ell_{\mathbf{P}}) \times N_{\mathbf{P}} \times W \hookrightarrow \overline{A_{\mathbf{P}}}(t) \times N_{\mathbf{P}} \times W \hookrightarrow \overline{X}^{BS}$$

descends to define a stratum preserving embedding

$$C^t(\mathscr{L}_{\mathbf{P}}) \times W \hookrightarrow \overline{X}^{RBS}$$

which restricts to the identity on $\{*\} \times W$ where $*$ is the apex of $C^t(\mathscr{L}_{\mathbf{P}})$ — note that as $\mathscr{L}_{\mathbf{P}}$ is non-compact, the stratified topological cone is different from the stratified cone $C(\mathscr{L}_{\mathbf{P}})$. Let U denote the image of the above map, so that we have a stratum preserving homeomorphism

$$\varphi\colon C^t(\mathscr{L}_{\mathbf{P}}) \times W \xrightarrow{\cong} U.$$

Our choice of W and t imply that

$$\{\gamma \in \Gamma \mid \gamma.U \cap U \neq \emptyset\} = \Gamma_{\mathbf{N_P}}.$$

Moreover, φ is $\Gamma_{\mathbf{N_P}}$-equivariant, when we let $\Gamma_{\mathbf{N_P}}$ act on $\mathscr{L}_{\mathbf{P}}$ by acting on the second factor of the $\mathbf{Q}$'th stratum $\ell_{\mathbf{PQ}} \times N_{\mathbf{Q}} \backslash N_{\mathbf{P}}$ via the quotient $\Gamma_{\mathbf{N_Q}} \backslash \Gamma_{\mathbf{N_P}}$. The quotient $\Gamma_{\mathbf{N_P}} \backslash \mathscr{L}_{\mathbf{P}}$ is compact as it factors through $\ell_{\mathbf{P}} \times \Gamma_{\mathbf{N_P}} \backslash N_{\mathbf{P}}$. Hence, the conditions of Corollary 3.14 are satisfied.

For a collection of compatible exit paths, we may choose the ones coming from the partial Borel–Serre compactification. If $x, y \in \overline{X}^{BS}$ are connected by a morphism in $\Pi_1^{\mathrm{exit}}(\overline{X}^{BS}) \simeq \mathscr{P}$, then the morphism is unique, and we choose an exit path $p_{x \to y}$ in $\overline{X}^{BS}$ representing this morphism (for $x = y$, we choose the trivial loop). Let $\mu \colon \overline{X}^{BS} \to \overline{X}^{RBS}$ denote the quotient map, and fix basepoints $x_{\mathbf{P}} \in e(\mathbf{P})$ for all $\mathbf{P}$. For any $\mathbf{P}, \mathbf{Q}$ and $\gamma \in \Gamma$ with ${}^{\gamma}\mathbf{P} \leq \mathbf{Q}$, we choose the path $\mu \circ p_{x_{\mathbf{P}} \to \gamma^{-1}.x_{\mathbf{Q}}}$ in $\overline{X}^{RBS}$. Then Corollary 3.15 applies and we are done. ☐

Remark 4.4. The uniform construction of [BJ06] gives rise to a conically stratified space equipped with an action of Γ whose quotient space agrees with $\Gamma \backslash \overline{X}^{RBS}$. We believe that one can apply Theorem 3.9 to this space directly, but by using the Borel–Serre compactification to define a collection of compatible exit paths, we save ourselves the trouble of having to analyse this topology in detail. ○

Remark 4.5. We wish to remark that the identification of neighbourhoods and link spaces in the proof of Theorem 4.3 make no claim to originality (see [JMSS15], [GHM94], [Zuc86], [BJ06]). We just make a detailed analysis in order to verify the conditions of Corollary 3.14.○

Observation 4.6. The equivalences

$$\mathscr{C}_{\Gamma}^{BS} \to \Pi_1^{\mathrm{exit}}(\Gamma \backslash \overline{X}^{BS}) \quad \text{and} \quad \mathscr{C}_{\Gamma}^{RBS} \to \Pi_1^{\mathrm{exit}}(\Gamma \backslash \overline{X}^{RBS})$$

of the theorems above can be defined compatibly as follows: for any rational parabolic subgroup $\mathbf{P}$, choose a basepoint $x_{\mathbf{P}} \in e(\mathbf{P})$ in the Borel–Serre boundary component — note that it in fact suffices to make a choice of basepoint $x_0 \in X$, i.e. a choice of maximal compact subgroup $K \leq G$, as this gives canonical choices of basepoints in the boundary components. For any two points $x, x' \in \overline{X}^{BS}$, if there is a morphism $x \to x'$ in $\Pi_1^{\mathrm{exit}}(\overline{X}^{BS})$, then it is unique, and we denote it by $p_{x \to x'}$.

Recall the commutative diagram of quotient maps below. We denote by $(-)_*$ the induced map of exit path categories whenever this makes sense (the exit path simplicial set of the partial reductive Borel–Serre compactification is not necessarily an ∞-category).

$$\begin{array}{ccc} \overline{X}^{BS} & \xrightarrow{\mu} & \overline{X}^{RBS} \\ \pi \downarrow & & \downarrow \rho \\ \Gamma \backslash \overline{X}^{BS} & \xrightarrow[\nu]{} & \Gamma \backslash \overline{X}^{RBS} \end{array}$$

With respect to the basepoints $x_{\mathbf{P}} \in e(\mathbf{P})$, the equivalences

$$F^{BS} \colon \mathscr{C}_{\Gamma}^{BS} \to \Pi_1^{\mathrm{exit}}(\Gamma \backslash \overline{X}^{BS}), \quad \text{and} \quad F^{RBS} \colon \mathscr{C}_{\Gamma}^{RBS} \to \Pi_1^{\mathrm{exit}}(\Gamma \backslash \overline{X}^{RBS})$$

are given by

$$F^{BS}(\mathbf{P}) = \pi(x_{\mathbf{P}}) \quad \text{and} \quad F^{RBS}(\mathbf{P}) = \rho(\mu(x_{\mathbf{P}})),$$

on objects, and on morphisms by

$$\begin{aligned} F^{BS}(\gamma\colon \mathbf{P} \to \mathbf{Q}) &= \pi_*(p_{x_{\mathbf{P}} \to \gamma^{-1}.x_{\mathbf{Q}}}), \\ F^{RBS}([\gamma]\colon \mathbf{P} \to \mathbf{Q}) &= (\rho \circ \mu)_*(p_{x_{\mathbf{P}} \to \gamma^{-1}.x_{\mathbf{Q}}}). \end{aligned}$$

The following diagram commutes

$$\begin{array}{ccccc} \mathscr{P} & \longrightarrow & \mathscr{C}_\Gamma^{BS} & \longrightarrow & \mathscr{C}_\Gamma^{RBS} \\ \downarrow & & \downarrow F^{BS} & & \downarrow F^{RBS} \\ \Pi_1^{\text{exit}}(\overline{X}^{BS}) & \xrightarrow{\pi_*} & \Pi_1^{\text{exit}}(\Gamma\backslash\overline{X}^{BS}) & \xrightarrow{\nu_*} & \Pi_1^{\text{exit}}(\Gamma\backslash\overline{X}^{RBS}) \end{array}$$

when $\mathscr{P} \to \mathscr{C}_\Gamma^{BS}$ is the inclusion as a subcategory sending the unique morphism $\mathbf{P} \leq \mathbf{Q}$ to the morphism $\mathbf{P} \to \mathbf{Q}$ given by the identity element in Γ; the functor $\mathscr{C}_\Gamma^{BS} \to \mathscr{C}_\Gamma^{RBS}$ is given by the obvious quotients on the hom-sets; and $\mathscr{P} \xrightarrow{\sim} \Pi_1^{\text{exit}}(\overline{X}^{BS})$ is given by sending $\mathbf{P}$ to $x_{\mathbf{P}}$. ◦

5. Consequences: homotopy type and the constructible derived category

We derive some immediate corollaries to Theorem 4.3, the identification of the exit path ∞-category of the reductive Borel–Serre compactification. We determine the homotopy type of the reductive Borel–Serre compactification and in particular the fundamental group, and we review the classification of constructible sheaves and the constructible derived category.

Let $\mathbf{G}$ be a connected reductive linear algebraic group over $\mathbb{Q}$ of positive $\mathbb{Q}$-rank whose centre is anisotropic over $\mathbb{Q}$. For a given neat arithmetic group $\Gamma \leq \mathbf{G}(\mathbb{Q})$, let $\Gamma\backslash\overline{X}^{RBS}$ denote the reductive Borel–Serre compactification of the associated locally symmetric space $\Gamma\backslash X$ as defined in Section 4.1. Let $\mathscr{C}_\Gamma^{RBS}$ be the category defined in Theorem 4.3.

Since the inclusion of the exit path ∞-category into the singular set is a weak homotopy equivalence of simplicial sets, we recover the homotopy type of the reductive Borel–Serre compactification (Corollary 3.18).

Corollary 5.1. *The reductive Borel–Serre compactification* $\Gamma\backslash\overline{X}^{RBS}$ *is weakly homotopy equivalent to the geometric realisation of* $\mathscr{C}_\Gamma^{RBS}$.

The fundamental group of the geometric realisation of a small category is the localisation of the category at all morphisms ([Qui73b, Proposition 1]). We thus recover the following result of Ji–Murty–Saper–Scherk ([JMSS15, Corollary 5.3]).

Corollary 5.2. *The fundamental group of the reductive Borel–Serre compactification* $\Gamma\backslash\overline{X}^{RBS}$ *is isomorphic to the group* Γ/E_Γ*, where* $E_\Gamma \lhd \Gamma$ *is the normal subgroup generated by the subgroups* $\Gamma_{\mathbf{N_P}} \leq \Gamma$ *as* $\mathbf{P}$ *runs through all rational parabolic subgroups of* $\mathbf{G}$.

Remark 5.3. One should think of E_Γ as the subgroup of "elementary matrices", cf. the case $\Gamma \leq \mathrm{GL}_n(\mathbb{Z})$, $n \geq 3$. ○

Having determined the exit path ∞-category, we get a classification of constructible sheaves on $\Gamma\backslash\overline{X}^{RBS}$ as representations of $\mathscr{C}_\Gamma^{RBS}$ (Theorem 2.21).

Corollary 5.4. *For any compactly generated ∞-category $\mathscr{C}$, there is an equivalence of ∞-categories*

$$\Psi_X\colon \mathrm{Fun}(N(\mathscr{C}_\Gamma^{RBS}), \mathscr{C}) \to \mathrm{Shv}_{\mathrm{cbl}}(\Gamma\backslash\overline{X}^{RBS}, \mathscr{C}).$$

Now, as the exit path ∞-category of $\Gamma\backslash\overline{X}^{RBS}$ is equivalent to the nerve of its homotopy category, we can apply Theorem 2.28 and Corollary 2.29 to express the constructible derived category as a derived functor category.

Proposition 5.5. *Let R be an associative ring. There is an equivalence of ∞-categories*

$$\mathrm{Shv}_{\mathrm{cbl}}(\Gamma\backslash\overline{X}^{RBS}, \mathrm{LMod}_R) \simeq \mathscr{D}(\mathrm{Fun}\left(\mathscr{C}_\Gamma^{RBS}, \mathrm{LMod}_R^1\right)),$$

which restricts to an equivalence

$$\mathrm{Shv}_{\mathrm{cbl,cpt}}(\Gamma\backslash\overline{X}^{RBS}, \mathrm{LMod}_R) \simeq \mathscr{D}_{\mathrm{cpt}}(\mathrm{Fun}\left(\mathscr{C}_\Gamma^{RBS}, \mathrm{LMod}_R^1\right)),$$

where $\mathscr{D}_{\mathrm{cpt}}(\mathrm{Fun}\left(\mathscr{C}_\Gamma^{RBS}, \mathrm{LMod}_R^1\right)) \subset \mathscr{D}(\mathrm{Fun}\left(\mathscr{C}_\Gamma^{RBS}, \mathrm{LMod}_R^1\right))$ is the full subcategory spanned by the complexes of functors $F_\bullet$ such that the complex $F_\bullet(x)$ is perfect for all $x \in X$.

Corollary 5.6. *Let R be an associative ring. There is an equivalence of categories*

$$D_{\mathrm{cbl}}(\mathrm{Shv}_1(\Gamma\backslash\overline{X}^{RBS}, R)) \simeq D(\mathrm{Fun}(\mathscr{C}_\Gamma^{RBS}, \mathrm{LMod}_R^1))$$

which restricts to an equivalence

$$D_{\mathrm{cbl,cpt}}(\mathrm{Shv}_1(\Gamma\backslash\overline{X}^{RBS}, R)) \simeq D_{\mathrm{cpt}}(\mathrm{Fun}(\mathscr{C}_\Gamma^{RBS}, \mathrm{LMod}_R^1)),$$

where $D_{\mathrm{cpt}}(\mathrm{Fun}(\mathscr{C}_\Gamma^{RBS}, \mathrm{LMod}_R^1)) \subset D(\mathrm{Fun}(\mathscr{C}_\Gamma^{RBS}, \mathrm{LMod}_R^1))$ is the full subcategory spanned by the complexes of functors $F_\bullet$ such that the complex $F_\bullet(x)$ is perfect for all $x \in X$.

As mentioned earlier (Example 2.27), both intersection cohomology of $\Gamma\backslash\overline{X}^{RBS}$ and weighted cohomology of Γ are examples of constructible compact-valued complexes of sheaves on $\Gamma\backslash\overline{X}^{RBS}$ taking values in complex vector spaces, i.e. they are objects of $D_{\mathrm{cbl,cpt}}(\Gamma\backslash\overline{X}^{RBS}, \mathbb{C})$ ([GM83, §3] and [GHM94, Theorem 17.6]). In [Sap05a] and [Sap05b], Saper introduced the theory of $\mathcal{L}$-modules, a combinatorial analogue of constructible complexes of sheaves on the reductive Borel–Serre compactification. The theory is used this to prove a conjecture of Rapoport and Goresky–MacPherson relating the intersection cohomology of certain Satake compactifications with that of the reductive Borel–Serre compactification ([Rap86, GM88]). This allows one to transfer cohomological calculations from the more singular spaces, Satake compactifications, to the reductive Borel–Serre compactification.

If one thinks of $\mathscr{L}$-modules as a combinatorial analogue of constructible complexes of sheaves, then the equivalence of Corollary 5.6 can be interpreted as providing an actual combinatorial incarnation.

6. Generalisation

The category $\mathscr{C}_\Gamma^{RBS}$ was defined in Section 4.2 in terms of the poset of rational parabolic subgroups of a reductive algebraic group, their unipotent radicals and the conjugation action of Γ on this poset — it is a special case of the category $\mathscr{C}_{G,X}$ defined in Theorem 3.9 in terms of stabilisers and poset relations for a group acting on a stratified space. The object of interest was the stratified space $\Gamma\backslash\overline{X}^{RBS}$ or in the general case a stratified orbit space $G\backslash X$, but it is easy to see that the categories make sense in a more general setting. We make this precise in this section.

6.1. Construction and examples.

We define a category associated to a group action on a poset equipped with some additional data.

Construction 6.1. Let G be a group acting on a poset I. Let G_i denote the stabiliser of $i \in I$, and suppose we have a choice of subgroup $G_i^\ell \leq G_i$ for every $i \in I$ such that the following conditions hold:

(i) $G_j^\ell \leq G_i^\ell$ for all $i \leq j$.
(ii) ${}^gG_i^\ell = G_{g.i}^\ell$ for all $i \in I$, $g \in G$.

We call G_i^ℓ the *link subgroup* at i. Define a category $\mathscr{C}_{G,I}$ with objects the elements of I and hom-sets

$$\mathscr{C}(i,j) = \{g \in G \mid g.i \leq j\}/G_i^\ell,$$

where G_i^ℓ acts by right multiplication, and with composition given by multiplication of representatives in G. Properties (i) and (ii) imply that this is well-defined. ◦

Example 6.2.

(i) Let $X \to I$ be a Hausdorff stratified space such that $X_i \subset \overline{X_j}$ for all $i \leq j$. Suppose G is a discrete group acting on $X \to I$. Let for all $i \in I$, $G_i^\ell \leq G_i$ denote the subgroup which fixes X_i pointwise. This recovers the category $\mathscr{C}_{G,X}$ in the situations of Theorem 3.9 and Corollary 3.15.
(ii) For any group G and any collection of subgroups $\mathcal{C}$ which is closed under conjugation, we can view $\mathcal{C}$ as a poset and consider the action of G on $\mathcal{C}$ by conjugation and choose the trivial subgroups as the link subgroups. This recovers the transport category on the collection $\mathcal{C}$.
(iii) Let $\mathbf{G}$ be a connected linear algebraic group defined over a field k and let $\mathscr{P}$ denote the poset of k-parabolic subgroups of $\mathbf{G}$. The group $\mathbf{G}(k)$ acts on $\mathscr{P}$ by conjugation. Let for all $\mathbf{P} \in \mathscr{P}$, $\mathbf{N_P} \leq \mathbf{P}$ denote the unipotent radical and choose the k-points of these as the link subgroups: $(\mathbf{G}(k))_\mathbf{P}^\ell = \mathbf{N_P}(k) \leq \mathbf{P}(k) \leq \mathbf{N_G}(\mathbf{P})(k)$.
(iv) As an extension of the previous example, we can also consider the action of a subgroup $\Gamma \leq \mathbf{G}(k)$ and the restricted subgroups $\Gamma_\mathbf{P}^\ell = \Gamma_{\mathbf{N_P}} = \Gamma \cap \mathbf{N_P}(k)$. A special case of this recovers the category $\mathscr{C}_\Gamma^{RBS}$ of Section 4.2. If we choose the trivial subgroups $e \leq \Gamma_\mathbf{P}$ as the link subgroups, then we recover $\mathscr{C}_\Gamma^{BS}$.

(v) Let $G = (G, B, N, S, U)$ be a finite group with a split BN-pair of characteristic p (see [CR87, §69] for details). Let $\mathscr{P}$ denote the collection of parabolic subgroups of G, i.e. the subgroups P containing some conjugate of B. Then $G_P = P$ for all $P \in \mathscr{P}$ ([CR87, Theorem 65.19]). As link subgroups, consider the maximal normal p-subgroups, $O_p(P) \leq P$ (the analogue of the unipotent radical). This recovers (the opposite of) the orbit category on the p-radical subgroups of G, an object of great interest in finite group theory (see Section 6.2 below).

(vi) We can generalise (iii) and (iv) to the case of reductive group schemes: for a reductive group scheme $\mathbf{G}$ over a scheme S, consider the poset $\mathscr{P}$ of parabolic subgroups and for each $\mathbf{P} \in \mathscr{P}$, let $\mathbf{N_P}$ denote the unipotent radical of $\mathbf{P}$ ([Con14, §5.2]). Any subgroup $\Gamma \leq \mathbf{G}(S)$ acts on $\mathscr{P}$ by conjugation and we can choose the link subgroups $\Gamma^\ell_{\mathbf{P}} = \Gamma_{\mathbf{N_P}} = \Gamma \cap \mathbf{N_P}(S)$ given by the unipotent radicals. ◦

6.2. **Orbit categories and p-radical subgroups.** Let $G = (G, B, N, S, U)$ be a finite group with a split BN-pair, and consider the categories $\mathscr{C}^{BS}_G$ respectively $\mathscr{C}^{RBS}_G$ obtained from Construction 6.1 by considering the poset of parabolic subgroups of G and as link subgroups the trivial subgroups $e \leq P$ respectively the largest normal p-groups $O_p(P) \leq P$ (cf. Example 6.2 (ii) respectively (v)). There is a canonical functor $\mathscr{C}^{BS}_G \to \mathscr{C}^{RBS}_G$ which is the identity on objects and is given on hom-sets by the quotient maps

$$\{g \in G \mid {}^gP \leq Q\} \longrightarrow \{g \in G \mid {}^gP \leq Q\}/O_p(P).$$

Remark 6.3. These categories generalise the exit path categories of the Borel–Serre respectively reductive Borel–Serre compactifications and the functor generalises the one induced by the quotient map $\Gamma\backslash\overline{X}^{BS} \to \Gamma\backslash\overline{X}^{RBS}$ as found in Observation 4.6. ◦

Definition 6.4. Let G be any finite group and p a prime. A subgroup $U \leq G$ is called *p-radical* if the greatest normal p-subgroup of the normaliser of U in G is U itself, i.e. if $O_p(N_G(U)) = U$ or equivalently $O_p(N_G(U)/U) = e$. ◁

Remark 6.5. The p-radical subgroups have been studied extensively in finite group theory: they play an important role in Alperin's weight conjecture [Alp87, AF90] and the poset of p-radical subgroups and the orbit category on this collection turn out to be of great significance in group cohomology and homotopy theory of classifying spaces ([Bou84, JMO92a, JMO92b, Gro02, Gro18]). ◦

For G a finite group with a split BN-pair of characteristic p, let $\mathscr{O}(G)$ denote the orbit category of G-orbits and G-maps and denote by $\mathcal{B}^e_p(G)$ the collection of p-radical subgroups of G. Consider the transport category $\mathscr{T}_{\mathcal{B}^e_p(G)}(G)$ on the collection of p-radical subgroups of G, and the full subcategory $\mathscr{O}_{\mathcal{B}^e_p(G)}(G) \subseteq \mathscr{O}(G)$ spanned by the G-orbits whose isotropy group is a p-radical subgroup of G. There is a canonical functor

$$\mathscr{T}_{\mathcal{B}^e_p(G)}(G) \to \mathscr{O}_{\mathcal{B}^e_p(G)}(G)$$

which sends P to $G/O_p(P)$ and on hom-sets is given by inversion and taking quotients, $g \mapsto [g^{-1}]$:

$$\{g \in G \mid {}^gO_p(Q) \leq O_p(P)\} \to \{g \in G \mid O_p(Q)^g \leq O_p(P)\}/O_p(P),$$

where we use that $\mathrm{Hom}_G(G/H, G/K) \xrightarrow{\cong} \{g \in G \mid H^g \leq K\}/K$ by sending a G-map to its value on the identity coset.

The poset of parabolic subgroups of G is G-equivalent to the (opposite) poset of p-radical subgroups of G by taking normaliser and O_p respectively. This is a well-known fact and a consequence of the Borel–Tits Theorem, which says that if a closed unipotent subgroup U of a connected algebraic group $\mathbf{H}$ is equal to the unipotent radical of its normaliser, then $N_{\mathbf{H}}(U)$ is a parabolic subgroup of $\mathbf{H}$ (see [BT71, Corollary 3.2] for the general case and [BW76] for the analogous result for finite Chevalley groups). The following proposition is a simple application of this fact — we spell out the steps for clarity (see also for example [Gro02, Remark 4.3]).

Proposition 6.6. *There is a commutative diagram*

$$\begin{array}{ccc}
\mathscr{C}_G^{BS} & \longrightarrow & \mathscr{C}_G^{RBS} \\
\Psi \big\downarrow & & \big\downarrow \Phi \\
\mathscr{T}_{\mathcal{B}_p^e(G)}(G)^{op} & \longrightarrow & \mathscr{O}_{\mathcal{B}_p^e(G)}(G)^{op}
\end{array}$$

where the horizontal functors are the canonical ones and the vertical ones are isomorphisms given by

$$\Psi(P) = O_p(P) \qquad \Psi(g\colon P \to Q) = g^{-1}\colon O_p(Q) \to O_p(P),$$
$$\Phi(P) = G/O_p(P) \qquad \Phi([g]\colon P \to Q) = [g]\colon G/O_p(Q) \to G/O_p(P).$$

Proof. The functors Φ and Ψ are well-defined as $N_G(O_p(P)) = P$ for all parabolic subgroups P ([CR87, Theorem 69.10]). They are bijective on objects by the Borel–Tits theorem. To see that they are bijective on hom-sets, note that

$$\{g \in G \mid O_p(Q)^g \leq O_p(P)\} = \{g \in G \mid {}^gP \leq Q\}, \tag{1}$$

since in the case where $P \leq Q$, both sets are equal to Q (this is seen in the proof of [Gro02, Lemma 4.2] and also in [CR87, Theorem 65.19]). □

Appendix A. Homotopy links and fibrations

We provide proofs of the two fundamental results on homotopy links used in Section 2. These are elementary point-set topological proofs and the results are well-known. We include them for the sake of self-containment, and since the proofs that we have been able to locate in the literature work in much more general or slightly different settings. It also clarifies why we impose metrisability conditions on the stratified spaces.

Throughout this appendix, we write $I = [0,1]$ to ease notation. This should not be confused with the posets I appearing in the main body of the paper.

We recall the definition of the homotopy link, also given in Section 2: let X be a topological space and $Y \subseteq X$ a subspace. The *homotopy link* of Y in X is defined as the following subspace of paths

$$H(X,Y) = \{\gamma\colon I \to X \mid \gamma(0) \in Y,\ \gamma((0,1]) \subseteq X - Y\} \subset C(I,X)$$

equipped with the compact-open topology.

This is a notion from the theory of homotopically stratified sets introduced by Quinn in [Qui88] in order to study purely topological stratified phenomena: a filtered space

$$X_0 \subset X_1 \subset \cdots \subset X_n$$

is *homotopically stratified* if for all $k > i$, the subspace $X_i - X_{i-1}$ has a "homotopically well-behaved" neighbourhood in $(X_k - X_{k-1}) \cup (X_i - X_{i-1})$ (i.e. is tame, see Remark A.5) and the evaluation at zero map from the homotopy link of this pair is a fibration. These conditions provide a homotopical replacement of mapping cylinder neighbourhoods, the homotopy link being an analogue of the frontier of such a mapping cylinder neighbourhood (see also [Qui02]).

A.1. **End point evaluation fibrations.** We show that for a suitably nice pair of spaces (X,Y), the end point evaluation map $H(X,Y) \to X \times Y$ is a fibration. The following lemma explains our need to impose metrisability conditions on the stratified spaces that we consider.

Lemma A.1. *Let X be a metrisable space, $Y \subseteq X$ a subspace, and U an open neighbourhood of Y in X. There is a continuous map $\delta\colon H(X,Y) \to (0,1)$ such that for all $\gamma \in H(X,Y)$, $\gamma([0,\delta(\gamma)]) \subseteq U$.*

Proof. The homotopy link $H(X,Y)$ admits partitions of unity, being a subspace of a metrisable space $C(I,X)$ and thus itself metrisable. For any $\gamma \in H(X,Y)$, let $\delta_\gamma \in (0,1)$ such that $\gamma([0,\delta_\gamma]) \subseteq U$. The subset

$$U_\gamma = C([0,\delta_\gamma],U) \cap H(X,Y) = \{\eta \in H(X,Y) \mid \eta([0,\delta_\gamma]) \subseteq U\}$$

is an open neighbourhood of γ in $H(X,Y)$. Let $\{\rho_\gamma\}$ be a partition of unity subordinate to the cover $\{U_\gamma\}$ of $H(X,Y)$, and define $\delta\colon H(X,Y) \to (0,1)$ as $\delta = \sum_\gamma \delta_\gamma \rho_\gamma$. □

Proposition A.2. *Let X be a metrisable space, $Y \subseteq X$ a subspace, and suppose there is an open neighbourhood N of Y in X such that the evaluation at zero map $H(N,Y) \to Y$ is a fibration. Then the evaluation at zero map $e_0\colon H(X,Y) \to Y$ is a fibration.*

Proof. Let A be a topological space, and let α_0 and α as in the following diagram.

$$\begin{array}{ccc} A & \xrightarrow{\ \alpha_0\ } & H(X,Y) \\ \big\downarrow & \overset{\widetilde{\alpha}}{\nearrow} & \big\downarrow e_0 \\ A \times I & \xrightarrow[\ \alpha\ ]{} & Y \end{array}$$

Let $\delta\colon H(X,Y) \to (0,1)$ be as in Lemma A.1 for $U = N$. For any $\gamma \in H(X,Y)$ and any $0 \leq r < s \leq 1$, let $\gamma_{[r,s]}\colon I \to X$ denote the reparametrisation of the restriction of γ to $[r,s]$ and define continuous maps

$$\begin{aligned} R\colon H(X,Y) &\to H(N,Y), & \gamma &\mapsto \gamma_{[0,\delta(\gamma)]}, \\ \overline{R}\colon H(X,Y) &\to C(I, X-Y), & \gamma &\mapsto \gamma_{[\delta(\gamma),1]}. \end{aligned}$$

By assumption, $e_0\colon H(N,Y) \to Y$ is a fibration, so there is a map $\hat{\alpha}\colon A \times I \to H(N,Y)$ such that $e_0 \circ \hat{\alpha} = \alpha$ and $\hat{\alpha}(-,0) = R(\alpha_0(-))$.

Consider the diagram below with η given by

$$\eta(-,-,0) = \hat{\alpha}(-,-)(1), \quad \text{and} \quad \eta(-,0,-) = \overline{R} \circ \alpha_0,$$

where $\overline{R} \circ \alpha_0\colon A \to C(I, X-Y)$ is viewed as a map $A \times I \to X - Y$ via the exponential law. The map $\hat{\eta}$ is an extension of η using that the pair $(A \times I, A \times \{0\})$ has the homotopy extension property.

$$\begin{array}{ccc} A \times (I \times \{0\} \cup \{0\} \times I) & \xrightarrow{\eta} & X - Y \\ \iota \big\downarrow & \overset{\hat{\eta}}{\nearrow} & \\ A \times I \times I & & \end{array}$$

We can view $\hat{\eta}$ as a map $A \times I \to C(I, X-Y)$ by applying the exponential law to the second factor of I, and we define $\widetilde{\alpha}\colon A \times I \to H(X,Y)$ as the vertical concatenation of $\hat{\alpha}$ and $\hat{\eta}$:

$$\widetilde{\alpha}(a,s)(t) = \begin{cases} \hat{\alpha}(a,s)\left(\frac{t}{\delta(\alpha_0(a))}\right) & t \in [0,\, \delta(\alpha_0(a))] \\ \hat{\eta}(a,s)\left(\frac{t-\delta(\alpha_0(a))}{1-\delta(\alpha_0(a))}\right) & t \in [\delta(\alpha_0(a)),\, 1] \end{cases}$$

This is the desired lift. □

Corollary A.3. *Let X be a metrisable space, $Y \subseteq X$ a subspace, and suppose there is a neighbourhood N of Y in X such that the evaluation at zero map $H(N,Y) \to Y$ is a fibration. Then the end point evaluation map $e = e_0 \times e_1$ is a fibration:*

$$e\colon H(X,Y) \to Y \times X, \quad \gamma \mapsto (\gamma(0), \gamma(1)).$$

We leave the proof of this as an exercise for someone wanting to practice concatenation and reparametrisation of homotopies. See [Woo09] for more related fibrations.

A.2. **Homotopy links and mapping cylinder neighbourhoods.** We show that when we are only interested in the homotopical information, the homotopy link provides an adequate replacement for the link space or link bundle. For more details, see [Qui88], in particular Lemma 2.4 and its corollary, or [Qui02].

Definition A.4. Let (N,Y) be a pair of spaces. A map $r\colon N \times I \to N$ is a *nearly strict deformation retraction* into Y if it satisfies

(i) $r(-,1) = \mathrm{id}$,
(ii) $r(N,0) \subseteq Y$,
(iii) $r(N-Y,t) \subseteq N-Y$ for all $t > 0$
(iv) $r(y,t) = y$ for all $y \in Y$, $t \in I$. ◁

Remark A.5. The 'nearly strict' refers to the fact that r preserves the pair (N,Y) until the very last moment at $t = 0$, when everything is pushed into Y. In the setting of homotopically stratified sets, a subspace $Y \subseteq X$ is called tame if there exists a neighbourhood of Y in X equipped with a nearly strict deformation retraction ([Qui88]). ○

A nearly strict deformation retraction $r\colon N \times I \to N$ into a subspace $Y \subset N$ defines a continuous map $\Psi\colon N-Y \to H(N,Y)$, sending a point $x \in N-Y$ to the path $p_x\colon t \mapsto r(x,t)$ tracing the image of x under r.

Lemma A.6. *Let $Y \subseteq N$ be a pair of topological spaces and suppose there is a nearly strict deformation retraction $r\colon N \times I \to N$ into Y. Then the map $\Psi\colon N-Y \to H(N,Y)$, $x \mapsto p_x$, defined above is a homotopy equivalence with homotopy inverse given by evaluation at 1, $e_1\colon H(N,Y) \to N-Y$.*

Proof. We first note that $e_1 \circ \Psi = \mathrm{id}$. The map $H\colon H(N,Y) \times I \to H(N,Y)$ given by

$$H(\gamma,s)(t) = \begin{cases} r(\gamma(t), 1+2s(t-1)), & s \leq \frac{1}{2} \\ r(\gamma(2(t-ts+s)-1), t), & s \geq \frac{1}{2} \end{cases}$$

provides a homotopy $\mathrm{id} \sim \Psi \circ e_1$. □

Lemma A.7. *Let X be a metrisable space, $Y \subseteq X$ a subspace, and N an open neighbourhood of Y. The inclusion $\iota\colon H(N,Y) \to H(X,Y)$ is a homotopy equivalence.*

Proof. Let $\delta\colon H(X,Y) \to (0,1)$ be as in Lemma A.1 for $U = N$, and define a map

$$G\colon H(X,Y) \times I \to H(X,Y), \qquad G(\gamma,s)(t) = \gamma(t(s\delta(\gamma)+1-s)) = \gamma_{[0,s\delta(\gamma)+1-s]}(t).$$

Then $G_1 = G(-,1)$, $\gamma \mapsto \gamma_{[0,\delta(\gamma)]}$, is a homotopy inverse to ι with G providing the desired homotopies $G\colon \mathrm{id}_{H(X,Y)} \sim \iota \circ G_1$ and $G \circ \iota\colon \mathrm{id}_{H(N,Y)} \sim G_1 \circ \iota$. □

Composing the homotopy equivalences from the above two lemmas, we have the following result.

Proposition A.8. *Let X be a metrisable space, and $Y \subseteq X$ a subspace. Suppose there is an open neighbourhood N of Y equipped with a nearly strict deformation retraction $r\colon N \times I \to N$. Then the map $N-Y \to H(X,Y)$, $x \mapsto p_x$, is a homotopy equivalence.*

Appendix B. Constructible $\mathscr{C}$-valued sheaves

In this appendix we show that the equivalence

$$\Psi_X\colon \operatorname{Fun}(\Pi_\infty^{\mathrm{exit}}(X), \mathcal{S}) \xrightarrow{\simeq} \operatorname{Shv}_{\mathrm{cbl}}(X, \mathcal{S})$$

of [Lur17, Theorem A.9.3] can be generalised to $\mathscr{C}$-valued sheaves for compactly generated $\mathscr{C}$. Here $\mathcal{S}$ denotes the ∞-category of spaces. The equivalence is used in Section 2.4 to give an expression of the constructible derived category of sheaves (of R-modules) in terms of the exit path ∞-category.

To anyone with a reasonable grasp of ∞-categories, this will be quite rudimentary, but we hope that the level of detail will make the results more accessible to any reader without a background in ∞-categories.

B.1. **Sheaves valued in compactly generated ∞-categories.** We give a very brief recap of the necessary definitions. We refer to [Lur11, §1.1] for details on $\mathscr{C}$-valued sheaves (see also [Tan19, §8.5]). Let $\mathscr{C}$ be a compactly generated ∞-category. Then $\mathscr{C} \simeq \mathrm{Ind}(\mathscr{C}_0)$ for a small ∞-category $\mathscr{C}_0$ admitting small colimits (see comment at the beginning of [Lur09, §5.5.7]).

Definition B.1. Let X be a topological space and let $\mathscr{U}(X)$ denote the category of open sets of X. The ∞-category of *$\mathscr{C}$-valued presheaves* on X is the functor ∞-category

$$\mathrm{Fun}(N(\mathscr{U}(X))^{\mathrm{op}}, \mathscr{C}).$$

A presheaf $\mathcal{F}\colon N(\mathscr{U}(X))^{\mathrm{op}} \to \mathscr{C}$ is a *$\mathscr{C}$-valued sheaf* on X if for any $U \in \mathscr{U}(X)$ and any covering sieve $\{U_\alpha\}$ of U, the map

$$\mathcal{F}(U) \to \varprojlim \mathcal{F}(U_\alpha)$$

is an equivalence. We denote the full subcategory of $\mathrm{Fun}(N(\mathscr{U}(X))^{\mathrm{op}}, \mathscr{C})$ spanned by the $\mathscr{C}$-valued sheaves by $\mathrm{Shv}(X, \mathscr{C})$. ◁

Remark B.2. We say that a sheaf $\mathcal{F} \in \mathrm{Shv}(X, \mathscr{C})$ is hypercomplete if it satisfies descent with respect to any hypercovering not just covering sieves (see [Lur09, §6.5.3]). ∘

Lemma B.3. *For any topological space X and any compactly generated $\mathscr{C} \simeq \mathrm{Ind}(\mathscr{C}_0)$, there is an equivalence of ∞-categories*

$$\mathrm{Shv}(X, \mathscr{C}) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{lex}}(\mathscr{C}_0^{\mathrm{op}}, \mathrm{Shv}(X, \mathcal{S})).$$

Proof. For any ∞-category $\mathscr{D}$, there is an equivalence

$$\mathrm{Fun}(\mathscr{D}^{\mathrm{op}}, \mathrm{Ind}(\mathscr{C}_0)) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{lex}}(\mathscr{C}_0^{\mathrm{op}}, \mathrm{Fun}(\mathscr{D}^{\mathrm{op}}, \mathcal{S})), \tag{2}$$

where the right hand side is the full subcategory spanned by the functors which preserve finite limits — this is in fact an isomorphism of simplicial sets identifying both sides with subcategories of $\mathrm{Fun}(\mathscr{D}^{\mathrm{op}} \times \mathscr{C}_0^{\mathrm{op}}, \mathcal{S})$. We apply this to (the nerve of) the category of open sets of X, $\mathscr{D} = N(\mathscr{U}(X))$, and note that by [Lur09, Corollary 5.1.2.3], the sheaf condition on the left hand side of the equivalence translates to the sheaf condition on the codomain of the right hand side. □

Remark B.4. Let $f\colon X \to Y$ be a map of topological spaces and consider the pushforward and pullback functors of $\mathcal{S}$-valued sheaves

$$\mathrm{Shv}(X, \mathcal{S}) \underset{f_*}{\overset{f^*}{\leftrightarrows}} \mathrm{Shv}(Y, \mathcal{S}).$$

Since both f^* and f_* preserve finite limits, postcomposition with these define an adjunction

$$\mathrm{Fun}^{\mathrm{lex}}(\mathscr{C}_0^{\mathrm{op}}, \mathrm{Shv}(X, \mathcal{S})) \underset{f_*}{\overset{f^*}{\leftrightarrows}} \mathrm{Fun}^{\mathrm{lex}}(\mathscr{C}_0^{\mathrm{op}}, \mathrm{Shv}(Y, \mathcal{S})).$$

Precomposition with the induced functor $\mathscr{U}(Y) \to \mathscr{U}(X)$ defines a pushforward map

$$f_*\colon \mathrm{Shv}(X, \mathscr{C}) \to \mathrm{Shv}(Y, \mathscr{C}),$$

and we have a commutative diagram as below.

$$\begin{array}{ccc} \mathrm{Shv}(X, \mathscr{C}) & \xrightarrow{\simeq} & \mathrm{Fun}^{\mathrm{lex}}(\mathscr{C}_0^{\mathrm{op}}, \mathrm{Shv}(X, \mathcal{S})) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ \mathrm{Shv}(Y, \mathscr{C}) & \xrightarrow[\simeq]{} & \mathrm{Fun}^{\mathrm{lex}}(\mathscr{C}_0^{\mathrm{op}}, \mathrm{Shv}(Y, \mathcal{S})) \end{array}$$

Therefore the left adjoint f^* on the right hand side defines a left adjoint f^* to the pushforward map of $\mathscr{C}$-valued sheaves on the left hand side. See also [Lur11, Remark 1.1.8]. ○

Definition B.5. Let X be a topological space and let $\rho\colon X \to *$ denote the unique map to a point. A sheaf $\mathcal{F} \in \mathrm{Shv}(X, \mathscr{C})$ is *constant* if it is in the essential image of the pullback functor $\rho^*\colon \mathrm{Shv}(*, \mathscr{C}) \to \mathrm{Shv}(X, \mathscr{C})$. A sheaf $\mathcal{F} \in \mathrm{Shv}(X, \mathscr{C})$ is *locally constant* if there is an open cover $\{U_\alpha\}$ of X such that the pullback of $\mathcal{F}$ to each U_α is constant. ◁

Definition B.6. Let X be an I-stratified space. A sheaf $\mathcal{F} \in \mathrm{Shv}(X, \mathscr{C})$ is *constructible* if the restriction to each X_i is locally constant. We denote by $\mathrm{Shv}_{\mathrm{cbl}}(X, \mathscr{C})$ the full subcategory of constructible sheaves. ◁

Since the equivalence of Lemma B.3 commutes with pullback functors, it descends to an equivalence of the subcategories of constructible sheaves.

Lemma B.7. *For any I-stratified space X and any compactly generated $\mathscr{C} \simeq \mathrm{Ind}(\mathscr{C}_0)$, there is an equivalence*

$$\mathrm{Shv}_{\mathrm{cbl}}(X, \mathscr{C}) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{lex}}(\mathscr{C}_0^{\mathrm{op}}, \mathrm{Shv}_{\mathrm{cbl}}(X, \mathcal{S}))$$

Definition B.8. Let X be an I-stratified space. A constructible $\mathscr{C}$-valued sheaf is *compact-valued* if its stalks are compact objects of $\mathscr{C}$. We denote by $\mathrm{Shv}_{\mathrm{cbl,cpt}}(X, \mathscr{C})$ the full subcategory of constructible compact-valued sheaves. ◁

B.2. **Exit path ∞-categories and constructible $\mathscr{C}$-valued sheaves.** Using Lemma B.7, we can generalise Lurie's classification of space-valued constructible sheaves as representations of the exit path ∞-category ([Lur17, Theorem A.9.3]) to sheaves taking values in compactly generated ∞-categories. The result is well-known and not hard to prove, but we have been unable to locate a proof in the literature. See [Tan19, §8.6] for a sketch of how to generalise the proof of [Lur17, Theorem A.9.3] to $\mathscr{C}$-valued sheaves.

