FREIE UNIVERSITÄT BERLIN
VORLESUNG WS 2016-2017

Étale Cohomology

Lei Zhang

February 8, 2017

INTRODUCTION

In this course we are going to follow closely SGA1 and SGA4 to develop an abstract framwork of fundamental groups and cohomology theory. To do this we first need a generalization of a topological space, and this would be the Grothendieck topology. The notion of sheaves on a topological space would be generalized to the notion of topos. The sheaf cohomology will be replaced by the derived category of a ringed topos. This general framwork serves like a machine: whenever one puts in a concrete Grothendieck topology one gets the corresponding cohomology theory out, and after some further work one may also get the corresponding fundamental group. In this course we are going to put in the étale topology in, and study the output, namely the étale cohomology and the étale fundamental group, which are also the most important output of this machine.

REFERENCES

[AM] M. Atiyah, I.G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley Publishing Company, 1969.

[Fu] L. Fu, *Etale Cohomology Theory*, Revised Edition, World Scientific, 2015.

[Fu2] L. Fu, *Algebraic Geometry*, Tsinghua University Press and Springer-Verlag, 2006.

[Mil] J. S. Milne, *Étale Cohomology*, Princeton University Press, Princeton, New Jersey, 1980.

[Gir] J. Giraud, *Cohomologie non abélienne*, Springer-Verlag, 1971.

[SGA1] A. Grothendieck, *Revêtements Étale et Groupe Fondamental*, Lecture Notes in Math 224, Springer-Verlag (1971).

[SGA4] A. Grothendieck, M. Artin, and J. L. Verdier, *Théorie des Topos et Cohomologie Étale des Schémas*, I, II, III, Lecture Notes in Math 269, 270, 305, Springer-Verlag (1972-1973).

1 FINITE MORPHISMS OF SCHEMES (19/10/2016)

Definition 1. Let A be a commutative ring. We define $\operatorname{Spec}(A)$ to be the set of prime ideas of A. We equip $\operatorname{Spec}(A)$ with a topology (the Zariski topology) by defining a closed subset to be a subset of the form $V(I)$, where $I \subseteq A$ is an ideal and $V(I)$ is the collection of primes of A containing I. One can show that the subsets $\{\operatorname{Spec}(A_f)\}_{f\in A}$ form a topological basis of $\operatorname{Spec}(A)$. We define $\mathcal{O}_{\operatorname{Spec}(A)}$ to be the sheaf of rings sending each open of the form $\operatorname{Spec}(A_f)$ to A_f. Note that to define a sheaf it is enough to define it on a open basis. By abuse of notation we often write $\operatorname{Spec}(A)$ for the pair $(\operatorname{Spec}(A),\mathcal{O}_{\operatorname{Spec}(A)})$.

Definition 2. Let $(X,\mathcal{O}_X)$ be a ringed space, i.e. a pair with X a topological space and $\mathcal{O}_X$ a sheaf of rings on X. The ringed space $(X,\mathcal{O}_X)$ is called a scheme if there exists an open covering $\{U_i\}_{i\in I}$ of X such that $(X,\mathcal{O}_X)|_{U_i} \cong (\operatorname{Spec}(A),\mathcal{O}_{\operatorname{Spec}(A)})$. By abuse of notation we often write X for the pair $(X,\mathcal{O}_X)$.

Definition 3. Let X, Y be two schemes. A morphism of schemes $f: X \to Y$ is just a morphism of ringed spaces $(X,\mathcal{O}_X) \to (Y,\mathcal{O}_Y)$ such that for each $x \in X$ the induced map of rings $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a local homomorphism, i.e. a morphism which sends the maximal ideal of $\mathcal{O}_{Y,f(x)}$ to the maximal ideal of $\mathcal{O}_{X,x}$.

Lemma 1.1. *Let X be a scheme and let A be a commutative ring. We have*

$$\operatorname{Hom}_{\mathrm{Sch}}(X,\operatorname{Spec}(A)) = \operatorname{Hom}_{\mathrm{Ring}}(A,\Gamma(X,\mathcal{O}_X))$$

Proof. We may assume that $X = \operatorname{Spec}(B)$ is an affine scheme. Giving a ring morphism $h: A \to B$ we get a morphism of topological spaces $f: \operatorname{Spec}(B) \to \operatorname{Spec}(A)$. Also for any canonical open subset $\operatorname{Spec}(A_a) \subseteq \operatorname{Spec}(A)$. It is clear that $f^{-1}(\operatorname{Spec}(A_a)) = \operatorname{Spec}(B_{h(a)})$. The maps $A_a \to B_{h(a)}$ for all $a \in A$ define a map $\mathcal{O}_{\operatorname{Spec}(A)} \to f_*\mathcal{O}_{\operatorname{Spec}(B)}$, which together with f define a map of schemes $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$.
On the other hand given a map of schemes $f: \operatorname{Spec}(B) \to \operatorname{Spec}(A)$, we take the global sections of the map of sheaves $\mathcal{O}_{\operatorname{Spec}(A)} \to f_*\mathcal{O}_{\operatorname{Spec}(B)}$. This gives us a map $h: A \to B$. Let $y \in \operatorname{Spec}(B)$ and let $x := f(y) \in \operatorname{Spec}(A)$. Using the map of sheaves we get a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ \downarrow & & \downarrow \\ A_x & \longrightarrow & B_y \end{array}$$

This shows that the map of topological spaces $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ induced by h is precisely f. Using this and the universality of localization we can see easily that the map of sheaves $\mathcal{O}_{\operatorname{Spec}(A)} \to f_*\mathcal{O}_{\operatorname{Spec}(B)}$ also coincides with the one given by f. □

Definition 4. Let $f : X \to Y$ be a morphism of schemes. The map f is called affine if there is an open affine covering $\{V_i\}_{i\in I}$ of Y such that the inverse image $f^{-1}V_i$ is affine for each i.

Lemma 1.2. *A morphism of schemes $f : X \to Y$ is affine iff for any open affine $V \subseteq Y$, $f^{-1}V$ is affine.*

Proof. One quickly reduces the problem to the case when $Y = \mathrm{Spec}\,(A)$ is affine. Suppose that $\{V_i\}_{i\in I}$ is a covering of Y such that all $U_i = f^{-1}V_i$ are affine, where $V_i = \mathrm{Spec}\,(A_{a_i})$ for $a_i \in A$. Let $B = \Gamma(X, \mathscr{O}_X)$. Then we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad h \quad} & \mathrm{Spec}\,(B) \\ & {}_{f}\searrow \quad \swarrow_{g} & \\ & \mathrm{Spec}\,(A) & \end{array}$$

Since $\Gamma(U_i, \mathscr{O}_X) = B_{a_i}$ and U_i is affine, we have $U_i = \mathrm{Spec}\,(B_{a_i})$. At the same time $g^{-1}V_i = \mathrm{Spec}\,(B_{a_i})$. Thus $h|_{U_i}$ is an isomorphism for all i, and is therefore an isomorphism. So X is affine. □

Definition 5. Let $f : X \to Y$ be a morphism of schemes. The map f is called finite if there is an open affine covering $\{V_i = \mathrm{Spec}\,(A_i)\}_{i\in I}$ of Y such that the inverse image $f^{-1}V_i = \mathrm{Spec}\,(B_i)$ is affine for each i, and B_i is a finite A_i-module.

Lemma 1.3. *A morphism of schemes $f : X \to Y$ is finite iff for any open affine $V = \mathrm{Spec}\,(A) \subseteq Y$, $f^{-1}V = \mathrm{Spec}\,(B)$ is affine and B is a finite A-module.*

Proof. Clearly we may assume that $Y = \mathrm{Spec}\,(A)$, $X = \mathrm{Spec}\,(B)$ are affine, and that there exist $\{a_i\}_{1\le i\le n}$ in A which generate the unit ideal of A and B_{a_i} are finitely generated A_{a_i}-modules. We have to show that B is a finitely generated A-module. Now for each i we choose a finite set of generators of B_{a_i} over A_{a_i} which are liftable to B. Then we let i vary, and collect all the lifts of the local generators to get a finite subset $\{b_i\}_{1\le i\le m}$. We claim that this is a set of generators of B over A. Suppose $x \in B$, then set

$$I := \{\, a \in A \mid ax \text{ is a linear combination of } \{b_i\}_{1\le i\le m}\}$$

Clearly I is an ideal of A, and it contains all a_is, so it must be the unit ideal. Hence x is a linear combination of $\{b_i\}_{1\le i\le m}$. □

Lemma 1.4.

1. *A closed immersion is finite;*
2. *The composite of two finite (affine) morphisms is finite;*
3. *Any base change of a finite (an affine) morphism is finite (affine);*

Proof. Let $i : Y \hookrightarrow X$ be a closed embedding of schemes, i.e. a morphism $(Y, \mathscr{O}_Y) \to (X, \mathscr{O}_X)$ in which the topological map embeds Y as a closed subspace of X and the map of sheaves $\mathscr{O}_X \to i_*\mathscr{O}_Y$ is surjective. We have to show that i is finite. To do this we may assume that $X = \mathrm{Spec}\,(A)$ is affine. Since the open subsets $X_a = \mathrm{Spec}\,(A_a) \subseteq Y$, with a run over all elements in A, form a topological basis of X, $X_a \bigcap Y$ also form an open basis of Y. As Y is a scheme, it is covered by open affine subsets $\{U_i\}_{i\in I}$. Now each $U_i = \bigcup_{1\le t_i \le n_i} Y \cap X_{a_{t_i}}$. Since $U_i \to X$ is affine and $Y \cap X_{a_{t_i}}$ is the inverse image of $X_{a_{t_i}}$ in U_i, $Y \cap X_{a_{t_i}}$ is affine. Thus $\{X_{a_{t_i}}\}_{i\in I, 1\le t_i\le n_i}$ is a covering of X whose inverse images are affine. This means i is affine. Let $Y = \mathrm{Spec}(B)$. The condition that $\mathcal{O}_X \to i_*\mathcal{O}_Y$ is surjective implies that $A \to B$ is surjective. Thus B is a finitely generated A module. The rest claims are completely trivial. □

Definition 6. A morphism $f : X \to Y$ is called separated if the diagonal $\Delta : X \to X \times_Y X$ is a closed embedding.

Example 1.5. If f is affine, then f is separated. To see this one just have to reduce the problem to the case that $Y = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(B)$ are affine. In this case Δ corresponds to the surjective ring map $B \otimes_A B \twoheadrightarrow B$, so it is a closed embedding.

Remark 1.6. The notion of separated in algebraic geometry corresponds to that of Hausdorff space in topology. Let X be a topological space then X is Hausdorff if and only if the diagonal $X \to X \times X$ is a closed subspace.

Definition 7. A morphism $f : X \to Y$ is called closed if for any closed subset $D \subseteq X$, $f(D)$ is closed. The map f is called universally closed if for any morphism $T \to Y$ the base change map $X \times_Y T \to T$ is a closed morphism.

Definition 8. A morphism $f : X \to Y$ is called proper if it is separated, of finite type and universally closed.

Lemma 1.7. *Any finite $f : X \to Y$ morphism is proper.*

Proof. We know that f is separated and of finite type. We only have to show that it is universally closed. Since base change of a finite morphism is still finite, we only have to show that finite morphisms are closed. For this we may assume that $Y = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$ and f corresponds to a morphism $\phi : A \to B$. Since ϕ factors as $A \twoheadrightarrow A' \hookrightarrow B$ and $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ is a closed embedding, replacing $\mathrm{Spec}(A)$ by $\mathrm{Spec}(A')$ we may assume that ϕ is injective. Now let $I \subseteq B$ be an ideal, it is enough to show that $f(V(I)) = V(I \bigcap A)$. Clear that $f(V(I)) \subseteq V(I \bigcap A)$, so we have to show the converse. Let $p \in V(I \bigcap A)$. Consider the following diagram

$$\begin{array}{ccc} A & \xrightarrow{\;a\;} & A/(I\bigcap A) \\ \Big\downarrow{\scriptstyle \phi} & & \Big\downarrow{\scriptstyle b} \\ B & \xrightarrow{\;c\;} & B/I \end{array}$$

Let $\bar{p}$ be the ideal of $A/(I \bigcap A)$ such that $a^{-1}(\bar{p}) = p$. As b is integral there is an ideal $\bar{q} \in B/I$ such that $b^{-1}(\bar{q}) = \bar{p}$. Then $q := c^{-1}(\bar{q})$ is an ideal in $V(I)$ such that $\phi^{-1}(q) = p$. □

Lemma 1.8. *Let k be a field. Let $X \to \mathrm{Spec}(k)$ be a morphism of finite type. The the following statements are equivalent.*

1. *X is affine and $\Gamma(X, \mathcal{O}_X)$ is an artinian local ring;*

2. $X \to \operatorname{Spec}(k)$ *is finite;*

3. *The underlying topological space of* X *is discrete.*

Proof. Clear! □

Definition 9. A morphism of schemes $f : X \to Y$ is called quasi-finite if it is of finite type and for any $y \in Y$ the fibre $f^{-1}(y)$ is finite as a set.

Example 1.9. A finite morphism is quasi-finite.

Theorem 1.10. *Let* $f : X \to Y$ *be a morphism of noehterian schemes. Then the following conditions are equivalent.*

1. f *is finite;*

2. f *is proper and affine;*

3. f *is proper and quasi-finite.*

Proof. 1⇒2 and 1⇒3 are clear. 2⇒1 and 3⇒1 goes as follows. For any proper morphism f we have that $f_*\mathcal{O}_X$ is a coherent $\mathcal{O}_Y$-module. This gives us a diagram

$$\begin{array}{ccc} X & \xrightarrow{\ a\ } & X' := \operatorname{Spec}_{\mathcal{O}_Y}(f_*\mathcal{O}_X) \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle b} & \\ & Y & \end{array}$$

where $\mathcal{O}'_X \to a_*\mathcal{O}_X$ is an isomorphism, $b_*\mathcal{O}_{X'} = f_*\mathcal{O}_X$ and b is finite. For example, if f is affine, then a is an isomorphism. So f is finite. □

2 Flat Morphisms (26/10/2016)

Definition 10. Let $f : A \to B$ be a morphism of commutative rings. We say that f is flat if B is a flat A-module, i.e. for any injective map of A-modules $M \hookrightarrow M'$ the induced map $M \otimes_A B \to M' \otimes_A B$ is injective. A map of schemes $f : X \to Y$ is called flat if for any point $x \in X$ the induced map $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,f(x)}$ is flat.

Example 2.1. Let S be a multiplicative subset of A then the localization map $A \to S^{-1}A$ is flat.

Lemma 2.2. *Let* M *be an* A*-module. Then the following are equivalent.*

1. *The module* M *is a flat* A*-module;*

2. *The module* $M_{\mathfrak{p}}$ *is a flat* $A_{\mathfrak{p}}$*-module for all* $\mathfrak{p} \in \operatorname{Spec}(A)$*;*

3. *The module* $M_{\mathfrak{m}}$ *is a flat* $A_{\mathfrak{m}}$*-module for all maximal ideals* $\mathfrak{m}$ *in* A*.*

Proof. The point is that a module M is 0 if and only if $M_{\mathfrak{m}} = 0$ for all maximal ideal $\mathfrak{m}$ of A. So if we have an injective A-linear map $N \hookrightarrow N'$ then $N \otimes_A M \to N' \otimes_A M$ is injective if and only if $N_{\mathfrak{m}} \otimes_{A_{\mathfrak{m}}} M_{\mathfrak{m}} \hookrightarrow N'_{\mathfrak{m}} \otimes_{A_{\mathfrak{m}}} M_{\mathfrak{m}}$ is injective for all maximal ideal $\mathfrak{m}$ in A. □

Lemma 2.3. *Let $f : A \to B$ be a morphism of commutative rings. Then the following are equivalent.*

1. *The map f is flat.*
2. *For any prime ideal $\mathfrak{q} \in B$, $A_{\mathfrak{p}} \to B_{\mathfrak{q}}$ is flat, where $\mathfrak{p} := f^{-1}(\mathfrak{q})$.*
3. *For any maximal ideal $\mathfrak{m} \in B$, $A_{\mathfrak{n}} \to B_{\mathfrak{m}}$ is flat, where $\mathfrak{n} := f^{-1}(\mathfrak{m})$.*

Proof. The point is that a B-module M is 0 if and only if $M_{\mathfrak{m}} = 0$ for all maximal ideal $\mathfrak{m}$ of B. So if we have an injective A-linear map $N \hookrightarrow N'$ then $N \otimes_A B \hookrightarrow N' \otimes_A B$ is injective if and only if $N_{\mathfrak{n}} \otimes_{A_{\mathfrak{n}}} B_{\mathfrak{m}} \hookrightarrow N'_{\mathfrak{n}} \otimes_{A_{\mathfrak{n}}} B_{\mathfrak{m}}$ is injective for all maximal ideal $\mathfrak{m}$ in A. □

In light of 2.3, we have the following:

Lemma 2.4. *A map of commuative rings $A \to B$ is flat if and only if the corresponding map* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *is flat.*

Lemma 2.5. *Let $f : X \to Y$ be a morphism of schemes. Then*

1. *The map f is flat;*
2. *There exists an affine open covering $\{U_i\}_{i \in I}$ of X such that for each $U_i = \mathrm{Spec}(A_i)$ there is an affine open $\mathrm{Spec}(B_i) = V_i \subseteq Y$ satisfying $f(U_i) \subseteq V_i$ and $A_i \to B_i$ is flat.*
3. *For any open affine $\mathrm{Spec}(A) = U \subseteq X$ and any open affine $\mathrm{Spec}(B) = V \subseteq Y$ with $f(U) \subseteq V$ the corresponding map $A \to B$ is flat.*

Proof. "$1 \Leftrightarrow 2 \Leftrightarrow 3$" follows from the definition. □

Lemma 2.6.

1. *An open immersion is flat.*
2. *The composite of flat morphisms is flat.*
3. *base change of flat morphisms is still flat.*

Proof. Leave as an exercise. □

Theorem 2.7. *Let A be a commuative ring, and let M be an A-module of finite presentation. The following statements are equivalent:*

1. *M is a flat A-module;*
2. *M is a projective A-module;*
3. *$M_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module for each $\mathfrak{p} \in \mathrm{Spec}(A)$;*

4. *There exist $\{a_i\}_{i\in I} \subseteq A$ with $\langle a_i\rangle_{i\in I} = A$ such that M_{a_i} is a free A_{a_i}-module;*

Proof. 4⇒3⇒2⇒1 is easy, maybe the only point to think about is that a finitely generated module is projective if and only if it so at each prime ideal. 1⇒2 too technical therefore omitted. 2⇒3: For this we may assume that A is local with maximal ideal $\mathfrak{p}$ and M is an A-module of finite presentation. Since $M/\mathfrak{p}M$ is a finite dimensional $A/\mathfrak{p}$-vector space. Lifting a basis of $M/\mathfrak{p}M$ to M we get a surjection $\phi : A^{\oplus n} \twoheadrightarrow M$ whose mod $\mathfrak{p}$ reduction is an isomorphism. Since M is projective, we get a split exact sequence

$$0 \to N \to A^{\oplus n} \xrightarrow{\phi} M \to 0$$

Thus $N = A^{\oplus n}/M$ is finitely generated. But we have $N/\mathfrak{p}N = 0$ (because $\phi\otimes_A A/\mathfrak{p}$ is an isomorphism). By Nakayma's lemma $N = 0$. So ϕ is an isomorphism. 3⇒4: The point is that if N is a finitely generated A module then $N_\mathfrak{p} = 0$ for some $\mathfrak{p} \in \mathrm{Spec}(A)$ implies that there exists $a \in A$ such that $N_a = 0$. Now choose a morphism $\phi : A^{\oplus n} \to M$ so that the induced map $A_\mathfrak{p}^{\oplus n} \to M_\mathfrak{p}$ is an isomorphism. Since the cokernel is finitely generated, after some localization we may assume that ϕ is surjective. Since M is projective $\mathrm{Ker}(\phi)$ is finitely generated. Thus $\exists\ a \in A$ such that ϕ_a is an isomorphism. □

Theorem 2.8. *Let $f : X \to Y$ be a flat morphism locally of finite presentation, then f is open, i.e. it sends open subsets of X to open subsets of Y.*

Proof. The proof uses Chevalley's theorem on constructible sets. We leave it as an exercise. □

Corollary 2.9. *Let $\mathscr{F}$ be a coherent sheaf on a Noetherian scheme X. Then $\mathscr{F}$ is locally free, i.e. there is an open covering $\{U_i\}_{i\in I}$ of X such that $\mathscr{F}|_{U_i}$ is a free $\mathscr{O}_{U_i}$-module, if and only if $\mathscr{F}$ is flat, i.e. $\mathscr{F}_x$ is a flat $\mathscr{O}_{X,x}$-module for each $x \in X$.*

Definition 11. Let $f : A \to B$ be a morphism of commutative rings. We say that f is faithfully flat if B is a faithfully flat A-module, i.e. for any map of A-modules $M \hookrightarrow M'$ the induced map $M \otimes_A B \to M' \otimes_A B$ is injective if and only if $M \hookrightarrow M'$ is injective.

Lemma 2.10. *Let $f : A \to B$ be a morphism of commutative rings. The following statements are equivalent.*

1. *The map f is faithfully flat;*
2. *The map f is flat and for any non-zero A-module M, $M \otimes_A B$ is non-zero;*
3. *The map f is flat and the induced map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective;*
4. *The map f is flat and any maximal ideal $\mathfrak{p} \in \mathrm{Spec}(A)$ is an inverse image of a maximal ideal $\mathfrak{q} \in \mathrm{Spec}(B)$;*
5. *The map f is flat injective and $B/f(A)$ is a flat A-module.*

Proof. (1) $\Rightarrow$ (2) $M = 0 \Leftrightarrow 0 \to M \to 0$ is exact $\Leftrightarrow 0 \to M \otimes_A B \to 0$ is exact $\Leftrightarrow M \otimes_A B = 0$ (2) $\Rightarrow$ (1) Let $M' \xrightarrow{f} M \xrightarrow{g} M"$ be a sequence of A-modules such that $M' \otimes_A B \to M \otimes_A B \to M" \otimes_A B$ is exact. This means that knowing that the two submodules $\mathrm{Ker}(g)$ and $\mathrm{Im}(f)$ of M are equal after tensoring with B we have to show that $\mathrm{Ker}(g) = \mathrm{Im}(f)$. Since $\mathrm{Im}(f) \otimes_A B = \mathrm{Ker}(g) \otimes_A B$, we have that $(\mathrm{Ker}(g) + \mathrm{Im}(f)/\mathrm{Im}(f)) \otimes_A B = 0$. By (2) we see that $\mathrm{Ker}(g) + \mathrm{Im}(f)/\mathrm{Im}(f) = 0$, i.e. $\mathrm{Ker}(g) + \mathrm{Im}(f) = \mathrm{Im}(f)$. So $\mathrm{Ker}(g) \subseteq \mathrm{Im}(f)$. (2) $\Rightarrow$ (3) Take $\mathfrak{p} \in \mathrm{Spec}(A)$. Since $\mathfrak{p}A_{\mathfrak{p}} \subsetneq A_{\mathfrak{p}} \Leftrightarrow A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \neq 0 \Leftrightarrow A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \otimes_A B \neq 0 \Leftrightarrow B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}} \neq 0 \Leftrightarrow \mathfrak{p}B_{\mathfrak{p}} \subsetneq B_{\mathfrak{p}}$. Now we take any maximal ideal $\mathfrak{q} \in B_{\mathfrak{p}}$ containing $\mathfrak{p}B_{\mathfrak{p}}$. Then the inverse image of $\mathfrak{q}$ under $B \to B_{\mathfrak{p}}$ is a maximal ideal lying over $\mathfrak{p} \in A$. (3) $\Rightarrow$ (4) Trivial. (4) $\Rightarrow$ (2) Let $N \neq 0$ be an A-module. Then $\exists x \in N$ such that $x \neq 0$. Consider the exact sequence $0 \to I \to A \xrightarrow{g} N$, where g is the map sending $1 \to x$. We have $A/I \subseteq N$ and $I \subsetneq A$. But $A/I \otimes_A B \subseteq N \otimes_A B$. It is enough to show that $A/I \otimes_A B \neq 0$, i.e. $IB \neq B$. Take any maximal ideal $A \supsetneq \mathfrak{p} \supset I$. Then there exists $\mathfrak{q} \in \mathrm{Spec}(B)$ such that $f^{-1}(\mathfrak{q}) = \mathfrak{p}$. Thus we have $IB \subseteq \mathfrak{p}B \subseteq \mathfrak{q} \subsetneq B$. (1) $\Rightarrow$ (5) Claim: For any A-module M the sequence $0 \to M \to B \otimes_A M$ is exact. To check this one just has to check the exactness for the pullback

$$0 \to B \otimes_A M \to B \otimes_A B \otimes_A M$$

But the pullback has a retraction, namely the map $B \otimes_A B \otimes_A M \to B \otimes_A M$ sending $b_1 \otimes b_2 \otimes m \mapsto b_1 b_2 \otimes m$. This shows the injectivity. To show that $B/f(A)$ is flat we consider the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I & \longrightarrow & A & \longrightarrow & A/I & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I \otimes_A B & \longrightarrow & B & \longrightarrow & A/I \otimes_A B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & I \otimes_A B/f(A) & \longrightarrow & B/f(A) & \longrightarrow & A/I \otimes_A B/f(A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where I is any ideal of A. Since the upper vertical arrows are all injective, we have that $I \otimes_A B/f(A) \to B/f(A)$ is injective. Thus $B/f(A)$ is flat. (5) $\Rightarrow$ (1) If $B/f(A)$ is flat, then the left arrow in the above diagram is injective, so it follows that $I \otimes_A B \to B$ is injective. Thus f is flat. But this also implies that $A/I \to A/I \otimes_A B$ is injective. So $A/I \neq 0$ implies that $B/IB \neq 0$. Thus f is faithfully flat. □

Definition 12. A morphism of schemes $f : X \to Y$ is called faithfully flat if it is flat and surjective.

