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Abstract

In this study, we were interested in exploring the extent to which advanced mathematics lecturers provide students opportunities to play a role in considering or generating course content. To do this, we examined the questioning practices of 11 lecturers who taught advanced mathematics courses at the university level. Because we are unaware of other studies examining advanced mathematics lecturers' questioning, we first analyzed the data using an open coding scheme to categorize the types of content lecturers solicited and the opportunities they provided students to participate in generating course content. In a second round of analysis, we examined the extent to which lecturers provide students opportunities to generate mathematical contributions and to engage in reasoning researchers have identified as important in advanced mathematics. Our findings highlight that although lecturers asked many questions, lecturers did not provide substantial opportunities for students to participate in generating mathematical content and reasoning. Additionally, we provide several examples of lecturers providing students some opportunities to generate important contributions. We conclude by providing implications and areas for future research.

Key words: Questioning, Teaching of advanced mathematics courses, Opportunities to participate

Teacher questioning and invitations to participate in advanced mathematics lectures

In the United States and internationally, lecture is the most common form of teaching in undergraduate advanced mathematics courses (Artemeva & Fox, 2011; Blair, Kirkman, & Maxwell, 2013; Fukawa-Connelly, Johnson, & Keller, 2016). Recent literature reviews suggest these lectures have been unsuccessful in promoting student learning. After taking advanced mathematics courses, most students develop neither understandings of course content (e.g., Rasmussen & Wawro, 2017) nor the skills required to be successful mathematicians, such as proof-writing (e.g., Stylianides, Stylianides, & Weber, 2017). In a recent meta-analysis, Chi and Wiley (2014) provided a theoretical framework of engagement by categorizing overt behaviors students displayed. By synthesizing the results of several empirical studies, Chi and Wiley argued that students are more likely to learn if they are making inferences about course content as compared to if they are only recording what their lecturers say. Because questioning is a common way lecturers can engage students in knowledge construction (e.g., Gabel & Dreyfus, 2017; Lew, Fukawa-Connelly, Mejia-Ramos, & Weber, 2016), in this study we explore the extent to which advanced mathematics lecturers provide students opportunities to generate course content via questions. To do this, we examined 11 lecturers' questions in advanced mathematics courses. By advanced mathematics courses, we mean pure mathematics courses for university students with a focus on proofs instead of computations.[1] Specifically, we address the research questions:

1. What mathematical content do lecturers' questions solicit in advanced mathematics courses?

[1] In the United States, mathematics students usually take these courses after completing a calculus sequence and an introduction to proof course. In other countries, university mathematics students take these courses in their first year.

2. In what ways do lecturers' use of questions promote (or fail to promote) opportunities for students to participate in advanced mathematics lecturers?

In what follows, we first synthesize relevant areas of research: teaching in advanced mathematics and teacher questioning in K-14 mathematics. We then present our analytical framework describing how we conceptualize *opportunities to participate* and *mathematical contributions in advanced mathematics*. Next, we provide information about the lecturers and describe how we coded the data. We then present results from two rounds of coding, highlighting how lecturers provided students limited opportunities to consider or generate course content. We conclude by summarizing our results, relating our results to the extant literature, and providing implications and areas for future research.

1. Literature Review

Because we are unaware of other studies of advanced mathematics lecturers' use of questions, we situate our study in two related areas of research. We first present central results from the literature examining teaching in advanced mathematics. We then summarize the literature on teachers' use of questions in K-14 mathematics.

1. 1. Teaching in advanced mathematics

For brevity's sake, we do not provide a comprehensive review of literature on teaching advanced mathematics[2] but instead present several central findings that are relevant to this paper. First, researchers using both large-scale surveys (Blair, Kirkman, & Maxwell, 2013; Fukawa-Connelly, Johnson, & Keller, 2016) and observations (Artemeva & Fox, 2011) have found that lecture is the predominant mode of instruction in advanced mathematics courses in the United States and internationally. These lectures largely consist of what Artemeva and Fox (2011) call

[2] We point the reader to Gabel and Dreyfus (2016) and Lew et al. (2016) for such reviews.

"chalk talk". "Chalk talk" occurs when the lecturer writes mathematics on the blackboard, provides an oral commentary of the meaning of the written mathematics, and occasionally presents a metacommentary on the reasoning processes involved in doing mathematics.

Second, lecturers in advanced mathematics attempt to convey more than just formal mathematics in their lectures (i.e., definitions, theorems, and proofs). For instance, researchers have shown some lecturers frequently provide examples (e.g., Fukawa-Connelly & Newton, 2014; Fukawa-Connelly, Weber, & Mejia-Ramos, 2017; Mills, 2014) and informal representations of mathematical concepts (e.g., Fukawa-Connelly, Weber, & Mejia-Ramos, 2017; Weber, 2004) during their lectures. Lecturers also have been observed describing informal reasoning processes they use to construct a proof (e.g., Gabel & Dreyfus, 2016; Lew et al., 2016; Weber, 2004). However, students often do not recognize the point the lecturer is attempting to make (e.g., Lew et al., 2016) and typically leave their advanced mathematics courses without the understandings and skills expected of them (e.g., Rasmussen & Wawro, 2017; Stylianides, Stylianides, & Weber, 2017).