Theorem B.9. *Let X be a conically I-stratified space which is paracompact and locally contractible, and where I satisfies the ascending chain condition. Let $\mathscr{C}$ be a compactly generated ∞-category. Then there is an equivalence of ∞-categories*

$$\Psi_X\colon \operatorname{Fun}(\Pi_\infty^{\text{exit}}(X), \mathscr{C}) \xrightarrow{\simeq} \operatorname{Shv}_{\text{cbl}}(X, \mathscr{C}).$$

Proof. Let $\mathscr{C}_0$ denote the ∞-category of compact objects of $\mathscr{C}$. We get a sequence of equivalences

$$\begin{aligned}\operatorname{Fun}(\Pi_\infty^{\text{exit}}(X), \operatorname{Ind}(\mathscr{C}_0)) &\simeq \operatorname{Fun}^{\text{lex}}(\mathscr{C}_0^{\text{op}}, \operatorname{Fun}(\Pi_\infty^{\text{exit}}(X), \mathcal{S})) \\ &\xrightarrow{\simeq} \operatorname{Fun}^{\text{lex}}(\mathscr{C}_0^{\text{op}}, \operatorname{Shv}_{\text{cbl}}(X, \mathcal{S})) \simeq \operatorname{Shv}_{\text{cbl}}(X, \operatorname{Ind}(\mathscr{C}_0))\end{aligned}$$

by applying (2) from the proof of Lemma B.3 to $\mathscr{D} = \Pi_\infty^{\text{exit}}(X)$ and combining this with the equivalences of Lemma B.7 and [Lur17, Theorem A.9.3]. ☐

We have the following naturality statement generalising [Lur17, Proposition A.9.16].

Proposition B.10. *Let $X \to I$ and $Y \to J$ be paracompact, locally contractible conically stratified spaces with $J \subset I$ and where I satisfies the ascending chain condition. Let $\mathscr{C}$ be a compactly generated ∞-category. For any stratum preserving map $f\colon Y \to X$ which on posets is given by the inclusion, there is an equivalence $\varphi_{Y,X}\colon \Psi_Y \circ f^* \Rightarrow f^* \circ \Psi_X$. In particular, the diagram below commutes up to homotopy.*

$$\begin{array}{ccc} \operatorname{Fun}(\Pi_\infty^{\text{exit}}(X), \mathscr{C}) & \xrightarrow{\Psi_X} & \operatorname{Shv}_{\text{cbl}}(X, \mathscr{C}) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \operatorname{Fun}(\Pi_\infty^{\text{exit}}(Y), \mathscr{C}) & \xrightarrow[\Psi_Y]{} & \operatorname{Shv}_{\text{cbl}}(Y, \mathscr{C}) \end{array}$$

Proof. The equivalence of [Lur17, Theorem A.9.3] is the composite of three equivalences, the first two of which are natural. For the third one, Proposition A.9.16 of [Lur17] provides the desired equivalence of functors for $\mathcal{S}$-valued sheaves:

$$\begin{array}{ccc} \operatorname{Fun}(\Pi_\infty^{\text{exit}}(X), \mathcal{S}) & \xrightarrow{\Psi_X} & \operatorname{Shv}_{\text{cbl}}(X, \mathcal{S}) \\ {\scriptstyle f^*}\downarrow & \nearrow & \downarrow{\scriptstyle f^*} \\ \operatorname{Fun}(\Pi_\infty^{\text{exit}}(Y), \mathcal{S}) & \xrightarrow[\Psi_Y]{} & \operatorname{Shv}_{\text{cbl}}(Y, \mathcal{S}) \end{array}$$

Applying $\mathrm{Fun}^{\mathrm{lex}}(\mathscr{C}_0^{\mathrm{op}}, -)$ to this diagram and noting that the equivalence of Lemma B.7 commutes with pullbacks (by definition of the pullback functor) and the equivalence (2) in the proof of Lemma B.3 is natural, we obtain the desired equivalence $\varphi_{Y,X} \colon \Psi_Y \circ f^* \Rightarrow f^* \circ \Psi_X$ for $\mathscr{C}$-valued sheaves. □

Corollary B.11. *Let X be a paracompact, locally contractible conically I-stratified space with I satisfying the ascending chain condition, and let $\mathscr{C}$ be a compactly generated ∞-category. Let $F \colon \Pi_\infty^{\mathrm{exit}}(X) \to \mathscr{C}$ and $\mathscr{F} := \Psi_X(F) \in \mathrm{Shv}_{\mathrm{cbl}}(X, \mathscr{C})$. For all $x \in X$, there is an equivalence $\mathscr{F}_x \xrightarrow{\simeq} F(x)$ in $\mathscr{C}$.*

Proof. For the one point space $*$, the equivalence $\Psi_* \colon \mathrm{Fun}(\Pi_\infty^{\mathrm{exit}}(*), \mathcal{S}) \to \mathrm{Shv}(*, \mathcal{S})$ sends a Kan complex Y to the Kan complex $\mathrm{Fun}(*, Y) \cong Y$ (see [Lur17, Construction A.9.2]). It follows that

$$\Psi_* \colon \mathrm{Fun}(\Pi_\infty^{\mathrm{exit}}(*), \mathscr{C}) \to \mathrm{Shv}(*, \mathscr{C})$$

is equivalent to the identity on $\mathscr{C}$. Applying Proposition B.10 to the map $x \colon * \to X$ sending $*$ to $x \in X$ provides an equivalence $\mathscr{F}_x \xrightarrow{\simeq} F(x)$ in $\mathscr{C}$. □

The following is an immediate consequence of Corollary B.11.

Corollary B.12. *Suppose X is a conically I-stratified space which is paracompact and locally contractible, and where I satisfies the ascending chain condition. Let $\mathscr{C}$ be a compactly generated ∞-category and let $\mathscr{C}_0$ denote the subcategory of compact objects. Then the equivalence of Theorem B.9 restricts to an equivalence*

$$\Psi_X \colon \mathrm{Fun}(\Pi_\infty^{\mathrm{exit}}(X), \mathscr{C}_0) \xrightarrow{\simeq} \mathrm{Shv}_{\mathrm{cbl,cpt}}(X, \mathscr{C}).$$

References

[AF90] J. L. Alperin and P. Fong. Weights for symmetric and general linear groups. *J. Algebra*, 131(1):2–22, 1990.

[Alp87] J. L. Alperin. Weights for finite groups. In *The Arcata Conference on Representations of Finite Groups (Arcata, Calif., 1986)*, volume 47 of *Proc. Sympos. Pure Math.*, pages 369–379. Amer. Math. Soc., Providence, RI, 1987.

[BJ06] Armand Borel and Lizhen Ji. *Compactifications of symmetric and locally symmetric spaces.* Mathematics: Theory & Applications. Birkhäuser Boston, Inc., Boston, MA, 2006.

[Bor69] Armand Borel. *Introduction aux groupes arithmétiques.* Publications de l'Institut de Mathématique de l'Université de Strasbourg, XV. Actualités Scientifiques et Industrielles, No. 1341. Hermann, Paris, 1969.

[Bor74] Armand Borel. Stable real cohomology of arithmetic groups. *Ann. Sci. École Norm. Sup. (4)*, 7:235–272 (1975), 1974.

[Bou84] Serge Bouc. Homologie de certains ensembles ordonnés. *C. R. Acad. Sci. Paris Sér. I Math.*, 299(2):49–52, 1984.

[BS73] A. Borel and J.-P. Serre. Corners and arithmetic groups. *Comment. Math. Helv.*, 48:436–491, 1973. Avec un appendice: Arrondissement des variétés à coins, par A. Douady et L. Hérault.

[BT71] A. Borel and J. Tits. Éléments unipotents et sous-groupes paraboliques de groupes réductifs. I. *Invent. Math.*, 12:95–104, 1971.

[BW76] N. Burgoyne and C. Williamson. On a theorem of Borel and Tits for finite Chevalley groups. *Arch. Math. (Basel)*, 27(5):489–491, 1976.

[CL15] Jiaming Chen and Eduard Looijenga. The homotopy type of the Baily-Borel and allied compactifications. Preprint: arXiv:1508.05599, 2015.

[CØJ20] Dustin Clausen and Mikala Ørsnes Jansen. The reductive Borel–Serre compactification as a model for unstable algebraic K-theory. Preprint, 2020.

[Con14] Brian Conrad. Reductive group schemes. In *Autour des schémas en groupes. Vol. I*, volume 42/43 of *Panor. Synthèses*, pages 93–444. Soc. Math. France, Paris, 2014.

[CR87] Charles W. Curtis and Irving Reiner. *Methods of representation theory. Vol. II*. Pure and Applied Mathematics (New York). John Wiley & Sons, Inc., New York, 1987. With applications to finite groups and orders, A Wiley-Interscience Publication.

[GHM94] M. Goresky, G. Harder, and R. MacPherson. Weighted cohomology. *Invent. Math.*, 116(1-3):139–213, 1994.

[GM80] Mark Goresky and Robert MacPherson. Intersection homology theory. *Topology*, 19(2):135–162, 1980.

[GM83] Mark Goresky and Robert MacPherson. Intersection homology. II. *Invent. Math.*, 72(1):77–129, 1983.

[GM88] Mark Goresky and Robert MacPherson. Weighted cohomology of Satake compactifications. Centre de recherches mathématiques, preprint # 1593, 1988.

[GM92] Mark Goresky and Robert MacPherson. Lefschetz numbers of Hecke correspondences. In *The zeta functions of Picard modular surfaces*, pages 465–478. Univ. Montréal, Montreal, QC, 1992.

[GM03] Mark Goresky and Robert MacPherson. The topological trace formula. *J. Reine Angew. Math.*, 560:77–150, 2003.

[Gro02] Jesper Grodal. Higher limits via subgroup complexes. *Ann. of Math. (2)*, 155(2):405–457, 2002.

[Gro18] Jesper Grodal. Endotrivial modules for finite groups via homotopy theory. Preprint: arXiv:1608.00499, 2018.

[Hat02] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[HTWW00] Bruce Hughes, Laurence R. Taylor, Shmuel Weinberger, and Bruce Williams. Neighborhoods in stratified spaces with two strata. *Topology*, 39(5):873–919, 2000.

[JM02] L. Ji and R. MacPherson. Geometry of compactifications of locally symmetric spaces. *Ann. Inst. Fourier (Grenoble)*, 52(2):457–559, 2002.

[JMO92a] Stefan Jackowski, James McClure, and Bob Oliver. Homotopy classification of self-maps of BG via G-actions. I. *Ann. of Math. (2)*, 135(1):183–226, 1992.

[JMO92b] Stefan Jackowski, James McClure, and Bob Oliver. Homotopy classification of self-maps of BG via G-actions. II. *Ann. of Math. (2)*, 135(2):227–270, 1992.

[JMSS15] Lizhen Ji, V. Kumar Murty, Leslie Saper, and John Scherk. The fundamental group of reductive Borel-Serre and Satake compactifications. *Asian J. Math.*, 19(3):465–485, 2015.

[Lur09] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.

[Lur11] Jacob Lurie. Derived Algebraic Geometry V: Structured Spaces. www.math.ias.edu/ lurie/papers/DAG-V.pdf, 2011.

[Lur17] Jacob Lurie. Higher algebra. www.math.ias.edu/ lurie/papers/HA.pdf, 2017.

[mat] mathoverflow. mathoverflow.net/questions/265557.

[Mil56] John Milnor. Construction of universal bundles. II. *Ann. of Math. (2)*, 63:430–436, 1956.

[Qui73a] Daniel Quillen. Finite generation of the groups K_i of rings of algebraic integers. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 179–198. Lecture Notes in Math., Vol. 341, 1973.

[Qui73b] Daniel Quillen. Higher algebraic K-theory. I. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 85–147. Lecture Notes in Math., Vol. 341, 1973.

[Qui88] Frank Quinn. Homotopically stratified sets. *J. Amer. Math. Soc.*, 1(2):441–499, 1988.

[Qui02] Frank Quinn. Lectures on controlled topology: mapping cylinder neighborhoods. 9:461–489, 2002.

[Rap86] Michael Rapoport. Letter to Borel. 1986.

[Sap05a] Leslie Saper. $\mathcal{L}$-modules and micro-support. Preprint: arXiv:0112251, 2005. To appear in Annals of Mathematics.

[Sap05b] Leslie Saper. $\mathcal{L}$-modules and the conjecture of Rapoport and Goresky-MacPherson. Number 298, pages 319–334. 2005. Automorphic forms. I.

[Tan19] Hiro Lee Tanaka. Cyclic structure and broken cycles. Preprint: arXiv:1907.03301, 2019.

[Tre09] David Treumann. Exit paths and constructible stacks. *Compos. Math.*, 145(6):1504–1532, 2009.

[Woo09] Jon Woolf. The fundamental category of a stratified space. *J. Homotopy Relat. Struct.*, 4(1):359–387, 2009.

[Zuc86] Steven Zucker. L_2-cohomology and intersection homology of locally symmetric varieties. II. *Compositio Math.*, 59(3):339–398, 1986.

[Zuc01] Steven Zucker. On the reductive Borel-Serre compactification: L^p-cohomology of arithmetic groups (for large p). *Amer. J. Math.*, 123(5):951–984, 2001.

[Zuc83] Steven Zucker. L_2-cohomology of warped products and arithmetic groups. *Invent. Math.*, 70(2):169–218, 1982/83.

Department of Mathematical Sciences, University of Copenhagen, 2100 Copenhagen, Denmark.
Email address: mikala@math.ku.dk

PAPER II

The reductive Borel–Serre compactification as a model for unstable algebraic K-theory

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THE REDUCTIVE BOREL–SERRE COMPACTIFICATION AS A MODEL FOR UNSTABLE ALGEBRAIC K-THEORY

DUSTIN CLAUSEN AND MIKALA ØRSNES JANSEN

Abstract. Let A be an associative ring and M a finitely generated projective A-module. We introduce a category $\mathrm{RBS}(M)$ and prove several theorems which show that its geometric realisation functions as a well-behaved unstable algebraic K-theory space. These categories $\mathrm{RBS}(M)$ naturally arise as generalisations of the exit path ∞-category of the reductive Borel–Serre compactification of a locally symmetric space, and one of our main techniques is to find purely categorical analogues of some familiar structures in these compactifications.

Contents

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1. Introduction

1.1. **Unstable algebraic K-theory.** Let A be a ring. The algebraic K-theory space $K(A)$ is an invariant of A which is built from the concrete linear algebra of finitely generated projective modules over A. But $K(A)$ has a subtle nature. In fact, there are several different ways of defining $K(A)$ as a CW-complex, and they are all different up to homeomorphism; however, they are nonetheless canonically homotopy equivalent. Thus the true $K(A)$ is this common homotopy type, or *anima*. An anima is elusive and difficult to grasp, but it anchors itself to reality via concrete invariants such as homotopy groups. The homotopy groups of $K(A)$ are abelian groups known as the higher K-groups, and they have myriad connections to other invariants of A arising in different contexts.

To every finitely generated projective module M corresponds a point in $K(A)$. Moreover, this association is functorial for isomorphisms, so one obtains a map

$$BGL(M) \to K(A).$$

This is very far from being an isomorphism, for two reasons: first, these anima have very different nature (one is a $K(\pi,1)$ for a generally non-abelian π and the other is a simple space), and second, $K(A)$ takes into account all finitely generated projective modules, not just M. We would like to mitigate the first reason while keeping the second. More precisely, we want to define an intermediary anima $\overline{BGL(M)}$, a sort of "closure" of $BGL(M)$ in $K(A)$, which is similar to $K(A)$ in terms of its nature and properties, but whose definition only uses linear algebra internal to M. Such an intermediary anima is called an *unstable algebraic K-theory.*

There have already been several proposed definitions for unstable algebraic K-theory in the literature, mostly in the the special case $M = A^n$. In contrast to the stable situation of $K(A)$, all of these definitions are in general pairwise inequivalent, even as anima. Our definition will be yet another one which is generally inequivalent to the others, see below for more remarks on the comparisons. It will be denoted

$$|\operatorname{RBS}(M)|.$$

The notation foreshadows that this anima arises as the geometric realisation of an explicit category $\operatorname{RBS}(M)$ built from linear algebra internal to M. We will say more about the definition and origins of this category later. But first let's state the main results, which all concern the question of how close the natural maps

$$BGL(M) \to |\operatorname{RBS}(M)| \to K(A)$$

are to being isomorphisms.

Our arguments are based on an inductive strategy, and for carrying many of them out it is at the very least convenient to impose the following condition on our module M:

Definition 1.1. We say that a finitely generated projective A-module M is *split noetherian* if every decreasing chain of splittable submodules of M stabilises. ◁

If the ring A is either noetherian or commutative with connected spectrum, then every finitely generated projective A-module is split noetherian.

Concerning the map $BGL(M) \to |\operatorname{RBS}(M)|$, both anima are connected, so the first question is what happens on π_1. Let $E(M) \subset GL(M) = \pi_1 BGL(M)$ denote the subgroup generated by those automorphisms of M which induce the identity on the associated graded of some splittable flag of submodules. We think of $E(M)$ as the subgroup of those elements which map to zero in $K_1(A)$ for reasons purely internal to M. It is a variant of the usual subgroup $E_n(A) \subset GL_n(A)$ generated by elementary matrices; there is a containment $E_n(A) \subset E(A^n)$ which is in general strict, but often an equality, for example $E_n(A) = E(A^n)$ if $n \geq 2 + \operatorname{sr}(A)$ so that $E_n(A) = \ker(GL_n(A) \to K_1(A))$, see [Vas69].

Our first result is actually fairly straightforward to prove from the definition, but it already gives a good indication of the nature of $|\operatorname{RBS}(M)|$.

Theorem 1.2. *Let A be a ring and M a split noetherian finitely generated projective A-module. The map $GL(M) = \pi_1 BGL(M) \to \pi_1|\operatorname{RBS}(M)|$ is surjective with kernel $E(M)$, so*

$$\pi_1|\operatorname{RBS}(M)| = GL(M)/E(M).$$

Our next result says that for a large class of rings A this π_1 calculation completely captures the difference between $BGL(M)$ and $|\operatorname{RBS}(M)|$. It is based on the work of Nesterenko-Suslin [NS90] who found a broadly satisfied hypothesis on a ring which guarantees that one can ignore the difference between block upper-triangular and block diagonal matrices when calculating group homology.

Theorem 1.3. *Let A be a ring with many units in the sense of* [NS90]*, and let M be a split noetherian finitely generated projective A-module. Then the comparison map*

$$c\colon BGL(M) \to |\operatorname{RBS}(M)|$$

is a $\mathbb{Z}$-homology isomorphism.
Suppose furthermore that every summand of M is free. Then c is an isomorphism on homology with all local coefficient systems. In particular, $E(M)$ is a perfect group, and

$$|\operatorname{RBS}(M)| \simeq BGL(M)^{+},$$

the plus-construction taken with respect to $E(M) \subset \pi_1 BGL(M)$. Equivalently, $|\operatorname{RBS}(M)|$ is the initial anima with a map from $BGL(M)$ which kills $E(M) \subset \pi_1 BGL(M)$.

Thus, for such rings $|\operatorname{RBS}(M)|$ provides an explicit linear-algebraic model for the plus-construction, which is otherwise a slightly esoteric homotopy-theoretic construction. There are lots of rings A with many units, for example any algebra over a commutative local ring with infinite residue field. A commutative local ring also satisfies the hypothesis that every finitely generated projective module is free.

The simplest non-example is a finite field, and our third theorem analyses this case to see the difference with the plus construction. As we will explain below, the resulting theorem should properly be attributed to Jesper Grodal, since in [Gro16] he proved a more general result in the context of arbitrary finite groups. However, we do give an independent proof based on the general machinery for analysing $\operatorname{RBS}(M)$ categories that we develop.

Theorem 1.4. *Let k be a finite field of characteristic p and V a finite-dimensional k-vector space. Then:*

(1) $|\operatorname{RBS}(V)|$ is a simple space;
(2) The map $|\operatorname{RBS}(V)| \to *$ is an $\mathbb{F}_p$-homology isomorphism;
(3) The map $BGL(V) \to |\operatorname{RBS}(V)|$ is a $\mathbb{Z}[1/p]$-homology isomorphism.

In particular, $|\operatorname{RBS}(V)|$ is the $\mathbb{Z}[1/p]$-homology localisation of $BGL(V)$.

We recall that the $\mathbb{Z}[1/p]$-homology of $BGL(V)$ was completely calculated by Quillen in the early days, [Qui72]. On the other hand, the $\mathbb{F}_p$-homology is nontrivial, rather complicated, and still largely unknown, [MP87], [LS18]. However, Quillen in [Qui72] also showed that in the stable range the $\mathbb{F}_p$-homology vanishes, so the complicated part does not contribute to $K(\mathbb{F}_q)$. Thus, compared to existing models such as the plus-construction, our new model for unstable algebraic K-theory exactly removes the complicated unknown part which anyway dies on stabilisation. Actually we can rephrase the above theorem as giving an identification

$$|\operatorname{RBS}(V)| \simeq (BU(n)')^{h\psi^q}$$

as the homotopy fixed points for the unstable q-Adams operation on the prime-to-p completion of $BU(n)$ for $n = \dim(V)$. This is the evident unstable analogue of Quillen's identification of the 0-component of the K-theory space

$$K(\mathbb{F}_q)_0 \simeq (BU')^{h\psi^q}.$$

Remark 1.5. The crucial point is part 2, that $|\operatorname{RBS}(V)|$ has the $\mathbb{F}_p$-homology of a point. This can also be deduced from a more general theorem of Jesper Grodal, [Gro16]. Indeed, Grodal's Theorem 4.3 says that for any finite group G and prime p, if $\mathcal{C}$ denotes the *p-radical* *orbit category* of G, then $|\mathcal{C}|$ has the $\mathbb{F}_p$-homology of a point. For $G = GL(V)$ it is a (not entirely obvious) matter of comparing definitions to see that $\mathcal{C} = \mathrm{RBS}(V)$, see the discussion in [ØJ20], and hence our theorem follows from Grodal's. ○

Now we turn to the relation between $|\,\mathrm{RBS}(M)|$ and $K(A)$. Our last theorem gives a sense in which the $|\,\mathrm{RBS}(M)|$ stabilise to $K(A)$.

Theorem 1.6. *Let A be a ring. Let $\mathcal{M}$ denote a set of representatives for the isomorphism classes of finitely generated projective A-modules. Then there is a natural structure of a monoidal category on $\coprod_{M\in\mathcal{M}} \mathrm{RBS}(M)$ and an identification*

$$K(A) \simeq \left| \coprod_{M\in\mathcal{M}} \mathrm{RBS}(M) \right|^{gp}$$

of $K(A)$ with the group completion of the realisation of this monoidal category.

We also prove a more general version of this theorem which describes in similar terms the K-theory of an arbitrary exact category in the sense of Quillen, [Qui73b].

The version of Theorem 1.6 with $BGL(M)$ instead of $\mathrm{RBS}(M)$ is essentially Segal's definition of algebraic K-theory, [Seg74]. However, there is a very important technical difference between the two situations, in that $\coprod BGL(M)$ forms a symmetric monoidal category, whereas $\coprod \mathrm{RBS}(M)$ really only forms a monoidal category. This means that as it stands we cannot use the "group completion theorem" of [MS76] to relate this group completion to the more naive procedure of taking the limiting object

$$\varinjlim_n |\,\mathrm{RBS}(A^n)|$$

along the natural stabilisation maps. Indeed, the group completion theorem requires some commutativity hypothesis which we don't know whether is satisfied for $|\coprod \mathrm{RBS}(M)|$ for general A and M.

1.2. **The reductive Borel–Serre category.** Perhaps the most important aspect of our model is that it is given as the geometric realisation of an explicit category $\mathrm{RBS}(M)$. Although we were led to this category by other means which we will discuss below, one can motivate it in terms of the following key property of algebraic K-theory: if M is a finitely generated projective A-module and

$$\mathcal{F} = (M_1 \subsetneq \ldots \subsetneq M_{d-1})$$

is a splittable flag in M, so that each graded piece M_i/M_{i-1} is nonzero and finitely generated projective (we set $M_0 = 0$ and $M_d = M$), then there is a canonically determined path

$$[M] \sim [\oplus_{i=1}^d M_i/M_{i-1}]$$

in $K(A)$. Thus, in the eyes of K-theory, every filtration is split. One can also say this in a different way. Let $P_{\mathcal{F}} \subset GL(M)$ denote the stabiliser of the flag $\mathcal{F}$ and $U_{\mathcal{F}} \subset P_{\mathcal{F}}$ the subgroup consisting of those elements which induce the identity on associated graded. Then the restriction of $BGL(M) \to K(A)$ to $BP_{\mathcal{F}}$ naturally factors through $B(P_{\mathcal{F}}/U_{\mathcal{F}})$. There are also a host of compatibilities satisfied by these canonical paths relating their functoriality under automorphisms and their behaviour under refinement of flags. This leads to the following.

Definition 1.7. Let A be a ring and M a finitely generated projective A-module. Define the category $\mathrm{RBS}(M)$ to have:

(1) objects the splittable flags of submodules of M

$$\mathcal{F} = (M_1 \subsetneq \ldots \subsetneq M_{d-1});$$

(2) morphisms $\mathcal{F} \to \mathcal{F}'$ the set

$$\{g \in GL(M) : g\mathcal{F} \leq \mathcal{F}'\}/U_{\mathcal{F}},$$

where the partial order $\leq$ is the relation of refinement: $\mathcal{F} \leq \mathcal{F}'$ when the modules occurring in $\mathcal{F}'$ are a subset of those occurring in $\mathcal{F}$;

(3) composition induced by multiplication in $GL(M)$. ◁

The empty flag $[\emptyset]$ has automorphism group $GL(M)$ in $\mathrm{RBS}(M)$. This produces a map $BGL(M) \to |\mathrm{RBS}(M)|$, and the preceding discussion hopefully makes it plausible that the natural map $BGL(M) \to K(A)$ factors through it:

$$BGL(M) \to |\mathrm{RBS}(M)| \to K(A).$$

But to prove this and the more refined Theorem 1.6, it's useful to look at $\mathrm{RBS}(M)$ from a more intrinsic perspective. For a splittable flag $\mathcal{F}$, the automorphism group of $\mathcal{F}$ in $\mathrm{RBS}(M)$ identifies not with the automorphisms of $\mathcal{F}$ as a flag, but with the automorphisms of its associated graded. Thus one should think that the objects of $\mathrm{RBS}(M)$ are not really flags, since giving a flag over-specifies the object. Rather the objects should be some abstract ordered list

$$(N_1, \ldots, N_d)$$

of nonzero finitely generated projective modules, which we imagine as the associated graded of some undetermined flag. The flags themselves only really come in to play when describing the morphsims: namely a map $(N_1, \ldots, N_d) \to (N'_1, \ldots, N'_e)$ can only exist when $d \geq e$, and then is the data of a flag on each N'_j together with an isomorphism of the total associated graded of this list of flags with the N_i, in order. There is an equivalent model for $\mathrm{RBS}(M)$ of exactly this form, and it is this model that is the most useful for giving the comparison with $K(A)$.

Going in a different direction, when A is a commutative there is yet another interpretation of $\mathrm{RBS}(M)$, this time in terms of the group $GL(M)$ viewed now as a reductive group scheme over A instead of just the abstract group of its A-valued points. This is simplest to state when $\mathrm{Spec}(A)$ is connected. Then splittable flags $\mathcal{F}$ in M are in bijection with *parabolic subgroups* of $GL(M)$ via assigning to $\mathcal{F}$ its stabiliser $P_{\mathcal{F}}$. Moreover the subgroup $U_{\mathcal{F}} \subset P_{\mathcal{F}}$ of those automorphisms of the flag inducing the identity on associated graded is recovered as the *unipotent radical* of $P_{\mathcal{F}}$. Thus one can also describe $\mathrm{RBS}(M)$ in reductive group terms: the objects are the parabolic subgroups, and the maps are the transporters of these subgroups taken modulo the unipotent radical of the source parabolic subgroup.

This ties in to our initial motivation for defining RBS(M). Let G be a connected reductive linear algebraic group defined over $\mathbb{Q}$, and $X = K\backslash G(\mathbb{R})/A_G$ the usual associated contractible symmetric space. For a neat arithmetic group $\Gamma \leq G(\mathbb{Q})$, the locally symmetric space $\Gamma\backslash X$ is a model for the classifying space $B\Gamma$ — unfortunately, it is very rarely compact. The Borel–Serre and reductive Borel–Serre compactification are two important compactifications of such locally symmetric spaces. The Borel–Serre compactification $\Gamma\backslash\widehat{X}$ is a compact smooth manifold with corners with the same homotopy type as $\Gamma\backslash X$. It was introduced in 1973 by Borel and Serre ([BS73]) and it was used crucially in Borel's calculation of the ranks of the K-groups $K_i(O_F)$ of the ring of integers O_F in a number field F ([Bor74]). It was also used by Quillen to show that these same K-groups are finitely generated ([Qui73a]). The reductive Borel–Serre compactification $\widehat{Y_\Gamma}$ was introduced by Zucker in 1982 as a quotient of the Borel–Serre compactification ([Zuc82]). Zucker was originally motivated by an interest in L^2-cohomology, but the reductive Borel–Serre compactification has since come to play a prominent and diverse role in the theory of compactifications.

The Borel–Serre compactification is naturally stratified as a manifold with corners, and this stratification descends to define a natural stratification of the reductive Borel–Serre compactification. In [ØJ20], the exit path ∞-category of the reductive Borel–Serre compactification $\widehat{Y_\Gamma}$ is identified as a 1-category RBS_Γ whose objects are the rational parabolic subgroups of G and whose morphisms are given by transporters of these subgroups by elements in Γ modulo an action of the unipotent radicals. The category RBS(M) introduced in this paper is a direct generalisation of the category RBS_Γ, cf. the reductive group approach above.

In fact, we also provide a proof of the identification of the exit path ∞-category of the reductive Borel–Serre compactification in this paper. Our proof uses entirely different methods to the one given in [ØJ20], and we find that the two different proofs complement each other nicely, as they provide very different insights into the structure of the reductive Borel–Serre compactification. Moreover, both methods are quite general in nature and have the potential to be useful for studying the exit path ∞-categories of other stratified spaces, so we think it is worthwhile to have them both explained. Whereas the method in [ØJ20] is based on the idea of calculating mapping spaces in the exit path ∞-category in terms of the homotopy-theoretic data embodied in the links of the strata, the method in this paper is based on the idea of finding a way to glue our stratified space from simpler pieces, whose exit path ∞-categories are equivalent to posets. If the gluing is robust enough, this reduces the determination of the exit path ∞-category of our space to the calculation of a colimit in the ∞-category of ∞-categories.

As we will see, the proof we present here has advantages with respect to the broader aim of this paper, namely in comparing with unstable K-theory, as the proof strategy by gluing can be transported over to the context of RBS(M) and exploited to make the necessary homology calculations. In order to make the various calculations and identifications in this paper, we start out by developing a variety of tools for identifying and calculating colimits of

∞-categories. This allows us to exploit the inductive nature of the reductive Borel–Serre compactification, namely the fact that its boundary admits a closed cover by "smaller" reductive Borel–Serre compactifications, and we can mimick this when working with the generalisations $\mathrm{RBS}(M)$.

Finally, we would like to note that the existence of a relationship between compactifications of locally symmetric spaces and algebraic K-theory is not original to this article. As Dan Petersen pointed out to us, Charney and Lee wrote an article [CL83] in 1982 in which they established such a relationship for the Satake compactification of the Siegel modular variety. They show that the homotopy type of the Satake compactification is rationally equivalent to the geometric realisation of a category W_n whose stable version W fits into a fiber sequence

$$K(\mathbb{Z}) \to K^{sympl}(\mathbb{Z}) \to |W|,$$

and therefore describes the difference between K-theory and symplectic K-theory of the integers. What we have, then, is an analogue of the Charney–Lee result for the reductive Borel–Serre compactification and plain algebraic K-theory. Moreover, the modern notion of exit path ∞-category lets us make a much more refined statement of the relationship, showing that not just the (rational) homotopy type, but the whole stratified homotopy type, as well as the theory of constructible sheaves, are determined by the associated category.

1.3. **Comparison with previous approaches.** There have been several previous approaches to unstable algebraic K-theory. Here we'd like to point out the ones we know about and say what we can about how our definition compares.

First, there is the plus construction definition. If $n \geq 3$, the subgroup $E_n(A) \subset GL_n(A)$ generated by elementary matrices is perfect, [Wei13] Lemma 1.3.2, so one can form the plus construction on $BGL_n(A)$ which kills the normal subgroup generated by $E_n(A)$. By Theorem 1.3 above, this agrees with our $|\,\mathrm{RBS}(A^n)|$ provided that A is commutative and local with infinite residue field. On the other hand, our Theorem 1.4 shows that for finite fields, the two definitions differ, and ours yields an unstable algebraic K-theory space which is much simpler and closer in nature to the stable K-theory.

Second, there is the Volodin definition, see [Sus82]. At first glance this looks quite similar, since it is based on the same idea of contracting away unipotent matrix groups. But the contraction happens in a very different way in Volodin's model: one considers all of the Σ_n-conjugates of the strict upper-triangular group and simultaneously collapses them, compatibly along their various intersections. Already in unstable K_1 one sees a difference, in that the Volodin K_1 is the quotient $GL_n(A)/E_n(A)$, which is not necessarily a group in general but just a pointed set. It also seems from our (albeit limited) experience that arguments which work for Volodin K-theory do not work for our model and vice-versa, so the nature of the two models really is quite different.

Finally, there is Allen Yuan's quite recent *partial K-theory*, [Yua19]. This had not yet appeared when we were proving our results, but it indeed seems very similar to our proposed model. Partial K-theory is defined essentially so as to make the analogue of our Theorem 1.6

a tautology. (Whereas for us the proof takes many pages of simplicial manipulations!) That is, Yuan takes Waldhausen's S-dot construction, and instead of freely making a group-like E_1-anima out of it, which produces $K(A)$, he freely makes an E_1-anima without the group-like condition, and this is the definition of $K^\partial(A)$. It is clear that partial K-theory should be similar to our E_1-anima $|\coprod \mathrm{RBS}(M)|$, because the S-dot construction exactly encodes filtrations and their associated gradeds with all compatibilities, and this was the essence of our RBS categories as well. But it turns out that when Yuan unravels K^∂ into something concrete, it ends up being slightly more combinatorially intricate, in that the basic objects are not lists of finitely generated projective modules, but lists of lists of finitely generated projective modules. The two models for unstable K-theory unwind to the same thing when all flags on M have length ≤ 2, but in other cases they are a priori different and it's not clear whether or not the anima are nonetheless equivalent. This would be interesting to investigate, because Yuan shows by an Eckmann–Hilton argument that $K^\partial(A)$ actually is E_∞, which means the group completion theorem does apply to it. Yuan also proves the analogue of our Theorem 1.4 part 2 for K^∂ of finite fields, and crucially uses this result in his work giving a new model for unstable homotopy theory. Moreover, his proof has the same rough outline as ours: after some combinatorial shuffling one reduces to the fact that the $\mathbb{F}_p$-homology of the Steinberg representation of $GL_n(k)$ vanishes.

1.4. **Conventions and notation.** We let $\mathcal{S}$ denote the ∞-category of anima, and Cat_∞ the ∞-category of (small) ∞-categories. We often view $\mathcal{S}$ as the full subcategory of Cat_∞ consisting of the ∞-groupoids. For a topological space X, if we write $\mathrm{Sh}(X)$ or talk about sheaves on X without specifying further, we mean to consider sheaves of anima, i.e. sheaves with values in the ∞-category $\mathcal{S}$. The same goes for presheaves on a category or ∞-category. We view posets as categories with at most one morphism between any two objects: $x \leq y$ means there is a map $x \to y$.

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2. Colimits in Cat_∞

In this section we will describe how to calculate certain colimits in the ∞-category of small ∞-categories. We note right away that there is a general description of such colimits as a localisation of the total space of the cartesian fibration classified by the diagram of ∞-categories, see [Lur09a] 3.3.4; but this is not what we're after. Rather we want simple criteria for showing that a given co-cone diagram is a colimit diagram.

In the cases we care about all the ∞-categories in our colimit diagram will actually be 1-categories, but still it being a colimit diagram in Cat_∞ is stronger than it being a colimit diagram in Cat_1, and we need this stronger fact to get our desired consequences.

2.1. **Some consequences of having a colimit in** Cat_∞**.** We start by explaining why we care about colimits in Cat_∞. First, they let you decompose both colimits and limits.

Proposition 2.1. *Let K be an ∞-category and $d\colon K \to \mathrm{Cat}_\infty$ a K-diagram in Cat_∞, with colimit $\mathcal{D} := \varinjlim_K d$. Suppose given an ∞-category $\mathcal{E}$ and a functor*

$$F\colon \mathcal{D} \to \mathcal{E}.$$

(1) We have

$$\varprojlim F \xrightarrow{\sim} \varprojlim_{k\in K^{op}} \varprojlim F|_{d(k)}$$

in the sense that if the limits on the right exist then so does the limit on the left, and the map is an equivalence.

(2) We have

$$\varinjlim F \xleftarrow{\sim} \varinjlim_{k\in K} \varinjlim F|_{d(k)}$$

in the sense that if the colimits on the right exist then so does the colimit on the left, and the map is an equivalence.

The natural comparison maps in play above will be constructed in the course of the proof.

Proof. Since $\mathcal{C} \mapsto \mathcal{C}^{op}$ is an equivalence of Cat_∞ with itself, it preserves colimits. Thus 2 follows from 1 by replacing every ∞-category with its opposite. We can always Yoneda-embed $\mathcal{E} \hookrightarrow \mathrm{Fun}(\mathcal{E}^{op}, \mathcal{S})$ and therefore reduce to $\mathcal{E}$ being a presheaf ∞-category; thus to construct the comparison maps in general it suffices to construct them functorially in the case $\mathcal{E} = \mathcal{S}$, and similarly to prove they are equivalences it suffices to treat that case.

We note that

$$\mathrm{Fun}(\mathcal{D}, \mathcal{S}) \xrightarrow{\sim} \varprojlim_{k\in K^{op}} \mathrm{Fun}(d(k), \mathcal{S})$$

by taking maps out of our colimit diagram to $\mathrm{Fun}(\Delta^n, \mathcal{S})$ and using adjunction. Now given an $F \in \mathrm{Fun}(\mathcal{D}, \mathcal{S})$ we can simply evaluate maps from the terminal functor $*$ to F via the above equivalence to deduce the required equivalence. □

Let $|\cdot|\colon \mathrm{Cat}_\infty \to \mathcal{S}$ denote the left adjoint to the inclusion of anima into ∞-categories. There are many ways of describing this functor; see Section 2.3. But in any case, it commutes with colimits, and so from a colimit diagram in Cat_∞ we obtain a colimit diagram in $\mathcal{S}$, that is a homotopy colimit diagram in the classical language:

Proposition 2.2. *Let K be an ∞-category and $d\colon K \to \mathrm{Cat}_\infty$ a K-diagram in Cat_∞, with colimit $\mathcal{D} := \varinjlim_K d$. Then*

$$\varinjlim_{k\in K} |d(k)| \xrightarrow{\sim} |\mathcal{D}|.$$

In particular, this means we have a spectral sequence for computing homology of local systems on $|\mathcal{D}|$ in terms of the homology of their pullback to the $|d(k)|$, and in the case where K is the poset $(1 > 0 < 1')$ this means a Mayer–Vietoris sequence. (These consequences could also be obtained from the previous proposition by taking $\mathcal{E} = D(\mathrm{Ab})$).

2.2. **Testing by applying** $\mathrm{Fun}(-,\mathcal{S})$**.** Here we prove the following basic result.

Theorem 2.3. *Let K be a small ∞-category, and $d\colon K^{\triangleright} \to \mathrm{Cat}_\infty$ a co-cone diagram of small ∞-categories indexed by K. Then d is a colimit diagram if and only if the following two conditions are satisfied:*

(1) As k runs over the objects of K, the functors $d(k) \to d(\infty)$ are jointly essentially surjective.
(2) The cone diagram of ∞-categories obtained by applying $\mathrm{Fun}(-,\mathcal{S})$ to d is a limit diagram.

To prove this we will need some preliminaries on presentable ∞-categories. First, recall that for every $\mathcal{C} \in \mathrm{Cat}_\infty$, there is a presentable ∞-category $\mathcal{P}(\mathcal{C})$ with a fully faithful functor $\mathcal{C} \to \mathcal{P}(\mathcal{C})$ uniquely characterised by the universal property that colimit-preserving functors $\mathcal{P}(\mathcal{C}) \to \mathcal{D}$ are equivalent, via restriction, to arbitrary functors $\mathcal{C} \to \mathcal{D}$. In fact, $\mathcal{P}(\mathcal{C})$ can be taken to be the ∞-category $\mathrm{Fun}(\mathcal{C}^{op},\mathcal{S})$ of presheaves on $\mathcal{C}$ and $\mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{op},\mathcal{S})$ to be the Yoneda embedding h, see [Lur09a] 5.1, though we would rather not emphasise this description.

Let us characterise the presentable ∞-categories of the form $\mathcal{P}(\mathcal{C})$.

Definition 2.4. An object X of a presentable ∞-category $\mathcal{D}$ is called *atomic* if the functor $\mathrm{Map}(X,-)\colon \mathcal{D} \to \mathcal{S}$ commutes with all colimits. Write $\mathcal{D}^{atom} \subset \mathcal{D}$ for the full subcategory of atomic objects. ◁

We refer to [Lur09a] 4.4.5 for the notions of idempotent-complete ∞-category and the operation of idempotent completion.

Lemma 2.5. *For $\mathcal{D} \in \mathrm{Pr}^L$, the ∞-category $\mathcal{D}^{atom}$ is essentially small and idempotent-complete.*

Proof. Since $\mathcal{D}$ is presentable, every object $X \in \mathcal{D}$ is a colimit of objects each of which lies in some fixed small idempotent-complete full subcategory of $\mathcal{D}$ (namely, the full subcategory of κ-small objects, if $\mathcal{D}$ is κ-accessible). If X is atomic, then this means the identity map on X factors through an object of that full subcategory, hence X lies in that full subcategory. Thus $\mathcal{D}^{atom}$ is essentially small. It is also idempotent-complete since $\mathcal{D}$ is (being co-complete), and a retract of an atomic object is clearly atomic. □

Lemma 2.6.

