Taylor’s Theorem and some Applications, Including one in Finance

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Introduction

Calculus curriculum tends to provide numerous applications of its content. However, some literature on mathematics education argues that these applications tend to be biased towards those in engineering and physical sciences – often leaving out the social sciences such as economics as well as business disciplines such as Finance (628). Schroder notes this may be due to the fact applications in these disciplines tend to require much more mathematical maturity to work through then one has while in a Calculus sequence (629). While it is true that some applications (particularly in economics) require more formal training in mathematics to work through, there are many topics in which examples appropriate for Calculus students can be found for those with interests outside of engineering and physical sciences. Taylor's Theorem, which has extensive usage in finance (both at a relatively basic level and at an advanced level) constitutes such a topic. To demonstrate this point, this paper is organized as follows: A review of Taylor's Theorem and its proof is provided followed by a survey of its typical applications (specifically, numerical approximation, derivation of inequalities, extrema, and convexity). Then, the paper shifts to provide an example of how Taylor's Theorem can be used to derive an important result concerning the change in a bond price.

Taylor's Theorem

In informal terms, Taylor's Theorem posits that a function with $n+1$ derivatives (the first n derivatives being continuous on some interval) can be written as an infinite series – this infinite series being the *Taylor Series*. A more formal definition of Taylor's Theorem, taken from Bartle and Sherbert appears below:

Taylor's Theorem: $Let\ n \in \mathbb{N}, let\ I \coloneqq [a,b], and\ let\ f: I \to \mathbb{R}\ be\ such\ that\ f\ and\ its\ derivatives\ f', f'', \dots, f^n\ are\ continious\ on\ I\ and\ that\ f^{n+1} exists\ on\ (a,b). If\ x_0 \in I, then\ for\ any\ x\ in\ I\ there\ exists\ a\ point\ c\ between\ x\ and\ x_0\ such\ that$

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^n(x_0)}{n!}(x - x_0)^n + \frac{f^{n+1}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

Recall that $\frac{f^{n+1}(c)}{(n+1)!}$ constitutes the *error term* while the first $n+1$ terms constitute the *Taylor Polynomial*. Thus, for shorthand, let P_n denote the nth Taylor Polynomial and let R_n denote the remainder term. Hence, we have that $f(x) = P_n(x) + R_n(x)$. The corresponding proof (which again is based off Bartle and Sherbert) appears on the following page.

Proof. Let x_0 and x be given and let I be the closed interval with endpoints x_0 and x. First, we define a function F as appears below. Note that F is essentially equivalent to R_n in the theorem.

$$F(t) \coloneqq f(x) - f(t) - (x-t)f'(t) - \cdots - \frac{(x-t)^n}{n!} f^n(t)$$

for $t \in I$. Differentiating with respect to t will yield

$$F'(t) = -f'(t) - (x-t)f''(t) + f'(t) - \frac{f'''(t)}{2!}(x-t)^2 + f''(t)(x-t)$$
$$- \cdots - \frac{f^{n+1}(t)}{n!}(x-t)^n + \frac{f^n(t)}{(n-1)!}(x-t)^{n+1}.$$

Canceling out the like terms leaves

$$F'(t) = -\frac{f^{n+1}(t)}{n!}(x-t)^n.$$

Now define G on I as follows:

$$G(t) = F(t) - \left(\frac{x-t}{x-x_0}\right)^{n+1} F(x_0)$$

for $t \in I$. Note that

$$G(x_0) = F(x_0) - \left(\frac{x-x_0}{x-x_0}\right)^{n+1} F(x_0) = F(x_0) - F(x_0) = 0$$

and

$$G(x) = F(x) - \left(\frac{x-x}{x-x_0}\right)^{n+1} F(x_0) = F(x) = f(x) - f(x) - \sum_{i=1}^{n} \frac{x-x}{i!} f^n = 0$$

Hence, $G(x_0) = G(x) = 0$. Recall Rolle's Theorem, which states that if f is continuous on a closed interval $J = [a, b]$ and if the derivative f' exists at every point of the open interval (a, b), and that $f(a) = f(b) = 0$, then there exists at least one point c in (a, b) such that $f'(c) = 0$.

