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TEACHERS' PEDAGOGICAL CONTENT KNOWLEDGE OF STUDENTS' PROBLEM SOLVING IN ELEMENTARY ARITHMETIC

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This study investigated 40 first-grade teachers' pedagogical content knowledge of children's solutions of addition and subtraction word problems. Most teachers could identify many of the critical distinctions between problems and the primary strategies that children used to solve different kinds of problems. But this knowledge generally was not organized into a coherent network that related distinctions between problems, children's solutions, and problem difficulty. The teachers' knowledge of whether their own students could solve different problems was significantly correlated with student achievement.

During the last decade, there has been an increasing emphasis on teachers' thought processes in the study of teaching (Clark & Peterson, 1986; Shavelson & Stern, 1981). This research indicates that teachers are reflective, thoughtful individuals and that teaching is a complex, cognitively demanding process involving problem solving and decision making. Given the critical role that specific content knowledge plays in performance in complex domains (Chi, Glaser, & Rees, 1982), it follows that teachers' knowledge should be a primary variable in the study of teaching.

Researchers have conducted a number of studies of teachers' knowledge. Most of them have focused on teachers' knowledge of mathematics content, often measured by the number of college mathematics courses completed or a score on a standardized test. In general, these studies have not found a strong relationship between teachers' knowledge of mathematics and student achievement. One limitation of many of these studies is that they have employed global measures of teachers' knowledge that are not directly related to instruction in the teachers' classrooms (Romberg & Carpenter, 1986).

Recently, Shulman (1986a, 1986b) proposed a framework for analyzing

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teachers' knowledge that distinguished between different categories of knowledge that may play very different roles in instruction. The investigation reported here concerned what Shulman (1986b) called "pedagogical content knowledge," which he defined as follows:

> The understanding of how particular topics, principles, strategies, and the like in specific subject areas are comprehended or typically misconstrued, are learned and likely to be forgotten. Such knowledge includes the categories within which similar problem types or conceptions can be classified (what are the ten most frequently encountered types of algebra word problems? least well-grasped grammatical constructions?), and the psychology of learning them. (Shulman, 1986a, p.26)

Pedagogical content knowledge includes knowledge of the conceptual and procedural knowledge that students bring to the learning of a topic, the misconceptions about the topic that they may have developed, and the stages of understanding that they are likely to pass through in moving from a state of having little understanding of the topic to mastery of it. It also includes knowledge of techniques for assessing students' understanding and diagnosing their misconceptions, knowledge of instructional strategies that can be used to enable students to connect what they are learning to the knowledge they already possess, and knowledge of instructional strategies to eliminate the misconceptions they may have developed.

The study reported here focused on dimensions of pedagogical content knowledge that are different from those examined in other recent studies of teacher knowledge such as those by Leinhardt and Greeno (1986) and Leinhardt and Smith (1985). Those studies were concerned with teachers' understanding of the computational procedures they taught and their knowledge of lesson structure and teaching routines. Our research focused on teachers' understanding of how children think about mathematics and on teachers' knowledge of their own students' thinking.

Research on children's thinking and problem solving has documented that children bring a great deal of knowledge to almost any learning situation, which significantly influences what they learn from instruction (Carpenter & Peterson, 1988). This evidence suggests that teachers' knowledge of students' concepts and misconceptions could seriously influence their instruction, but there has been relatively little research to investigate that hypothesis. Putnam (1987) and Putnam and Leinhardt (1986) proposed that the assessment of students' knowledge is not a primary goal of most teachers. They argued that keeping track of the knowledge of 25 students would create an overwhelming demand on the teachers' cognitive resources. They hypothesized that teachers follow curriculum scripts in which they make only minor adjustments based on student feedback. The evidence is far from conclusive, however, to support the belief that teachers do not or cannot monitor students' knowledge and use that information in instruction. Furthermore, Lampert (1987) has argued that a concern for monitoring students' knowledge may be related to a teacher's goals for instruction. Shortterm computational goals may be achieved without attending to students' knowledge, but there are higher level goals that may be related to teachers' attempts to understand students' thinking.

A more detailed analysis of the nature of teachers' knowledge about students and their goals for instruction may be needed to resolve this issue. Research on children's cognition and problem solving has provided a framework for evaluating teachers' knowledge and their ability to assess their students' thinking. Using this framework as a reference may yield more explicit insights regarding the nature of teachers' knowledge of students' thinking than have emerged from previous studies.

