Of Primary Interest

Teaching Math in the Primary Grades

The Learning Trajectories Approach

Julie Sarama and Douglas H. Clements

Two kindergarten teachers sit down for lunch during a professional development workshop. One says, "I think it's ridiculous. The children are still babies. They're trying to teach them too much." Her friend nods. Soon they are joined by a colleague from another school, who bubbles, "Isn't this great? The children are going to know so much more!"

Most of us can sympathize with both perspectives. What *should* we be teaching in the early grades? Three research findings provide some guidance in mathematics instruction.

1. Learning substantial math is critical for primary grade children.

The early years are especially important for math development. Children's knowledge of math in these years predicts their math achievement for later years—and throughout their school career. Furthermore, what they know in math predicts their later reading achievement as well (Duncan et al. in press). Given that early math learning predicts later math and reading achievement, math appears to be a core component of learning and thinking.

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2. All children have the potential to learn challenging and interesting math.

Primary grade children have an often surprising ability to do abstract math—that is, math that is done by reasoning mentally, without the need for concrete objects. Listen to the worries of this first-grader.

> "I find it easier not to do it [simple addition] with my fingers because sometimes I get into a big muddle with them [and] I find it much harder to add up because I am not concentrating on the sum. I am concentrating on getting my fingers right . . . It can take longer to work out the sum [with fingers] than it does to work out the sum in my head." [In her head, Emily imagined dot arrays. Why didn't she just use those?] "If we don't use our fingers, the teacher is going to think, 'Why aren't they using their fingers . . . they are just sitting there thinking' . . . We are meant to be using our fingers because it is easier . . . which it is not." (Gray & Pitta 1997, 35)

Should the teacher encourage Emily to use concrete objects to solve math problems? Or should she encourage children like Emily to use arithmetic reasoning?

Primary grade children often know, and can definitely learn, far more challenging and interesting math than they are taught in most U.S. classrooms. That does not necessarily mean math pushed down from higher grades. It means letting children invent their own strategies for solving a variety of types of problems. How can teachers best support creative thinking in mathematics?

3. Understanding children's mathematical development helps teachers be knowledgeable and effective in teaching math.

Children's thinking follows natural developmental paths in learning math. When teachers understand these paths and offer activities based on children's progress along them, they build math learning environments that are developmentally appropriate and particularly effective. A useful tool in understanding and supporting the development of children's mathematical reasoning is a *math learning trajectory.* There are learning trajectories for mathematics at all age levels, from birth throughout the school years, and for learning all kinds of content—from specific math concepts such as number and operations to specific science concepts like understanding electricity.

Learning trajectories

Math learning trajectories have three parts: a mathematical **goal**, a **developmental path** along which children's math knowledge grows to reach that goal, and a set of **instructional tasks**, or activities, for each level of children's understanding along that path to help them become proficient in that level before moving on to the next level. Let's examine each of these three parts.

Goal. The first part of a learning trajectory is the goal. Goals should include the big ideas of math, such as "numbers can be used to tell us how many, describe order, and measure" and "geometry can be used to understand and to represent the objects, directions, and locations in our world, and the relationship between them" (Clements, Sarama, & DiBiase 2004). In this article, we look at the goal of knowing how to solve a variety of addition and subtraction problems.

Developmental path. The second part of a learning trajectory consists of levels of thinking, each more sophisticated than the last, leading to achieving the mathematical goal. That is, the developmental path describes a typical learning route children follow in developing understanding of and skill in a particular mathematics topic.

Learning trajectories are important because young children's ideas and their interpretations of situations are different from those of adults. Teachers must interpret what the child is doing and thinking and attempt to see the situation from the child's viewpoint. Knowledge of developmental paths enhances teachers' understanding of children's thinking, helping teachers assess children's level of understanding and offer instructional activities at that level. Similarly, effective teachers consider the instructional tasks from the child's perspective.

Instructional tasks. The third part of a learning trajectory consists of sets of instructional tasks or activities matched to each level of thinking in a developmental progression.

The National Association of Early Childhood Specialists in State Departments of Education (NAECS/SDE) works to improve instruction, curriculum, and administration in education programs for young children and their families. Of Primary Interest is written by members of NAECS/SDE for kindergarten and primary teachers. The column appears in March, July, and November issues of *Young Children* and Beyond the Journal (online at **www.journal.naeyc.org/btj**).

The tasks are designed to help children learn the ideas and practice the skills needed to master that level. Teachers use instructional tasks to promote children's growth from one level to the next

Teaching challenging and interesting math

The three research findings—the importance of math learning in the primary grades, all children's potential to learn math, and teachers' need to understand children's learning development—have implications for teaching primary grade math well. We suggest the following approach:

- Know and use learning trajectories.
- Include a wide variety of instructional activities. The learning trajectories provide a guide as to which activities are likely to challenge children to invent new strategies and build new knowledge.
- Use a combination of teaching strategies. One effective approach is to (a) discuss a problem with a group, (b) follow up by having children work in pairs, and then (c) have the children share solution strategies back with the group. Discuss strategies with children in pairs and individually. Differentiate instruction by giving groups or individual children different problem types.

Alexander and Entwisle state that "the early grades may be precisely the time that schools have their strongest effects" (1988, 114). Math is so important to children's success in school, in the primary grades and in future learning, that it is critical to give children motivating, substantive educational experiences. Learning trajectories are a powerful tool to engage all children in creating and understanding math.

References

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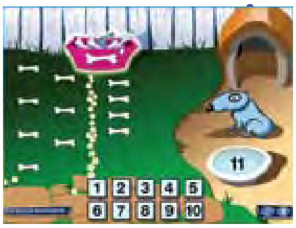
Learning Trajectory for Addition and Subtraction: Sample Levels of the Developmental Path and Examples of Instructional Tasks

This chart gives simple labels and a sampling of levels in the developmental learning progressions for ages 5 through 7 years. The ages in the first column are not exact indications—children in challenging educational environments often create strategies that are surprisingly sophisticated. The second column describes four main levels of thinking in the addition and subtraction learning trajectory. These levels are samples—there are many levels in between them (for full learning trajectories, see Clements & Sarama 2009 and Sarama & Clements 2009). The third column briefly describes examples of instructional tasks.

