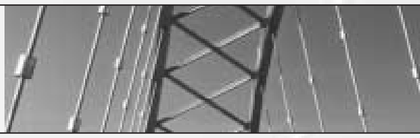


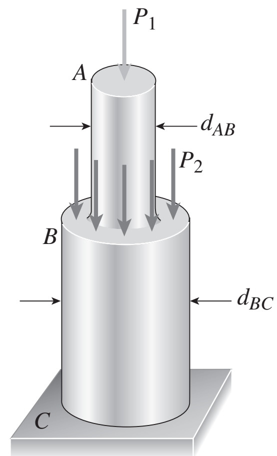
1

Tension, Compression, and Shear

Normal Stress and Strain

Problem 1.2-1 A solid circular post *ABC* (see figure) supports a load $P_1 = 2500$ lb acting at the top. A second load P_2 is uniformly distributed around the shelf at *B*. The diameters of the upper and lower parts of the post are $d_{AB} = 1.25$ in. and $d_{BC} = 2.25$ in., respectively.

(a) Calculate the normal stress σ_{AB} in the upper part of the post.
(b) If it is desired that the lower part of the post have the same compressive stress as the upper part, what should be the magnitude of the load P_2?



Solution 1.2-1 **Circular post in compression**

$P_1 = 2500$ lb

$d_{AB} = 1.25$ in.

$d_{BC} = 2.25$ in.

(a) Normal stress in part *AB*

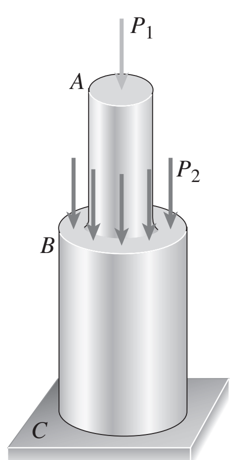
$$\sigma_{AB} = \frac{P_1}{A_{AB}} = \frac{2500 \text{ lb}}{\frac{\pi}{4}(1.25 \text{ in.})^2} = 2040 \text{ psi} \quad \longleftarrow$$

(b) Load P_2 for equal stresses

$$\sigma_{BC} = \frac{P_1 + P_2}{A_{BC}} = \frac{2500 \text{ lb} + P_2}{\frac{\pi}{4}(2.25 \text{ in.})^2}$$

$$= \sigma_{AB} = 2040 \text{ psi}$$

Solve for P_2: $P_2 = 5600$ lb $\longleftarrow$



Alternate Solution for Part (b)

$$\sigma_{BC} = \frac{P_1 + P_2}{A_{BC}} = \frac{P_1 + P_2}{\frac{\pi}{4} d_{BC}^2}$$

$$\sigma_{AB} = \frac{P_1}{A_{AB}} = \frac{P_1}{\frac{\pi}{4} d_{AB}^2} \qquad \sigma_{BC} = \sigma_{AB}$$

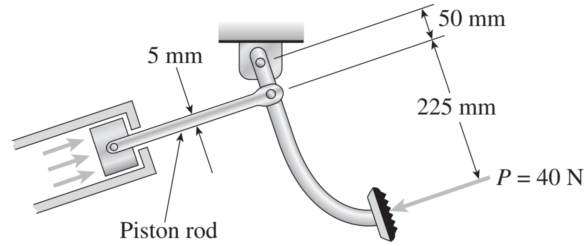
$$\frac{P_1 + P_2}{d_{BC}^2} = \frac{P_1}{d_{AB}^2} \quad \text{or} \quad P_2 = P_1\left[\left(\frac{d_{BC}}{d_{AB}}\right)^2 - 1\right]$$

$$\frac{d_{BC}}{d_{AB}} = 1.8$$

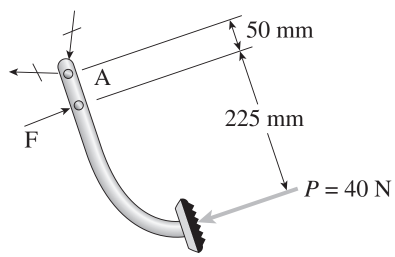
$$\therefore \; P_2 = 2.24 P_1 = 5600 \text{ lb} \quad \longleftarrow$$

Problem 1.2-2 Calculate the compressive stress σ_c in the circular piston rod (see figure) when a force $P = 40$ N is applied to the brake pedal.

Assume that the line of action of the force P is parallel to the piston rod, which has diameter 5 mm. Also, the other dimensions shown in the figure (50 mm and 225 mm) are measured perpendicular to the line of action of the force P.