Thus, applying Rolle's Theorem here gives a point c between x and x_0 such that

$$0 = G'(c) = F'(c) + (n+1)\frac{(x-c)^n}{(x-x_0)^{n+1}}F(x_0).$$

The final step requires solving for $F(x_0)$:

$$F(x_0)(n+1)\frac{(x-c)^n}{(x-x_0)^{n+1}} = -F'(c)$$

$$\rightarrow F(x_0)\ \frac{(x-c)^n}{(x-x_0)^{n+1}} = -\frac{F(c)}{n+1}$$

$$\rightarrow F(x_0) = -\frac{1}{n+1}\frac{(x-x_0)^{n+1}}{(x-c)^n}F'(c)$$

Recall that $F'(t) = -\frac{f^{n+1}(t)}{n!}(x-t)^n$. Thus,

$$F(x_0) = \frac{1}{n+1}\frac{(x-x_0)^{n+1}}{(x-c)^n}\frac{f^{n+1}(c)}{n!}(x-c)^n$$

$$\rightarrow F(x_0) = \frac{f^{n+1}(c)}{(n+1)!}(x-x_0)^{n+1}.$$

But, recall that F is equivalent to R_n. Hence, $R_n(x) = F(x_0)$. This completes the proof. ■

As a side note to the above formulation of Taylor's Theorem as well as its proof pertains to the fact that Taylor's Theorem has numerous forms. In fact, Taylor's Theorem has numerous forms. For instance, *Numerical Mathematics and Computing* by Cheney and Kincaid provides four variations of Taylor's Theorem. The differences simply relate to the presentation of the error term, R_n. Similarly, many more proof techniques for Taylor's Theorem exist in addition to the technique presented above. For example, a common alternative utilizes multiple invocations of L'Hopital's rule to reach the required result.

Mathematical Applications

Taylor's Theorem, as alluded to in the introduction, has a plethora of applications. In addition to its applications in finance, four mathematical applications are discussed below: numerical approximation, inequalities, extrema, and convexity.

Numerical Approximation. Out of the four mathematical applications, numerical approximation is likely the one that gets utilized the most. To understand the motivation, consider estimating the value of the following three functions at $x = 0.039$.

$$f(x) = x$$

$$g(x) = \sqrt{x}$$

$$h(x) = \sqrt[3]{1+x}$$

Estimating f is, of course, trivial. $g(x)$ may not be immediately obvious, but if one notes that $\sqrt{0.04} = 0.02$, one might reasonably guess that $\sqrt{0.039} \approx 0.198$. Using a calculator, one can see that this quick mental estimation is not that far off: $\sqrt{0.039} \approx 0.19748$. $h(x)$, on the other hand, might prove to be a challenge to estimate mentally since there is no good "mental anchor." However, by utilizing Taylor's Theorem a relatively precise estimate can be obtained without the need to resort to a calculator with a cube root function!

For $h(x)$, apply Taylor's Theorem with $n = 3$ and $x_0 = 0$. To do so, first take the necessary derivatives of $h(x)$:

$$h(x) = \sqrt[3]{1+x} = (1+x)^{\frac{1}{3}}$$

$$h'(x) = \frac{1}{3}(1+x)^{-\frac{2}{3}}$$

$$h''(x) = -\frac{2}{9}(1+x)^{-\left(\frac{5}{3}\right)}$$

$$h'''(x) = \frac{10}{27}(1+x)^{-\left(\frac{8}{9}\right)}$$

Plugging this into Taylor's Theorem yields the following:

$$h(x) = h(0) + h'(0)x + \frac{h''(0)}{2}x^2 + \frac{h'''(0)}{6}x^3 + R_3$$
$$\to h(x) = 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 + R_3$$

Thus,

$$h(0.039) \approx 1.012835.$$

One should note the precision of this estimate:

$$R_3(x) = \frac{f^{iv}(c)}{24}x^4 = -\frac{10}{243}(1+c)^{-\frac{11}{3}}x^4$$

Since $c \in (x_0, x) = (0, 0.039)$, then $(1+c)^{-\frac{11}{3}} < 1$. Hence, a bound for the error will be $|R_3(0.039)| \leq \frac{10}{243}(0.039)^4 = 9.52\times10^{-8}$. In other the words, the error is quite miniscule. Moreover, using a calculator, one will get 1.01283. Hence, Taylor's Theorem provides a very precise estimate.

Inequalities. Determining bounds for various functions can be very useful. For instance, the bounds on $\sin(x)$ and $\cos(x)$ often prove to be quite useful in a variety of situations:

$$-x \leq \sin(x) \leq x$$
$$-x \leq \cos(x) \leq x$$

Various techniques exist to derive such inequalities. One technique, of course, is Taylor's Theorem. For a more interesting example, consider the following two numbers: π^e and e^π. Which one is greatest? At first it might seem to be impossible to determine without a calculator. However, a quick application of Taylor's Theorem makes the answer easy to derive.

To begin, first establish the following: $e^x > 1 + x$ for $x \neq 0$. This can be done via Taylor's Theorem with $n = 1$ and $x_0 = 0$:

$$e^x = 1 + x + R_1(x)$$

Note that

$$R_1(x) = \frac{1}{2}e^c x^2 > 0 \text{ for } x \neq 0.$$

Also note that

$$\pi > e.$$

Taking $x = \frac{\pi}{e} - 1 > 0$, then $e^{\frac{\pi}{e}-1} > 1 + \left(\frac{\pi}{e} - 1\right) = \frac{\pi}{e}$. So, $e^{\frac{\pi}{e}} > \left(\frac{\pi}{e}\right)e = \pi$. It follows that

$e^\pi > \pi^e$.