Because lecture is the predominate mode of instruction and students are not developing the understandings expected of them, there is a need to better understand the opportunities students are given to develop understandings of course content in advanced mathematics lectures. As giving students opportunities to consider or generate course content has been shown to improve student learning (Chi & Wiley, 2014), there is a need to examine the extent to which lecturers provide students opportunities to make mathematical contributions in these courses.

1. 2. Inviting student participation in K-14 mathematics

There is a large body of research examining teacher questioning in K-14 mathematics.[3] Due to potential similarities between advanced mathematics lectures and K-14 teachers' use of questions, we synthesize the literature examining the content K-14 teachers' questions solicit and the wait time they provide their students to respond to questions.

1. 2. 1. Categorizing teachers' questions

Across studies categorizing teachers' questions in K-14 mathematics, we synthesized four broad categories of questions: factual, probing, generative, and orienting questions. Factual questions ask students to provide something already known including facts, rules, or procedures (e.g., Boaler & Brodie, 2004; Hiebert & Wearne, 1993; Moyer & Milewicz, 2002; Nathan & Kim, 2009; Sahin & Kulm, 2008; Viirman, 2015). Probing questions ask students to explain or elaborate on their thinking (e.g., Boaler & Brodie, 2004; Franke et al., 2009; Hiebert & Wearne, 1993; Moyer & Milewicz, 2002; Sahin & Kulm, 2008). Generative questions ask students to provide mathematical information or a next step that is not factual (e.g., Boaler & Brodie, 2004; Hiebert & Wearne, 1993). Finally, orienting questions direct students' attention to specific ideas or solution strategies (e.g., Wood, 1998). Some researchers such as Boaler and Brodie (2004) and Hiebert and Wearne (1993) have elaborated on these broad categories, looking at different types of factual questions, probing questions and so on, but these more nuanced taxonomies are beyond the scope of this paper.

A shared finding across primary mathematics (e.g., Franke et al., 2009), secondary mathematics (e.g., Boaler & Brodie, 2004; Sahin & Kulm, 2008), and undergraduate

[3] Researchers have examined teacher questioning in K-12 classrooms (e.g., Boaler & Brodie, 2004), community college classrooms (Mesa 2010; Mesa, Celis, & Lande, 2013), and calculus classrooms (Viirman, 2015). As the pedagogical goals of these courses tend to be similar (i.e., helping students develop conceptual understanding and/or procedural competence) and this literature has produced findings that are largely consistent, we collapse our review to K-14 mathematics.

calculus (Viirman, 2015) is that teachers predominately pose factual questions. This occurs even though researchers claim that factual questions limit students engagement and do not promote students' developing robust understandings (e.g., Stein, Remillard, & Smith, 2007).

1. 2. 2. Examining teachers' wait time

Researchers have indicated that longer wait times can invite greater student participation, contribute to classroom discourse, and improve student achievement (e.g., Duell, Lynch, Ellsworth, & Moore, 1992; Mesa, 2010; Tobin, 1986). For example, Tobin (1986) found that increasing middle school teachers' wait time to between three and five seconds produced positive changes in classroom discourse and student achievement in both mathematics and language arts. As a general rule, researchers have posited students should be given at least three seconds to respond to a question (e.g., Rowe, 1974; Swift & Gooding, 1983; Tobin, 1986) on the grounds that with wait times under three seconds, "students are not encouraged to voice an answer or contribution and therefore opportunities for participation are shut down" (Mesa, 2010, p. 67). However, in the K-14 classrooms where studies have been conducted both in the United States (e.g., Aizikovitsh-Udi & Star, 2011; Duell et al., 1992; Mesa, 2010; Rowe, 1986) and internationally (e.g., Heinze & Erhard, 2006), teachers usually do not provide students with three seconds to answer a question. Hence, when teachers ask questions in K-14 classrooms, these questions often do not provide students with significant opportunities to participate; the extent to which such findings are also true in advanced mathematics courses is an open question.

2. Analytical Framework

Based on the aforementioned literature, we consider a lecturer's question as a *participation opportunity* for students either if a student provides an answer to the question or if the lecturer gives students at least three seconds to respond. Because of our interest in lecturers verbally prompting students to consider or generate course content, we considered direct statements not phrased in the form of a question (e.g., “[Student name] give me an example of a non-commutative ring.”) as questions in our analysis.