Solution 1.2-2 **Free-body diagram of brake pedal**



F = compressive force in piston rod

d = diameter of piston rod

= 5 mm

EQUILIBRIUM OF BRAKE PEDAL

$\Sigma M_A = 0$ ↺+ ↻

$$F(50 \text{ mm}) - P(275 \text{ mm}) = 0$$

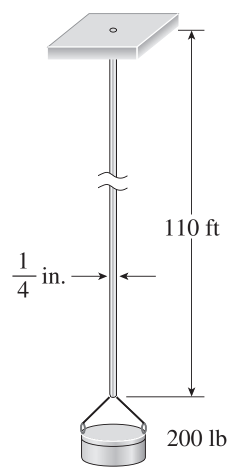
$$F = P\left(\frac{275 \text{ mm}}{50 \text{ mm}}\right) = (40 \text{ N})\left(\frac{275}{50}\right) = 220 \text{ N}$$

COMPRESSIVE STRESS IN PISTON ROD ($d = 5$ mm)

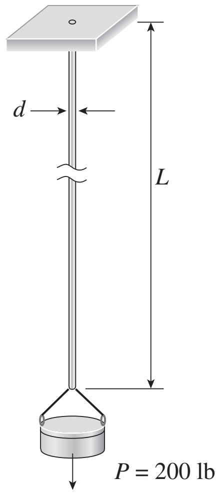
$$\sigma_c = \frac{F}{A} = \frac{220 \text{ N}}{\frac{\pi}{4}(5 \text{ mm})^2} = 11.2 \text{ MPa} \quad \longleftarrow$$

Problem 1.2-3 A steel rod 110 ft long hangs inside a tall tower and holds a 200-pound weight at its lower end (see figure).

If the diameter of the circular rod is ¼ inch, calculate the maximum normal stress σ_{max} in the rod, taking into account the weight of the rod itself. (Obtain the weight density of steel from Table H-1, Appendix H.)



Solution 1.2-3 **Long steel rod in tension**



$P = 200$ lb

$L = 110$ ft

$d = \frac{1}{4}$ in.

Weight density: $\gamma = 490$ lb/ft^3

$W =$ Weight of rod

$= \gamma(\text{Volume})$

$= \gamma AL$

$$\sigma_{\max} = \frac{W + P}{A} = \gamma L + \frac{P}{A}$$

$$\gamma L = (490 \text{ lb/ft}^3)(110 \text{ ft})\left(\frac{1}{144}\frac{\text{ft}^2}{\text{in.}^2}\right)$$

$$= 374.3 \text{ psi}$$

$$\frac{P}{A} = \frac{200 \text{ lb}}{\frac{\pi}{4}(0.25 \text{ in.})^2} = 4074 \text{ psi}$$

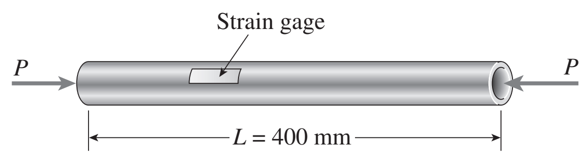
$$\sigma_{\max} = 374 \text{ psi} + 4074 \text{ psi} = 4448 \text{ psi}$$

Rounding, we get

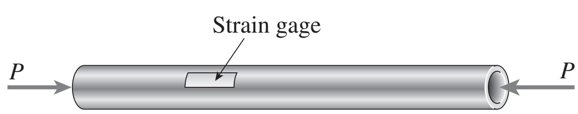
$$\sigma_{\max} = 4450 \text{ psi} \quad \longleftarrow$$

Problem 1.2-4 A circular aluminum tube of length $L = 400$ mm is loaded in compression by forces P (see figure). The outside and inside diameters are 60 mm and 50 mm, respectively. A strain gage is placed on the outside of the bar to measure normal strains in the longitudinal direction.

(a) If the measured strain is $\epsilon = 550 \times 10^{-6}$, what is the shortening δ of the bar?
(b) If the compressive stress in the bar is intended to be 40 MPa, what should be the load P?