(1) For $\mathcal{C} \in \mathrm{Cat}_\infty$, an object of $\mathcal{P}(\mathcal{C})$ is atomic if and only if it is a retract of an object in the image of $\mathcal{C} \to \mathcal{P}(\mathcal{C})$. In particular, $\mathcal{P}(\mathcal{C})$ is generated under colimits by its atomic objects (as it is generated under colimits by objects in the Yoneda image, [Lur09a] *5.1).*
(2) Conversely, if $\mathcal{D} \in \mathrm{Pr}^L$ is generated under colimits by its atomic objects, then the induced colimit-preserving functor $\mathcal{P}(\mathcal{D}^{atom}) \to \mathcal{D}$ is an equivalence.
(3) For a colimit-preserving functor $f\colon \mathcal{C} \to \mathcal{D}$ between presentable ∞-categories generated under colimits by their atomic objects, we have $f(\mathcal{C}^{atom}) \subset \mathcal{D}^{atom}$ if and only if the right adjoint of f commutes with colimits.

Proof. First of all we note that each $X \in \mathcal{P}(\mathcal{C})$ in the Yoneda image is atomic, since $\mathrm{Map}(h_c, -) = (-)(c)$ and colimits are computed objectwise in presheaf categories, [Lur09a] 5.1. Since the collection of atomic objects is closed under retracts, this shows one direction of 1. For the other direction, suppose X is atomic. Then we can write X as a colimit of objects in the Yoneda image. By definition of atomic the identity map $X \to X$ factors through some stage of this colimit, so X is a retract of an object in the Yoneda image.

Now we show 2. The functor $\mathcal{P}(\mathcal{D}^{atom}) \to \mathcal{D}$ is fully faithful for general $\mathcal{D} \in \mathrm{Pr}^L$, as we see by writing each object in $\mathcal{P}(\mathcal{D}^{atom})$ as a colimit of objects in $\mathcal{D}^{atom}$. Then the assumption exactly guarantees that it's also essentially surjective.

Finally, 3 is immediate by adjunction. □

We note that the universal property of $\mathcal{P}(-)$ gives a covariant functoriality, more specifically $\mathcal{P}\colon \mathrm{Cat}_\infty \to \mathrm{Pr}^L$.

Proposition 2.7. *The functor $\mathcal{C} \mapsto \mathcal{P}(\mathcal{C})$ gives an equivalence from the ∞-category of idempotent-complete small ∞-categories to the subcategory of Pr^L whose objects are the presentable ∞-categories generated by atomic objects and whose morphisms are the colimit-preserving functors whose right adjoint also preserves colimits.*

Proof. We claim that an inverse functor is given by $\mathcal{D} \mapsto \mathcal{D}^{atom}$. This is well-defined on the subcategory by part 3 of the lemma above. From part 1 of the lemma above, we know that if $\mathcal{C}$ is idempotent-complete, then $\mathcal{C} \xrightarrow{\sim} \mathcal{P}(\mathcal{C})^{atom}$. On the other hand, from part 2 we know that if $\mathcal{D}$ lies in the subcategory then $\mathcal{P}(\mathcal{D}^{atom}) \xrightarrow{\sim} \mathcal{D}$. □

Now, our desired Theorem 2.3 follows by combining parts 1 and 2 of the following.

Proposition 2.8. *Let K be a small ∞-category, and $d\colon K^{\triangleright} \to \mathrm{Cat}_\infty$ a co-cone diagram of small ∞-categories indexed by K. Then:*

(1) the map $\varinjlim d|_K \to d(\infty)$ is an equivalence after applying idempotent completion if and only if applying $\mathrm{Fun}(-, \mathcal{S})$ to d gives a limit diagram of ∞-categories;
(2) $\varinjlim d|_K \to d(\infty)$ is an equivalence if and only if it is an equivalence after applying idempotent completion and the $d(k) \to d(\infty)$ are jointly essentially surjective, $k \in K$.

Proof. Let's prove 1. The direction $\Rightarrow$ follows by mapping out to $\mathrm{Fun}(\Delta^n, \mathcal{S})$ for all n. For $\Leftarrow$, suppose we have a limit diagram on functors out to $\mathcal{S}$. Mapping in from an arbitrary small ∞-category, we deduce that we have a limit diagram on functors out to $\mathcal{P}(\mathcal{D})$ for all $\mathcal{D}$. Therefore applying $\mathcal{P}(-)$ gives a colimit diagram in the subcategory of Pr^L identified in Proposition 2.7, whence the conclusion follows.

Now for 2, first suppose $\varinjlim d|_K \xrightarrow{\sim} d(\infty)$. Then certainly we also have an equivalence on idempotent completion. Let $\mathcal{C} \subset d(\infty)$ denote the union of the essential images of the $d(k) \to d(\infty)$. Then by the universal property of colimits we deduce that this inclusion $\mathcal{C} \subset d(\infty)$ has a section, whence it's an equality, as desired. Now suppose we have an equivalence on idempotent completion. Since every ∞-category embeds fully faithfully in its idempotent completion, it follows that $\varinjlim d|_K \to d(\infty)$ is fully faithful. But the other condition gives essential surjectivity, whence the conclusion. □

2.3. **Inverting all arrows.** In the following sections we will need to use several different "formulas" for the functor left adjoint to the inclusion $\mathcal{S} \to \mathrm{Cat}_\infty$. The purpose of this section is to collect them.

Theorem 2.9. *For a functor $F\colon \mathrm{Cat}_\infty \to \mathcal{S}$, the following properties are equivalent:*

(1) F is left adjoint to the inclusion $\mathcal{S} \subset \mathrm{Cat}_\infty$.
(2) F preserves all colimits, $F(*) = *$, and $F(\Delta^1) = *$.
(3) F preserves all colimits, and $F(\Delta^n) = *$ for all n.

Moreover, the ∞-category of all such functors is equivalent to the terminal ∞-category $*$. (In particular, the implicit data of the adjunction in 1 is unique.)

Proof. This is a simple consequence of the complete Segal space presentation of Cat_∞ ([Rez01], and see [Lur09b] for a natively ∞-categorical account). First, from that presentation (more specifically, from the fact that it realises Cat_∞ as a localisation of $\mathcal{P}(\Delta)$), one sees that any colimit-preserving functor out of Cat_∞ is the left Kan extension of its restriction to Δ. Since the ∞-category of terminal functors from any ∞-category to $\mathcal{S}$ is always $*$, this shows the last claim holds if we take equivalent condition 3. On the other hand the complete Segal space presentation (more specifically, the Segal condition) also shows that Δ^n is the colimit of n copies of Δ^1 placed end-to-end, which implies that $2 \Leftrightarrow 3$. Note that the functor in 1 is uniquely characterised up to equivalence, and so is the functor in 3, again by the complete Segal space presentation. Thus, to see that 1 is equivalent to 2 and 3, we just need to see that the left adjoint F to the inclusion indeed satisfies $F(*) = *$ and $F(\Delta^1) = *$. The first claim is tautological as $* \in \mathcal{S}$. For the second claim, it exactly corresponds to the completeness criterion in complete Segal spaces. □

From now on we will write $\mathcal{C} \mapsto |\mathcal{C}|$ for the functor F characterised by the previous theorem. For a given ∞-category $\mathcal{C}$, to verify a proposed description of $|\mathcal{C}|$, one has, generally speaking, two options: either make that description functorial in $\mathcal{C}$ and verify condition 2 of the theorem, or else produce a comparison map $\mathcal{C} \to |\mathcal{C}|$ and argue that it satisfies the universal property implicit in condition 1, namely that maps to ∞-groupoids from $|\mathcal{C}|$ are the same as from $\mathcal{C}$.

Corollary 2.10. *For each ∞-category $\mathcal{C}$, the following are descriptions of the ∞-groupoid $|\mathcal{C}|$:*

(1) $|\mathcal{C}| = \mathcal{C}[\mathrm{Ar}\,\mathcal{C}^{-1}]$, the ∞-category obtained by inverting all arrows.
(2) $|\cdot|$ is the left Kan extension of the terminal functor $*\colon \Delta \to \mathcal{S}$ along the inclusion $\Delta \subset \mathrm{Cat}_\infty$.
(3) $|\mathcal{C}|$ is the colimit of the simplicial space $\Delta^n \mapsto \mathrm{Map}_{\mathrm{Cat}_\infty}(\Delta^n, \mathcal{C})$ (the complete Segal space associated to $\mathcal{C}$).
(4) $|\mathcal{C}| = \varinjlim_{\mathcal{C}} *$.

(5) $|\mathcal{C}| = \mathcal{P}^{lc}(\mathcal{C})^{atom}$, *where* $\mathcal{P}^{lc}(\mathcal{C}) \subset \mathcal{P}(\mathcal{C})$ *is the full subcategory on those* $\mathcal{C}^{op} \to \mathcal{S}$ *which are constant on every simplex, or equivalently send every morphism to an isomorphism.*
(6) *If a simplicial set* $X \in \mathrm{sSet}$ *localises to* $\mathcal{C}$ *in the Joyal presentation* $\mathrm{Cat}_\infty \simeq \mathrm{sSet}[ce^{-1}]$, *then* X *localises to* $|\mathcal{C}|$ *in the Kan presentation* $\mathcal{S} \simeq \mathrm{sSet}[we^{-1}]$.

Proof. 1 is tautologically a description of the left adjoint to the inclusion $\mathcal{S} \subset \mathrm{Cat}_\infty$. For 2, note again from the complete Segal space picture that every colimit-preserving functor out of Cat_∞ is left Kan extended from Δ, so this is a rephrasing of condition 3 in the above theorem. For 3, the colimit of the simplicial space is by definition the left adjoint to the inclusion of constant simplicial spaces into all simplicial spaces, and this restricts to the desired adjunction on complete Segal spaces. For 4, note that the colimit in question is by definition determined by

$$\mathrm{Map}(\varinjlim_{\mathcal{C}} *, X) = \mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{S})}(*, X),$$

the mapping space between the constant functor on $*$ and the constant functor on X. As $\mathcal{C} \to |\mathcal{C}|$ is a localisation, the pullback map on functors to $\mathcal{S}$ is fully faithful, so we deduce $\varinjlim_{\mathcal{C}} * \xrightarrow{\sim} \varinjlim_{|\mathcal{C}|} *$. So it suffices to assume $\mathcal{C}$ is an ∞-groupoid. But then there is an equivalence $\mathrm{Fun}(\mathcal{C}, \mathcal{S}) \simeq \mathcal{S}_{/\mathcal{C}}$ given by pulling back along the forgetful functor $\mathcal{S}_* \to \mathcal{S}$, see [Lur09a] 3.3.2.7, and in these terms we see that $\mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{S})}(*, X)$ identifies with the space of sections of the projection $\mathcal{C} \times X \to \mathcal{C}$. But this is just $\mathrm{Map}(\mathcal{C}, X)$, as desired. Finally, we can calculate $\mathcal{P}(|\mathcal{C}|)$ using the universal property of $|\cdot|$ to see that 5 holds. Finally, 6 follows from the fact that the identity functor exhibits sSet with the Kan model structure as a Bousfield localisation of sSet with the Quillen model structure, which verifies criterion 1 of the theorem. □

We also can generate more descriptions by applying 1 of the following:

Corollary 2.11.

(1) *There is a unique functorial equivalence* $|\mathcal{C}| \simeq |\mathcal{C}^{op}|$. *Even more,* $|\cdot|\colon \mathrm{Cat}_\infty \to \mathcal{S}$ *is uniquely* C_2*-equivariant with respect to the action of passing to opposite categories on the left and the trivial action on the right.*
(2) $|\cdot|$ *preserves finite products.*
(3) *If two functors* $\mathcal{C} \to \mathcal{D}$ *are related by a natural transformation, they induce homotopic maps* $|\mathcal{C}| \to |\mathcal{D}|$.
(4) *If a functor admits an adjoint in either direction, it induces an equivalence on* $|\cdot|$.

Proof. Claim 1 follows from the theorem because the terminal functor $\Delta \to \mathcal{S}$ obviously has the required invariance properties. Claim 2 follows from description 3 of the above corollary, since Δ^{op} is sifted, [Lur09a] 5.5.8.4. Claim 3 follows from claim 2 and $|\Delta^1| = *$. Claim 4 follows from claim 3. □

2.4. **Topological analogue: proper maps, proper base change, and proper descent.** In the next section we will discuss the notion of proper functors between ∞-categories, and the associated proper base-change and proper descent theorems. But for motivation, and because we will later use it, we start by recalling the more familiar topological analogue. A map of locally compact Hausdorff topological spaces $f\colon X \to Y$ is called *proper* if the preimage of every compact subset is compact. A crucial fact about proper maps is the *tube lemma*: If $y \in Y$, then the $f^{-1}(U)$ for U an open neighbourhood of y form a cofinal system of open neighbourhoods of the fibre X_y.

We start by recalling the version of the proper base-change theorem proved by Lurie in [Lur09a] 7.3.

Theorem 2.12. *Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a pullback diagram of locally compact Hausdorff spaces with f proper. Then the induced commutative diagram of ∞-categories gotten by applying $\mathrm{Sh}(-)$

$$\begin{array}{ccc} \mathrm{Sh}(X') & \xleftarrow[g'^*]{} & \mathrm{Sh}(X) \\ {\scriptstyle f'^*}\uparrow & & {\scriptstyle f^*}\uparrow \\ \mathrm{Sh}(Y') & \xleftarrow[g^*]{} & \mathrm{Sh}(Y) \end{array}$$

is right adjointable *(or* right Beck–Chevalley*): the vertical maps f^* and f'^* have right adjoints f_* and f'_* respectively, and the natural comparison map is an equivalence*

$$g^* f_* \xrightarrow{\sim} f'_* g'^*.$$

As observed by Deligne in [Del74], this proper base-change can be used to give "proper descent" results. We start with "cdh descent":

Corollary 2.13. *Suppose given a pullback square of locally compact Hausdorff spaces*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

such that:

(1) f is proper;
(2) g is the inclusion of a closed subspace;
(3) the pullback of f to the open complement $Y \smallsetminus Y'$ is an isomorphism.

Then applying $\mathrm{Sh}(-)$ *with pullback functoriality gives a pullback diagram, so*

$$\mathrm{Sh}(Y) \xrightarrow{\sim} \mathrm{Sh}(X) \times_{\mathrm{Sh}(X')} \mathrm{Sh}(Y').$$

Proof. Recall that if we have a closed subset $T' \subset T$ of a topological space T, then equivalences of sheaves on T can be detected by pullback to T' and $T \setminus T'$, [Lur09a] 7.3.2. Furthermore, pullback functors on ∞-categories of sheaves associated to maps of topological spaces preserve finite limits, because they correspond to geometric morphisms of ∞-topoi. It then follows from [Lur12] 5.2.2.37 and the proper base-change theorem that we can test the conclusion of this corollary after pulling back to Y' and $Y \setminus Y'$, compare with the proof of Theorem 2.28 But on pullback to Y' the horizontal maps become equivalences and on pullback to $Y \setminus Y'$ the vertical maps become equivalences, and in either case the conclusion is tautological. □

Corollary 2.14. *Let T be a locally compact Hausdorff space, and let P be a finite set of closed subsets of T. Suppose that for all $P' \subset P$ the intersection $\cap_{S \in P'} S$ admits a cover by elements of P. (In particular, taking $P' = \emptyset$, we deduce $\cup_{S \in P} S = T$.) Then*

$$\mathrm{Sh}(T) \xrightarrow{\sim} \varprojlim_{S \in P^{op}} \mathrm{Sh}(S)$$

via pullback, viewing P as a poset under inclusion.

Proof. When P has ≤ 3 elements this reduces to the special case of Corollary 2.13 in which the proper map f is also a closed inclusion. The general case follows by induction. □

Remark 2.15. The locally compact Hausdorff hypothesis is unnecessary here. Indeed, the proper base-change theorem holds for general topological spaces when the proper map f is a closed inclusion, see [Lur09a] 7.3.2. ○

If one uses open covers instead of closed covers, the finiteness requirements can be removed.

Theorem 2.16. *Let X be a topological space and $\{X_i \to X\}_{i \in I}$ a set of maps to X such that for all $x \in X$, there is an open U containing x and an $i \in I$ such that the pullback of $X_i \to X$ to U has a section. Let $\mathcal{U} \subset \mathrm{Top}_{/X}$ denote the sieve generated by the X_i, so $Y \to X$ lies in $\mathcal{U}$ if and only if it factors through some X_i. Then*

$$\mathrm{Sh}(X) \xrightarrow{\sim} \varprojlim_{(Y \to X) \in \mathcal{U}} \mathrm{Sh}(Y)$$

via the pullback functors.

Proof. Let us define a covering family in Top to be a family of maps $(U_i \to X)_{i \in I}$ whose images cover X. Then the axioms of a pretopology are clearly satisfied, so we get a Grothendieck topology on Top for which the covering sieves over X are those sieves which contain some open cover of X. Our sieve $\mathcal{U}$ is clearly such a sieve, so it suffices to show that $X \mapsto \mathrm{Sh}(X)$ satisfies descent for this Grothendieck topology. However, because the open subsets of X are closed under finite intersection we see that the sieve generated by an open cover in $\mathrm{Open}(X)$ is cofinal in the sieve generated by that open cover in $\mathrm{Top}_{/X}$, meaning our desired descent is equivalent to saying that $U \mapsto \mathrm{Sh}(U)$ is a sheaf of ∞-categories on X. But this is a general property of sheaf categories, [Lur09a] 6.1.3. □

Corollary 2.17. *Let X be a topological space and P a collection of open subsets of X with $U, V \in P \Rightarrow U \cap V \in P$ and $\cup_{U \in P} U = X$. Then*

$$\mathrm{Sh}(X) \xrightarrow{\sim} \varprojlim_{U \in P^{op}} \mathrm{Sh}(U)$$

via pullback, where we view P as a poset under inclusion.

Proof. Let $\mathcal{U}$ denote the sieve of those open subsets of X contained in some $U \in P$. This is a covering sieve by the second condition, so it suffices to show that the inclusion $P \subset \mathcal{U}$ is cofinal, or a $\varinjlim$-equivalence in the language we will use in this paper, see the following section. The right fibre over an element $V \in \mathcal{U}$ is the poset of those $U \in P$ containing V. This is nonempty by definition and is closed under intersection as P is by construction, therefore it is filtered and hence contractible. □

Recall that if Γ is a discrete group, then a Γ-action on an object of a category (or ∞-category) $\mathcal{C}$ is a functor $B\Gamma \to \mathcal{C}$; the underlying object $X \in \mathcal{C}$ is the image of the unique object of $B\Gamma$. The Γ-fixed point (or homotopy fixed point) object, if it exists, is the limit over this $B\Gamma$-diagram in $\mathcal{C}$ and is abusively denoted X^Γ.

Corollary 2.18. *Let X be a topological space with a free proper left action of a discrete group Γ, meaning for all $x \in X$ there is an open neighbourhood U of x such that all the $\gamma \cdot U$ are disjoint, $\gamma \in \Gamma$. Then*

$$\mathrm{Sh}(\Gamma\backslash X) \xrightarrow{\sim} \mathrm{Sh}(X)^\Gamma$$

via pullback.

Proof. Let $Y = \Gamma\backslash X$. The condition implies that the quotient map $X \to Y$ admits local sections; thus we have descent for the sieve $\mathcal{U}$ on $\mathrm{Top}_{/Y}$ generated by $X \to Y$. Thus it suffices to show that the functor $B\Gamma \to \mathcal{U}$ classifying the Γ-action on X is a $\varinjlim$-equivalence. The right fibre over an element $X' \to Y$ is the category whose objects are the factorisations of $X' \to Y$ through $X \to Y$, and the morphisms are induced by composition with elements of Γ acting on X. By the assumptions the set of such factorisations forms a free Γ-orbit, whence this category is equivalent to $*$ and is therefore contractible, as required. □

2.5. **Proper functors, proper base change, and proper descent.** The basic concepts in this section were picked up from reading Grothendieck's *Pursuing stacks*. We recall the following theorem/definition, Joyal's ∞-categorical generalisation of Quillen's theorem A; see [Lur09a] 4.1.3. Although Joyal (and later Lurie) prove this theorem in the quasi-category model using combinatorial arguments, if we take the ∞-categorical Yoneda lemma and related results for granted, we can give a quick non-combinatorial proof.

Theorem 2.19. *Let $f\colon \mathcal{C} \to \mathcal{D}$ be a functor of small ∞-categories. The following properties are equivalent:*

(1) For any functor $X\colon \mathcal{D} \to \mathcal{E}$ to an ∞-category $\mathcal{E}$, the comparison map of limits

$$\varprojlim X \to \varprojlim (f^* X)$$

is an equivalence (in the sense that if one limit exists so does the other and the map is an equivalence).

(2) *Same condition, but just with $\mathcal{E} = \mathcal{S}$.*
(3) *For any $d \in \mathcal{D}$, the left fibre $\mathcal{C}_{/d}$ is contractible in the sense that $|\mathcal{C}_{/d}| \simeq *$.*

If these properties are satisfied, we say that f is a $\varprojlim$-equivalence*. If the dual properties are satisfied, meaning if the above conditions are satisfied for $f^{op}\colon \mathcal{C}^{op} \to \mathcal{D}^{op}$, we say f is a* $\varinjlim$-equivalence*. (The usual terminology for $\varinjlim$-equivalence is "cofinal functor", see* [Lur09a] *4.1.)*

Here the left fibre $\mathcal{C}_{/d}$ stands for the ∞-category given as the left pullback $\mathcal{C} \overset{\rightarrow}{\times}_{\mathcal{D}} \{d\}$ as defined in [Tam18]. Informally, an object of $\mathcal{C}_{/d}$ is an object $c \in \mathcal{C}$ together with a map $f(c) \to d$.

Proof. In condition 1 we may as well assume $\mathcal{E} = \mathcal{S}$, because limits in $\mathcal{E}$ can be tested on applying $\mathrm{Map}(e, -)$ for all $e \in \mathcal{E}$. So 1 and 2 are equivalent.

Note that the pullback functor $f^*\colon \mathrm{Fun}(\mathcal{D}, \mathcal{S}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{S})$ has a left adjoint $f_!$ given by Kan extension, [Lur09a] 4.3. By adjunction, 2 holds if and only if the (unique) map

$$f_!(*) \to *$$

in $\mathrm{Fun}(\mathcal{D}, \mathcal{S})$ is an equivalence. By the objectwise formula for left Kan extensions, this amounts to the assertion that for all $d \in \mathcal{D}$ the map

$$\varinjlim_{(\mathcal{C}_{/d})^{op}} * \to *$$

is an equivalence. By Corollaries 2.10 and 2.11, this is equivalent to condition 3. □

Remark 2.20. If $f\colon \mathcal{C} \to \mathcal{D}$ is a $\varprojlim$-equivalence, then it induces an equivalence on $|\cdot|$. Indeed, we need to see that if $K \in \mathcal{S}$ then $\mathrm{Map}(|\mathcal{D}|, K) \xrightarrow{\sim} \mathrm{Map}(|\mathcal{C}|, K)$; but this is the special case of 2 where X is the constant functor with value K. This is why Joyal's theorem A is a generalisation of Quillen's. (Recall Quillen's says that condition 3 implies that f induces an equivalence on geometric realisation.) ◦

Example 2.21.

(1) Any left adjoint functor is a $\varprojlim$-equivalence. Indeed, being a left adjoint is equivalent to each left fibre admitting a terminal object.
(2) If f is a localisation, i.e. if it is of the form $\mathcal{C} \to \mathcal{C}[S^{-1}]$ for some collection of arrows S in $\mathcal{C}$, then f is a $\varprojlim$-equivalence. Indeed, in this case the map on functors out to $\mathcal{S}$ is fully faithful, so the comparison map in 2 is an equivalence. ◦

Definition 2.22. Let $f\colon \mathcal{C} \to \mathcal{D}$ be a functor of small ∞-categories. We say that f is *proper* if for every $d \in \mathcal{D}$, the inclusion $\mathcal{C}_d \to \mathcal{C}_{d/}$ of the fibre into the right fibre is a $\varprojlim$-equivalence.◁

Here the fibre $\mathcal{C}_d$ means the pullback of $\mathcal{C} \xrightarrow{f} \mathcal{D} \leftarrow \{d\}$ in the ∞-category of ∞-categories; it is the ∞-category of objects $c \in C$ together with an equivalence $d \simeq f(c)$. For the right fibre, we have an arbitrary map $d \to f(c)$ instead of an equivalence. This definition is some sort of analogue of the tube lemma for proper maps in the topological context.

We will soon show that the class of proper maps is closed under composition and base change. Here are also some general examples.

Example 2.23.

(1) Let $\mathcal{C} \xrightarrow{f} \mathcal{D} \xleftarrow{g} \mathcal{B}$ be arbitrary functors of ∞-categories as indicated. Then the projection
$$\mathcal{C} \overset{\rightarrow}{\times}_{\mathcal{D}} \mathcal{B} \to \mathcal{C}$$
from the left pullback (∞-category of tuples $(c \in \mathcal{C}, b \in \mathcal{B}, f(c) \to g(b))$, see [Tam18]) is proper. Indeed, for fixed $c \in \mathcal{C}$, the fibre is the ∞-category of $(b \in \mathcal{B}, f(c) \to g(b))$ whereas the right fibre is the ∞-category of $(x \in \mathcal{C}, b \in \mathcal{B}, c \to x, f(x) \to g(b))$. The inclusion of the former into the latter has right adjoint given by sending this latter data to the object in the fibre given by $b \in \mathcal{B}$ and the composite $f(c) \to f(x) \to g(b)$.

(2) For any $c \in \mathcal{C}$, the projection $\mathcal{C}_{/c} \to \mathcal{C}$ is proper. This is a special case of 1.

(3) Suppose $f \colon \mathcal{C} \subseteq \mathcal{D}$ is the inclusion of a full subcategory closed under isomorphisms. Then f is left proper if and only if $\mathcal{C}$ is *left closed*: $x \to y$ and $y \in \mathcal{C}$ implies $x \in \mathcal{C}$. (In site-theoretic terminology, this means $\mathcal{C}$ is a *sieve* in $\mathcal{D}$.) Indeed, if $y \in \mathcal{D}$ lies in $\mathcal{C}$ then the condition that $\mathcal{C}_y \to \mathcal{C}_{y/}$ be a $\varprojlim$-equivalence is automatic as it identifies with the inclusion of an initial object, whereas when $y \notin \mathcal{C}$ we exactly need that $\mathcal{C}_{y/}$ be empty.

(4) If every morphism in $\mathcal{D}$ is invertible then any functor $\mathcal{C} \to \mathcal{D}$ is proper, as the fibre identifies with the right fibre.

(5) Encompassing all the above examples, any *locally cartesian fibration* is proper. Indeed, by [AF17] Lemma 2.20 the locally cartesian fibrations are characterised up to equivalence by $\mathcal{C}_d \to \mathcal{C}_{d/}$ being a left adjoint for all $d \in \mathcal{D}$. ○

Proposition 2.24. *Let $f \colon \mathcal{C} \to \mathcal{D}$ be a functor. The following are equivalent:*

(1) f is proper.

(2) For any functor $\Delta^1 \to \mathcal{D}$, the pullback $\mathcal{C}' \to \Delta^1$ of f satisfies the condition that the inclusion $\mathcal{C}'_0 \to \mathcal{C}'$ of the fibre above 0 is a $\varprojlim$-equivalence.

Proof. First, we remark that, in the situation of a functor $\mathcal{C}' \to \Delta^1$, to test whether the inclusion $\mathcal{C}'_0 \to \mathcal{C}'$ is a $\varprojlim$-equivalence, it suffices to prove that the left fibre above any object of $\mathcal{C}'_1$ is contractible. Indeed, the left fibre over an object of $\mathcal{C}'_0$ has a terminal object, hence will automatically be contractible.

By definition, 1 holds if and only if for any $d \in \mathcal{D}$ and any $x = (c, d \to f(c)) \in \mathcal{C}_{d/}$, the left fibre of $\mathcal{C}_d \to \mathcal{C}_{d/}$ above x is contractible. On the other hand, consider an arbitrary functor $\Delta^1 \to \mathcal{D}$ classifying a map $d_0 \to d_1$. Then 2 holds if and only if the left fibre of $\mathcal{C}_{d_0}$ including into $\Delta^1 \times_{\mathcal{D}} \mathcal{C}$, taken at some c lying above d_1, is contractible. However, the data of $d_0 \to d_1$ and c is the same as the data of x, and the corresponding left fibres are equivalent. Thus the conditions in the proposition are equivalent. □

Corollary 2.25. *The class of proper functors between small ∞-categories is closed under pullbacks.*

Proof. The other equivalent condition from the proposition manifestly satisfies this closure property. $\square$

The following is the proper base change theorem in this context.

Theorem 2.26. *Let*

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow{g'} & \mathcal{C} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathcal{D}' & \xrightarrow{g} & \mathcal{D} \end{array}$$

be a pullback diagram of small ∞-categories, and let $\mathcal{E}$ be an ∞-category with all limits. Suppose f is proper. Then the induced commutative diagram gotten from applying $\mathrm{Fun}(-,\mathcal{E})$

$$\begin{array}{ccc} \mathrm{Fun}(\mathcal{C}',\mathcal{E}) & \xleftarrow[g'^*]{} & \mathrm{Fun}(\mathcal{C},\mathcal{E}) \\ {\scriptstyle f'^*}\uparrow & & {\scriptstyle f^*}\uparrow \\ \mathrm{Fun}(\mathcal{D}',\mathcal{E}) & \xleftarrow[g^*]{} & \mathrm{Fun}(\mathcal{D},\mathcal{E}) \end{array}$$

is right adjointable *(or* right Beck–Chevalley*): the vertical maps f^* and f'^* have right adjoints f_* and f'_* respectively, and the natural comparison map*

$$g^*f_* \to f'_*g'^*$$

is an equivalence.

Conversely, suppose that the functor f, as well as all its pullbacks, satisfies the condition that the commutative diagram gotten by applying $\mathrm{Fun}(-,\mathcal{E})$ *is right adjointable, even just in the special case $\mathcal{E} = \mathcal{S}$. Then f is proper.*

Proof. The fact that f^* and f'^* admit right adjoints is purely formal and does not require the left properness. Indeed, the right adjoints are given by right Kan extension. Now assume f proper and choose $F \in \mathrm{Fun}(\mathcal{C},\mathcal{E})$; we want to see that

$$g^*f_*F \to f'_*g'^*F$$

is an equivalence in $\mathrm{Fun}(\mathcal{D}',\mathcal{E})$.

First assume that $\mathcal{D}'$ is the terminal category $*$, so that the functor g classifies an object $d \in \mathcal{D}$. For a functor $F \in \mathrm{Fun}(\mathcal{C},\mathcal{E})$, the objectwise description of the right Kan extension shows that the value $g^*f_*F \in \mathcal{E}$ identifies with the limit of F over the right fibre $\mathcal{C}_{d/}$. Meanwhile the value $f'_*g'^*F$ identifies with the limit of F over the fibre $\mathcal{C}_d$. By definition of properness this comparison map is an equivalence. This handles the case $\mathcal{D}' = *$.

To deduce the general case, note that a map in $\mathrm{Fun}(\mathcal{D}',\mathcal{E})$ is an equivalence if and only if it is so after evaluating on any object, or in other words after pulling back along any functor from $*$. As the pullback of a proper map is proper and Beck–Chevalley comparison maps compose, this reduces us to the case of a point.

For the converse, suppose all base-changes of f satisfy the proper base change theorem for $\mathcal{E} = \mathcal{S}$. Consider the special case of the base-change by a map $\Delta^1 \to \mathcal{D}$, and then apply the proper base change theorem to the pullback of that base-changed map $\mathcal{C}' \to \Delta^1$ along $0 \to \Delta^1$. For a functor $\Delta^1 \to \mathcal{S}$ its limit is the same as its evaluation at the initial object 0, so we see exactly the equivalent condition for properness enunciated in Proposition 2.24. □

Corollary 2.27. *The composition of proper functors is proper. The class of proper functors is closed under colimits in* $\mathrm{Fun}(\Delta^1, \mathrm{Cat}_\infty)$.

Proof. This follows from the above converse to the proper base change theorem, as right adjointability composes and is preserved by limits [Lur12] 4.7.4.18. □

Now we discuss proper descent, or how to identify colimits of ∞-categories along proper maps.

Theorem 2.28. *Let K be a small ∞-category and d a functor $K^{\triangleright} \to \mathrm{Cat}_\infty$, viewed as a K-shaped diagram of small ∞-categories together with a co-cone for this diagram. Suppose that:*

(1) The functor $d(f)$ is proper for all maps f in $K^{\triangleright}$.
(2) For all functors $f\colon * \to d(\infty)$ from the terminal category to the co-cone point of f, the pullback $f^{-1}d$ is a colimit diagram. (Here $f^{-1}d$ is the functor $K^{\triangleright} \to \mathrm{Cat}_\infty$ defined by $(f^{-1}d)(k) = * \times_{d(\infty)} d(k)$.)

Then d is a colimit diagram.

Proof. First, we note that the collection of functors $d(k) \to d(\infty)$, ranging over all $k \in K$, is jointly essentially surjective. Indeed, if an object were not in the joint essential image, the pullback of d along the functor $* \to d(\infty)$ classifying that object would have empty restriction to K, whence empty colimit, contradicting the assumption. Thus by Theorem 2.3 it suffices to see that applying $\mathrm{Fun}(-, \mathcal{S})$ to our diagram d yields a limit diagram of ∞-categories, assuming the same for every pullback $f^{-1}d$ along a functor $f\colon * \to d(\infty)$.

Consider $f\colon \sqcup_I * \to d(\infty)$, a disjoint union of terminal categories indexed by the isomorphism classes of objects in $d(\infty)$, mapping to $d(\infty)$ by selecting an object in each isomorphism class. Consider the induced natural transformation

$$\mathrm{Fun}(d(-), \mathcal{S}) \to \mathrm{Fun}((f^{-1}d)(-), \mathcal{S})$$

of diagrams $(K^{op})^{\triangleleft} \to \mathrm{CAT}_\infty$. We want to see that the source is a limit diagram. Using the criterion of [Lur12] 5.2.2.37, it suffices to check the following four conditions:

(1) The target is a limit diagram. This holds because it is a product of limit diagrams by assumption.
(2) For each $k \in (K^{op})^{\triangleleft}$, the induced functor $\mathrm{Fun}(d(k), \mathcal{S}) \to \mathrm{Fun}((f^{-1}d)(k), \mathcal{S})$ is conservative. This holds because the functor $(f^{-1}d)(k) \to d(k)$ is a pullback of the essentially

surjective functor f hence is itself essentially surjective, and equivalences in presheaf categories are detected objectwise.

(3) The ∞-category $\mathrm{Fun}(d(\infty), \mathcal{S})$ admits K-indexed limits, and these are preserved by

$$\mathrm{Fun}(d(\infty), \mathcal{S}) \to \mathrm{Fun}((f^{-1}d)(\infty), \mathcal{S}).$$

In fact functor categories to $\mathcal{S}$ admit all limits and these are preserved by all pullbacks, since limits are calculated objectwise in functor categories.

(4) For every morphism α in $(K^{op})^{\triangleleft}$, the commutative square of ∞-categories gotten by applying our natural transformation to α is right adjointable. This holds by the proper base change theorem and our assumption that $d(\alpha)$ is proper.

Thus the conditions of [Lur12] 5.2.2.37 apply and finish the proof. □

Corollary 2.29. *Every colimit in* Cat$_\infty$ *produced by the above theorem is* universal*: stable under pullback (via an arbitrary map to the co-cone object).*

Proof. Clear, since the two conditions are stable under pullback. □

Here are some special cases. First we have Cech descent along a covering map.

Corollary 2.30. *Let $f\colon \mathcal{C} \to \mathcal{D}$ be an essentially surjective proper functor in* Cat$_\infty$. *Then $\mathcal{D}$ identifies with the colimit of the Cech nerve of f.*

Proof. Recall that the nondegenerate simplex category is cofinal in Δ, [Lur09a] 6.5.3.7, so in calculating the colimit of the Cech nerve we can restrict to the functors induced by non-degenerate maps in Δ. But all such functors are pullbacks of f, hence are also proper by Corollary 2.25. Hence by Theorem 2.28 we can reduce to the case $\mathcal{D} = *$. But then f admits a section and hence gives a colimit diagram, [Lur09a] 6.1.3.16. □

Here is the analogue of "cdh descent" in algebraic geometry.

Corollary 2.31. *Suppose given a pullback square σ in* Cat$_\infty$

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow{g'} & \mathcal{C} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathcal{D}' & \xrightarrow{g} & \mathcal{D} \end{array}$$

such that:

(1) f is proper;
(2) g is the inclusion of a left closed full subcategory;
(3) the pullback of f to the full subcategory given by the complement $\mathcal{D} \smallsetminus \mathcal{D}'$ is an equivalence.

Then σ is also a pushout square.

Proof. Note that the conditions are closed under base-change along any functor $\mathcal{X} \to \mathcal{D}$. Furthermore, all of f, g, f', g' are proper, as they are pullbacks of proper maps. Therefore, by the proper descent theorem, it suffices to prove this when $\mathcal{D} = *$. But then either $\mathcal{D}' = \emptyset$, in which case f and f' are equivalences and hence the square is a pushout, or $\mathcal{D}' = *$, in which case g and g' are equivalences and hence the square is a pushout. □

The following is "descent for left-closed covers".

Corollary 2.32. *Let $\mathcal{C}$ be a small ∞-category. Suppose given a collection P of left-closed full subcategories of $\mathcal{C}$, viewed as a poset under inclusion, such that for all $x \in \mathcal{C}$ the subposet of those elements of $\mathcal{D}$ containing x is contractible. Then*

$$\mathcal{C} = \varinjlim_{\mathcal{D} \in P} \mathcal{D}.$$

Proof. As every inclusion of left closed full subcategories is itself a left closed inclusion, it is proper. Therefore, by the proper descent theorem, it suffices to show that the pullback of our diagram along any functor $* \to \mathcal{C}$, classifying an object $c \in \mathcal{C}$, has colimit $*$. But as the elements of P are full subcategories, every term in this pullback is either $*$ (when c lies in the corresponding full subcategory) or $\emptyset$ (otherwise). Thus we see exactly the condition that the poset of those $\mathcal{D} \in P$ containing c should be contractible, in the form that the colimit of the terminal diagram is terminal. (Note that colimits in $\mathcal{S}$ are automatically also colimits in Cat_∞, as the inclusion has a right adjoint given by neglecting the non-invertible morphisms.) □

Here we calculate homotopy orbits for a group action. We stick to the special case that's relevant for us.

Corollary 2.33. *Let P be a poset and G a group acting on P. We can encode this action by a functor $\mathcal{P}\colon BG \to$ Posets $\subseteq \mathrm{Cat}_\infty$. Then the colimit in Cat_∞*

$$\varinjlim_{BG} \mathcal{P}$$

naturally identifies with the action category $G\backslash\backslash P$ whose objects are the $p \in P$ and whose morphisms $p \to p'$ are the $g \in G$ with $gp \leq p'$.

Proof. Let's make the comparison map. Note that $G\backslash\backslash * = BG$, and the functor $* \to BG$ is tautologically G-invariant for the trivial G-action on $*$. Now consider the functor $G\backslash\backslash P \to BG$ induced by $P \to *$. The pullback of $* \to BG$ along this functor recovers P together with its G-action, which gives the desired comparison map.

To show the comparison map is an isomorphism, because proper descent is universal it suffices to use proper descent to establish $\varinjlim_{BG} * \xrightarrow{\sim} BG$. But after we pull back along $* \to BG$ we find that what we need is $\varinjlim_{BG} G = *$ where G is promoted to a G-object by the translation action. But G with the translation action is the same as the left Kan extension of the terminal functor along $* \to BG$, so this follows because left Kan extensions preserve colimits. □

Note that the set of isomorphism classes of objects in $G\backslash\backslash P$ is the quotient set $G\backslash P$. In general, this quotient set does not have a poset structure making the quotient $P \to G\backslash P$ a map of posets, but under a suitable regularity hypothesis this holds.

Lemma 2.34. *Let P be a poset and G a group acting on P. Suppose that for $x \in P$ and $g \in G$ we have the implication $x \leq gx \Rightarrow x = gx$. Then:*

1. *Every endomorphism in $G\backslash\backslash P$ is an isomorphism.*
2. *There is a poset structure on the quotient set $G\backslash P$ defined by $X \leq Y$ if and only if there exists an $x \in X$ and $y \in Y$ with $x \leq y$.*
3. *This poset structure on $G\backslash P$ serves as the quotient of G acting on P in the category of posets.*

Proof. If $g \in G$ gives a map $x \to x$ in the action category, then $gx \leq x$, so by hypothesis (applied to $x \leq g^{-1}x$) we deduce that $gx = x$, and it follows that g^{-1} gives an inverse map. Thus part 1 holds. Part 2 is a consequence: in general, if $\mathcal{C}$ is a category where every endomorphism is an isomorphism, then the set of isomorphism classes of objects in $\mathcal{C}$ forms a poset with $[x] \leq [y]$ iff there exists a map $x \to y$. Finally, part 3 is clear once we know that the quotient set is indeed a poset. □

The last corollary is an almost tautological though fairly fundamental colimit diagram.

Corollary 2.35. *Let $\mathcal{C}$ be a small ∞-category. Then*

$$\varinjlim_{x \in \mathcal{C}} \mathcal{C}_{/x} \xrightarrow{\sim} \mathcal{C},$$

and this colimit diagram is universal (still gives a colimit after arbitrary pullback).

Proof. All the functors in the co-cone diagram are of the form $\mathcal{D}_{/y} \to \mathcal{D}$, hence are proper by Example 2.23. Thus, by the proper descent theorem, it suffices to see that we have a colimit diagram after pullback along any $* \to \mathcal{C}$ classifying an object $c \in \mathcal{C}$. Then the claim becomes that $\varinjlim_{x \in \mathcal{C}} \mathrm{Map}_{\mathcal{C}}(c, x) = *$. Note that the functor $x \mapsto \mathrm{Map}_{\mathcal{C}}(c, x)$ under consideration here is the left Kan extension of the terminal functor along the projection $\mathcal{C}_{c/} \to \mathcal{C}$, thus the value of the colimit is equivalently $\varinjlim_{\mathcal{C}_{c/}} * = |\mathcal{C}_{c/}|$. So we need that $\mathcal{C}_{c/}$ is contractible; but indeed it has an initial object. □

Remark 2.36. One can also give many other proofs of this result. For example, one can directly check that applying $\mathrm{Map}(\Delta^n, -)$ gives a colimit diagram for all n, so that we have an a priori stronger statement: in the complete Segal anima world, we even have a colimit of simplicial anima. Or else one can use ∞-topos theory: in $\mathrm{PSh}(\mathcal{C})$ we have $* = \varinjlim_{x \in \mathcal{C}} h_x$ because maps out of either side calculates the limit over a $\mathcal{C}^{op}$-diagram; then the conclusion follows by descent. ○

3. Miscellaneous background on constructible sheaves

Let $\pi \colon X \to P$ be a stratified topological space in the sense of Lurie, [Lur12] Appendix A: a continuous map from a topological space X to a poset P equipped with the Alexandroff topology. The $X_p := \pi^{-1}(\{p\})$ are the *strata* of the stratified space. Recall that in the Alexandroff topology, every point $p \in P$ has a minimal open neighbourhood, namely the set of q with $q \geq p$. The stratum X_p is a closed subspace of the open subspace $U_p := \pi^{-1}(\{q \geq p\})$ of X, the *open star* around the p-stratum.