2. 1. Mathematical contributions in advanced mathematics

We are interested in the *types* of mathematical contributions lecturers solicit via questions. We broadly define mathematical contributions to include mathematical content and ways of reasoning. By *mathematical content*, we are referring to formal (e.g. proofs, definitions) and informal (e.g., heuristics) content that is often stated or drawn to represent a mathematical idea. By *ways of reasoning* we are referring to identifying productive directions for approaching mathematical problems and evaluating whether a particular approach or assertion is valid or productive. By broadly defining mathematical contributions, we intend to capture as many opportunities as possible for students to offer contributions in response to lecturers’ questions.

Fukawa-Connelly, Weber, and Mejia-Ramos (2017) synthesized the research literature on lecturing in advanced mathematics to characterize mathematical contributions commonly made in these lectures. They noted that although lectures were typically structured around formal products—specifically *definitions, propositions,* and *proofs*—lecturers also discussed examples, informal representations of mathematical concepts, and heuristics for solving mathematical problems including mathematical methods. An *example* of a concept is a specific object that satisfies the concept’s definition (e.g., 5 is an example of a prime number). An *informal representation* is a diagrammatic or graphical representation of a concept, or a description of a concept in colloquial English (e.g., a strictly increasing function is a graph that goes up from left to right). A *mathematical method* is a non-algorithmic approach to accomplish a general task or conditions under which a technique is likely to be useful (e.g., a useful way to prove a sequence is convergent when you do not have a limit candidate is to show that the sequence is a Cauchy sequence).[4]

2. 1. 1. Ways of thinking particular to proving in advanced mathematics

Given the significance of proofs in advanced mathematics courses (e.g., Weber, 2001), we view the extent to which lecturers provide students opportunities to engage in the reasoning processes used in proof construction to be particularly important. Alcock (2010) identified four modes of reasoning successful provers use. *Structural thinking* involves generating a proof by using the formal structure of a statement. *Instantiation* involves understanding statements by considering examples to which a statement applies. *Creative thinking* involves identifying mathematical properties or manipulations that may lead in productive proof directions. Finally, *critical thinking* involves checking the correctness of assertions within a proof, either by checking if the assertion holds for particular example objects or examining if expected properties are preserved by the inference.

3. Participants, Data Collection, And Methods

We recruited participants by sending e-mails to every lecturer at three doctoral-granting institutions in the eastern United States who was teaching an advanced mathematics course. We asked to observe and audio-record one of their lectures. Lecturers were not told the purpose of the study. Eleven lecturers agreed. The content of their courses is summarized in Table 1 below.

Table 1

Description of lecturer, course content, and content in the observed lesson

[4] Fukawa-Connelly, Weber, and Mejia-Ramos (2017) also discussed *modeling mathematical behavior* as occurring when a lecturer emphasizes the types of practices, questions, and appraisals that are common or natural in mathematics. However, this is not relevant for the purposes of this paper.

Lecturer	Overarching Course-content	Description of content in the lesson
L1	Number Theory	Transfinite arithmetic, counting
L2	Real Analysis	Infinite series, convergence, examples of sequences that do and do not converge
L3	Number Theory	Prime number theorem, approximations of the prime number theorem
L4	Linear algebra	Jordan Canonical form, T-invariant subspaces
L5	Abstract algebra	Exam problems, permutations, cycle notation, operations on permutations, order
L6	Number theory	Reduced residue systems, Euler's theorem, multiplicative functions, Euler phi function
L7	Geometry	Isometries and similarities
L8	Abstract algebra	Ideals, principal ideals, how congruence mod n is similar to congruence in polynomials
L9	Abstract Algebra	Ideals, congruence modulo an ideal, well-defined operations
L10	Real Analysis	Partitions, Riemann integration, Riemann integral
L11	Differential Geometry	Gaussian curvature, eigenvalues & eigenvectors, principal curvature

All lectures were approximately 80 minutes in length. All professors gave "chalk talk" lectures (Artemeva & Fox, 2011). Each class had between seven and 30 students enrolled, with a mean of approximately 18 students. A researcher audio-recorded the lecture while transcribing everything that the lecturer wrote on the board.

3. 1. Analyzing for student contributions and wait time

In order to examine students' opportunities to participate, each lecture was transcribed and a member of the research team identified every time a lecturer solicited a mathematical contribution.[5] We coded each question for wait time and who responded to the question. To do this, we listened to the audio-recording and noted the number of seconds that passed after a lecturer asked a question before someone spoke, truncating wait time of partial seconds (e.g., if the next utterance was between two and three seconds, we recorded a wait time of two seconds). We then noted if the lecturer or a student spoke next.

3.2. Analyzing the data using an open coding scheme

We engaged in two rounds of coding to categorize the content lecturers solicited via questions. Because research on teacher questioning in advanced mathematics lectures is limited, we initially used an open coding scheme, engaging in thematic analysis (Braun & Clarke, 2006), to categorize the mathematical contributions lecturers' questions solicited. Although our knowledge of the mathematics education literature with respect to proof (e.g., Alcock & Weber, 2005; Selden & Selden, 2003) and questioning (e.g., Boaler & Brodie 2004) informed this analysis, we aimed to let the data create our categories rather than force the data into pre-existing categories. After going through all of the lectures, we synthesized our preliminary codes into operationalized categories presented in Table 2 (for more details regarding the context of each example in Table 2, see Appendix A in the Extra Supplementary Materials).