A FRAMEWORK FOR THE ANALYSIS OF PEDAGOGICAL CONTENT KNOWLEDGE

Much of the recent research on children's thinking and problem solving has focused on performance in specific, semantically rich content domains. This research has generated detailed descriptions of children's knowledge and problem-solving processes that can serve as a basis for analyzing teachers' pedagogical content knowledge. In particular, work on students' learning of addition and subtraction concepts has provided an explicit framework for analyzing problems and the processes children use to solve them. This analysis served as the basis of the examination of teachers' knowledge that is reported here.

There are a number of different perspectives from which researchers have examined children's learning of addition and subtraction concepts and skills (Carpenter, Moser, & Romberg, 1982). One major strand of research has focused on the development of addition and subtraction concepts and procedures as reflected in children's solutions of different types of word problems. For reviews of this research, see Carpenter (1985); Carpenter and Moser (1983); and Riley, Greeno, and Heller (1983). Although there are differences in details and emphasis, researchers have found remarkably consistent results over a number of studies; and there is general consensus about how children solve different problems that provides a basis for evaluating teachers' knowledge of their students' thinking.

Earlier studies of arithmetic were concerned primarily with problem difficulty, and research on word problems attempted to specify distinctions between problems in terms of linguistic variables that could be related to problem difficulty (cf. Jerman & Mirman, 1974). Current research on addition and subtraction word problems focuses on the processes that children use to solve different problems. Most recent research has been based on an analysis of verbal problem types that distinguishes between different classes of problems on the basis of their semantic characteristics. There are minor differences in how problems are categorized, and some researchers include additional categories, but the central distinctions included in almost all categorization schemes are illustrated by the problems in Table l. Although all the problems in Table 1 can be solved by solving the mathematical sentences $5 + 8 = ?$ or $13 - 5 = ?$, each provides a distinct interpretation of addition and subtraction.

Table 1
Classification of Word Problems

Type	Problem		
	Result unknown	Change unknown	Start unknown
Join	1. Connie had 5 marbles. Jim gave her 8 more marbles. How many does Connie have altogether?	2. Connie has 5 marbles. How many more marbles does she need to win to have 13 marbles altogether?	3. Connie had some marbles. Jim gave her 5 more marbles. Now she has 13 marbles. How many marbles did Connie have to start with?
Separate	4. Connie had 13 marbles. She gave 5 marbles to Jim. How many marbles does she have left?	5. Connie had 13 marbles. She gave some to Jim. Now she has 5 marbles left. How many marbles did Connie give to Jim?	6. Connie had some marbles. She gave 5 to Jim. Now she has 8 marbles left. How many marbles did Connie have to start with?
Combine	7. Connie has 5 red marbles and 8 blue marbles. How many marbles does she have?		8. Connie has 13 marbles. Five are red and the rest are blue. How many blue marbles does Connie have?
Compare	9. Connie has 13 marbles. Jim has 5 marbles. How many more marbles does Connie have than Jim?	10. Jim has 5 marbles. Connie has 8 more than Jim. How many marbles does Connie have?	11. Connie has 13 marbles. She has 5 more than Jim. How many marbles does Jim have?

The *join* and *separate* problems in the first two rows of Table 1 involve two distinct types of action, whereas the *combine* and *compare* problems in the third and fourth rows describe static relationships. The combine problems involve part-whole relationships within a set, and the compare problems involve the comparison of two distinct sets. For each type of action or relation, distinct problems can be generated by varying the quantity that is unknown, as is illustrated by the distinctions between problems within each row in Table 1. As can be seen from these examples, a number of semantically distinct problems can be generated by varying the structure of the problem, even though most of the same words appear in each problem.

These distinctions between problems are reflected in children's solutions. Most young children invent informal modeling and counting strategies for solving addition and subtraction problems that have a clear relationship to the structure of the problems. At the initial level of solving addition and subtraction problems, children are limited to solutions involving direct representations of the problem. They must use fingers or physical objects to represent each quantity in the problem, and they can represent only the specific action or relationship described. For example, to solve Problem 2 in Table l, they construct a set of 5 objects, add more until there is a total of 13 objects, and count the number of objects added. To solve Problem 4, they make a set of 13 objects, remove 5, and count the remaining objects. Problem 9 might be solved by matching two sets and counting the unmatched elements. Children at this level cannot solve problems like Problem 6, because the initial quantity is the unknown and, therefore, cannot be represented directly with objects.