Solution 1.2-4 **Aluminum tube in compression**



$\varepsilon = 550 \times 10^{-6}$

$L = 400$ mm

$d_2 = 60$ mm

$d_1 = 50$ mm

(a) SHORTENING δ OF THE BAR

$$\delta = \varepsilon L = (550 \times 10^{-6})(400 \text{ mm})$$

$$= 0.220 \text{ mm} \quad \longleftarrow$$

(b) COMPRESSIVE LOAD P

$\sigma = 40$ MPa

$$A = \frac{\pi}{4}[d_2^2 - d_1^2] = \frac{\pi}{4}[(60 \text{ mm})^2 - (50 \text{ mm})^2]$$

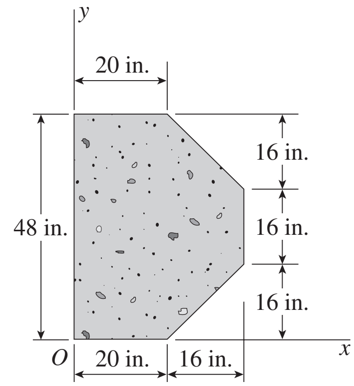
$$= 863.9 \text{ mm}^2$$

$$P = \sigma A = (40 \text{ MPa})(863.9 \text{ mm}^2)$$

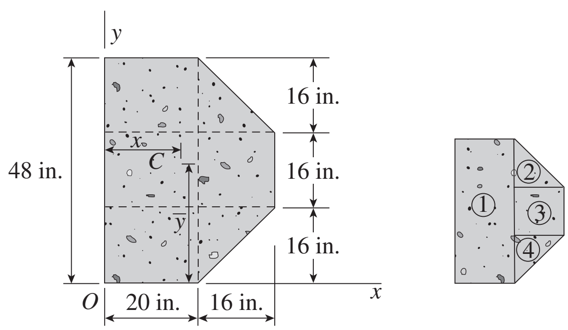
$$= 34.6 \text{ kN} \quad \longleftarrow$$

Problem 1.2-5 The cross section of a concrete pier that is loaded uniformly in compression is shown in the figure.

(a) Determine the average compressive stress σ_c in the concrete if the load is equal to 2500 k.
(b) Determine the coordinates $\bar{x}$ and $\bar{y}$ of the point where the resultant load must act in order to produce uniform normal stress.



Solution 1.2-5 **Concrete pier in compression**



USE THE FOLLOWING AREAS:

$$A_1 = (48\text{ in.})(20\text{ in.}) = 960\text{ in.}^2$$

$$A_2 = A_4 = \frac{1}{2}(16\text{ in.})(16\text{ in.}) = 128\text{ in.}^2$$

$$A_3 = (16\text{ in.})(16\text{ in.}) = 256\text{ in.}^2$$

$$\begin{aligned} A &= A_1 + A_2 + A_3 + A_4 \\ &= (960 + 128 + 256 + 128)\text{ in.}^2 \\ &= 1472\text{ in.}^2 \end{aligned}$$

(a) AVERAGE COMPRESSIVE STRESS σ_c

$P = 2500\text{ k}$

$$\sigma_c = \frac{P}{A} = \frac{2500\text{ k}}{1472\text{ in.}^2} = 1.70\text{ ksi} \quad \longleftarrow$$

(b) COORDINATES OF CENTROID C

From symmetry, $\bar{y} = \frac{1}{2}(48\text{ in.}) = 24\text{ in.} \quad \longleftarrow$

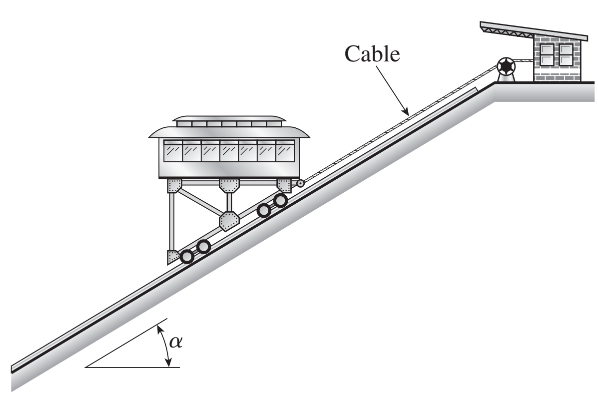
$$\bar{x} = \frac{\Sigma\, \bar{x}_i A_i}{A} \quad \text{(see Chapter 12, Eq. 12-7a)} \quad \longleftarrow$$

$$\bar{x} = \frac{1}{A}(\bar{x}_1 A_1 + 2\bar{x}_2 A_2 + \bar{x}_3 A_3)$$

$$\begin{aligned} &= \frac{1}{1472\text{ in.}^2}[(10\text{ in.})(960\text{ in.}^2) \\ &\quad + 2(25.333\text{ in.})(128\text{ in.}^2) \\ &\quad + (28\text{ in.})(256\text{ in.}^2)] \\ &= 15.8\text{ in.} \quad \longleftarrow \end{aligned}$$

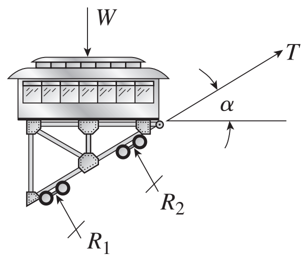
Problem 1.2-6 A car weighing 130 kN when fully loaded is pulled slowly up a steep inclined track by a steel cable (see figure). The cable has an effective cross-sectional area of 490 mm^2, and the angle α of the incline is 30°.