Example 2.11. The inclusion $\mathbb{Z} \to \mathbb{Q}$ is flat but not faithfully flat. In fact a localization map $A \to S^{-1}A$ is faithfully flat if and only if it is an isomorphism. The reason is that if the map is faithfully flat then S is not contained in any maximal ideal of A, thus elements in S are invertible. Also open embeddings are faithfully flat if and only if they are isomorphisms. Any morphism from a non-empty scheme X to a spectrum of a field k is faithfully flat.

Definition 13. Let X be a scheme. A sheaf $\mathscr{F}$ of $\mathscr{O}_X$-modules is called quasi-coherent if only if for any point $x \in X$ there is an open neighborhood U of x and an exact sequence

$$\mathscr{O}_U^{\oplus I} \to \mathscr{O}_U^{\oplus J} \to \mathscr{F}|_U \to 0$$

A sheaf $\mathscr{F}$ of $\mathscr{O}_X$-modules is called of finite type (resp. of finite presentation) if only if for any point $x \in X$ there is an open neighborhood U of x and an exact sequence

$$\mathscr{O}_U^{\oplus n} \to \mathscr{F}|_U \to 0 \qquad (\text{ resp. } \mathscr{O}_U^{\oplus m} \to \mathscr{O}_U^{\oplus n} \to \mathscr{F}|_U \to 0\,)$$

We denote the category of quasi-coherent sheaves by $\mathrm{Qcoh}(X)$ and the category of sheaves of finite type by $\mathrm{Coh}(X)$.

Definition 14. Let $f : X \to Y$ be a morphism of schemes. Let $\mathrm{Qcoh}(f)$ be the category whose objects are pairs $(\mathscr{F}, \phi)$, where $\mathscr{F}$ is in $\mathrm{Qcoh}(X)$, ϕ is an isomorphism $p_1^*\mathscr{F} \xrightarrow{\cong} p_2^*\mathscr{F}$ satisfying the cocycle condition $p_{23}^*\phi \circ p_{12}^*\phi = p_{13}^*\phi$, where $p_1, p_2, p_{12}, p_{23}, p_{13}$ are the projection maps:

$$X \times_Y X \times_Y X \overset{p_{12}, p_{23}, p_{13}}{\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}} X \times_Y X \times_Y \underset{p_2}{\overset{p_1}{\rightrightarrows}} X \xrightarrow{f} Y$$

The morphisms in $\mathrm{Qcoh}(f)$ are morphisms in $\mathrm{Qcoh}(X)$ which are compatible with the given isomorphisms in a natural way. The isomorphism ϕ with the cocycle condition is called descent data of $\mathscr{F}$.

Theorem 2.12. *Let $f : X \to Y$ be a faithfully flat and quasi-compact or faitfhully flat and locally of finite presentation morphism of schemes. Then there is a canonical equivalence between* $\mathrm{Qcoh}(Y)$ *and* $\mathrm{Qcoh}(f)$.

Proof. By some general non-sense, for example in Notes on Grothendieck topologies, fibered categories and descent theory, Chapter 4, Lemma 4.25, pp. 89, we may assume that $X = \mathrm{Spec}\,(B)$ and $Y = \mathrm{Spec}\,(A)$ are affine. In this case $\mathrm{Qcoh}(f)$ is equivalent to the category $\mathrm{Mod}_{A\to B}$ of B-modules with descent data. Given $M \in \mathrm{Mod}_A$ we get a B-module $B \otimes_A M$ with an isomorphism $\phi : (B \otimes_A M) \otimes_A B \to B \otimes_A (B \otimes_A M)$ sending $b_1 \otimes m \otimes b_2 \mapsto b_1 \otimes b_2 \otimes m$. It is an easy calculation that ϕ satisfies cocycle condition. In this way we get a functor $F : \mathrm{Mod}_A \to \mathrm{Mod}_{A\to B}$. Conversely given a pair (N, ϕ), where N is a B-module and $\phi : N \otimes_A B \to B \otimes_A N$ is a $B \otimes_A B$-linear isomorphism, we define N^ϕ to be the kernel of $N \xrightarrow{\phi\circ\lambda_1 - \lambda_2} B \otimes_A N$, where $\lambda_1 : N \to N \otimes_A B$ sends $n \mapsto n \otimes 1$ and $\lambda_2 : N \to B \otimes_A N$ sends $n \mapsto 1 \otimes n$. This defines a functor $G : \mathrm{Mod}_{A\to B} \to \mathrm{Mod}_A$. Then one can check that F and G are quasi-inverse to each other. For details see Notes on Grothendieck topologies, fibered categories and descent theory, Chapter 4, Theorem 4.21, pp. 80. □

Corollary 2.13. *Let $f : X \to Y$ be a faithfully flat and quasi-compact or faitfhully flat and locally of finite presentation morphism of schemes. Then the pullback functor $F : \mathrm{Qcoh}(Y) \to \mathrm{Qcoh}(f)$ induces an equivalence between* $\mathrm{Coh}(Y)$ *and* $\mathrm{Coh}(f)$.

Proof. Clearly F sends the full subcategory $\mathrm{Coh}(Y)$ to $\mathrm{Coh}(f)$. We have to show that if $\mathscr{F} \in \mathrm{Qcoh}(Y)$ and if $F(\mathscr{F}) \in \mathrm{Coh}(f)$, then $\mathscr{F} \in \mathrm{Coh}(f)$. For this we may assume that $Y = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(B)$. Suppose M is an A-module and $M \otimes_A B$ is finitely generated, say by a family $\{n_i = \sum_{j\in J} m_{ij} \otimes b_{ij}\}_{i\in I}$, where I, J are finite sets. Then $\{m_{ij} \otimes 1\}_{i\in I, j\in J}$ also generates $M \otimes_A B$. Let $\lambda : A^{\oplus I\times J} \to M$ be a map sending the free basis $e_{ij} \mapsto m_{ij}$. Then we have $\phi \otimes_A B$ is surjective. Thus ϕ is also surjective, so M is finitely generated. □

Corollary 2.14. *Let $f : X \to Y$ be a faithfully flat and quasi-compact or faitfhully flat and locally of finite presentation morphism of schemes. Then the pullback functor f^* induces an equivalence between* $\mathrm{Aff}(Y)$ *(resp.* $\mathrm{Fin}(Y)$*) and* $\mathrm{Aff}(f)$ *(resp.* $\mathrm{Fin}(Y)$*), where* $\mathrm{Aff}(Y)$ *(resp.* $\mathrm{Fin}(Y)$*) is the category of affine (resp. finite) schemes on Y and* $\mathrm{Aff}(f)$ *(resp.* $\mathrm{Fin}(Y)$*) is the category of affine (resp. finite) schemes on X equipped with descent data.*

Remark 2.15. The category $\mathrm{Aff}(Y)$ (resp. $\mathrm{Fin}(Y)$) is defined to be the category of affine (resp. finite) morphisms with target Y. The category $\mathrm{Aff}(f)$ (resp. $\mathrm{Fin}(f)$) is the category of pairs $(X' \to X, \phi)$, where $X' \to X$ is an affine (resp. finite) morphism and $\phi : p_1^* X' \to p_2^* X'$ is a morphism of $X \times_Y X$-schemes satisfying cocycle condition. One can check that $\mathrm{Aff}(Y)$ is equivalent to the category of quasi-coherent $\mathscr{O}_Y$-algebras (see the exercise). This is the starting point of the proof of this corollary.

Proof. Since we have that $\mathrm{Aff}(Y)$ is equivalent to the category of quasi-coherent $\mathscr{O}_Y$-algebras, i.e. the category with the following data $\{\mathscr{A} \in \mathrm{Qcoh}(Y), \mathscr{A} \otimes \mathscr{A} \xrightarrow{m_A} \mathscr{A}, \mathscr{O}_Y \xrightarrow{u_A} \mathscr{A}\}$, where $m_A, u_A \in \mathrm{Qcoh}(Y)$ satisfies obvious conditions which make $\mathscr{A}$ an $\mathscr{O}_Y$-algebra. By the equivalence between $\mathrm{Qcoh}(Y)$ and $\mathrm{Qcoh}(f)$ we see that $\{\mathscr{A} \in \mathrm{Qcoh}(Y), \mathscr{A} \otimes \mathscr{A} \xrightarrow{m_A} \mathscr{A}, \mathscr{O}_Y \xrightarrow{u_A} \mathscr{A}\}$ is equivalent to the category $\{\mathscr{B} \in \mathrm{Qcoh}(X), \mathscr{B} \otimes \mathscr{B} \xrightarrow{m_B} \mathscr{B}, \mathscr{O}_Y \xrightarrow{u_B} \mathscr{A}\}$ with $m_B, u_B \in \mathrm{Qcoh}(f)$ satisfies obvious conditions which make $\mathscr{B}$ an $\mathscr{O}_X$-algebra. Now one checks that the category $\{\mathscr{B} \in \mathrm{Qcoh}(X), \mathscr{B} \otimes \mathscr{B} \xrightarrow{m_B} \mathscr{B}, \mathscr{O}_Y \xrightarrow{u_B} \mathscr{A}\}$ is literally the written up version of $\mathrm{Aff}(f)$. □

Theorem 2.16. *Let $f : X \to Y$ be a morphism of schemes, and let $Y' \to Y$ be a faithfully flat and quasi-compact or faitfhully flat and locally of finite presentation morphism of schemes. If the base change $f' : X' = X \times_Y Y' \to Y'$ has one of the following properties:*

1. *separated;*
2. *quasi-compact;*
3. *locally of finite presentation;*
4. *proper;*
5. *affine;*
6. *finite;*
7. *flat;*
8. *smooth;*

9. *unramified;*

10. *étale;*

11. *an open embedding;*

12. *a closed embedding;*

13. *injective;*

14. *surjecgtive,*

then f has the same property.

Proof. You can find the proof in ***EGA IV2***, *Proposition 2.7.1*. Assuming 3, let's do 1,2,4,7 11, 12, 13, 14 as examples. First a small lemma:

Lemma 2.17. *For any some subset $T \subseteq X$, we have $g^{-1}(f(T)) = f'(h^{-1}(T))$ inside Y'.*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

Proof. Clearly $g(f'(h^{-1}(T))) \subseteq f(T)$. This implies that $f'(h^{-1}(T)) \subseteq g^{-1}(f(T))$. Suppose that $y' \in g^{-1}(f(T))$, $t \in T$ such that $y' = f(t)$. We have to show that $\exists\, x' \in X'$ such that $f'(x') = y'$ and $h(x') = t$. We have maps $\mathrm{Spec}\,(\kappa(t)) \to X$, $\mathrm{Spec}\,(\kappa(y')) \to Y$. Now

$$\mathrm{Spec}\,(\kappa(t)) \times_Y \mathrm{Spec}\,(\kappa(y')) = \mathrm{Spec}\,(\kappa(t)) \times_{\kappa(y)} \mathrm{Spec}\,(\kappa(y')) \neq \emptyset$$

Thus any point $x' \in \mathrm{Spec}\,(\kappa(t)) \times_Y \mathrm{Spec}\,(\kappa(y'))$ would do the job. □

Let's prove 13: Take $x_1, x_2 \in X$ and assume that $f(x_1) = f(x_2) = y$. Choose $y' \in Y'$ so that $g(y') = y$. If we take T to be $\{x_1\}$, then we get $y' \in f'(h^{-1}(T))$, i.e. there exists $x_1' \in X'$ such that $y' = f'(x_1')$ and $h(x_1') = x_1$. To the same for x_2, we find x_2' such that $y' = f'(x_2')$ and $h(x_2') = x_2$. Since f' is injective, we must have $x_1' = x_2'$. Therefore we have $x_1 = h(x_1') = h(x_2') = x_2$.

14 is a direct check: $g \circ f'$ is surjective implies that f is surjective.

Let's show that f' is universally closed implies that f is universally closed. Clearly, to prove that it is enough to show that f' is closed implies that f is closed. Let $T \subseteq X$ be a closed subset. By 2.17 we have $f'(h^{-1}(T)) = g^{-1}(f(T))$. By the continuity of h and the assumption that f' is closed, we have that $g^{-1}(f(T)) = f'(h^{-1}(T))$ is closed in Y'. We have $Y \setminus f(T) = g(g^{-1}(Y \setminus f(T))) = g(X \setminus g^{-1}(f(T)))$. Since g is flat, by ***EGA IV2***, *Proposition 2.4.6* and our Ex 2.3, we see that $Y \setminus f(T)$ is open, i.e. $f(T)$ is closed.

Let's show 7. For this we only need to show that if we have successive maps $X' \xrightarrow{h} X \xrightarrow{f} Y$, and if we know that h is faithfully flat and $f \circ h$ is flat then f is flat. For this we may assume that $X' = \operatorname{Spec}(B')$, $X = \operatorname{Spec}(B)$ and $Y = \operatorname{Spec}(A)$. Let $0 \to M' \to M$ be an exact sequence of A-modules. Then $0 \to M' \otimes_A B' \to M' \otimes_A B'$ is exact. But $M \otimes_A B' = M \otimes_A B \otimes_B B'$ ($M \otimes_A B' = M \otimes_A B \otimes_B B'$), and $B \to B'$ is faithfully flat, thus $0 \to M' \otimes_A B \to M' \otimes_A B$ is exact.

Now we are in the position to prove 11 (resp. 12). Since by 3 and 7 (resp. by the proof for universally closed morphisms) f is an open map (resp. a closed map). As f is injective by 13, f embeds X as an open (resp. a closed) subset of Y. So we only have to show that for each point $x \in X$, the map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,y}$ is an isomorphism (resp. surjective). The problem is local, we may assume that $Y = \operatorname{Spec}(A)$, $X = \operatorname{Spec}(B)$, and $Y' = \operatorname{Spec}(A')$. Assume further that A, B, A' are local rings. Now the claim follows as $A \to B$ is an isomorphism (resp. surjective) if and only if $A' \to A' \otimes_A B$ is an isomorphism (resp. surjective).

Let's show 1: For this we just have to consider the cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{\Delta'} & X' \times_{Y'} X' \\ \downarrow & & \downarrow \\ X & \xrightarrow{\Delta} & X \times_Y X \end{array}$$

Using 12 we see that Δ' is a closed embedding implies that Δ is a closed embedding.

4 follows from 1, 3 and the proof for univerally closed maps.

Let's show 2: For this we may assume that $Y = \operatorname{Spec}(A)$ is affine. We can also take a finitely many open affines $\{U_i\}_{1 \le i \le n}$ of Y' so that $\bigcup_{0 \le i \le n} g(U_i) = Y$. Since quasi-compact maps are stable under base change, replacing Y' by $\coprod_{0 \le i \le n} U_i$ we may assume that Y' is affine. Then by Ex 3.2 X' is quasi-compact. But $X' \twoheadrightarrow X$ is surjective, so X is quasi-compact. □

Corollary 2.18. *Let $f : X \to Y$ be a morphism of S-schemes. Let $S' \to S$ be a faithfully flat and quasi-compact or faitfhully flat and locally of finite presentation morphism of schemes. Then $f' : X \times_S S' \to Y \times_S S'$ is an isomorphism if and only if f is an isomorphism.*

Proof. The problem is local on Y, we may assume that S, Y are affine. Then by 2.16 we see that X is also affine. Then the claim follows from 2.12. □

3 Étale Morphisms (02/11/2016)

Definition 15. A morphism $f : X \to Y$ is called unramified if it is locally of finite presentation and if for any $x \in X$, $\mathcal{O}_{X,x}/\mathfrak{m}_{Y,f(x)}\mathcal{O}_{X,x}$ is a field, where $\mathfrak{m}_{Y,f(x)} \subseteq \mathcal{O}_{Y,f(x)}$ is the maximal ideal, and the residue field extension $\kappa(y) \subseteq \kappa(x)$ is a separable field extension.

Remark 3.1. In other words a locally of finite presentation morphism f is unramified iff $\mathfrak{m}_{Y,f(x)}\mathcal{O}_{X,x} = \mathfrak{m}_{X,x}$ and $\kappa(y) \subseteq \kappa(x)$ is a separable field extension. By Hilbert's Nullstellensatz $\kappa(y) \subseteq \kappa(x)$ is automatically a finite field extension.

Proposition 3.2. *Let $f : X \to Y$ be a locally of finite presentation morphism of schemes. Then the following are equivalent.*

1. *The map f is unramified;*
2. *For all $y \in Y$, $f^{-1}(y) \to \mathrm{Spec}(\kappa(y))$ is unramified;*
3. *For all $y \in Y$, $f^{-1}(\bar{y}) \to \mathrm{Spec}(\overline{\kappa(y)})$ is unramified;*
4. *For all $y \in Y$, $f^{-1}(y)$ is a disjoint union of $\mathrm{Spec}(K_i)$ where K_i/k is a finite separable extension;*
5. *For all $y \in Y$, $f^{-1}(\bar{y})$ is a disjoint union of $\mathrm{Spec}(\overline{\kappa(y)})$.*

Here $f^{-1}(y) := \mathrm{Spec}(\kappa(y)) \times_Y X$ and $f^{-1}(\bar{y}) := \mathrm{Spec}(\overline{\kappa(y)}) \times_Y X$.

Proof. Step 1: We first reduce to the case when $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(A)$.

Step 2: Assume that A is a local ring with maximal ideal $\mathfrak{p}$.

Step 3: Replacing A by $A/\mathfrak{p}$ and B by $B/\mathfrak{p}B$ we may assume that A is a field.

Step 4: In this case we have: f is unramified $\Leftrightarrow$ the localization at each maximal ideal $\mathfrak{m}$ of B is a field which is a field separable extension of $k \Leftrightarrow B$ is a reduced Artinian ring all of whose residue fields are finite separable extensions of $k \Leftrightarrow$ (2), (3), (4), (5). Everything follows from Ex 1.5. □

Proposition 3.3. *Let $f : X \to Y$ be a morphism locally of finite presentation. Then the following are equivalent.*

1. *f is unramified;*
2. *$\Omega^1_{X/Y} = 0$;*
3. *$\Delta : X \to X \times_Y X$ is an open embedding.*

Proof. $1 \Rightarrow 2$ Step 1. Assume $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(A)$. We can do this because unramified is a local property and $\Omega^1_{X/Y}$ can be computed locally, i.e. for $V \subseteq Y$, $U \subseteq X$ such that $f(U) \subseteq Y$ we have $\Omega^1_{X/Y}|_U = \Omega_{U/V}$.
Step 2. Assume that A is a local ring with maximal ideal $\mathfrak{m}$. We can do this because for any $\mathfrak{p}\mathrm{Spec}(A)$, we have $(\Omega_{B/A})_{\mathfrak{p}} = \Omega_{B_{\mathfrak{p}}/A_{\mathfrak{p}}}$.
Step 3. Assume that A is a field. We can do this because for any A-algebra C we have $\Omega_{B/A} \otimes_A C = \Omega_{B \otimes C/C}$. If C is chosen to be $A/\mathfrak{m}$, then $\Omega_{B \otimes C/C} = 0$ would imply that $\Omega_{B/A} = \mathfrak{m}\Omega_{B/A}$. Since B is finitely generated over A, $\Omega^1_{B/A}$ is of finite type, thus by Nakayama $\Omega_{B/A} = 0$.
Step 4. Assume that B is a field. We can do this because in this case by 3.2 B is a finite product of fields which are finite separable extensions of A.

Step 5. We have $\Omega^1_{K/k} = 0$ if K/k is finite separable. Suppose that $K = k[X]/(f(X))$ with $f(X)$ a separable polynomial. We have the following exact sequence

$$(f(X))/(f(X))^2 \to K \otimes_{k[X]} \Omega^1_{k[X]/k} \to \Omega^1_{K/k} \to 0$$

Here $K \otimes_{k[X]} \Omega^1_{k[X]/k}$ is a 1-dimensional K-vector space generated by dX. The image of $f(X)$ in $K \otimes_{k[X]} \Omega^1_{k[X]/k}$ is $f'(X)dX$, so the image of $(f(X))/(f(X))^2$ is the subspace of KdX generated by $f'(X)dX$. But $f'(X)$ is invertible in $K = k[X]/(f(X))$, as $f'(X)$ is a separable polynomial. Thus $(f(X))/(f(X))^2 \to K \otimes_{k[X]} \Omega^1_{k[X]/k}$ is surjective and $\Omega^1_{K/k}$ is therefore 0 by the exact sequence.
$2 \Rightarrow 3$ Recall the definition of $\Omega^1_{X/Y}$. We have the exact sequence

$$0 \to \mathscr{I} \to \Delta^{-1}\mathscr{O}_{X \times_Y X} \to \mathscr{O}_X \to 0$$

Since $f : X \to Y$ is finitely presented, $\mathscr{I}$ is finitely generated. By Nakayama $\mathscr{I}/\mathscr{I}^2 = 0$ implies that $\mathscr{I} = 0$. This implies that for any $x \in X$, $\mathscr{O}_{X\times_Y X,\Delta(x)} \to \mathscr{O}_{X,x}$ is an isomorphism. In particular Δ is flat. The claim now follows from the following general phenomenon:

Lemma 3.4. *If A is a ring, and if $I \subseteq A$ is a finitely generated ideal, then $A \to A/I$ is flat iff* $\mathrm{Spec}(A/I) \to \mathrm{Spec}(A)$ *is an open embedding.*

Proof. Consider the sequence $0 \to I \to A$. Since $A \to A/I$ is flat, $0 \to I/I^2 \to A/I$ is exact. Since $I/I^2 \to A/I$ is the 0 map, $I = I^2$. Thus for any $\mathfrak{p} \in \mathrm{Spec}(A/I)$, we have $I_\mathfrak{p} = I_p p^2$. Thus $I_\mathfrak{p} = 0$. If $\mathfrak{p} \notin \mathrm{Spec}(A/I)$ then $I_\mathfrak{p} = A$. But $\mathrm{Spec}(A/I) = \{\mathfrak{p} \in \mathrm{Spec}(A) | I_\mathfrak{p} = 0\}$ is also open as I is finitely generated. Thus $\mathrm{Spec}(A/I) \subseteq \mathrm{Spec}(A)$ is an open embedding. □

$3 \Rightarrow 1$ We may assume that $Y = \mathrm{Spec}(k)$ and $k = \bar{k}$ is algebraically closed. We need to show that X is a disjoint union of $\mathrm{Spec}(k)$. Replacing X by a connected component, we may assume that X is connected. Now take $x : \mathrm{Spec}(k) \to X$ a point, and consider the following cartesian diagram

$$\begin{array}{ccc} \mathrm{Spec}(k) & \xrightarrow{x} & X \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle (\mathrm{id},x)} \\ X & \xrightarrow{\Delta} & X \times_k X \end{array}$$

This diagram implies that x is an open embedding. Since X is connected and the point x is both open and closed, X is a one point scheme. Thus $x : \mathrm{Spec}(k) \to X$ is an isomorphism. □

Definition 16. A morphism $f : X \to Y$ is called étale if f is unramified and flat.