Extrema. In Calculus settings, the term extrema likely brings to mind something like the First Derivative Test (as well as other similar tests). These "tests" are actually theorems – the majority of which utilize Taylor's Theorem in their respective proofs. To demonstrate, consider the more generalized theorem (statement based off of presentation in Bartle and Sherbert) for determining relative extrema:

Generalized Derivative Test. *Let I be an interval, let x_0 be an interior point of I, and let $n \geq 2$. Suppose that the derivatives $f', f'', \ldots\ f^n$ exist and are continous in a neighborhood of x_0 and that $f'(x_0) = \cdots = f^{n-1}(x_0) = 0$, but $f^n(x_0) \neq 0$. Then,*

(i) If n is even and $f^n(x_0) > 0$, then f has a relative minmum at x_0.
(ii) If n is even and $f^n(x_0) < 0$, then f has a relative maximum at x_0.
(iii) If n is odd, then f has neither a n a relative minimum nor relative maximum at x_0.

A proof of the above theorem appears on the following page (and like the proof of Taylor's Theorem is partially based off of the proof provided in Bartle and Sherbert).

Proof. For x_0 and x both $\in I$, Taylor's Theorem yields the following:

$$f(x) = f(x_0) + \frac{f^n(c)}{n!}(x - x_0)^n$$

where $c \in (x_0, x)$. Note that if $f^n(x_0) \neq 0$, then, since f^n is continuous, there must be an interval U containing x_0 such that $f^n(x)$ will have the same sign as $f^n(x_0)$ for $x \in U$. Since $x \in U$, c must also belong to U (since $c \in (x_0, x)$). Thus, $f^n(c)$ and $f^n(x_0)$ will have the same sign.

First, suppose that n is even and consider the two cases:

(i) $f^n(x_0) > 0$. Since $f^n(c)$ has the same sign, then $f^n(c) > 0$. Moreover, $(x - x_0)^n \geq 0$ for $x \in U$, meaning that $R_{n-1}(x) \geq 0$. In other words, $f(x) \geq f(x_0)$ for $x \in U$, which means that f has a relative minimum at x_0.

(ii) $f^n(x_0) < 0$. This implies that $R_{n-1}(x) \leq 0$ for $x \in U$. Thus, $f(x) \leq f(x_0)$ for $x \in U$, meaning that f has a relative maximum at x_0.

Now suppose that n is odd.

Note that $(x - x_0)^n > 0$ for $x > x_0$ and that $(x - x_0)^n < 0$ for $x < x_0$. If $x \in U$, then $R_{n-1}(x)$ will have different signs to the left and right of x_0. This means that f does not have a relative minimum nor a relative maximum at x_0. ■

One should note that this just one of many various tests. Many more exist – and the results of nearly all of them can be derived with a quick use for Taylor's Theorem. A survey of many such tests can be found in the paper by Sheldon Gordon.

Convexity. Recall that a function f is *convex* on some interval if for any t such that $0 < t < 1$ and points x_1, x_2 in the interval then

$$f\big((1 - t)x_1 + x_2\big) \leq (1 - t)f(x_1) + tf(x_2).$$

At first glance, Taylor's Theorem may not appear to be applicable here. However, Taylor's Theorem is actually quite useful in proving some results regarding convexity. Moreover, Taylor's Theorem provides an useful tool for quickly seeing whether a function is convex or not.

To see this, first recall that an inflection point, in informal terms, is a point at which a function changes from being concave (i.e., concave downward) to convex (i.e., concave upward). One of the "tests" in the paper by Gordon mentioned above allows for determining whether a function has an inflection point at a certain point (89).

Third Derivative Test for Inflection Points. *Suppose that a function* f *is such that* $f'(x_0) = 0$ *and* $f''(x_0) = 0$. *Then,*

$(i)\ If\ f'''(x_0) \neq 0, then\ f\ has\ an\ inflection\ point\ at\ x_0.$
$(ii) If\ f'''(x_0) = 0, then\ the\ test\ is\ inconclusive.$

Rather than prove this result, an example demonstrating the applicability of Taylor's Theorem follows.

Gordon provides a nice example (: $f(x) = e^{-x^3}$ (91). At first glance, it is likely not clear where an inflection point might fall (or if f even has an inflection point). But, consider applying Taylor's Theorem with $n = 3$ and $x_0 = 0$. Note that $f'(0) = f''(0) = 0$.

$$e^{-x^3} \approx 1 - 6\left(\frac{x^3}{6}\right) = 1 - x^3$$

From this approximation, it more readily apparent that f has inflection point (specifically, at $x_0 = 0$). Indeed, $f'''(0) \neq 0$. By the above theorem, then, f has an inflection point at 0.