It will be handy to be able to test equivalences of constructible sheaves by restricting to strata. Some hypothesis on $X \to P$ is necessary for this to be possible. We prefer to impose the hypothesis only on P, and we will take the condition singled out by Lurie in his Theorem A.9.3: P satisfies the ascending chain condition, meaning there is no infinite chain $p_0 < p_1 < p_2 < \dots$ of strict inequalities in P.

Lemma 3.1. *Suppose $X \to P$ is a stratified space with P a poset satisfying the ascending chain condition. Then a map of constructible sheaves on X is an equivalence if and only if its pullback to the stratum X_p is an equivalence for all $p \in P$.*

Proof. Suppose we have a map f of sheaves which is an equivalence on each stratum. Since the U_p cover X, it suffices to show that f induces an isomorphism on restriction to each U_p. To prove this by noetherian induction on p, it suffices to show that if it holds for all $q > p$, then it holds for p. But $U_p \smallsetminus X_p$ is covered by the U_q for $q > p$, so we deduce f gives an isomorphism there. Since f gives an isomorphism on the closed complement $X_p \subset U_p$ by assumption, it follows from the gluing formalism, [Lur12] A.8, that f gives an isomorphism on U_p, as desired. □

Now we prove a version of the homotopy invariance of constructible sheaves.

Proposition 3.2. *Let $X \to P$ be a stratified topological space such that P satisfies the ascending chain condition. Consider the projection $f\colon X \times [0,1] \to X$. Then the pullback functor*

$$f^*\colon \mathrm{Sh}^{constr}(X) \to \mathrm{Sh}^{constr}(X \times [0,1])$$

is an equivalence. Here $X \times [0,1]$ is stratified by the composition $X \times [0,1] \to X \to P$.

Proof. Recall from [Lur12] A.2.10 that for arbitrary topological spaces T, the pullback

$$f^*\colon \mathrm{Sh}(T) \to \mathrm{Sh}(T \times [0,1])$$

is fully faithful and admits a left adjoint $f_\natural$ which commutes with pullbacks in the T variable; and for future reference we recall this also holds for open and half-open intervals replacing $[0,1]$. We deduce that the f^* in our statement is fully faithful, and that an $\mathcal{F} \in \mathrm{Sh}^{constr}(X \times [0,1])$ lies in the essential image if and only if $f_\natural\mathcal{F}$ is constructible and $\mathcal{F} \xrightarrow{\sim} f^* f_\natural \mathcal{F}$. By the lemma and compatibility of $f_\natural$ with pullbacks, we therefore reduce to the case where $\mathcal{F}$ is locally constant, provided we also show $f_\natural\mathcal{F}$ is locally constant.

Thus suppose $\mathcal{F} \in \mathrm{Sh}(X \times [0,1])$ is locally constant. By refining an open cover of $X \times [0,1]$, we find there is an open cover $\{U_i\}$ of X and open subintervals $I_{1i}, \dots, I_{n_i i}$ covering $[0,1]$ with empty intersection except for $I_{ji} \cap I_{j+1,i} \neq \emptyset$, such that $\mathcal{F}|_{U_i \times I_{ji}}$ is constant for all j; in particular $\mathcal{F}|_{U_i \times I_{ji}}$ is pulled back from a constant sheaf on U_i, hence by full faithfulness of pullbacks along intervals, the constant sheaf on U_i from which it's pulled back is uniquely and functorially determined. Thus, working our way from 1 to n_i along the intersections, we can identify all these constant sheaves on U_i with one another, hence $\mathcal{F}|_{U_i \times [0,1]}$ is pulled back from this same constant sheaf on U_i. But again by compatibility of $f_\natural$ with pullbacks, the desired claims are local on X, so this suffices as the $\{U_i\}$ cover. □

Corollary 3.3. *Suppose given a map $f\colon X \to Y$ of topological spaces, compatibly stratified by a map $Y \to P$ to a P satisfying the ascending chain condition.*

If f is a stratified homotopy equivalence,[1] *then pullback induces*

$$f^*\colon \mathrm{Sh}^{constr}(Y) \xrightarrow{\sim} \mathrm{Sh}^{constr}(X),$$

and in particular for any constructible sheaf $\mathcal{F}$ on Y the natural map is an equivalence

$$\Gamma(Y, \mathcal{F}) \xrightarrow{\sim} \Gamma(X, f^*\mathcal{F}).$$

Proof. The lemma implies that any two stratified-homotopic maps induce the same pullback functor on constructible sheaves. This gives the first claim, and the second claim follows by taking mapping spaces from the constant sheaf on $*$ to the sheaf $\mathcal{F}$. □

Let us return to the general situation of a stratified space $X \to P$. The association $p \mapsto U_p$ gives a contravariant functor from the poset P to the poset of open subsets of X. Thus there is an induced geometric morphism of ∞-topoi

$$\pi_*\colon \mathrm{Sh}(X) \to \mathrm{Fun}(P, \mathcal{S}),$$

defined by $(\pi_*\mathcal{F})(p) = \mathcal{F}(U_p)$. The left adjoint π^* lands inside the full subcategory $\mathrm{Sh}^{constr}(X)$ of constructible sheaves. Thus we have a comparison functor

$$\pi^*\colon \mathrm{Fun}(P, \mathcal{S}) \to \mathrm{Sh}^{constr}(X).$$

Theorem 3.4. *Let $\pi\colon X \to P$ be a stratified topological space with π surjective and P satisfying the ascending chain condition. Suppose there is a collection $\mathcal{B}$ of open subsets of X such that:*

1. *the representable sheaves h_U for $U \in \mathcal{B}$ generate the ∞-topos $\mathrm{Sh}(X)$;*[2]
2. *for all $U \in \mathcal{B}$, there is a $p \in P$ such that U includes into U_p by a stratified homotopy equivalence.*

Then the pullback map

$$\pi^*\colon \mathrm{Fun}(P, \mathcal{S}) \to \mathrm{Sh}(X)$$

preserves all limits and colimits and is fully faithful with essential image $\mathrm{Sh}^{constr}(X)$. Moreover every constructible sheaf on X is the limit of its Postnikov tower, and hence is hypercomplete, compare [Lur12] *A.5.9.*

Proof. Let $\mathcal{F} \in \mathrm{Sh}(X)$. We claim the following are equivalent:

[1] This means there is a stratum-preserving map backwards and stratum-preserving homotopies making both composites homotopic to the identity. In particular, the restriction to each stratum is a homotopy equivalence, but also more.

[2] This condition implies that $\mathcal{B}$ is a basis for the topology. If every sheaf is hypercomplete, the converse holds. In general it's enough for $\mathcal{B}$ to be a basis closed under finite intersections, or even just a collection such that every open subset admits a truncated hypercover by elements of $\mathcal{B}$.

(1) $\mathcal{F}$ is constructible;
(2) For all $U \in \mathcal{B}$, we have $\mathcal{F}(U_p) \xrightarrow{\sim} \mathcal{F}(U)$ where U_p is as in hypothesis 2 (and is uniquely determined as $U_p = \pi^{-1}(\pi(U))$);
(3) $\pi^*\pi_*\mathcal{F} \xrightarrow{\sim} \mathcal{F}$.

Indeed, $1 \Rightarrow 2$ follows from Corollary 3.3 and $3 \Rightarrow 1$ holds because π^* lands inside the constructible sheaves. For $2 \Rightarrow 3$, note that 2 says that $\mathcal{F}|_{\mathcal{B}}$ is the presheaf pullback of the presheaf $\pi_*\mathcal{F}$. Now take an arbitrary sheaf $\mathcal{G}$ on X. By hypothesis 1 in our theorem, we can calculate $\mathrm{Map}(\mathcal{F}, \mathcal{G})$ as maps of presheaves on $\mathcal{B}$. Thus we deduce $\mathrm{Map}(\mathcal{F}, \mathcal{G}) = \mathrm{Map}_{\mathrm{PSh}(P^{op})}(\pi_*\mathcal{F}, \pi_*\mathcal{G})$, which says exactly that $\pi^*\pi_*\mathcal{F} \xrightarrow{\sim} \mathcal{F}$.

From $1 \Leftrightarrow 2$, we already see that the full subcategory of constructible sheaves is closed under all limits. Furthermore, from $1 \Leftrightarrow 3$ we see that if $\mathcal{F}$ is constructible then $\pi^*\pi_*\mathcal{F} \xrightarrow{\sim} \mathcal{F}$, so to see the equivalence $\pi^*\colon \mathrm{Fun}(P, \mathcal{S}) \xrightarrow{\sim} \mathrm{Sh}^{constr}(X)$, which also gives preservation under colimits, we only need the other direction $\varphi \xrightarrow{\sim} \pi_*\pi^*\varphi$. However, by adjunction identities and the previous direction it suffices to show that if a map $\varphi \to \varphi'$ is an equivalence on π^*, then it is an equivalence. But $\varphi(p)$ is recovered as the pullback of $\pi^*\varphi$ to any point in the stratum X_p, so this follows from the surjectivity of π.

The final claim about Postnikov towers follows, because the analogous claim in $\mathrm{Fun}(P, \mathcal{S})$ is clear as Postnikov truncations and limits are computed objectwise. □

Remark 3.5. Consider the special case of $X = P$ mapping to P via the identity. If P satisfies the ascending chain condition, then Lemma 3.1 shows that the U_p generate the ∞-topos $\mathrm{Sh}(P)$; hence the collection of them forms a $\mathcal{B}$ as required. Moreover, every sheaf on P is constructible.

We deduce that if the poset P satisfies the ascending chain condition, then

$$\mathrm{Fun}(P, \mathcal{S}) \xrightarrow{\sim} \mathrm{Sh}(P) = \mathrm{Sh}^{hyp}(P).$$

For arbitrary P, we still have

$$\mathrm{Fun}(P, \mathcal{S}) \xrightarrow{\sim} \mathrm{Sh}^{hyp}(P),$$

as the U_p form a basis for the topology of P on which the induced Grothendieck topology is trivial: every covering of U_p is refined by $U_p = U_p$. ◦

We can interpret the above theorem in light of the following definition.

Definition 3.6.

(1) We say that a stratified space $X \to P$ *admits an exit path ∞-category* if the following conditions hold:
 (a) The full subcategory $\mathrm{Sh}^{constr}(X) \subset \mathrm{Sh}(X)$ is closed under all limits and colimits;
 (b) The ∞-category $\mathrm{Sh}^{constr}(X)$ is generated under colimits by a set of atomic objects (see Lemma 2.6).
 (c) $\pi^*\colon \mathrm{Fun}(P, \mathcal{S}) \to \mathrm{Sh}^{constr}(X)$ preserves all limits (and colimits, but that is automatic);

(2) If $X \to P$ admits an exit path ∞-category, we define its exit path ∞-category $\Pi(X \to P)$ to be the opposite category of the full subcategory $\left[\mathrm{Sh}^{constr}(X)\right]^{atom}$ of atomic constructible sheaves.
(3) If $f\colon (X \to P) \to (Y \to Q)$ is a map of stratified spaces, we say that f *respects exit path ∞-categories* if $f^*\colon\ \mathrm{Sh}^{constr}(Y) \to \mathrm{Sh}^{constr}(X)$ preserves limits (and colimits, but that is automatic). ◁

If $X \to P$ admits an exit path ∞-category, it follows from Proposition 2.7 that there is an induced "exodromy" equivalence (cf. [BGH18] for the terminology)

$$\mathrm{Fun}(\Pi(X \to P), \mathcal{S}) \xrightarrow{\sim} \mathrm{Sh}^{constr}(X),$$

and it follows from Proposition 2.7 that, in terms of the exodromy equivalence, the pullback map $\pi^*\colon\ \mathrm{Fun}(P, \mathcal{S}) \to \mathrm{Sh}^{constr}(X)$ is recovered as composition along a uniquely determined functor

$$\Pi(X \to P) \to P.$$

Similarly, the condition that $f\colon (X \to P) \to (Y \to Q)$ respect exit path ∞-categories is equivalent to the condition that the induced pullback functor on constructible sheaves is given, via exodromy, by composition with a functor

$$\Pi(X \to P) \to \Pi(Y \to Q).$$

Moreover, this functor is then uniquely determined as the restriction to atomic objects of the right adjoint to $f^*\colon\ \mathrm{Sh}^{constr}(Y) \to \mathrm{Sh}^{constr}(X)$, see Proposition 2.7.

Theorem 3.4 already gives examples of stratified spaces admitting an exit path ∞-category; indeed, in those cases the exit path ∞-category is the stratifying poset P itself. Note that if $(X \to P)$ and $(Y \to Q)$ are stratified spaces whose exit path ∞-category identifies with the stratifying poset, then every map $(X \to P) \to (Y \to Q)$ respects exit path ∞-categories simply because we are given the required map $P \to Q$ as part of the data. This simple observation, together with the following permanence properties, will be enough for us to identify the exit path ∞-categories we need in the next section.

Proposition 3.7.

(1) Let $f\colon X \to P$ be a stratified space admitting an exit path ∞-category. Then for every locally closed subset $Q \subset P$, the stratified space $f^{-1}(Q) \to Q$ admits an exit path ∞-category, the inclusion $(f^{-1}Q \to Q) \to (X \to P)$ respects exit path ∞-categories, and

$$\Pi(f^{-1}(Q) \to Q) \xrightarrow{\sim} \Pi(X \to P) \times_P Q.$$

(2) Let K be a small ∞-category and $\{X_k \to P_k\}_{k \in K}$ a K-shaped diagram of stratified spaces, equipped with a co-cone $X_\infty \to P_\infty$. Suppose:
(a) Each object $X_k \to P_k$ admits an exit path ∞-category for $k \in K$.
(b) Each map $(X_k \to P_k) \to (X_{k'} \to P_{k'})$ respects exit path ∞-categories for $k \to k'$.
(c) We have $\mathrm{Sh}(X_\infty) \xrightarrow{\sim} \varprojlim_{k \in K^{op}} \mathrm{Sh}(X_k)$ and $\mathrm{Sh}^{constr}(X_\infty) \xrightarrow{\sim} \varprojlim_{k \in K^{op}} \mathrm{Sh}^{constr}(X_k)$ via pullback.

Then:
(a) $X_\infty \to P_\infty$ *admits an exit path* ∞*-category.*
(b) Each map $(X_k \to P_k) \to (X_\infty \to P_\infty)$ *respects exit path* ∞*-categories,* $k \in K$*.*
(c) $\Pi(X_\infty \to P_\infty) \overset{\sim}{\leftarrow} \varinjlim_{k\in K} \Pi(X_k \to P_k)$*.*

Proof. For part 1, by factoring a locally closed inclusion as a closed inclusion followed by an open inclusion, it suffices to treat those cases separately. For an open subset $U \subset P$, we have that $\mathrm{Sh}(U) \overset{\sim}{\to} \mathrm{Sh}(X)_{/h_U}$ via the left adjoint to the pullback, see [Lur09a] 7.3.2. Moreover, this description is compatible with base change, hence it passes to constructible sheaves: $\mathrm{Sh}^{constr}(U) \overset{\sim}{\to} (\mathrm{Sh}^{constr}(X))_{/h_U}$. This gives the conclusion in that case. For a closed subset $Z \subset P$, we argue similarly but using $\mathrm{Sh}(Z) \overset{\sim}{\to} \ker(\mathrm{Sh}(X) \to \mathrm{Sh}(U))$, [Lur09a] 7.3.2, meaning those sheaves on X which restrict to $\emptyset$ on $U = X \setminus Z$, the equivalence being induced by pushforward. Part 2 is straightforward from Proposition 2.7. □

We also note that if we take $P = *$ then constructible sheaf means locally constant sheaf, and by comparing with [Lur12] A.1.5 we find that $X \to *$ admits an exit path ∞-category if and only if $\mathrm{Sh}(X)$ is locally of constant shape in the sense of [Lur12], and the exit path ∞-category is the ∞-groupoid given by the shape.

To finish, let us also recall from [ØJ20] that exodromy for constructible sheaves with values in $\mathcal{S}$ automatically extends to sheaves with values in an arbitrary compactly generated ∞-category.

Proposition 3.8. *Let* $X \to P$ *be a stratified space which admits an exit path* ∞*-category, and let* $\mathcal{E}$ *be a compactly generated* ∞*-category. Then there is a natural equivalence*

$$\mathrm{Fun}(\Pi(X \to P), \mathcal{E}) \overset{\sim}{\to} \mathrm{Sh}^{constr}(X; \mathcal{E})$$

where the constructible full subcategory of $\mathrm{Sh}(X;\mathcal{E})$ *is again defined as the full subcategory of those sheaves whose pullback to each stratum is locally constant.*

This equivalence is essentially determined from the version where $\mathcal{E} = \mathcal{S}$ *as follows: a functor* $\varphi \in \mathrm{Fun}(\Pi(X \to P), \mathcal{E})$ *and a constructible sheaf* $\mathcal{F} \in \mathrm{Sh}^{constr}(X;\mathcal{E})$ *correspond under the above equivalence if and only if for all* $x \in \mathcal{E}$*, the functor* $\mathrm{Map}(x, \varphi(-))$ *and the sheaf* $\mathrm{Map}(x, \mathcal{F}(-))$ *correspond under the equivalence for* $\mathcal{E} = \mathcal{S}$*.*

4. Borel–Serre and reductive Borel–Serre compactifications

We start with a recap of some material from Borel–Serre's article, taken from a slightly different perspective.

4.1. **The canonical homogeneous space over $\mathbb{R}$.** Let $G = G_\mathbb{R}$ be a connected reductive group over $\mathbb{R}$. The canonical homogeneous space X is a transitive $G(\mathbb{R})$-space whose isotropy groups are exactly the maximal compact subgroups of $G(\mathbb{R})$. It exists and is unique up to isomorphism as $G(\mathbb{R})$ admits a unique conjugacy class of maximal compact subgroups ([Mos55]). This does not quite justify calling it "canonical" because it is not in general unique up to unique isomorphism. But we can always fix a choice for G, say $X = G(\mathbb{R})/K$ for some choice of maximal compact subgroup K, and as we discuss later this determines a choice for each Levi factor of G as well, and that will be enough canonicity for us.

4.2. **The Borel–Serre corners.** Now, switching notation, let us take a connected reductive group G over $\mathbb{Q}$ giving rise to a $G_{\mathbb{R}}$ as in the previous section by extension of scalars. We will be interested in the restriction of the $G(\mathbb{R})$-action on X to arithmetic subgroups $\Gamma \subset G(\mathbb{Q})$. The basic problem to "fix" is that the Γ-action on X, while properly discontinuous, is not cocompact. It turns out the explanation for this non-cocompactness lies in the parabolic subgroups of G, and we start with a brief recap on those and their relation to relative root systems.

For a parabolic subgroup P of G, let S_P denote the maximal split torus in the centre of the Levi factor P/U_P, where U_P is the unipotent radical of P. If $P \subset Q$ then there is an induced natural injection $S_Q \hookrightarrow S_P$. Moreover, if P is conjugate to P' then any choice of conjugating element induces the *same* isomorphism $S_P \xrightarrow{\sim} S_{P'}$. In this sense $S_P = S_{[P]}$ only depends on the conjugacy class $[P]$ of P, and all the $S_{[P]}$ can be compatibly viewed as subtori of the "abstract maximal split torus", which is $S := S_{[P_0]}$ for minimal parabolic P_0.

We recall also that there is a canonical finite subset $\Delta \subset X^*(S) = \operatorname{Hom}(S, \mathbb{G}_m)$ such that if we choose a maximal split torus S_0 inside a minimal parabolic P_0, determining an isomorphism $S_0 \simeq S$ via projection to P_0/U_{P_0}, then Δ corresponds, via this isomorphism, to the basis of the relative root system $\Phi(S_0, G)$ occuring as weights in $\operatorname{Lie}(U_{P_0})$, compare [BT65, §5], [BJ06, III.1.14]. Then there is an inclusion-reversing bijective correspondence between conjugacy classes of parabolic subgroups $P \subset G$ and subsets of Δ, determined as follows: if $\Delta_P \subset \Delta$ is the subset corresponding to $[P]$, then $S_{[P]} = (\cap_{\chi \in \Delta \smallsetminus \Delta_P} \ker(\chi))^\circ$, see [BS73, 4.1]. Another way of describing the situation is that the restriction of Δ_P to $S_{[P]}$ gives a basis of $\ker(X^*(S_{[P]}) \otimes \mathbb{Q} \to X^*(S_{[G]}) \otimes \mathbb{Q})$.

Let us note the following consequence of this discussion of conjugacy classification of parabolic subgroups. It will be used over and over again.

Proposition 4.1. *Let $P' \subset P$ be an inclusion of parabolic subgroups of G. If $\gamma \in G(\mathbb{Q})$ conjugates P' back inside P, then $\gamma \in P(\mathbb{Q})$.*

Proof. The classification of conjugacy classes of parabolic subgroups recalled above, applied to both G and the Levi factor P/U_P, implies in particular that two parabolic subgroups of P/U_P are P/U_P-conjugate if and only if their preimages are G-conjugate. We deduce that there is a $\rho \in P(\mathbb{Q})$ with $\rho P' \rho^{-1}/U_P = \gamma P' \gamma^{-1}/U_P$, which implies $\rho P' \rho^{-1} = \gamma P' \gamma^{-1}$. (Note that $U_P \subset U_{P'} \subset P'$.) Thus $\rho^{-1}\gamma$ normalises P'. But every parabolic is its own normaliser, [BT65, 4.3], so $\gamma \in P(\mathbb{Q})$ as desired. □

Now, again for a parabolic subgroup $P \subset G$, let $A_P = S_P(\mathbb{R})^\circ$. This group plays a crucial role in the story. Namely, on the one hand there is a natural proper and free right action $\bullet_P \colon X \times A_P \to X$ of A_P on X, the *geodesic action* of [BS73, §3]. But on the other hand the discussion above provides natural *root coordinates*

$$A_P/A_G \xrightarrow{\sim} (\mathbb{R}_{>0})^{\Delta_P},$$

see [BS73, 4.2]. For $P \subset Q$, the geodesic actions of A_P and A_Q are compatible under the natural injection $A_Q \hookrightarrow A_P$, and the root coordinates are too, in that they make this injection correspond to the inclusion $(\mathbb{R}_{>0})^{\Delta_Q} \subset (\mathbb{R}_{>0})^{\Delta_P}$ of the coordinate hypersurface corresponding to $\Delta_Q \subset \Delta_P$.

Take care that while A_P, Δ_P, and the root coordinates only depend on the conjugacy class $[P]$, the geodesic action depends on P itself. In fact, if $\gamma \in G(\mathbb{Q})$ and $a \in A_{[P]}$, then

$$x \bullet_{\gamma P \gamma^{-1}} a = \gamma x \bullet_P a,$$

see [BS73, 5.6]. Note that this in particular says that the $P(\mathbb{Q})$-action on X commutes with the geodesic action by A_P; but in fact the whole $P(\mathbb{R})$-action does. (This is explained by the fact that one can also define an analogous geodesic action for an arbitrary parabolic subgroup of $G_{\mathbb{R}}$, and then the previous formula holds for all $\gamma \in G(\mathbb{R})$.)

Loosely speaking, the point of all this is that the geodesic actions by parabolic subgroups give enough directions via which a point in X can "wander off to ∞" to fully account for the non-cocompactness of the Γ-action on X. Actually, with the conventions of [BS73], it is the limit as $t \to 0$ in $\mathbb{R}_{>0}$ that corresponds, under the root coordinates and the geodesic action, to wandering off to ∞ in X. This leads to the following definition.

Definition 4.2. Let G be a reductive group over $\mathbb{Q}$, and recall the canonical homogeneous space X associated to $G_{\mathbb{R}}$ as above, whose stabilisers are the maximal compact subgroups of $G(\mathbb{R})$. Let $P \subset G$ be a parabolic subgroup.

(1) The *Borel–Serre corner* is the topological space defined by

$$\widehat{X}_{\geq P} := X \times^{A_P} (\mathbb{R}_{\geq 0})^{\Delta_P},$$

the quotient of $X \times (\mathbb{R}_{\geq 0})^{\Delta_P}$ which equalises the right geodesic action on X and the left action by componentwise multiplication on $(\mathbb{R}_{\geq 0})^{\Delta_P}$ via the root coordinates $A_P/A_G \simeq (\mathbb{R}_{>0})^{\Delta_P}$.

(2) The *combinatorial Borel–Serre corner* is the partially ordered set

$$\{0 < 1\}^{\Delta_P},$$

which we will identify with the poset $\mathcal{P}_{P/}$ of parabolic subgroups containing P under containment, by matching $Q \supseteq P$ with the indicator function of the subset $\Delta_P \smallsetminus \Delta_Q \subseteq \Delta_P$.

(3) The *stratified Borel–Serre corner* is the continuous projection map

$$\widehat{X}_{\geq P} \to \{0 < 1\}^{\Delta_P}$$

induced by the map $\mathbb{R}_{\geq 0} \to \{0 < 1\}$ sending 0 to 0 and $t \neq 0$ to 1. ◁

We have a stratified homeomorphism $\widehat{X}_{\geq P} \simeq \mathbb{R}^d \times (\mathbb{R}_{\geq 0})^{\Delta_P} \to \{0 < 1\}^{\Delta_P}$ for some $d \geq 0$. Indeed, if we fix a point x on X determining a maximal compact subgroup of $G(\mathbb{R})$, then the Langlands decomposition of $P(\mathbb{R})$ and the fact that $P(\mathbb{R})$ acts transitively on X (by the Iwasawa decomposition) give an isomorphism $X \simeq \mathbb{R}^d \times A_P$ via which the geodesic action is the right action on the second coordinate, see [BS73, 5.4].

Lemma 4.3. *The Borel–Serre corner $\widehat{X}_{\geq P}$ admits an exit path ∞-category, which identifies with its stratifying poset $\mathcal{P}_{P/}$, see Section 3. In particular, the pullback functor*

$$\mathrm{Fun}(\mathcal{P}_{P/}, \mathcal{S}) \to \mathrm{Sh}^{constr}(\widehat{X}_{\geq P})$$

is an equivalence of ∞-categories.

Proof. We need to produce the neighbourhood bases as in condition 1 and 2 of Theorem 3.4, for $X = \mathbb{R}^d \times (\mathbb{R}_{\geq 0})^n \to \{0 < 1\}^n$. It suffices to take the open boxes $(a_1, b_1) \times \ldots \times (a_{d+n}, b_{d+n})$ in $\mathbb{R}^{d+n}$ and intersect with X. □

4.3. **The Borel–Serre compactification.** In order to define the Borel–Serre compactification, we need to discuss the functoriality of the Borel–Serre corners. There are two types of functoriality:

(1) First of all, if $P \subset Q$ is an inclusion of parabolics, then the compatibility of the geodesic action and root coordinates with the inclusion $A_Q \hookrightarrow A_P$ gives a natural open inclusion
$$\widehat{X}_{\geq Q} \hookrightarrow \widehat{X}_{\geq P}$$
lying above the combinatorial analogue
$$\mathcal{P}_{Q/} \hookrightarrow \mathcal{P}_{P/}$$
coming from including the poset of parabolics containing Q into that of those containing P.

(2) Second, for $\gamma \in G(\mathbb{Q})$ the action of γ on X induces a natural homeomorphism
$$\widehat{X}_{\geq P} \xrightarrow{\sim} \widehat{X}_{\geq \gamma P \gamma^{-1}},$$
lying above the combinatorial analogue
$$\mathcal{P}_{P/} \xrightarrow{\sim} \mathcal{P}_{\gamma P \gamma^{-1}/}$$
coming from conjugating a parabolic containing P by γ.

The first functoriality is more formally a functor $P \mapsto \widehat{X}_{\geq P}$ from the poset $\mathcal{P}^{op}$ of parabolic subgroups under reverse inclusion to topological spaces, lying over an analogous functor $P \mapsto \mathcal{P}_{P/}$ from $\mathcal{P}^{op}$ to posets.

Definition 4.4. For G a reductive group over $\mathbb{Q}$, the *Borel–Serre partial compactification* (of X/A_G) is the topological space defined as the colimit

$$\widehat{X} := \varinjlim_{P \in \mathcal{P}^{op}} \widehat{X}_{\geq P},$$

viewed as a stratified space over

$$\mathcal{P} = \varinjlim_{P \in \mathcal{P}^{op}} \mathcal{P}_{P/},$$

the poset of parabolic subgroups of G under inclusion. ◁

Note that the entire structure of the colimit defining $\widehat{X}$ is recovered from the output stratified space $\pi\colon \widehat{X} \to \mathcal{P}$, because $\widehat{X}_{\geq P}$ identifies with the open star $\pi^{-1}(\mathcal{P}_{P/})$ around the P-stratum of $\widehat{X}$.

Proposition 4.5. *Let G be a reductive group over $\mathbb{Q}$. Then via the natural pullback functors we have:*

(1) $\mathrm{Sh}(\widehat{X}) \xrightarrow{\sim} \varprojlim_{P \in \mathcal{P}^{op}} \mathrm{Sh}(\widehat{X}_{\geq P})$.
(2) $\mathrm{Sh}^{constr}(\widehat{X}) \xrightarrow{\sim} \varprojlim_{P \in \mathcal{P}^{op}} \mathrm{Sh}^{constr}(\widehat{X}_{\geq P})$.
(3) $\mathrm{Fun}(\mathcal{P}, \mathcal{S}) \xrightarrow{\sim} \varprojlim_{P \in \mathcal{P}^{op}} \mathrm{Fun}(\mathcal{P}_{P/}, \mathcal{S})$.

Proof. Part 1 follows from Corollary 2.17 and part 3 follows from Corollary 2.35. To deduce 2 from 1, we need to know that a sheaf on $\widehat{X}$ is constructible if its pullback to each $\widehat{X}_{\geq P}$ is. For that it suffices to note that the P-stratum is fully contained in $\widehat{X}_{\geq P}$. □

Corollary 4.6. *The stratified space $\widehat{X} \to \mathcal{P}$ admits an exit path ∞-category which identifies with its stratifying poset $\mathcal{P}$; in particular, the comparison functor gives an equivalence*

$$c^*\colon \mathrm{Fun}(\mathcal{P}, \mathcal{S}) \xrightarrow{\sim} \mathrm{Sh}^{constr}(\widehat{X}).$$

Proof. The comparison functor is natural in the stratified space by construction, so Proposition 3.7 parts 2 and 3 reduce us to the analogous claim for the $\widehat{X}_{\geq P}$, which is Lemma 4.3. □

Now it is time to consider the second functoriality on the Borel–Serre corners. In terms of the glued space $\widehat{X}$, this simply manifests itself in a continuous action of the discrete group $G(\mathbb{Q})$, extending the natural action on the interior X/A_G and covering the conjugation action of $G(\mathbb{Q})$ on $\mathcal{P}$. The main result of Borel–Serre is that if $\Gamma \subset G(\mathbb{Q})$ is a torsionfree arithmetic subgroup, then Γ acts properly, freely, and cocompactly on $\widehat{X}$. Thus the quotient space $\Gamma\backslash\widehat{X}$ is a compact Hausdorff space compactifying its interior $\Gamma\backslash X/A_G$, which has the homotopy type of $B\Gamma$.

Definition 4.7. Let G be a reductive group over $\mathbb{Q}$ and let $\Gamma \subset G(\mathbb{Q})$ be a torsionfree arithmetic subgroup. Then the *Borel–Serre compactification* (of $\Gamma\backslash X/A_G$) is the quotient space

$$\Gamma\backslash\widehat{X}$$

of the Borel–Serre partial compactification by the natural Γ-action, viewed as a stratified space over the quotient poset $\Gamma\backslash\mathcal{P}$ of Γ-conjugacy classes of parabolic subgroups under the relation induced by inclusion. ◁

By Proposition 4.1 and Lemma 2.34, the quotient $\Gamma\backslash\mathcal{P}$ in the category of sets gets an induced poset structure from $\mathcal{P}$. However, this is not the same as the (homotopy) quotient in the ∞-category of ∞-categories; rather that is the action category $\Gamma\backslash\backslash\mathcal{P}$, see Corollary 2.33, whose objects are the $P \in \mathcal{P}$ and whose morphisms $P \to Q$ are the $\gamma \in \Gamma$ with $\gamma P \gamma^{-1} \subseteq Q$.

Proposition 4.8. *Let G be a reductive group over $\mathbb{Q}$ and $\Gamma \subset G(\mathbb{Q})$ a torsionfree arithmetic subgroup. Then:*

(1) $\mathrm{Sh}(\Gamma\backslash\widehat{X}) \xrightarrow{\sim} \mathrm{Sh}(\widehat{X})^{\Gamma}$.
(2) $\mathrm{Sh}^{constr}(\Gamma\backslash\widehat{X}) \xrightarrow{\sim} \mathrm{Sh}^{contr}(\widehat{X})^{\Gamma}$
(3) $\mathrm{Fun}(\Gamma\backslash\backslash\mathcal{P}, \mathcal{S}) \xrightarrow{\sim} \mathrm{Fun}(\mathcal{P}, \mathcal{S})^{\Gamma}$.

Proof. Part 1 follows from Corollary 2.18, and part 3 follows from Corollary 2.33. To deduce part 2 from part 1, we need to know that a sheaf on $\Gamma\backslash\widehat{X}$ is constructible if its pullback to $\widehat{X}$ is. This follows because for any parabolic P, the projection from the P-stratum in $\widehat{X}$ to the $[P]$-stratum in $\Gamma\backslash\widehat{X}$ has local sections; in fact it is the quotient by the proper free action of $\Gamma_P = \Gamma \cap P(\mathbb{Q})$. □

Corollary 4.9. *Let G be a reductive group over $\mathbb{Q}$ and $\Gamma \subset G(\mathbb{Q})$ a torsionfree arithmetic subgroup. Then $\Gamma\backslash\widehat{X}$ admits an exit path ∞-category which identifies $\Gamma\backslash\backslash\mathcal{P}$. In particular, there is a natural equivalence*

$$\mathrm{Fun}(\Gamma\backslash\backslash\mathcal{P}, \mathcal{S}) \xrightarrow{\sim} \mathrm{Sh}^{constr}(\Gamma\backslash\widehat{X}).$$

Proof. Follows by combining the previous proposition and Proposition 3.7. □

4.4. **The reductive Borel–Serre compactification.** To motivate the reductive Borel–Serre compactification, we start by taking a closer look at the Borel–Serre compactification. For a parabolic subgroup P, the P-stratum of $\widehat{X}$ identifies with $X \times^{A_P} * = X/A_P$, and the $[P]$-stratum of $\Gamma\backslash\widehat{X}$ identifies with the quotient

$$\Gamma_P\backslash X/A_P.$$

Thus all the strata are of a similar form as the open stratum $\Gamma\backslash X/A_G$, except with the reductive group G replaced by the non-reductive group P.

(One may be bothered by the fact that this description of the $[P]$-stratum, on the face of it, depends on the chosen representative P. But this is an illusion: if P is conjugate to P' via γ, then the induced homeomorphism $\Gamma_P\backslash X/A_P \simeq \Gamma_{P'}\backslash X/A_P$ is independent of γ. This follows from the fact that parabolic subgroups are their own normalisers.)

To get a better inductive structure we would like to replace the parabolic subgroup by its Levi quotient $L = P/U_P$, which is reductive. Let $\Gamma_L \subset L(\mathbb{Q})$ denote the quotient Γ_P/Γ_{U_P}. If Γ is not just torsionfree but *neat*, [Ji06], then Γ_L is also a neat, and in particular torsionfree, arithmetic subgroup of $L(\mathbb{Q})$. Moreover, there is a natural map

$$\Gamma_P\backslash X/A_P \to \Gamma_L\backslash X_L/A_L$$

from the $[P]$-stratum of the Borel–Serre compactification for $\Gamma \subset G$ to the open stratum of the Borel–Serre compactification for $\Gamma_L \subset L$, which is in fact a fibre bundle with compact fibre $U_P(\mathbb{R})/\Gamma_{U_P}$. Indeed, this map comes from the canonical identification

$$X_L = U_P(\mathbb{R})\backslash X$$

of the canonical homogeneous space of $L_{\mathbb{R}}$ with the indicated quotient of the canonical homogeneous space of $G_{\mathbb{R}}$, together with the fact that A_L acting on X_L identifies with A_P acting on $U_P(\mathbb{R})\backslash X$. More generally if $P' \subset P$ is parabolic, then the geodesic action of A_{P'/U_P} on X_L identifies with the geodesic action of $A_{P'}$ on $U_P(\mathbb{R})\backslash X$. This leads to the following definition made by Zucker.

Definition 4.10. Let G be a reductive group over $\mathbb{Q}$ and $\Gamma \subset G(\mathbb{Q})$ a neat arithmetic subgroup. The *reductive Borel–Serre compactification* (of $\Gamma\backslash X/A_G$) is the quotient topological space

$$\widehat{Y_\Gamma} = (\Gamma\backslash\widehat{X})/\sim$$

obtained from $\Gamma\backslash\widehat{X}$ by collapsing the $[P]$-stratum to $\Gamma_L\backslash X_L/A_L$ via the above quotient map, for all parabolic subgroups P (or just one representative from each Γ-conjugacy class).

We view $\widehat{Y_\Gamma}$ as stratified over the poset $\Gamma\backslash\mathcal{P}$ of Γ-conjugacy classes of parabolic subgroups by the unique factoring

$$\widehat{Y_\Gamma} \to \Gamma\backslash\mathcal{P}$$

of the stratifying map $\Gamma\backslash\widehat{X} \to \Gamma\backslash\mathcal{P}$ of the Borel–Serre compactification. ◁

Zucker checked that that $\widehat{Y_\Gamma}$ is Hausdorff, hence it is a compact Hausdorff space. An important aspect of the reductive Borel–Serre compactification is its inductive nature, based on the following:

Proposition 4.11. *Let G be a reductive group over $\mathbb{Q}$ and $\Gamma \subset G(\mathbb{Q})$ a neat arithmetic subgroup. Then:*

1. *The projection map $\Gamma\backslash\widehat{X} \to \widehat{Y_\Gamma}$ is proper and restricts to an isomorphism over the open stratum $\Gamma\backslash X/A_G$ of $\widehat{Y_\Gamma}$;*
2. *For $P \subset G$ parabolic with Levi factor L, there is a canonical closed inclusion*

$$\widehat{Y_{\Gamma_L}} \hookrightarrow \widehat{Y_\Gamma}$$

identifying $\widehat{Y_{\Gamma_L}}$ with $\widehat{Y_\Gamma} \times_{\Gamma\backslash\mathcal{P}} (\Gamma\backslash\mathcal{P})_{/[P]}$, the closure of the $[P]$-stratum in $\widehat{Y_\Gamma}$.

Proof. The first claim is obvious from the definition (and the fact that $\widehat{Y_\Gamma}$ is Hausdorff). For the second claim, the existence of the map follows from the identification of symmetric spaces $X_L = U_P(\mathbb{R})\backslash X_G$ and the compatibility of the geodesic actions, as discussed above. The map clearly restricts to a homeomoprhism on each stratum, so to see it is an inclusion it suffices to recall the fact that if $P', P'' \subset P$ are parabolic subgroups and $\gamma \in G(\mathbb{Q})$ conjugates P' to P'', then γ actually lies in $P(\mathbb{Q})$, see Proposition 4.1. □

Let $\partial\widehat{Y}_\Gamma = \widehat{Y}_\Gamma \smallsetminus (\Gamma\backslash X/A_G)$ denote the *boundary* of the reductive Borel–Serre compactification: the complement of the open stratum. Similarly set $\partial\Gamma\backslash\widehat{X} = (\Gamma\backslash\widehat{X}) \smallsetminus (\Gamma\backslash X/A_G)$. Then it follows from the above proposition that, in the category of topological spaces, we have:

(1) $\widehat{Y}_\Gamma = (\Gamma\backslash\widehat{X}) \coprod_{\partial\Gamma\backslash\widehat{X}} \partial\widehat{Y}_\Gamma$, and
(2) $\partial\widehat{Y}_\Gamma = \varinjlim_{[P]\in(\Gamma\backslash\mathcal{P})^{op},[P]\neq[G]} \widehat{Y}_{\Gamma_{P/U_P}}$,

giving a sense in which reductive Borel–Serre compactifications for a given group G are built up from Borel–Serre compactifications together with reductive Borel–Serre compactifications for proper Levi factors of G. In fact, this inductive nature of the reductive Borel–Serre compactification is very robust: these colimit diagrams turn into limit diagrams on categories of sheaves of all sorts. This follows not from the bare statement about colimits in topological spaces, but from the more primitive Proposition 4.11:

Corollary 4.12. *We have:*

(1) $\mathrm{Sh}(\widehat{Y}_\Gamma) \xrightarrow{\sim} \mathrm{Sh}(\Gamma\backslash\widehat{X}) \times_{\mathrm{Sh}(\partial\Gamma\backslash\widehat{X})} \mathrm{Sh}(\partial\widehat{Y}_\Gamma)$*, and similarly for constructible sheaves;*
(2) $\mathrm{Sh}(\partial\widehat{Y}_\Gamma) \xrightarrow{\sim} \varprojlim_{[P]\in\Gamma\backslash\mathcal{P},[P]\neq[G]} \mathrm{Sh}(\widehat{Y}_{\Gamma_{P/U_P}})$ *and similarly for constructible sheaves.*

Proof. Part 1 for sheaves follows from the topological cdh descent, Corollary 2.13. To deduce the claim for constructible sheaves, we need to know that a sheaf on $\widehat{Y}_\Gamma$ is constructible if its pullback to the other three terms is. This is clear because the only stratum not in the boundary is the open stratum, and the projection from $\Gamma\backslash\widehat{X}$ is an isomoprhism over the open stratum. Part 2 follows for sheaves from the descent for closed covers, Corollary 2.14. To deduce the claim for constructible sheaves it suffices to note that every stratum is contained in its closure which is some $\widehat{Y}_{\Gamma_{P/U_P}}$. □

In principle, this corollary inductively yields an identification of the exit path ∞-category of $\widehat{Y}_\Gamma$, based on the case of $\Gamma\backslash\widehat{X}$ treated in the previous section. But for technical reasons we will need to make a comparison functor before we can use the inductive description to prove it's an equivalence. To accomplish that we will describe the reductive Borel–Serre compactification in terms of stratified spaces whose exit path ∞-categories are equivalent to posets; on such stratified spaces the required comparison functor comes for free, see Section 3, and then we deduce the correct comparison functor for $\widehat{Y}_\Gamma$ by passing to colimits.