Age	Developmental path—Sample levels	Instructional tasks
5	**Find Change.** Children find the missing addend (5 + _ = 7) by adding on objects.	**Word Problems.** For example, say to the children, "You have 5 balls and then get some more. Now you have 7 balls in all. How many more balls did you get?" Children use balls in 2 colors to solve such problems.
5½	**Counting Strategies.** Children find sums for joining problems ("You have 8 apples and get 3 more . . .") and part-part-whole problems ("6 girls and 5 boys . . .") with finger patterns [*counting using fingers and quickly recognizing the quantity*] and/or by counting on.	**How Many Now? Problems.** For example, have the children count objects as you place them in a box. Ask, "How many are in the box now?" Add 1, repeating the question, then check the children's responses by counting all the objects. Repeat, checking occasionally. When children are ready, sometimes add 2, and eventually more, objects.
	Counting On. The teacher asks, "How much is 4 and 3 more?" A child replies, "4 . . . 5, 6, 7 [*uses a rhythmic or finger pattern to keep track*]. 7!"	
	Counting Up. A child may solve a missing addend (3 + _ = 7) or compare problems by counting up; for example, the child counts "4, 5, 6, 7" while putting up fingers, and then counts or recognizes the 4 fingers raised. Or the teacher asks, "You have 6 balls. How many more do you need to have 8 balls?" The child says, "6, 7 [*puts up a finger*], 8 [*puts up a second finger*]. 2!"	**Double Compare.** Children compare sums of 2 cards to determine which sum is greater. Encourage the children to use more sophisticated strategies, such as counting on.  **Bright Idea.** Using a numeral and a frame with dots, children count on from the numeral to identify the total amount. They then move forward a corresponding number of spaces on a game board.
6	**Part-Whole.** The child has an initial part-whole understanding and can solve all the preceding problem types using flexible strategies (may use some known combinations, such as "5 + 5 is 10").	**Hidden Objects.** Hide 4 counters under a dark cloth and show children 7 counters. Tell them that 4 counters are hidden and challenge them to tell you how many counters there are in all. Or tell the children there are 11 counters in all, and ask how many are hidden. Have the children discuss their solution strategies. Repeat with different sums.  **Barkley's Bones.** Children determine the missing addend in problems such as 4 + _ = 7.
7	**Deriver.** The child uses flexible strategies and derived combinations ("7 + 7 is 14, so 7 + 8 is 15") to solve all types of problems.	**Twenty-one.** Play this card game, whose object is to have the sum of one's cards be 21 or as close as possible without exceeding 21. An ace is worth either 1 or 11, and cards for 2 through 10 are worth their face value. A child deals everyone 2 cards, including herself. • In each round, if a player's sum is less than 21, the player can request another card or stand pat, saying, "Hold." • If the new card makes the player's sum greater than 21, the player is out. • Play continues until everyone holds. The player whose sum is closest to 21 wins. **Multidigit Addition and Subtraction.** "What's 28 + 35?"

Adapted from D.H. Clements and J. Sarama, *Learning and Teaching Early Math: The Learning Trajectories Approach* (New York: Routledge, 2009), and J. Sarama and D.H. Clements, *Early Childhood Mathematics Education Research: Learning Trajectories for Young Children* (New York: Routledge, 2009).

Activity images from D.H. Clements and J. Sarama, *Building Blocks* [Computer software] (Columbus, OH: SRA/McGraw-Hill, 2007). Used with permission from SRA/McGraw-Hill.

Learning Trajectory
Developmental Levels for "Counting"

The ability to count with confidence develops over the course of several years. Beginning in infancy, children show signs of understanding number. With instruction and number experience, most children can count fluently by age 8, with much progress in counting occurring in kindergarten and first grade. Most children follow a natural developmental progression in learning to count with recognizable stages or levels. This developmental path can be described as part of a learning trajectory.

Level	Level Name	Description	Notes
1	Pre-Counter	A child names some number words in no apparent order and without meaning.	
2	Chanter	A child sing-songs numbers often in some order, but it is a song and without meaning of quantity or counting.	
3	Reciter	A child verbally recites number names as separate words with the intention to count, but does not necessarily recite the correct order.	
4	Reciter (10)	A child verbally counts to 10 with some correspondence with objects. The child may point to objects to count a few items but then often loses track.	
5	Corresponder	A child can keep one-to-one correspondence between counting words and objects—at least for small groups of objects laid in a line. When asked "how many," the child often recounts the objects starting over with one each time.	
6	Counter (Small Numbers)	A child begins to count meaningfully. The child accurately counts a given set of objects to 5 and answers the "how many" question with the last number counted without needing to recount the objects.	
7	Producer—Counter To (Small Numbers)	When asked to show a specific number of objects, a child can accurately produce or make a set of objects up to 5.	
8	Counter (10)	A child accurately counts structured arrangements of objects to 10. He or she may be able to draw representations for quantities up to 10. The child can also find the number just after or just before another number, but only by counting up from 1.	
9	Counter and Producer Counter to (10+)	Child accurately counts and produces sets to 10 and beyond to 30, keeping track of objects that have and have not been counted. Child draws representations to 10, then to 20 and 30, and can find the next number to 20 or 30. Child recognizes errors in others' counting and can eliminate most errors in one's own counting.	

10	Counter Backward from 1	The child is able to count backwards from 10.	
11	Counter from N (N+1, N-1)	The child begins to count on from numbers other than one, either in verbal counts or with objects. The child can determine the number just before or just after another number quickly without having to start counting back at one.	
12	Skip-Counting by 10s to 100	The child can count by tens to 100.	
13	Counter to 100	The child can count by ones through 100, including knowing the decade transitions from 39 to 40, 49 to 50, and so on, starting at any number.	
14	Counter on using patterns	The child keeps track of counting acts by using numerical patterns or movements, such as tapping as he or she counts.	
15	Skip Counter	The child can count by five and twos with understanding.	
16	Counter of Imagined Items	The child can count mental images of hidden objects.	
17	Counter On Keeping Track	The child can keep track of counting acts numerically with the ability to count on (one to four counts) from a given number.	
18	Counter of Quantitative Units	The child can count unusual units such as "wholes" when shown combinations of wholes and parts. For example when shown three whole plastic eggs and four halves, a child at this level will say there are five whole eggs.	
19	Counter to 200	The child counts accurately to 200 and beyond, recognizing the patterns of ones, tens, and hundreds.	
20	Number Conserver	The child demonstrates the ability to conserve number. She or he understands that a number is unchanged even if a group of objects is rearranged. For example, if there is a row of ten buttons, the child understands there are still ten without recounting, even if they are rearranged in a long row or a circle.	

Source: Sarama, J., & Clements, D. H. (2009). *Early childhood mathematics education research: Learning trajectories for young children*. Routledge.

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Patterns	Counting	Algebra	Organized by Grade
Sorting	Addition	Negative Integers	Organized by Subject
Measurement	Subtraction	Exponents	
Data & Probability	Multiplication	Cartesian	
Shape & Symmetry	Fractions	Pencil & Paper Games	
	Primes & Composites	Commercial Games	
	Division		

http://mathpickle.com/counting/

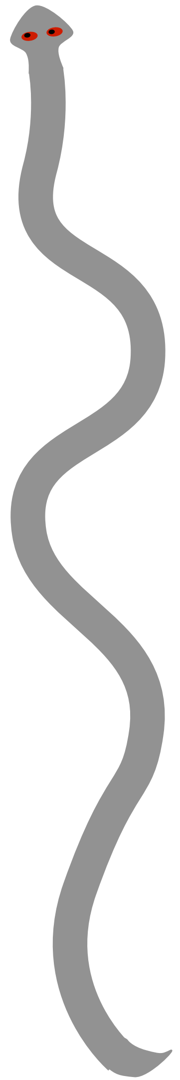
Counting

Students practice counting and writing numbers from 1 to 25. Download printable puzzle-sheets here. Some are purposefully impossible.