Table 2

Categories, descriptions, and examples of question types we identified.

Category	Description	Example
Fact	Questions asking for a closed form mathematical response that did not ask for a course of action.	Do you remember what Cauchy means, for a sequence to be Cauchy?
Next Step	Questions asking students to recommend a course of action that	A'C' is equal to *k*AC and B'C' is equal to *k*BC. Therefore,

[5] Because we focus on the mathematical contributions, questions that were non-mathematical (e.g., Do you know when your exam is?) were excluded from our analysis.

	would continue the logical progression of a proof or example.	therefore now what?
Proof Framework	Questions addressing higher level logical structures of a proof.	What will I start [the proof] out assuming?
Warrant	Questions asking for a justification for a statement or claim.	Why is that true?
Evaluation	Questions asking students to provide a truth-value for a statement.	So, claim $R_{L2}(A')$ is A'. Wouldn't that be nice. Is it true?
Convention	Questions addressing a convention or notation.	How do you write the identity in cycle notation?
Other	Questions that do not fit into the other categories	Here's a field Q which is in R, we talked about compatibility of operations, and this was a pedagogical risk, of how, well, what are we to make of the operations here?

In coding for mathematical contribution, we focused on what type of student contribution would constitute a literal answer to the question. For instance, if a lecturer asked "is this claim true?," we coded this question as asking students to evaluate the truth-value of a claim because a "yes" or "no" response would constitute an answer to the question. If a student responded, "yes, the claim is true because…" and proceeded to provide a warrant for the claim, we still coded the question as an evaluation question, not a warrant question.[6]

After we operationalized our categories, each lecture was assigned to one of the original coders to code each question using these categories. Subsequently, a randomly chosen 20-minute segment of each 80-minute lecture was re-coded by another member of the research team to

[6] It is possible that due to the socio-mathematical norms in the class, lecturer questions were soliciting contributions that went beyond what was literally stated in the question. However, we observed few instances of students providing answers that went beyond what was explicitly solicited in the question.

check for inter-rater reliability. The coders agreed on 236 of the 258 recoded questions (91.5% agreement, Cohen's Kappa = .90), representing a very high level of agreement.

3. 3. Analyzing the data in relation to our analytic framework

In the second phase of analysis, we recoded the data to examine the extent lecturers' questions solicited the mathematical contributions described in the analytic framework. Before examining teachers' questioning, another research team documented each instance in which a definition, proposition, proof, example, informal representation, and mathematical method were presented during a lecture. The results of this analysis were presented in Fukawa-Connelly, Weber, and Mejia-Ramos (2017). The purpose of this second round of analysis was to explore how often lecturers provided students opportunities to consider or generate this mathematical content via questioning. For instance, for example contributions, we (the authors of the current paper) flagged instances in which a lecturer asked students to provide an example of a concept. For proposition contributions, we included any question that asked students to state a theorem, proposition, lemma, or corollary as well as any statement outside the context of proofs with hypotheses and claims. Because questions regarding most of these contributions were rare, we did not compute inter-rater reliability, but instead resolved disputes through discussion.

Within any of the content that Fukawa-Connelly, Weber, and Mejia-Ramos (2017) coded as proof, we coded questions that pertained to Alcock's (2010) four modes of reasoning. We coded a lecturer's question as soliciting structural thinking when the lecturer asked students to provide a part of a proofs framework, or, a type of symbolic unpacking of a definition or other known result (e.g., "when we see the statement for all epsilon, what do we do?"). To code for an instantiation, we identified any instance where, during a proof, a lecturer referenced an example that would illustrate part of the proof. We coded a lecturer's question as soliciting creative thinking if the lecturer asked students to identify examples, properties, or manipulations that solicited the crux of the proof. Finally, we coded for critical thinking when a lecturer either solicited a counter-example or asked for implications of claims that went beyond an algebraic manipulation in a proof. Questions soliciting each of Alcock's (2010) four modes of reasoning were rare, so again we resolved disputes through discussion.

4. Results

Lecturer questioning was common in our data set; across the 11 lectures, there were 619 questions that solicited mathematical contributions. On average, the lecturers posed 56 questions per 80-minute lecture. There was substantial variance in how many questions lecturers posed. Nine lecturers asked between 15 and 90 questions. However, one lecturer (L10) asked only four questions and another lecturer (L8) asked 202. In what follows, we first present the results from the open coding and then from the coding done in relation to our analytic framework. We incorporate data relevant to the ways in which lecturers' provided students' opportunities to participate throughout.