Example 3.5.
1. A closed embedding with a finitely generated ideal sheaf is unramified and not étale in general.
2. An open embedding is étale.
3. A disjoint union: $\coprod_{i\in I} X \to X$ is étale.
4. A finite separable field extension is étale.

Lemma 3.6. 1. *The composition of unramified (resp. étale) morphisms is unramified (resp. étale).*

2. *The base change of an unramified (resp. étale) is still unramified (resp. étale).*

3. *Let $f : X \to Y$, $g : Y \to X$ be morphisms of schemes. If $g \circ f$ is unramified (resp. étale) and g is arbitrary (resp. unramified), then f is unramified (resp. étale).*

Proof. See exercises. □

Lemma 3.7. *Any finite étale surjective morphism $f : X \to Y$ there is a finite étale surjective $Y' \to Y$ such that $X' := X \times_Y Y'$ is isomorphic to $\coprod_{1 \leq i \leq n} Y'$ as an Y'-scheme.*

Proof. See the exercise. □

Lemma 3.8. *Let $f, g : X \to Y$ be two morphisms between two S-schemes. If X is connected and Y/S is étale separated, then $f = g$ if and only if $\exists$ a geometric point $x : \mathrm{Spec}\,(k) \to X$ such that the two compositions*

$$\mathrm{Spec}\,(\bar{k}) \xrightarrow{x} X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

are equal.

Proof. Consider the following diagram

$$\begin{array}{ccc} X & \xrightarrow{\Gamma_f (\text{resp. } \Gamma_g)} & X \times_S Y \\ {\scriptstyle f (\text{resp. } g)} \downarrow & & \downarrow {\scriptstyle (f, \mathrm{id}) (\text{resp. } (g, \mathrm{id}))} \\ Y & \xrightarrow{\Delta} & Y \times_S Y \end{array}$$

Since Δ is a both open and closed embedding, Γ_f (resp. Γ_g) is also a both open and closed embedding. The point x ensures that $\Gamma_f(X) = \Gamma_g(X)$ as two connected components of $X \times_S Y$. Let Γ denote $\Gamma_f(X) = \Gamma_g(X)$. Since Γ_f is an isomorphism from $X \to \Gamma$, the restriction map $\phi : \Gamma \subseteq X \times_S Y \to X$ is an isomorphism, and it is also the inverse of Γ_f. The same holds for Γ_g. Thus $\Gamma_f = \Gamma_g$ as they are both the inverse of ϕ. Finally $f = g$ because they are all compositions of $\Gamma_f = \Gamma_g$ with the second projection $X \times_S Y \to Y$. □

Definition 17. Let $f : X \to Y$ be a morphism of schemes. We call f an étale cover or a cover if it is finite étale and surjective.

Remark 3.9. If Y is connected and f is finite étale, then f is a cover if and only if X is non-empty.

Lemma 3.10. *Let $f : X \to Y$ be a finite étale morphism of schemes, and let Y be a connected scheme. Then for each two geometric point $y_1, y_2 : \mathrm{Spec}\,(\Omega) \to Y$ with Ω algebraically closed field, we have*

$$\#(f^{-1}(y_1)) = \#(f^{-1}(y_2)) = \dim_\Omega H^0(\mathcal{O}_{f^{-1}(y_1)}) = \dim_\Omega H^0(\mathcal{O}_{f^{-1}(y_2)}) = \mathrm{rank}_{\mathcal{O}_Y}(f_* \mathcal{O}_X)$$

Remark 3.11. Since f is finite, $f_*\mathcal{O}_X$ is a finite $\mathcal{O}_Y$-module. As f is flat and locally of finite presentation, by 2.7, $f_*\mathcal{O}_X$ is a locally free $\mathcal{O}_Y$-module of finite rank. Since Y is connected, the rank is constant.

Proof. It is enough to show that $\dim_\Omega(H^0(\mathcal{O}_{f^{-1}(y_1)})) = \operatorname{rank}_{\mathcal{O}_Y}(f_*\mathcal{O}_X)$. Now consider the following cartesian diagram

$$\begin{array}{ccc} f^{-1}(y_1) & \xrightarrow{g} & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ \operatorname{Spec}(\Omega) & \xrightarrow{y_1} & Y \end{array}$$

We have a canonical morphism $y_1^* f_*\mathcal{O}_X \to h^* f_*\mathcal{O}_X$, and it is an isomorphism because f is affine. But then

$$\operatorname{rank}_{\mathcal{O}_Y}(f_*\mathcal{O}_X) = \operatorname{rank}_\Omega(y_1^* f_*\mathcal{O}_Y) = \dim_\Omega(y_1^* f_*\mathcal{O}_Y) = \dim_\Omega(h^* f_*\mathcal{O}_X) = \dim_\Omega(H^0(\mathcal{O}_{f^{-1}(y_1)}))$$

□

Definition 18. A finite étale morphism $f : X \to Y$ is called of degree $n \in \mathbb{N}$ if all its geometric fibres have cardinality n.

Remark 3.12. A finite étale morphism of degree 0 is the empty morphism. A degree 1 morphism $f : X \to Y$ is an isomorphism: Assume that $Y = \operatorname{Spec}(A)$ then $Y = \operatorname{Spec}(B)$. Since f is faithfully flat we know that $A \to B$ is injective. But by the assumption, i.e. $\operatorname{rank}_{\mathcal{O}_Y}(f_*\mathcal{O}_X)$ we know that B is generated by $1 \in B$ as an A-module, so $A \to B$ is also surjective.

Corollary 3.13. *Let Y be a connected locally Noetherian scheme, and let $y : \operatorname{Spec}(\Omega) \to Y$ be a geometric point. Suppose that we have a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{g} & X' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f'} & \\ & Y & \end{array}$$

where f, f' are étale covers. If g induces an isomorphism $f^{-1}(y) \to f'^{-1}(y)$, then g is an isomorphism.

Proof. We may assume that X' is connected. In this case g is also finite étale and the degree is 1 (because $f^{-1}(y) \to f'^{-1}(y)$ is an isomorphism). Therefore g is an isomorphism. □

Definition 19. A morphism $f : X \to Y$ is called formally unramified (resp. formally étale) if and only if for any diagram

$$\begin{array}{ccc} T_0 & \xrightarrow{a} & X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\ T & \xrightarrow{b} & Y \end{array}$$

where T_0, T are schemes affine over Y and i is closed embedding whose ideal sheaf is square 0, there exists at most one (resp. exactly one) broken arrow which makes all the triangles commutative.

Remark 3.14. It is easy to see that formally unramified and formally étale are locally properties of morphisms, that is $f : X \to Y$ is formally unramified or formally étale if and only if for an open covering $\{U_i \subseteq X\}_{i\in I}$ and $\{V_j \subseteq Y\}_{j\in J}$, with the property that each $f(U_i)$ is contained in some V_j, the restriction $U_i \to V_j$ is formally unramified or formally étale.

Theorem 3.15. *Let $f : X \to Y$ be a morphism locally of finite presentation. Then f is formally unramified (resp. formally étale) if and only if f is unramified (resp. étale).*

Proof. By 3.14 we may assume that $Y = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$. Let's first look at formally unramified:
Consider the following diagram

$$\begin{array}{ccc} A & \xrightarrow{a} & C \\ {\scriptstyle c}\downarrow & \nearrow & \downarrow{\scriptstyle d} \\ B & \xrightarrow[b]{} & C/N \end{array}$$

We have to show that the following are equivalent:

1. There exists at most one the broken arrow in the middle making everything commutative.

2. The module $\Omega^1_{B/A} = 0$

$1 \Rightarrow 2$: Take $C := B \otimes_A B/I^2$, where $I := \mathrm{Ker}(B \otimes_A B \to B)$, and take $N := \mathrm{Ker}(B \otimes_A B/I^2 \to B)$. Let $b : B \to B$ be the identity. There are two broken arrows $\lambda_1, \lambda_2 : B \to B \otimes_A B/I^2$ sending b to $b \otimes 1$ and $1 \otimes b$ respectively. By the assumption $d = \lambda_1 - \lambda_2 = 0$. But $d : B \to B \otimes_A B/I^2$ factors through $\Omega^1_{B/A} \subseteq B \otimes_A B/I^2$, and $d : B \to \Omega^1_{B/A}$ is by definition the derivation map. Since $\Omega^1_{B/A}$ is generated by $\{dx | x \in B\}$, $\Omega^1_{B/A} = 0$ as desired.

$2 \Rightarrow 1$: Suppose that λ_1, λ_2 are two broken arrows. Let $\lambda := \lambda_1 - \lambda_2$. Clearly $\lambda : B \to C$ is an A-linear map which factors through $N \subseteq C$ and which kills A. The $B \otimes_A B$-module N is also a B-module defined by $bn = \lambda_1(b)n$ for all $b \in B$ and $n \in N$. Note that since $N^2 = 0$, we have $\lambda_1(b)n = \lambda_2(b)n$. We will show that λ also satisfies the Leibniz rule, so λ is an A-derivation, which is necessarily 0 as $\Omega^1_{B/A} = 0$.
Now $\lambda(bb') = (\lambda_1 - \lambda_2)(bb') = \lambda_1(b)\lambda_1(b') - \lambda_2(b)\lambda_2(b') = \lambda_1(b')(\lambda_1(b) - \lambda_2(b)) + \lambda_2(b)(\lambda_1(b') - \lambda_2(b')) = b'\lambda(b) + b\lambda(b')$ as desired.

If f is étale, then for any diagram

$$\begin{array}{ccc} T_0 & \xrightarrow{a} & X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\ T & \xrightarrow{b} & Y \end{array}$$

we have to show that the broken arrow exists uniquely. Replacing X by $X \times_T Y$ we may assume that $Y = T$ and b is the identity. Now consider the diagram

$$\begin{array}{ccccc} T_0 & \xrightarrow{\Gamma_a} & X \times_T T_0 & \xrightarrow{(\mathrm{id}, i)} & X \times_T T = X \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle (\mathrm{id}, a)} & & \\ X & \xrightarrow{\Delta} & X \times_T X & & \end{array}$$

From the diagram it is clear that Γ_a is an open embedding and (id, i) is a square 0 closed embedding. Let $\Gamma := (\mathrm{id}, i) \circ \Gamma_a(T_0)$. This is an open subset of X and we equip it with the open subscheme structure. Now $\Gamma \to X \to T$ becomes a degree 1 étale morphism, so it is an isomorphism. We take the broken arrow to be the inverse of this isomorphism.
The other direction is a little technique, we omit it. □

Lemma 3.16. *Let k be a field. Let $P(X) \in k[T]$. then the following statements are equivalent:*

1. *The k-algebra $k[T]/(P(T))$ is étale.*
2. *The polynomial $P(T)$ has no multiple roots in $\bar{k}$.*
3. *The formal derivative $P'(T)$ is coprime to $P(T)$, i.e. $P(T)$ and $P'(T)$ generates the trivial ideal $k[T]$.*
4. *There exist $u(T), v(T)$ and an equation $u(T)P(T) + v(T)P'(T) = 1$.*
5. *The element $P'(T)$ is invertible in $k[T]/(P(T))$.*

Proof. 2,3,4,5 are clearly equivalent. We just have to show $1 \Leftrightarrow 2$. For this we just have to assume it in the case when $k = \bar{k}$ is algebraically closed. In this case everything follows from 3.2. □

Definition 20. Let A be a ring. Let $P(T) \in A[T]$ be a polynomial. We say that $P(T)$ is separable if $P(T)$ and $P'(T)$ generated the unit ideal $A[T]$.

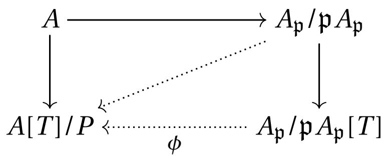
Lemma 3.17. *The following are equivalent:*

1. *The polynomial $P(T)$ is separable.*
2. *There exist $u(T), v(T)$ and an equation $u(T)P(T) + v(T)P'(T) = 1$.*
3. *The element $P'(T)$ is invertible in $A[T]/(P(T))$.*

Proof. This is obvious. □

Lemma 3.18. *Let $g(T), P(T) \in A[T]$. Then $P'[T]$ is invertible in $A[T]_{g(T)}/(P(T))$ if and only if it is so seeing as a polynomial in $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}[T]$ for all $\mathfrak{p} \in \mathrm{Spec}(A)$. In particular if $g(T) = 1$, then $P(T)$ is separable if and only if it is so seeing as a polynomial in $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}[T]$ for all $\mathfrak{p} \in \mathrm{Spec}(A)$.*

Proof. If we have a relation $u(T)P(T)+v(T)P'(T)=1$ then the relation still holds when we go to $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$. Conversely let $I:=(P(T),P'(T))$. If $I\neq A[T]_{g(T)}$, then take a maximal ideal $P\supseteq I$ of $A[T]_{g(T)}$. Consider the diagram



where the broken arrows exist uniquely by the universal property. Now the relation $\phi(u(T))\phi(P(T))+\phi(v(T))\phi(P'(T))=1$ provides an equation $0=1$ in $A[T]/P$ which is a contradiction. □

Lemma 3.19. *Let $g(T),P(T)\in A[T]$. Then $P'[T]$ is invertible in $A[T]_{g(T)}/(P(T))$ if and only if $A[T]_{g(T)}/(P(T))$ is étale over A.*

Proof. In light of 3.18 we may assume that $A=k$ is a field. Let $I:=(P(T),P'(T))\subseteq A[T]_{g(T)}/(P(T))$. By Lemma 2.10 (2) $I=A[T]_{g(T)}/(P(T))$ if and only if $I\otimes_k\bar{k}=A[T]_{g(T)}/(P(T))\otimes_k\bar{k}$. Thus we may assume that $k=\bar{k}$. In this case $g(T),P(T)$ split into linear factors. Let $Q(T)\in k[T]$ be the factor of $P(T)$ removing all the factors $(T-a_i)$ where a_i is a root of $g(T)$. Then $A[T]_{g(T)}/(P(T))=A[T]/(Q(T))$. In this case the statement is clear. □

Definition 21. Let $f:X\to Y$ be an étale morphism of schemes. We call f a standard étale morphism if $Y=\mathrm{Spec}(A)$, $X=\mathrm{Spec}(A[T]_{g(T)}/(P(X)))$ and f is the canonical projection.

Theorem 3.20. *Let $f:X\to Y$ be a morphism of schemes. Let $x\in X$ be a point which is étale over Y. Then there is an open affine $U=\mathrm{Spec}(B)\subseteq X$ containing x and an open affine $V=\mathrm{Spec}(A)\subseteq Y$ containing $f(x)$, such that $f(U)\subseteq V$ and $f|_U:U\to V$ is standard étale.*

Proof. See Stack Project, Lemma 28.34.14. □

Corollary 3.21. *Let $f:X\to Y$ be a morphism of schemes. Let $x\in X$ be a point which is étale over Y. Then there is an open affine $U=\mathrm{Spec}(B)\subseteq X$ containing x and an open affine $V=\mathrm{Spec}(A)\subseteq Y$ containing $f(x)$, such that $f(U)\subseteq V$ and B as an A-algebra is of the form $A[T_1,\cdots,T_n]/(P_1,\cdots,P_n)$ with the property that $\det(\frac{\partial P_i}{\partial T_j})_{ij}$ is invertible in B.*

Proof. " $\Rightarrow$ " By 3.20 we get $B=A[T]_{g(T)}/(P(T))=A[T,S]/(P(T),Sg(T)-1)$ with $P'(T)$ being invertible in B. Compute the determinant of the matrix we get $\det(\frac{\partial P_i}{\partial T_j})_{ij}=g(T)P'(T)$ which is clearly invertible in B.

" $\Leftarrow$ " Let $I:=(P_1,P_2,\cdots,P_n)$. Look at the exact sequence of B-modules:

$$I/I^2\to\Omega^1_{A[T_1,\cdots,T_n]/A}\otimes_{A[T_1,\cdots,T_n]}B\to\Omega^1_{B/A}\to 0$$

We see that $\Omega^1_{B/A}$ is the free module $\langle T_1,\cdots,T_n\rangle$ devided by the relation

$$\langle\sum_{1\le j\le n}\frac{\partial P_1}{\partial T_j},\cdots,\sum_{1\le j\le n}\frac{\partial P_n}{\partial T_j}\rangle$$

This means that $\Omega^1_{B/A} = 0$ if and only if $\det(\frac{\partial P_i}{\partial T_j})_{ij}$ is invertible in B. The algebra B is always flat over A (24.6.G, 24.6.7), so f being étale is equivalent to $\Omega^1_{B/A} = 0$, whence the proof. □

Corollary 3.22. *Let $i : X_0 \hookrightarrow X$ be a nilpotent closed embedding, i.e. a closed embedding which is a homeomorphism on the topological spaces. The the pullback functor sending $Y \to X$ to $Y \times_{X_0} X$ induces an equivalence between the category of étale X-schemes and the category of étale X_0-schemes.*

Proof. We first show the fully faithfulness. In fact to give a morphism in $\mathrm{Hom}_X(Y, Z)$ is equivalent to giving a morphism $\mathrm{Hom}_Y(Y, Y \times_X Z)$. The correspondence is just taking a morphism to its graph. Since $Y \times_X Z$ is étale over Y, any element in $\mathrm{Hom}_Y(Y, Y \times_X Z)$ is étale. This plus the fact that any element in $\mathrm{Hom}_Y(Y, Y \times_X Z)$ is a locally closed embedding allow us to conclude that any element in $\mathrm{Hom}_Y(Y, Y \times_X Z)$ is an open embedding. Thus $\mathrm{Hom}_X(Y, Z)$ is in one to one correspondence with open subschemes of $Y \times_X Z$ which map isomorphically to Z via the second projection.

Let $Y_0 := Y \times_X X_0$ and $Z_0 := Z \times_X X_0$. Given an open subscheme $U_0 \subseteq Y_0 \times_{X_0} Z_0$ which maps isomorphically to Z_0 we get an open subscheme $U \subseteq Y \times_X Z$ such that the second projection induces a universal homeomorphism $U \to Z$. By Exercise 5.5 we see that $U \to Z$ is an isomorphism. This means that the open subschemes of $Y \times_X Z$ which map isomorphically to Z via the second projection is in one-to-one correspondence with the open subschemes of $Y_0 \times_{X_0} Z_0$ which map isomorphically to Z_0 via the second projection. Hence we have $\mathrm{Hom}_X(Y, Z) = \mathrm{Hom}_{X_0}(Y_0, Z_0)$.

Thanks to the fully faithfulness, to prove the essential surjectivity it is enough to work Zariski locally. Now suppose $Y_0 \to X_0$ is an étale morphism of schemes we want to find $Y \to X$ étale such that its restriction to X_0 is Y_0. Since we can work Zariski locally, we may assume that $X = \mathrm{Spec}(A)$, $X_0 = \mathrm{Spec}(A_0)$ and $Y_0 = \mathrm{Spec}(B_0)$. Assume further that $B_0 = A_0[T]_{g_0(T)}/(P_0(T))$ is the standard étale algebra. Now we can lift $g_0(T)$ (resp. $P_0(T)$) to a polynomial (resp. monic polynomial) in $A[T]$. In this case $Y := \mathrm{Spec}(A[T]_{g(T)}/(P(T))$ would be a lift of Y_0, and it is easy to see that Y is a standard étale algebra, i.e. $P'(T)$ is invertible in $\mathrm{Spec}(A[T]_{g(T)}/(P(T))$. □

4 Henselian Rings (09/11/2016)

In this section let $(R, \mathfrak{m}, \kappa)$ be a local ring with $\mathfrak{m}$ the maximal ideal and $\kappa := R/\mathfrak{m}$ the residue field.

Definition 22.

1. We say R is *henselian* if for every monic $f \in R[T]$ and every root $a_0 \in \kappa$ of $\bar{f}$ such that $\bar{f}'(a_0) \neq 0$ there exists an $a \in R$ such that $f(a) = 0$ and $a_0 = \bar{a}$.
2. We say that R is *strictly henselian* if R is henselian and its residue field is separably closed.

Theorem 4.1. *The following are equivalent.*

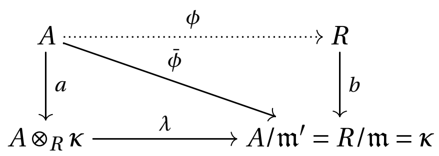
1. *The ring R is henselian.*

2. *For any étale morphism $f : Y \to \operatorname{Spec}(R)$ with a κ−point $y \in Y$ such that $f(y)$ corresponds to the maximal ideal $\mathfrak{m} \in \operatorname{Spec}(R)$.*

3. *For any monic $f \in R[T]$ and any factorization $\bar{f} = g_0 h_0$ with g_0, h_0 coprime, there exists monic polynomials g, h such that $f = gh$ and $\bar{g} = g_0$, $\bar{h} = h_0$.*

4. *Any finite R-algebra A is of the form $\prod_{1\le i\le n} A_i$ where A_i are local R-algebras.*

Proof. $1 \Rightarrow 2$ We may assume that $Y = \operatorname{Spec}(A)$ equipped with a maximal ideal $\mathfrak{m}' \subseteq A$ (i.e. $y \in Y$) lying over $\mathfrak{m}$ with residue field $A/\mathfrak{m}' = \kappa$. By 3.20 we may assume that $A = R[T]_{g(T)}/(P(T))$ is standard étale over R. Consider the commutative diagram



where a, b are $\mod \mathfrak{m}$ reduction maps, $\bar{\phi}$ is the $\mod \mathfrak{m}'$ reduction map, and λ is the unique map induced by $\bar{\phi}$. We need to find the R-algebra map ϕ which is indicated by the broken arrow in the above diagram. But the map λ provides an element $\bar{x} \in \kappa$ satisfying $\bar{P}(\bar{x}) = 0$, $\bar{g}(\bar{x}) \neq 0$. Since A is standard étale we know that $u(T)P(T) + v(T)P'(T) = g(T)^n$. Thus $\bar{P}'(\bar{a}) \neq 0$. By 1 we can find an element $x \in R$ lifting $\bar{x}$ with the property that $P(x) = 0$. Since $\bar{g}(\bar{x}) \neq 0$, $g(x)$ is invertible in R. Thus we get the desired map ϕ defined by x.

$2 \Rightarrow 3$ Follows readily from 4.3.

$3 \Rightarrow 1$ Trivial.

$4 \Rightarrow 3$ Let $A := R[T]/(f(T))$. Then A is a free R module whose rank is equal to the degree of $f(T)$. By 4, $A = A_1 \times A_2 \times \cdots \times A_n$ where A_i are finite local R-algebras. Since $A/\mathfrak{m}A = \kappa[T]/(\bar{f}(T)) = \kappa[T]/(g_0(T)) \times \kappa[T]/(h_0(T))$. Since $A_i/\mathfrak{m}A_i$ are connected components of $A/\mathfrak{m}A$, after reordering we may assume that

$$\kappa[T]/(g_0(T)) = \prod_{1\le i\le r} A_i/\mathfrak{m}A_i \qquad \text{and} \qquad \kappa[T]/(h_0(T)) = \prod_{r+1\le i\le n} A_i/\mathfrak{m}A_i$$

By 2.7 $\prod_{1\le i\le r} A_i$ is a free R-module of rank equal to $\deg(g_0(T))$. Let $g(T)$ be the characteristic polynomial of the R-linear map

$$T_0 : \prod_{1\le i\le r} A_i \to \prod_{1\le i\le r} A_i$$

where T_0 is the image of T under $R[T]/(f(T)) = A \twoheadrightarrow \prod_{1\le i\le r} A_i$. Clearly we have $g(T)$ is monic, $\bar{g}(T) = g_0(T)$, and $g(T_0) = 0$ by Hamilton-Cayley. Thus there is a surjective morphism

$$R[T]/(g(T)) \twoheadrightarrow \prod_{1\le i\le r} A_i$$

sending $T \mapsto T_0$, which is also an isomorphism because both sides are free R-modules of rank equal to $\deg(g_0(T)) = \deg(g(T))$. Now the commutative diagram

$$\begin{array}{ccc} R[T]/(f(T)) & \longrightarrow & R[T]/(g(T)) \\ & \searrow \quad \swarrow_{\cong} & \\ & \prod_{1\le i\le r} A_i & \end{array}$$

tells us that $f(T)$ is 0 in $R[T]/(g(T))$, i.e. $f(T) = g(T)h(T)$. Clearly $h(T)$ is monic and $\bar{h}(T) = h_0(T)$.