Make your math class active by teaching real skipping. In this video we go forward, backward and stand still on the number line.

Give your students practice in skip counting. Download the puzzle-sheets here.

1				



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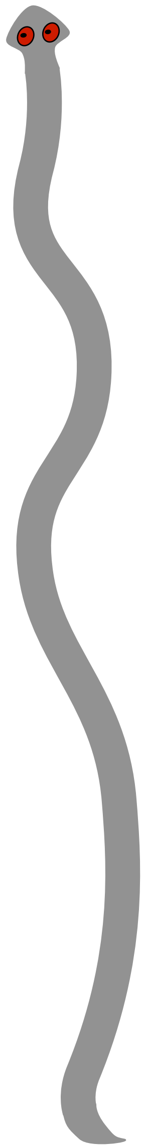

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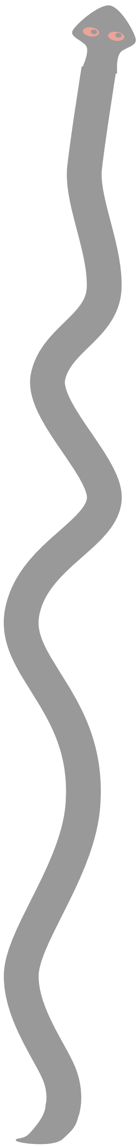
25			22	



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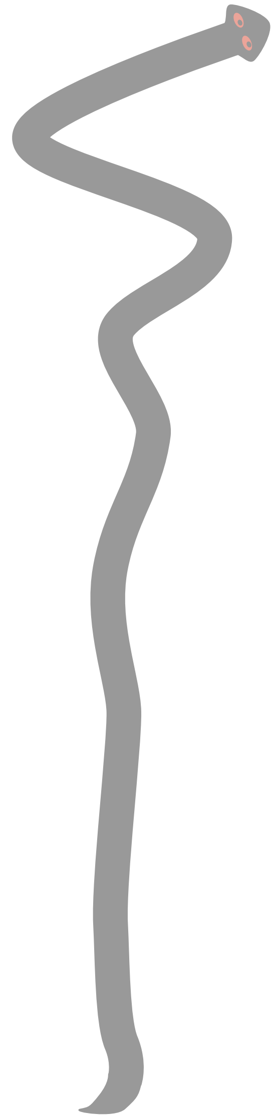

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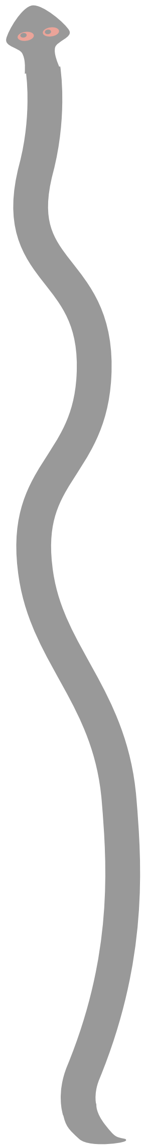

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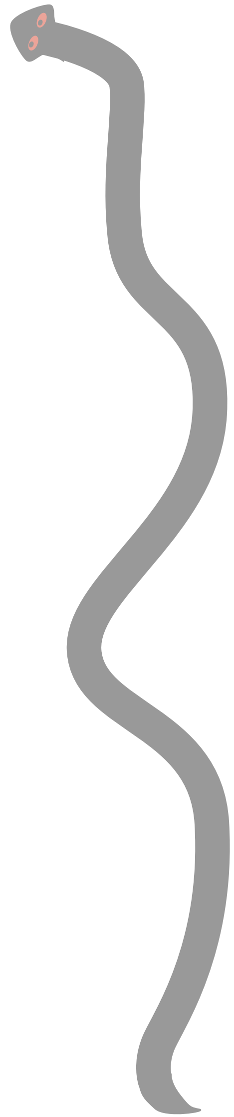

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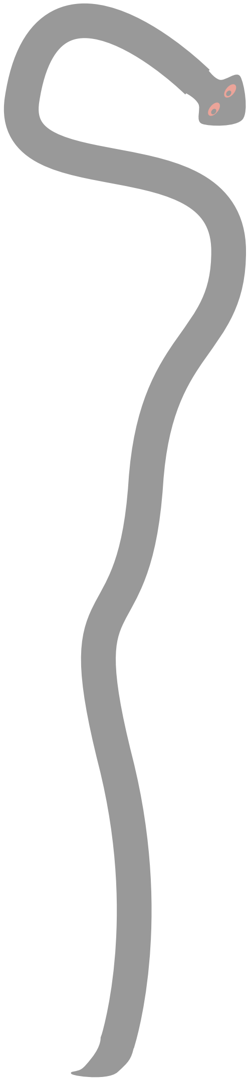

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For teachers only: these puzzles are from different worksheets.

Three are possible to solve.

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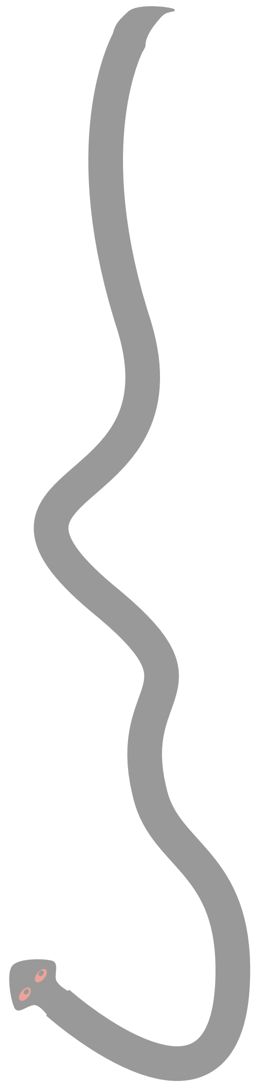

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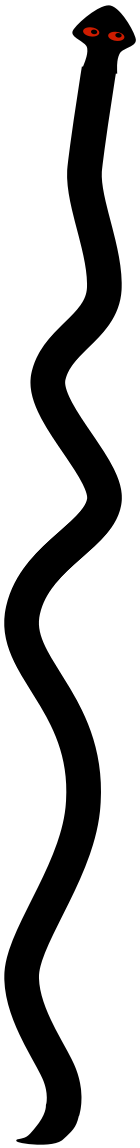