A Financial Application of Taylor's Theorem

Finance has made great use of Taylor's Theorem. The reason primarily corresponds to the ability of Taylor's Theorem in making relatively precise approximations. Indeed, the application that follows is just a numerical approximation problem in a more practical setting.

This application will concern approximating the change in the price of a bond (note that the logic can extend to other types of assets as well). First, some preliminaries. An overarching assumption is that bond prices as functions of interest rates. The goal is to understand how bond prices respond to a change in the interest rate r. In other words, the following needs to be approximated:

$$p(r) - p(r_0)$$

For clarity, note that the first term in any series Taylor approximation is $f(x_0)$. In the above, this term has simply been moved to the left-hand side. All that is left to do is to determine what is on the right-hand side. In order to do so, the following concepts are need: *bond duration* and *bond convexity*.

In informal terms, bond duration constitutes the amount of time (on average) that a bond holder must wait before receiving cash flows. Note the price of a bond is simply the present value of its cash flows c_i at time t_i:

$$p(r) = \sum_{i=1}^{n} c_i e^{-rt_i}$$

In more precise terms, bond duration can be defined as "a weighted average of the times when payments are made" (Hull 92). Mathematically, this is

$$D(r) = \sum_{i=1}^{n} t_i \left[\frac{c_i e^{-rt_i}}{p(r)}\right]$$

An important connection to make here concerns the fact that duration closely relates to the first derivative of $p(r)$. To see this, first note the following approximation:

$$p(r) - p(r_0) \approx p'(r)(r_0 - r)$$

Going a step further, note that by actually taking the first derivative of $p(r)$, the following is obtained:

$$p(r) - p(r_0) \approx (r - r_0) \sum_{i=1}^{n} c_i e^{-rt_i}$$

Note the change in sign of $(r_0 - r)$, which implies that there is a negative relationship between the price of the bond and the interest rate. The above can be rewritten as follows:

$$p(r) - p(r_0) \approx (r - r_0) p(r) D(r).$$

With this, the relationship between the duration and the first derivative can be seen:

$$\frac{p(r) - p(r_0)}{p(r)} \approx (r - r_0) D(r)$$

One should immediately note that the right-hand side is just the second term of the Taylor Series Expansion!

This approximation can be made more precise by utilizing bond convexity. Unlike duration which had a very precise financial defintion behind it, the concept of bond convexity is very much just the standard mathematical idea of convexity: it "measures the curvature" of $p(r)$" (95). Hence, in mathematical terms, convexity can be written as follows:

$$C(r) = \frac{1}{p(r)} p''(r) = \frac{\sum_{i=1}^{n} c_i e^{-rt_i}}{p(r)}$$

Putting everything together, Taylor's Theorem gives the following relationships:

$$p(r) - p(r_0) \approx p'(r)(r_0 - r) + \frac{1}{2}p''(r)(r_0 - r)^2$$

Rewriting, one can see that

$$\frac{p(r) - p(r_0)}{p(r)} \approx (r - r_0)D(r) + \frac{1}{2}C(r)(r - r_0)^2$$

Hence, Taylor's Theorem makes an important result readily apparent: *the change in the price of a bond in response to a change in the interest rate is related to a bond's duration and convexity*. Without Taylor's Theorem, this result would be very difficult to see. With that said, this application of Taylor's Theorem is quite straightforward – as stated earlier, this is simply numerical approximation in more general terms. Given the mathematical defintion of $p(r)$ and $D(r)$, the above result is simple to derive as shown. The only real "trick" is finding duration in the first derivative of $p(r)$.

One might wonder: what use is this since we could just plug in r and r_0 in the left-hand side of the result above? While true, the right-hand side of this result is nice for two reasons. First, note that the left-hand side is not all that "calculation friendly" (recall what $p(r)$ represents!). In contrast, one can make a quick, yet quite precise, approximation in a few seconds with the right-hand side. The second reason pertains to the discussion above: the right-hand side reveals some of the inner mechanics of what is driving the change in price in response to change in interest rate.

Conclusion

What makes Taylor's Theorem such a powerful tool is the fact that it makes relationships that may initially be unobservable readily apparent. From inequalities to extrema to inflection points, quick applications of Taylor's Theorem can make many results apparent in seconds. More importantly, Taylor's Theorem is an excellent topic to introduce business and finance related applications. Indeed, as shown above, a crucial result relating the change in a bond price in response to a change in the interest rate to duration and convexity can be derived quite easily with Taylor's Theorem. In fact, the process of doing so, is just an exercise in numerical approximation.

References

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