Since the inductive description is based on closed subsets and not open subsets, the suitable building blocks will be the closures of strata in Borel–Serre compactifications, for all Levi factors of G. We therefore start with a discussion of these.

Let G be a reductive group over $\mathbb{Q}$, and let P be a parabolic subgroup of G. Denote by

$$\widehat{X}_{\leq P} := \widehat{X} \times_{\mathcal{P}} \mathcal{P}_{/P}$$

the closure of the P-stratum in $\widehat{X}$, which we view as stratified over the poset $\mathcal{P}_{/P}$ of parabolic subgroups contained in P.

Proposition 4.13. *Let G be a reductive group over $\mathbb{Q}$, let P be a parabolic subgroup of G, and set $L = P/U_P$. There is a natural stratum-preserving free and proper $U_P(\mathbb{R})$-action on $\widehat{X}_{\leq P}$ extending the $U_P(\mathbb{Q})$-action, and the quotient*

$$U_P(\mathbb{R})\backslash\widehat{X}_{\leq P}$$

identifies with the Borel–Serre partial compactification associated to the reductive group L.

More generally, if Q is a parabolic subgroup containing P, then $U_Q(\mathbb{R})\backslash\widehat{X}_{\leq P}$ identifies with the closure of the P/U_Q-stratum in the Borel–Serre partial compactification associated to Q/U_Q.

Proof. Note that $\widehat{X}_{\leq P}$ is glued from its open subsets $\widehat{X}_{[P',P]}$ indexed by parabolic $P' \subset P$, which are in turn closed subsets of the Borel–Serre corners $\widehat{X}_{\geq P'}$, namely

$$\widehat{X}_{[P',P]} = X \times^{A_{P'}} (\mathbb{R}_{\geq 0})^{\Delta_{P'} \smallsetminus \Delta_P} \subset X \times^{A_{P'}} (\mathbb{R}_{\geq 0})^{\Delta_{P'}} = \widehat{X}_{\geq P'},$$

coming from the inclusion of the coordinate hypersurface $(\mathbb{R}_{\geq 0})^{\Delta_{P'} \smallsetminus \Delta_P} \subset (\mathbb{R}_{\geq 0})^{\Delta_{P'}}$. We recall that the $P'(\mathbb{R})$-action on X commutes with the geodesic action by $A_{P'}$. Since

$$U_P(\mathbb{R}) \subset U_{P'}(\mathbb{R}) \subset P'(\mathbb{R})$$

when $P' \subset P$, this induces compatible $U_P(\mathbb{R})$-actions on all the $\widehat{X}_{[P',P]}$, whence the required action on $\widehat{X}_{\leq P}$. For the identification of the quotient with the Borel–Serre partial compactification associated to Γ_L, it follows from comparing the definitions using the identification $X_L = U_P(\mathbb{R})\backslash X_G$ discussed above. The more general claim is completely analogous. ☐

Corollary 4.14. *The exit path ∞-category of $U_Q(\mathbb{R})\backslash\widehat{X}_{\leq P}$ identifies with its stratifying poset $\mathcal{P}_{\leq P}$.*

Proof. By Proposition 3.7 this follows from the identification of the exit path ∞-category of the Borel–Serre partial compactification, Corollary 4.6. ☐

The following category will end up being the exit path ∞-category of $\widehat{Y}_\Gamma$, see [ØJ20].

Definition 4.15. Let G be a reductive group over $\mathbb{Q}$ and $\Gamma \subset G(\mathbb{Q})$ a neat arithmetic subgroup. Let RBS_Γ denote the category whose objects are the parabolic subgroups $P \subset G$, and where the set of maps $P \to Q$ is the quotient

$$\{\gamma \in \Gamma : \gamma P \gamma^{-1} \subset Q\}/\Gamma_{U_P},$$

composition being induced by multiplication in Γ, which is well-defined as $\Gamma_{U_Q} \subset \Gamma_{U_{\gamma P \gamma^{-1}}}$. ◁

Recall the *twisted arrow category* $\mathrm{Tw}(\mathcal{C})$ of a category $\mathcal{C}$: its objects are the maps $X \to Y$ in $\mathcal{C}$, and a map $(f\colon X \to Y) \to (f'\colon X' \to Y')$ is a factorisation of the latter through the former, namely maps $a\colon X' \to X$ and $b\colon Y \to Y'$ such that $f' = bfa$. We will identify the exit path ∞-category of $\widehat{Y}_\Gamma$ with RBS_Γ by expressing $\widehat{Y}_\Gamma$ as a colimit indexed by $\mathrm{Tw}(\mathrm{RBS}_\Gamma)^{op}$ of the spaces $U_Q(\mathbb{R})\backslash\widehat{X}_{\leq P}$ discussed above. To make this work we need some lemmas.

Lemma 4.16. *Let $\mathcal{C}$ be a category. The projection functor $\mathrm{Tw}(\mathcal{C})^{op} \to \mathcal{C}$ sending $x \to y$ to x is a $\varinjlim$-equivalence.*

Proof. The right fibre over an object $c \in \mathcal{C}$ identifies with the category whose objects are the composable maps

$$c \to x \to y$$

emanating from c, and whose morphisms $(c \to x \to y) \to (c \to x' \to y')$ are pairs of maps

$$(x \to x', y' \to y)$$

making the evident diagram commute. The full subcategory on those objects for which $c \overset{=}{\to} x$ is equivalent to $(\mathcal{C}_{c/})^{op}$ and is therefore contractible; but on the other hand there is a retraction to the inclusion of this full subcategory given by composition, $c \to x \to y \mapsto (c = c \to y)$, and an obvious natural transformation from this retraction to the identity. Thus the right fibre is homotopy equivalent to $(\mathcal{C}_{c/})^{op}$ and is therefore also contractible, hence by Theorem 2.19 our functor is a $\varinjlim$-equivalence as desired. □

Lemma 4.17. *The category $\mathrm{Tw}(\mathrm{RBS}_\Gamma)^{op}$ is equivalent to the category whose objects are the inclusions $P \subset Q$ of parabolic subgroups, and whose set of maps $(P \subset Q) \to (P' \subset Q')$ is given by*

$$\Gamma_{U_{Q'}} \backslash \{\gamma \in \Gamma : \gamma P \gamma^{-1} \subset P', \gamma Q \gamma^{-1} \supset Q'\}$$

with composition induced by multiplication in Γ.

More precisely, the equivalence is given by the functor which on objects sends the inclusion $P \subset Q$ to the map $P \overset{[id]}{\to} Q$ in RBS_Γ, and on maps sends $[\gamma]\colon (P \subset Q) \to (P' \subset Q')$ to the pair of maps $P \overset{[\gamma]}{\to} P'$ and $Q' \overset{[\gamma^{-1}]}{\to} Q$ in RBS*.*

Proof. The functor is well-defined because $\Gamma_{U_{Q'}} \subset \Gamma_{U_{P'}} \subset \Gamma_{U_{\gamma P \gamma^{-1}}}$ so that $\Gamma_{U_{Q'}}\gamma \subset \gamma \Gamma_{U_P}$. To give the identification, we should show that every map $P \to Q$ in RBS_Γ is equivalent to an inclusion, meaning a map given by $\gamma = id$, and that maps in $\mathrm{Tw}(\mathrm{RBS}_\Gamma)^{op}$ between inclusions are given by the set posited in the statement.

For the first claim, if $P \to Q$ is induced by $\gamma \in \Gamma$ then we can factor it as $P \overset{\sim}{\to} \gamma P \gamma^{-1} \overset{\subseteq}{\to} Q$ where the first map is induced by γ and the second map is an inclusion. For the second claim, maps $(P \subset Q) \to (P' \subset Q')$ in $\mathrm{Tw}(\mathrm{RBS}_\Gamma)^{op}$ are by definition given by

$$\{\gamma_a, \gamma_b \in \Gamma : \gamma_a P \gamma_a^{-1} \subset P', \gamma_b Q' \gamma_b^{-1} \subset Q, \gamma_b \gamma_a \in \Gamma_{U_P}\}/\sim$$

where $(\gamma_a, \gamma_b) \sim (\rho_a, \rho_b)$ iff $\gamma_a \Gamma_{U_P} = \rho_a \Gamma_{U_P}$ and $\gamma_b \Gamma_{U_{Q'}} = \rho_b \Gamma_{U_{Q'}}$. It follows that we can uniquely specify γ_a in terms of γ_b by setting $\gamma_a = \gamma_b^{-1}$, which leads to the claim. □

Lemma 4.18. *For P a parabolic subgroup, the natural functor $\mathcal{P}_{/P} \to (\mathrm{RBS}_\Gamma)_{/P}$ induced by $(P' \subset P) \mapsto (P' \overset{[id]}{\to} P)$ is an equivalence.*

Proof. This follows similarly: by direct calculation, the functor is essentially surjective and fully faithful. □

In terms of the equivalence of Lemma 4.17, we find that the spaces $U_Q(\mathbb{R})\backslash\widehat{X}_{\leq P}$ organise into a functor

$$\widehat{Y}\colon \operatorname{Tw}(\mathrm{RBS}_\Gamma)^{op} \to \mathrm{Top}$$

which sends $(P \subset Q) \mapsto U_Q(\mathbb{Q})\backslash\widehat{X}_{\leq P}$ on objects, and on morphisms is induced by the Γ-action on $\widehat{X}$. This lies above the combinatorial analogue $\operatorname{Tw}(\mathrm{RBS}_\Gamma)^{op} \to \mathrm{Posets}$ defined by

$$(P \subset Q) \mapsto \mathcal{P}_{\leq P}$$

on objects, and induced by Γ-conjugation on maps. Thus we promote $\widehat{Y}$ to a functor from $\operatorname{Tw}(\mathrm{RBS}_\Gamma)^{op}$ to stratified spaces.

A rephrasing of Zucker's definition of the reductive Borel–Serre compactification is that

$$\widehat{Y_\Gamma} = \varinjlim \widehat{Y}.$$

Indeed, in both cases we are gluing together Γ-orbits and collapsing unipotent fibres, we just do it in a different order. But this new description is more robust, in that it promotes to a statement about categories of sheaves:

Theorem 4.19. *Let G be a reductive group over $\mathbb{Q}$ and $\Gamma \subset G(\mathbb{Q})$ a neat arithmetic subgroup. Then*

$$\mathrm{Sh}(\widehat{Y_\Gamma}) \xrightarrow{\sim} \varprojlim_{(P\subset Q)\in \operatorname{Tw}(\mathrm{RBS}_\Gamma)} \mathrm{Sh}(U_Q(\mathbb{R})\backslash\widehat{X}_{\leq P}),$$

and similarly for constructible sheaves.

We will prove this theorem shortly, but for now let us deduce the following consequence.

Corollary 4.20. *The stratified space $\widehat{Y_\Gamma}$ admits an exit path ∞-category, and this exit path ∞-category identifies with the category RBS_Γ.*

Proof. By Corollary 4.14, the exit path ∞-category of $U_Q(\mathbb{R})\backslash\widehat{X}_{\leq P}$ identifies with its stratifying poset $\mathcal{P}_{/P}$. Thus, by Proposition 3.7, it suffices to calculate that in Cat_∞ we have

$$\varinjlim_{(P\subset Q)\in \operatorname{Tw}(\mathrm{RBS}_\Gamma)^{op}} \mathcal{P}_{/P} \simeq \mathrm{RBS}_\Gamma\,.$$

However, for a parabolic subgroup P the natural functor $\mathcal{P}_{/P} \to (\mathrm{RBS}_\Gamma)_{/P}$ sending $P' \subset P$ to $[id]\colon P' \to P$ is an equivalence by Lemma 4.18. Thus this will follow from the more general claim that for any category $\mathcal{C}$ we have

$$\varinjlim_{(c\to d)\in \operatorname{Tw}(\mathcal{C})^{op}} \mathcal{C}_{/c} \xrightarrow{\sim} \mathcal{C}.$$

But we have $\varinjlim_{c\in\mathcal{C}} \mathcal{C}_{/c} \xrightarrow{\sim} \mathcal{C}$ by Corollary 2.35, so this follows from Lemma 4.16. □

To prove Theorem 4.19, we need the following diagrammatic analogue of the inductive structure of the reductive Borel–Serre compactification.

Proposition 4.21. *Let G be a reductive group over $\mathbb{Q}$ and $\Gamma \subset G(\mathbb{Q})$ a subgroup. Consider the following full subcategories of* $\mathrm{Tw}(\mathrm{RBS}_\Gamma)^{op}$*:*

(1) *For a Γ-conjugacy class of parabolic subgroups $[P]$ of G, we write*

$$\mathrm{Tw}(\mathrm{RBS}_\Gamma)^{op}_{\leq [P]}$$

for the full subcategory of $\mathrm{Tw}(\mathrm{RBS}_\Gamma)^{op}$ *on those inclusions $P' \subset Q'$ with P' contained in a representative of $[P]$.*

(2) *For a parabolic subgroup P, we view*

$$\mathrm{Tw}(\mathrm{RBS}_{\Gamma_{P/U_P}})^{op}$$

as the further full subcategory of those $P' \subset Q'$ with $Q' \subset P$.

(3) *We view $\Gamma\backslash\backslash\mathcal{P}$ as a full subcategory of* $\mathrm{Tw}(\mathrm{RBS}_\Gamma)^{op}$ *by the embedding $P \mapsto (P \subset G)$.*

(4) *For $P \subset Q$, view $B\Gamma_{P/U_Q}$ as the full subcategory spanned by $(P \subset Q)$.*

Then:

(1) *The full subcategories* $\mathrm{Tw}(\mathrm{RBS}_\Gamma)^{op}_{\leq [P]}$ *and $\Gamma\backslash\backslash\mathcal{P}$ are left-closed.*

(2) *The union of the subcategories* $\mathrm{Tw}(\mathrm{RBS}_\Gamma)^{op}_{\leq [P]}$ *for $[P] \neq [G]$ is equal to the complement* $\mathrm{Tw}(\mathrm{RBS}_\Gamma)^{op} \smallsetminus B\Gamma$*, and for all* $(P' \subset Q') \in \mathrm{Tw}(\mathrm{RBS}_\Gamma)^{op} \smallsetminus B\Gamma$ *the collection of those* $\mathrm{Tw}(\mathrm{RBS}_\Gamma)^{op}_{\leq [P]}$ *containing $P' \subset Q'$ has a minimal element, namely* $\mathrm{Tw}(\mathrm{RBS}_\Gamma)^{op}_{\leq [P']}$*.*

(3) *For a parabolic subgroup $P \subset G$, the inclusion*

$$B\Gamma_P \subset (\Gamma\backslash\backslash\mathcal{P})_{\leq [P]} = (\Gamma\backslash\backslash\mathcal{P}) \cap \mathrm{Tw}(\mathrm{RBS}_\Gamma)^{op}_{\leq [P]}$$

has a left adjoint.

(4) *For a parabolic subgroup $P \subset G$, the inclusion*

$$\mathrm{Tw}(\mathrm{RBS}_{\Gamma_{P/U_P}})^{op} \subset \mathrm{Tw}(\mathrm{RBS}_\Gamma)^{op}_{\leq [P]}$$

has a left adjoint.

Proof. Claims 1 and 2 are obvious. For claim 3, note that $(\Gamma\backslash\backslash\mathcal{P})_{\leq [P]}$ is equivalent to its full subcategory on those parabolic P' such that $P' \subset P$. Recall that any $\gamma \in \Gamma$ which conjugates such a $P' \subset P$ back inside P must necessarily lie in Γ_P, see Proposition 4.1. Thus we can identify $(\Gamma\backslash\backslash\mathcal{P})_{\leq [P]} \simeq \Gamma_P\backslash\backslash\mathcal{P}_{/P}$, and the projection to $B\Gamma_P$ provides a left adjoint proving the claim. Claim 4 follows similarly: we can replace $\mathrm{Tw}(\mathrm{RBS}_\Gamma)^{op}_{\leq [P]}$ with the equivalent full subcategory of those $(P' \subset Q')$ such that $P' \subset P$, and then the functor backwards given by $(P' \subset Q') \mapsto (P' \subset P \cap Q')$ on objects and $[\gamma] \mapsto [\gamma]$ on morphisms provides a left adjoint to the inclusion. $\square$

Corollary 4.22. *Let $F\colon \mathrm{Tw}(\mathrm{RBS}_\Gamma) \to \mathcal{C}$ be a functor to an arbitrary ∞-category $\mathcal{C}$ with all limits. Then:*

(1) $\varprojlim F \xrightarrow{\sim} \varprojlim F|_{\mathrm{Tw}(\mathrm{RBS}_\Gamma) \smallsetminus B\Gamma} \times_{\varprojlim F|_{(\Gamma\backslash\backslash\mathcal{P})^{op} \smallsetminus B\Gamma}} \varprojlim F|_{(\Gamma\backslash\backslash\mathcal{P})^{op}}$.

(2) $\varprojlim F|_{\mathrm{Tw}(\mathrm{RBS}_\Gamma) \smallsetminus B\Gamma} \xrightarrow{\sim} \varprojlim_{[P] \in (\Gamma\backslash\mathcal{P})^{op} \smallsetminus [G]} \varprojlim F|_{\mathrm{Tw}(\mathrm{RBS}_\Gamma)_{\leq [P]}}$, *and*

$$\varprojlim F|_{\mathrm{Tw}(\mathrm{RBS}_\Gamma)_{\leq [P]}} \xrightarrow{\sim} \varprojlim F|_{\mathrm{Tw}(\mathrm{RBS}_{\Gamma_{P/U_P}})}.$$

(3) $\varprojlim F|_{(\Gamma\backslash\backslash\mathcal{P})^{op}\smallsetminus B\Gamma} \xrightarrow{\sim} \varprojlim_{[P]\in(\Gamma\backslash\mathcal{P})^{op}\smallsetminus[G]} \varprojlim F|_{((\Gamma\backslash\backslash\mathcal{P})_{\leq[P]})^{op}}$*, and for all parabolic* $P \subset G$ *we have*

$$\varprojlim F|_{((\Gamma\backslash\backslash\mathcal{P})_{\leq[P]})^{op}} \xrightarrow{\sim} \varprojlim F|_{B\Gamma_P}.$$

Proof. This follows by descent for closed covers, Corollary 2.32 which lets one decompose limits by Proposition 2.1, and the fact that a left adjoint functor is a $\varprojlim$-equivalence, Example 2.21. □

Now we can prove Theorem 4.19.

Proof. Consider the functor $F\colon \operatorname{Tw}(\mathrm{RBS}_\Gamma) \to \mathrm{Cat}_\infty$ defined by

$$F = \mathrm{Sh}\circ\widehat{Y},$$

so $F(P \subset Q) = \mathrm{Sh}(U_Q(\mathbb{R})\backslash\widehat{X}_{\leq P})$ with pullback functoriality. For any full subcategory $\mathcal{D} \subset \operatorname{Tw}(\mathrm{RBS}_\Gamma)$, we have the associated comparison map

$$\mathrm{Sh}(\varinjlim \widehat{Y}|_{\mathcal{D}^{op}}) \to \varprojlim F|_{\mathcal{D}}.$$

We want to prove that this is an equivalence for $\mathcal{D} = \operatorname{Tw}(\mathrm{RBS}_\Gamma)$. Proceeding by induction on the $\mathbb{Q}$-rank of G, we can assume it is an equivalence for $\mathcal{D} = \operatorname{Tw}(\mathrm{RBS}_{\Gamma_{P/U_P}})$ for any proper parabolic $P \subset G$. Then comparing part 2 of the corollary above with part 2 from Propositoin 4.12 we get that it is an equivalence for $\mathcal{D} = \operatorname{Tw}(\mathrm{RBS}_\Gamma) \smallsetminus B\Gamma$; other the other hand part 3 of the corollary above plus descent for the closed cover of $\partial\Gamma\backslash\widehat{X}$ by the $(\Gamma\backslash\widehat{X})_{\leq[P]}$ for $[P] \neq [G]$ shows that it is an equivalence for $\mathcal{D} = (\Gamma\backslash\backslash\mathcal{P})^{op}\smallsetminus B\Gamma$. Then comparing part 1 of the corollary with part 1 of Propostion 4.12 gives the desired claim, finishing the proof of Theorem 4.19 for sheaves, without the constructibility condition. But since the maps from the strata of the Borel–Serre compactification to those of the reductive Borel–Serre compactification are fibre bundles and hence have local sections, a sheaf on $\widehat{Y}_\Gamma$ is constructible if and only if its pullback to $\Gamma\backslash\widehat{X}$ is constructible, which shows that the variant with constructible sheaves follows. □

5. *RBS*(M) as unstable algebraic K-theory

As an intermediary step, let us transport some of the above discussion into the general context of reductive groups over commutative rings. We recall the definition from [ØJ20].

Definition 5.1. Let G be a reductive group over a commutative ring R. Define the category RBS_G to have objects the parabolic subgroups $P \subset G$ and morphisms $P \to P'$ the set

$$\{g \in G(R) : gPg^{-1} \subset P'\}/U_P(R),$$

composition being induced by multiplication in $G(R)$. ◁

Thus, if we take $R = \mathbb{Q}$ and further restrict to the subcategory specified by the choice of an arithmetic subgroup $\Gamma \subset G(\mathbb{Q})$, this recovers the category RBS_Γ of the previous section, which we identified with the exit path ∞-category of the reductive Borel–Serre compactification $\widehat{Y}_\Gamma$. But now we want to consider general R and forget about Γ.

For connections to algebraic K-theory, we restrict to $G = GL_n$, or more generally $G = GL(M)$ for a finitely generated projective R-module M. For the classification of parabolic subgroups of reductive group schemes we refer to [DG70] Exposé XXVI. We see that if $\mathrm{Spec}(R)$ is connected, then parabolic subgroups of $GL(M)$ correspond to splittable flags $\mathcal{F}$ of submodules of M: chains of inclusions

$$\mathcal{F} = (M_1 \subsetneq \ldots \subsetneq M_{d-1})$$

such that each quotient M_i/M_{i-1} is nonzero and projective (we set $M_0 = 0$ and $M_d = M$). We call d the *length* of the flag; it is the number of these associated graded pieces M_i/M_{i-1}. The corresponding parabolic subgroup $P_{\mathcal{F}}$ represents the automorphisms g of M preserving the flag, meaning $g(M_i) = M_i$ for all i, and its unipotent radical $U_{\mathcal{F}} \subset P_{\mathcal{F}}$ represents those automorphisms preserving the flag and inducing the identity on each M_i/M_{i-1}. Note that the Levi factor $L_{\mathcal{F}} = P_{\mathcal{F}}/U_{\mathcal{F}}$ identifies with the product

$$L_{\mathcal{F}} = \prod_{i=1}^{d} GL(M_i/M_{i-1}).$$

Furthermore, the partial order of inclusion of parabolic subgroups translates into the partial order of refinement of flags, defined by setting $\mathcal{F} \leq \mathcal{G}$ if and only if the set of submodules occurring in $\mathcal{G}$ is a subset of the set of submodules occurring in $\mathcal{F}$. Note that the inclusion on unipotent radicals goes the opposite direction: if $\mathcal{F} \leq \mathcal{G}$ then while $P_{\mathcal{F}} \subset P_{\mathcal{G}}$, we have $U_{\mathcal{F}} \supset U_{\mathcal{G}}$.

This discussion of flags $\mathcal{F}$ of splittable submodules of M and their associated subgroups $P_{\mathcal{F}} \subset GL(M)$ quotients $L_{\mathcal{F}} = P_{\mathcal{F}}/U_{\mathcal{F}}$ makes no use of the commutativity of R. Thus we arrive at the following.

Definition 5.2. Let A be an associative ring and M a finitely generated projective A-module. Define the category $\mathrm{RBS}(M)$ to have objects the splittable flags $\mathcal{F}$ of submodules of M, with set of maps $\mathcal{F} \to \mathcal{F}'$ given by

$$\{g \in GL(M) : g\mathcal{F} \leq \mathcal{F}'\}/U_{\mathcal{F}},$$

with composition induced by multiplication in $GL(M)$ (it is well-defined because $\mathcal{F} \leq \mathcal{G}$ implies that $U_{\mathcal{G}} \subset U_{\mathcal{F}}$).

Let also $\mathcal{P}$ denote the poset of splittable flags of submodules of M with partial order $\leq$ given by refinement, as above. ◁

First we show that $\mathrm{RBS}(M)$ "behaves" like an exit path ∞-category with stratifying poset $GL(M)\backslash\mathcal{P}$ and $K(\pi,1)$ strata.

Lemma 5.3. *Let A be an associative ring and M a finitely generated projective A-module. Then:*

(1) The quotient set $GL(M)\backslash\mathcal{P}$ inherits the poset structure from $\mathcal{P}$.

(2) *There is a functor*

$$\pi\colon \mathrm{RBS}(M) \to GL(M)\backslash\mathcal{P}$$

given by $\pi(\mathcal{F}) = [\mathcal{F}]$.

(3) *For a point* $x \in GL(M)\backslash\mathcal{P}$ *the fibre* $\pi^{-1}(x)$ *is a connected groupoid.*

(4) *For a splittable flag* $\mathcal{F}$*, the automorphism group of* $\mathcal{F}$ *in* RBS(M) *identifies as*

$$\mathrm{Aut}_{\mathrm{RBS}(M)}(\mathcal{F}) = P_{\mathcal{F}}/U_{\mathcal{F}} = L_{\mathcal{F}}.$$

Proof. For part 1, by Lemma 2.34 we need to check that $g\mathcal{F} \leq \mathcal{F}$ implies $g\mathcal{F} = \mathcal{F}$. But the $GL(M)$-action preserves length, and a refinement between flags of equal length is necessarily an identity. Then parts 2, 3 and 4 are immediate. ☐

Now we describe the basic inductive structure of RBS(M). For this we will impose the split noetherian hypothesis on M described in the introduction: that there are no infinite descending sequences split submodules of M. This has the following consequence.

Lemma 5.4. *Let* A *be a ring and* M *a split noetherian finitely generated projective* A*-module. Then:*

(1) *The posets* $\mathcal{P}$ *and* $GL(M)\backslash\mathcal{P}$ *satisfy the descending chain condition.*

(2) *If* $N \subset M$ *is a split submodule and we have* $g \in GL(M)$ *with* $gN \subset N$*, then* $gN = N$.

Proof. For part 1, it suffices to show that $\mathcal{P}$ satisfies the descending chain condition. If not, we would get an infinite chain of split submodules of M, so there would either be an infinite descending sequence of split submodules or an infinite ascending sequence. But as the submodules are split we can convert one situation to the other so both are ruled out by our split noetherian hypothesis. For 2, if $gN \subsetneq N$ then we get the infinite chain

$$\ldots \subsetneq g^k N \subsetneq g^{k-1} N \subsetneq \ldots \subsetneq N$$

which contradicts our assumption. ☐

Furthermore, it is often the case that every M is split noetherian.

Lemma 5.5. *Let* A *be a ring. If either:*

(1) A *is noetherian, or*

(2) A *is commutative and* Spec(A) *has only finitely many connected components,*

then every finitely generated projective A*-module* M *is split noetherian.*

Proof. The claim is clear if A is noetherian. If A is commutative, then the dimension function $x \mapsto \dim(M \otimes_A k(x))$ on Spec(A) is locally constant as every finitely generated projective module is locally free; moreover if N is a proper split submodule of M then the dimension of N must be strictly less than that of M on at least one connected component, as otherwise M/N would be a finitely generated projective module of dimension 0 everywhere, whence $M/N = 0$ by Nakayama's lemma. Thus the claim reduces to the fact that a finite product of copies of the poset $(\mathbb{N}, \leq)$ satisfies the descending chain condition, which is clear. ☐

Definition 5.6. Let A be an associative ring and M a finitely generated projective A-module. For $[\mathcal{F}] \in GL(M)\backslash\mathcal{P}$ an orbit of splittable flags, denote by $\mathrm{RBS}(M)_{\leq[\mathcal{F}]}$ the full subcategory of those objects which admit a map to $\mathcal{F}$, and similarly for $(GL(M)\backslash\backslash\mathcal{P})_{\leq[\mathcal{F}]}$. ◁

The following will be the basis of many inductive arguments.

Proposition 5.7. *Let A be an associative ring and M a split noetherian finitely generated projective A-module.*

(1) *For a splittable flag $\mathcal{F}$ in M with associated graded $gr(\mathcal{F}) = (M_1, \ldots, M_d)$ we have identifications*
$$\mathrm{RBS}(M)_{\leq[\mathcal{F}]} = \prod_{i=1}^{d} \mathrm{RBS}(M_i)$$
and
$$(GL(M)\backslash\backslash\mathcal{P})_{\leq[\mathcal{F}]} = P_{\mathcal{F}}\backslash\backslash\mathcal{P}_{\leq\mathcal{F}},$$
where $P_{\mathcal{F}} \subset GL(M)$ is the stabiliser group of $\mathcal{F}$. Moreover, the natural functor $BP_{\mathcal{F}} \to P_{\mathcal{F}}\backslash\backslash\mathcal{P}_{\leq\mathcal{F}}$ is a right adjoint and in particular induces an isomorphism on anima.

(2) *The natural functor $p\colon GL(M)\backslash\backslash\mathcal{P} \to \mathrm{RBS}(M)$ is proper and an isomorphism over $BGL(M)$, the full subcategory spanned by the empty flag.*

Proof. For claim 1, note that every splittable flag $\mathcal{G}$ with a map to $\mathcal{F}$ is $GL(M)$-equivalent to a splittable flag with $\mathcal{G} \leq \mathcal{F}$, so one can replace the left-hand categories by their full subcategories on such flags. This lets one match up the objects, and then one has to calculate maps, where one needs the claim that if $g \in GL(M)$ satisfies $g\mathcal{G} \leq \mathcal{F}$, then necessarily $g \in P_{\mathcal{F}}$.[3] But this follows from Lemma 5.4 part 2. The last claim, about $BP_{\mathcal{F}} \to P_{\mathcal{F}}\backslash\backslash\mathcal{P}_{\leq\mathcal{F}}$ being a right adjoint, is immediate to verify by taking the left adjoint to be the projection map backwards.

For claim 2, note that $GL(M)\backslash\backslash\mathcal{P}$ identifies with the left pullback
$$\mathrm{RBS}(M) \overset{\rightarrow}{\times}_{\mathrm{RBS}(M)} BGL(M).$$
Indeed, $\mathcal{P}$ identifies with $\mathrm{RBS}(M)_{/\emptyset}$ by sending $\mathcal{F}$ to the map $\mathcal{F} \to \emptyset$ given by the identity $e \in GL(M)$; then when we factor in the automorphisms of $\emptyset$ we get the claim. □

In terms of colimits in Cat_∞, or colimits in $\mathcal{S}$ after applying geometric realisation, we have the following.

Corollary 5.8. *Let A be a ring and M a split noetherian finitely generated projective A-module. There are the following colimits in Cat_∞:*

(1)
$$GL(M)\backslash\backslash\mathcal{P} \sqcup_{GL(M)\backslash\backslash\mathcal{P}\smallsetminus BGL(M)} (\mathrm{RBS}(M) \smallsetminus BGL(M)) \xrightarrow{\sim} \mathrm{RBS}(M).$$

[3] This is the analogue of the key lemma about parabolic subgroups, Proposition 4.1.

(2)

$$\mathrm{RBS}(M) \smallsetminus BGL(M) = \varinjlim_{[\mathcal{F}]\in GL(M)\backslash\mathcal{P},[\mathcal{F}]\neq[\emptyset]} \mathrm{RBS}(M)_{\leq[\mathcal{F}]}$$

and

$$(GL(M)\backslash\backslash\mathcal{P}) \smallsetminus BGL(M) = \varinjlim_{[\mathcal{F}]\in GL(M)\backslash\mathcal{P},[\mathcal{F}]\neq[\emptyset]} (GL(M)\backslash\backslash\mathcal{P})_{\leq[\mathcal{F}]}.$$

Proof. This follows from the cdh descent, Corollary 2.31, and descent for covers, Corollary 2.32. □

There is a natural comparison map

$$BGL(M) \to |\,\mathrm{RBS}(M)|$$

coming from the empty flag, and we want to understand the extent to which this is an equivalence. First of all, it is clear that $\mathrm{RBS}(M)$ is connected, as every flag maps to the empty flag. Thus the first thing to look at is π_1. This turns out to not be so difficult to analyse.

Theorem 5.9. *Let A be an associative ring and M a split noetherian finitely generated projective A-module. Denote by $E(M) \subset GL(M)$ the subgroup generated by the $U_{\mathcal{F}}$ as $\mathcal{F}$ runs through all splittable flags in M. Then the map*

$$GL(M) = \pi_1 BGL(M) \to \pi_1|\,\mathrm{RBS}(M)|$$

is surjective with kernel $E(M)$.

Proof. First let us note that $E(M)$ is in the kernel. Indeed, if $\mathcal{F}$ is a splittable flag and $g \in U_{\mathcal{F}}$, then the refinement $\mathcal{F} \leq \emptyset$ is invariant under the g action on $\emptyset$, which produces a nullhomotopy of the image of g in $\pi_1|\,\mathrm{RBS}(M)|$.

Next let us produce a map $\pi_1|\,\mathrm{RBS}(M)| \to GL(M)/E(M)$ such that the composition with $GL(M) \to \pi_1|\,\mathrm{RBS}(M)|$ is the natural quotient. For this define a functor

$$\mathrm{RBS}(M) \to B(GL(M)/E(M))$$

by sending each flag to the basepoint and the map $\mathcal{F} \to \mathcal{F}'$ induced by an element $g \in GL(M)$ with $g\mathcal{F} \leq \mathcal{F}'$ to the image of g in $GL(M)/E(M)$. This is clearly well-defined and functorial.

To finish the proof, it suffices to show that the map $BGL(M) \to |\,\mathrm{RBS}(M)|$ is surjective on π_1. For this, recall that a map of anima $X \to Y$ is an isomorphism on π_0 and surjective on π_1 if and only if it is left orthogonal to the class of 0-truncated maps, meaning those maps each of whose homotopy fibres is 0-truncated. It follows that the collection of maps $X \to Y$ which are isomorphism on π_0 and surjective on π_1 is closed under colimits in $\mathrm{Fun}(\Delta^1, \mathcal{S})$. It is also clearly closed under products and composition.

Using these permanence properties, let us now prove the claim by noetherian induction on M. Thus, we can assume the claim holds for all proper splittable submodules of M, hence it holds for all associated graded pieces of nonempty flags in M. But then part 2 of Proposition 5.7 shows that

$$BL_{\mathcal{F}} \to |\,\mathrm{RBS}(M)_{\leq[\mathcal{F}]}|$$

is an isomorphism on π_0 and surjective on π_1. It follows that the same is true for the composition $BP_{\mathcal{F}} \to BL_{\mathcal{F}} \to |\,\mathrm{RBS}(M)_{\leq[\mathcal{F}]}|$, which is equivalent to saying that the same is true for

$$|(GL(M)\backslash\backslash\mathcal{P})_{\leq[\mathcal{F}]}| \to |\,\mathrm{RBS}(M)_{\leq[\mathcal{F}]}|,$$

since $(GL(M)\backslash\backslash\mathcal{P})_{\leq[\mathcal{F}]} = P_{\mathcal{F}}\backslash\backslash\mathcal{P}_{\leq\mathcal{F}}$ by part 2 of Proposition 5.7.

But parts 1 and 2 of Corollary 5.8 show that our map $BGL(M) \to |\,\mathrm{RBS}(M)|$ is an iterated colimit of such maps, so we deduce the desired claim. □

In particular, if the subgroup $E(M)$ happens to be perfect, we can perform the plus construction and obtain a comparison map

$$BGL(M)^+ \to |\,\mathrm{RBS}(M)|.$$

which is an isomorphism on π_0 and π_1. In the next section we will see that if A *has many (central) units* in the technical sense introduced by Nesterenko–Suslin, and every finitely generated projective A-module is free, then this map is an equivalence. For the proof we use the inductive structure explained in this section to reduce to proving a certain homology isomorphism for matrix groups. This is a close analogue to the homology isomorphism proved by Nesterenko–Suslin in [NS90], and our proof is based on theirs. We do have to take care to ensure that we get the desired result with local coefficient systems as well, though.

5.1. **Comparison with the plus-construction.**

Lemma 5.10. *Let A be an associative ring and M a split noetherian finitely generated projective A-module. Let $\mathcal{L}$ be a local system of abelian groups on $|\,\mathrm{RBS}(M)|$, viewed also as a local system on $BL_{\mathcal{F}}$ for any splittable flag $\mathcal{F}$ of submodules of M, by pullback to the full subcategory on $\mathcal{F}$. Suppose that for all $\mathcal{F} \leq \mathcal{G}$ the quotient map $BP_{\mathcal{F}} \to B(P_{\mathcal{F}}/U_{\mathcal{G}})$ induces an isomorphism on homology with $\mathcal{L}$ coefficients. Then the map*

$$BGL(M) \to |\,\mathrm{RBS}(M)|$$

also induces an isomorphism on homology with $\mathcal{L}$-coefficients.

Proof. Using the inductive nature of the RBS categories, Proposition 5.7 and Corollary 5.8, lets us prove by noetherian induction on a splittable flag $\mathcal{G}$ that the map

$$BL_{\mathcal{G}} \to |\,\mathrm{RBS}(M)_{\leq[\mathcal{G}]}|$$

is an isomorphism on homology with $\mathcal{L}$-coefficients. Thus, assume the claim holds for all finer flags. Let $(M_1, \ldots, M_d)$ denote the associated graded of $\mathcal{G}$, and consider the proper functor

$$\prod_{i=1}^{n} GL(M_i)\backslash\backslash\mathcal{P}_i \to \prod_{i=1}^{n} \mathrm{RBS}(M_i) = \mathrm{RBS}(M)_{\leq[\mathcal{G}]},$$

where $\mathcal{P}_i$ denotes the poset of splittable flags in M_i. We want to show it's an isomorphism on $\mathcal{L}$-homology. By cdh descent, it suffices to show the same for its pullback to $\mathrm{RBS}(M)_{\leq[\mathcal{F}]}$ for any finer flag $\mathcal{F} \leq \mathcal{G}$. But if we write $\mathcal{F}_i$ for the image of $\mathcal{F}$ in M_i, this pullback gives

$$\prod_{i=1}^{n} P_{\mathcal{F}_i}\backslash\backslash(\mathcal{P}_i)_{\leq[\mathcal{F}_i]} \to \mathrm{RBS}(M)_{\leq[\mathcal{F}]},$$

so it suffices to see that $B(P_{\mathcal{F}}/U_{\mathcal{G}}) = \prod_{i=1}^{n} BP_{\mathcal{F}_i} \to \mathrm{RBS}(M)_{\leq[\mathcal{F}]}$ gives an isomorphism on $\mathcal{L}$-homology. But now this follows from the inductive hypothesis, our hypothesis, and the 2 out of 3 property for isomorphisms. □

Lemma 5.11. *Let k be a prime field, let $1 \to U \to P \to L \to 1$ be a short exact sequence of groups and let $\mathcal{L}$ be a local system of k-modules on BL. Suppose there exist:*

(1) A normal subgroup $D \subset L$;
(2) A map $s\colon D \to P$ giving a splitting of the pullback of $P \to L$ to D;

such that:

(1) The local system $\mathcal{L}$ is constant when restricted to BD;
(2) For all $i \geq 1$ the k-module $H_i(BU;k)$, equipped with D-action induced by the conjugation action of $s(D)$ on U, has vanishing D-homology in all degrees.

Then the map

$$BP \to BL$$

induces an isomorphism on homology with $\mathcal{L}$-coefficients.

Proof. By the Serre spectral sequence, it suffices to show that $H_p(BL; H_q(BU;\mathcal{L})) = 0$ for $p \geq 0$ and $q \geq 1$. By the Serre spectral sequence for $BD \to BL \to B(D/L)$, for this it suffices to show that $H_p(BD; H_q(BU;\mathcal{L})) = 0$ for $p \geq 0$ and $q \geq 1$. But now by hypothesis 1 the local system is constant so it suffices to show $H_p(BD; H_q(BU;k)) = 0$ for $p \geq 0$ and $q \geq 1$. But using the splitting s the action of D on $H_q(BU;k)$ is induced by the conjugation action of $s(D)$ on U, so this is handled by hypothesis 2. □

Lemma 5.12. *Let A be an associative ring, let $\lambda \in Z(A)^\times$ be a central unit, and let N and N' be finitely generated projective A-modules. Fix $p, q \in \mathbb{N}$ and let D_λ denote the element of $GL(N) \times GL(N') \subset GL(N \oplus N')$ given by multiplication by λ^p in the first factor and multiplication by λ^{-q} in the second factor. Then:*

(1) D_λ lies in the centre of $GL(N) \times GL(N')$.
(2) For a homomorphism $f\colon N' \to N$, let $U_f \in GL(N \oplus N')$ denotes the map which fixes N and sends $N' \to N \oplus N'$ via (f, id). Then

$$D_\lambda \cdot U_f \cdot (D_\lambda)^{-1} = U_{\lambda^{p+q} f}.$$

(3) If $N \simeq A^n$ and $N' \simeq A^{n'}$ and we choose $p = n'$ and $q = n$, then D_λ lies in $E(N \oplus N')$.

Proof. Parts 1 and 2 are simple calculations. For part 3, note that if we consider D_λ as an $(n + n') \times (n + n')$ matrix, then it has determinant 1. Thus it suffices to show in general that a $d \times d$ diagonal matrix with entries lying in $Z(A)$ and determinant 1 necessarily lies in $E_d(A)$. We can clearly assume $A = R$ commutative. In the world of 2×2 matrices, a standard calculation shows that $(\lambda, 0; 0, \lambda^{-1})$ lies in $E_2(A)$. Hence a diagonal matrix with determinant one and only two nontrivial adjacent entries lies in $E_d(A)$. But by multiplying by such matrices we can inductively arrange to make a matrix in $E_d(A)$ which agrees with our given diagonal matrix in its first $d-1$ diagonal entries. Then the last diagonal entries have to also be the same because of the determinant condition. □

Let us adopt the following notation. If V is an A-module and $n \in \mathbb{Z}$, write $V(n)$ for V considered as an additive abelian group, equipped with $Z(A)^\times$-action described by

$$\lambda \cdot m := \lambda^n m.$$

Theorem 5.13. *Let k be a prime field, and let A be an associative ring with centre $R = Z(A)$. Suppose that for all A-modules V isomorphic to A^d for some $d \geq 0$ we have*

$$H_p(BR^\times; H_q(BV(n); k)) = 0$$

for all $p \geq 0$, all $q \geq 1$, and all $n \geq 1$. Here the $V(n)$-action on k is trivial and the $R^\times$-action on $H_q(BV(n); k)$ comes by functoriality from its action on $V(n)$.