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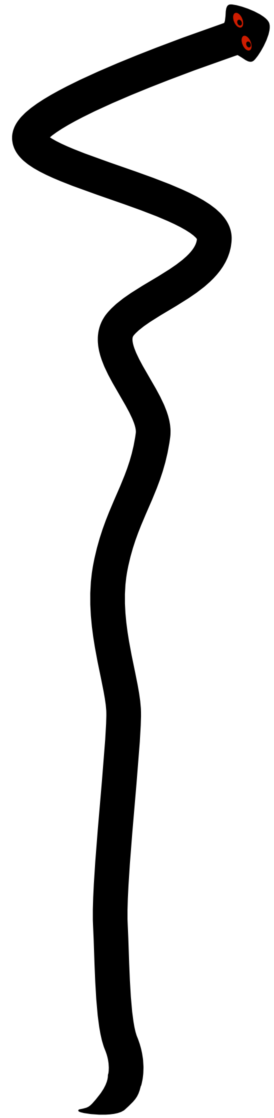

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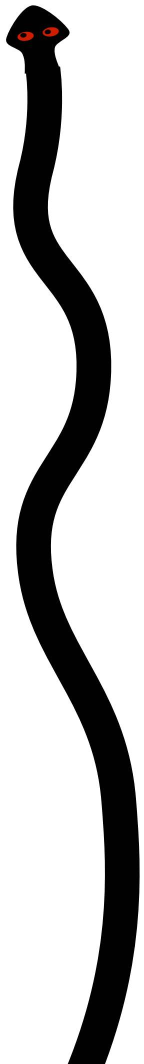

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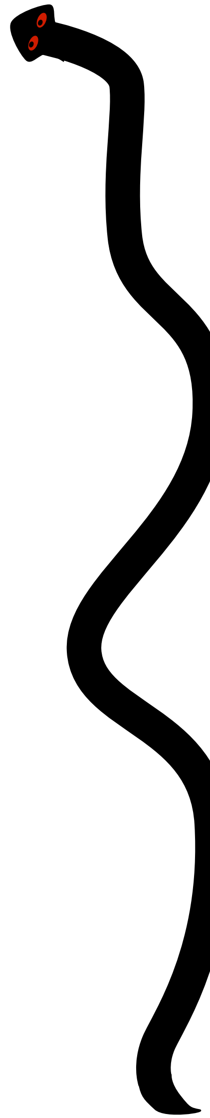

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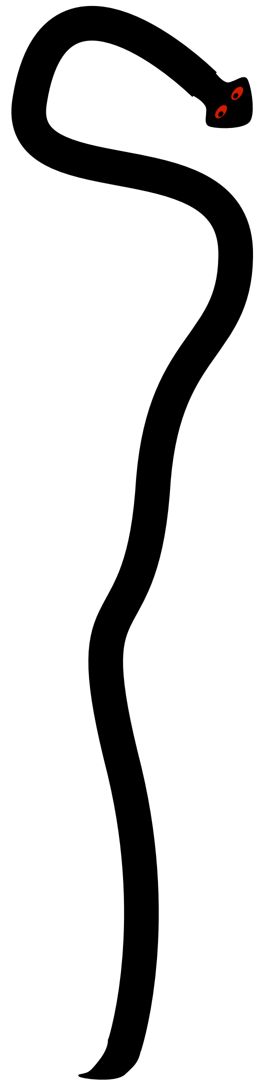

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Racing Bears

Materials

Racing Bears Game board
dot cube (dice)
four teddy bear counters (or another type of counter)
4 counters (any objects-pennies, chips, etc.)

Object

Play is over when players together collect 10 counters

To Play

1. Place a teddy bear at the beginning of each of the four tracks and a counter in the circle at the end of each track
2. Take turns rolling the dot cube and moving any of the bears that number of spaces.
3. The object is to work together to get a bear to the tenth space on any track. When a bear lands exactly on the tenth space, the players take the counter off.
4. Continue playing until all counters have been collected.

The game can be played individually, a group of two, or with the whole family.

Name Date

How Many Do You Have?

Racing Bears Gameboard

10	○	○	○	○
9				
8				
7				
6				
5				
4				
3				
2				
1				
0	☆ Start	☆ Start	☆ Start	☆ Start

Bear Tracks

Bear Tracks is a board game that provides very young children with opportunities to think about number, in particular cardinality and one-to-one correspondence.

Directions for Playing

Children play in pairs. Each player chooses a track on the game board (Appendix A). Alternately the players roll the foam number cube and take a matching number of plastic teddy bear counters or connecting cubes, placing them on their track. Play is collaborative with players helping each other. Play continues until both tracks are filled with counters or cubes.

Noting the Mathematical Landscape: Openings and Possibilities

Young children who are learning to count often confuse the number they end on as the name of the last object, rather than the quantity in the set. Rather than moving a counter along a board as in most board games, Bear Tracks is designed to encourage children to examine number as a quantity—cardinal number. The quantity that is rolled is the amount placed on the board. Because the teddy bears (and connecting cubes) come in several colors, small sets appear within the larger ones. Noticing these sets, children may comment after taking five bears, "I have three yellow and two green." This noticing of sets within sets is to be encouraged. Also note how children determine how many pieces to take after they roll the number cube. Do they count the dots and know that the number they end on is the total quantity of dots? Do they double-tag or skip some dots? Or do they need to place a teddy bear on each dot to determine how many (this is why the game is played with a large foam cube).

Materials Needed

- Large foam number cubes—one per pair of children
- Bear Tracks game board (Appendix A)—one per pair of children
- Plastic teddy bears (or other counters such as cubes)

Playing Bear Tracks

Author's Notes

Karen (the teacher): Inez, you roll the cube first. And remember we read the top to tell us how many bears to put on the track.

Inez: It's 1, 2, 3, 4, 5, 6.

Karen: So how many teddy bears do you need to take?

Inez: *(Places one teddy bear on each dot.)* This many.

Karen: How many is that?

Inez: It's 1, 2, 3, 4, 5, 6.

Karen asks how many after Inez counts to see if she realizes that the result of her counting is the number contained in the set represented by "6." She does not. In fact, she counts again even after she places the bears one-to-one on the dots.

Karen: OK, it's 6 teddy bears. Let's put them on your track. *(Inez places them on the track, one on each square.)* You took red and blue teddy bears. How many red ones do you have?

Karen asks about smaller quantities (the red and blue) to see if Inez subitizes smaller amounts. She does not need to count three.

Inez: I have 3.

