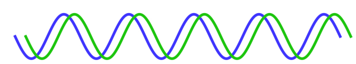
Physics 121H
Fall 2015

Homework #15
23-November-2015
Due Date : 2-December-2015

Reading : Chapters 16 and 17.

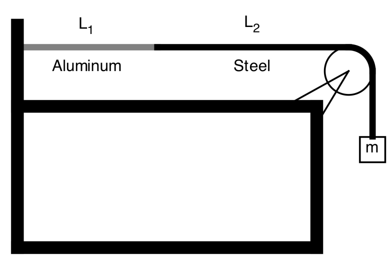
Note: This is a two-week homework assignment that will be worth 2 homework grades. It is due on **Wednesday** 2-Dec, the last class meeting except for the lab on Friday 4-Dec. We will use Wed. 2-Dec to discuss problems from this HW set and review for the final exam.

Reminder: The final exam will be on Wed. 9-Dec-2015 from 7:30-9:20 am. You can bring three one-page 'cheat sheets' for use during the exam. You may not use a calculator during the exam. The exam will cover material from the entire semester, with an emphasis on material covered since the third one-hour exam. Lab 8 is due at the beginning of the final exam session.



1. An infinite stretched string lies along the *x*-axis. The tension in the string is 100N and its density is 10 g/m. When a sinusoidal wave propagates along the string, the time for a particular point on the string to move from maximum displacement to zero is 0.2 sec. The maximum displacement of any point on the string is 0.2 mm. What is the (a) angular frequency and (b) the wavelength of the wave. The picture shows snapshots of the wave at two different times, t_b (blue) and t_g (green). (c) If $t_b > t_g$, write an equation for the displacement from equilibrium of any point along the string at any time. (d) If $t_b < t_g$, write an equation for the displacement of any point on the string at any time. (e) What is the maximum transverse velocity of any point on the string? (f) What is the maximum transverse acceleration of any point on the string?

2. A sinusoidal wave of angular frequency 1200 rad/s and amplitude 3.00 mm is sent along a cord with linear density 2.00 g/m and tension 1200 N. (a) What is the average rate at which energy is transported by the wave to the opposite end of the cord? (b) If simultaneously, an identical wave travels along an adjacent, identical cord, what is the total average rate at which energy is transported to the opposite ends of the two cords by the waves? If, instead, those two waves are sent along the same cord simultaneously, what is the total average rate at which they transport energy when their phase difference is (c) 0, (d) 0.4 π rad, and (e) π rad?

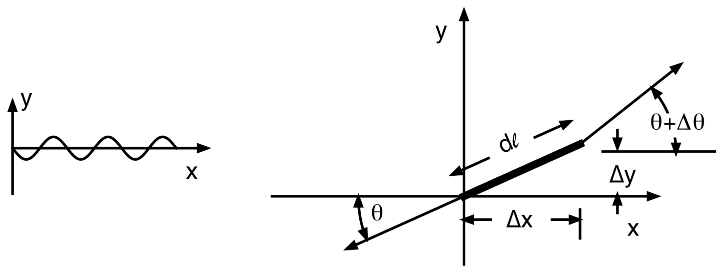


3. In the figure, an aluminum wire, of length $L_1 = 60.0$ cm, cross-sectional area 1.00×10^{-2} cm^2, and density 2.60 g/cm^3,is joined to a steel wire, of density 7.80 g/cm^3 and the same cross-sectional area. The compound wire, loaded with a block of mass $m = 10.0$ kg, is arranged so that the distance L_2 from the joint to the supporting pulley is 86.6 cm. Transverse waves are set up on the

wire by an external source of variable frequency; a node is located at the pulley. (a) Find the lowest frequency that generates a standing wave having the joint as one of the nodes. (b) How many nodes are observed at this frequency?

4. A transverse sinusoidal wave is generated at one end of a long, horizontal string by a bar that moves up and down through a distance of 1.00 cm. The motion is continuous and is repeated regularly 120 times per second. The string has linear density 120 g/m and is kept under a tension of 90.0 N. Find the maximum value of (a) the transverse speed u and (b) the transverse component of the tension τ. (c) Show that the two maximum values calculated above occur at the same phase values for the wave. What is the transverse displacement y of the string at these phases? (d) What is the maximum rate of energy transfer along the string? (e) What is the transverse displacement y when this maximum transfer occurs? (f) What is the minimum rate of energy transfer along the string? (g) What is the transverse displacement y when this minimum transfer occurs?