Then for all split noetherian finitely generated projective A-modules M, the natural map

$$BGL(M) \to |\operatorname{RBS}(M)|$$

is an isomorphism on homology with k-coefficients.

If furthermore we assume that either:

(1) every split submodule of M is free, or
(2) $H_q(BV; k) = 0$ for all $q \geq 1$ and all A-modules V isomorphic to A^d for some d,

then it is an isomorphism on homology with all k-module local coefficients.

Proof. By Lemma 5.10, it suffices to show that for all splittable flags $\mathcal{F} \leq \mathcal{G}$ on M, the map

$$BP_{\mathcal{F}} \to B(P_{\mathcal{F}}/U_{\mathcal{G}})$$

induces an isomorphism on homology with the correct coefficients. Let's prove this by induction on d, the length of $\mathcal{G} = (M_1 \subsetneq \ldots \subsetneq M_{d-1})$. Write $\mathcal{G}_1$ for the flag of length 2 given just by just M_1. Then we can factor the map in two steps:

$$BP_{\mathcal{F}} \to B(P_{\mathcal{F}}/U_{\mathcal{G}_1}) \to B(P_{\mathcal{F}}/U_{\mathcal{G}}).$$

The second map is the product with $BGL(M_1)$ of an instance of our comparison map with length $d-1$, thus it gives an isomorphism. For the first map, if we set $N = M_1$ and let N' be the image of a splitting of $M \to M/M_1$, then the subgroup

$$D \subset P_{\mathcal{F}}/U_{\mathcal{G}_1} \subset GL(M_1) \times GL(M/M_1) = GL(N) \times GL(N')$$

formed by the D_λ as in Lemma 5.12 verifies the conditions of Lemma 5.11 and proves the claim. □

The following is Nesterenko–Suslin's key observation, see [NS90].

Lemma 5.14. *Let R be a commutative ring* with many units*: for any $n \geq 1$ there exist $r_1, \dots r_n \in R^\times$ such that for all nonempty $I \subset \{1, \dots, n\}$, the sum $\sum_{i \in I} r_i$ is also a unit.*

Then for $n > 0$, all prime fields k, and all R-modules M, we have

$$H_p(BR^\times; H_q(BM(n); k)) = 0$$

for all $p \geq 0$ and $q \geq 1$.

Proof. This is explained in [NS90] when $n = 1$, and the same argument works in general. Recall Nesterenko–Suslin's result: for all $n \geq 1$, if we let $S_n(R)$ denote the ring $(R^{\otimes n})^{S_n}$ and take the diagonal embedding $R^\times \to S_n(R)^\times$, then for every $S_n(R)$-module N we have $H_p(BR^\times; N) = 0$ for all $p \geq 0$. Here all tensor products are over $\mathbb{Z}$. Now, if $k = \mathbb{Q}$ we have

$$H_q(BM(n); \mathbb{Q}) = \Lambda^q M_{\mathbb{Q}},$$

with $R^\times$-action given by

$$\lambda \cdot (m_1 \wedge \dots \wedge m_q) = (\lambda^n m_1) \wedge \dots \wedge (\lambda^n m_q).$$

We can put an $S_{nq}(R)$-module structure on $\Lambda^q M$ by viewing M as an $R^{\otimes n}$-module via restriction along the multiplication map $R^{\otimes n} \to R$, then using this to view $\otimes^q M$ as an $(R^{\otimes n})^{\otimes q}$-module, hence as an $S_{nq}(R)$-module by restriction, and passing to the quotient $\Lambda^q M$. The correct $R^\times$-action is recovered and so Nesterenko–Suslin's result proves the desired vanishing in this case.

If $k = \mathbb{F}_p$, then we have to use the fact that there is functorial filtration on $H_q(M; \mathbb{F}_p)$ with associated graded pieces given by $\Lambda^{q-2j}(M/pM) \otimes \Gamma_j(M[p])$, and then we can similarly argue for vanishing of $H_*(R^\times; -)$ on these pieces, hence on $H_q(M; \mathbb{F}_p)$, by equipping $\Lambda^{q-2j}(M/pM) \otimes \Gamma_j(M[p])$ with appropriate $S_{n \cdot (q-j)}$-action. ☐

Theorem 5.15. *Let A be an associative ring with many units, meaning that its centre $R = Z(A)$ has many units in the sense described in the above lemma, and let M be a split noetherian finitely generated projective A-module. Then*

$$BGL(M) \to |\operatorname{RBS}(M)|$$

is an isomorphism on $\mathbb{Z}$-homology, and if every split submodule of M is free then it is also an isomorphism on homology with all local coefficient systems, and hence $E(M) \subset GL(M)$ is perfect and for the associated plus construction we have

$$BGL(M)^+ \xrightarrow{\sim} |\operatorname{RBS}(M)|.$$

Proof. We already saw that the map

$$c \colon BGL(M) \to |\operatorname{RBS}(M)|$$

is an isomorphism on π_0, and on π_1 it identifies $\pi_1|\operatorname{RBS}(M)|$ with $GL(M)/E(M)$. Thus it suffices to show the homology isomorphism statements. But these follow from Lemma 5.14 and Theorem 5.13. ☐

5.2. **The case of finite fields.** The simplest example of a ring not having many units is a finite field $\mathbb{F}_q$, and here we will see that for a finite dimensional $\mathbb{F}_q$-vector space the anima $|\operatorname{RBS}(V)|$ is "better" than the plus construction in the sense that it can be computed and identified with the absolute most naive unstable analogue of the K-theory $K(\mathbb{F}_q)$ as computed by Quillen.

First we note that with $\mathbb{Q}$-coefficients or $\mathbb{F}_\ell$ coefficients for $\ell \neq p$, there is no difference. So all the interest lies in $\mathbb{F}_p$-coefficients.

Lemma 5.16. *Let k be a prime field and A be an associative ring with $A \otimes_{\mathbb{Z}} k = 0$. Then for any split noetherian finitely generated projective A-module M the map*

$$BGL(M) \to |\operatorname{RBS}(M)|$$

is an isomorphism on homology with local coefficients a k-module.

Proof. By Theorem 5.13, it suffices to show that $H_p(BA^r; k) = 0$ for all $r \geq 0$, and $p \geq 1$; actually we can take $r = 1$ by the Kunneth theorem. But the description of homology of abelian groups with k-coefficients recalled in the proof of Lemma 5.14 shows that each $H_p(BA; k)$ admits an $A \otimes_{\mathbb{Z}} k$-module structure, hence vanishes. □

Also, the π_1 is easy to identify.

Lemma 5.17. *Let A be a local commutative ring and M a finitely generated projective A-module. Then*

$$\pi_1|\operatorname{RBS}(M)| = GL(M)/SL(M) = A^\times.$$

Proof. By Nakayama's lemma it follows that $M \simeq A^n$ is free. Recall that M is automatically split noetherian as $\operatorname{Spec}(A)$ is connected, Lemma 5.5. Then by Theorem 5.9 it suffices to see that $E(A^n) = SL_n(A)$. But this follows by induction because $GL_n(A) = E_n(A) \cdot GL_{n-1}(A)$, see [Wei13] III.1.4. □

Now we handle $\mathbb{F}_p$-coefficients.

Theorem 5.18. *Let k be a finite field with $q = p^r$ elements, p prime, and let V be a finite dimensional k-vector space. Then the Postnikov truncation map $|\operatorname{RBS}(V)| \to Bk^\times$ induces an isomorphism on homology with coefficients in any local system $\mathcal{L}$ of $\mathbb{F}_p$-modules.*

Proof. Let us prove this by induction on the dimension d of V. For $d = 1$ we have that $\operatorname{RBS}(V) = Bk^\times$ and the claim is tautological. Now, since $k^\times$ has order prime to p its $\mathcal{L}$-homology vanishes in positive degrees, and its degree zero part is $\mathcal{L}_{k^\times}$, so we have to show the same for the $\mathcal{L}$-homology of $|\operatorname{RBS}(V)|$. We can assume $n > 1$ and use induction.

Note that the inductive hypothesis and the decomposition in part 2 of Proposition 5.7 imply that for any nonempty flag $\mathcal{F}$ in V the pullback of $\mathcal{L}$ to any $\operatorname{RBS}(V)_{\leq[\mathcal{F}]}$ has vanishing homology in positive degrees and is $\mathcal{L}_{k^\times}$ in degree 0. By part 2 of Corollary 5.8 it follows that the $\mathcal{L}$-homology of the boundary $|\operatorname{RBS}(V) \smallsetminus BGL(V)|$ identifies with the homology of the constant local system with value $\mathcal{L}_{k^\times}$ on the poset $(GL(V)\backslash\mathcal{P}) \smallsetminus [\emptyset]$; but this poset has a minimal element given by the full flags and hence is contractible, so this homology is just

$\mathcal{L}_{k^\times}$ in degree 0. Thus it suffices to show that the $\mathcal{L}$-homology of RBS(V) relative to the boundary vanishes.

However, by Corollary 5.8 part 1 we can replace RBS(V) by $GL(V)\backslash\backslash\mathcal{P}$ for this question. Now, $|GL(V)\backslash\backslash\mathcal{P}|$ is the (homotopy) quotient of $|\mathcal{P}|$ by the $GL(V)$-action, Corollary 2.33, and similarly $|\partial\, GL(V)\backslash\backslash\mathcal{P}|$ is the analogous quotient of $|\mathcal{P}\smallsetminus\{\emptyset\}|$ of nonempty flags, also known as the Tits building. Thus $|\operatorname{RBS}(V)|/|\partial \operatorname{RBS}(V)|$ identifies with the homotopy quotient

$$(|\mathcal{P}|/|\mathcal{P}\smallsetminus\{\emptyset\}|)_{hGL(V)}.$$

Now, recall the Solomon–Tits theorem: $|\mathcal{P}|/|\mathcal{P}\smallsetminus\{\emptyset\}| \simeq \Sigma|\mathcal{P}\smallsetminus\{\emptyset\}|$ has the pointed homotopy type of a wedge of $q^{\frac{n^2-n}{2}}$ many $(n-1)$-spheres. Thus what need to show is that for the *Steinberg representation* of $GL(V)$ with $\mathbb{F}_p$-coefficients, defined as

$$\mathrm{St}_{\mathbb{F}_p} := \widetilde{H}_{n-1}(|\mathcal{P}|/|\mathcal{P}\smallsetminus\{\emptyset\}|;\mathbb{F}_p),$$

we have that $\mathrm{St}_{\mathbb{F}_p}\otimes_{\mathbb{F}_p}\mathcal{L}$ has vanishing $GL(V)$-homology. We refer to the survey [H+87] for the Steinberg representation. Character computations and the Brauer–Nesbitt theorem show it is projective and irreducible, hence so is its tensor with $\mathcal{L}$. Furthermore it has dimension $q^{\frac{n^2-n}{2}} > 1$ and is therefore nontrivial, whence the conclusion. □

It follows from these lemmas that each homotopy group of $|\operatorname{RBS}(V)|$ is prime-to-p-torsion, and is all accounted for by the homology of $GL(V)$ with mod ℓ-coefficients, $\ell \neq p$, which Quillen has computed, [Qui72]. In fact we can be more precise:

Corollary 5.19. *Let k be a finite field with $q = p^r$ elements and V a k-vector space of dimension $n \in \mathbb{N}$. Then*

$$|\operatorname{RBS}(V)| \simeq ((B|U(n)|)')^{\psi^q\sim\mathrm{id}},$$

the homotopy fixed space for the unstable q-Adams operation on the prime-to-p completion of the delooping of the group anima underlying the compact Lie group $U(n)$.[4]

Proof. The Friedlander–Quillen argument, see [Fri82] Theorem 12.2, uses etale homotopy theory and the Lang isogeny to produce a map

$$BGL(V) \to ((B|U(n)|)')^{\psi^q\sim\mathrm{id}},$$

which is an equivalence on homology with $\mathbb{F}_\ell$-coefficients for $\ell \neq p$, hence identifies the target as the prime-to-p completion of the source. Since Lemma 5.16 implies that the map $BGL(V) \to |\operatorname{RBS}(V)|$ is a $\mathbb{Z}[1/p]$-equivalence, we deduce a comparison map

$$|\operatorname{RBS}(V)| \to ((B|U(n)|)')^{\psi^q\sim\mathrm{id}}$$

which is an isomorphism with $\mathbb{Z}[1/p]$-coefficients, and on π_0 and π_1 as we see from Lemma 5.17. But the homotopy groups of the target are prime-to-p, so the map from the target to its $\tau_{\leq 1}$-Postnikov truncation induces an isomorphism on homology with coefficients an $\mathbb{F}_p$-module. The same was checked for the left hand side in the previous theorem, so we conclude our

[4]Thus, ψ^q restricted to $B|U(1)^n|'$ is induced by the endomorphism $x \mapsto x^q$ of $U(1)$. See [JMO94] for unstable Adams operations.

comparison map is an isomorphism both on $\tau_{\leq 1}$ and on homology with all local coefficient systems, hence is an isomorphism of anima. □

6. Monoidal categories and actions: a lemma

In the next section, Section 7, we relate the categories $RBS(M)$ for finitely generated projective modules M over an associative ring A to the algebraic K-theory space $K(A)$. The present section is in some sense just a long and technical lemma that we need to make the final comparison. To motivate the work to be done in this section, we will describe in broad terms what we do in the following section and in slightly more detail what will happen in this section.

For an associative ring A and the exact category $\mathcal{P}(A)$ of finitely generated projective A-modules, we consider a monoidal category $M_{\mathcal{P}(A)}$ whose objects are finite ordered lists $(M_1, \dots, M_d)$ of objects in $\mathcal{P}(A)$ and a morphism $(M_1, \dots, M_d) \to (N_1, \dots, N_e)$ is the data of a flag on each N_i together with an isomorphism of the total associated graded of this list of flags with the M_i, in order. In particular, such a morphism can only exist if $e \leq d$. The relationship between $M_{\mathcal{P}(A)}$ and the categories $\mathrm{RBS}(M)$ should be thought of as an analogue of the relationship between the (symmetric) monoidal category $i\mathcal{P}(A)$ and the $BGL(M)$. If $\mathcal{M}$ is a set of representatives of isomorphism classes of finitely generated projective A-modules, then, in the same way that $i\mathcal{P}(A) \simeq \coprod_{M \in \mathcal{M}} BGL(M)$ with monoidal product given by direct sum, we have $M_{\mathcal{P}(A)} \simeq \coprod_{M \in \mathcal{M}} \mathrm{RBS}(M)$ with monoidal product also induced by direct sum, but this time it is not symmetric.

In Section 7, we do all this in much greater generality, defining a monoidal category $M_{\mathscr{E}}$ for any exact category $\mathscr{E}$ (and in fact, a little more generally than that), and the main theorem of that section is the following (see Theorem 7.38).

Theorem. *For any exact category $\mathscr{E}$, the geometric realisation of Quillen's Q-construction $Q(\mathscr{E})$ is homotopy equivalent to the classifying space $B|M_{\mathscr{E}}|$ of the topological monoid $|M_{\mathscr{E}}|$. In particular, $\Omega B|M_{\mathscr{E}}| \simeq K(\mathscr{E})$.*

Now, to make this comparison we introduce an intermediary Q-construction, which we on the one hand can compare with Quillen's Q-construction and on the other hand can compare with the classifying space of our monoidal category. This latter comparison is what we prepare for in this section.

Recall Segal's classical result in which he uses edgewise subdivision to prove that the classifying space of a monoid M is homeomorphic to the geometric realisation of the category $\mathscr{C}(M)$ with objects the elements of M and morphisms $(a, b)\colon m \to m'$ where $a, b \in M$ such that $amb = m'$ ([Seg73, Proposition 2.5]). The intermediary Q-construction that we introduce is a 2-categorical version of Segal's $\mathscr{C}(M)$ associated to a monoidal category instead of a monoid, and the comparison of the classifying space with the geometric realisation also goes through edgewise subdivision. The extra categorical level means, however, that we have to go through a wealth of simplicial manipulations to make the comparison.

To ease notation, we work in greater generality and introduce a 2-categorical Q-construction $Q(M, X)$ encoding the action of a monoidal category M on a category X, and we then compare the geometric realisation of this 2-category with the total realisation of the simplicial category whose category of n-simplices is $M^n \times X$ and whose structure maps are given by the action, multiplication and projection maps. The case that we will then be ultimately interested in is the following: for a monoidal category M, the product $M \times M^{\otimes\text{op}}$ acts on M by left and right multiplication, and in this case our 2-categorical Q-construction generalises Segal's category, and the simplicial category $(M \times M^{\otimes\text{op}})^\bullet \times M$ identifies with the edgewise subdivision of the usual bar construction on M.

In both this and the following section, we work in a slightly different setting than in the first part of this paper: we will not be working with ∞-categories but exploit instead a whole range of lower-categorical objects including monoidal categories, 2-categories, double categories and various simplicial and bisimplicial constructions. We will assume that all categories and 2-categories in this section are small. See Appendix A for relevant definitions of double categories, 2-categories, nerves and geometric realisations.

Outline and proof strategy. Let M be a strict monoidal category acting strictly on a category X via a functor $M \times X \to X$ satisfying the necessary coherency axioms. We will compare various constructions which encode this action in different ways, but before we begin we sketch the outline of the section so as to make the ideas easier to follow. The proofs given here are straightforward but the technicalities build up as we enlarge different structures by incorporating "redundant" data in order to compare them, so it is easy to loose sight of the bigger picture.

We consider the double category $\mathscr{M} \times X = [M \times X \rightrightarrows X]$ encoding the action of M on X and whose geometric realisation is a model for the homotopy quotient $|X|_{|M|}$ of the topological monoid $|M|$ acting on the geometric realisation $|X|$. We define another double category $\mathscr{M} \ltimes X$, which is in some sense a lax version of $\mathscr{M} \times X$, and a double functor $\mathscr{M} \times X \to \mathscr{M} \ltimes X$ and we show that this induces a homotopy equivalence of geometric realisations.

We then define $Q(M, X)$ to be the vertical 2-category of $\mathscr{M} \ltimes X$, that is, the sub-double category whose only horizontal morphisms are the identities. More precisely, the objects are those of $\mathscr{M} \ltimes X$, the morphisms are the vertical morphisms and the 2-cells are the 2-cells whose source and target horizontal morphisms are identities. The proof of the following theorem will take up the most of this section.

Theorem 6.12. *The inclusion $Q(M, X) \to \mathscr{M} \ltimes X$ induces a homotopy equivalence of geometric realisations.*

The horizontal morphisms of $\mathscr{M} \ltimes X$ are special cases of the vertical morphisms, so this result sounds like a double categorical version of Waldhausen's swallowing lemma ([Wal85, Lemma 1.6.5]). The proof that we present is, although much more involved, inspired by Waldhausen's proof.

The proof strategy is as follows. We define a bisimplicial category $\mathscr{B}_{\bullet\bullet}$ which horisontally collapses to a simplicial double category $\mathscr{B}_\bullet$ and vertically collapses to a simplicial 2-category $\mathscr{A}_\bullet$. For every $n \geq 0$, we define a double functor $\mathscr{M} \ltimes X \to \mathscr{B}_n$ and a pseudofunctor $Q(M, X) \to \mathscr{A}_n$, and we show that these induce homotopy equivalences of geometric realisations. In both cases, the proofs are analogous to that of Waldhausen's swallowing lemma, in that we have adjunctions given by inclusion at zero and retraction to zero. It follows that we have a zig-zag of homotopy equivalences

$$|\mathscr{M} \ltimes X| \xrightarrow{\simeq} |\mathscr{B}_{\bullet\bullet}| \xleftarrow{\simeq} |Q(M, X)|,$$

and one can then verify on the diagonals that this is given by the inclusion $Q(M, X) \to \mathscr{M} \ltimes X$.

The idea behind the construction of $\mathscr{B}_{\bullet\bullet}$ is to incorporate the 2-cells of $\mathscr{M} \ltimes X$ both horizontally and vertically. When we collapse $\mathscr{B}_{\bullet\bullet}$ vertically, we "swallow" the vertical 2-cell structure into the horizontal 2-cell structure, and vice versa for in the other direction.

Note that the proof as it is written up runs through this procedure backwards. We define double categories $\mathscr{B}_n$ and 2-categories $\mathscr{A}_n$ for all $n \geq 0$ and establish the desired homotopy equivalences, and we then incorporate the $\mathscr{A}_n$'s and $\mathscr{B}_n$'s into a bisimplicial category $\mathscr{B}_{\bullet\bullet}$ at the end.

6.1. **Lax action double category.** Let M be a strict monoidal category acting strictly on a category X, i.e. via a functor $M \times X \to X$ satisfying the necessary coherency axioms. To ease notation, we denote the monoidal product and the action by juxtaposition, and we also write $m\varphi$ in place of $\mathrm{id}_m \varphi$ for an object m in M and a morphism φ in X.

The action of M on X gives rise to a double category

$$\mathscr{M} \times X = [M \times X \rightrightarrows X]$$

where the source map is projection to X, the target map is given by the action map, the identity section is the section at the identity element e of M, and vertical composition is given by the product in M. In other words, the objects and horizontal morphisms are those of X, the vertical morphisms are of the form $m\colon x \to mx$ with m in M and x in X and with composition given by the monoidal product, and finally the 2-cells are morphisms $(\alpha, f)\colon (m, x) \to (m', x')$ in $M \times X$, which can be interpreted as a commutative diagram

$$\begin{array}{ccc} x & \xrightarrow{f} & x' \\ m\downarrow & & \downarrow m' \\ mx & \underset{\alpha f}{\to} & m'x' \end{array}$$

We now define another double category, which can be interpreted as a "lax" version of $\mathscr{M} \times X$.

Construction 6.1. Let $M.X$ be the following category. The objects are of the form

$$(x, y, m, \varphi\colon mx \to y),$$

where x, y are objects of X, m is an object of M and φ is a morphism in X. The morphisms are tuples

$$(x, y, m, \varphi) \xrightarrow{(f,g,\alpha)} (x', y', m', \varphi')$$

where $f\colon x \to x'$, $g\colon y \to y'$ in X and $\alpha\colon m \to m'$ in M such that the following diagram commutes

$$\begin{array}{ccc} mx & \xrightarrow{\alpha f} & m'x' \\ \varphi\downarrow & & \downarrow\varphi' \\ y & \xrightarrow[g]{} & y' \end{array}$$

Composition is given by coordinatewise composition.

Define a double category

$$\mathscr{M} \ltimes X = [M.X \rightrightarrows X]$$

whose source and target maps are the projections

$$s\colon (x, y, m, \varphi) \mapsto x, \quad (f, g, \alpha) \mapsto f, \quad \text{and} \quad t\colon (x, y, m, \varphi) \mapsto y, \quad (f, g, \alpha) \mapsto g.$$

Vertical composition is given by the monoidal product: for morphisms

$$(y, z, n, \psi) \circ_v (x, y, m, \varphi) = (x, z, nm, \psi \circ n\varphi),$$

and for 2-cells

$$(g, h, \beta) \circ_v (f, g, \alpha) = (f, h, \beta\alpha).$$

The identity section is the functor $X \to M.X$, $x \mapsto (x, x, e, \mathrm{id}_x)$. ◦

Definition 6.2. The *action Q-construction* of the action of M on X is the vertical 2-category $Q(M, X)$ of $\mathscr{M} \ltimes X$. More precisely, the objects are those of $\mathscr{M} \ltimes X$, the morphisms are the vertical morphisms and the 2-cells are the 2-cells whose source and target morphisms are identities. We denote the hom-categories of $Q(M, X)$ by $M(x, y)$. ◦

Remark 6.3. Recall that a strict monoidal category M can be viewed as a 2-category $\mathscr{M}$ with one object, whose morphisms are the objects of M, with composition given by the monoidal product, and whose 2-cells are the morphisms of M with the usual composition. The double category $\mathscr{M} \ltimes X$ can be viewed as a "double categorical Grothendieck construction" for the functor $\mathscr{M} \to \mathrm{Cat}$ from the 2-category associated to M into the 2-category of small categories which sends the unique object to the category X, a morphism m to the map $m\colon X \to X$ given by the action of M, and a 2-cell $m \to m'$ to the corresponding natural transformation.◦

Remark 6.4. The proof of the "$Q = +$" Theorem uses an intermediary $S^{-1}S$-construction (see [Gra76]) which is a category $\langle S \times S, S \rangle$ defined for a monoidal category S and an action of $S \times S$ acting on S. The construction $\langle M, X \rangle$ is defined more generally in [Gra76] for an action of a monoidal M on a category X. The construction $Q(M, X)$ is related to the category

$\langle M, X\rangle$ in the following way: $\langle M, X\rangle$ is the 1-category obtained by taking isomorphism classes of objects in the hom-categories of $Q(M, X)$. ○

Consider the functor $\Phi_1\colon M \times X \to M.X$ given by

$$\Phi_1(m, x) = (x, mx, m, \mathrm{id}_{mx}) \quad \text{and} \quad \Phi_1(\alpha, f) = (f, \alpha, f\alpha)$$

and consider the double functor

$$\Phi\colon \mathscr{M} \times X \to \mathscr{M} \ltimes X$$

restricting to the identity on the object category X and given by Φ_1 on morphism categories.

Lemma 6.5. *The double functor $\Phi\colon \mathscr{M} \times X \to \mathscr{M} \ltimes X$ induces a homotopy equivalence of geometric realisations.*

Proof. The double functor Φ induces a morphism of the vertical nerves

$$\Phi_n\colon N^v_\bullet(\mathscr{M} \times X) \to N^v_\bullet(\mathscr{M} \ltimes X).$$

Recall that the vertical nerve of a double category $\mathscr{C} = [C_1 \rightrightarrows C_0]$ is a simplicial category $N^v_\bullet(\mathscr{C})$, where $N^v_n(\mathscr{C})$ has as objects sequences of vertical morphisms

$$c_0 \xrightarrow{\varphi_1} c_1 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_n} x_n$$

and a morphism from $c_0 \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_n} c_n$ to $d_0 \xrightarrow{\psi_1} \cdots \xrightarrow{\psi_n} d_n$ is a collection of 2-cells

$$\alpha_i\colon \varphi_i \Rightarrow \psi_i$$

satisfying $t(\alpha_i) = s(\alpha_{i+1})$ for all i (see also Appendix A).

We show that for all n, the functor Φ_n admits a right adjoint, which proves the claim in view of the realisation lemma. Consider the functor $\Psi_n\colon N^v_n(\mathscr{M} \ltimes X) \to N^v_n(\mathscr{M} \times X)$ which sends an object

$$x_0 \xrightarrow{(x_0,x_1,m_1,\varphi_1)} x_1 \xrightarrow{(x_1,x_2,m_2,\varphi_2)} \cdots \xrightarrow{(x_{n-1},x_n,m_n,\varphi_n)} x_n$$

to

$$x_0 \xrightarrow{m_1} m_1x_0 \xrightarrow{m_2} m_2m_1x_0 \xrightarrow{m_3} \cdots \xrightarrow{m_n} m_n \cdots m_1x_0$$

and a morphism

$$(f_{i-1}, f_i, \alpha_i)\colon (x_{i-1}, x_i, m_i, \varphi_i) \Rightarrow (x'_{i-1}, x'_i, m'_i, \varphi'_i), \quad i = 1, \ldots, n,$$

to

$$(\alpha_i, \alpha_{i-1} \cdots \alpha_1 f_0)\colon (m_i, m_{i-1} \cdots m_1x_0) \Rightarrow (m'_i, m'_{i-1} \cdots m'_1x'_0), \quad i = 1, \ldots, n.$$

We claim that Ψ_n is right adjoint to Φ_n. Indeed, the unit transformation is the identity and for the counit transformation $\Phi_n \circ \Psi_n \Rightarrow \mathrm{id}$, we take the morphisms

$$(*, \varphi_i \circ m_i(*))\colon (m_{i-1} \cdots m_1x_0, m_i \cdots m_1x_0, m_i, \mathrm{id}) \Rightarrow (x_{i-1}, x_i, m_i, \varphi_i)$$

where $* = \varphi_{i-1} \circ m_{i-1}(\varphi_{i-2} \circ m_{i-2}(\cdots m_3(\varphi_2 \circ m_2\varphi_1)))$. □

6.2. **Enlarging the lax action double category.** Fix $n \geq 0$. We define a double category $\mathscr{B}_n$ and a double functor $\mathscr{M} \ltimes X \to \mathscr{B}_n$ which induces a homotopy equivalence of geometric realisations. There will be quite a bit of redundant data in our notation, but we keep this to make the comparison that we are building up to clearer.

Construction 6.6. We define a double category $\mathscr{B}_n = [\operatorname{mor} \mathscr{B}_n \rightrightarrows X]$ whose morphism category $\operatorname{mor} \mathscr{B}_n$ is given as follows. The object set is

$$\coprod_{x,y \in \operatorname{ob} X} N_n M(x,y),$$

where $M(x,y)$ is the hom-category of vertical morphisms from x to y in $M.X$. In other words, an object in $\operatorname{mor} \mathscr{B}_n$ is a sequence

$$(x,y,m_0,\varphi_0) \xrightarrow{(\mathrm{id}_x,\mathrm{id}_y,\beta_1)} (x,y,m_1,\varphi_1) \xrightarrow{(\mathrm{id}_x,\mathrm{id}_y,\beta_2)} \cdots \xrightarrow{(\mathrm{id}_x,\mathrm{id}_y,\beta_n)} (x,y,m_n,\varphi_n)$$

in $M.X$. The morphisms in $\operatorname{mor} \mathscr{B}_n$ are given by commutative diagrams in $M.X$ as pictured below.

$$\begin{array}{ccccccc}
(x,y,m_0,\varphi_0) & \xrightarrow{(\mathrm{id}_x,\mathrm{id}_y,\beta_1)} & (x,y,m_1,\varphi_1) & \xrightarrow{(\mathrm{id}_x,\mathrm{id}_y,\beta_2)} & \cdots\cdots & \xrightarrow{(\mathrm{id}_x,\mathrm{id}_y,\beta_n)} & (x,y,m_n,\varphi_n) \\
\downarrow{\scriptstyle (f,g,\alpha_0)} & & \downarrow{\scriptstyle (f,g,\alpha_1)} & & & & \downarrow{\scriptstyle (f,g,\alpha_n)} \\
(x,y,m'_0,\varphi'_0) & \xrightarrow[(\mathrm{id}_{x'},\mathrm{id}_{y'},\beta'_1)]{} & (x',y',m'_1,\varphi'_1) & \xrightarrow[(\mathrm{id}_{x'},\mathrm{id}_{y'},\beta'_2)]{} & \cdots\cdots & \xrightarrow[(\mathrm{id}_{x'},\mathrm{id}_{y'},\beta'_n)]{} & (x',y',m'_n,\varphi'_n)
\end{array}$$

Composition is given by composition of the (f,g,α_i) in $M.X$.

To ease notation, we denote objects by

$$\Big[(x,y,m_{i-1},\varphi_{i-1}) \xrightarrow{(\mathrm{id}_x,\mathrm{id}_y,\beta_i)} (x,y,m_i,\varphi_i)\Big]_{1\leq i\leq n}$$

or simply $(\mathrm{id}_x,\mathrm{id}_y,\beta_i)_{1\leq i\leq n}$ if the m_i and φ_i are implicit. A morphism will be denoted by

$$\left[\begin{array}{ccc}
(x,y,m_{i-1},\varphi_{i-1}) & \xrightarrow{(\mathrm{id}_x,\mathrm{id}_y,\beta_i)} & (x,y,m_i,\varphi_i) \\
\downarrow{\scriptstyle (f,g,\alpha_{i-1})} & & \downarrow{\scriptstyle (f,g,\alpha_i)} \\
(x',y',m'_{i-1},\varphi'_{i-1}) & \xrightarrow[(\mathrm{id}_{x'},\mathrm{id}_{y'},\beta'_i)]{} & (x',y',m'_i,\varphi'_i)
\end{array}\right]_{1\leq i\leq n}$$

or simply by $(f,g,\alpha_i)_{0\leq i\leq n}$ if the source and target are implicit.

The double category $\mathscr{B}_n = [\operatorname{mor} \mathscr{B}_n \rightrightarrows X]$ has the following structure maps: the source and target maps are the obvious projections

$$s(\mathrm{id}_x,\mathrm{id}_y,\beta_i) = x, \quad s(f,g,\alpha_i) = f \quad \text{and} \quad t(\mathrm{id}_x,\mathrm{id}_y,\beta_i) = y, \quad t(f,g,\alpha_i) = g,$$

the identity section sends an object x in X to the sequence

$$\Big[(x,x,e,\mathrm{id}_x) \xlongequal{(\mathrm{id}_x,\mathrm{id}_x,\mathrm{id}_e)} (x,x,e,\mathrm{id}_x)\Big]_{1\leq i\leq n}$$

and it sends a morphism $h\colon x \to y$ to the morphism $(h, h, \mathrm{id}_e)_{0\le i\le n}$.

Vertical composition is given by vertical composition in $\mathscr{M}$. More precisely, the vertical composite of

$$\Big[(x, y, m_{i-1}, \varphi_{i-1}) \xrightarrow{(\mathrm{id}_x, \mathrm{id}_y, \beta_i)} (x, y, m_i, \varphi_i)\Big]_{1\le i\le n}$$

and

$$\Big[(y, z, m'_{i-1}, \varphi'_{i-1}) \xrightarrow{(\mathrm{id}_y, \mathrm{id}_z, \beta'_i)} (y, z, m'_i, \varphi'_i)\Big]_{1\le i\le n}$$

is the sequence

$$\Big[(x, z, m'_{i-1}m_{i-1}, \varphi'_{i-1} \circ m'_{i-1}\varphi_{i-1}) \xrightarrow{(\mathrm{id}_x, \mathrm{id}_z, \beta'_i\beta_i)} (x, z, m'_i m_i, \varphi'_i \circ m'_i\varphi_i)\Big]_{1\le i\le n}.$$

The vertical composite of 2-cells is given by

$$(f, g, \alpha_i)_{0\le i\le n} \circ_v (g, h, \alpha'_i)_{0\le i\le n} = (f, h, \alpha'_i\alpha_i)_{0\le i\le n}$$

for composable 2-cells. ◦

We define a functor $\iota_n\colon M.X \to \operatorname{mor}\mathscr{B}_n$ sending an object (x, y, m, φ) of $M.X$ to the sequence

$$\Big[(x, y, m, \varphi) \xlongequal{(\mathrm{id}_x, \mathrm{id}_y, \mathrm{id}_m)} (x, y, m, \varphi)\Big]_{1\le i\le n}, \tag{1}$$

and a morphism $(f, g, \alpha)\colon (x, y, m, \varphi) \to (x', y', m', \varphi')$ to $(f, g, \alpha)_{0\le i\le n}$. The functor ι_n can be thought of as inclusion at zero and it identifies $M.X$ with the full subcategory of $\operatorname{mor}\mathscr{B}_n$ on the objects of the form (1). In fact, for $n = 0$, this is an equality $M.X = \operatorname{mor}\mathscr{B}_0$.

Consider the double functor $I_n\colon \mathscr{M} \ltimes X \hookrightarrow \mathscr{B}_n$ which restricts to the identity on object categories and is given by the embedding $\iota_n\colon M.X \hookrightarrow \operatorname{mor}\mathscr{B}_n$ on morphism categories.

The proof of the following lemma resembles the proof of Waldhausen's swallowing lemma ([Wal85, Lemma 1.6.5]). We show that $\mathscr{B}_n$ retracts to $\mathscr{M} \ltimes X$.

Lemma 6.7. *The double functor $I_n\colon \mathscr{M} \ltimes X \hookrightarrow \mathscr{B}_n$ induces a homotopy equivalence of geometric realisations.*

Proof. Define a functor $\rho_n\colon \operatorname{mor}\mathscr{B}_n \to M.X$ given by retraction to zero, that is, an object

$$\Big[(x, y, m_{i-1}, \varphi_{i-1}) \xrightarrow{(\mathrm{id}_x, \mathrm{id}_y, \beta_i)} (x, y, m_i, \varphi_i)\Big]_{1\le i\le n}$$

is mapped to the (x, y, m_0, φ_0), and a morphism

$$\left[\begin{array}{ccc} (x,y,m_{i-1},\varphi_{i-1}) & \xrightarrow{(\mathrm{id}_x,\mathrm{id}_y,\beta_i)} & (x,y,m_i,\varphi_i) \\ {\scriptstyle (f,g,\alpha_{i-1})}\downarrow & & \downarrow{\scriptstyle (f,g,\alpha_i)} \\ (x',y',m'_{i-1},\varphi'_{i-1}) & \xrightarrow[(\mathrm{id}_{x'},\mathrm{id}_{y'},\beta'_i)]{} & (x',y',m'_i,\varphi'_i) \end{array}\right]_{1\le i\le n}$$

is mapped to $(x,y,m_0,\varphi_0) \xrightarrow{(f,g,\alpha_0)} (x',y',m'_0,\varphi'_0)$.

The composite $\rho_n \circ \iota_n$ is the identity and the morphisms

$$\left[\begin{array}{ccc} (x,y,m_0,\varphi_0) & \overset{(\mathrm{id}_x,\mathrm{id}_y,\mathrm{id}_{m_0})}{=\!=\!=\!=} & (x,y,m_0,\varphi_0) \\ {\scriptstyle (\mathrm{id}_x,\mathrm{id}_y,\beta_{i-1}\circ\cdots\circ\beta_1)}\downarrow & & \downarrow{\scriptstyle (\mathrm{id}_x,\mathrm{id}_y,\beta_i\circ\cdots\circ\beta_1)} \\ (x,y,m_{i-1},\varphi_{i-1}) & \xrightarrow[(\mathrm{id}_x,\mathrm{id}_y,\beta_i)]{} & (x,y,m_i,\varphi_i) \end{array}\right]_{1\le i\le n}$$

define a counit transformation $\iota_n \circ \rho_n \Rightarrow \mathrm{id}$. The functor ρ_n and the unit and counit transformations sit above the identity on X, and it follows that ρ_n induces a morphism of vertical nerves which at each simplicial level is right adjoint to the morphism induced by ι_n. This proves the claim. □

6.3. **Enlarging the action 2-category.** Fix $n \geq 0$. Consider the vertical 2-category $Q(M,X)$ of the double category $\mathscr{M} \ltimes X = [M.X \rightrightarrows X]$ (Definition 6.2). We define a 2-category $\mathscr{A}_n$ and a pseudofunctor $Q(M,X) \to \mathscr{A}_n$ which induces a homotopy equivalence of geometric realisations. As with the double category $\mathscr{B}_n$, there will be some redundant data in the notation.

Construction 6.8. The 2-category $\mathscr{A}_n$ is given as follows. The object set is $N_n(X)$ and the morphisms are elements in $N_n(M.X)$ with source and target maps inherited from $\mathcal{M} \ltimes X$. In other words, the objects are sequences $x_0 \to x_1 \to \cdots \to x_n$ in X, and a morphism

$$(x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} x_n) \longrightarrow (y_0 \xrightarrow{g_1} y_1 \xrightarrow{g_2} \cdots \xrightarrow{g_n} y_n)$$

is a sequence

$$(x_0,y_0,m_0,\varphi_0) \xrightarrow{(f_1,g_1,\alpha_1)} (x_1,y_1,m_1,\varphi_1) \xrightarrow{(f_2,g_2,\alpha_2)} \cdots \xrightarrow{(f_n,g_n,\alpha_n)} (x_n,y_n,m_n,\varphi_n)$$

in $M.X$. Finally, the 2-cells are given by commutative diagrams in $M.X$ as pictured below.

$$\begin{array}{ccccccc} (x_0,y_0,m_0,\varphi_0) & \xrightarrow{(f_1,g_1,\alpha_1)} & (x_1,y_1,m_1,\varphi_1) & \xrightarrow{(f_2,g_2,\alpha_2)} & \cdots\cdots & \xrightarrow{(f_n,g_n,\alpha_n)} & (x_n,y_n,m_n,\varphi_n) \\ {\scriptstyle (\mathrm{id}_{x_0},\mathrm{id}_{y_0},\beta_0)}\downarrow & & {\scriptstyle (\mathrm{id}_{x_1},\mathrm{id}_{y_1},\beta_1)}\downarrow & & & & {\scriptstyle (\mathrm{id}_{x_n},\mathrm{id}_{y_n},\beta_n)}\downarrow \\ (x_0,y_0,m'_0,\varphi'_0) & \xrightarrow[(f_1,g_1,\alpha'_1)]{} & (x_1,y_1,m'_1,\varphi'_1) & \xrightarrow[(f_2,g_2,\alpha'_2)]{} & \cdots\cdots & \xrightarrow[(f_n,g_n,\alpha'_n)]{} & (x_n,y_n,m'_n,\varphi'_n) \end{array}$$

To ease notation, we will denote a morphism by

$$\left[(x_{i-1}, y_{i-1}, m_{i-1}, \varphi_{i-1}) \xrightarrow{(f_i, g_i, \alpha_i)} (x_i, y_i, m_i, \varphi_i)\right]_{1\leq i\leq n}$$

or simply $(f_i, g_i, \alpha_i)_{1\leq i\leq n}$ if the objects are implicit. Similarly, we denote a 2-cell by

$$\left[\begin{array}{ccc} (x_{i-1}, y_{i-1}, m_{i-1}, \varphi_{i-1}) & \xrightarrow{(f_i, g_i, \alpha_i)} & (x_i, y_i, m_i, \varphi_i) \\ {\scriptstyle (\mathrm{id}_{x_{i-1}}, \mathrm{id}_{y_{i-1}}, \beta_{i-1})}\downarrow & & \downarrow{\scriptstyle (\mathrm{id}_{x_i}, \mathrm{id}_{y_i}, \beta_i)} \\ (x_{i-1}, y_{i-1}, m'_{i-1}, \varphi'_{i-1}) & \xrightarrow[(f'_i, g'_i, \alpha'_i)]{} & (x_i, y_i, m'_i, \varphi'_i) \end{array}\right]_{1\leq i\leq n}$$

or simply by $(\mathrm{id}_{x_i}, \mathrm{id}_{y_i}, \beta_i)_{0\leq i\leq n}$ if the objects and morphisms are implicit.