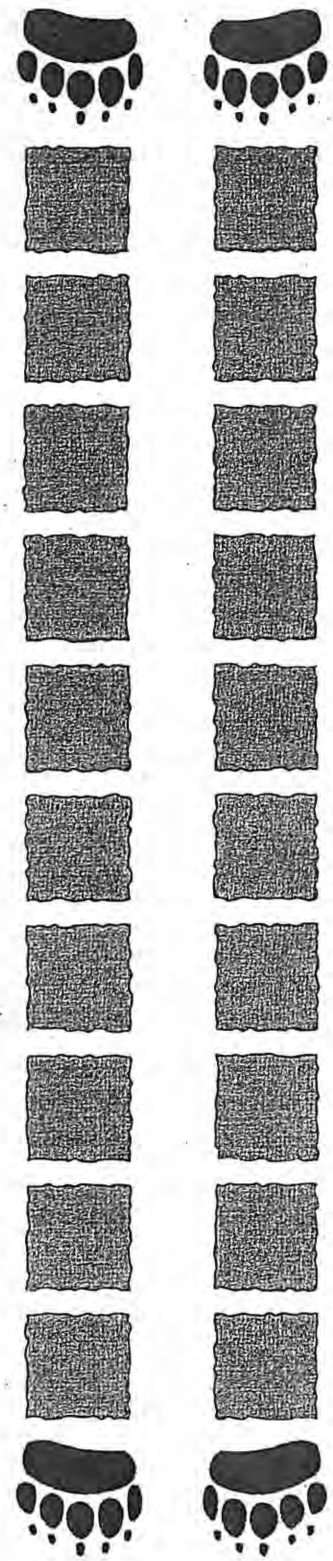
Karen: And how many blue ones?

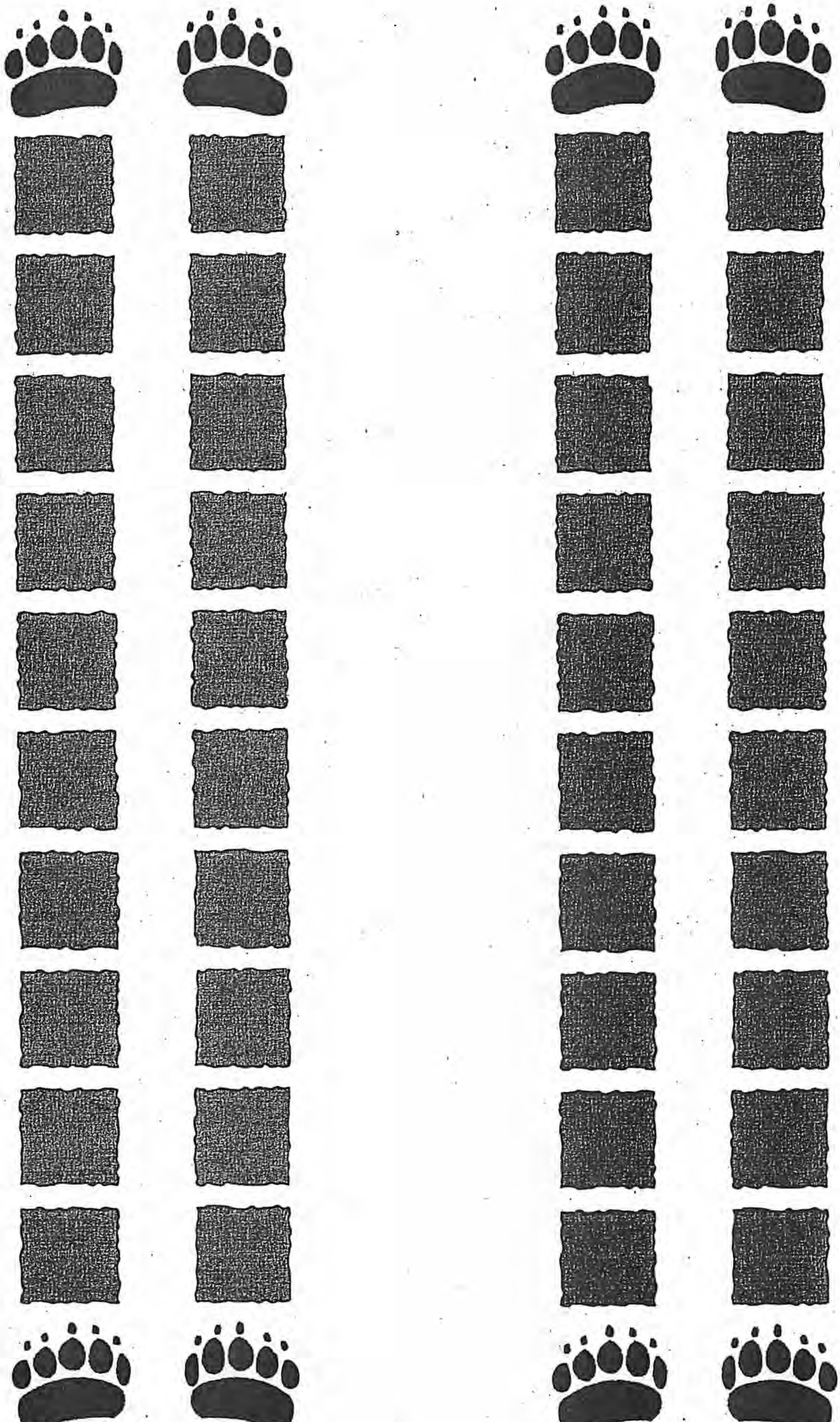
Inez: Also 3.