3 $\Rightarrow$ 4 First suppose that $A = R[T]/(f(T))$ where $f(T) \in R[T]$ is a monic polynomial. In this case we use induction on the degree of $f(T)$. If A is not local, then $A/\mathfrak{m} = \kappa[T]/(\bar{f}(T))$ is not local either, as all maximal ideals of A lie over $\mathfrak{m}$ (because A is finite over R). But then $\bar{f}(T)$ is not irreducible, so $\bar{f}(T) = g_0(T)h_0(T)$ for non-trivial monic polynomials $g_0(T), h_0(T) \in \kappa[T]$. By 4 we are able to lift $g_0(T), h_0(T)$ to monic polynomials $g(T), h(T) \in R[T]$ so that $f(T) = g(T)h(T)$. But $g(T), h(T)$ generate the unit ideal in A (If not then there is a maximal ideal $P \subseteq A$ which contains $g(T), h(T)$ and which lies over $\mathfrak{m} \subseteq R$. Then we would have $0 = (A/P) \times_R \kappa = A/P$.), so by Chinese reminder theorem $A = R[T]/(g(T)) \times R[T]/(h(T))$. Now we can apply the induction hypothesis. If A is arbitrary finite R-algebra, and if A is not local, then $\mathrm{Spec}\,(A/\mathfrak{m})$ is not connected. Take a non-trivial idempotent $\bar{b} \in A/\mathfrak{m}$. We can lift $\bar{b}$ to $b \in A$. As A is finite over R we can find a monic polynomial $f(T) \in R[T]$ such that $f(b) = 0$. Now consider the following diagram

$$\begin{array}{ccc} R[T]/(f(T)) & \xrightarrow{\phi} & A \\ \downarrow & & \downarrow \\ \kappa[T]/(f(T)) & \twoheadrightarrow B \hookrightarrow & A/\mathfrak{m} \end{array}$$

where B is the image and ϕ is the map which sends $T \mapsto b$. Applying what we just discussed we get $R[T]/(f(T)) = C_1 \times C_2 \times \cdots \times C_n$ and $\kappa[T]/(f(T)) = C_1/\mathfrak{m}C_1 \times C_2/\mathfrak{m}C_2 \times \cdots \times C_n/\mathfrak{m}C_n$ where C_i (and hence $C_i/\mathfrak{m}C_i$) are all local rings. Now the non-trivial idempotent $\bar{b} \in A/\mathfrak{m}$ is contained in B as it is the image of $T \in \kappa[T]/(f(T))$. By Chinese reminder theorem we have $B = B/\bar{b} \times B/(1-\bar{b})$. Since $\mathrm{Spec}\,(B) \hookrightarrow \mathrm{Spec}\,(\kappa[T]/(f(T)))$ is a closed embedding, we have surjections (after a reodering) $C_1 \times \cdots \times C_r \twoheadrightarrow B/\bar{b}$ and $C_{r+1} \times \cdots \times C_n \twoheadrightarrow B/(1-\bar{b})$. Thus we find $a \in R[T]/(f(T))$ an idempotent which maps to $\bar{b}$, so $\phi(a) \in A$ is also an idempotent. Applying Chinese reminder theorem we get $A = A/\phi(a) \times A/(1-\phi(a))$. Now we can use induction on the κ-dimension of $A/\mathfrak{m}$. □

Lemma 4.2. *Let $n, m \ge 1$ be integers. Consider the ring map:*

$$R = \mathbb{Z}[A_1, \cdots, A_{n+m}] \longrightarrow A = \mathbb{Z}[B_1, \cdots, B_n, C_1, \ldots, C_m]$$

which sends $A_k \mapsto A_k(B_i, C_j) := \sum_{i+j=k} B_i C_j$. Clearly we have A as an R-algebra can be written as

$$A = R[B_1, \cdots, B_n, C_1, \ldots, C_m]/(A_k(B_i, C_j) - A_k)_{1\le k\le m+n}$$

Then we have a matrix

$$M := (\frac{\partial(A_k(B_i, C_j) - A_k)}{\partial T_l})_{kl}$$

where $T_l = B_l$ *when* $1 \le l \le n$ *and* $T_l = C_{l-n}$ *if* $1 + n \le l \le m+n$. *Let*

$$g(T) := T^n + B_1 T^{n-1} + \cdots + B_n \qquad \text{and} \qquad h(T) := T^m + C_1 T^{m-1} + \cdots + C_m$$

The determinant of M *is usually denoted by* Δ *or* $\mathrm{Res}(g, h)$ *and called the resultant of* g, h. *Now suppose that* $\mathfrak{q} \subseteq A$ *is a prime ideal. Then the following statements are equivalent:*

1. *The map* $R \to A$ *is étale at* $\mathfrak{q}$.
2. *The element* $\Delta \in A$ *is not contained in* $\mathfrak{q}$.
3. *The polynomials* $\bar{g}(T), \bar{h}(T)$ *as reductions of* $g(T), h(T)$ *in* $\kappa(\mathfrak{q})$ *have no common factor.*

Proof. By 3.21 we have $1 \Leftrightarrow 2$. For $1 \Leftrightarrow 2$ one just have to notice that M is the transpose the matrix of the A-linear map

$$\lambda : A[T]_{<m} \bigoplus A[T]_{<n} \longrightarrow A[T]_{<m+n}$$

sending $a[T] \oplus b[T]$ to $a[T]g[T] + b[T]h[T]$. But $\Delta \notin \mathfrak{q} \Leftrightarrow M \otimes_A \kappa(\mathfrak{q})$ is invertible $\Leftrightarrow \lambda \otimes_A \kappa(\mathfrak{q})$ is an isomorphism $\Leftrightarrow \lambda \otimes_A \kappa(\mathfrak{q})$ is an injective $\Leftrightarrow \bar{g}(T), \bar{h}(T)$ have no common factor. □

Lemma 4.3. *Let* R *be a ring. Let* $f(T) \in R[T]$ *be a monic polynomial. Let* $\mathfrak{p}$ *be a prime ideal of* R. *Let* $\bar{f} = g_0 h_0$ *be a factorization of monic polynomials in* $\kappa(\mathfrak{p})[T]$. *If* $g_0(T)$ *and* $h_0(T)$ *are coprime, then there exist*

1. *an étale ring map* $R \to R'$,
2. *a prime* $\mathfrak{p}' \subseteq R'$ *lying over* $\mathfrak{p}$, *and*
3. *a factorization* $f(T) = g(T)h(T) \in R'[T]$

such that

1. $\kappa(\mathfrak{p}) = \kappa(\mathfrak{p}')$,
2. $\bar{g}(T) = g_0(T), \bar{h}(T) = h_0(T)$ *and*
3. *The polynomials* $g(T), h(T)$ *generate the unit ideal in* $R'[T]$.

Proof. Suppose that $g_0(T) = T^n + \bar{b}_1 T^{n-1} + \cdots + \bar{b}_n$ and $h_0(T) = T^m + \bar{c}_1 T^{m-1} + \cdots + \bar{c}_m$ with $\bar{b}_i, \bar{c}_j \in \kappa(\mathfrak{p})$. Write $f(T) = T^{m+n} + a_1 T^{m+n-1} + \cdots + a_{m+n}$ with $a_k \in R$. Now define

$$S := R \otimes_{\mathbb{Z}[A_1, \cdots, A_{n+m}]} \mathbb{Z}[B_1, \cdots, B_n, C_1, \ldots, C_m]$$

where $\mathbb{Z}[A_1, \cdots, A_{n+m}] \to R$ sends A_i to a_i and $\mathbb{Z}[A_1, \cdots, A_{n+m}] \to \mathbb{Z}[B_1, \cdots, B_n, C_1, \ldots, C_m]$ is the map defined in 4.2. By the assumption we have a map

$$\lambda : S = R[B_1, \cdots, B_n, C_1, \ldots, C_m]/(A_k(B_i, C_j) - a_k)_{1 \le k \le m+n} \to \kappa(\mathfrak{p})$$

sending $B_i \mapsto \bar{b}_i$ and $C_j \mapsto \bar{c}_j$. Let $\mathfrak{p}' \subseteq S$ be the kernel of λ, which is the prime ideal. Let $g(T) := T^n + B_1 T^{n-1} + \cdots + B_n$ and $h(T) := T^m + C_1 T^{m-1} + \cdots + C_m$. Since the reduction of$g(T)$ (resp. $h(T)$) in $\kappa(\mathfrak{p}') = \kappa(\mathfrak{p})$ is $g_0(T)$ (resp. $h_0(T)$) and since $g_0(T)$ is coprime with $h_0(T)$, the resultant $\Delta = \mathrm{Res}(g,h)$ is not in $\mathfrak{p}'$ by 4.2, and $R \to S$ is therefore étale at $\mathfrak{p}'$. Now let $R' := S[\frac{1}{\Delta}]$. By 3.21 $R \to R'$ is étale, and by construction $f(T) = g(T)h(T)$. Since the R'-linear map

$$\lambda : R'[T]_{<m} \bigoplus R'[T]_{<n} \longrightarrow R'[T]_{<m+n}$$

is now invertible, there exists $a(T), b(T) \in R'[T]$ such that $a(T)g(T) + b(T)h(T) = 1$. □

Lemma 4.4. *If $(R, \mathfrak{m}, \kappa)$ is a Henselian local ring, then all finite local A-algebra and any quotient A/I with I a proper ideal of A are Henselian.*

Proof. Clear! □

Lemma 4.5. *All complete local rings are Henselian.*

Proof. See Atiyah-Macdonald *Introduction to Commutative Algebra*, Exercise 9, pp. 115. □

Theorem 4.6. *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. There exists a local ring map $R \to R^h$ with the following properties*

1. *R^h is henselian,*
2. *R^h is a filtered colimit of étale R-algebras,*
3. *$\mathfrak{m}R^h$ is the maximal ideal of R^h, and $\kappa = R^h/\mathfrak{m}R^h$.*

Proof. We would like to take the category I consisting of pairs $(A, \mathfrak{p})$ where A is an étale R-algebra and $\mathfrak{p}$ is a prime ideal of A with the property that $\mathfrak{p}$ lies over $\mathfrak{m}$ and $\kappa(\mathfrak{p}) \subseteq \kappa(\mathfrak{m})$ is an isomorphism. We would like to define

$$R^h := \varinjlim_{i \in I} A_i$$

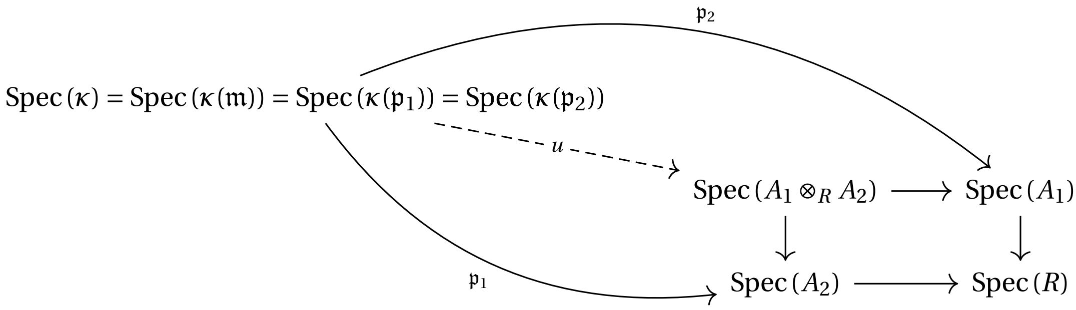
But to do that we need to justify the definition, i.e. we have to check that I is filtered. Recall that a category is called cofiltered iff

(i) the category I has at least one object,

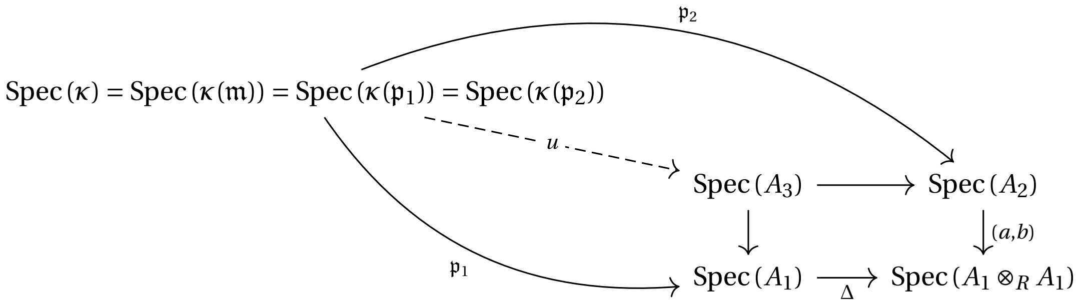
(ii) for every pair of objects x, y of I there exists an object z and morphisms $x \to z$, $y \to z$, and

(iii) for every pair of objects $x, y \in I$ and every pair of morphisms $a, b : x \to y \in I$ there exists a morphism $c : y \to z \in I$ such that $c \circ a = c \circ b$ as morphisms in I.

(i) is given by $(R, \mathfrak{m}, \kappa)$. (ii) is given by the diagram



(iii) is given by the following diagram



Next we show that R^h is a local ring. Suppose $(A, \mathfrak{p}) \in I$ is an object. By definition we have a morphism $(R, \mathfrak{m}) \xrightarrow{f} (A, \mathfrak{p})$ such that $f^{-1}(\mathfrak{p}) = \mathfrak{m}$. Let $\mathfrak{p}_1, \cdots, \mathfrak{p}_n \in \mathrm{Spec}(A)$ be the other prime ideals (different from $\mathfrak{p}$) in A which lie over $\mathfrak{m}$. We can choose $s \in \mathfrak{p}_1 \bigcap \cdots \bigcap \mathfrak{p}_n$ but $s \notin \mathfrak{p}$. Then $(A_s, \mathfrak{p}A_s) \in I$ and it has exactly one maximal ideal lying over $\mathfrak{m}$, i.e. $\mathfrak{p}A_s$. Since $A_s/\mathfrak{m}A_s$ is unramified over $R/\mathfrak{m}R$ and has only one prime ideal $\mathfrak{p}/\mathfrak{m}A$, we have $\mathfrak{p}A_s/\mathfrak{m}A_s = 0$, i.e. $\mathfrak{p}A_s = \mathfrak{m}A_s$. Suppose $x \in R^h$, then there exists $(A, \mathfrak{p})$ with $\mathfrak{m}A = \mathfrak{p}$ and x comes from $x_A \in A$. If $x \notin \mathfrak{m}R^h$, then $x_A \notin \mathfrak{m}A = \mathfrak{p}$. Thus $\frac{x_A}{1} \in (A_{x_A}, \mathfrak{p}A_{x_A})$ is invertible and its image in R^h is x. So x is invertible R^h. This shows that R^h is local with maximal ideal $\mathfrak{m}R^h$. Clearly $R^h/\mathfrak{m}R^h = \kappa$. The fact that R^h is Henselian follows from 4.3. □

Theorem 4.7. *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Let $\kappa \subseteq \kappa^{\mathrm{sep}}$ be a separable algebraic closure. There exists a commutative diagram*

$$\begin{array}{ccccc} R & \longrightarrow & R^h & \longrightarrow & R^{sh} \\ \downarrow & & \downarrow & & \downarrow \\ \kappa & \longrightarrow & \kappa & \longrightarrow & \kappa^{\mathrm{sep}} \end{array}$$

with the following properties

1. *the map $R^h \to R^{sh}$ is local,*
2. *R^{sh} is strictly henselian,*

3. R^{sh} *is a filtered colimit of étale R-algebras,*

4. $\mathfrak{m}R^{sh}$ *is the maximal ideal of* R^{sh}*, and* $\kappa^{\text{sep}} = R^{sh}/\mathfrak{m}R^{sh}$.

Proof. The proof is roughly the same as before, but here we take I as pairs (A,ϕ) with A an étale R-algebra with a fixed point $\phi : \operatorname{Spec}(k^{\text{sep}}) \to \operatorname{Spec}(A)$. □

5 The Étale Fundamental Group (I) (16/11/2016)

In this lecture we would like to define the notion of *Galois Category.*

Definition 23. Let $\mathscr{C}$ be a category, and let F be a functor $\mathscr{C} \to$ (Fsets), where (Fsets) denotes the category of finite sets. We call the pair $(\mathscr{C}, F)$ a Galois category if it satisfy the following axioms.

1. The category $\mathscr{C}$ has a final object and fibered products. (This is equivalent to saying that $\mathscr{C}$ has finite projective limits.)

2. The category $\mathscr{C}$ has finite coproducts, and for any $A \in \mathscr{C}$ the quotient by a finite subgroup $G \subseteq \operatorname{Aut}(A)$ exists.

3. Let $u : Y_1 \to Y_2$ be a morphism in $\mathscr{C}$. Then u factorises as a strict epimorphism followed by a monomorphism
$$Y_1 \twoheadrightarrow Y \hookrightarrow Y_2 = Y \coprod (Y_2 \setminus Y)$$
which embeds the image Y as a direct summand of Y_2,

4. The functor $F : \mathscr{C} \to$ (Fsets) is left exact, i.e. it takes final object to final object and fibered products to fibered products.

5. The functor $F : \mathscr{C} \to$ (Fsets) commutes with finite direct sums, translates strict epimorphisms to epimorphisms, and commutes with quotient by finite subgroups of the automorphism group.

6. If $u : Y_1 \to Y_2$ induces an isomorphism $F(u) : F(Y_1) \to F(Y_2)$, then u is an isomorphism.

Theorem 5.1. *Let X be a locally Noetherian scheme, and let $x : \operatorname{Spec}(k) \to X$ be a geometric point. Let* $\text{Ét}(X)$ *be the category of finite étale morphisms with target X, and let F_x be the functor* $\text{Ét}(X) \to$ (Fsets) *sending a finite étale morphism $Y \to X$ to the k-points of the k-scheme* $f^{-1}(x) := Y \times_X \operatorname{Spec}(k)$*. Then* $(\text{Ét}(X), F_x)$ *is a Galois category.*

Proof. For axiom 1, the final object in $\text{Ét}(X)$ is X equipped with the identity structure map. For the fibred product we just take it in the category of schemes and that would work.

For axiom 2, we take the disjoint union in the category of schemes. The quotient condition follows from the theorem of quotients by finite flat group schemes, see for example Abelian Varities, Chapter 4.

For axiom 3, we just have to notice that $u(Y_1)$ is a both open and closed subscheme of Y_2. then we just have to take Y to be $u(Y_1)$. We know from the discussion of faithfully flat morphisms of scheme that any faithfully flat morphism is a strict epimorphism.

For axiom 4, the final object in $\acute{\mathrm{E}}\mathrm{t}(X)$ is X equipped with the identity structure map, but in this case $f^{-1}(x)$ is just $\mathrm{Spec}(k)$. For the fibred products we just have to notice that the category of finite sets is equivalent to the category of finite étale schemes over $k = \bar{k}$. So if one has a fibred product in $\acute{\mathrm{E}}\mathrm{t}(X)$, then by taking fibered at x one gets a fibred product in $\acute{\mathrm{E}}\mathrm{t}(\mathrm{Spec}(k))$ which gives, via the equivalence, a fibred product in (Fsets).

For axiom 5, the claims for finite diect sum and strict epimorphism are clear. The claim about quotient follows again from the theorem of quotients by finite flat group schemes.

For axiom 6, we just have to remind you of 3.13. □

Theorem 5.2. *Let $(\mathscr{C}, F)$ be a Galois category. Then there is a profinite group Π, and if we denote $\Pi-$Fsets the category of finite sets equipped with a continuous action from Π (we always give finite sets the discrete topology), then there is an equivalence of categories*

$$G : \mathscr{C} \longleftarrow \Pi - \mathrm{Fsets}$$

such that $f \circ G = F$, where $f : \Pi - \mathrm{Fsets} \to \mathrm{Fsets}$ is the forgetful functor.

Remark 5.3. A topological group Π is called profinite if it is isomorphic as a topological group to $\varprojlim_{i\in I} G_i$ where I is a cofiltered category with G_i a finite group. Here the topology on $\varprojlim_{i\in I} G_i$ is the coarsest topology so that all the projections $\varprojlim_{i\in I} G_i \to G_i$ are continuous.

Proof. Step 1. The functor F is representable. We start with a definition:

Definition 24. Let $\mathscr{D}$ be any category. We can define $\mathrm{Pro}(\mathscr{D})$ to be the category whose objects are functors $I \to \mathscr{D}$ where I is a small cofiltered category. One can write an object $P \in \mathrm{Pro}(\mathscr{D})$ in the form $\{P_i\}_{i\in I}$. A morphism between $\{P_i\}_{i\in I}$ and $\{Q_j\}_{j\in J}$ is defined as the following set

$$\varprojlim_{j\in J} \varinjlim_{i\in I} \mathrm{Hom}_{\mathscr{C}}(P_i, Q_j)$$

Indeed one can also embed $\mathrm{Pro}(\mathscr{D})$ into the presheaves of $\mathscr{D}$ by seeing $\{P_i\}_{i\in I}$ as the projective limit of presheaves $\varprojlim_{i\in I} \underline{P}_i$, where $\underline{P}_i$ denotes the presheaf on $\mathscr{D}$ defined by $P_i \in \mathscr{D}$.

What we mean here is that there is an object $P := \{P_i\}_{i\in I}$ equipped with the following isomorphism of functors

$$F(-) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{Pro}(\mathscr{C})}(P, -)$$

In fact we can choose P so that all the transition maps $\phi_{ji} : P_j \to P_i$ are epimorphism, and we can also assume that for any epimorphism $\lambda : P_j \to Q$ there is some $i \in I$ such that $\lambda \cong \phi_{ji}$. We denote the canonical projection map $P \to P_i$ in $\mathrm{Pro}(\mathscr{C})$ as ϕ_i.

Step 2. The previously constructed P_i are connected and not equal to the initial object $\emptyset_{\mathscr{C}}$.

Definition 25. An object $P \in \mathscr{C}$ is called connected if there is no isomorphism $P \cong A \coprod B$ with both A and B are not $\emptyset_{\mathscr{C}}$.

Step 3. Any morphism $u : X \to Y \in \mathscr{C}$ with $X \neq \emptyset_{\mathscr{C}}$ and Y connected is a strict epimorphism. All endomorphism of a connected object is an automorphism.
Indeed, we can factorize u as $X \twoheadrightarrow Y' \hookrightarrow Y$ where the first is a strict epimorphism and the second is an embedding of a direct summand. Since $X \neq \emptyset_{\mathscr{C}}$, we have $F(X) \neq \emptyset$, so $F(Y') \neq \emptyset$, and therefore $Y' \neq \emptyset_{\mathscr{C}}$. But Y is connected, so $Y \setminus Y' = \emptyset_{\mathscr{C}}$, i.e. $Y' \hookrightarrow Y$ is an isomorphism. For the second claim we have to show that if $u : X \to X$ is any morphism, then u is an isomorphism. For this we may suppose that $X \neq \emptyset$. Then $F(u)$ is a surjective map of sets by the first claim and axiom 5. But this implies that $F(u)$ is an isomorphism as $F(X)$ is a finite set. Thus by axiom 6 u is an isomorphism.

Step 4. The following conditions are equivalent.

(i) The map $\mathrm{Hom}_{\mathscr{C}}(P_i, P_i) \to \mathrm{Hom}_{\mathrm{Pro}(\mathscr{C})}(P, P_i) = F(P_i)$ is surjective.

(ii) The map $\mathrm{Hom}_{\mathscr{C}}(P_i, P_i) \to \mathrm{Hom}_{\mathrm{Pro}(\mathscr{C})}(P, P_i) = F(P_i)$ is bijective.

(iii) The group $\mathrm{Aut}_{\mathscr{C}}(P_i)$ acts transitively on $F(P_i)$.

(iv) The group $\mathrm{Aut}_{\mathscr{C}}(P_i)$ acts transitively and freely on $F(P_i)$.

Indeed (i) $\Leftrightarrow$ (iii) and (ii) $\Leftrightarrow$ (iv) follow from the fact that both (i) and (iii) are equivalent to the following: For any $\phi : P \to P_i$, $\exists\, P_i \xrightarrow{u} P_i \in \mathrm{Aut}_{\mathscr{C}}(P_i)$ such that $\phi_i = \phi \circ u$. (iii) $\Leftrightarrow$ (iv) follows from the fact that $\phi_i : P \to P_i$ is, in the obvious sense, an epimorphism. Thus if we have $u, v : P_i \to P_i$ and if $u \circ \phi_i = v \circ \phi_i$ then $u = v$.

Definition 26. An object P_i is called Galois if it satisfies one of the above conditions.