Children's problem-solving strategies become increasingly abstract as direct modeling gives way to counting strategies like counting on and counting back. For example, to solve Problem 2 in Table 1, a child using a counting-on strategy would recognize that it is unnecessary to construct the set of 5 objects and instead would simply count on from 5 to 13, keeping track of the number of counts. The same child might solve Problem 4 by counting back from 13. Virtually all children use counting strategies before they learn number facts at a recall level.

Number facts are learned over an extended period of time during which some recall of number facts is used concurrently with counting strategies. Children learn certain number combinations earlier than others. Before all the addition facts are completely mastered, many children use a small set of memorized facts to derive solutions for problems involving other number combinations. These solutions usually are based on doubles or numbers whose sum is 10. For example, to find $6 + 8 = ?$, a child might recognize that $6 + 6 = 12$ and $6 + 8$ is just 2 more than 12. Derived facts are not used just by a handful of bright students, and it appears that derived facts play an important role for many children in the learning of number facts.

The analysis above of problem types and solution strategies provides a highly structured framework for analyzing teachers' pedagogical content knowledge. On the basis of that analysis, this study addressed the following questions regarding the status of teachers' pedagogical content knowledge and the relationship between their knowledge and student achievement:

1. What do teachers know about the distinctions between different addition and subtraction problem types?

2. What do teachers know about the strategies that children use to solve different problems?

3. How successful are teachers in predicting their own students' success in solving different types of problems and in identifying the strategies used by children to solve problems of different types?

4. What is the relation between different measures of teachers' pedagogical content knowledge and their students' achievement?

METHOD

Subjects

The subjects for the study were 40 first-grade teachers in 27 schools located in Madison, Wisconsin, and four smaller communities near Madison. The schools included 3 Catholic schools and 24 public schools. All the teachers in the sample had volunteered to participate in a month-long in-service program in mathematics the following summer and to be included in a study of classroom instruction the following year. The mean number of years of teaching elementary school for the teachers in the sample was 10.90, and the mean number of years of teaching first grade was 5.62. Two of the teachers were in their first year of teaching. Thirty-four of the teachers had participated in in-service courses in the last 3 years, and 9 of them had participated in courses dealing with mathematics. None of the teachers reported participating in any training in which recent research in addition and subtraction was discussed. The 40 teachers used 11 different textbooks.

Measures of Teachers' Knowledge

The measures of teachers' knowledge focused on their knowledge of distinctions between problem types, general knowledge of the types of strategies children use to solve different problems, and teachers' ability to predict the performance of specific students in their classes on different problems.

Distinctions between problem types. Two measures were used to test the teachers' ability to distinguish between problem types: Writing Word Problems and Relative Problem Difficulty. For the Writing Word Problems test, the teachers were asked to write six word problems that would be best represented by six given number sentences ($5 + 7 = ?$, $6 + ? = 11$, $? + 4 = 12$, $13 - 4 = ?$, $15 - ? = 9$, and $? - 3 = 9$). These number sentences correspond to the six join and separate problem types presented in Table 1 (Carpenter, Moser, & Bebout, 1988). The test was scored by assigning 2 points to appropriate word problems that corresponded to the given number sentence and 1 point to word problems that did not directly match the given number sentence but had the same answer. No points were awarded to word problems that had different answers, were incomplete, or made no sense.

The Relative Problem Difficulty test measured teachers' knowledge of the relative difficulty of problem types. The teachers were given 16 pairs of word problems and asked to identify which of the two problems would be more difficult for first-grade children. The teachers were told to assume that the problems were read aloud and that the children had counters available to help them solve the problems. In 4 of the pairs, the problems were of the same type with relatively minor changes in context and wording. The other

12 items consisted of pairs of problems for which there is a well-established hierarchy of problem difficulty (Carpenter & Moser, 1983; Riley et al., 1983). Six pairs included a separate-result-unknown problem (Table 1, Problem 4), and 6 included a join-change-unknown problem (Table 1, Problem 2). These problems were each paired with a more difficult combine, compare, or separate-start-unknown problem. Within each pair the same number combinations were used, and problem length and other factors that might affect problem difficulty were held constant. After the teachers had responded to all 16 pairs, they were asked to explain their responses for 5 of the pairs selected in advance to represent different relationships between problems.