4. 1. Trends in lecturers' questioning in regards to general content and opportunities to participate

Table 3 presents the average number of times per lecture each question type we identified in our open coding was posed. Table 3 also includes the standard deviation, median, and maximum number of questions for each type. Although there was substantial variance in each of these categories, Table 3 reveals some general trends across the lectures. Most importantly, the majority of questions asked students to recall information or procedures (Fact) or provide the next step in a computation or proof (Next Step). There was an average of 37 questions of these two types per lecture, or 66% of the questions asked (see Extra Supplementary Materials Appendix B Table 1B and Figure 1B for counts of each question type by lecturer).

Table 3

The average number of questions, standard deviation, median, and maximum number of questions by category.

Questioning Category	Average number of questions per lecture	Standard Deviation	Median	Max
Fact	26.2	25.6	18	80
Next Step	10.7	16.2	5	56
Proof Framework	2.8	4.9	0	13
Warrant	5.5	5.6	4	18
Evaluation	3.1	2.9	2	9
Convention	2.5	3.5	1	11
Other	5.4	5.9	2	18

Table 4 presents the extent that lecturers provided students opportunities to respond to questions. Recall, we categorized a question as a *participation opportunity* either if a student provided an answer to the question or if the lecturer gave students at least three seconds to respond. We highlight that students did not respond to the majority of questions. Further, when students did not respond to a question, lecturers infrequently provided three seconds of wait time.

Table 4

The average number of questions, student responses, and participation opportunities for each questioning category.

Question Type	Average number of questions per lecture	Average number of student responses per lecture (as a % of that question type)	Average number of participation opportunities (as a % of that question type)
Fact	26.4	10.5 (40%)	14.1 (54%)
Next Step	10.5	4.1 (39%)	5.1 (49%)
Proof Framework	2.9	1.4 (48%)	1.5 (51%)
Warrant	5.6	2.5 (44%)	3.1 (55%)
Evaluation	3.1	1.0 (32%)	1.4 (44%)

Convention	2.5	1.0 (41%)	1.2 (48%)
Other	5.4	1.3 (24%)	2.0 (38%)
Total	56.3	21.6 (38%)	28.2 (50%)

With respect to lecturers' use of wait time, students were given limited time to respond to questions. Across all categories (Table 5), 80% of questions had a wait time of less than three seconds and 51% of the questions had a wait time of less than one second.

Table 5

Average number of questions per lecture by category and wait time

Question Type	Avg.	0 s. (%)	1-2 s. (%)	3-4 s. (%)	>5 s. (%)
Fact	26.4	13.1 (50%)	6.8 (26%)	4.0 (15%)	2.5 (9%)
Next Step	10.5	4.8 (46%)	3.8 (36%)	1.4 (13%)	0.5 (5%)
Proof Framework	2.9	1.5 (52%)	0.9 (32%)	0.3 (10%)	0.2 (7%)
Warrant	5.6	2.8 (50%)	1.7 (31%)	0.5 (8%)	0.6 (11%)
Evaluation	3.1	2.0 (65%)	0.6 (21%)	0.4 (12%)	0.1 (3%)
Convention	2.5	1.2 (48%)	0.9 (37%)	0.3 (11%)	0.1 (4%)
Other	5.4	3.2 (59%)	1.5 (27%)	0.5 (9%)	0.4 (7%)
Total	56.3	28.5 (51%)	16.3 (29%)	7.2 (13%)	4.4 (8%)

4. 2. Questions pertaining to the mathematical contributions in advanced mathematics lecturers

In the sub-sections that follow we highlight how lecturers posed questions pertaining to the mathematical contributions in advanced mathematics synthesized by Fukawa-Connelly, Weber, and Mejia-Ramos (2017). For each contribution we provide the number of questions asked by lecturers pertaining to the content, the number of participation opportunities, and examples of ways in which lecturers promoted opportunities for students to consider or generate these mathematical contributions.

4. 2. 1. Definitions

The lecturers presented a total of 21 definitions in the lectures.[7] Three lecturers each asked a single question in which they solicited either a definition or a critical piece of a definition. While L1 did not present any formal definitions, L1's use of questions to solicit definitions was notable. L1's set theory lecture was devoted to inviting students to consider what definitions for cardinality, cardinal addition, cardinal multiplication, and cardinal exponentiation should look like without ever providing a formal definition for the latter three.[8] For example, when introducing cardinal multiplication L1 asked, "What should a definition of multiplication be? What does it come out to be in familiar cases? What laws does it satisfy?" He then described how cardinal multiplication mimicked the Cartesian product and provided a numeric example. In this example, as with cardinal addition and exponentiation, L1 asked students to consider how to define cardinal operations and what properties need to be maintained in order for these operations to be well defined. Hence, L1 provided an example of how a lecturer can solicit definitions via questions in an advanced mathematics course.