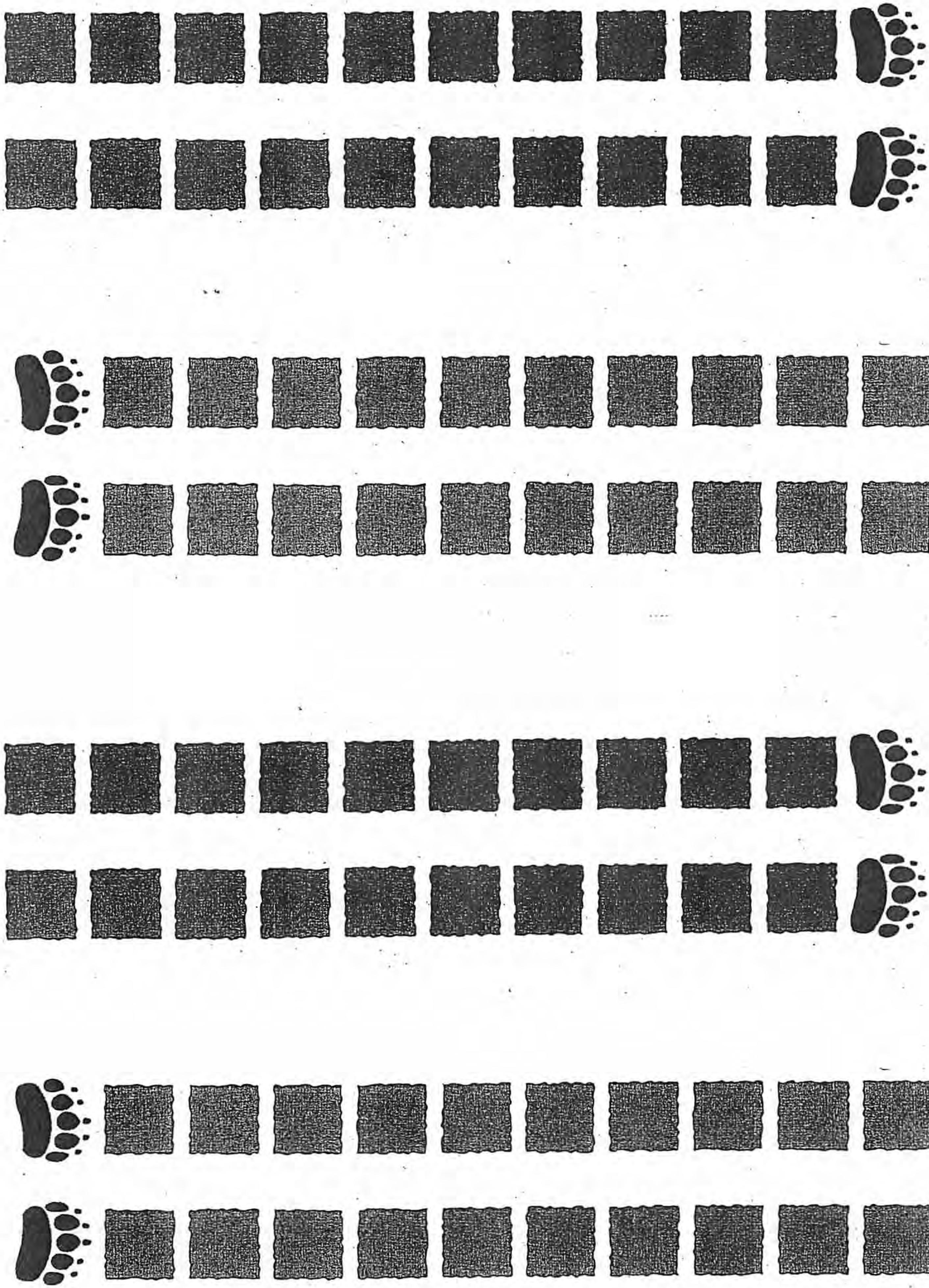
Karen: So you have 3 red ones and 3 blue ones and 6 altogether. OK, it's your turn, Shareema.

By summarizing, Karen provides a focus for reflection.

Make two copies and glue together with paws showing only at the top and bottom of tracks.







CCSSM Counting and Cardinality and Number and Operations in Base Ten Standards Progression

Grade: Kindergarten **Domain: Counting and Cardinality**	**Use teacher friendly language to interpret concepts, key terms or phrases within the standard**	**Provide examples of the mathematics in the standard**
Cluster: Know number names and the count sequence		
Standard: K.CC.1 Count to 100 by ones and tens		
Standard: K.CC.2 Count forward beginning from a given number within the known sequence (instead of having to begin at 1)		

Grade: Kindergarten **Domain: Counting and Cardinality**	**Use teacher friendly language to interpret concepts, key terms or phrases within the standard**	**Provide examples of the mathematics in the standard**
Cluster: Count to tell the number of objects		
Standard: K.CC.4 Understand the relationship between numbers and quantities; connect counting to cardinality a. When counting objects, say the number names in the standard order, pairing each object with one and only one number name and each number name with one and only one object		

b. Understand the last number name tells the number of the objects counted. The number of objects is the same regardless of their arrangement or the order in which they were counted.		
Standard: K.CC.5 Count to answer "how many?" questions about as many as 20 things arranged in a line, a rectangular array, or a circle, or as many as 10 things in a scattered configuration; given a number from 1-20, count out that many objects.		

Grade: First Grade **Domain: Number and Operations in Base Ten**	**Use teacher friendly language to interpret concepts, key terms or phrases within the standard**	**Provide examples of the mathematics in the standard**
Cluster: Extend the Counting Sequence		
Standard: 1.OA.1 Count to 120, starting at any number less than 120. In this range, read and write numerals and represent a number of objects with a written numeral		

Grade: 2nd Grade **Domain: Number and Operations in Base Ten**	**Use teacher friendly language to interpret concepts, key terms or phrases within the standard**	**Provide examples of the mathematics in the standard**
Cluster: Understand place value		
Standard: 2.NBT.2 Count within 1000; skip count by 5s, 10s and 100s		

Counting Collections

Lakeisha has lined up paintbrushes across a table and is rolling them one by one to the side. Tyler and Auveen are wrapping pencils into bundles. Maya is organizing toy kangaroos, and her partner, Max, is drawing a picture of how she is doing it. What is going on in this classroom?

Why Count Collections?

At the beginning of every school year, the five- to seven-year-olds at Corinne A. Seeds University Elementary School (UES) spend several weeks "counting collections." UES, the laboratory school of the Graduate School of Education and Information Studies at UCLA, serves a socioeconomically and ethnically diverse student population from urban and suburban Los Angeles. The classes are multiage, and the five- to seven-year-old classes include children who would be considered kindergarten and first-grade students.

Our work in counting collections was inspired by Megan Franke, a parent at our school and a researcher in mathematics education and children's thinking who has often worked in our classrooms. Megan encouraged us to try counting collections of objects with our young children, believing this would provide children with rich opportunities to practice oral counting, develop efficient counting strategies, group objects in strategic ways, record numbers, and represent their thinking. Research shows that although counting is one of the best ways we know to help children develop number sense and other important mathematical ideas, we do not do nearly enough of it in elementary schools. Children need lots of experience with counting to learn which number comes next, how this number sequence is related to the objects in front of them, and how to keep track of which ones have been counted and which still need to be counted (Fuson 1988). Experience with counting provides a solid foundation for future experience with addition, subtraction, multiplication, and division (National Research Council 2001).

Convinced by the literature as well as the outcomes we have seen with our students, we have made counting collections a fundamental part of what we do with young children at UES beginning the first week of school each fall. We hope this article will provide a window on the process of counting collections in our classrooms as well as evidence that every child in our classrooms can build his or her mathematical skills by counting collections.

By Julie Kern Schwerdtfeger and Angela Chan

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Identifying Collections and Beginning the Counting Process

At first, as the children are getting to know the classroom and we are getting to know them, we have them take an inventory of objects they find in the room, such as buckets of markers, pattern blocks, and Legos. We have also accumulated boxes of shells, keys, coins, bottle caps, and the

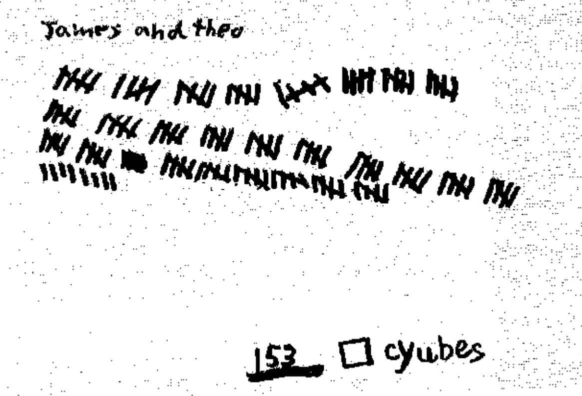
like, all of which are available for the children to count. Later on, to add variety to what we have on hand, children bring collections from home, including objects such as hair clips, sugar packets, toy cars, beads, seed pods, acorns, pennies, and corks.