General knowledge of strategies. The teachers' general knowledge of children's strategies for solving addition and subtraction problems was assessed by showing them a videotape of three first-grade children solving different problems and asking the teachers to describe how these children would solve related problems. The first child solved four problems by direct modeling. He correctly solved a join-result-unknown problem, a separate-result-unknown problem, and a join-change-unknown problem but was unable to solve a separate-start-unknown problem. The teacher was asked to describe how the child would respond to seven problems that were similar to the four worked on the videotape. The most critical distinction was between the join-change-unknown problem, which could be modeled, and the separate-start-unknown problem, which could not.

The second child on the videotape solved one join-result-unknown problem by counting on. The child clearly articulated the counting sequence as she visibly extended fingers to keep track of where she was in the counting sequence. Each teacher was asked how the child would solve four additional problems. Two of the problems could be solved by a straightforward application of the counting procedures that the child used in the videotape. The other two problems required the teachers to recognize that counting on from the larger number and counting back were strategies that the child might use to solve appropriate problems.

The third child on the videotape solved three problems using derived facts based on doubles. Each teacher was asked to describe how the child would solve four additional problems. The teacher was expected to generate solutions based on doubles where appropriate. Each videotaped episode was repeated twice before a teacher was asked to demonstrate how the child would solve other problems, and each teacher responded to all questions about a given child before the next episode was played.

Teachers' knowledge of their own students. To test teachers' knowledge of their own students, each teacher was asked to demonstrate how each of six students, randomly selected from the teacher's class, would solve six different addition and subtraction word problems. The target students had solved the same problems in individual interviews that took place 1 or 2 days before the teacher interview. The teachers' and students' responses were coded by trained coders using a system developed by Carpenter and Moser (1984). The analysis of teachers' knowledge of their own students was based on the match between a teacher's predictions about each student's performance for a given item and the student's actual performance.

Two scores were generated from the interviews. A Knowledge of Students' Correct Answer score was based on the teacher's success in predicting whether a student solved a given problem correctly irrespective of whether the teacher correctly predicted the strategy the student used. A Knowledge of Students' Strategy score was based on the teacher's success in accurately predicting the strategy the student would use to solve the problem.

Student Performance Measures

Two measures of student performance were administered: Number Facts and Problem Solving.

Number Facts. The Number Facts test was administered to all first-grade students of the teachers in the study and consisted of 20 addition and subtraction basic number facts. Ten problems involved sums less than 10, and ten involved sums between 10 and 18. Sixteen problems were written vertically, and four were written horizontally. Addition and subtraction problems were intermixed. The students were given 2 minutes to complete the test.

Problem Solving. The Problem Solving test consisted of 17 word problems. Nine problems were addition and subtraction problems representing a range of problem types from Table 1, four problems involved several operations or included extraneous numbers, and four involved grouping and partitioning. All of the numbers in the problems were less than 20. Each problem was printed on a separate page of a test booklet. The problems were read to the students, and the students were instructed not to turn to the next page until instructed to do so.

Procedure

All instruments were administered in the spring of 1986. The teacher instruments were administered individually to each teacher by one of five trained interviewers in the following sequence, which was designed to minimize interference from prior tests: Writing Word Problems, Relative Problem Difficulty, Teachers' Knowledge of Their Own Students, and General Knowledge of Strategies. The student Number Facts and Problem Solving tests were administered to all students in the participating teachers' classrooms. The tests were administered by trained testers who followed written protocols.

RESULTS

Knowledge of Problems

The results for the tests of Writing Word Problems and Relative Problem Difficulty are summarized in Table 2. The mean score on the Writing Word Problems test was 11 out of a possible score of 12. Twenty-three teachers had perfect scores. There was only 1 error on the two result-unknown items, 9 on the two change-unknown items, and 17 on the two start-unknown items.