Step 5. For any $X \in \mathscr{C}$ there exists P_i Galois such that for all $u \in \mathrm{Hom}_{\mathrm{Pro}(\mathscr{C})}(P, X)$ there is a factorization $P \xrightarrow{\phi_i} P_i \to X$ of u. In particular for $P \xrightarrow{\phi_j} P_j$ there exists P_i Galois and a morphism $\phi_{ij} : P_i \to P_j$. Now let $I' \subseteq I$ be the full subcategory so that P_i is Galois for all $i \in I'$. Then we have $\mathrm{Hom}_{\mathrm{Pro}(\mathscr{C})}(P, P) = \varprojlim_{i \in I} \mathrm{Hom}_{\mathrm{Pro}(\mathscr{C})}(P, P_i) = \varprojlim_{i \in I'} \mathrm{Hom}_{\mathrm{Pro}(\mathscr{C})}(P, P_i) = \varprojlim_{i \in I} \mathrm{Hom}_{\mathscr{C}}(P_i, P_i) = \varprojlim_{i \in I} \mathrm{Aut}_{\mathscr{C}}(P_i)$. The equality also reveals that

$$\mathrm{Hom}_{\mathrm{Pro}(\mathscr{C})}(P, P) = \mathrm{Aut}_{\mathrm{Pro}(\mathscr{C})}(P)$$

Step 6. Define Π to be $\mathrm{Aut}(F) = \mathrm{Aut}_{\mathrm{Pro}(\mathscr{C})}(P) = \varprojlim_{i \in I} \mathrm{Aut}_{\mathscr{C}}(P_i)$. Clearly Π acts on $F(X)$ for each $X \in \mathscr{C}$. Step 5 actually shows that the action of $\mathrm{Aut}(F)$ on $F(X)$ factors through a finite quotient $\mathrm{Aut}(F) \to \mathrm{Aut}_{\mathscr{C}}(P_i)$. Thus the action is continuous. Now we obtain a functor $\mathscr{C} \to \Pi - \mathrm{Fsets}$.

Step 7. The above functor is an equivalence. □

Definition 27. Let X be a connected locally Noetherian scheme. Let $x : \mathrm{Spec}(k) \to X$ be a geometric point. We call $\pi_1^{\text{ét}}(X,x)$ the étale fundamental group of (X,x) if $\pi_1^{\text{ét}}(X,x)$ is the profinite group associated with the Galois category $(\text{Ét}(X), F_x)$.

Example 5.4. If $X = \mathrm{Spec}(k)$ and $x : \mathrm{Spec}(\bar{k}) \to X$, then $\{P_i\}_{i\in I'}$ is just the system of finite Galois extensions of k inside the fixed algebraic closure $\bar{k}$, so

$$\pi_1^{\text{ét}}(X,x) = \mathrm{Aut}(F_x) = \varprojlim_{i\in I'} \mathrm{Aut}(P_i) = \varprojlim_{i\in I'} \mathrm{Gal}(P_i/k) = \mathrm{Gal}(k)$$

6 The Étale Fundamental Group (II) (23/11/2016)

The construction of the projective system $P = \{P_i\}_{i\in I} \in \mathrm{Pro}(\mathscr{C})$.
Let I be the category of pairs (P,ξ) where $P \in \mathscr{C}$ is connected and $P \neq \emptyset_{\mathscr{C}}$, and $\xi \in F(P)$. A typical example would be when $\mathscr{C} = \text{Ét}(X)$ and $x : \mathrm{Spec}(k) \to X$ a geometric point. Then a pair (P,ξ) is just a commutative diagram

$$\begin{array}{ccc} & & Y \\ & \overset{\xi}{\nearrow} & \downarrow \\ \mathrm{Spec}(k) & \xrightarrow{x} & X \end{array}$$

A morphism $(P_1,\xi_1) \to (P_2,\xi_2)$ is a morphism $P_1 \to P_2$ in $\mathscr{C}$ such that $F(P_1) \to F(P_2)$ sends $\xi_1 \to \xi_2$. If $\mathscr{C} = \text{Ét}(X)$, then a morphism is just a morphism of étale schemes over X which preserves the prescribed point.

Lemma 6.1. *1. If $(P,\xi) \in I$, if $P' \hookrightarrow P$ is a monomorphism in $\mathscr{C}$ and $\xi' \in F(P')$ such that $F(\xi') = \xi$, then $P' = P$.*

2. If $(P,\xi) \in I$, (P',ξ') is a pair with $P' \in \mathscr{C}$, then there is at most one morphism $(P,\xi) \to (P',\xi')$.

3. If $(P,\xi) \in I$, $(P',\xi') \xrightarrow{u} (P,\xi)$ with $P' \in \mathscr{C}$, then $u : P' \to P$ is an epimorphism.

Proof. For 1, $P' \to P$ factorizes as $P' \twoheadrightarrow Q \hookrightarrow P$, where $\twoheadrightarrow$ is a strict epimorphism and $\hookrightarrow$ embeds Q as a direct summand of P. Since $P' \to P$ is monomorphism, $P' \to Q$ is also a monomorphism. This implies that $F(P') \to F(Q)$ is injective. But $F(P') \to F(Q)$ is already surjective by axiom 4, so it is an isomorphism. By axiom 6, $P' \to P$ is an isomorphism.

For 2, Suppose $u_1, u_2 : (P,\xi) \to (P',\xi')$ are two morphisms. Since we have finite limits in $\mathscr{C}$, we can take the kernel Q of u_1 and u_2, so that we get the exact sequence

$$Q \longrightarrow P \underset{u_2}{\overset{u_1}{\rightrightarrows}} P'$$

Since $u_1(\xi) = \xi' = u_2(\xi)$, we see that $\xi \in \mathbb{F}(P)$ is contained in $F(Q) \subseteq F(P)$. Thus $(Q,\xi) \subseteq (P,\xi)$, and this implies that $Q = P$ by 1). Thus $u_1 = u_2$.

The third is trivial. □

Now we define the forgetful functor $P : I \to \mathscr{C}$ sending $(P,\xi) \mapsto P$ to be the pro-object $\{P_i\}_{i\in I} \in \mathrm{Pro}(\mathscr{C})$.

Lemma 6.2. *1. For any pair (P,ξ) with $P \in \mathscr{C}$, $\xi \in F(P)$, there exists $(Q,\eta) \in I$ and a map $(Q,\eta) \to (P,\xi)$.*

2. The category I is a cofiltered category.

3. The transition maps $P_j \to P_i$ are epimorpohisms.

4. For any connected object $P \in \mathscr{C}$, there exists $i \in I$ such that $P_i \cong P$.

Proof. Only 1 deserves an argument. 2 follows from 1, and 3,4 are trivial. Let $P \in \mathscr{C}$ If $P = P_1 \coprod P_2$ and $P_i \neq \emptyset_{\mathscr{C}}$, then $\xi \in F(P) = F(P_1)\coprod F(P_2)$. Say, $\xi \in F(P_1)$. Then we have $(P_1,\xi) \subseteq (P,\xi)$, and $F(P_1) \subsetneq F(P_2)$. Using induction on the number of elements in $F(P)$ we can conclude the proof. □

Proposition 6.3. *The object $P := \{P_i\}_{i\in I}$ represents F.*

Proof. By Yoneda lemma a pair (T,ξ) is equivalent to a morphism of functors $\underline{T} \to F$, where $\underline{T}(-) = \mathrm{Hom}(T,-)$. Thus the system $\{\underline{P}_i\}_{i\in I}$ defines a system of compatible maps of functors

$$\begin{array}{ccc} \underline{P}_i & \longrightarrow & F \\ \downarrow & \nearrow & \\ \underline{P}_j & & \end{array}$$

This defines a map of functors $\phi : \varprojlim_{i\in I} \underline{P}_i \to F$. Since for any $T \in \mathscr{C}$ we have

$$(\varprojlim_{i\in I} \underline{P}_i)(T) = \varprojlim_{i\in I} \mathrm{Hom}_{\mathscr{C}}(P_i, T) = \mathrm{Hom}_{\mathrm{Pro}(\mathscr{C})}(P,T)$$

Thus it is enough to show that ϕ is an isomorphism.

The map ϕ is surjective. Take a pair (T,η) with $T \in \mathscr{C}$ and $\eta \in F(T)$ we need to show that η comes from $(\varprojlim_{i\in I} \underline{P}_i)(T)$. Since there exists $(Q,\xi) \to (T,\eta)$ with Q connected and if the claim works for (Q,ξ) then we are done, we could replace (T,η) by (Q,ξ). Now let $i = (Q,\xi) \in I$. Then $P_i = Q$ and we have a diagram

$$\begin{array}{ccc} \underline{P}_i & & \\ \downarrow & \overset{\xi}{\searrow} & \\ \varprojlim_{i\in I} \underline{P}_i & \xrightarrow{\ \phi\ } & F \end{array}$$

Thus clearly we have id $\mapsto \xi$ under $\underline{P}_i(P_i) = \mathrm{Hom}(P_i, P_i) \to F(P_i)$.

The map ϕ is injective. Suppose that $\xi_1, \xi_2 \in (\varprojlim_{i\in I} \underline{P}_i)(T)$ such that $\phi(\xi_1) = \phi(\xi_2)$. We need to show that $\xi_1 = \xi_2$. Since

$$(\varprojlim_{i\in I} \underline{P}_i)(T) = \varprojlim_{i\in I} \mathrm{Hom}_{\mathscr{C}}(P_i, T)$$

We may assume that ξ_1, ξ_2 come from $\eta_1, \eta_2 \in \mathrm{Hom}_{\mathscr{C}}(P_i, T)$ for i large. Since $\phi(\eta_1) = \phi(\eta_2)$, we have a morphism

$$(P_i, \xi) \underset{\eta_2}{\overset{\eta_1}{\rightrightarrows}} (T, \phi(\eta_1) = \phi(\eta_2))$$

By 6.1 we have $\eta_1 = \eta_2$ □

Recall that an object P_i in $\{P_i\}_{i\in I}$ is called Galois if it satisfies one of the following equivalent conditions.

(i) The map $\mathrm{Hom}_{\mathscr{C}}(P_i, P_i) \to \mathrm{Hom}_{\mathrm{Pro}(\mathscr{C})}(P, P_i) = F(P_i)$ is surjective.

(ii) The map $\mathrm{Hom}_{\mathscr{C}}(P_i, P_i) \to \mathrm{Hom}_{\mathrm{Pro}(\mathscr{C})}(P, P_i) = F(P_i)$ is bijective.

(iii) The group $\mathrm{Aut}_{\mathscr{C}}(P_i)$ acts transitively on $F(P_i)$.

(iv) The group $\mathrm{Aut}_{\mathscr{C}}(P_i)$ acts transitively and freely on $F(P_i)$.

Definition 28. An object $A \in \mathscr{C}$ is called Galois if there is a Galois object P_i in the projective system such that $A \cong P_i$.

Proposition 6.4. *Let $f : Y \to X \in \acute{\mathrm{E}}\mathrm{t}(X)$ be an object. The following statements are equivalent.*

1. *The object $f : Y \to X \in \acute{\mathrm{E}}\mathrm{t}(X)$ is Galois.*
2. *The object $f : Y \to X \in \acute{\mathrm{E}}\mathrm{t}(X)$ is connected and $\#(\mathrm{Aut}_X(Y))$ is equal to the degree of f.*
3. *The object $f : Y \to X \in \acute{\mathrm{E}}\mathrm{t}(X)$ is connected and f is a torsor under the abstract group $G := \mathrm{Aut}_X(Y)$, i.e. $Y/G = X$.*

Proof. One just has to notice that in $\acute{\mathrm{E}}\mathrm{t}(X)$ the fiber functor F_x takes f to its fiber at x. So the degree of f is equal to the cardinality of $F_x(Y)$. □

Recall that For $i = (P_i, \xi_i) \in I$ there exists (P_j, ξ_j) with P_j Galois and a morphism $(P_j, \xi_j) \to (P_i, \xi_i)$. We take $I' \subseteq I$ to be the full subcategory consisting of Galois objects. Clearly $P = \{P_i\}_{i\in I} = \{P_i\}_{i\in I'}$.

Theorem 6.5. *Let $(\mathscr{C}, F)$ be a Galois category, and let Π be the corresponding profinite group.*

1. *There is a one-to-one correspondence between the set of isomorphic classes of pairs (P, ξ), where $P \in \mathscr{C}$ is connected and $\xi \in F(P)$, and the open subgroups of Π.*

2. *There is a one-to-one correspondence between the set of isomorphic classes of pairs* (P,ξ)*, where* $P \in \mathscr{C}$ *is connected and* $\xi \in F(P)$*, and the open normal subgroups of* Π.

Proof. For the first statement one just have to notice that $P \in \mathscr{C}$ Galois $\Leftrightarrow F(P) \in \Pi - \text{Fsets}$ is connected $\Leftrightarrow \Pi$ acts transitively on $F(P)$.
For the second statement we have to show that the open subgroup $H \subseteq \Pi$ corresponding to (P,ξ) is normal if and only if P is Galois. First suppose that $P \in \mathscr{C}$ is Galois, so $\Pi/H = \{H, g_1H, g_2H, \cdots, g_nH\}$ is also Galois as an object in $\Pi - \text{Fsets}$. Suppose $a \in \text{Aut}_{\Pi-\text{Fsets}}(\Pi/H)$, then $a(H) = g_iH$, and for any $g \in \Pi$ we have $a(gH) = a(g(H)) = g(a(H)) = gg_iH$. Thus we know that a is determined by its value on H, i.e. $\text{Aut}_{\Pi-\text{Fsets}}(\Pi/H) \subseteq \Pi/H$. But since both sides have the same cardinality we know that the $\subseteq$ is actually $=$. Thus for any g_iH there exists $a \in \text{Aut}_{\Pi-\text{Fsets}}(\Pi/H)$ such that $a(H) = g_iH$. Thus there exists $h \in H$ such that $g_iH = hg_iH$. Hence $g_iHg_i^{-1} \subseteq H$, i.e. $H \subseteq \Pi$ is normal.
Conversely if $H \subseteq \Pi$ is normal, then Π/H is a group. Then we have

$$\text{Aut}_{\Pi-\text{Fsets}}(\Pi/H) = \text{Aut}_{\Pi/H-\text{Fsets}}(\Pi/H) = \Pi/H$$

□

Theorem 6.6. *The functor sending an object in* $\text{Pro}(\Pi - \text{Fsets})$ *to its projective limit in the category of topological spaces equipped with a continuous* Π*-action induces an equivalence between* $\text{Pro}(\Pi - \text{Fsets})$ *and the category of Hausdorf, quasi-compact, totally disconnected topological spaces equipped with a continuous* Π*-action.*

Proof. See [SGA1, Proposition 5.2, pp.127]. □

Corollary 6.7. *The isomorphism class of pairs* (P,ξ)*, where* P *is a connected object in* $\text{Pro}(\Pi - \text{Fsets})$ *and* ξ *is an element in the corresponding compact totally disconnected topological space on which* Π*-acts continuously, is in one-to-one correspondence with closed subgroups of* Π.

Proof. This follows readily from 6.6. □

Corollary 6.8. *If* F, G *are two fiber functors of* $\mathscr{C}$*, then* $F \cong G$.

Proof. Let $(\mathscr{C}, F) \cong (\Pi - \text{Fsets}, \mathscr{F})$. Since there exists a connected object $P := \{P_i\}_{i\in I}$ in $\text{Pro}(\mathscr{C})$ with P_i Galois such that $G(-) = \text{Hom}_{\text{Pro}(\mathscr{C})}(P, -)$. Then there exists $H \subseteq \Pi$ closed which corresponds to P with an arbitrarily chosen point $\xi \in F(P)$ (Here we identify P via 6.6 to a Π-space and F is the forgetful functor). Now for any normal open subgroup $N \subseteq \Pi$ there is an isomorphism $(P_i, \xi) \cong (\Pi/N, e)$, where $e \in \Pi/N$ is the unit. Thus the base point preserving map $P \to P_i$ provides a map $\Pi/H \to \Pi/N$ which sends $e \mapsto e$. In this way we get an inclusion $H \subseteq N$. Thus H is contained in all normal open subgroups of Π, and is therefore trivial. Now (P,ξ) corresponds to (Π, e), and the pointed pro-object which represents F also corresponds to (Π, e), thus they have to be isomorphic. This shows that $F \cong G$. □

Corollary 6.9. *If* X *is a locally Noetherian connected scheme, and if* $x_1, x_2 : \text{Spec}(k) \to X$ *are two geometric points, then* $F_{x_1} \cong F_{x_2}$*. Thus we have an isomorphism* $\pi_1^{\text{ét}}(X, x_1) = \text{Aut}(F_{x_1}) \cong \text{Aut}(F_{x_2}) = \pi_1^{\text{ét}}(X, x_2)$.

Proposition 6.10. *If $(\mathscr{C}, F)$, $(\mathscr{C}', F')$ are Galois categories, and if $H : (\mathscr{C}, F) \to (\mathscr{C}', F')$ is a functor, then H induces a continuous map $\Pi_{(\mathscr{C},F)} \to \Pi_{(\mathscr{C}',F')}$.*

Proof. See [SGA1, Proposition 6.1, pp. 134]. □

Theorem 6.11. *(Riemann existence theorem) Let X be a normal connected scheme of finite type over $\mathbb{C}$. Then the association*

$$(Y \to X) \mapsto (Y^{an} \to X^{an})$$

induces an equivalence between $\text{Ét}(X)$ and finite topological covers of X^{an}. Thus if x is a $\mathbb{C}$ rational point of X, then $\pi_1^{\text{ét}}(X, x)$ is the profinite completion of the topological fundamental group $\pi_1^{\text{top}}(X^{an}, x)$.

Proof. See Hartshorne's *Algebraic Geometry*, Appendix B, pp. 44, Theorem 3.2. □

Theorem 6.12. *Let $f : X \to Y$ be a proper separable morphism with Y a locally Noetherian connected scheme. Suppose that $\mathscr{O}_Y \to f_*\mathscr{O}_X$ is an isomorphism. Then for any geometric point $x : \text{Spec}(k) \to X$ with $y := f(x)$, we have an exact sequence*

$$\pi_1^{\text{ét}}(X_y, x) \to \pi_1^{\text{ét}}(X, x) \to \pi_1^{\text{ét}}(Y, y) \to 1$$

where X_y is the fiber of f at y.

Proof. See [SGA1, Exposé X, Corollaire 1.4, pp. 263]. □

Theorem 6.13. *Let X be a geometrically connected k-scheme with a geometric point*

$$x : \text{Spec}(\bar{k}) \to X$$

Suppose that $X \times_k \bar{k}$ is Noetherian. Then we have an exact sequence

$$1 \to \pi_1^{\text{ét}}(\bar{X}, \bar{x}) \to \pi_1^{\text{ét}}(X, x) \to \pi_1^{\text{ét}}(k, x) = \text{Gal}(k) \to 1$$

where $\bar{x}$ is any lift of x.

Proof. See [SGA1, Exposé IX, Théorèm 6.1, pp. 253]. □

7 The Grothendieck Topology, Sites, Sheaves (30/11/2016)

In this section we fix a category E, and set $\hat{E}$ the category of presheaves on E, i.e. the contravariant functors from E to the category of sets.

Definition 29. A topology on E is an association, to each $S \in E$ we associate a non-empty set $J(S)$, which is the subset of the following set

$$\{\, R \mid R \subseteq \underline{S} \text{ an inclusion in the category } \hat{E} \,\}$$

i.e. the set of sub presheaves of $\underline{S}$. The association $S \mapsto J(S)$ has to satisfy the following axioms:

1. For any arrow $T \to S \in E$ and any $R \in J(S)$, $\underline{T} \times_{\underline{S}} R$ as a sub presheaf of $\underline{T}$ is in $J(T)$.

2. For any subsheaf $R' \subseteq \underline{S}$ and any $R \subseteq \underline{S}$ which is contained in $J(S)$, one has $R' \in J(S)$ as long as for each $\underline{T} \to R \subseteq \underline{S}$ (where $T \in E$) the pullback $\underline{T} \times_{\underline{S}} R$ is $J(T)$.

We call an element of $J(S)$ a refinement of S. We call the pair (E, J) a site.

Lemma 7.1. *(i) The intersection of two refinements of S is still a refinement;*

(ii) A sub presheaf of $\underline{S}$ which contains a refinement of S is a refinement.
In particular the biggest sub presheaf $\underline{S} = \underline{S}$ is in $J(S)$.

Proof. For (i) suppose that $R_1, R_2 \in J(S)$. Then for any $\underline{T} \to R_2 \subseteq \underline{S}$ we have $\underline{T} \times_{\underline{S}} (R_1 \cap R_2) = \underline{T} \times_{\underline{S}} R_1$. By 29 axiom 1, $\underline{T} \times_{\underline{S}} R_1$ is in $J(T)$, and by 29 axiom 2, we have $R_1 \cap R_2 \in J(S)$.
For (ii) suppose that $R_1 \subseteq R$ and $R_1 \in J(S)$. We have for any $\underline{T} \to R_1 \subseteq \underline{S}$, $\underline{T} \times_{\underline{S}} R = \underline{T} \times_{\underline{S}} R_1$. By axiom (1) $\underline{T} \times_{\underline{S}} R \in J(T)$ thus by axiom (2) $R \in J(S)$. □

Definition 30. A pretopology on E is an association: To each $S \in E$ we associate a set Cov(S) of covers of S whose elements are of the form $\mathscr{S} = (S_i \to S)_{i \in I}$. The association has to satisfy the following axioms

1. For any $S \in E$, any $(S_i \to S)_{i \in I} \in \mathrm{Cov}(S)$ and any $T \to S \in E$, $T_i := T \times_S S_i$ exist and $(T_i \to T)_{i \in I}$ is in Cov(T).

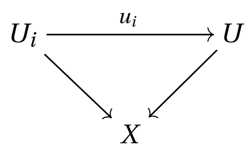
2. If $(S_i \to S)_{i \in I} \in \mathrm{Cov}(S)$ and $(S_{ij} \to S_i)_{j \in J} \in \mathrm{Cov}(S_i)$, then we have $(S_{ij} \to S)_{i \in I, j \in J} \in \mathrm{Cov}(S)$.

3. If $S \in E$, then $\mathrm{id}_S : S = S$ is in Cov(S).

Construction. Given a cover $(S_i \xrightarrow{u_i} S)_{i \in I} \in \mathrm{Cov}(S)$ set $u_i(\underline{S}_i) \subseteq \underline{S}$ the image of $\underline{S}_i$ under u_i. Let $R \subseteq \underline{S}$ be the union of $u_i(\underline{S}_i)$ of all i inside $\underline{S}$. We collect all these $R \subseteq \underline{S}$ which are defined by elements of Cov(X) and denote it by $J'(S)$. The association $S \mapsto J'(S)$ is in general not a topology for E. But we take the smallest topology J on E containing $J'(S)$ for each $S \in E$ and call it the topology generated by the pretopology Cov.

Example 7.2. 1. Let X be a topological space. Let E be the category of open subsetes $U \subseteq X$ with inclusions as morphisms. For each $U \in E$, a cover of U is just a familiy of open subsets $U_i \subseteq U$ such that $\cup_{i \in I} U_i = U$.

2. Let X be a scheme. Let E be the category of étale morphisms $U \to X$ with maps of X-schemes as morphisms. For each $U \to X \in E$, a cover of $U \to X$ is just a familiy of arrows



such that $\cup_{i \in I} u_i(U_i) = U$.

Definition 31. Let (E, J) be a site, and let $F \in \hat{E}$ be a presheaf. We call F a sheaf (resp. a separated presheaf) if for all $S \in E$ and all $R \in J(S)$ the natural morphism

$$\mathrm{Hom}_{\hat{E}}(\underline{S}, F) \to \mathrm{Hom}_{\hat{E}}(R, F)$$

is an isomorphism (resp. injective).

Proposition 7.3. *Let $F \in \hat{E}$ be a presheaf. For all $S \in E$, let $J(S)$ be the set of sub presheaves of $R \subseteq \underline{S}$ such that for any $T \to S$ the natural map*

$$\mathrm{Hom}_{\hat{E}}(\underline{T}, F) \to \mathrm{Hom}_{\hat{E}}(\underline{T} \times_{\underline{S}} R, F)$$

is an isomorphism (resp. injective). Then the association $S \mapsto J(S)$ is a topology on E.