4. 2. 2. Propositions

The lecturers presented a total of 59 propositions. Five lecturers each asked students to pose or complete exactly one proposition; all five of these questions were coded as a participation opportunity. As an example, after finishing a proof of the maximum number of primes between 0 and x, L3 intended to prove Euler's Theorem of convergence. He prompted his students,

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[7] For more on how each contribution was counted across the lectures we refer the reader to Fukawa-Connelly, Weber, and Mejia-Ramos (2017).

[8] Because L1 did not provide a formal definition, cardinal addition, multiplication, and exponentiation are not part of the 21 total definitions presented by the lecturers.

L3: There is another thing that I wanted to prove… this is called Euler's theorem… it says the following, *the sum over all primes p of 1/p, yeah, what can you say about it?* Primes. What's the sum over all integers *n*, of 1/*n*?

S: It's, that's divergent.

The italicized question solicited the claim of Euler's theorem, and was therefore coded as pertaining to a proposition. As the student provided a response to this question, this was coded as an opportunity for a student to state a proposition (i.e., that the series in question was divergent).

4. 2. 3. Examples

The lecturers presented a total of 65 examples. Two lecturers asked questions that solicited a total of five examples; three of these questions were coded as participation opportunities. In an abstract algebra lecture about rings and ideals, L8 was continuing her introduction of ideal and explaining that some ideals are described as left-ideals. While doing so she solicited an example asking, "What's our classic example of non-commutative rings?" Multiple students responded, "matrices," which L8 claimed required sided-ideals. Later in the lecture, L8 solicited an example of an ideal asking, "What was an example of an ideal in the ring Z?" She then prompted a student to respond to this question. Hence, L8 provided students some opportunity to generate examples.

4. 2. 4. Informal Representations

The lecturers presented a total of 157 informal representations. Two lecturers asked questions that solicited four informal representations. L7 asked three of these questions and we coded each question as a participation opportunity. After presenting a definition of dilation, L7 indicated there were several cases to consider depending on the value of the ratio of dilation, *k*.

> L7: So k could be between 0 and 1, that's one case, or k can be bigger than 1, but we also can consider the possibility that k is negative, right? So let's start out with some point O and P, so O is gonna be the center of our dilation. Let's consider another point, maybe Q… *Will somebody give us a sketch of what it looks like if k is between 0 and 1? [Student 1 name], you'll do one? Who will do it for k bigger than one? [Student 2 name], I feel like you want to do it*.

A third student volunteered to produce a sketch for negative k values and the students produced the requested representations. Hence, L7 provided students opportunities to generate informal representations.

4. 2. 5. Mathematical Method

The lecturers presented a total of 61 mathematical methods (i.e., heuristics for accomplishing certain types of tasks or when certain techniques would be useful). Four lecturers each solicited exactly one method from their students; one question was a participation opportunity. As an example of a question we coded as pertaining to mathematical method, consider L7 beginning to prove that all isometries can be written as the composition of three reflections.

> L7: So, in some sense, reflections are like our little generators, right? I think this is pretty cool, and the proof is actually much easier than you might expect. So the proof is this. *We said that if you want to know an isometry, it's enough to know what the isometry does to what?*
>
> S: Three points.

We described the italicized question as pertaining to mathematical method as the question solicited a critical piece of information that contributes to a non-algorithmic means of understanding a particular transformation. Essentially L7 was asking for an efficient way to characterize an isometry. The student's response stated this critical approach.

4. 2. 6. Summary

Table 6 presents the total number of each contribution across the 11 lectures, the number of contributions lecturers solicited via questions, and the number of participation opportunities to generate the contribution. Although the lecturers frequently presented definitions, propositions, proofs, and so on, these lecturers rarely solicited a contribution via questions and almost never provided students participation opportunities to generate a contribution.

Table 6

The total number of contributions in this data set as identified by Fukawa-Connelly, Weber, and Mejia-Ramos (2017), the number of contributions lecturers solicited via questions, and the number of participation opportunities across the lectures.

Mathematical Contribution	Total number of contributions	Number of questions soliciting a contribution	Number of participation opportunities to provide the contribution
Definitions	21	3	1
Propositions	59	5	5
Examples	65	5	3
Informal Representation	157	4	3
Method	61	4	1
Total	363	21	13

4. 3. Lecturer questioning when presenting proofs

Proofs played an important part in the lectures that we observed; proofs accounted for 28% of the total lecture time. Further, 26% of the questions asked occurred within the context of a proof. Proofs also differed from the other contributions as the average proof took six minutes to produce whereas the other contributions described by Fukawa-Connelly, Weber, and Mejia-Ramos (2017) were usually provided by the lecturer or student within seconds. In this section, we present our findings regarding lecturer questioning when presenting proofs with respect to our initial analysis and to Alcock's (2010) description of important reasoning processes used by successful provers.