Table 2
Results for Tests of Distinctions Between Problem Types and General Knowledge of Strategies ($N = 40$)

Scale	Maximum possible	M	SD	Highest score	Lowest score
Distinctions between problem types					
Writing word problems	12	11.00	1.32	12	8
Relative problem difficulty					
Equivalent problems	4	3.67	0.66	4	2
Separate result unknown	6	5.40	1.10	6	2
Join change unknown	6	2.13	1.71	6	0
General knowledge of strategies					
Direct modeling	7	6.83	0.50	7	5
Counting	4	2.85	0.62	4	2
Derived facts	4	2.88	0.88	4	1

On the Relative Problem Difficulty test, performance was extremely high for comparisons of equivalent problems and comparisons involving separate-result-unknown problems. However, many teachers overestimated the difficulty of the join-change-unknown problems, and fewer than half of the comparisons involving such problems were correct.

When the teachers were asked to explain their responses, they generally had difficulty articulating the distinctions between problems. The research on addition and subtraction problems summarized in the introduction has shown that problem difficulty is a function of the processes children generally use to solve different problems. Problems that are easily modeled with counters are easier than problems that are not. Only eight teachers mentioned the processes that children would likely use to justify their decisions regarding relative difficulty. The following paraphrased response from one teacher illustrates a judgment based on the processes a child would use to solve the problems:

- For the [join-change-unknown] problem, they would easily use the number line and count up from 5 to 13. . . . For the [separate-start-unknown] problem, there is no specific number to begin with.

Some teachers were able to identify relevant differences between problems, but most of them were unable to articulate why those differences were important. Although a total of 18 teachers specifically referred to the fact that the unknown appeared at the beginning of the separate-start-unknown problem, 10 of them did not relate the difficulty of the problem to the difficulty of representing it directly.

Eighteen teachers focused on the language in the problem or key words in evaluating the relative difficulty of the problems. The following responses illustrate judgments based on language or key words:

- "How many more" throws kids off. They think of adding when they see "more."
- Because of the word "gave" they would realize it was take away.

Many of the responses based on language or key words suggested that some teachers tended to group the problems into two classes based on the operation they would use to solve the problems. These teachers indicated that children would solve problems by deciding whether to add or subtract. Eleven teachers consistently discussed the relative difficulty of problems in terms of how difficult it would be to decide whether to add or subtract. The following responses illustrate this perspective:

- It would be hard to set the problem up as to addition or subtraction.
- Subtraction would be clear in [the separate-result-unknown problem]. They would be unsure of the operation in [the combine problem].

Twelve teachers appeared to group the join-result-unknown and separate-result-unknown problems in one class and the other problems in another. Some mentioned how the action in result-unknown problems (like birds flying away) could be visualized, whereas the nonaction problems could not easily be visualized. Others talked about "straightforward subtraction problems" in contrast to all other subtraction problems. These responses suggested that the teachers could distinguish between standard result-unknown problems and other problems, but they did not clearly distinguish the dimensions on which the problems differed. The distinctions between problems was unclear in many of the responses, and a number of teachers had difficulty giving coherent explanations for their choices. The following response illustrates these difficulties:

- Because it sounded harder. Even if it was read over and over, it would be hard to sort through.

Thus, most teachers in the study appeared to be able to discriminate between problem types. Most of them could write appropriate problems that corresponded to different number sentences. But distinctions between problem types were not usually the primary considerations when the teachers decided on the relative difficulty of different problems. Many teachers tended to focus on syntactic features like key words rather than on the semantics of the problems, which are reflected in children's solutions. They failed to recognize the importance of the differences between problem types like join change unknown and separate start unknown. As a result, many of the teachers overestimated the difficulty of join-change-unknown problems.

General Knowledge of Strategies

The results for each video episode of the test of General Knowledge of Strategies are summarized in Table 2. Almost all teachers could characterize the direct modeling strategies used by the first child on the videotape and could associate the strategies with appropriate problems. Although many of the teachers failed to distinguish between join-change-unknown and separate-start-unknown problems on the test of Relative Problem Difficulty, they were able to pick up the distinction when the child on the videotape was able to model one problem but not the other.