Proof. The axiom 1 of 29 follows from the construction, so we only have to show axiom 2. Axiom 2 follows from the following lemma:

Lemma 7.4. *(a) If $R_1 \subseteq R_2 \subseteq \underline{S}$, if for any $\underline{T} \to R_2 \subseteq \underline{S}$ we have $\underline{T} \times_{\underline{S}} R_1 \in J(T)$ and $R_2 \in J(S)$.*

(b) If $R_1 \subseteq R_2 \subseteq \underline{S}$, and if $R_1 \in J(S)$, then $R_2 \in J(S)$.

Proof. Let's prove (a), (b) simultaneously. Note that for both (a) and (b) we have for any $\underline{T} \to R_2$, the following morphism

$$\mathrm{Hom}_{\hat{E}}(\underline{T}, F) \to \mathrm{Hom}_{\hat{E}}(\underline{T} \times_{\underline{S}} R_1, F) = \mathrm{Hom}_{\hat{E}}(\underline{T} \times_{R_2} R_1, F)$$

is an isomorphism (resp. injective). This would imply that the natural map

$$\lambda : \mathrm{Hom}_{\hat{E}}(R_2, F) \to \mathrm{Hom}_{\hat{E}}(R_1, F)$$

is an isomorphism (resp. injective). For example, let's prove that λ is injective. Suppose $a, b \in \mathrm{Hom}_{\hat{E}}(R_2, F)$ be two different elements. Since $a \neq b$, there exist $T \in E$ and $x \in R(T)$ such that $a(x) \neq b(x)$. By Yoneda's Lemma x corresponds to a map $x^* : \mathrm{Hom}_{\hat{E}}(R_2, F) \to \mathrm{Hom}_{\hat{E}}(\underline{T}, F)$ which sends a, b to the functors that correspond to $a(x), b(x)$. Now look at the commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\hat{E}}(R_2, F) & \xrightarrow{\lambda} & \mathrm{Hom}_{\hat{E}}(R_1, F) \\
\big\downarrow{\scriptstyle x^*} & & \big\downarrow{\scriptstyle \phi} \\
\mathrm{Hom}_{\hat{E}}(\underline{T}, F) & \xrightarrow{\cong} & \mathrm{Hom}_{\hat{E}}(\underline{T} \times_{R_2} R_1, F)
\end{array}$$

Since $a(x) \neq b(x)$, $x^*(a) \neq x^*(b)$. Thus $\phi \circ \lambda(a) \neq \phi \circ \lambda(b)$. So we have $\lambda(a) \neq \lambda(b)$. Now look at the diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\hat{E}}(R_2, F) & \xrightarrow{\lambda} & \mathrm{Hom}_{\hat{E}}(R_1, F) \\
{\scriptstyle \beta}\nwarrow & & \nearrow{\scriptstyle \alpha} \\
& \mathrm{Hom}_{\hat{E}}(S, F) &
\end{array}$$

Since λ is an isomorphism α is an isomorphism iff β is an isomorphism. Note that our argument is stable by base change. Thus $R_1 \in J(S)$ iff $R_1 \in J(S)$ for both (a) and (b). □

For the proposition, we take $R \cap R' \subseteq R$ and $R' \cap R \subseteq R'$. Claim (a) implies that $R \cap R' \in J(S)$, claim (b) then implies that $R' \in J(S)$. □

Corollary 7.5. *Let* Cov *be a pretopology on E, and let $F \in \hat{E}$. We have F is a sheaf (resp. separated presheaf) on E if and only if for all cover $\mathscr{S} := (S_i \to S)_{i \in I}$ the sequence*

$$(*) \qquad F(S) \to \Pi_{i \in I} F(S_i) \rightrightarrows \Pi_{i,j \in I} F(S_i \times_S S_j)$$

is exact.

Proof. Let's just prove the sheaf case because the other case is actually the same. Let $R \subseteq \underline{S}$ be the sub presheaf corresponding to the cover $\mathscr{S}$. Then we have an exact sequence

$$\mathrm{Hom}(R, F) \to \Pi_{i \in I} \mathrm{Hom}(\underline{S_i}, F) \rightrightarrows \Pi_{i,j \in I} \mathrm{Hom}(\underline{S_i \times_S S_j}, F)$$

By Yoneda lemma we have that $(*)$ is exact iff $\mathrm{Hom}(R, F) = \mathrm{Hom}(\underline{S}, F)$. Thus $(*)$ is exact iff all elements in $J'(S)$ are contained in the topology constructed in 7.3, but the last condition is equivalent to that the topology generated by Cov is contained in the topology constructed in 7.3, i.e. F is a sheaf on the topology generated by E. □

Corollary 7.6. *If $\{F_i\}_{i \in I}$ is a collection of elements of $\hat{E}$, then the association*

$$S \mapsto J(S) := \{R \subseteq \underline{S} | \forall T \to S,\ \mathrm{Hom}(\underline{T} \times_{\underline{S}} R, F_i) \xrightarrow{\cong} \mathrm{Hom}(\underline{T}, F_i) \text{ for all } i\}$$

defines a topology on E. In particular, if F_i are taken to be the collection of all representable presheaves on E, then the resulting topology is called the canonical topology on E.

Proof. Clear. □

Theorem 7.7. *Let (E, J) be a site. Let $\tilde{E} \subseteq \hat{E}$ be the full subcategory consisting of presheaves which are sheaves. Then the forgetful functor $i : \tilde{E} \to \hat{E}$ admits a left adjoint $a : \hat{E} \to \tilde{E}$ which is compatible with finite projective limits.*

Proof. For a proof we refer to [SGA4, Exposé II, §3]. But we actually recommend a proof in the case when J comes from a pretopology Cov. One can find the details of the proof in Notes on Grothendieck topologies, fibered categories and descent theory, 2.3.7, pp. 39. Note that by exercise 10.5 any topology comes from a pretopology. Thus this proof does not reduce the generality. □

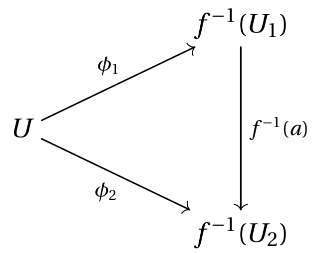
8 Topoi (07/12/2016)

In this section we fix the following notations: If $\mathscr{C}$ is a category, we denote $\hat{\mathscr{C}}$ the category presheaves on $\mathscr{C}$. In $(\mathscr{C}, J)$ is a site, we denote $\tilde{\mathscr{C}}$ the category of sheaves on $(\mathscr{C}, J)$. We have natural functors $\eta : \mathscr{C} \to \hat{\mathscr{C}}$ sending an object to a representable presheaf. There is also a functor $i : \tilde{\mathscr{C}} \to \hat{\mathscr{C}}$ and a functor $a : \hat{\mathscr{C}} \to \tilde{\mathscr{C}}$, the forgetful functor and the sheafification functor, which are adjoint to each other.

Definition 32. Let X, Y be two sites. A functor $f^{-1} : Y \to X$ is called continuous if for any $\mathscr{F} \in \tilde{X}$ the presheaf $\tilde{f}_* \mathscr{F} \in \tilde{Y}$ which is defined as $\mathscr{F} \circ f^{-1}$ is a sheaf.

Proposition 8.1. *Suppose that $f^{-1} : Y \to X$ is a functor. Then the functor $\hat{f}_* : \hat{X} \to \hat{Y}$ sending $\mathscr{F} \mapsto \mathscr{F} \circ f^{-1}$ has a left adjoint $\hat{f}^*$.*

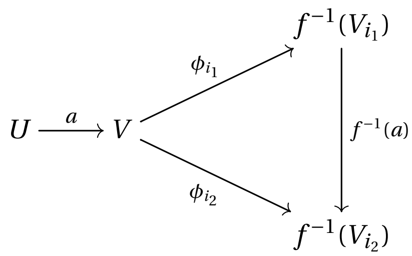
Proof. Define $\hat{f}^* : \hat{Y} \to \hat{X}$ as follows. Given $U \in X$ we define the category I_U to be the category of pairs (U', ϕ) with $U' \in Y$ and $\phi : U \to f^{-1}(U')$. A morphism between two objects $(U_1, \phi_1) \to (U_2, \phi_2)$ consists of a morphism $a : U_1 \to U_2$ in Y and a commutative diagram



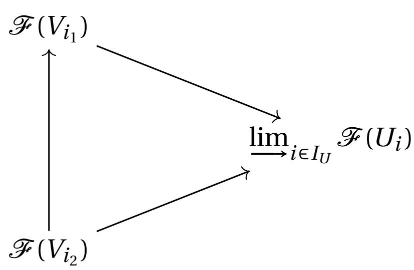
Given $\mathscr{F} \in \hat{Y}$ we define $\hat{f}^* \mathscr{F}$ to be the association $U \mapsto \varinjlim_{i \in I_U} \mathscr{F}(U_i)$, where $i = (U_i, \phi_i)$. Now consider $a : U \to V$ in Y. We want to define

$$\hat{f}^* \mathscr{F}(V) = \varinjlim_{i \in I_V} \mathscr{F}(V_i) \longrightarrow \varinjlim_{i \in I_U} \mathscr{F}(V_i) = \hat{f}^* \mathscr{F}(U)$$

This is obtained as follows. Given an index $(V_i, \phi_i) \in I_V$, we obtain an index $(V_i, \phi_i \circ a) \in I_U$. This index induces a map of sets $\mathscr{F}(V_i) \longrightarrow \varinjlim_{i \in I_U} \mathscr{F}(U_i)$. If we have a morphism $(V_{i_1}, \phi_{i_1}) \to (V_{i_2}, \phi_{i_2})$, then there is a commutative diagram



that is a morphism $(V_{i_1}, \phi_{i_1} \circ a) \to (V_{i_2}, \phi_{i_2} \circ a)$ in I_U. Thus we have a commutative diagram



This induces the desired map by the universal property of the inductive limit. Thus $\hat{f}^*\mathscr{F} \in \hat{X}$. Now let's show the adjointness. First consider $V \in Y$ then $(V, \mathrm{id}_{f^{-1}(V)}) \in I_{f^{-1}(V)}$, where $\mathrm{id}_{f^{-1}(V)} : f^{-1}(V) \xrightarrow{=} f^{-1}(V)$. This index defines a map $\mathscr{F}(V) \longrightarrow \hat{f}^*\mathscr{F}(f^{-1}(V)) = \varinjlim_{i \in I_{f^{-1}(V)}} \mathscr{F}(V_i)$. If we have $a : U \to V$, the diagram

$$\begin{array}{ccc} f^{-1}(V) & = & f^{-1}(V) \\ {\scriptstyle f^{-1}(a)}\downarrow & & \downarrow{\scriptstyle f^{-1}(a)} \\ f^{-1}(U) & = & f^{-1}(U) \end{array}$$

induces the commutative diagram

$$\begin{array}{ccc} \mathscr{F}(V) & \longrightarrow & \hat{f}^*\mathscr{F}(f^{-1}(V)) = \varinjlim_{i \in I_{f^{-1}(V)}} \mathscr{F}(V_i) \\ \downarrow & & \downarrow \\ \mathscr{F}(U) & \longrightarrow & \hat{f}^*\mathscr{F}(f^{-1}(U)) = \varinjlim_{i \in I_{f^{-1}(U)}} \mathscr{F}(U_i) \end{array}$$

Thus we get a map $\psi : \mathscr{F} \to \hat{f}_*\hat{f}^*\mathscr{F}$. Given $\mathscr{G} \in \hat{X}$, $U \in X$, $(U', \phi) \in I_U$, there is a morphism $\mathscr{G}(f^{-1}(U')) \to \mathscr{G}(U)$ which is induced by ϕ, and if we have $(U_1, \phi_1) \to (U_2, \phi_2) \in I_U$, then there will be a commutative diagram

$$\begin{array}{ccc} \mathscr{G}(f^{-1}(U_1)) & \searrow & \\ \uparrow & & \mathscr{G}(U) \\ \mathscr{G}(f^{-1}(U_2)) & \nearrow & \end{array}$$

Thus we have a unique map $\varinjlim_{i \in I_U} \mathscr{G}(f^{-1}(U_i)) = \varinjlim_{i \in I_U} \hat{f}_*\mathscr{G}(U_i) \to \mathscr{G}(U)$. If $a : U \to V$ is a morphism in X, then we have a commutative diagram

$$\begin{array}{ccc} \varinjlim_{i \in I_V} \hat{f}_*\mathscr{G}(V_i) & \longrightarrow & \mathscr{G}(V) \\ \downarrow & & \downarrow \\ \varinjlim_{i \in I_U} \hat{f}_*\mathscr{G}(U_i) & \longrightarrow & \mathscr{G}(U) \end{array}$$

In this way we get a map $\varphi : \hat{f}^* \hat{f}_* \mathscr{G} \to \mathscr{G}$. Now we get maps

$$\lambda_1 : \mathrm{Hom}_{\hat{X}}(\hat{f}^* \mathscr{F}, \mathscr{G}) \to \mathrm{Hom}_{\hat{Y}}(\mathscr{F}, \hat{f}_* \mathscr{G}) \qquad (\hat{f}^* \mathscr{F} \to \mathscr{G}) \mapsto (\mathscr{F} \xrightarrow{\psi} \hat{f}_* \hat{f}^* \mathscr{F} \to \hat{f}_* \mathscr{G})$$

$$\lambda_2 : \mathrm{Hom}_{\hat{Y}}(\mathscr{F}, \hat{f}_* \mathscr{G}) \to \mathrm{Hom}_{\hat{X}}(\hat{f}^* \mathscr{F}, \mathscr{G}) \qquad (\mathscr{F} \to \hat{f}_* \mathscr{G}) \mapsto (\hat{f}^* \mathscr{F} \to \hat{f}^* \hat{f}_* \mathscr{G} \xrightarrow{\varphi} \mathscr{G})$$

One checks that $\lambda_1 \circ \lambda_2$ and $\lambda_2 \circ \lambda_1$ are identities, so they are isomorphisms. □

Theorem 8.2. *Suppose that $f^{-1} : Y \to X$ is a continuous functor between two sites. Then the functor $\tilde{f}_* : \tilde{X} \to \tilde{Y}$ sending $\mathscr{F} \mapsto \mathscr{F} \circ f^{-1}$ has a left adjoint $\tilde{f}^*$.*

Proof. Given $\mathscr{F} \in \tilde{Y}$ we define $\tilde{f}^* \mathscr{F} := a \hat{f}^* i(\mathscr{F})$. Now by the adjointness of a, i and $\hat{f}^*, \hat{f}_*$ we have the following equations

$$\begin{aligned} \mathrm{Hom}_{\tilde{X}}(a\hat{f}^* i(\mathscr{F}), \mathscr{G}) &= \mathrm{Hom}_{\hat{X}}(\hat{f}^* i(\mathscr{F}), i(\mathscr{G})) \\ &= \mathrm{Hom}_{\hat{Y}}(i(\mathscr{F}), \hat{f}_* i(\mathscr{G})) \\ &= \mathrm{Hom}_{\hat{Y}}(i(\mathscr{F}), i(\tilde{f}_* \mathscr{G})) \\ &= \mathrm{Hom}_{\tilde{Y}}(\mathscr{F}, \tilde{f}_* \mathscr{G}) \end{aligned}$$

which finishes the proof. □

Definition 33. Let X, Y be two sites. A *morphism of sites*

$$f : X \to Y$$

consists of a continuous functor $f^{-1} : Y \to X$ such that $\tilde{f}_* : \tilde{X} \to \tilde{Y}$ admits a left adjoint $\tilde{f}^* : \tilde{Y} \to \tilde{X}$ which commutes with finite projective limits.

Lemma 8.3. *If $\mathscr{F} \in \hat{Y}$, then we have $a\hat{f}^*(\mathscr{F}) = \tilde{f}^* a(\mathscr{F})$.*

Proof. Let $\mathscr{G} \in \tilde{X}$. We have the following equations

$$\begin{aligned} \mathrm{Hom}_{\tilde{X}}(\tilde{f}^* a(\mathscr{F}), \mathscr{G}) &= \mathrm{Hom}_{\tilde{Y}}(a(\mathscr{F}), \tilde{f}_*(\mathscr{G})) \\ &= \mathrm{Hom}_{\hat{Y}}(\mathscr{F}, i(\tilde{f}_* \mathscr{G})) \\ &= \mathrm{Hom}_{\hat{Y}}(\mathscr{F}, \hat{f}_* i(\mathscr{G})) \\ &= \mathrm{Hom}_{\hat{Y}}(\hat{f}^* \mathscr{F}, i(\mathscr{G})) \\ &= \mathrm{Hom}_{\tilde{Y}}(a\hat{f}^* \mathscr{F}, \mathscr{G}) \end{aligned}$$

which finish the proof. □

Lemma 8.4. *If $f : X \to Y$ is a morphism of sites, then we have a commutative diagram:*

$$\begin{array}{ccc} X & \xrightarrow{a\eta} & \tilde{X} \\ {\scriptstyle f^{-1}}\uparrow & & \uparrow{\scriptstyle \tilde{f}^*} \\ Y & \xrightarrow{a\eta} & \tilde{Y} \end{array}$$

Proof. Let $V \in Y$. We have $\underline{V} \in \hat{Y}$. For any $\mathscr{G} \in \tilde{X}$ we have the following equations.

$$\begin{aligned}
\mathrm{Hom}_{\tilde{X}}(\tilde{f}^* a(\underline{V}), \mathscr{G}) &= \mathrm{Hom}_{\tilde{X}}(a(\hat{f}^*(\underline{V})), \mathscr{G}) \\
&= \mathrm{Hom}_{\hat{X}}(\hat{f}^*(\underline{V}), i(\mathscr{G})) \\
&= \mathrm{Hom}_{\hat{Y}}(\underline{V}, \hat{f}_* i(\mathscr{G})) \\
&= \hat{f}_* i(\mathscr{G})(V) \\
&= \mathscr{G}(f^{-1}(V)) \\
&= \mathrm{Hom}_{\hat{X}}(\underline{f^{-1}(V)}, i(\mathscr{G})) \\
&= \mathrm{Hom}_{\tilde{X}}(a(\underline{f^{-1}(V)}), \mathscr{G})
\end{aligned}$$

Thus we have $\tilde{f}^* a(\eta(V)) = a(\eta(f^{-1}(V)))$, and this finishes the proof. □

Example 8.5. 1. Let $f : X \to Y$ be a map of topological spaces. Let E_X (resp. E_Y) be the category of opens of X (resp. Y). Then we have a map $f^{-1} : E_Y \to E_X$ sending $V \in E_Y$ to $f^{-1}(V)$. It is obviously continuous and hence admits a left adjoint. See Exercise 11 that this defines a morphism of sites.

2. Let $f : X \to Y$ be a map of schemes. Then the pullback functor $f^{-1} : Y_{\text{ét}} \to X_{\text{ét}}$ defines a continuous functor. Thus we get a pair of adjoint functors. See Exercise 11 that this defines a morphism of sites.

Definition 34. A site is called *standard* if it is coarser than the canonical topology and finite fibred products exist in the category.

Theorem 8.6. *Let T be a category, then the following are equivalent.*

1. *There exists a site (E, J) such that $T \cong \tilde{E}$.*

2. *There exists a standard a site (E, J) such that $T \cong \tilde{E}$.*

3. *Equip T with its canonical topology T becomes a site in which all the sheaves are representable.*

4. *Equip T with its canonical topology T becomes a site, and it satisfies the following axioms*

   a) *Projective limits exist.*

   b) *Direct sums exist and are universal and disjoint.*

   c) *All equivalence relations are effective and universal.*

Proof. For an explaination of the terminologies see [Gir, 2.6.2, pp. 8]. For a proof the theorem see [SGA4, Exposé IV, §1]. □

Definition 35. A category T is called a topos if it satisfies one of the above four conditions.

Theorem 8.7. *Let E be a category. Then there exists a bijection between the topologies on E and the full subcategory $i : T \subseteq \hat{E}$ which is a topos such that the inclusion i admits a left adjoint which commutes with finite projective limits.*

Proof. We have seen the association in the two directions. If we have a topology J on E, then $T := \tilde{E} \subseteq \hat{E}$. If we have a topos with the embedding, then we take the finest topology so that all the presheaves in T are sheaves. □

Theorem 8.8. *Let X, Y be topoi, and let $f* : Y \to X$ a functor. Then the following are equivalent.*

1. *The functor f^* commutes with finite projective limit and arbitrary colimits.*
2. *The functor f^* commutes with finite projective limit and has a right adjoint.*

Definition 36. A *morphism of topoi $X \to Y$* is a pair of adjoint functors (f^*, f_*) such that f^* commutes with finite projective limits.

9 Ringed Topoi (14/12/2016)

Let T be a topos. Then by definition there is a site (E, J) and T is equivalent to $\tilde{E}$. The category $\tilde{E}$ is the sheaf of sets on (E, J). Now we are going to define sheaves of abelian groups and sheaf of rings. It turns out that once $\tilde{E}$ is fixed the category of sheaves of abelian groups or rings does not depend on (E, J) any more.

Definition 37. Let $\mathscr{C}$ be a category which admits finite projective limits. An object $\mathscr{O} \in \mathscr{C}$ is called a ring object if there are operations $m, a : \mathscr{O} \times \mathscr{O} \to \mathscr{O}$, a zero morphism $o : \text{pt} \to \mathscr{O}$, a unit $u : \text{pt} \to \mathscr{O}$, and an additive inverse $i : \mathscr{O} \to \mathscr{O}$. The morphisms (m, a, o, u, i) subject to the unique axioms so that when we see $\mathscr{O}$ as an object $\underline{\mathscr{O}} \in \hat{\mathscr{C}}$ it becomes a presheaf of rings on $\mathscr{C}$ with respect to (m, a, o, u, i). The ringed objects of $\mathscr{C}$ form a category in which a morphism between two ringed objects $(\mathscr{O}_1, m_1, a_1, o_1, u_1, i_1), (\mathscr{O}_2, m_2, a_2, o_2, u_2, i_2)$ is just a morphism $\mathscr{O}_1 \to \mathscr{O}_2$ in $\mathscr{C}$ which is compatible with all the prescribed morphisms. We use $\text{Ring}(\mathscr{C})$ to denote the category of ring objects in $\mathscr{C}$. Similarly one can define group (resp. abelian group) objects in $\mathscr{C}$.

Definition 38. Let T be a topos, and let $\mathscr{O} \in \text{Ring}(T)$. We call the pair $(T, \mathscr{O})$ a ringed topos.

Example 9.1.

1. If T is the category of sets, then $\text{Ring}(T)$ is the category of rings.
2. If (E, J) is a site, then the constant functor $A \mapsto \mathbb{Z}$ for all $A \in E$ defines a constant presheaf of rings on E. The associated sheaf $\mathscr{O}_{\mathbb{Z}}$ with all its multiplication, addition, 0,unit, inverse structures is a ring object in $\tilde{E}$. The pair $(\tilde{E}, \mathscr{O}_{\mathbb{Z}})$ is a ringed topos.
3. If $(X, \mathscr{O}_X)$ is a ringed space, then this is a ringed topos.

Definition 39. Let $\mathscr{C}$ be a category which admits finite projective limits. Let $\mathscr{O}$ be a ringed object in $\mathscr{C}$. We define an $\mathscr{O}$-module object an abelian group object $\mathscr{F}$ in $\mathscr{C}$ together with a morphism $\rho : \mathscr{O} \times \mathscr{F} \to \mathscr{F}$ such that for each $U \in \mathscr{C}$, $\mathscr{F}(U)$ is a module under the ring $\mathscr{O}(U)$ with the action $\rho(U)$. If $(T,\mathscr{O})$ is a ringed topos, then an $\mathscr{O}$-module is just an $\mathscr{O}$-module object in T. A morphism between two $\mathscr{O}$-module objects $(\mathscr{F},\rho_1),(\mathscr{G},\rho_2)$ is just a morphism of abelian group objects $\mathscr{F} \to \mathscr{G}$ which is compatible with ρ_1,ρ_2. The category of $\mathscr{O}$-module objects in T is denoted by $\mathrm{Mod}(T,\mathscr{O})$.

Example 9.2. It is clear that an $\mathscr{O}_{\mathbb{Z}}$-module object in $\tilde{E}$ is just an abelian group object in $\tilde{E}$.