Lecturers asked 163 questions while presenting proofs with 93 participation opportunities. Table 7 provides counts from the initial analysis for questions posed during proofs. Lectures provided students with considerably more participation opportunities with respect to proofs (9 per lecture) as compared to the other five contributions described by Fukawa-Connelly, Weber, and Mejia-Ramos (2017). Consistent with the lectures as a whole, Fact and Next Step questions comprised the majority of these participation opportunities. Proof Framework and Warrant questions occurred on average only two times per lecture.[9]

Table 7

The average number of questions per lecture, average number of questions students responded to, and average number of participation opportunities for each questioning category during proof presentations.

Question Type	Average number of questions during a proof per lecture	Average number of questions students responded to per lecture	Average number of participation opportunities per lecture
Fact	5.7	2.2	2.9
Next Step	4.0	2.3	2.5
Proof Framework	2.0	1.2	1.3
Warrant	1.8	0.8	0.9
Evaluation	0.4	0.1	0.2
Convention	0.1	0.0	0.0

[9] The number of Proof Framework questions per lecturer is less than those counted for the entirety of the lectures as some Proof Framework questions occurred outside of the context of formally writing a proof (e.g., considering how one would start a proof without ever actually writing the proof).

Other	0.8	0.5	0.7
Total	14.8	7.3	8.7

We provide an atypical example to show how one lecturer provided students opportunities to make important mathematical contributions when presenting a proof. Throughout his real analysis lecture, L2 provided students many opportunities make mathematical contributions. The following excerpt shows L2 calling on students without them necessarily volunteering to engage the students in proving that multiplying a converging series by a constant generates a convergent series (i.e. Given that $\{S_N\}$ converges to A where $S_N = a_1 + \ldots + a_N$ then $\{T_N\}$ converges to cA where $T_N = (ca_1) + \ldots + (ca_N)$). For each question, we provide the question code and wait time using the convention [Code; wait time in seconds].

[1] L2: [S1[10]], you want to start, how do you prove this one? [Proof Framework; 1].

[2] S1: Start with what you know.

[3] L2: Always a good strategy. What do you know? [Proof Framework; 1].

[4] S1: We know that a_n converges to A.

[5] L2: [*Writes "Given that* $\{S_N\}$ *converges to A, where* $S_N = a_1 + \ldots + a_N$*"*] Okay, [S2], that's what we know.[11] [Next Step; 1].

[6] S2: cS_n equals the [inaudible].

[7] L2: Okay, so. Write what you know, write what you're going to show. Now [S3], I guess you're next. So this is what you know, this is what you're trying to show. What would you do next? [Next Step; 1].

[8] S3: I'd grab a c from each of the terms.

[9] L2: [*Writes "*$T_N = cS_N$*"*] Like that? [Non-Mathematical].

[10] S1 was the first student to respond in this excerpt but was the 18th student called on to this point in the lecture.
[11] We coded L2's statement as an invitation to participate as S19 responded to this statement by providing the next step in the proof.

[10] S3: Yeah.

[11] L2: So what is true then? [Next Step; 1].

[12] S3: It converges to *c* times whatever S_N converges to. [*L2 writes "But* $T_N = cS_N$*, so this converges to clim* $S_N = cA$*."*]

[13] L2: [S4], why is that true? [Warrant; 2].

[14] S4: [inaudible]

[15] L2: The theorems we proved – you're absolutely right by the theorem that we proved.

We note two of the six questions L2 asked were Proof Framework questions (Turns 1 and 3) and one was a Warrant question (Turn 13). These questions went beyond recalling facts or giving suggestions for next steps, and were particular to contributing to the proof L2 and his students were generating. Hence, L2 provided students opportunities to make important mathematical contributions while writing a proof.

In terms of Alcock's (2010) ways of thinking, we coded the questions in Turns 1 and 3 as asking students to engage in structural thinking as the questions asked students to generate a proof by considering the formal structure of the proof; both questions were participation opportunities for students. Across all 11 lectures, there were 44 questions that pertained to structural thinking and 29 of these questions were participation opportunities. Hence, lecturers did provide students some opportunities to participate in structural thinking.

There were an additional three questions that solicited critical thinking, all by L7. For example, during a proof that any isometry can be written as the composition of three reflections, L7 was discussing the transformation of particular points under reflections. After defining a reflection that sent A to A' she defined a second reflection (R_{L2}) that would send B to B'. She then said, "So claim, R_{L2}(A') is A'. Wouldn't that be nice. Is it true? Is A' equidistant from these two? So this is true." L7 asked students to consider the correctness of her assertion that $R_{L2}(A')$ is A'. Hence, we coded these two questions as soliciting critical thinking. In the sample of 11 lectures, there were no questions pertaining to Alcock's (2010) descriptions of creative thinking or instantiation during the construction of a proof. Hence, lecturers rarely solicited critical thinking, creative thinking, or instantiation via questions when presenting proofs.