The teachers were somewhat less successful on the Counting Strategies subtest. Almost all teachers recognized that the second child would be likely to solve join-result-unknown and join-change-unknown problems by counting on as was done on the videotape. However, only 21 of the teachers identified counting back as a possible strategy for a separate-result-unknown problem, and only 15 identified counting on from the larger number as a potential strategy when the smaller number appeared first in the problem. Thus, almost all errors occurred on the two problems in which teachers had to describe counting strategies they had not actually seen done by the child on the videotape. In other words, the teachers were generally successful in identifying the features of strategies they observed but had difficulty in identifying how counting strategies could be modified for other problems.

Their performance was also somewhat lower on the Derived Facts subtest. Most teachers recognized that the third child was using some sort of derived fact strategy, but many of them did not identify doubles as the basis for the derived facts.

Teachers' Knowledge of Their Own Students

The analysis of teachers' specific knowledge about their own individual students was based on their success in predicting each student's performance on specific problems. These results are summarized in Table 3. The teachers accurately predicted whether a student could solve a particular problem approximately three fourths of the time, and they accurately predicted the strategy students would use approximately half of the time. The standard deviation was much higher for the strategy measure than for the success measure.

The results for individual problems are presented in Table 4. There was a great deal of variability between problems. The teachers were more successful in predicting student success for the join-result-unknown and separate-result-unknown problems than for the other four problems. These two problems are the problems most commonly included in first-grade mathematics curriculum materials, and most students solved them correctly (Table 5).

Table 3
Results for Teachers' Knowledge of Students' Success in Solving Word Problems and the Strategies Used (N = 40)

Scale	Maximum possible	M	SD	Highest score	Lowest score
Success	36	27.48	2.72	32	18
Strategy	36	16.73	3.96	28	9

Table 4
Item Means and Standard Deviations for Teachers' Knowledge of Students' Success in Solving Word Problems and the Strategies Used (N = 40)

Problem	Success M	Success SD	Strategy M	Strategy SD
Join result unknown	5.65	0.53	2.75	1.31
Separate result unknown	5.28	0.88	3.58	1.36
Join change unknown	4.05	1.41	2.23	1.44
Separate start unknown	3.48	1.40	2.75	1.51
Compare	4.00	0.96	1.98	1.23
Join result unknown without counters	5.03	1.01	3.45	1.30

Note. For each scale, maximum possible score = 6.

Table 5
Percent Correct for Teachers' Estimates of Students' Success and for Actual Student Performance by Problem Type

Problem	Teachers' estimate	Students' performance
Join result unknown	98	96
Separate result unknown	97	89
Join change unknown	68	78
Separate start unknown	46	42
Compare	67	64
Join result unknown without counters	90	89

The results reported in Tables 3 and 4 concern the match between teachers' predictions and students' performance. In Tables 5 and 6, the teachers' predictions of student performance and the students' actual performance are reported separately. The contrast between predictions and performance provides some insights as to whether the teachers as a group tended to overestimate or underestimate either the difficulty of a given problem (Table 5) or the frequency with which a given strategy was used (Table 6).

The data in Table 5 indicate that as a group, the teachers were remarkably accurate in estimating the students' overall success on given items. They overestimated performance on the separate-result-unknown problem and underestimated performance on the join-change-unknown problem, but on the other four items, their aggregated predictions were within several percentage points of actual student performance. In predicting strategies children would use to solve different problems (Table 6), the teachers consistently overestimated the use of direct modeling and recall of number facts and underestimated the use of counting strategies.

Table 6
Frequency in Percent for Teachers' Estimates of Students' Strategies and for Students' Use of Strategies

Strategy	Teachers	Students
Join result unknown		
Direct model	55	43
Count on	26	41
Number fact	16	2
Separate result unknown		
Direct model	75	61
Count down	10	19
Number fact	9	3
Join change unknown		
Direct model	27	24
Count on	27	41
Number fact	5	3

Note. Percents do not sum to 100 because errors and infrequently used strategies have been omitted.

Teachers' Knowledge and Students' Achievement

The correlation between the measures of teachers' knowledge and the measures of student achievement are reported in Table 7. The Writing Word Problem test was not included in the analysis because almost all teachers were near the ceiling for the test, and there was not sufficient variability to warrant its inclusion. Several correlations were of primary interest: the correlations between the general measures of teachers' knowledge and their specific knowledge of their own students and the correlations between teachers' knowledge of their own students and the measures of student achievement. One might expect that the teachers' general knowledge of problem difficulty would be related to their ability to predict their students' success in solving different problems, but the correlation between those variables was not significant. Similarly there was not a significant correlation between the teachers' general knowledge of strategies and their ability to predict the strategies that their own students would use to solve different problems.