Proposition 9.3. *Let (E,J) be a site, and let $\mathscr{O} \in \mathrm{Ring}(\tilde{E})$. Then the adjoint pair $i : \tilde{E} \to \hat{E}$ and $a : \hat{E} \to \tilde{E}$ induces an adjoint pair between the category of presheaves of $\mathscr{O}$-modules and the category $\mathrm{Mod}(\tilde{E},\mathscr{O})$, which we still denote by i, a. Moreover the functor a commutes with finite projective limits.*

Proof. Let $\mathscr{F}$ be an $\mathscr{O}$-module in $\hat{E}$, and let $\mathscr{G} \in \mathrm{Mod}(\tilde{E},\mathscr{O})$. We have shown that

$$\mathrm{Hom}_{\tilde{E}}(a(\mathscr{F}),\mathscr{G}) \underset{\phi}{\overset{\varphi}{\Longleftrightarrow}} \mathrm{Hom}_{\hat{E}}(\mathscr{F}, i(\mathscr{G}))$$

Since i, a all commute with finite projective limits, the functors for sets induce the corresponding functors for modules. Note that the set of morphisms of $\mathscr{O}$-modules is a subset of morphisms of (pre)sheaves of sets. It is enough to show that map of rings goes to map of rings in both directions. The maps φ, ϕ were defined by the adjunctions

$$\mathscr{F} \to i(a(\mathscr{F}))$$

and

$$a(i(\mathscr{G})) \to \mathscr{G}$$

But since the adjunctions are morphisms of $\mathscr{O}$-modules, ϕ, φ send modules maps to modules maps. The fact that a commutes with finite projective limits follows from the fact that the canonical morphism between sheaves of sets is an isomorphism iff it is an isomorphism as sheaves of modules. □

Proposition 9.4. *Let E be a category. Then $\mathrm{Mod}(\tilde{E},\mathscr{O})$ is an abelian category.*

Proof. Follows readily from the above proposition and the fact that presheaves of $\mathscr{O}$-modules is an abelian category. □

Definition 40. Let $(T_1,\mathscr{O}_1),(T_2,\mathscr{O}_2)$ be two ringed topoi. A morphism of ringed topoi is a morphism of topoi $f := (f^*, f_*)$ plus a map of ring objects $f^\#\colon f^*\mathscr{O}_2 \to \mathscr{O}_1$, or equivalently $\mathscr{O}_2 \to f_*\mathscr{O}_1$. So it is a pair $(f, f^\#)$.

Proposition 9.5. *Let $(f, f^\#) : (T_1,\mathscr{O}_1) \to (T_2,\mathscr{O}_2)$ be a morphism of ringed topoi. Then the map $(f^*, f_*) : T_1 \to T_2$ of topoi induces two adjoint functors*

$$\mathrm{Mod}(T_1,\mathscr{O}_1) \Longleftrightarrow \mathrm{Mod}(T_2,\mathscr{O}_2)$$

which are still denoted by f^* and f_*. Moreover f_* is left exact and f^* is right exact.

Proof. $f_* : \mathrm{Mod}(T_1, \mathscr{O}_1) \Longrightarrow \mathrm{Mod}(T_2, \mathscr{O}_2)$ is defined using the obvious map $f1^\#$ and the pull-back

$$f^* : \mathrm{Mod}(T_2, \mathscr{O}_2) \Longleftrightarrow \mathrm{Mod}(T_1, \mathscr{O}_1)$$

is defined by $\mathscr{F} \mapsto \mathscr{F} \otimes_{f^{-1}\mathscr{O}_2} \mathscr{O}_1$. Note that there is a confusion between the ringed topoi pull-back and the set topoi pullback. Whenever such a confusion exists we will use f^{-1} to denote the set topoi pullback. The proof of the adjointness is basically the same as 9.3. The last statement follows from the general property of adjoint functors. □

Example 9.6.

1. If $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a morphism of ringed spaces, then we have the usual adjoint pairs (f^*, f_*).
2. If X is a scheme, then the functor $X_{\text{ét}} \to$ (Rings) sending $U \to X$ to $\Gamma(U, \mathscr{O}_U)$ is a sheaf of rings. We call this sheaf of rings $\mathscr{O}_{X_{\text{ét}}}$. If $X \to Y$ is a map of schemes then there will be a map of ringed topoi $(\tilde{X}_{\text{ét}}, \mathscr{O}_{X_{\text{ét}}}) \to (\tilde{Y}_{\text{ét}}, \mathscr{O}_{Y_{\text{ét}}})$.

10 Cohomology of ringed topos (04/01/2017)

Let $\mathscr{A}$ be an abelian category.

Definition 41. An object $I \in \mathscr{A}$ is called an injective object if for any injective morphism $f : A \to B$ the induced map of groups

$$\mathrm{Hom}_{\mathscr{A}}(B, I) \longrightarrow \mathrm{Hom}_{\mathscr{A}}(A, I)$$

is surjective. We say that the category $\mathscr{A}$ has enough injectives if any object $A \in \mathscr{A}$ admits an injection $A \hookrightarrow I$ where I is an injective object.

Theorem 10.1. *If $(T, \mathscr{O})$ is a ringed topos, then* $\mathrm{Mod}(T, \mathscr{O})$ *has enough injectives.*

Proof. Suppose (E, J) is a site such that $T = \tilde{E}$. By Stack Project $(\tilde{E}, \mathscr{O}_{\mathbb{Z}})$ has enough injectives. Note that the forgetful functor

$$\psi : \mathrm{Mod}(T, \mathscr{O}) \longrightarrow \mathrm{Mod}(T, \mathscr{O}_{\mathbb{Z}})$$

has a left adjoint $- \otimes_{\mathscr{O}_{\mathbb{Z}}} \mathscr{O}$

$$\varphi : \mathrm{Mod}(T, \mathscr{O}_{\mathbb{Z}}) \longrightarrow \mathrm{Mod}(T, \mathscr{O})$$

and a right adjoint

$$\phi : \mathrm{Mod}(T, \mathscr{O}_{\mathbb{Z}}) \longrightarrow \mathrm{Mod}(T, \mathscr{O})$$

sending an object $M \in \mathrm{Mod}(T, \mathscr{O}_{\mathbb{Z}})$ to $\mathscr{H}om_{\mathrm{Mod}(T,\mathscr{O}_{\mathbb{Z}})}(\mathscr{O}, M)$. Now let M be in $\mathrm{Mod}(T, \mathscr{O})$, then $\delta : \psi(M) \hookrightarrow I$, where I is an injective object in $\mathrm{Mod}(T, \mathscr{O}_{\mathbb{Z}})$. Applying ϕ to δ we get

$$0 \to \phi \circ \psi(M) \to \phi(I)$$

Since for any injection $A \hookrightarrow B \in \mathrm{Mod}(T, \mathscr{O})$ we want to show that

$$\mathrm{Hom}_{\mathrm{Mod}(T,\mathscr{O})}(B, \phi(I)) \longrightarrow \mathrm{Hom}_{\mathrm{Mod}(T,\mathscr{O})}(A, \phi(I))$$

is surjective. But the above is equal to

$$\mathrm{Hom}_{\mathrm{Mod}(T,\mathcal{O}_{\mathbb{Z}})}(\psi(B), I) \longrightarrow \mathrm{Hom}_{\mathrm{Mod}(T,\mathcal{O}_{\mathbb{Z}})}(\psi(A), I)$$

which is a surjection because ψ is exact (it admits a left adjoint) and I is injective. Therefore $\phi(I)$ is injective. Since the adjunction map $M \to \phi\circ\psi(M)$ is clearly injective, we can conclude the proof. □

In general if $\mathscr{A}$ is an abelian category with enough injectives. Then for any object $A \in \mathscr{A}$ we have an *injective resolution*

$$0 \to A \to I_0 \to I_1 \to I_2 \to \cdots$$

where I_n are injective objects in $\mathscr{A}$. Moreover if there is a left exact functor $F : \mathscr{A} \to \mathscr{B}$ where $\mathscr{B}$ is another abelian category, then

$$F(I_0) \to F(I_1) \to \cdots$$

is a complex in $\mathscr{B}$. The i-th cohomology group is denoted by $R^iF(A)$ and R^iF is called the i-th derived functor. We have $R^0F = F$. Indeed the value $R^iF(A)$ does not depend on the choice of the injective resolution and for any map $A \to B$ we could get a natural map $R^iF(A) \to R^iF(B)$ this makes R^iF a functor.

Definition 42. Let $(T,\mathcal{O})$ be a ringed topos. Then we get a left exact functor

$$\mathrm{Hom}_{\mathrm{Mod}(T,\mathcal{O})}(\mathcal{O}, -) : \mathrm{Mod}(T,\mathcal{O}) \to (\mathrm{AbelianGroups})$$

For any $M \in \mathrm{Mod}(T,\mathcal{O})$ the value of the i-th right derived functors are called the cohomology groups of M.

Example 10.2. If $(T,\mathcal{O}) = (\tilde{X}_{\text{ét}}, \mathcal{O}_{X_{\text{ét}}})$, then for any $M \in \mathrm{Mod}(T,\mathcal{O})$ we have that

$$\mathrm{Hom}_{\mathrm{Mod}(T,\mathcal{O})}(\mathcal{O}, M) = M(X)$$

So the right derived functors are the derived functors of the global section functor. For each M, the i-th cohomology group of M is called the i-th étale cohomology of M, and this is often denoted by $H^i(X_{\text{ét}}, M)$.

11 TRIANGULATED CATEGORIES (I) (11/01/2017)

We fix $\mathscr{A}$ an abelian category.

Definition 43. Let $f, g : C^{\cdot} \to D^{\cdot}$ be two maps of cochain complexes in $\mathscr{A}$. We say f is *homotopic* to g if there are maps $s^n : C^n \to D^{n-1}$ such that the following equation

$$f^n - g^n = d^{n-1}s^n + s^{n+1}d^n$$

holds. If $g = 0$, then we call f null homotopic. We call f a *homotopy equivalence* if there exists $h : D^{\cdot} \to C^{\cdot}$ such that $f \circ g$ and $g \circ f$ are all homotopic to the identities.

Lemma 11.1. *Let $f,g : C^\cdot \to D^\cdot$ be two maps of cochain complexes in $\mathscr{A}$. If f and g are homotopic, then the maps $H^n(C^\cdot) \to H^n(D^\cdot)$ induced by f and g are the same.*

Proof. This is a direct computation. □

Definition 44. Let $f : B^\cdot \to C^\cdot$ be a map of cochain complexes in $\mathscr{A}$. The *mapping cone* if the f is the cochain complex whose n-th degree part is $B^{n+1} \oplus C^n$, and whose differentials are given by the following formula:

$$d(b,c) = (b,c)\begin{pmatrix} -d_B & -f \\ 0 & d_C \end{pmatrix} = (-d_B(b), d_C(c) - f(b))$$

for all $b \in B^{n+1}, c \in C^n$. Note that in this way we get a complex:

$$d(d(b,c)) = d(-d_B(b), d_C(c) - f(b)) = (d_B d_B(c), d_C d_C(c) - d_C(f(b)) + f(d_B(b))) = (0,0)$$

The new complex is denoted by $\mathrm{Cone}(f)$.

Lemma 11.2. 1. *If f is taken to be the identity map $\mathrm{id} : C^\cdot \to C^\cdot$, then the complex $\mathrm{Cone}(C^\cdot) :=$ $\mathrm{Cone}(\mathrm{id})$ is split exact, that is, it is exact and there are maps $s^n : C^n \to C^{n-1}$ with the property that $d^{n-1}s^n d^{n-1} = d^{n-1}$.*

2. *Let $f : C^\cdot \to D^\cdot$ be a map of cochain complexes. Then f is null homotopic if and only if f extends to a map of complexes $(-s, f) : \mathrm{Cone}(C^\cdot) \to D^\cdot$.*

3. *There is a short exact sequence*

$$0 \to C^\cdot \xrightarrow{\lambda} \mathrm{Cone}(f) \xrightarrow{\delta} B^\cdot[1] \to 0$$

where $\lambda(c) = (0,c)$ and $\delta(b,c) = -b$. In this way we get a long exact sequence

$$\cdots \to H^{n-1}(\mathrm{Cone}(f)) \to H^n(B^\cdot) \xrightarrow{\partial} H^n(C^\cdot) \to H^n(\mathrm{Cone}(f)) \to H^{n+1}(B^\cdot) \to \cdots$$

where ∂ is exactly the map induced by f.

Proof. For 1, we first show that $\mathrm{Cone}(C^\cdot)$ is exact. Consider the sequence

$$C^n \oplus C^{n-1} \xrightarrow{d} C^{n+1} \oplus C^n \xrightarrow{d} C^{n+2} \oplus C^{n+1}$$

Suppose that $(b,c) \in C^{n+1} \oplus C^n$ and if $d(b,c) = (-d_C(b), d_C(c) - b) = 0$, then we have $d_C(c) = b$. Thus for $(-c, 0) \in C^n \oplus C^{n-1}$ we have $d(-c,0) = (d_C(c), c) = (b,c)$. Thus the sequence is exact. The splitting is given by $s^n(x,y) = (-y, 0)$. We have

$$d^{n-1}s^n d^{n-1}(b,c) = d^{n-1}s^n(-d^n_C(b), d^{n-1}_C(c) - b) = d^{n-1}(b - d^{n-1}_C(c), 0) = (-d_C(b), d^{n-1}_C(c) - b)$$

For 2, first notice that f extends to $(-s, f)$ if and only if the following diagram is commutative:

$$\begin{array}{ccc} C^{n+1} \oplus C^n & \xrightarrow{(-s^n, f^n)} & D^n \\ \downarrow{\scriptstyle d^n} & & \downarrow{\scriptstyle d^n_D} \\ C^{n+2} \oplus C^{n+1} & \xrightarrow{(-s^{n+1}, f^{n+1})} & D^{n+1} \end{array}$$

This is true if and only if $d_D^n \circ (-s^n, f^n) = (-s^{n+1}, f^{n+1}) \circ d^n$. Now take $(b,c) \in C^{n+1} \oplus C^n$, we have $-d_D^n \circ s^{n+1} + d_D^n \circ f^n(c) = s^{n+2}(d^{n+1}(b)) + f^{n+1}(d^n(c)) - f^{n+1}(b)$. The the above equality holds if and only if $f^{n+1} = s^{n+2} \circ d^{n+1} + d_D^n \circ s^{n+1}$, that is, f is null homotopic.

For 3, If $b \in B^{n+1}$ is a cocycle, then $(-b, 0) \in B^{n+1} \oplus C^n$ is a lift of b. Applying the differential we get $(d_B^{n+1}(b), f^{n+1}(b)) = (0, f^{n+1}(b))$. Thus $\partial(b) = f^{n+1}(b)$. □

Corollary 11.3. *A map $f\colon B^\cdot \to C^\cdot$ be a map of cochain complexes is a quasi-isomorphism if and only if* Cone(f) *is exact.*

Definition 45. Let $f : B^\cdot \to C^\cdot$ be a map of cochain complexes in $\mathscr{A}$. A *mapping cylinder* Cyl(f) is defined as follows: The degree n part is $B^n \oplus B^{n+1} \oplus C^n$ and the differential is

$$d(b,b',c) = (b,b',c)\begin{pmatrix} -d_B & 0 & 0 \\ \mathrm{id}_B & d_B & -f \\ 0 & 0 & d_C \end{pmatrix} = (d_B(b) + b', -d_B(b'), d_C(c) - f(b'))$$

In this way we get a complex because

$$\begin{pmatrix} -d_B & 0 & 0 \\ \mathrm{id}_B & d_B & -f \\ 0 & 0 & d_C \end{pmatrix}^2 = \begin{pmatrix} d_B^2 & 0 & 0 \\ d_B - d_B & d_B^2 & f d_B - d_C f \\ 0 & 0 & d_C^2 \end{pmatrix}$$

Proposition 11.4. *Let $f\colon B^\cdot \to C^\cdot$ be a map of cochain complexes.*

1. *If f is taken to be the identity map* $\mathrm{id} : C^\cdot \to C^\cdot$, *then we have the complex* $\mathrm{Cyl}(C^\cdot) := \mathrm{Cyl}(\mathrm{id})$. *Two cochain maps $f, g\colon B^\cdot \to C^\cdot$ are homotopic if and only if they extend to a map* $(f,s,g) : \mathrm{Cyl}(B^\cdot) \to C^\cdot$.

2. *The natural map $\alpha : C^\cdot \to$* Cyl(f), *$c \mapsto (0,0,c)$ is a chain homotopy, and the map to the other direction is $\beta : (b,b',c) \mapsto f(b) + c$.*

3. *There is a commutative diagram with exact rows*

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & C^\cdot & \xrightarrow{\lambda} & \mathrm{Cone}(f) & \xrightarrow{\delta} & B^\cdot[1] & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle \alpha} & & \| & & & & \\ 0 & \longrightarrow & B^\cdot & \xrightarrow{\varphi} & \mathrm{Cyl}(f) & \xrightarrow{\phi} & \mathrm{Cone}(f) & \longrightarrow & 0 & & \end{array}$$

   where $\varphi(b) = (b,0,0)$ and $\phi(b,b',c) = (b',c)$. In this way we get a commutative diagram with exact rows.

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & H^{n-1}(B^\cdot[1]) & \xrightarrow{f_*} & H^n(C^\cdot) & \xrightarrow{\lambda_*} & H^n(\mathrm{Cone}(f)) & \xrightarrow{-\delta_*} & H^n(B^\cdot[1]) & \longrightarrow & \cdots \\ & & \| & & \downarrow{\scriptstyle \alpha_*} & & \| & & \| & & \\ \cdots & \longrightarrow & H^n(B^\cdot) & \xrightarrow{\varphi_*} & H^n(\mathrm{Cyl}(f)) & \xrightarrow{\phi_*} & H^n(\mathrm{Cone}(f)) & \xrightarrow{\partial} & H^{n+1}(B^\cdot) & \longrightarrow & \cdots \end{array}$$

Proof. For 1, the commutativity of the diagram

$$\begin{CD} B^n \oplus B^{n+1} \oplus B^n @>{(f^n, s^{n+1}, g^n)}>> C^n \\ @V{d^n}VV @VV{d_C^n}V \\ B^{n+1} \oplus B^{n+2} \oplus B^{n+1} @>{(f^{n+1}, s^{n+2}, g^{n+1})}>> C^{n+1} \end{CD}$$

is equivalent to that given $(b, b', c) \in B^n \oplus B^{n+1} \oplus B^n$, we have

$$\begin{aligned}(b, b', c) &\mapsto (d_B^n(b) + b', -d_B^{n+1}(b'), d_B^n(c) - b') \\ &\mapsto f^{n+1}(d_B^n(b)) + f^{n+1}(b') - s^{n+2}(d_B^{n+1}(b')) + g^{n+1}(d_B^n(c)) - g^{n+1}(b')\end{aligned}$$

is equal to

$$(b, b', c) \mapsto f^n(b) + s^{n+1}(b') + g^n(c) \mapsto d_C^n(f^n(b)) + d_C^n(s^{n+1}(b')) + d_C^n(g^n(c))$$

Thus we have $f^{n+1}(b') - g^{n+1}(b') = s^{n+2}(d_B^{n+1}(b')) + d_C^n(s^{n+1}(b'))$ if and only if the diagram is commutative.

For 2, first notice that $\beta \circ \alpha = \mathrm{id}_{C^\cdot}$. Since $\alpha \circ \beta(b, b', c) = (0, 0, f(b) + c)$,

$$(\alpha \circ \beta - \mathrm{id}_{\mathrm{Cyl}})(b, b', c) = (-b, -b', f(b))$$

On the other hand, let $s^n \colon B^n \oplus B^{n+1} \oplus C^n \to B^{n+1} \oplus B^{n+2} \oplus C^{n+1}$ sending (b, b', c) to $(0, b, 0)$. Then $(d^{n-1} \circ s^n + s^{n+1} d^n)(b, b', c) = (b, -d^n(b), -f(b)) + (0, d^n(b) + b', 0) = (b, b', -f(b))$. This means that $\alpha \circ \beta$ is homotopic to $\mathrm{id}_{\mathrm{Cyl}}$.

For 3, it is enough to show that $-\delta_* = \partial$. Let (b, c) be an n-cocycle in $\mathrm{Cone}(f)$, so $d_B^{n+1}(b) = 0$ and $f^{n+1}(b) = d_C^n(c)$. Now $(0, b, c)$ is a lift of (b, c). Applying the differential we get

$$d(0, b, c) = (0 + b, -d_B^{n+1}(b), d_C^n(c) - f^{n+1}(b)) = (b, 0, 0)$$

Thus $\partial(b, c) = b$, but $-\delta_*(b, c)$ is, by its very definition, also b. □

Definition 46. Let $\mathbf{Ch}(\mathscr{A})$ or simply $\mathbf{Ch}$, when $\mathscr{A}$ is clear from the context, be the category of cochain complexes in $\mathscr{A}$. Let $\mathbf{K}(\mathscr{A})$ or simply $\mathbf{K}$, when $\mathscr{A}$ is clear from the context, be the *homotopy category* of cochain complexes in $\mathscr{A}$, whose objects are precisely those in $\mathbf{Ch}$ and whose morphisms between two objects $\mathrm{Hom}_{\mathbf{K}}(A, B)$ is the set $\mathrm{Hom}_{\mathbf{Ch}}(A, B)$ modulo the equivalence relation $f \sim g$ if and only if f and g are homotopic. In fact $\sim$ is an equivalent relation because $f - g = ds + sd$ and $g - h = dt + td$ imply that $f - h = d(s + t) + (s + t)d$, and we also have that if $f \sim f'$, $g \sim g'$, then $f \circ g \sim f' \circ g \sim f' \circ g'$.

Proposition 11.5. (Universal property) *Let $F : \mathbf{Ch}(\mathscr{A}) \to \mathscr{D}$ be any functor that sends a cochain homotopy to an isomorphism. Then F factors uniquely through $\mathbf{K}(\mathscr{A})$:*

$$\begin{array}{ccc} \mathbf{Ch}(\mathscr{A}) & \xrightarrow{\quad F \quad} & \mathscr{D} \\ \big\downarrow & \nearrow_{\exists !} & \\ \mathbf{K}(\mathscr{A}) & & \end{array}$$

Proof. By 11.4, (2) we have α is a homotopy equivalence, and since $\beta \circ \alpha = \mathrm{id}$, we have $F(\alpha)$ is an isomorphism whose inverse is $F(\beta)$. As $\alpha' : B^{\cdot} \to \mathrm{Cyl}(B^{\cdot})$, $b \mapsto (b, 0, 0)$ satisfies $\beta \circ \alpha' = \mathrm{id}$, we have $F(\alpha) = F(\alpha')$.
Now if $f, g \colon B^{\cdot} \to C^{\cdot}$ be two cochain homotopies, then by 11.4, (1) there is an extension $\psi \colon =$ $(f, s, g) \colon \mathrm{Cyl}(B^{\cdot}) \to C^{\cdot}$. Moreover $\psi\alpha' = f$ and $\psi\alpha = g$. Hence we have $F(f) = F(g)$. □

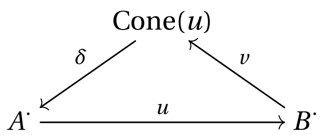
12 Triangulated Categories (II) (18/01/2017)

Let $\mathscr{A}$ be an abelian category, and let $\mathbf{K}(\mathscr{A})$ be the homotopy category of chain complexes. In this lecture we will prove that $\mathbf{K}(\mathscr{A})$ is a triangulated category.

Definition 47. Let $u : A \to B$ be a morphism in $\mathbf{Ch}(\mathscr{A})$. Then we get a split exact sequence

$$0 \to B^{\cdot} \xrightarrow{v} \mathrm{Cone}(u) \xrightarrow{\delta} A^{\cdot}[1] \to 0$$

This data provides a triangle of maps



in $\mathbf{Ch}(\mathscr{A})$. Now if we have any triangle of maps in $\mathbf{K}(\mathscr{A})$, i.e. maps $u : A^{\cdot} \to B^{\cdot}$, $v : B^{\cdot} \to C^{\cdot}$ and $w : C^{\cdot} \to A^{\cdot}[1]$, then we will call the triple (u, v, w) a *distinguished triangle* if there is a triple $u' : A'^{\cdot} \to B'^{\cdot}$, $v' : B'^{\cdot} \to C'^{\cdot}$ and $w' : C'^{\cdot} \to A'^{\cdot}[1]$ and a commutative diagram in $\mathbf{K}(\mathscr{A})$

$$\begin{array}{ccccccc} A^{\cdot} & \xrightarrow{u} & B^{\cdot} & \xrightarrow{v} & C^{\cdot} & \xrightarrow{w} & A^{\cdot}[1] \\ \downarrow f & & \downarrow g & & \downarrow h & & \downarrow f[1] \\ A'^{\cdot} & \xrightarrow{u'} & B'^{\cdot} & \xrightarrow{v'} & C'^{\cdot} & \xrightarrow{w'} & A'^{\cdot}[1] \end{array}$$

in which f, g, h are isomorphisms in $\mathbf{K}(\mathscr{A})$.