Initially, we simply ask the children to choose a collection, count with a partner, and record what they counted and what their total was (see **fig. 1**). Sooner rather than later we increase the complexity of the task by asking the children to also show us *how* they counted. **Figure 2a** shows Tess and Ashley's recording of how they counted chains of ten links each. While Ashley moved the chains two at a time from the table to the floor, Tess counted out loud by twenties (**fig. 2b**). Their recording not only helped them keep track of their collection but also clearly represents their strategy.

For children who have had ample practice counting by ones, we find that packages of supplies from the supply closet make excellent collections for counting by groups. We challenge the children to count without opening the shrink-wrapped packages. Paper clips come in boxes of 100. If ten boxes are shrink-wrapped together, how many paper clips are in the whole package? How many paper clips do we have if we include the half-used box in the classroom? Resealable plastic bags come 125 to a box, so how many bags are in a case of ten boxes? Watercolor paints have eight pans in a box. How can we add those up without peeking into the boxes? These challenges push children beyond counting by ones. No longer able to touch each object or even decide on how to group them, our older children begin thinking in multiples and invent impressive strategies for counting their collections.

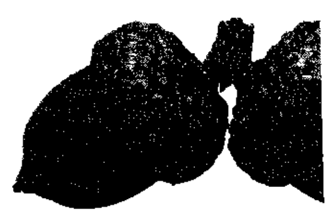
Figure 1

James and Theo used tally marks to keep track of the collection they counted.



Noticing What Children Do and Helping Them Notice What Others Are Doing

Counting collections early in the school year serves as an important assessment for us: We get to know the children as we watch them count. We find out who can make one-to-one correspondence and count consistently, who can "count on," who remembers what comes after 59, and who can



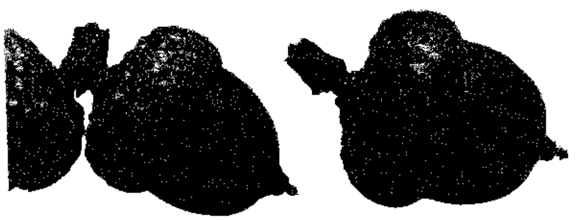
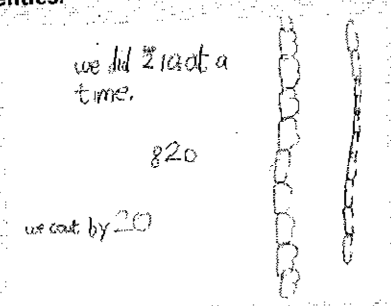
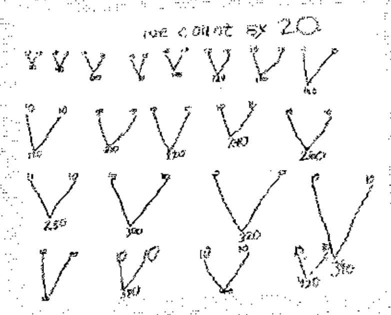


Figure 2

Tess and Ashley made chains of links to facilitate counting by tens and then by twenties.



a. Tess and Ashley's record of how they counted chains of ten links each



b. Their recording of moving the chains two at a time from the table to the floor

record the numbers they can count to. We watch to see if anyone is counting by twos or grouping objects into tens.

Always, when we let the children develop their own strategies that make sense to them, we are surprised and delighted by their thinking. We name each child's counting strategy and encourage the other children to give the various strategies a try. The "Jackson way" might be to move objects to one side to help keep track of what has been counted and what has not. Natalie's idea might be to use Dixie cups to group beads; she puts five beads in a cup and then counts up by fives. Jimmy might line up his collection of wooden people into "armies of ten." Once he has covered a table with little armies, he might notice that some of the armies are larger than others. He then makes adjustments as necessary so that each grouping has five rows of two and then counts up his groups by tens. Alex might use a sticky note to write down how many blocks she counted today so that she does not have to count them over tomorrow. She will come back to her tray labeled "not cownted" and count on from where she left off (**fig. 3**). All these strategies, coming straight from the children, become topics of group conversation as we gather on the carpet at the end of mathematics time.

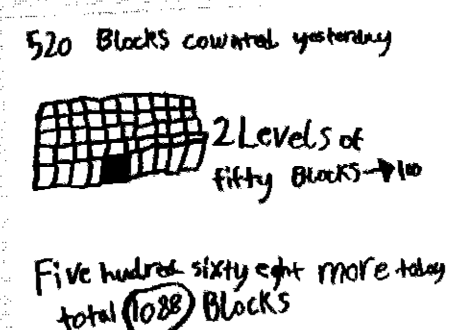
By the end of each hour-long session spent counting collections, some students have counted more than one collection and have multiple recording sheets, while others are still in the middle of their first counting project. We wrap up the session by gathering on the carpet and sharing the children's counting strategies as a group. We begin the next counting collections session by revisiting these strategies before we send the students off to continue counting their current collections or begin counting new ones.

Something for Everyone: Multiple Entry Points

At any given time, the students in any UES classroom are at different points in their development of counting skills and number sense. In our multiage classrooms, where students from two different grade levels (for example, second and third graders) are combined, teachers are continually challenged to create curriculum with multiple entry points. Perhaps the most convincing evidence for teachers

Figure 3

Alex's strategy for noting which blocks she has counted and which remain to be counted



that counting collections is valuable in our multiage classrooms is the ease with which we can engage and challenge children at different points in their development. By getting to know the children as we watch them count, we can figure out where they are on the trajectory of their counting development and stretch them forward.

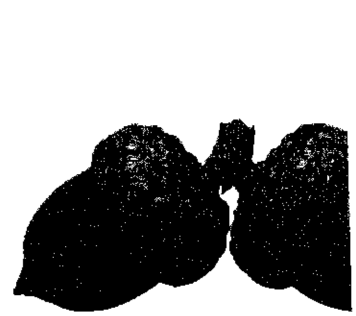
As we walk around the room, we appreciate how many different ways the children are approaching the job at hand. Leo, who is just beginning to make one-to-one correspondence between the objects he touches and the counting words he says, has brought a collection of pebbles from home. He loses track of which pebbles he has counted and which he has not. We stop to talk about how he could help himself keep track. Instead of just moving the pebbles to the side as he counts them, he puts a bowl in his lap; then he pushes the pebbles to the edge of his table and counts them as they drop into the bowl. At the end of mathematics time, he will ask for help writing "56" in his mathematics journal and will draw a picture labeled "pbls" to show what he counted.