Table 7
Correlations Between Teacher Knowledge and Student Achievement

Variable	2	3	4	5	6
1. Relative problem difficulty	−.16	.01	−.10	−.23	−.15
2. General knowledge of strategies	—	.09	.06	.24	.25
3. Knowledge of students' success		—	.33*	.32*	.31*
4. Knowledge of students' strategies			—	−.01	−.22
5. Student computation				—	.47*
6. Student problem solving					—

*$p < .05$.

The teachers' ability to predict students' success in solving different problems was significantly correlated with student performance on both the number-fact and problem-solving tasks. However, the teachers' ability to predict the strategies that students would use to solve different problems was not significantly correlated with either measure of students' performance.

DISCUSSION

The teachers in this study could distinguish some of the basic differences between the major types of addition and subtraction problems. Most of them could write word problems to represent different joining and separating situations, and they could identify that the first child on the videotape was having difficulty solving one type of problem but not the other. However, their recognition of differences between problems appeared to operate on an ad hoc problem-by-problem basis. Most of the teachers did not appear to have a coherent framework for classifying problems, and they frequently could not articulate the basis for the distinctions they made between problems. Most teachers were familiar with the most frequently used strategies for solving addition and subtraction problems, and they could successfully identify strategies when they observed children using them on videotape. However, they generally did not categorize problems in terms of the strategies that children use to solve them. Many teachers did not seem to recognize the general principle that problems that can be directly modeled are easier than problems that cannot. As a consequence, they consistently overestimated the difficulty of join-change-unknown problems.

In general, most of the teachers in this study were reasonably successful in identifying many of the critical distinctions between problems and the primary strategies that children use in solving addition and subtraction problems. However, this knowledge generally was not organized by the teachers into a coherent network that related distinctions between problems, children's solutions, and problem difficulty to one another. It is not surprising that most teachers do not focus on these relationships considering that it took researchers many years to specify them clearly.

None of the measures of teachers' general knowledge of problems, problem difficulty, or strategies were significantly correlated with student achievement or even with teachers' ability to predict either their own students' success in solving different problems or the strategies the students would use to solve them. One would expect that a general knowledge of problem difficulty would be a prerequisite for successfully predicting students' performance on different problems and that a general knowledge of strategies would be related to teachers' success in identifying strategies specific children in their classes would use. The lack of success in identifying expected relationships may have resulted from the lack of variability on the measures of teachers' general knowledge of problems and strategies. Teachers' performances on the measures of general knowledge of problems and strategies were uniformly high.

It appears that most teachers had the general knowledge needed to predict their own students' problem-solving performance. For example, most teachers were familiar with the basic strategies that children use to solve addition and subtraction problems and could successfully identify them when they observed them on videotape. It appears that the variability in teachers' success in predicting the strategies that their own students would use to solve different problems was not based on differences in ability to identify the strategies. Most teachers seemed to be capable of identifying students' strategies when they saw them.

The teachers' ability to predict their students' success in solving different problems was significantly correlated with both measures of students' achievement, but their ability to predict the strategies that students would use was not correlated with either achievement measure. One would expect the success measure and the strategy measure to follow the same pattern, given that knowing the strategies students use would help teachers to make judgments about students' ability to solve different problems. However, there are some difficulties in measuring teachers' knowledge of the strategies students can use. Many students do not consistently use a single strategy (Carpenter & Moser, 1984). A teacher may have identified a strategy that a student used frequently, but the student chose an alternative strategy when he or she was interviewed. Given the variability in students' choice of strategy, it would be inappropriate to conclude that teachers' knowledge of the strategies students can use is not related to student achievement. On the other hand, these results could be interpreted to suggest that teachers do not traditionally make instructional decisions based on the strategies that children use to solve different problems, whereas they do make decisions about whether to include particular problems based on their assessment of whether the problems would be too difficult for their students. From this line of reasoning, teachers' knowledge of students' success may be related to achievement because instructional decisions are based upon that knowledge, and teachers' knowledge of the strategies children use may not be related to achievement because instructional decisions are not based upon knowledge of strategy use. Such a hypothesis is extremely speculative at this point.