Lemma 12.1. *Let $A^{\cdot} \in \mathbf{Ch}(\mathscr{A})$. The complex $A^{\cdot}$ is split exact, i.e. it is acyclic and $\exists$ a splitting s such that $d = dsd$, if and only if the identity morphism $\mathrm{id}_A \in \mathbf{Ch}(\mathscr{A})$ is null homotopic.*

Proof. Let $B'^n \subseteq A^n$ be the image of the composition $A^{n+1} \xrightarrow{s^{n+1}} A^n \xrightarrow{d^n} A^{n+1} \xrightarrow{s^{n+1}} A^n$. Since $d = dsd$, we have $d^n s^{n+1}(A^{n+1}) = d^n(B'^n) \subseteq A^{n+1}$. This implies that $d^n s^{n+1} : A^{n+1} \to A^{n+1}$ induces the identity map on $d^n(B'^n) \subseteq A^{n+1}$. Let $Z^n \subseteq A^n$ be the cocycle. Then $Z^n \bigcap B'^n = 0$ because $d^n s^{n+1}(x) = 0$ implies $x = 0$ and hence $s^{n+1}(x) = 0$ for all $x \in d^n(B'^n)$. On the other hand, we have

$$d^n(A^n) = d^n s^{n+1} d^n(A^n) \subseteq d^n s^{n+1}(A^{n+1}) = d^n(B'^n) \subseteq d^n(A^n)$$

So $d^n(A^n) = d^n(B'^n)$, and this implies that $Z^n \bigoplus B'^n = A^n$. By definition $A^n \to A^{n+1}$ sends Z^n to 0, and by exactness B'^n goes to $Z^{n+1} = d^n(A^n) = d^n(B'^n)$. Moreover $B'^n \longrightarrow Z^{n+1}$ is an isomorphism.

Now we define the splittings $t^{n+1} : A^{n+1} \to A^n$ via the composition

$$A^{n+1} = Z^{n+1} \bigoplus B'^{n+1} \twoheadrightarrow Z^{n+1} \xrightarrow{\cong} B'^{n} \subseteq A^n$$

It is now clear that $\mathrm{id}_{A^n} = d^{n-1}s^n + s^{n+1}d^n$. □

Lemma 12.2. *Every morphism $u : A^\cdot \to B^\cdot$ in $\mathbf{K}(\mathscr{A})$ can be embedded in a distinguished triangle (u, v, w). If u is $A^\cdot = A^\cdot$ the identity and if $C^\cdot = 0^\cdot$, then the triangle $(\mathrm{id}_A, 0, 0)$ is a distinguished triangle. Any triangle which is isomorphic to a distinguished triangle is a distinguished triangle.*

Proof. We only have to show that the triangle $(\mathrm{id}_A, 0, 0)$ is a distinguished triangle. But by 12.1 $\mathrm{Cone}(A^\cdot)$ which is split exact (11.2) is null homotopic. Thus $\mathrm{Cone}(A^\cdot)$ is isomorphic to 0 in $\mathbf{K}(\mathscr{A})$. Now the claim follows from the following commutative diagram

$$\begin{array}{ccccccc}
A^\cdot & \xrightarrow{\mathrm{id}_A} & A^\cdot & \longrightarrow & 0 & \longrightarrow & A^\cdot[1] \\
\downarrow{\scriptstyle \mathrm{id}_A} & & \downarrow{\scriptstyle \mathrm{id}_A} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}_{A[1]}} \\
A^\cdot & \xrightarrow{\mathrm{id}_A} & A^\cdot & \longrightarrow & \mathrm{Cone}(\mathrm{id}_A) & \longrightarrow & A^\cdot[1]
\end{array}$$

□

Lemma 12.3. *Let (u, v, w) be a distinguished triangle on $(A^\cdot, B^\cdot, C^\cdot)$ in $\mathbf{K}(\mathscr{A})$, then both of its rotates $(v, w, -u[1])$ and $(-w[-1], u, v)$ are distinguished triangles on $(B^\cdot, C^\cdot, A^\cdot[1])$ and $(C^\cdot[-1], A^\cdot, B^\cdot)$ respectively.*

Proof. We can suppose that we have a sequence in $\mathbf{Ch}(\mathscr{A})$:

$$A^\cdot \xrightarrow{u} B^\cdot \xrightarrow{v} \mathrm{Cone}(u) \xrightarrow{w} A^\cdot[1]$$

Let's first prove the lemma for $(v, w, -u[1])$. Look at the cochain complex

$$\mathrm{Cone}(v)^n = B^{n+1} \oplus \mathrm{Cone}(u)^n = B^{n+1} \oplus A^{n+1} \oplus B^n$$

whose differential is given by

$$(b', a, b) \mapsto (-d_B^{n+1}(b'), d^n_{\mathrm{Cone}(u)}(a, b) - (0, b')) = (-d_B^{n+1}(b'), -d_A^{n+1}(a), d_B^n(b) - u^n(a) - b')$$

In this way we get a morphism of complexes $\phi : \mathrm{Cone}(v) \to A^\cdot[1]$ via the second projection. We want to show that it is an homotopy equivalence. The homotopy inverse $\varphi : A^\cdot[1] \to \mathrm{Cone}(v)$ is given by $a \mapsto (-u(a), a, 0)$. This defines a map in $\mathbf{Ch}(\mathscr{A})$ because the following two compositions are identical.

$$a \mapsto -d_A(a) \mapsto (u(d_A(a)), -d_A(a), 0)$$

$$a \mapsto (-u(a), a, 0) \mapsto (d_A(u(a)), -d_A(a), 0)$$

Clearly $\phi \circ \varphi = \mathrm{id} \in \mathbf{Ch}(\mathscr{A})$, so we only have to show that $\varphi \circ \phi$ is homotopic to the identity. The homotopy $s^{n+1} : \mathrm{Cone}(v)^{n+1} \to \mathrm{Cone}(v)^n$ is given by $(b', a, b) \mapsto (b, 0, 0)$. Since we have

$$\begin{aligned}
& d^{n-1}_{\mathrm{Cone}(v)}(b,0,0) + s^{n+1}(-d_B^{n+1}(b'), -d_A^{n+1}(a), d_B^n(b) - u^n(a) - b') \\
=& (-d_B^n(b), 0, -b) + (d_B^n(b) - u^n(a) - b', 0, 0) \\
=& (-u^n(a) - b', 0, -b) \\
=& (-u^n(a), a, 0) - (b', a, b)
\end{aligned}$$

there is an equation $\varphi \circ \phi - \mathrm{id} = sd + ds$. Now consider the following diagrams

$$\begin{array}{ccccccc}
B^{\cdot} & \xrightarrow{v} & \mathrm{Cone}(u) & \longrightarrow & \mathrm{Cone}(v) & \xrightarrow{\delta} & B^{\cdot}[1] \\
\| & & \| & & \downarrow{\scriptstyle -\phi} & & \downarrow{\scriptstyle \mathrm{id}_B[1]} \\
B^{\cdot} & \xrightarrow{v} & \mathrm{Cone}(u) & \xrightarrow{w} & A^{\cdot}[1] & \xrightarrow{-u[1]} & B^{\cdot}[1]
\end{array}$$

It is enough to check that the last diagram is commutative in $\mathbf{K}(\mathscr{A})$, i.e. we have $\delta = u[1] \circ \phi$. But this is the same as checking $\delta \circ \varphi = u[1]$ which is obvious.

Now let's first prove the lemma for $(-w[-1], u, v)$. We can write the $\mathrm{Cone}(w[-1])$ as the complex $A^n \oplus A^{n+1} \oplus B^n$ with the differential $d^n : A^n \oplus A^{n+1} \oplus B^n \to A^{n+1} \oplus A^{n+2} \oplus B^{n+1}$ sending

$$(a, a', b) \mapsto (d_A(a) + a', -d_{\mathrm{Cone}(u)[-1]}(a', b)) = (d_A(a) + a', -d_A(a'), d_B(b) - u(a'))$$

Thus we have $\mathrm{Cone}(w[-1]) = \mathrm{Cyl}(u)$. Now by 11.4 (2) the following commutative diagrams

$$\begin{array}{ccccccc}
\mathrm{Cone}(u)[-1] & \xrightarrow{w[-1]} & A^{\cdot} & \longrightarrow & \mathrm{Cone}(w[-1]) = \mathrm{Cyl}(u) & \xrightarrow{\delta} & \mathrm{Cone}(u) \\
\downarrow{\scriptstyle -\mathrm{id}} & & \| & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle -\mathrm{id}_B[1]} \\
\mathrm{Cone}(u)[-1] & \xrightarrow{-w[-1]} & A^{\cdot} & \xrightarrow{u} & B^{\cdot} & \xrightarrow{v} & \mathrm{Cone}(u)
\end{array}$$

with the same trick as above provides the desired distinguished triangle. □

Definition 48. Suppose that $\mathbf{K}$ is an additive category equipped with an automorphism $T : \mathbf{K} \to \mathbf{K}$. We call $\mathbf{K}$ a *triangulated category* if $\mathbf{K}$ is equipped with a class of distinguished triangles which are subject to the following four axioms:

1. (TR1) Any triangle isomorphic to a distinguished triangle is a distinguished triangle. Any triangle of the form $(X, X, 0, \mathrm{id}, 0, 0)$ is distinguished. For any morphism $f : X \to Y$ of $\mathbf{K}$ there exists a distinguished triangle of the form (X, Y, Z, f, g, h).

2. (TR2) The triangle (X, Y, Z, f, g, h) is distinguished if and only if the triangle $(Y, Z, T(X), g, h, -T(f))$ is.

3. (TR3) Given a solid diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & T(X) \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \exists \, \vdots \, {\scriptstyle c} & & \downarrow{\scriptstyle T(a)} \\
X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & T(X')
\end{array}$$

whose rows are distinguished triangles and which satisfies $b \circ f = f \circ a$, there exists a morphism $c : Z \to Z'$ such that (a, b, c) is a morphism of triangles.

4. (TR4) Given objects X, Y, Z of $\mathbf{K}$, and morphisms $f : X \to Y, g : Y \to Z$, and distinguished triangles (X, Y, Q_1, f, p_1, d_1), $(X, Z, Q_2, g \circ f, p_2, d_2)$, and (Y, Z, Q_3, g, p_3, d_3), there exist morphisms $a : Q_1 \to Q_2$ and $b : Q_2 \to Q_3$ such that
   a) $(Q_1, Q_2, Q_3, a, b, T(p_1)d_3)$ is a distinguished triangle,
   b) the triple (id_X, g, a) is a morphism of triangles $(X, Y, Q_1, f, p_1, d_1) \to (X, Z, Q_2, g \circ f, p_2, d_2)$, and
   c) the triple (f, id_Z, b) is a morphism of triangles
$$(X, Z, Q_2, g \circ f, p_2, d_2) \to (Y, Z, Q_3, g, p_3, d_3)$$

The category $\mathbf{K}$ is called a *pre-triangulated category* if TR1-TR3 hold.

Corollary 12.4. *Let* $\mathbf{K}$ *be a triangulated category, and let* (X, Y, Z, f, g, h) *be a distinguished triangle. Then* $g \circ f$*,* $h \circ g$ *and* $f[1] \circ h$ *are all 0.*

Proof. By TR1 $(X, X, 0, 1, 0, 0)$ is a distinguished triangle. Applying TR3 we have a commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{\mathrm{id}} & X & \xrightarrow{0} & 0 & \xrightarrow{\exists} & T(X) \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle f} & & \exists \vdots c & & \downarrow{\scriptstyle \mathrm{id}} \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & T(X) \end{array}$$

we get the dashed arrow c which has to be the unique arrow 0. Thus $g \circ f = 0$ and TR2 takes care of the others. □

Proposition 12.5. *The category* $\mathbf{K}(\mathscr{A})$ *is a triangulated category.*

Proof. TR1 and TR2 are proved in 12.2 and 12.3. TR3 is obvious from the functoriality of the mapping cone. For TR4 we recommend Stack Project. □

13 Derived Categories (25/01/2017)

Definition 49. Let S be a collection of morphisms in a category $\mathscr{C}$. A *localization of* $\mathscr{C}$ *with respect to* S is a category $S^{-1}\mathscr{C}$ together with a functor $q : \mathscr{C} \to S^{-1}\mathscr{C}$ such that

1. $q(S)$ is an isomorphism in $S^{-1}\mathscr{C}$ for every $s \in S$ and,
2. any functor $F : \mathscr{C} \to \mathscr{D}$ such that $F(s)$ is an isomorphism for all $s \in S$ factors in a unique way through q.

Example 13.1. If $\mathscr{A}$ is an abelian category, $\mathscr{C} := \mathbf{Ch}(\mathscr{A})$ and S is the collection of all the homotopy equivalences in $\mathbf{Ch}(\mathscr{A})$. We have seen in 11.5 that $\mathbf{K}(\mathscr{A})$ with the natural projection $\mathbf{Ch}(\mathscr{A}) \to \mathbf{K}(\mathscr{A})$ is the localization of $\mathbf{Ch}(\mathscr{A})$ at the homotopy equivalences.

Definition 50. A collection S of morphisms in a category $\mathscr{C}$ is called a *multiplicative system* in $\mathscr{C}$ if it satisfies the following axioms:

1. The collection S is closed under composition and contains all the identity morphisms.
2. If $t : Z \to Y$ is in S, then for every $g : X \to Y$ in $\mathscr{C}$ there is a commutative diagram $gs = tf$ in $\mathscr{C}$ with $s \in S$.
$$\begin{array}{ccc} W & \xrightarrow{f} & Z \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle t} \\ X & \xrightarrow{g} & Y \end{array} \tag{13.1}$$
Moreover, the symmetric statement holds for any s and f.
3. If f, g are parallel morphisms in $\mathscr{C}$, then the following two conditions are equivalent:
    a) $sf = sg$ for some $s \in S$ and,
    b) $ft = gt$ for some $t \in S$.

Example 13.2. Let R be a possibly non-commutative ring, and let $\mathscr{C}$ be the category with only one object $E \in \mathscr{C}$ whose morphisms are defined by the elements in R with its multiplicative structure. If $S \subseteq R$ is a subset which is contained in the center, then S is a multiplicative subset if and only if it is a multiplicative subset in the usual sense.

Construction. Let $\mathscr{C}$ be a category, and let S a multiplicative system of $\mathscr{C}$. We are going to construct $S^{-1}\mathscr{C}$ and $q : \mathscr{C} \to S^{-1}\mathscr{C}$.
Let $S^{-1}\mathscr{C}$ be the category whose objects are precisely those of $\mathscr{C}$, and whose morphisms $\mathrm{Hom}_{S^{-1}\mathscr{C}}(X, Y)$ are the equivalent classes of (left) fractions $fs^{-1} : X \xleftarrow{s} X_1 \xrightarrow{f} Y$ where $s \in S$, and $fs^{-1} \sim t^{-1}g$ for $gt^{-1} : X \xleftarrow{t} X_2 \xrightarrow{g} Y$ if and only if there exists a left fraction $\phi\alpha^{-1} : X \xleftarrow{\alpha} X_4 \xrightarrow{\phi} Y$ with morphisms $\phi\alpha^{-1} \Rightarrow fs^{-1}$ and $\phi\alpha^{-1} \Rightarrow gt^{-1}$ making all the diagrams commutative. To check that $\sim$ is an equivalence relation we have to show that if $hr^{-1} : X \xleftarrow{r} X_3 \xrightarrow{h} Y$ is equivalent to $fs^{-1} : X \xleftarrow{s} X_1 \xrightarrow{f} Y$ by a fraction $\phi\beta^{-1} : X \xleftarrow{\beta} X_5 \xrightarrow{\varphi} Y$ then $hr^{-1} \sim gt^{-1}$. It is enough to show that $a : \phi\alpha^{-1} \Rightarrow fs^{-1}$ and $a : \varphi\beta^{-1} \Rightarrow fs^{-1}$ implies that $\phi\alpha^{-1} \sim \varphi\beta^{-1}$. Apply 50, 2 to α, β we get $X \leftarrow X_6'$ with two arrows $X_6' \to X_4$ and $X_6' \to X_5$. Applying 50 3 to s and $X_6' \to X_4 \to X_1, X_6' \to X_5 \to X_1$ we get a morphism $X_6 \to X_6'$ in S so that the two morphisms $X_6 \to X_1$ coincide. Now there is a unique morphism $e : X_6 \to Y$ and a morphism $u : X_6 \to X$ which provides the left fraction $eu^{-1} : X \xleftarrow{u} X_6 \xrightarrow{e} Y$ and morphisms $eu^{-1} \Rightarrow \phi\alpha^{-1}, eu^{-1} \Rightarrow \varphi\beta^{-1}$.

The composition

$$\mathrm{Hom}_{S^{-1}\mathscr{C}}(X, Y) \times \mathrm{Hom}_{S^{-1}\mathscr{C}}(Y, Z) \to \mathrm{Hom}_{S^{-1}\mathscr{C}}(X, Z)$$

is given by the following commutative diagram applying 50, 2:

$$\begin{CD} @. W @>u>> W_2 @>g>> Z \\ @. @VvVV @VVtV @. \\ X @<<s< W_1 @>>f> Y \end{CD} \tag{13.2}$$

where $fs^{-1} \in \mathrm{Hom}_{S^{-1}\mathscr{C}}(X,Y)$ and $gt^{-1} \in \mathrm{Hom}_{S^{-1}\mathscr{C}}(Y,Z)$. We define the product of fs^{-1} and gt^{-1} to be $(gu)(sv)^{-1}$. Suppose that there is $f's'^{-1} : X \xleftarrow{s'} W_1' \xrightarrow{f'} Y$ and a map $f's'^{-1} \Rightarrow fs^{-1}$, then we apply 50, 2 to v and $W_1' \to W_1$ we get the desired equivalence. Suppose that there is $g't'^{-1} : Y \xleftarrow{t'} W_2' \xrightarrow{g'} Z$ and a map $g't'^{-1} \Rightarrow gt^{-1}$, then we apply 50, 2 to tu and $W_2' \to W_2 \xrightarrow{t} Y$ and use 50, 3 to get the desired equivalence. The other direction can be proved similarly.

For the associativity is indicated in the following diagrams

$$\begin{array}{ccccccc} Z & & & & & & \\ \uparrow & & & & & & \\ W_2 & \dashrightarrow & Y & & & & \\ \uparrow & & \uparrow & & & & \\ W_3 & \dashrightarrow & W_1 & \dashrightarrow & X & & \\ \uparrow & & \uparrow & & \uparrow & & \\ W_6 & \dashrightarrow & W_5 & \dashrightarrow & W_4 & \dashrightarrow & T \end{array}$$

where the dashed arrows are all in S. In this way $S^{-1}\mathscr{C}$ becomes a category and $q : \mathscr{C} \to S^{-1}\mathscr{C}$ is defined in an obvious way.

Theorem 13.3. *The category $\mathscr{C}$ with the functor $q : \mathscr{C} \to S^{-1}\mathscr{C}$ is the localization of $\mathscr{C}$ at S.*

Proof. The fact that elements of $q(S)$ are convertible in $S^{-1}\mathscr{C}$ can be seen via the diagrams

$$\begin{CD} @. X @= X @= X \\ @. @| @VVsV @. \\ X @= X @>>s> Y @. \end{CD} \qquad \begin{CD} @. X @= X @>s>> Y \\ @. @| @| @. \\ X @<<s< X @= Y @. \end{CD} \tag{13.3}$$

For any functor $F : \mathscr{C} \to \mathscr{D}$ which sends S to isomorphisms, we define $G : S^{-1}\mathscr{C} \to \mathscr{D}$ by sending

$$(X \xleftarrow{s} W \xrightarrow{f} Y) \mapsto (F(X) \xrightarrow{F(s)^{-1}} F(W) \xrightarrow{F(f)} F(Y))$$

This does not depend on the choice of the representative, and using the diagrams we drew it is clear that G is functor. We also have $G \circ q = F$. Clearly the functor G is unique, because for a morphism $fs^{-1} : X \xleftarrow{s} W \xrightarrow{f} Y \in \mathrm{Hom}_{S^{-1}\mathscr{C}}(X,Y)$ we can write $fs^{-1} = q(f) \circ q(s)^{-1}$ according to the following diagram

$$\begin{array}{ccccc} & & W & = & W & \xrightarrow{f} & Y \\ & & \| & & \| & & \\ X & \xleftarrow[s]{} & W & = & W & & \end{array}$$

□

Lemma 13.4. *If $\mathscr{C}$ is an additive category, then $S^{-1}\mathscr{C}$ is also an additive category, and q is an additive functor.*

Proof. The proof is routine and will be left as an exercise. □

Let $\mathbf{K}$ be triangulated category, and let $H : \mathbf{K} \to \mathscr{A}$ be a cohomological functor. We define S to be the collection of morphisms of $\mathbf{K}$ whose image under $H \circ T^i$ are all isomorphisms for $i \in \mathbb{Z}$.

Theorem 13.5. *Notations being as above, we have*

1. *S is a multiplicative system;*
2. *$S^{-1}\mathbf{K}$ is still a triangulated category and q is a morphism between triangulated categories.*

Proof. Let's prove 1. 50, 1 is obvious. For50, 2 we start with $f : X \to Y$ and $s : Y \to Z$, and assume that $s \in S$. Embed s into a distinguished triangle (s, u, δ, Y, Z, W) and complete $uf : X \to C$ to an distinguished triangle (t, uf, v, W, X, C). Now have the following commutative diagram by TR3:

$$\begin{array}{ccccccc} W & \xrightarrow{t} & X & \xrightarrow{uf} & C & \xrightarrow{v} & T(W) \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \| & & \downarrow{\scriptstyle T(g)} \\ Z & \xrightarrow{s} & Y & \xrightarrow{u} & C & \xrightarrow{\delta} & T(Z) \end{array}$$

The fact that $H^i(s)$ is an isomorphism implies that $H^i(C) = 0$, which implies that $H^i(s)$ is an isomorphism.

To see 50, 3 let's consider the difference $f - g : X \to Y$. Given $s : Y \to Y'$ in S with $sf = sg$, embed s in an exact triangle (u, s, δ, Z, Y, Y'). Note that $H^i(Z) = 0$. Since $\mathrm{Hom}_{\mathbf{K}}(X, -)$ is a cohomological functor,

$$\mathrm{Hom}_{\mathbf{K}}(X,Z) \xrightarrow{u} \mathrm{Hom}_{\mathbf{K}}(X,Y) \xrightarrow{s} \mathrm{Hom}_{\mathbf{K}}(X,Y')$$

is exact. Since $s(f - g) = 0$, there is a h such that $f - g = uh$. Embed h into a distinguished triangle (t, h, w, X', X, Z) we get $ht = 0$, so $(f - g)t = hut = 0$. Since $H^i(Z) = 0$ we have that $t \in S$, and this finishes the proof of 1.

For the second one define a distinguished triangle in $S^{-1}\mathbf{K}$ to be those which are isomorphic to an image of a distinguished triangle in $\mathbf{K}$ under q. One checks easily that TR1-TR3 are satisfied. For details see Stack Project. □

Corollary 13.6. *Notations and assumptions are as above. Let* $F : \mathbf{K} \to \mathbf{L}$ *be a morphism of triangulated categories such that* $F(s)$ *is an isomorphism for all* $s \in S$*. The induced functor* $G : S^{-1}\mathbf{K} \to \mathbf{L}$ *is morphism of triangulated categories.*

Example 13.7. Let $(T, \mathscr{O})$ be a ringed topos, and let $\mathscr{A}$ be the category of $\mathscr{O}$-module objects in T. Then we get triangulated categories $\mathbf{D}(\mathscr{A})$, $\mathbf{D}^b(\mathscr{A})$, $\mathbf{D}^+(\mathscr{A})$ and $\mathbf{D}^-(\mathscr{A})$ via localizing the homotopy categories $\mathbf{K}(\mathscr{A})$, $\mathbf{K}^b(\mathscr{A})$, $\mathbf{K}^+(\mathscr{A})$ and $\mathbf{K}^-(\mathscr{A})$ (unbounded, bounded, bounded from below, bounded from above).

14 The Étale Topos (01/02/2017)

15 The Étale Cohomology (08/02/2017)

16 The Six operations (15/02/2017)