Composition of morphisms is given by vertical composition in $\mathscr{M} \ltimes X$, that is, the composite of

$$\left[(x_{i-1}, y_{i-1}, m_{i-1}, \varphi_{i-1}) \xrightarrow{(f_i, g_i, \alpha_i)} (x_i, y_i, m_i, \varphi_i)\right]_{1\leq i\leq n}$$

and

$$\left[(y_{i-1}, z_{i-1}, m'_{i-1}, \varphi'_{i-1}) \xrightarrow{(g_i, h_i, \alpha'_i)} (y_i, z_i, m'_i, \varphi'_i)\right]_{1\leq i\leq n}$$

is the sequence

$$\left[(x_{i-1}, z_{i-1}, m'_{i-1}m_{i-1}, \varphi'_{i-1} \circ m'_{i-1}\varphi_{i-1}) \xrightarrow{(f_i, h_i, \alpha'_i\alpha_i)} (x_i, z_i, m'_i m_i, \varphi'_i \circ m'_i\varphi_i)\right]_{1\leq i\leq n}.$$

Composition of the 2-cells along morphisms (within the hom-categories) is given by horizontal composition in $\mathscr{M} \ltimes X$, that is, by composition of morphisms in M:

$$(\mathrm{id}_{x_i}, \mathrm{id}_{y_i}, \beta'_i)_{0\leq i\leq n} \circ_h (\mathrm{id}_{x_i}, \mathrm{id}_{y_i}, \beta_i)_{0\leq i\leq n} = (\mathrm{id}_{x_i}, \mathrm{id}_{y_i}, \beta'_i \circ \beta_i)_{0\leq i\leq n}$$

for composable morphisms.

Composition of the 2-cells along objects is given by vertical composition in $\mathscr{M} \ltimes X$, that is, by the product in M:

$$(\mathrm{id}_{y_i}, \mathrm{id}_{z_i}, \beta'_i)_{0\leq i\leq n} \circ_v (\mathrm{id}_{x_i}, \mathrm{id}_{y_i}, \beta_i)_{0\leq i\leq n} = (\mathrm{id}_{x_i}, \mathrm{id}_{z_i}, \beta'_i\beta_i)_{0\leq i\leq n}$$

for composable morphisms. ○

Consider the strict pseudofunctor $\Upsilon_n \colon Q(M, X) \to \mathscr{A}_n$ given by inclusion at zero; that is, it sends an object x to the sequence $x = x = \cdots = x$, a morphism (x, y, m, φ) to the sequence

$$\left[(x, y, m, \varphi) \overset{(\mathrm{id}_x, \mathrm{id}_y, \mathrm{id}_m)}{=\!=\!=\!=} (x, y, m, \varphi)\right]_{1\leq i\leq n}$$

and a 2-cell $(\mathrm{id}_x, \mathrm{id}_y, \beta) \colon (x, y, m, \varphi) \to (x, y, m', \varphi')$ to the 2-cell $(\mathrm{id}_x, \mathrm{id}_y, \beta)_{0\leq i\leq n}$. For $n = 0$, this is an equality $Q(M, X) = \mathscr{A}_0$.

As was the case for Lemma 6.5, the proof of the following lemma resembles the proof of Waldhausen's swallowing lemma ([Wal85, Lemma 1.6.5]). We show that $\mathscr{A}_n$ retracts onto $Q(M, X)$.

Lemma 6.9. *The pseudofunctor* $\Upsilon_n \colon Q(M, X) \to \mathscr{A}_n$ *admits a right* 2*-adjoint. In particular, it induces a homotopy equivalence of geometric realisations.*

Proof. Consider the strict pseudofunctor $R_n \colon \mathscr{A}_n \to Q(M, X)$ which sends an object

$$x_0 \to x_1 \to \cdots \to x_n$$

to x_0, a morphism

$$(x_0, y_0, m_0, \varphi_0) \to (x_1, y_1, m_1, \varphi_1) \to \cdots \to (x_n, y_n, m_n, \varphi_n)$$

to $(x_0, y_0, m_0, \varphi_0)$ and a 2-cell $(\mathrm{id}_{x_i}, \mathrm{id}_{y_i}, \beta_i)_{0 \le i \le n}$ to

$$(\mathrm{id}_{x_0}, \mathrm{id}_{y_0}, \beta_0) \colon (x_0, y_0, m_0, \varphi_0) \to (x_0, y_0, m_0', \varphi_0').$$

The composite $R_n \circ \Upsilon_n$ is equal to the identity, and for the other composite, we define an oplax natural transformation $\varepsilon \colon \Upsilon_n \circ R_n \Rightarrow \mathrm{id}$. Given an object $x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} x_n$, consider the morphism

$$\varepsilon_{f_i} \colon (x_0 = x_0 = \cdots = x_0) \longrightarrow (x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} x_n)$$

in $\mathscr{A}_n$ given by the sequence

$$\left[(x_0, x_{i-1}, e, f_{i-1} \circ \cdots \circ f_1) \xrightarrow{(\mathrm{id}_{x_0}, f_i, \mathrm{id}_e)} (x_0, x_i, e, f_i \circ \cdots \circ f_1) \right]_{1 \le i \le n}.$$

For a morphism $F \colon (x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} x_n) \longrightarrow (y_0 \xrightarrow{g_1} y_1 \xrightarrow{g_2} \cdots \xrightarrow{g_n} y_n)$ given by a sequence

$$\left[(x_{i-1}, y_{i-1}, m_{i-1}, \varphi_{i-1}) \xrightarrow{(f_i, g_i, \alpha_i)} (x_i, y_i, m_i, \varphi_i) \right]_{1 \le i \le n},$$

consider the 2-cell $A_F \colon \varepsilon_{g_i} \circ (I_n \circ R_n)(F) \to F \circ \varepsilon_{f_i}$ given by

$$\left[\begin{array}{ccc} (x_0, y_{i-1}, m_0, g_{i-1} \circ \cdots \circ g_1 \circ \varphi_0) & \xrightarrow{(\mathrm{id}_{x_0}, g_i, \mathrm{id}_{m_0})} & (x_0, y_i, m_0, g_i \circ \cdots \circ g_1 \circ \varphi_0) \\ \downarrow {\scriptstyle (\mathrm{id}_{x_0}, \mathrm{id}_{y_{i-1}}, \alpha_{i-1})} & & \downarrow {\scriptstyle (\mathrm{id}_{x_0}, \mathrm{id}_{y_i}, \alpha_i)} \\ (x_0, y_{i-1}, m_{i-1}, \varphi_{i-1} \circ m_{i-1}(f_{i-1} \circ \cdots \circ f_1)) & \xrightarrow[(\mathrm{id}_{x_0}, g_i, \alpha_i)]{} & (x_0, y_i, m_i, \varphi_i \circ m_i(f_i \circ \cdots \circ f_1)) \end{array} \right]_{1 \le i \le n}$$

The 2-cells A_F assemble to define natural transformations $(\varepsilon_{g_i})_* \circ (I_n \circ R_n) \Rightarrow (\varepsilon_{f_i})^*$, and they respect identities and composition. Hence, we have an oplax natural transformation $\varepsilon \colon \Upsilon_n \circ R_n \Rightarrow \mathrm{id}$. One can check that the triangle identities are satisfied. $\square$

6.4. **Comparing geometric realisations.** Consider the double category $\mathscr{M} \ltimes X$, its vertical 2-category $Q(M,X)$ and for all $n \geq 0$, the 2-category $\mathscr{A}_n$ and the double category $\mathscr{B}_n$ as constructed in the previous sections. For the remainder of this section, we consider $\mathscr{A}_n$ as a double category with only identity horizontal morphisms.

Construction 6.10. We can define two simplicial double categories $\mathscr{A}_\bullet$ and $\mathscr{B}_\bullet$ by defining the simplicial structure maps as below — they are defined in the "obvious" way, but to be precise we write them out.

For $\theta\colon [k] \to [n]$ in Δ, the structure map $\theta^*\colon \mathscr{A}_n \to \mathscr{A}_k$ is given by the usual structure map $\theta^*\colon N_n(X) \to N_k(X)$ on object categories (recall that we interpret it as a double category with discrete object category), and on the morphism category it is given on objects by the usual structure map

$$\theta^*\colon N_n(M.X) \to N_k(M.X)$$

and on morphisms by removing or repeating the β_i's accordingly.

The structure map $\theta^*\colon \mathscr{B}_n \to \mathscr{B}_k$ is given by the identity $X \to X$ on object categories and on morphism categories by the usual structure map

$$\theta^*\colon \coprod_{x,y} N_n(M(x,y)) \to \coprod_{x,y} N_k(M(x,y))$$

of objects and on morphisms by removing or repeating the α_i's accordingly.

Define two bisimplicial categories $\mathscr{A}_{\bullet\bullet}$ and $\mathscr{B}_{\bullet\bullet}$ by applying the horizontal nerve functor levelwise: $\mathscr{A}_{nk} = N_k^h(\mathscr{A}_n)$, and $\mathscr{B}_{nk} = N_k^h(\mathscr{B}_n)$. ○

Lemma 6.11. *The bisimplicial category $\mathscr{A}_{\bullet\bullet}$ is isomorphic to the transpose of the bisimplicial category $\mathscr{B}_{\bullet\bullet}$.*

Proof. This is a case of writing out the definitions. We verify that the objects and morphisms coincide and leave composition and simplicial structure maps to the reader. Let $n, k \geq 0$ and consider the categories $\mathscr{A}_{nk} = N_k^h(\mathscr{A}_n)$ and $\mathscr{B}_{kn} = N_n^h(\mathscr{B}_k)$.

Since we only have identity horizontal morphisms in $\mathscr{A}_n$, the object set of $\mathscr{A}_{nk}$ is in bijection with the object set $N_n(X)$ of $\mathscr{A}_n$. The object set of $\mathscr{B}_{kn}$ is the set of n-simplices in the nerve of the object category of $\mathscr{B}_k$, i.e. $N_n(X)$.

A morphism in $\mathscr{A}_{nk}$ is a sequence of k vertical morphisms in $\mathscr{A}_n$ connected by 2-cells: the diagram below is a morphism from $x_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} x_n$ to $y_0 \xrightarrow{g_1} \cdots \xrightarrow{g_n} y_n$, where the horizontal sequences come from the morphisms and the vertical sequences from the 2-cells. A morphism in $\mathscr{B}_{kn}$ is a sequence of n vertical morphisms in $\mathscr{B}_k$ connected by 2-cells: the diagram below is a morphism from $x_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} x_n$ to $y_0 \xrightarrow{g_1} \cdots \xrightarrow{g_n} y_n$, but now the vertical sequences come from the morphisms and the horizontal sequences from the 2-cells.

$$
\begin{array}{ccccc}
(x_0,y_0,m_0^0,\varphi_0^0) & \xrightarrow{(f_1,g_1,\alpha_1^0)} & (x_1,y_1,m_1^0,\varphi_1^0) & \xrightarrow{(f_2,g_2,\alpha_2^0)} \cdots \xrightarrow{(f_n,g_n,\alpha_n^0)} & (x_n,y_n,m_n^0,\varphi_n^0) \\
{\scriptstyle (\mathrm{id}_{x_0},\mathrm{id}_{y_0},\beta_0^1)}\downarrow & & {\scriptstyle (\mathrm{id}_{x_1},\mathrm{id}_{y_1},\beta_1^1)}\downarrow & & {\scriptstyle (\mathrm{id}_{x_n},\mathrm{id}_{y_n},\beta_n^1)}\downarrow \\
(x_0,y_0,m_0^1,\varphi_0^1) & \xrightarrow[(f_1,g_1,\alpha_1^1)]{} & (x_1,y_1,m_1^1,\varphi_1^1) & \xrightarrow[(f_2,g_2,\alpha_2^1)]{} \cdots \xrightarrow[(f_n,g_n,\alpha_n^1)]{} & (x_n,y_n,m_n^1,\varphi_n^1) \\
{\scriptstyle (\mathrm{id}_{x_0},\mathrm{id}_{y_0},\beta_0^2)}\downarrow & & {\scriptstyle (\mathrm{id}_{x_1},\mathrm{id}_{y_1},\beta_1^2)}\downarrow & & {\scriptstyle (\mathrm{id}_{x_n},\mathrm{id}_{y_n},\beta_n^2)}\downarrow \\
\vdots & & \vdots & \ddots & \vdots \\
{\scriptstyle (\mathrm{id}_{x_0},\mathrm{id}_{y_0},\beta_0^k)}\downarrow & & {\scriptstyle (\mathrm{id}_{x_1},\mathrm{id}_{y_1},\beta_1^k)}\downarrow & & {\scriptstyle (\mathrm{id}_{x_n},\mathrm{id}_{y_n},\beta_n^k)}\downarrow \\
(x_0,y_0,m_0^k,\varphi_0^k) & \xrightarrow[(f_1,g_1,\alpha_1^k)]{} & (x_1,y_1,m_1^k,\varphi_1^k) & \xrightarrow[(f_2,g_2,\alpha_2^k)]{} \cdots \xrightarrow[(f_n,g_n,\alpha_n^k)]{} & (x_n,y_n,m_n^k,\varphi_n^k)
\end{array}
$$

In both cases, composition is given by vertical composition of morphisms and 2-cells, and one can verify that these coincide. Likewise, one can check that the simplicial structure maps can be identified. □

We can combine this with the homotopy equivalences of the previous sections to show that the geometric realisations of $\mathscr{M} \times X$, $\mathscr{M} \ltimes X$ and $Q(M,X)$ are homotopy equivalent.

Theorem 6.12. *The inclusion $Q(M,X) \to \mathscr{M} \ltimes X$ induces a homotopy equivalence of geometric realisations.*

Proof. By Lemmas 6.7, 6.9 and 6.11, we have a diagram as below, where the homotopy equivalences on the left and right are induced by maps of simplicial double categories whose sources are constant simplicial objects:

$$|\mathscr{M} \ltimes X| \xrightarrow{\simeq} |\mathscr{B}_\bullet| \cong |\mathscr{B}_{\bullet\bullet}| \cong |\mathscr{A}_{\bullet\bullet}| \cong |\mathscr{A}_\bullet| \xleftarrow{\simeq} |Q(M,X)|.$$

To see that this homotopy equivalence is induced by the inclusion, we analyse the diagonal instead of collapsing to the horizontal and vertical axes. This leaves us with a zig-zag of simplicial categories, which levelwise fits into the diagram below, where the left vertical map is the one induced by $Q(M,X) \to \mathscr{M} \ltimes X$.

$$
\begin{array}{ccc}
N_n^h(\mathscr{M} \ltimes X) & \longrightarrow & N_n^h(\mathscr{B}_n) \\
\uparrow & & \| \\
N_n^h(Q(M,X)) & \longrightarrow & N_n^h(\mathscr{A}_n)
\end{array}
$$

Tracing through the definitions, we see that this diagram commutes for all $n \geq 0$ and the claim follows. □

Combined with the homotopy equivalence of Lemma 6.5, we have the following immediate corollary.

Corollary 6.13. *The zig-zag of double functors*

$$\mathscr{M} \times X \to \mathscr{M} \ltimes X \leftarrow Q(M, X)$$

induces a homotopy equivalence of geometric realisations.

The geometric realisation $|M|$ is naturally a topological monoid, and the action of M on X defines an action of $|M|$ on the geometric realisation $|X|$.

Corollary 6.14. *The homotopy quotient $|X|_{|M|}$ of the action of $|M|$ on $|X|$ is homotopy equivalent to the geometric realisation of $Q(M, X)$.*

Proof. The vertical nerve of the double category $\mathscr{M} \times X = [M \times X \rightrightarrows X]$ is the simplicial category whose category of n-simplices is $M^n \times X$ with the usual simplicial structure maps given by projection, action and product. Since geometric realisation commutes with finite products, the realisation of this is the homotopy quotient $|X|_{|M|}$. □

Finally, we consider the special case that we will need in following section.

Observation 6.15. Let $(M, \otimes)$ be a strict monoidal category and consider the monoidal category $M \times M^{\otimes^{\mathrm{op}}}$, where the second factor is the category M with the opposite product: $a \otimes^{\mathrm{op}} b = b \otimes a$. The monoidal category $M \times M^{\otimes^{\mathrm{op}}}$ acts on the category M by left and right multiplication: $(a, b).m = a \otimes m \otimes b$ for all objects a, b, m in M. We see that the 2-category

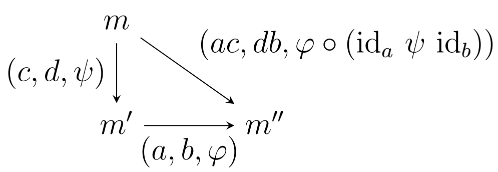
$$Q_2(M) := Q(M \times M^{\otimes^{\mathrm{op}}}, M)$$

is given as follows:

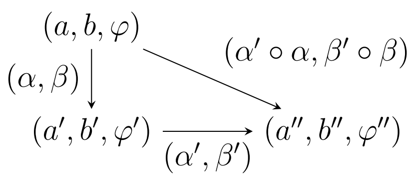
- the objects are those of M,
- a morphism $m \to m'$ is a tuple $(a, b, \varphi\colon amb \to m')$ where a and b are objects of M and φ is a morphism in M,
- and a 2-cell $(a, b, \varphi) \to (a', b', \varphi')$ is a pair of morphisms $\alpha\colon a \to a'$, $\beta\colon b \to b'$ in M such that $\varphi = \varphi' \circ (\alpha \; \mathrm{id}_m \; \beta)$.

Composition is given by:

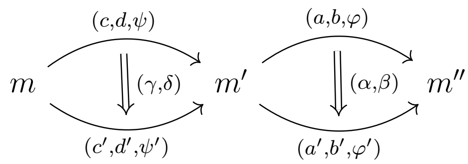
- for morphisms:



- composition of 2-cells along morphisms (i.e. within hom-categories) is given by coordinatewise composition:



∗ composition of 2-cells along objects is given by the monoidal product: the composite of the following 2-cells



is the 2-cell

$$(\alpha\gamma, \delta\beta)\colon (ac, db, \varphi \circ (\mathrm{id}_a\ \psi\ \mathrm{id}_b)) \longrightarrow (a'c', d'b', \varphi' \circ (\mathrm{id}_{a'}\ \psi'\ \mathrm{id}_{b'}))$$

Note that this also makes sense for M a non-small strict monoidal category. ○

Let M be a small monoidal category and $|M|$ the topological monoid given by the geometric realisation of M. Recall that the classifying space $B|M|$ of $|M|$ is the total geometric realisation of the standard bar construction $B_\bullet M$ whose category of n-simplices is M^n and whose simplicial structure maps are given by the monoidal structure of M. If M is an essentially small monoidal category, then we denote by $|M|$ and $B|M|$ the topological monoid and its classifying space defined as above for some equivalent small monoidal category.

Corollary 6.16. *Let M be an essentially small strict monoidal category. The classifying space $B|M|$ of the topological monoid $|M|$ is homotopy equivalent to the geometric realisation of $Q_2(M)$.*

Proof. We may assume M to be small. The vertical nerve of the double category

$$[(M \times M^{\otimes\mathrm{op}}) \times M \rightrightarrows M]$$

given by the action of $M \times M^{\otimes\mathrm{op}}$ on M is the edgewise subdivision of the bar construction $B_\bullet M$. The claim then follows from Corollary 6.13 together with the fact that the geometric realisation of the edgewise subdivision is homeomorphic to the geometric realisation of the original simplicial space ([Seg73, Proposition A.1]). □

Remark 6.17. The above result should be compared with the classical result of Segal: the classifying space of a topological monoid M is homeomorphic to the geometric realisation of a topological category $\mathscr{C}(M)$ with objects the objects of M and morphisms $(a, b)\colon m \to m'$ where a and b are objects of M such that $amb = m'$ ([Seg73, Proposition 2.5]). ○

7. Comparison with (stable) algebraic K-theory

In this last section of the paper, we compare the categories $\mathrm{RBS}(M)$ to the (stable) algebraic K-theory space. We have already remarked on this at the beginning of the previous section in order to motivate the results proved there. To recap, we associate to any exact category $\mathscr{E}$, a strict monoidal category $M_{\mathscr{E}}$ of flags and associated gradeds, and when $\mathscr{E} = \mathcal{P}(A)$ is the exact category of finitely generated projective modules over an associative ring A the monoidal category $M_{\mathcal{P}(A)}$ decomposes into a disjoint union of $\mathrm{RBS}(M)$'s. We show that the monoidal category $M_{\mathscr{E}}$ defines a model for the algebraic K-theory space $K(\mathscr{E})$ by comparing with Quillen's Q-construction $Q(\mathscr{E})$. We find that $B|M_{\mathscr{E}}| \simeq |Q(\mathscr{E})|$, so that in particular, $K(\mathscr{E}) \simeq \Omega B|M_{\mathscr{E}}|$.

In fact, we will work in slightly greater generality, namely with categories with filtrations as introduced below. This is a category with a distinguished class of short exact sequences satisfying a set of axioms enabling us to merge and split filtrations. Any exact category is a category with filtrations, but we do not need the full power of exact categories for our constructions, so we choose to work in this broader setting. It is also clear that Quillen's Q-construction can be defined verbatim for categories with filtrations. They have the advantage of including for example the category of vector spaces of dimension at most n.

7.1. **Categories with filtrations.**

Definition 7.1. Let $\mathscr{C}$ be a category with a zero object 0 and a distinguished class C of triples $a \to b \to c$ called *short exact sequences*. If a morphism appears as the first morphism in a short exact sequence, we call it an *admissible monomorphism* and denote it by $\rightarrowtail$; if it appears as the second, we call it an *admissible epimorphism* and denote it by $\twoheadrightarrow$. We say that $\mathscr{C}$ is a *category with filtrations* (with respect to the collection C) if it satisfies the following axioms:

(1) C is closed under isomorphisms,
(2) the sequences $0 \to a \xrightarrow{=} a$ and $a \xrightarrow{=} a \to 0$ are short exact sequences for all objects a,
(3) the composite of admissible monomorphisms (epimorphisms) is itself an admissible monomorphism (epimorphism),
(4) admissible monomorphisms are kernels of their corresponding admissible epimorphisms, and admissible epimorphisms are cokernels of their corresponding admissible monomorphisms,
(5) the pullback of an admissible epimorphism along an admissible monomorphism is an admissible epimorphism,
(6) the pushout of an admissible monomorphism along an admissible epimorphism is an admissible monomorphism. ◁

Existence of pullbacks and pushouts in axioms (5) and (6) comes for free, so we do not need to assume this — see Proposition 7.4 below.

Example 7.2. Let $\mathscr{A}$ be an abelian category, and let $\mathscr{C}$ be a full subcategory containing 0 which is closed under isomorphisms. Let C be the class of sequences $A \to B \to C$ in $\mathscr{C}$ which are exact in $\mathscr{A}$. Suppose the classes of admissible monomorphisms respectively admissible epimorphisms are closed under composition. Then $\mathscr{C}$ is a category with filtrations. ○

In view of this we have the following list of examples.

Example 7.3.

(1) Exact categories.
(2) Consider the abelian category $\mathrm{Vect}(k)$ of finite dimensional vector spaces over a field k. Fix $n \in \mathbb{N}$ and let $\mathrm{Vect}(k)_{\leq n}$ denote the strictly full subcategory spanned by the vector spaces of dimension less than or equal to n. This is a category with filtrations.

(3) Similarly, if R is a ring such that the rank of projective modules is well-defined, then the category $\mathcal{P}(R)_{\leq n}$ of projective R-modules of rank at most n is a category with filtrations. ○

The following proposition is an immediate consequence of the axioms. Note that the roles of monomorphisms and epimorphisms are swapped when comparing with axioms (5) and (6) of the definition.

Proposition 7.4. *The pullback of an admissible monomorphism along an admissible epimorphism exists and is an admissible monomorphism. The pushout of an admissible epimorphism along an admissible monomorphism exists and is an admissible epimorphism. Moreover, in both cases the squares are bicartesian.*

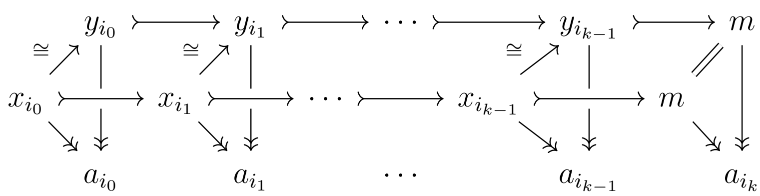
Remark 7.5. We implicitly assume that all categories with filtrations are essentially small, that is, equivalent to a small category with filtrations. ○

Let $\mathscr{C}$ be a category with filtrations. We now introduce the formalities of filtrations, flags and associated gradeds needed for our constructions.

Definition 7.6. Let $I = \{i_0 < \cdots < i_k\}$ be a finite linearly ordered set, let m be an object in $\mathscr{C}$ and let $(a_i)_{i\in I}$ be an I-graded object in $\mathscr{C}$. An *I-indexed filtration in m with associated graded* $(a_i)_{i\in I}$ is an equivalence class $[x^I, (\rho_i)_{i\in I}]$ of diagrams as below satisfying that $x_{i-1} \rightarrowtail x_i \overset{\rho_i}{\twoheadrightarrow} a_i$ is a short exact sequence for all $i \in I$, where $x_{i_0-1} := 0$.

$$\begin{array}{ccccccccc} x_{i_0} & \rightarrowtail & x_{i_1} & \rightarrowtail \cdots \rightarrowtail & x_{i_{k-1}} & \rightarrowtail & x_{i_k} = m \\ \downarrow{\scriptstyle \rho_{i_0}} & & \downarrow{\scriptstyle \rho_{i_1}} & & \downarrow{\scriptstyle \rho_{i_{k-1}}} & & \downarrow{\scriptstyle \rho_{i_k}} \\ a_{i_0} & & a_{i_1} & \cdots & a_{i_{k-1}} & & a_{i_k} \end{array}$$

Two such diagrams are equivalent, if there is a commutative diagram



In that case the isomorphisms $x_i \to y_i$ are necessarily unique, so a representing diagram is unique up to unique isomorphism.

In I-indexed filtration in m with associated graded $(a_i)_{i\in I}$ is called an *(I-indexed) flag (with associated graded)* if $a_i \neq 0$ for all $i \in I$. Equivalently, some (and thus any) representative is a sequence of non-invertible monomorphisms. ◁

We observe that any filtration has an underlying flag given by composing all invertible admissible monomorphisms with the succeeding morphism.

The existence of pullbacks of admissible monomorphisms along admissible epimorphisms and the fact that these are themselves admissible monomorphisms enable us to merge filtrations as in the definition below. The universal property of pullbacks implies that this is well-defined, that is, independent of the choice of representatives of the filtrations.

Definition 7.7. Let $\theta\colon I \to J$ be a surjective order preserving map. Suppose we are given a J-indexed filtration $[x^J, (\pi_j)_{j\in J}]$ in m with associated graded $(b_j)_{j\in J}$ and for every $j \in J$, a $\theta^{-1}(j)$-indexed filtration

$$[y^j, (\rho_i)] = [y^{\theta^{-1}(j)}, (\rho_i)_{i\in\theta^{-1}(j)}],$$

in b_j with associated graded $(a_i)_{i\in\theta^{-1}(j)}$.

The *merging of (the collection)* $[y^j, (\rho_i)]_{j\in J}$ *into* $[x^J, (\pi_j)]$ is the I-indexed filtration in m with associated graded $(a_i)_{i\in I}$ represented by a sequence $(\hat{y}^I, (\hat{\rho}_i))$ satisfying that for all $j \in J$, the restriction

$$(\hat{y}^{\theta^{-1}(j)}, (\hat{\rho}_i)_{i\in\theta^{-1}(j)})$$

to $\theta^{-1}(j) = \{i_0 < \cdots < i_k\}$ factors through $(y^j, (\rho_i))$ as indicated by the commutative diagram of pullbacks below

$$\begin{array}{ccccccc}
\hat{y}_{i_0} & \rightarrowtail & \hat{y}_{i_1} & \rightarrowtail \cdots \rightarrowtail & \hat{y}_{i_{k-1}} & \rightarrowtail & \hat{y}_{i_k} = x_j \\
\downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \pi_j \\
y_{i_0} & \rightarrowtail & y_{i_1} & \rightarrowtail \cdots \rightarrowtail & y_{i_{k-1}} & \rightarrowtail & b_j \\
\downarrow \rho_{i_0} & & \downarrow \rho_{i_1} & & \downarrow \rho_{i_{k-1}} & & \downarrow \rho_{i_k} \\
a_{i_0} & & a_{i_1} & \cdots & a_{i_{k-1}} & & a_j
\end{array}$$

where $\hat{\rho}_i$ is the composite of ρ_i with the admissible epimorphism $\hat{y}_i \twoheadrightarrow y_i$.

We write $[x^J, (\pi_j)] \circ \Big([y^j, (\rho_i)]_{j\in J}\Big) = [\hat{y}^I, (\hat{\rho}_i)]$. $\triangleleft$

Remark 7.8. The existence and uniqueness of a merging together with the observation that we can split flags should be interpreted as a generalisation of the following statement for vector spaces: for a surjective order preserving map $\theta\colon I \to J$ of finite linearly ordered sets, a J-indexed filtration $\{V_j\}_{j\in J}$ of V together with a $\theta^{-1}(j)$-indexed filtration of the cokernel V_j/V_{j-1} for all j is equivalent to an I-indexed filtration of V. $\circ$

7.2. **A monoidal category of flags and associated gradeds.** We now define a monoidal category encoding the data of flags with associated gradeds in a given category with filtrations. Intuitively, the objects should be thought of as associated gradeds, and the morphisms as those induced by flags where we allow refinement of flags. For example, a morphism from (a, b, c) to (m) is a 3-step filtration of m with associated graded (a, b, c).

Let $\mathscr{C}$ be a category with filtrations. The monoidal category $M_{\mathscr{C}}$ is defined in Constructions 7.9, 7.14 and 7.16 (see also Remark 7.17 for a different perspective in terms of multicategories).

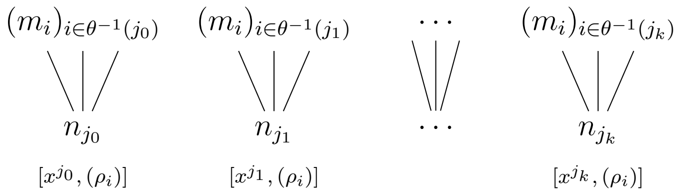
Construction 7.9 (Objects and morphisms)**.** The objects of $M_{\mathscr{C}}$ are tuples $(I, (m_i)_{i\in I})$, where I is a finite linearly ordered set and $(m_i)_{i\in I}$ is an I-graded object in $\mathscr{C}$ with $m_i \neq 0$ for all $i \in I$. We just write $(m_i)_{i\in I}$ and call such an object an *I-indexed list*, and we include the empty list $\emptyset$. A morphism $\varphi\colon (m_i)_{i\in I} \to (n_j)_{j\in J}$ consists of the following data

(1) a surjective order preserving map $\theta\colon I \to J$,
(2) for every $j \in J$, a $\theta^{-1}(j)$-indexed flag in n_j with associated graded $(m_i)_{i\in\theta^{-1}(j)}$:

$$[x^j, (\rho_i)] = [x^{\theta^{-1}(j)}, (\rho_i)_{i\in\theta^{-1}(j)}]$$

We write $\varphi = (\theta, [x^j, (\rho_i)]_{j\in J})\colon (m_i)_{i\in I} \to (n_j)_{j\in J}$. ◦

Remark 7.10. Diagramatically, one can picture a morphism φ as specified above as follows. Writing out the list of objects of the source in the top line, and the list of objects of the target in the bottom line, we connect the objects as specified by the order preserving map and label the target objects by the appropriate flags. Of course, this can be more or less detailed in order to emphasise the relevant data or structure.



We will use diagrams like this to picture an important decomposition below, but other than that, we only include this remark hoping that it might help the reader to detach themselves a little from the technical aspects and notation. ◦

Before defining composition, we observe that the concatenation operation on the objects of $M_{\mathscr{C}}$ can be extended to the morphisms. This will also be used to define a monoidal product in $M_{\mathscr{C}}$ (see Construction 7.16).

Construction 7.11 (Concatenation)**.** We denote the concatenation of linearly ordered sets $I = \{i_0 < \cdots < i_k\}$ and $J = \{j_0 < \cdots < j_l\}$ by

$$I \circledast J = \{i_0 < \cdots i_k < j_0 < \cdots < j_l\}.$$

Recall that the concatenation of graded objects $(m_i)_{i\in I}$ and $(n_j)_{j\in J}$ is the $(I \circledast J)$-graded object

$$(m_i)_{i\in I} \circledast (n_j)_{j\in J} = ((m \circledast n)_i)_{i\in I\circledast J} = (m_{i_0}, \ldots, m_{i_k}, n_{j_0}, \ldots, n_{j_l}).$$

For morphisms, we can likewise concatenate the data: the concatenation of

$$(\theta, [x^j, (\rho_i)])\colon (m_i)_{i\in I} \to (k_j)_{j\in J}, \quad \text{and} \quad (\sigma, [y^j, (\pi_i)])\colon (n_i)_{i\in I'} \to (l_j)_{j\in J'},$$

is the morphism

$$(\theta, [x^j, (\rho_i)]) \circledast (\sigma, [y^j, (\pi_i)])\colon ((m \circledast n)_i)_{i\in I\circledast I'} \to ((k \circledast l)_j)_{j\in J\circledast J'},$$

given by

(1) the surjective order preserving map $\theta \circledast \sigma \colon I \circledast I' \to J \circledast J'$ defined on I respectively I' by θ respectively σ.
(2) the flag $[x^j, (\rho_i)]$ for $j \in J$, and the flag $[y^j, (\pi_i)]$ for $j \in J'$.

We also write $(\theta, [x^j, (\rho_i)]) \circledast (\sigma, [y^j, (\pi_i)]) = (\theta \circledast \sigma, [x^j, (\rho_i)] \circledast [y^j, (\pi_i)])$. ◦

Definition 7.12. Let I be a finite linearly ordered set. An *interval* $I' \subseteq I$ is a subset satisfying that if $i < j < l$ and $i, l \in I'$, then $j \in I'$. A *partition* of I is a decomposition $I = \circledast_{t \in T} I_t$ for ordered intervals $I_t \subseteq I$, $t \in T$, where T is some linearly ordered set. ◁

We observe that any morphism in $M_{\mathscr{C}}$ can be completely decomposed as the concatenation of morphisms to one object lists.

Observation 7.13. Let $(\theta, [x^j, (\rho_i)]_{j \in J}) \colon (m_i)_{i \in I} \to (n_j)_{j \in J}$ be a morphism in $M_{\mathscr{C}}$. Then

$$(\theta, [x^j, (\rho_i)]_{j \in J}) = \circledast_{j \in J} (\theta^j, [x^j, (\rho_i)]_{j \in \{j\}}),$$

where $(\theta^j, [x^j, \rho_i]_{j \in \{j\}}) \colon (m_i)_{i \in \theta^{-1}(j)} \to (n_j)_{j \in \{j\}}$ is the morphism given by

(1) the surjective map $\theta^j = \theta|_{\theta^{-1}(j)} \colon \theta^{-1}(j) \to \{j\}$,
(2) the flag $[x^j, (\rho_i)]$ in n_j. ◦

We now define composition in $M_{\mathscr{C}}$.

Construction 7.14 (Composition)**.** Let

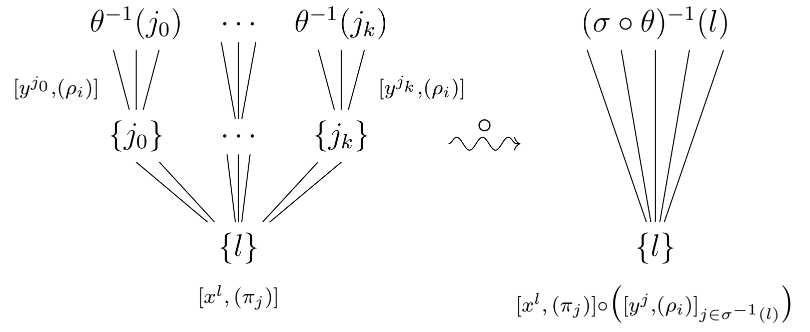
$$(\theta, [y^j, (\rho_i)]_{j \in J}) \colon (m_i)_{i \in I} \to (n_j)_{j \in J} \quad \text{and} \quad (\sigma, [x^l, (\pi_j)]_{l \in L}) \colon (n_j)_{j \in J} \to (k_l)_{l \in L}$$

be morphisms in $M_{\mathscr{C}}$. The composite is defined by merging the flags of the given morphisms for each $l \in L$:

$$\begin{aligned}(\sigma, [x^l, (\pi_j)]_{l \in L}) &\circ (\theta, [y^j, (\rho_i)]_{j \in J}) \\ &:= \Big(\sigma \circ \theta, \circledast_{l \in L} [x^l, (\pi_j)] \circ \Big([y^j, (\rho_i)]_{j \in \sigma^{-1}(l)}\Big)\Big),\end{aligned}$$

where $[x^l, (\pi_j)] \circ \Big([y^j, (\rho_i)]_{j \in \sigma^{-1}(l)}\Big)$ is the $(\sigma \circ \theta)^{-1}(l)$-indexed flag in k_l with associated graded $(a_i)_{i \in (\sigma \circ \theta)^{-1}(l)}$ as defined in Definition 7.7. ◦

Remark 7.15. For each $l \in L$, the composition can be pictured by the diagram below, where we have omitted the objects and just denote the finite linearly ordered sets and write $\sigma^{-1}(l) = \{j_0, \cdots, j_k\}$.



$\circ$

The concatenation operation defines a monoidal product in $M_{\mathscr{C}}$.

Construction 7.16 (Monoidal product)**.** Let

$$\circledast\colon M_{\mathscr{C}} \times M_{\mathscr{C}} \to M_{\mathscr{C}}$$

be the functor given by:

$$(m_i)_{i\in I} \circledast (n_j)_{j\in J} = ((m \circledast n)_i)_{i\in I\circledast J},$$

and

$$(\theta, [x^j, (\rho_i)]) \circledast (\sigma, [y^j, (\pi_i)]) = (\theta \circledast \sigma, [x^j, (\rho_i)] \circledast [y^j, (\pi_i)]).$$

One easily verifies that this defines a strict monoidal product with identity object the empty list $\emptyset$. $\circ$

Remark 7.17. The category $M_{\mathscr{C}}$ can also be interpreted as the strict monoidal category coming from a non-symmetric multicategory whose objects are those of $\mathscr{C}$ and where a morphism $(a_1, \ldots, a_n) \to (b)$ is a flag in b with associated graded $(a_1, \ldots, a_n)$. See [Lei04, Definition 2.1.1 and §2.3] or [GH15, Definitions 3.1.6 and 3.1.7]. $\circ$

Remark 7.18. We make a small remark relating this definition to the reductive Borel–Serre categories defined in Definition 5.2. Let R be an associative ring and let $\mathcal{P}(R)$ be the exact category of finitely generated R-modules. For $M \in \mathcal{P}(R)$, there is a fully faithful functor

$$F_M\colon \mathrm{RBS}(M) \to M_{\mathcal{P}(R)}$$

given by sending a splittable flag to its associated graded. For a morphism $gU_{\mathcal{F}}\colon \mathcal{F} \to \mathcal{F}'$ in $\mathrm{RBS}(M)$, the corresponding morphism is most easily described as a composite

$$\mathrm{gr}(\mathcal{F}) \xrightarrow{g} \mathrm{gr}(g\mathcal{F}) \xrightarrow{\text{incl}} \mathrm{gr}(\mathcal{F}'),$$

where the first map is the map of associated gradeds induced by the isomorphism $g\colon \mathcal{F} \to g\mathcal{F}$ and the second is the one induced by the refinement $g\mathcal{F} \subset \mathcal{F}'$. More precisely, the refinement $g\mathcal{F} \subset \mathcal{F}'$ defines a morphism as follows, where we write

$$\mathcal{F} = (M_1 \subsetneq \cdots \subsetneq M_{d-1}), \quad \mathcal{F}' = (N_1 \subsetneq \cdots \subsetneq N_{e-1})$$

and recall that there is an order-preserving injective map $f\colon \{1,\dots,e-1\} \to \{1,\dots,d-1\}$ such that $N_j = gM_{f(j)}$. The surjective order preserving map $\theta\colon \{1,\dots,d\} \to \{1,\dots,e\}$ is given by $i \mapsto \min\{j \mid i \leq f(j)\}$. For $j \in \{1,\dots,e\}$, we write $\theta^{-1}(j) = \{i,\dots,f(j)\}$ and we choose the flag in N_j/N_{j-1} given by the image of the flag

$$[gM_i \subset gM_{i+1} \subset \cdots \subset gM_{f(j)} = N_j].$$

For a set $\mathcal{M}$ of representatives of finitely generated projective R-modules, these functors provide an equivalence $M_{\mathcal{P}(R)} \simeq \coprod_{M\in\mathcal{M}} \mathrm{RBS}(M)$. ∘

We make the following useful observations.