Even if one were to accept the hypothesis, it does not imply that teachers cannot or should not base instructional decisions on the strategies children use to solve different problems or that deeper, more principled knowledge of problem types and strategies on the part of teachers would not lead to higher levels of achievement. The results of this study may reflect the fact that teachers traditionally have not had a sufficiently rich knowledge base to plan for instruction based on a careful assessment of the processes that students use to solve problems. The research on addition and subtraction provides a principled basis for differentiating between problems and children's processes for solving them. If teachers had this knowledge base, they might perform very differently than the teachers in this study.

REFERENCES

Carpenter, T. P. (1985). Learning to add and subtract: An exercise in problem solving. In E. A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 17–40). Hillsdale, NJ: Erlbaum.

Carpenter, T. P., & Moser, J. M. (1983). The acquisition of addition and subtraction concepts. In R. Lesh & M. Landau (Eds.), *The acquisition of mathematical concepts and processes* (pp. 7-14). New York: Academic Press.

Carpenter, T. P., & Moser, J. M. (1984). The acquisition of addition and subtraction concepts in grades one through three. *Journal for Research in Mathematics Education, 15,* 179–202.

Carpenter, T. P., Moser, J. M., & Bebout, H. C. (1988). Representation of addition and subtraction problems. *Journal for Research in Mathematics Education, 19,* 345–357.

Carpenter, T. P., Moser, J. M., & Romberg, T. A. (Eds.). (1982). *Addition and subtraction: A cognitive perspective.* Hillsdale, NJ: Erlbaum.

Carpenter, T. P., & Peterson, P. L. (Eds.). (1988). Learning mathematics from instruction [Special issue]. *Educational Psychologist.*

Chi, M. T. H., Glaser, R., & Rees, E. (1982). Expertise in problem solving. In R. Sternberg (Ed.), *Advances in the psychology of human intelligence* (pp. 7-75). Hillsdale, NJ: Erlbaum.

Clark, C. M., & Peterson, P. L. (1986). Teachers' thought processes. In M. W. Wittrock (Ed.), *Handbook of research on teaching* (3rd ed., pp. 255–296). New York: Macmillan.

Jerman, M., & Mirman, S. (1974). Linguistic and computational variables in problem solving in elementary mathematics. *Educational Studies in Mathematics, 5,* 317–362.

Lampert, M. (1987, April). *Attending to students' thinking in math classes: Teachers' perspectives.* Paper presented at the meeting of the American Educational Research Association, Washington, DC.

Leinhardt, G., & Greeno, J. G. (1986). The cognitive skill of teaching. *Journal of Educational Psychology, 78,* 75–95.

Leinhardt, G., & Smith, D. A. (1985). Expertise in mathematics instruction: Subject matter knowledge. *Journal of Educational Psychology, 77,* 247–271.

Putnam, R. T. (1987). Structuring and adjusting content for students: A study of live and simulated lecturing of addition. *American Educational Research Journal, 24,* 13–48.

Putnam, R. T., & Leinhardt, G. (1986, April). *Curriculum scripts and the adjustment of content in lessons.* Paper presented at the meeting of the American Educational Research Association, San Francisco.

Riley, M. S., Greeno, J. G., & Heller, J. I. (1983). Development of children's problem-solving ability in arithmetic. In H. Ginsburg (Ed.), *The development of mathematical thinking* (pp. 153–200). New York: Academic Press.

Romberg, T. A., & Carpenter, T. P. (1986). Research on teaching and learning mathematics: Two disciplines of scientific inquiry. In M. C. Wittrock (Ed.), *Handbook of research on teaching* (3rd ed., pp. 850–873). New York: Macmillan.

Shavelson, R. J., & Stern, P. (1981). Research on teachers' pedagogical thoughts, judgments, decisions, and behavior. *Review of Educational Research, 51,* 455–498.

Shulman, L. S. (1986a). Paradigms and research programs in the study of teaching: A contemporary perspective. In M. C. Wittrock (Ed.), *Handbook of research on teaching* (3rd ed., pp. 3-36). New York: Macmillan.

Shulman, L. S. (1986b). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4–14.

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