5. By considering Newton's 2nd law for an infinitessmal segment of a vibrating stretched string, we'll derive the wave equation describing transverse waves on a stretched string. We'll worry about solving this partial differential equation later. We will characterize the string using the tension, T, which is uniform along the string and its uniform mass per unit length, μ. The left-hand image in the figure shows a small part of an infinitely long string along which a wave is propagating. The right-hand image shows a highly magnified view of an infinitessmal segment, $d\ell$, of the string. This segment is so short that it is looks like a straight line. We will imagine that the segment immediately to its left pulls to the left with tension T at an angle directed θ below the x-axis. We will also imagine that the segment immediately to the right pulls to the right with the same tension T but at a slightly different angle wrt the x-axis, $\theta + \Delta\theta$. As we work through this derivation, we will keep in mind that the vibrations are very small. Think about a guitar or violin string. The amplitude of their vibrations are very small. This means that the angles θ (and especially $\Delta\theta$, which is the change in a small quantity) are very small and are << 1 when measured in radians. These facts will be used several times to approximate the values of trigonometric functions. As we work through the derivation, you will be asked to demonstrate several important points. These demonstrations are numbered 6.X. You are supposed to turn in these demonstrations with this week's HW.



Considering forces acting on this segment of the string Newton's 2nd law reads,

$$\sum F_y = T\sin(\theta + \Delta\theta) - T\sin(\theta) = ma_y$$
$$\sum F_x = T\cos(\theta + \Delta\theta) - T\cos(\theta) = ma_x \qquad (1)$$

Use trig identities to expand $\sin(\theta + \Delta\theta)$ and $\cos(\theta + \Delta\theta)$. Next, remember that we are considering *small* vibrations so that $\Delta\theta << \theta << 1$. In this case, we can approximate $\sin \Delta\theta = \Delta\theta$ and $\cos \Delta\theta = 1$; and $\sin \theta = \theta$ and $\cos \theta = 1$.

5.1 With these approximations, show that the x- and y- components of Newton's 2nd law become

$$\sum F_y = T\Delta\theta = \mu\, d\ell\, a_y$$
$$\sum F_x = -T\,\theta\,\Delta\theta = \mu\, d\ell\, a_x \qquad . \qquad (2)$$

Now we will compare the magnitude of a_x to the magnitude of a_y. Note that a_x is a factor of θ smaller than a_y. Since $\theta << 1$, we can safely ignore a_x compared to a_y and will do so from here on. This approximation is justified since we are considering transverse waves, for which the disturbance (in the y-direction) is perpendicular to the propagation (x-) direction.

Now, referring to the figure, $\Delta x = d\ell \cos\theta = d\ell$. So we will replace $d\ell$ with Δx in eqn. (2).

5.2 After rearranging, the y-component of Newton's 2nd law looks like

$$T\frac{\Delta\theta}{\Delta x} = T\frac{\partial\theta}{\partial x} = \mu\, a_y \qquad (3)$$

We have replaced 'Δ' with '∂' since the changes in θ and x are infinitessmal. We use '∂' and not 'd' since θ depends on both x and t. We need to know how θ changes when we change x. Here's how we'll figure it out. Referring to the figure, we see $(\Delta y)/(\Delta x) = (\partial y)/(\partial x) = \tan\theta$. So,

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left(\tan\theta\right) = \frac{1}{\cos^2\theta}\frac{\partial\theta}{\partial x} = \frac{\partial\theta}{\partial x} \qquad . \qquad (4)$$

We have again used $\cos\theta = 1$ in the last step. So, substituting (4) into (3) and remembering that a_y is just the 2nd time derivative of y with respect to time, we have

$$\frac{\partial^2\theta}{\partial x^2} = \frac{\mu}{T}\frac{\partial^2 y}{\partial t^2} \qquad . \qquad (5)$$

Since we discussed in class that solutions to the wave equation were $y(x,t) = y_m \sin(kx \pm \omega t)$ and $v_{ph} = \omega/k$, we can relate the phase velocity of a traveling wave to the constants μ and T.

5.3 Substitute the relations in the preceeding sentence into (5) to relate the phase velocity of a traveling wave to the constants μ and T. You should find $v_{ph} = (T/\mu)^{1/2}$. We now have partial differential equation for wave propagation on a vibrating stretched string.

$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{v_{ph}^2}\frac{\partial^2 y}{\partial t^2} \qquad . \qquad (6)$$

We'll find the solutions for traveling waves in another problem.

6. This is an exercise that demonstrates how to find solutions of the wave equation. We will use a method known as 'separation of variables' that is a powerful technique for solving many of the partial differential equations encountered in physics. Begin by guessing that the solution is a product of functions of only time or position:

$y(x,t) = X(x)T(t)$.