In the same classroom on the same day, and showing the same seriousness of purpose, Grace groups teddy bear counters into families of five while her partner, Lindsey, points to two families at a time to help count up by tens. At another table Spencer and his friends are making "one hundred" buildings out of one-inch cubes. They have agreed that each building will have two floors of fifty blocks each. The trouble comes when seven full buildings have been completed but two buildings are incomplete. The boys are puzzling over how to combine their numbers. Yet another group is figuring out how to count all the legs on the chairs in the classroom, a problem that requires an organizational strategy (the chairs are occupied and so cannot be moved) and poses the challenge of counting by groups of four.

Another way to increase the complexity of the counting task is to have children add collections or parts of collections together. For example, we often count how many books are in our classrooms, assigning small teams of children to various shelves and baskets. Of course, we want to keep the books in their places, so rather than bringing all the books to a central location and stacking them into piles of tens, for example, each shelf or basket ends up with a label indicating the number of books on that shelf or in that basket. As a group, we figure out how to add those numbers together to get our total.

Working Together and Pushing One Another's Mathematical Ideas

We are encouraged by the range of challenges that can so easily be created to meet the wide range of needs of the children in our classrooms. We also find compelling evidence that the children learn a great deal from one another's strategies. Leo and Carlos have the job of counting markers from the supply closet. Some of the markers come in boxes of ten and some in boxes of eight. Leo, who is just learning to read numbers, begins by separating boxes with an "8" on the cover from boxes with a "10" on the cover. As Leo makes a stack of the



boxes of tens, Carlos counts out loud by ten. When they get to the boxes of eight markers, Carlos hesitates. He spreads the boxes of eight out on the floor and stops to think. Meanwhile, he sees Leo lining up the loose markers, thinking about how he could count them. Carlos asks for some of the loose markers and places two markers next to each box of eight to make tens. Leo, by his side, counts the remaining loose markers into a basket. In the end, Leo has counted up to 28 and, with a teacher's help, has written that number on his paper so that he does not forget it. Carlos has reached a total of 160. Although Carlos quickly says the total is 188, the teacher asks Leo to help Carlos double-check to see if he is right. Leo sits by Carlos's side and takes one pen at a time out of the basket, hearing Carlos count by ones from 160 to 188 and joining in when he can.

What do we think each child got out of this partnership and this counting project? Leo took on a counting project bigger than one he could have handled by himself. He watched Carlos group pens in ways that he himself was not ready to do and heard Carlos count out loud several times. Leo had opportunities to read numbers (the "8"s and "10"s on the boxes of markers); to practice a newly acquired strategy for keeping track (moving markers into a basket as he counted them, just as he had moved pebbles into a bowl on a previous day); to record his total (28); and to count out loud (from 1 to 28 and from 160 to 188). And what did Carlos gain from the partnership? Perhaps the idea of adding the two pens to the boxes of eights came from watching Leo organize the loose markers. Leo also provided a reason for Carlos to count orally by ones from 160 to 188, a valuable opportunity he would otherwise have passed up. And Carlos, a whiz at counting but not necessarily at sharing his methods, took on a valuable mentoring role. Together, the boys accomplished a counting job in a way that they could not have on their own.

The evidence is abundant that children benefit from working together. The classroom is full of conversation and problem solving. Not everyone agrees on counting strategies. So what happens if William dumps out boxes of paper clips on the carpet while Cody starts counting the closed boxes by 100s? Partners have lots of negotiating to do as they decide on a collection, choose a place to work, and figure out a strategy. Will one person count while the other records? Will both children count all the tiles into cups of tens and then count up by tens together? Will the process work if the partners start counting from opposite sides of the pile until they meet in the middle? Opportunities to learn to work together are as abundant as opportunities to develop number sense and counting skills.

Notice, Question, Extend

So what is our job as adults as we watch children count collections and as we learn about the remarkable things children can already do? How do we as teachers help children move forward? We notice and name the strategies children are using; we ask questions that provoke children to think, articulate their thinking, and sometimes try a new strategy; and we extend their thinking.

We recently taped two hour-long mathematics sessions in a K–1 classroom in which children were counting collections. Here are some of the questions we heard the teachers ask:

"How do you know which ones you have counted and which ones you have not?"

"What were you doing yesterday to keep track?"

"Why did you switch strategies today?"

"Why did you decide to put those into cups of fifteens?"

"How many cups did it take to get up to 150?"

"If you made another set of 10 cups, would there be 150 in there too?"

"Can you explain to Cody what you are doing now?"

"Is it easier to count by tens than by eights?"

"What are you going to do with all those loose ones?"

"What will you do when you run out of counting cups?"

"By looking at Natalie's recording, what can you tell about how she counted?"

"It looks like the two of you are using different strategies. Do you have a plan for how you will add your totals together?"

"Why did that turn out to be a tricky collection to count?"

"How are the two of you working together?"

"What will you do differently next time?"

In addition to asking questions, our job as teachers is also to keep track of what the children can do. As we walk through the room, we jot down our observations on sticky notes, both to document what the children can do at that time of year and to remind ourselves of strategies worth talking about later as we all sit together on the carpet. Patterns emerge, and these are helpful for planning instruction. Several young children may say more than one counting word per object, and so we group some of these children together to practice touching and counting or dropping one item into each cup of an egg carton. Teachers interact with so many children in a given mathematics hour and over the course of a day that these little records we keep as we run become important at the end of the day to help us figure out what is going well and what the children need next.

Keeping the Counting Going

All of this negotiating, organizing, and counting takes time. Early in the year, for a period of about three weeks, we are likely to devote our full mathematics hour every day to counting collections. Teachers convinced that counting collections enhances their students' growth in number sense and counting strategies often revisit counting collections more than once during the year. They find that what the children are able to do in the spring is far different from what they could do in the fall, and this measure of accomplishment and growth is appreciated by both children and teachers.

To keep the counting going throughout the year, we make sure that counting collections is part of our weekly work, even when we are delving into other mathematics units. We send a large plastic jar home with a note requesting that a family provide some objects for us to count, and about once each week, for the first few minutes of our mathematics hour, we count out these objects together as a class. We save the objects counted—75 cotton balls, 28 pencils, 125 pasta shells, and so forth—and put them in a "number gallery," all labeled with their quantity.

And, of course, we begin each mathematics hour by counting orally—by ones starting at 85, by tens starting at 63, backwards from 50. But that is the subject of another article.

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Counting Collections

Names:______________________________

We counted _______________________________________

We counted __________________________ items altogether

This is how we counted:

We counted our items by:

____

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____

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Created Based on work by Lakeridge Elementary School together with the University of Washington 2012

Counting Collection Observations

Student Name	Counting Observations	Questions to Ask

(Created by Danielle Robinson, Milwaukee Public Schools)