Proposition 7.19. *All morphisms in $M_{\mathscr{C}}$ are monomorphisms.*

Proof. It suffices to show that morphisms to one object lists are monomorphisms. This amounts to showing that for given any flag $[x^J, (\rho_j)]$ in m with associated graded $(b_j)_{j\in J}$ and any given surjective order preserving maps $\theta, \sigma\colon I \to J$, and flags $[y^{\theta^{-1}(j)}, (\mu_i)]$ and $[z^{\sigma^{-1}(j)}, (\nu_i)]$ in b_j for all $j \in J$, we have

$$[x^J, (\rho_j)] \circ \Big([y^{\theta^{-1}(j)}, (\mu_i)]_{j\in J}\Big) = [x^J, (\rho_j)] \circ \Big([z^{\sigma^{-1}(j)}, (\nu_i)]_{j\in J}\Big),$$

if and only if $\theta = \sigma$ and $[y^{\theta^{-1}(j)}, (\mu_i)] = [z^{\sigma^{-1}(j)}, (\nu_i)]$ for all $j \in J$. The equality $\theta = \sigma$ follows directly from the fact that flags are defined by sequences of non-invertible admissible monomorphisms, and the equality of flags is then verified by the universal properties of pushouts. □

Proposition 7.20. *Let $\varphi\colon (a_i)_{i\in I_1} \circledast (m_i)_{i\in I_2} \circledast (b_i)_{i\in I_3} \to (n_j)_{j\in J}$ be a morphism in $M_{\mathscr{C}}$ given by an order preserving map $\theta\colon I_1 \circledast I_2 \circledast I_3 \to J$. Let $J = J_1 \circledast J_2 \circledast J_3$ be the partition given by $J_1 = \theta(I_1) - \theta(I_2 \cup I_3)$ and $J_3 = \theta(I_3) - \theta(I_2 \cup I_1)$. Then φ can be written on the form*

$$\varphi = \varphi_A \circledast (f \circ (\varphi_a \circledast \mathrm{id} \circledast \varphi_b)) \circledast \varphi_B,$$

for a morphism

$$f\colon a \circledast (m_i)_{i\in I_2} \circledast b \to (n_j)_{j\in J_2}$$

with a and b one object lists or the empty list, and morphisms

$$\varphi_A\colon (a_i)_{i\in I_A} \to (n_j)_{j\in J_1}, \quad \text{and} \quad \varphi_a\colon (a_i)_{i\in I_a} \to a,$$
$$\varphi_B\colon (b_i)_{i\in I_B} \to (n_j)_{j\in J_3}, \quad \text{and} \quad \varphi_a\colon (b_i)_{i\in I_b} \to b,$$

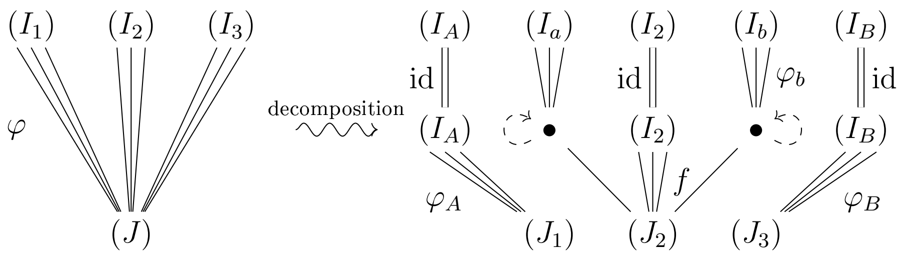
where $I_A \subset I_1$ is the preimage of J_1 and I_a is its complement, and $I_B \subset I_3$ is the preimage of J_3 and I_b its complement.

Moreover, if $\varphi = \varphi'_A \circledast (f' \circ (\varphi_{a'} \circledast \mathrm{id} \circledast \varphi_{b'})) \circledast \varphi'_B$ is another such decomposition, then $\varphi_A = \varphi'_A$, $\varphi_B = \varphi'_B$ and

$$\varphi_a = \alpha \circ \varphi_{a'}, \quad \varphi_b = \beta \circ \varphi_{b'}, \quad \text{and} \quad f' = f \circ (\alpha \circledast \mathrm{id}_m \circledast \beta)$$

for unique isomorphisms $\alpha\colon a' \to a$, $\beta\colon b' \to b$.

We do not provide a full proof of this, as it is a straightforward albeit technical observation. Instead we provide an example below in the case when $\mathscr{C} = \mathrm{Vect}(k)$ is the category of finite dimensional vector spaces over some field k, as this illustrates the intuition behind the decomposition better. The reader can readily verify that this generalises directly to the general case. The decomposition relies crucially on the fact that filtrations can be merged and split, mirroring the way filtrations of vector spaces behave (see Remark 7.8). The idea is to "collapse" the outer tuples to one-object (or empty) lists so that we find a "terminal decomposition" of the morphism — this will be vital to our arguments later on (see Proposition 7.26). Before explaining the example, we note that the decomposition can be illustrated by the diagram below (where we have replaced the objects by the finite linearly ordered sets for notational simplicity). The dashed arrows illustrate that the decompositions differ by (unique) isomorphisms of the one object lists that are interpolated.



Example 7.21. Let k be a field and $\mathscr{C} = \mathrm{Vect}(k)$, and consider a morphism

$$\varphi\colon (A_0, A_1, A_2, M_0, M_1, B_0, B_1, B_2, B_3) \longrightarrow (N_0, N_1, N_2)$$

in $M_{\mathscr{C}}$, which is given by the surjective order preserving map $[8] \cong [2] \circledast [1] \circledast [3] \to [2]$ which partitions $[8]$ into $\{0 < 1 < 2 < 3\}$, $\{4 < 5 < 6\}$ and $\{7 < 8\}$. Then φ is additionally given by three flags with associated graded:

* a flag $F_0 \subset F_1 \subset F_2 \subset F_3 = N_0$ together with an identification

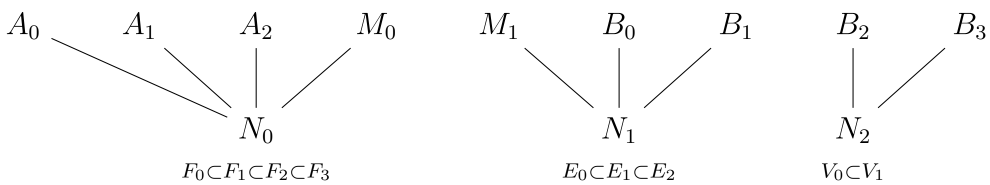
$$F_0 \oplus F_1/F_0 \oplus F_2/F_1 \oplus F_3/F_2 \cong A_0 \oplus A_1 \oplus A_2 \oplus M_0,$$

* a flag $E_0 \subset E_1 \subset E_2 = N_1$ together with an identification

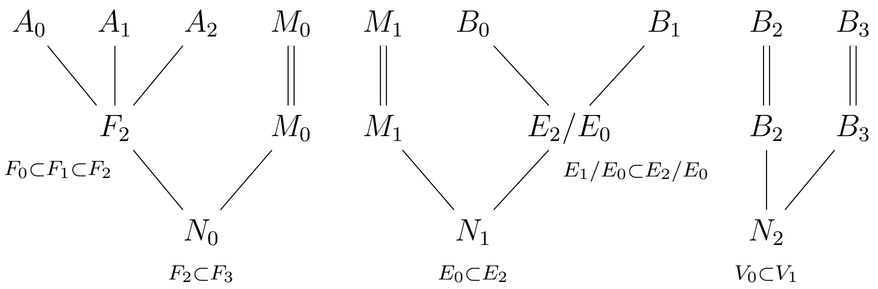
$$E_0 \oplus E_1/E_0 \oplus E_2/E_1 \cong M_1 \oplus B_0 \oplus B_1,$$

* a flag $V_0 \subset V_1 = N_2$ together with an identification $V_0 \oplus V_1/V_0 \cong B_2 \oplus B_1$.

We illustrate this morphism by the following diagram.



We obtain the desired decomposition by replacing the subset (A_0, A_1, A_2) of the source tuple by (F_2) and the subset (B_0, B_1) by (E_2/E_0) as indicated in the following diagram which illustrates the decomposition. The identification of the associated gradeds are the obvious ones inherited from the data defining φ.



In the general case, one needs to make a choice of subobject F_2 and a choice of quotient object E_2/E_0. These are unique up to unique isomorphism, so decompositions given by different choices differ by unique isomorphisms of these one object lists.

With the notation of the proposition, we have

- $J = [2] = \emptyset \circledast \{0 < 1\} \circledast \{2\} = J_1 \circledast J_2 \circledast J_3$,
- $\varphi_A \colon \emptyset \to \emptyset$ is the unique morphism, and $\varphi_B \colon (B_2, B_3) \to (N_3)$ is given by the flag $V_0 \subset V_1 = N_3$ and the identification of the associated graded coming from of φ,
- $\varphi_a \colon (A_0, A_1, A_2) \to (F_2)$ is given by the flag $F_0 \subset F_1 \subset F_2$ and the identification of the associated graded coming from φ,
- $\varphi_b \colon (B_0, B_1) \to (E_2/E_0)$ is given by the flag $E_1/E_0 \subset E_2/E_0$ and the identification of the associated graded coming from φ,
- $f \colon (F_2, M_0, M_1, E_2/E_0) \to (N_0, N_1)$ is the concatenation of two morphisms

$$f_1 \colon (F_2, M_0) \to (N_0) \quad \text{and} \quad f_2 \colon (M_1, E_2/E_0) \to (N_1)$$

  given by the flags $F_2 \subset F_3 = N_0$ respectively $E_0 \subset E_2 = N_1$ where the identifications of the associated gradeds are given by the identities on F_2 respectively E_2/E_0 and by the isomorphisms $F_3/F_2 \cong M_0$ respectively $E_0 \cong M_1$ given by φ.

○

7.3. **A 2-categorical Q-construction.** In this section, we associate a 2-category $Q_2(M)$ to any strict monoidal category M. See Appendix A for the basic notions of 2-categories needed here.

Let M be a strict monoidal category. We denote the monoidal product by juxtaposition. The following construction is simply the action 2-category $Q(M \times M^{\otimes \mathrm{op}}, M)$ of Observation 6.15, but we spell out the details for clarity.

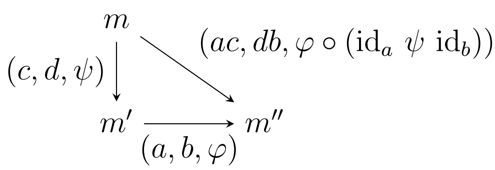
Construction 7.22. We define a 2-category $Q_2(M)$ as follows:

- the objects are those of M,

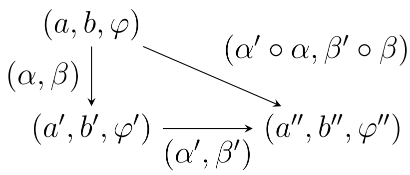
* a morphism $m \to m'$ is a tuple $(a, b, \varphi\colon amb \to m')$ where a and b are objects of M and φ is a morphism in M,
* and a 2-cell $(a, b, \varphi) \to (a', b', \varphi')$ is a pair of morphisms $\alpha\colon a \to a'$, $\beta\colon b \to b'$ in M such that $\varphi = \varphi' \circ (\alpha \ \mathrm{id}_m \ \beta)$.

Composition is defined as follows:

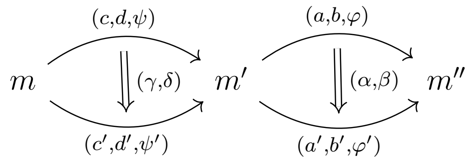
* for morphisms:



* composition of 2-cells along morphisms (i.e. within hom-categories) is given by coordinatewise composition:



* composition of 2-cells along objects is given by the monoidal product: the composite of the following 2-cells



is the 2-cell

$$(\alpha\gamma, \delta\beta)\colon (ac, db, \varphi \circ (\mathrm{id}_a \ \psi \ \mathrm{id}_b)) \longrightarrow (a'c', d'b', \varphi' \circ (\mathrm{id}_{a'} \ \psi' \ \mathrm{id}_{b'}))$$

We also define a 1-category $Q_1(M)$ by taking components of the hom-categories in $Q_2(M)$. More precisely, the objects of $Q_1(M)$ are those of $Q_2(M)$ and the hom-sets are given by taking equivalence classes of morphisms in $Q_2(M)$, where two morphisms are equivalent, if there is a zig-zag of 2-cells between them.

Interpreting $Q_1(M)$ as a 2-category with only identity 2-cells, there is a canonical lax pseudofunctor $\kappa_M\colon Q_2(M) \to Q_1(M)$. ◦

Observation 7.23. A strong monoidal functor $M \to N$ induces a pseudofunctor between the associated 2-categories $Q_2(M) \to Q_2(N)$, and a monoidal natural transformation of strong monoidal functors induces an oplax natural transformation of pseudofunctors. In particular, if M is essentially small, then $Q_2(M)$ and $Q_1(M)$ admit geometric realisations. ◦

Remark 7.24. This construction is related to Quillen's $S^{-1}S$-construction, an intermediary construction used to prove the "$Q = +$" Theorem ([Gra76]). For a monoidal category S, if the hom-categories in $Q_2(S)$ are groupoids, then $Q_1(S)$ is the category $\langle S \times S, S\rangle$ as defined in [Gra76]. In comparison, Quillen's $S^{-1}S$-construction is the category $S^{-1}S = \langle S, S \times S\rangle$. See also Remark 6.4. ◦

We observe that $Q_2(M)$ contracts onto $Q_1(M)$ in the following situation.

Proposition 7.25. *Let M be a strict monoidal category. If for all pairs of objects m, m' in M, the hom-category $\mathrm{Hom}_2(m, m')$ is a disjoint union of categories with terminal objects, then the pseudofunctor $\kappa_M \colon Q_2(M) \to Q_1(M)$ admits a right 2-adjoint. In particular, if M is essentially small, then κ_M induces a homotopy equivalence of geometric realisations.*

Proof. The adjoint is given by fixing a choice of terminal object in each component of all hom-categories in $Q_2(M)$. The unique 2-cells from morphisms to the terminal objects define a lax unit transformation. □

Let $\mathscr{C}$ be a category with filtrations and let $M_{\mathscr{C}}$ be the monoidal category of flags and associated gradeds as defined in Section 7.2. Consider the 2-categorical Q-construction $Q_2(M_{\mathscr{C}})$ as defined in the previous section. Since we implicitly assume that $\mathscr{C}$ is essentially small, so is $M_{\mathscr{C}}$, and thus the Q-constructions admit geometric realisations.

Proposition 7.26. *For any morphism*

$$((a_i)_{i\in I_a}, (b_i)_{i\in I_b}, \varphi) \colon (m_i)_{i\in I} \to (n_j)_{j\in J},$$

in $Q_2(M_{\mathscr{C}})$ there is a unique 2-cell to a morphism of the form

$$\Big((n_j)_{j\in J_1},\ (n_j)_{j\in J_3},\ \mathrm{id}_{(n_j)_{j\in J}}\Big) \ \circ\ (a, b, f),$$

for some partition $J = J_1 \circledast J_2 \circledast J_3$ and some $(a, b, f) \colon (m_i)_{i\in I} \to (n_j)_{j\in J_2}$ in $Q_2(M_{\mathscr{C}})$ with a and b one object lists or the empty list. Moreover, this morphism is unique up to a change of (a, b, f), and two such representatives given by (a, b, f) respectively (a', b', f') will be connected by a unique (necessarily invertible) 2-cell $(\alpha, \beta) \colon (a, b, f) \to (a', b', f')$.

Proof. In view of Proposition 7.20, any morphism

$$\varphi \colon (a_i)_{i\in I_1} \circledast (m_i)_{i\in I_2} \circledast (b_i)_{i\in I_3} \to (n_j)_{j\in J}$$

in $M_{\mathscr{C}}$ can be written on the form

$$\varphi = \varphi_A \circledast (f \circ (\varphi_a \circledast \mathrm{id} \circledast \varphi_b)) \circledast \varphi_B.$$

It follows that we have 2-cells in $Q_2(M_{\mathscr{C}})$

$$\begin{array}{c}
\Big((a_i)_{i\in I_A\circledast I_a},\ (b_i)_{i\in I_b\circledast I_B},\ \varphi_A\circledast\big(f\circ(\varphi_a\circledast \mathrm{id}_{(m_i)_{i\in I}}\circledast\varphi_b)\big)\circledast\varphi_B\Big)\\
\Big\downarrow (\varphi_A\circledast\varphi_a,\varphi_b\circledast\varphi_B)\\
\Big((n_j)_{j\in J_1}\circledast a,\ b\circledast(n_j)_{j\in J_3},\ \mathrm{id}_{(n_j)_{j\in J_1}}\circledast f\circledast \mathrm{id}_{(n_j)_{j\in J_3}}\Big)
\end{array}$$

which are unique by the uniqueness observation of Proposition 7.20 and the fact that f is a monomorphism (Proposition 7.19). The final statement also follows directly from the uniqueness of the decomposition. ☐

As a direct consequence of this, we can apply Proposition 7.25.

Corollary 7.27. *The hom-categories of $Q_2(M_{\mathscr{C}})$ are disjoint unions of categories with terminal objects. In particular, the pseudofunctor $\kappa_{M_{\mathscr{C}}}\colon Q_2(M_{\mathscr{C}})\to Q_1(M_{\mathscr{C}})$ admits a right* 2*-adjoint, and thus induces a homotopy equivalence of geometric realisations.*

Remark 7.28. The decomposition in the proof of Proposition 7.26 will be referred to as the *terminal decomposition* of a morphism. ○

7.4. **Comparing with Quillen's Q-construction.** Let $\mathscr{C}$ be a category with filtrations and let $M_{\mathscr{C}}$ be the monoidal category of flags and associated gradeds defined in Section 7.2. We want to compare the classifying space $B|M_{\mathscr{C}}|$ with Quillen's Q-construction $Q(\mathscr{C})$. To do this, we first of all compare $Q_1(M_{\mathscr{C}})$ with $Q(\mathscr{C})$ and then combine this with the results of the previous sections.

Recall that Quillen's Q-construction $Q(\mathscr{C})$ is the category with objects those of $\mathscr{C}$, and where a morphism $x\to y$ is given by an isomorphism class of diagrams of the form $x \twoheadleftarrow z \rightarrowtail y$, where two such diagrams are isomorphic if there is an isomorphism between the middle objects which commutes with the morphisms to x and y. Composition is given by pullbacks, that is, the composite of $[x_1 \twoheadleftarrow z_1 \rightarrowtail x_2]$ and $[x_2 \twoheadleftarrow z_2 \rightarrowtail x_3]$ is given by the sequence $x_1 \twoheadleftarrow z_1\times_{x_2} z_2 \rightarrowtail x_3$.

We define a functor $\Psi\colon Q(\mathscr{C})\to Q_1(M_{\mathscr{C}})$. On objects, it is given by

$$\Psi(x)=(x),\quad \text{for } x\neq 0,\quad \text{and}\quad \Psi(0)=\emptyset.$$

Defining Ψ on morphisms requires a little more work. For a fixed representative $x \twoheadleftarrow z \rightarrowtail y$ of a morphism in $Q(\mathscr{C})$, fix additionally an admissible monomorphism $a\rightarrowtail z$ corresponding to $z\twoheadrightarrow x$ and an admissible epimorphism $y\twoheadrightarrow b$ corresponding to $z\rightarrowtail y$. Consider the morphism $\varphi\colon \Psi(a)\circledast\Psi(x)\circledast\Psi(b)\to\Psi(y)$ in $M_{\mathscr{C}}$ given by the underlying flag of the filtration with associated graded represented by the diagram

$$\begin{array}{ccccc}
a & \rightarrowtail & z & \rightarrowtail & y\\
\| & & \big\downarrow\!\!\!\downarrow & & \big\downarrow\!\!\!\downarrow\\
a & & x & & b
\end{array}$$

The following lemma is easily verified by tracing through the definitions.

Lemma 7.29. *The morphism $[\Psi(a), \Psi(b), \varphi]$ in $Q_1(M_{\mathscr{C}})$ defined above is independent of the choice of representative of the morphism $[x \twoheadleftarrow z \rightarrowtail y]$ and of the choice of $a \rightarrowtail z$ and $y \twoheadrightarrow b$. Moreover, the morphism $(\Psi(a), \Psi(b), \varphi)$ in $Q_2(M_{\mathscr{C}})$ is a terminal representative of this morphism.*

In view of this, we set

$$\Psi([x \twoheadleftarrow z \rightarrowtail y]) = [\Psi(a), \Psi(b), \varphi].$$

One can check that this preserves composition and is associative, so that we have indeed defined a functor

$$\Psi\colon Q(\mathscr{C}) \to Q_1(M_{\mathscr{C}}).$$

Remark 7.30. To see that it preserves composition, one needs to identify the composite, i.e. the 5-step filtration given by the diagram of pullbacks below, and then determine a terminal representative of the resulting morphism by identifying the terminal decomposition of it as in the proof of Proposition 7.26. Doing this, we find that it is represented by the subfiltration with associated graded given by picking out the sequence $a' \rightarrowtail z_3 \rightarrowtail x_3$ below, which is easily seen to represent the image of the composite in $Q(\mathscr{C})$.

$$\begin{array}{ccccccccc}
c & \rightarrowtail & a' & \rightarrowtail & z_3 & \rightarrowtail & z_2 & \rightarrowtail & x_3 \\
\| & & \downarrow\!\!\downarrow & \lrcorner & \downarrow\!\!\downarrow & \lrcorner & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\
c & & a & \rightarrowtail & z_1 & \rightarrowtail & x_2 & & d \\
 & & \| & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \\
 & & a & & x_1 & & b & &
\end{array}$$

○

We will apply Quillen's Theorem A to the functor Ψ to show that it induces a homotopy equivalence of geometric realisations. First of all, we make the following observations.

Lemma 7.31. *The functor $\Psi\colon Q(\mathscr{C}) \to Q_1(M_{\mathscr{C}})$ is fully faithful.*

Proof. It is easy to see that it is full, since a 3-step filtration in y whose associated graded is (a, x, b) identifies x as a subquotient of y, which exactly corresponds to a morphism in $Q(\mathscr{C})$. To see that it is faithful, consider diagrams as below and assume that they define the same morphism in $Q_1(M_{\mathscr{C}})$.

$$\begin{array}{ccccc}
a & \rightarrowtail & z & \rightarrowtail & y \\
\| & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\
a & & x & & b
\end{array}
\qquad\qquad
\begin{array}{ccccc}
c & \rightarrowtail & z' & \rightarrowtail & y \\
\| & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\
c & & x & & d
\end{array}$$

Then there exist isomorphisms $\alpha\colon c \to a$ and $\beta\colon d \to b$ such that the composite of the associated graded of the right hand diagram with $(\alpha, \mathrm{id}_x, \beta)$ defines the same filtration with associated graded as the left hand diagram. In particular, there exists a (unique) isomorphism $z \to z'$ which commutes with the morphisms to x and y, i.e. $[x \twoheadleftarrow z \rightarrowtail y] = [x \twoheadleftarrow z' \rightarrowtail y]$. $\square$

The following proposition is an immediate consequence of Proposition 7.26 and Lemma 7.31.

Proposition 7.32. *Let x be an object in $Q(\mathscr{C})$ and $(n_j)_{j\in J}$ an object in $M_{\mathscr{C}}$. Any morphism in $Q_1(M_{\mathscr{C}})$ from $\Psi(x)$ to $(n_j)_{j\in J}$ can be written uniquely as a composite*

$$[(n_j)_{j<j_0},\ (n_j)_{j>j_0},\ \mathrm{id}_{(n_j)_{j\in J}}] \circ \Psi([x \twoheadleftarrow z \rightarrowtail n_{j_0}])$$

for some $j_0 \in J$ and some morphism $[x \twoheadleftarrow z \rightarrowtail n_{j_0}]$ in $Q(\mathscr{C})$.

We now show that the comma category $\Psi \downarrow \alpha$ has contractible geometric realisation for any object α in $Q_1(M_{\mathscr{C}})$.

Let $(m_i)_{i\in I}$ be an object in $M_{\mathscr{C}}$. The comma category $\Psi \downarrow (m_i)$ has objects

$$\Big(x,\ \Big[(a_i),\ (b_i),\ \varphi\colon (a_i) \circledast \Psi(x) \circledast (b_i) \to (m_i)\Big]\Big)$$

where x is an object in $\mathscr{C}$, and $[(a_i), (b_i), \varphi]\colon \Psi(x) \to (m_i)$ is a morphism in $Q_1(M_{\mathscr{C}})$. A morphism in $\Psi \downarrow (m_i)$ is of the form

$$(x,\ [(a_i), (b_i), \varphi]) \xrightarrow{[x \twoheadleftarrow z \rightarrowtail y]} (y,\ [(c_i), (d_i), \psi])$$

where $[x \twoheadleftarrow z \rightarrowtail y]$ is a morphism in $Q(\mathscr{C})$ such that

$$[(a_i), (b_i), \varphi] = [(c_i), (d_i), \psi] \circ \Psi([x \twoheadrightarrow z \rightarrowtail y]).$$

For every $i \in I$, set

$$m_{\le i} := (m_i)_{i\in I_{\le i}}, \quad m_{<i} := (m_i)_{i\in I_{<i}}, \quad m_{\ge i} := (m_i)_{i\in I_{\ge i}}, \quad m_{>i} := (m_i)_{i\in I_{>i}}.$$

For all $i_0 \in I$, consider the full subcategory $C_{i_0} \subseteq \Psi \downarrow (m_i)$ spanned by the objects of the form

$$\Big(x,\ [m_{<i_0}, m_{>i_0}, \mathrm{id}] \circ \Psi([x \twoheadleftarrow z \rightarrowtail m_{i_0}]\Big).$$

The following two lemmas are immediate consequences of Proposition 7.32.

Lemma 7.33. *Let $i_0 \in I$. The object $(m_{i_0}, [m_{<i_0}, m_{>i_0}, \mathrm{id}])$ is a terminal object in C_{i_0}. In particular, C_{i_0} has contractible geometric realisation.*

Lemma 7.34. *The subcategories C_{i_0}, $i_0 \in I$, cover $\Psi \downarrow (m_i)$.*

With this, we can make the final observation.

Proposition 7.35. *The comma category $\Psi \downarrow (m_i)$ has contractible geometric realisation.*

Proof. Note that for any $i, j \in I$,

$$C_i \cap C_j = \begin{cases} (0, [m_{<i_0}, m_{\geq i_0}, \mathrm{id}]) & \text{if } \{i,j\} = \{i_0, i_0+1\}, \\ C_i & \text{if } i = j, \\ \emptyset & \text{else,} \end{cases}$$

where $(0, [m_{<i}, m_{\geq i}, \mathrm{id}])$ denotes the terminal category on this object. Hence, $|\Psi \downarrow (m_i)|$ is contractible by Corollary 2.32, since the nerve of this cover is contractible (alternatively one can use the Nerve Theorem of [Bor48, Page 234], see also [Hat02, Exercise 4G.4 and Corollary 4G.3]). □

Then by Quillen's Theorem A ([Qui73a]) and Proposition 7.35, we have the following result.

Proposition 7.36. *The functor $\Psi\colon Q(\mathscr{C}) \to Q_1(M_{\mathscr{C}})$ induces a homotopy equivalence of geometric realisations.*

Combining this with Corollary 7.27, we have the following.

Corollary 7.37. *The zig-zag*

$$Q(\mathscr{C}) \xrightarrow{\Psi} Q_1(M_{\mathscr{C}}) \xleftarrow{\kappa_{M_{\mathscr{C}}}} Q_2(M_{\mathscr{C}})$$

induces a homotopy equivalence of geometric realisations, $|Q(\mathscr{C})| \simeq |Q_2(M_{\mathscr{C}})|$.

Now we can combine this with the results of Section 6 to show that the monoidal category of flags and associated gradeds produces a model for the algebraic K-theory space. More precisely, we apply Corollary 6.16 which says that for any strict monoidal category M, there is a homotopy equivalence $B|M| \simeq |Q_2(M)|$ between the classifying space of the topological monoid $|M|$ and the geometric realisation of the 2-categorical Q-construction.

Theorem 7.38. *For any category with filtrations $\mathscr{C}$, the geometric realisation of Quillen's Q-construction $Q(\mathscr{C})$ is homotopy equivalent to the classifying space $B|M_{\mathscr{C}}|$ of the topological monoid $|M_{\mathscr{C}}|$. In particular, for any exact category $\mathscr{E}$, the space $\Omega B|M_{\mathscr{E}}|$ is a model for the algebraic K-theory space $K(\mathscr{E})$.*

Appendix A. Nerves and geometric realisations

We give a quick recap of the definitions of double categories and 2-categories, their nerves and their geometric realisations. We only define the notions that we will need and we refrain from specifying the various coherency axioms; these can be found in any good source on the subject (see for example [Lei98]).

A.1. **Double categories.** Let Cat denote the category of small categories.

Definition A.1. A *double category* is a category internal in to Cat: it consists of an object category C_0 and a morphism category C_1 equipped with source and target maps $s, t\colon C_1 \to C_0$, an identity section $e\colon C_0 \to C_1$, and a vertical composition $c\colon C_1 \times_{C_0} C_1 \to C_1$ satisfying the necessary coherency axioms. We write $\mathscr{C} = [C_1 \rightrightarrows C_0]$, omitting the identity and vertical composition functors.

The objects of C_0 are called the *objects* of $\mathscr{C}$, the morphisms of C_0 are call the *horizontal morphisms* of $\mathscr{C}$, the objects of C_1 are called the *vertical morphisms* of $\mathscr{C}$ and the morphisms of C_1 are called 2-*cells*. ◁

Definition A.2. Let $\mathscr{C} = [C_1 \rightrightarrows C_0]$ be a double category. The *transpose* $\mathscr{C}^t$ of $\mathscr{C}$ is the double category obtained by interchanging vertical and horizontal morphisms. ◁

Definition A.3. Let $\mathscr{C} = [C_1 \rightrightarrows C_0]$ and $\mathscr{D} = [D_1 \rightrightarrows D_0]$ be double categories. A *double functor* $F\colon \mathscr{C} \to \mathscr{D}$ is a pair of functors $(F_1\colon C_1 \to D_1, F_0\colon C_0 \to D_0)$ which commute with the source, target, identity and vertical composition functors. ◁

Definition A.4. Let $\mathscr{C} = [C_1 \rightrightarrows C_0]$. The *horizontal nerve* of $\mathscr{C}$ is the simplicial category $N^h_\bullet(\mathscr{C})$ defined as follows: the category $N^h_n(\mathscr{C})$ has object set $N_n(C_0)$ and the morphism set $N_n(C_1)$ with the inherited source and target maps, i.e. a morphism from $c_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} c_n$ to $d_0 \xrightarrow{g_1} \cdots \xrightarrow{g_n} d_n$ is a sequence

$$\varphi_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} \varphi_n$$

in C_1 with $s(\varphi_i) = c_i$, $t(\varphi_i) = d_i$, $s(\alpha_i) = f_i$ and $t(\alpha_i) = g_i$ for all i. Composition is given by vertical composition in $\mathscr{C}$.

The *vertical nerve* of $\mathscr{C}$ is the simplicial category $N^v_\bullet(\mathscr{C}) = N^h_\bullet(\mathscr{C}^t)$ given by the horizontal nerve of the transpose double category. More precisely, the category $N^v_n(\mathscr{C})$ has as objects sequences of vertical morphisms

$$c_0 \xrightarrow{\varphi_1} c_1 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_n} x_n$$

and a morphism from $c_0 \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_n} c_n$ to $d_0 \xrightarrow{\psi_1} \cdots \xrightarrow{\psi_n} d_n$ is a collection of 2-cells

$$\alpha_i\colon \varphi_i \Rightarrow \psi_i$$

satisfying $t(\alpha_i) = s(\alpha_{i+1})$ for all i.

The *double nerve* of $\mathscr{C}$ is the bisimplicial set $N_{\bullet\bullet}(\mathscr{C})$ obtained by applying the usual 1-categorical nerve functor levelwise to the horizontal nerve of $\mathscr{C}$:

$$N_{nk}(\mathscr{C}) = N_n(N^h_k(\mathscr{C})) = N_k(N^v_n(\mathscr{C})).$$

The *geometric realisation* of $\mathscr{C}$, denoted $|\mathscr{C}|$, is the total geometric realisation of $N_{\bullet\bullet}(\mathscr{C})$. ◁

Observation A.5. A double functor induces a continuous map of geometric realisations. ○

A.2. **Strict 2-categories.** We will need to work with 2-categories which are not necessarily small, nor even locally small, that is, the hom-categories need not be small either. However, we may restrict our attention to strict 2-categories.

Definition A.6. A *strict* 2-*category* Q consists of a collection $\operatorname{ob} Q$ of objects and for each pair of objects $a, b \in \operatorname{ob} Q$, a hom-category $Q(a, b)$. It is equipped with composition functors $Q(b, c) \times Q(a, b) \to Q(a, c)$ and identities $\mathrm{id}_a \in Q(a, a)$ for all $a, b, c \in \operatorname{ob} Q$, and these must satisfy the necessary (strict) coherency axioms.

The objects of the hom categories are called *morphisms* and the morphisms are called 2-*cells*. We denote composition of 2-cells along morphisms (within the hom-categories) by $\circ$ and composition of 2-cells along objects (via the composition functors) multiplicatively, i.e. the composite of $\alpha \in Q(a,b)$ and $\beta \in Q(b,c)$ is written $\beta\alpha$.

Let Q, R be strict 2-categories. A *pseudofunctor* $F\colon Q \to R$ consists of the following data:

(1) an assignment $F\colon \operatorname{ob} Q \to \operatorname{ob} R$,
(2) for every pair of objects $a, b \in \operatorname{ob} Q$, a functor $F_{a,b}\colon Q(a,b) \to R(F(a), F(b))$,
(3) for any pair of composable morphism $f\colon a \to b$, $g\colon b \to c$ in Q, an invertible 2-cell $\hat{F}_{f,g}\colon F_{b,c}(g) \circ F_{a,b}(f) \to F_{a,c}(g \circ f)$,
(4) for all objects $a \in \operatorname{ob} Q$, an invertible 2-cell $\hat{F}_a\colon \operatorname{id}_{F(a)} \to F_{a,a}(\operatorname{id}_a)$,

subject to the necessary coherency axioms. ◁

Definition A.7. Let Q, R be strict 2-categories, and let $F, G\colon Q \to R$ be pseudofunctors. An *oplax natural transformation* $\alpha\colon F \Rightarrow G$ consists of the following data:

* for each $a \in \operatorname{ob} Q$, a morphism $\alpha_a\colon F(a) \to G(a)$,
* for all $a, b \in \operatorname{ob} Q$, a natural transformation

$$\hat{\alpha}\colon (\alpha_b)_* \circ F_{a,b} \Rightarrow (\alpha_a)^* \circ G_{a,b}$$

  of functors $Q(a,b) \to R(F(a), G(b))$,

satisfying the following conditions

(1) $\hat{\alpha}_{\operatorname{id}_a} \circ (\operatorname{id}_{\alpha_a} \hat{F}_a) = \hat{G}_a \operatorname{id}_{\alpha_a}$ for all $a \in \operatorname{ob} Q$,
(2) for all composable morphisms $g\colon a \to b$, $f\colon b \to c$ in Q,

$$\hat{\alpha}_{f \circ g} \circ (\operatorname{id}_{\theta_c} \hat{F}_{f,g}) = (\hat{G}_{f,g} \operatorname{id}_{\theta_a}) \circ (\operatorname{id}_{G(f)} \alpha_g) \circ (\alpha_f \operatorname{id}_{F(g)}).$$

A *lax natural transformation* is as above, but with the 2-cells reversed. ◁

Definition A.8. A strict 2-category Q is *small* if the hom-categories are small and $\operatorname{ob} Q$ is a set. It is *essentially small* if it is equivalent to a small 2-category. ◁

Observation A.9. Any small strict 2-category Q can be viewed as a double category with only identity horizontal morphisms:

$$\mathscr{Q} = \Big[\coprod_{a,b \in \operatorname{ob} Q} Q(a,b) \ \rightrightarrows \operatorname{ob} Q \Big]$$

with the obvious structure maps. ○

Definition A.10. Let Q be a small strict 2-category. The *geometric realisation* $|Q|$ of Q is the geometric realisation of $\mathscr{Q}$. ◁

Remark A.11. This definition agrees with the usual definition of the geometric realisation of Q via the double nerve (see for example [BC03]).

There are, however, various options for defining the nerve of a small (strict) 2-category. See [CCG10] for a comparison, in which it is also established that the ten different nerve constructions (of small bicategories) that they consider all have homotopy equivalent geometric realisations. ○

The following proposition is most easily proved by exploiting the fact that there is a natural homotopy equivalence between the geometric realisation of a small 2-category and the geometric realisation of its geometric nerve ([BC03, Theorem 1]), see for example [CCG10, Proposition 7.1].

Proposition A.12. *Lax and oplax natural transformations induce homotopies between the induced maps of geometric realisations.*

Definition A.13. For essentially small strict 2-category Q, we define the *geometric realisation* $|Q|$ of Q to be the geometric realisation of any equivalent small 2-category. ◁

References

[AF17] David Ayala and John Francis. Fibrations of ∞-categories. *arXiv preprint arXiv:1702.02681*, 2017.

[BC03] M. Bullejos and A. M. Cegarra. On the geometry of 2-categories and their classifying spaces. *K-Theory*, 29(3):211–229, 2003.

[BGH18] Clark Barwick, Saul Glasman, and Peter Haine. Exodromy. *arXiv preprint arXiv:1807.03281*, 2018.

[BJ06] Armand Borel and Lizhen Ji. *Compactifications of symmetric and locally symmetric spaces*. Mathematics: Theory & Applications. Birkhäuser Boston, Inc., Boston, MA, 2006.

[Bor48] Karol Borsuk. On the imbedding of systems of compacta in simplicial complexes. *Fund. Math.*, 35:217–234, 1948.

[Bor74] Armand Borel. Stable real cohomology of arithmetic groups. In *Annales scientifiques de l'École Normale Supérieure*, volume 7, pages 235–272, 1974.

[BS73] Armand Borel and Jean-Pierre Serre. Corners and arithmetic groups. *Commentarii Mathematici Helvetici*, 48(1):436–491, 1973.

[BT65] Armand Borel and Jacques Tits. Groupes réductifs. *Inst. Hautes Études Sci. Publ. Math.*, (27):55–150, 1965.

[CCG10] Pilar Carrasco, Antonio M. Cegarra, and Antonio R. Garzón. Nerves and classifying spaces for bicategories. *Algebr. Geom. Topol.*, 10(1):219–274, 2010.

[CL83] Ruth Charney and Ronnie Lee. Cohomology of the Satake compactification. *Topology*, 22(4):389–423, 1983.

[Del74] Pierre Deligne. Théorie de Hodge, III. *Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, 44(1):5–77, 1974.

[DG70] Michel Demazure and Alexandre Grothendieck. *Schémas en groupes: séminaire de géométrie algébrique du Bois Marie, 1962/64, SGA 3*, volume 1. Springer, 1970.

[Fri82] Eric M Friedlander. *Etale homotopy of simplical schemes*, volume 104. Princeton University Press, 1982.

[GH15] David Gepner and Rune Haugseng. Enriched ∞-categories via non-symmetric ∞-operads. *Adv. Math.*, 279:575–716, 2015.

[Gra76] Daniel Grayson. Higher algebraic K-theory. II (after Daniel Quillen). In *Algebraic K-theory (Proc. Conf., Northwestern Univ., Evanston, Ill., 1976)*, pages 217–240. Lecture Notes in Math., Vol. 551, 1976.

[Gro16] Jesper Grodal. Endotrivial modules for finite groups via homotopy theory. *arXiv preprint arXiv:1608.00499*, 2016.

[H+87] James E Humphreys et al. The Steinberg representation. *Bulletin (New Series) of the American Mathematical Society*, 16(2):247–263, 1987.
[Hat02] Allen Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.
[Ji06] Lizhen Ji. Lectures on locally symmetric spaces and arithmetic groups. In *Lie groups and automorphic forms*, volume 37 of *AMS/IP Stud. Adv. Math.*, pages 87–146. Amer. Math. Soc., Providence, RI, 2006.
[JMO94] Stefan Jackowski, James McClure, and Bob Oliver. Homotopy theory of classifying spaces of compact lie groups. In *Algebraic topology and its applications*, pages 81–123. Springer, 1994.
[Lei98] Tom Leinster. Basic bicategories. Preprint:arXiv:9810017, 1998.
[Lei04] Tom Leinster. *Higher operads, higher categories*, volume 298 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2004.
[LS18] Anssi Lahtinen and David Sprehn. Modular characteristic classes for representations over finite fields. *Advances in Mathematics*, 323:1–37, 2018.
[Lur09a] Jacob Lurie. *Higher topos theory.* Princeton University Press, 2009.
[Lur09b] Jacob Lurie. $(\infty, 2)$-Categories and the Goodwillie Calculus I. *arXiv preprint arXiv:0905.0462*, 2009.
[Lur12] Jacob Lurie. Higher algebra, 2012.
[Mos55] G. D. Mostow. Self-adjoint groups. *Ann. of Math. (2)*, 62:44–55, 1955.
[MP87] R James Milgram and Stewart B Priddy. Invariant theory and $H^*(\mathrm{gl}_n(\mathbf{f_p}); \mathbf{f_p})$. *Journal of Pure and Applied Algebra*, 44(1-3):291–302, 1987.
[MS76] Dusa McDuff and Graeme Segal. Homology fibrations and the "group-completion" theorem. *Inventiones mathematicae*, 31(3):279–284, 1976.
[NS90] Yu P Nesterenko and Andrei A Suslin. Homology of the full linear group over a local ring, and Milnor's K-theory. *Mathematics of the USSR-Izvestiya*, 34(1):121, 1990.
[ØJ20] Mikala Ørsnes Jansen. The stratified homotopy type of the reductive Borel–Serre compactification. Preprint: arXiv:2012.10777, 2020.
[Qui72] Daniel Quillen. On the cohomology and K-theory of the general linear groups over a finite field. *Annals of Mathematics*, pages 552–586, 1972.
[Qui73a] Daniel Quillen. Finite generation of the groups K_i of rings of algebraic integers. In *Higher K-theories*, pages 179–198. Springer, 1973.
[Qui73b] Daniel Quillen. Higher algebraic K-theory: I. In *Higher K-theories*, pages 85–147. Springer, 1973.
[Rez01] Charles Rezk. A model for the homotopy theory of homotopy theory. *Transactions of the American Mathematical Society*, 353(3):973–1007, 2001.
[Seg73] Graeme Segal. Configuration-spaces and iterated loop-spaces. *Invent. Math.*, 21:213–221, 1973.
[Seg74] Graeme Segal. Categories and cohomology theories. *Topology*, 13(3):293–312, 1974.
[Sus82] Andrei A Suslin. Stability in algebraic K-theory. In *Algebraic K-Theory*, pages 304–333. Springer, 1982.
[Tam18] Georg Tamme. Excision in algebraic K-theory revisited. *Compositio Mathematica*, 154(9):1801–1814, 2018.
[Vas69] L. N. Vaserstein. On the stabilization of the general linear group over a ring. *Math. USSR Sbornik*, 8:383–400, 1969.
[Wal85] Friedhelm Waldhausen. Algebraic K-theory of spaces. In *Algebraic and geometric topology (New Brunswick, N.J., 1983)*, volume 1126 of *Lecture Notes in Math.*, pages 318–419. Springer, Berlin, 1985.
[Wei13] Charles A Weibel. *The K-book: An introduction to algebraic K-theory*, volume 145. American Mathematical Society Providence, RI, 2013.
[Yua19] Allen Yuan. Integral models for spaces via the higher Frobenius. *arXiv preprint arXiv:1910.00999*, 2019.

[Zuc82] Steven Zucker. L_2-cohomology of warped products and arithmetic groups. *Inventiones mathematicae*, 70(2):169–218, 1982.