Now, substitute this $y(x,t)$ into the wave equation. Remember that partial differentiation means to differentiate with respect to that variable, keeping everything else constant. Here's an example:

$$\frac{\partial}{\partial t} y(x,t) = \frac{\partial}{\partial t}\left[X(x)T(t)\right] = X(x)\frac{d}{dt}T(t).$$

Note how the partial differential symbol 'changed' to a total differential symbol. Why? Demonstrate that the wave equation can be rewritten as

$$\frac{1}{v_{ph}^2} X(x)\frac{d^2}{dt^2}T(t) = T(t)\frac{d^2}{dx^2}X(x).$$

Now, divide both sides of this equation by $[X(x)T(t)]$ to get

$$\frac{1}{v_{ph}^2}\frac{1}{T(t)}\frac{d^2}{dt^2}T(t) = \frac{1}{X(x)}\frac{d^2}{dx^2}X(x).$$

Note that the LHS (left-hand side) depends only on x and the RHS depends only on t. We say that the variables have been separated. Now comes the central supposition of the separation of variables method: We are setting a function of only t (LHS) equal to a function of only x (RHS). The only way for this to be true for all values of x and t is for these two functions to be equal to the same constant. Let's choose this constant to be $-k^2$. Demonstrate that this leads to the following two equations:

$\frac{d^2}{dx^2}X(x) + k^2 X(x) = 0$ and $\frac{d^2}{dt^2}T(t) + \omega^2 T(t) = 0$. To obtain the second equation, remember that $\omega = kv_{ph}$. We already know how to solve these equations (they are the simple harmonic oscillator equation). By direct substitution, show that

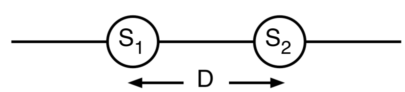
$T(t) = Ae^{i\omega t} + Be^{-i\omega t}$ and $X(x) = Ce^{ikx} + De^{-ikx}$ solve the two equations. Now multiply these together to recover $y(x,t) = X(x)T(t)$. Your solution should have exponentials with arguments $kx+\omega t$ or $kx-\omega t$. We could stop here, but the solutions don't look like right- and left-traveling waves yet. Use Euler's equation, $e^{\pm i\theta} = \cos\theta \pm i\sin\theta$ and re-define the constants multiplying the trigonometric functions to get

$$y(x,t) = \alpha\cos(kx+\omega t) + \beta\sin(kx+\omega t) + \gamma\cos(kx-\omega t) + \delta\sin(kx-\omega t).$$

What are α, β, γ and δ in terms of A, B, C and D? Now, set $\alpha = A\cos(\phi)$, $\beta = -A\sin(\phi)$, $\gamma = B\cos(\theta)$ and $\delta = -B\sin(\theta)$ and use a trigonometric identity to find the general solution, which is a sum of right and left traveling waves with arbitrary amplitudes and phases.

$$y(x,t) = A\cos(kx+\omega t+\phi) + B\cos(kx-\omega t+\theta).$$

7. The figure shows two isotropic point sources of sound, S_1 and S_2. The sources emit waves in phase at wavelength 0.50 m; they are separated by $D = 1.75$ m.



If we move a sound detector along a large circle centered at the midpoint between the sources, at how many points do waves arrive at the detector (a) exactly in phase and (b) exactly out of phase?

8. You have five tuning forks that oscillate at close but different frequencies. What are the (a) maximum and (b) minimum number of different beat frequencies you can produce by sounding the forks two at a time, depending on how the frequencies differ?

9. A sound source *A* and a reflecting surface *B* move directly toward each other. Relative to the air, the speed of source *A* is 29.9 m/s, the speed of surface *B* is 65.8 m/s, and the speed of sound is 329 m/s. The source emits waves at frequency 1200 Hz as measured in the source frame. In the reflector frame, what are the (a) frequency and (b) wavelength of the arriving sound waves? In the source frame, what are the (c) frequency and (d) wavelength of the sound waves reflected back to the source?

10. A plane flies at 1.25 times the speed of sound. Its sonic boom reaches a man on the ground 1.00 min after the plane passes directly overhead. What is the altitude of the plane? Assume the speed of sound to be 330 m/s.

11. Two identical tuning forks can oscillate at 440 Hz. A person is located somewhere on the line between them. Calculate the beat frequency as measured by this individual if (a) she is standing still and the tuning forks move in the same direction along the line at 3.00 m/s, and (b) the tuning forks are stationary and the listener moves along the line at 3.00 m/s.

12. A guitar player tunes the fundamental frequency of a guitar string to 440 Hz. (a) What will be the fundamental frequency if she then increases the tension in the string by 20%? (b) What will it be if, instead, she decreases the length along which the string oscillates by sliding her finger from the tuning key one-third of the way down the string toward the bridge at the lower end?