5. Discussion

Although we examined a new population, advanced mathematics lecturers, the results of our open coding are largely consistent with the K-14 literature. For instance, we found that lecturers predominately (66% of all questions) asked students to provide factual information or a next step (e.g., Boaler & Brodie, 2004; Franke et al., 2009; Viirman, 2015); in our population, lecturers infrequently asked questions seeking other types of mathematical contributions (e.g., warrants, evaluation). Such findings may indicate that regardless of grade level or content, questions soliciting a fact or next step tend to be the easiest questions for teachers to ask or the questions teachers believe students are most capable of answering.

Although, we identified several questioning categories consistent with the K-14 mathematics literature, we also identified novel questioning categories. Fact questions and Next Step questions are consistent with the factual and generative questions in the K-14 literature (e.g., Boaler & Brodie, 2004; Hiebert & Wearne, 1993). To our knowledge, the categories of Proof Framework and Convention were not discussed in papers categorizing questions in K-14 mathematics. The existence of Proof Framework questions is perhaps not surprising, as proofs are often central to advanced mathematics classes (e.g., Alcock & Weber, 2005). The emergence of Convention questions might signal the increased importance of notation and syntax in advanced mathematics.

Despite the fact that each mathematical contribution described by Fukawa-Connelly, Weber, and Mejia-Ramos (2017) was common across the lectures, with the exception of proofs, lecturers infrequently solicited this content via questions. Further, students almost never actually provided this content. With respect to Alcock's (2010) ways of thinking in proof construction, although students were given opportunities to address questions soliciting structural thinking, there were limited questions soliciting critical thinking and no opportunities to engage in creative thinking or instantiation. Further, lecturers asking Proof Framework and Warrant questions on average only two times per lecture is notable considering researchers (Alcock & Weber, 2005; Selden & Selden, 2003) have indicated identifying proof frameworks and warrants are important skills in understanding and producing proofs. Such findings may provide some insight into why students leave their advanced mathematics courses without the proving skills expected of them (Stylianides, Stylianides, & Weber, 2017).

Although lecturers generally did not use questions to promote opportunities for students to participate in advanced mathematics courses, we provided several examples of lecturers eliciting some contributions from students (e.g., L7 having students generate informal representations and mathematical methods). These examples empirically support Pinto's (2013) and Weber's (2004) observations that lecturing in advanced mathematics is a multifaceted practice and there may not be a single "traditional" lecture in advanced mathematics courses.

We are cautious about generalizing the findings of this study to all advanced mathematics lecturers for several reasons. First, the lecturers were all at research universities from the United States. Second, the lecturers all volunteered to participate in our study. Third, our sample was relatively small (11 lecturers). However, this study does highlight that even if lecturers frequently ask questions, they still might not be providing students with opportunities to contribute to course content. Further, our analysis identifies interesting reasons for why this might be the case. However, any claims about the general trends that we observed should be treated only as tentative hypotheses that require validation in further studies.

6. Implications and Directions for Future Research

Across our sample, students were given limited opportunities to generate content in their advanced mathematics courses. Based on their meta-analysis, Chi and Wiley (2014) concluded that providing students opportunities to generate course content can support students developing more sophisticated understandings. Students lack of opportunities to participate in their advanced mathematics lectures may be one reason they fail to develop the understandings and skills expected of them in these courses (e.g., Rasmussen & Wawro, 2017; Stylianides, Stylianides, & Weber, 2017).

We conjecture students may develop more sophisticated understandings if they are given more opportunities to generate course content. To test this conjecture, we propose one area for future research involving investigating if the number of opportunities students have to make (or at least consider making) mathematical contributions in a lecture correlates with student understanding and achievement. For instance, our findings indicate that students were willing to provide warrants, evaluate mathematical statements, or provide examples when these contributions were solicited (students' response rates to these questions were similar to their response rates to Fact and Next Step questions). If advanced mathematics lecturers provide students more opportunities to address (or at least consider via longer wait times) more of the former types of questions, would students develop better understandings of the course content?

In the previous section, we discussed limitations of our self-selected sample of 11 lecturers. Artemeva and Fox (2011) indicated that "chalk talk" in advanced mathematics is a relatively consistent practice in the U.S. and internationally but also highlighted some differences they attributed to different cultural-historical contexts. As Artemeva and Fox (2011) did not address lecturer questioning, it would be interesting to explore the extent that lecturers at other types of universities or in other countries used similar questioning practices. For instance, in the United States, smaller liberal arts colleges typically have smaller class sizes and are taught by faculty members whose institutional obligations are more geared toward teaching than research. In other countries, class sizes are typically much larger. In both cases lecturers questioning practices might differ from the lecturers in our sample. Further, future researchers may be interested in exploring the teaching practices of lecturers who have been involved in professional development specific to teaching or have won awards for their teaching versus lecturers who have not had such experiences or acknowledgements. Hence, we intend our findings to serve as a starting point for more robust investigations into advanced mathematics lecturers’ use of questions to solicit mathematical contributions from students.

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