Calculate the tensile stress σ_t in the cable.



Solution 1.2-6 Car on inclined track

FREE-BODY DIAGRAM OF CAR



W = Weight of car

T = Tensile force in cable

α = Angle of incline

A = Effective area of cable

R_1, R_2 = Wheel reactions (no friction force between wheels and rails)

EQUILIBRIUM IN THE INCLINED DIRECTION

$\Sigma F_T = 0 \quad \nearrow^{+} \swarrow^{-} \quad T - W \sin \alpha = 0$

$T = W \sin \alpha$

TENSILE STRESS IN THE CABLE

$$\sigma_t = \frac{T}{A} = \frac{W \sin \alpha}{A}$$

SUBSTITUTE NUMERICAL VALUES:

$W = 130 \text{ kN} \quad \alpha = 30°$

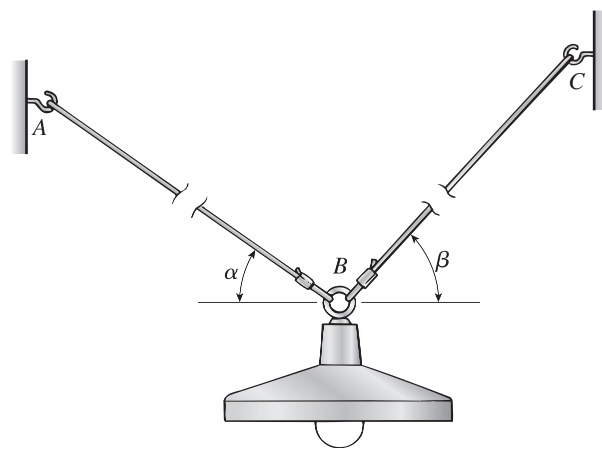
$A = 490 \text{ mm}^2$

$$\sigma_t = \frac{(130 \text{ kN})(\sin 30°)}{490 \text{ mm}^2}$$

$$= 133 \text{ MPa} \longleftarrow$$

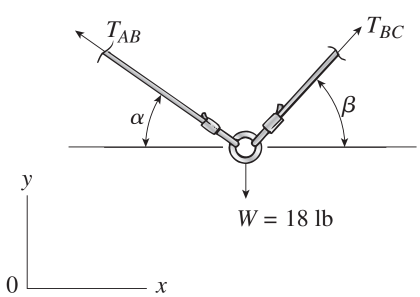
Problem 1.2-7 Two steel wires, AB and BC, support a lamp weighing 18 lb (see figure). Wire AB is at an angle $\alpha = 34°$ to the horizontal and wire BC is at an angle $\beta = 48°$. Both wires have diameter 30 mils. (Wire diameters are often expressed in mils; one mil equals 0.001 in.)

Determine the tensile stresses σ_{AB} and σ_{BC} in the two wires.



Solution 1.2-7 Two steel wires supporting a lamp

FREE-BODY DIAGRAM OF POINT B



$\alpha = 34° \qquad \beta = 48°$

$d = 30 \text{ mils} = 0.030 \text{ in.}$

$$A = \frac{\pi d^2}{4} = 706.9 \times 10^{-6} \text{ in.}^2$$

EQUATIONS OF EQUILIBRIUM

$\Sigma F_x = 0 \quad -T_{AB} \cos \alpha + T_{BC} \cos \beta = 0$

$\Sigma F_y = 0 \quad T_{AB} \sin \alpha + T_{BC} \sin \beta - W = 0$

SUBSTITUTE NUMERICAL VALUES:

$-T_{AB}(0.82904) + T_{BC}(0.66913) = 0$

$T_{AB}(0.55919) + T_{BC}(0.74314) - 18 = 0$

SOLVE THE EQUATIONS:

$T_{AB} = 12.163 \text{ lb} \quad T_{BC} = 15.069 \text{ lb}$

TENSILE STRESSES IN THE WIRES

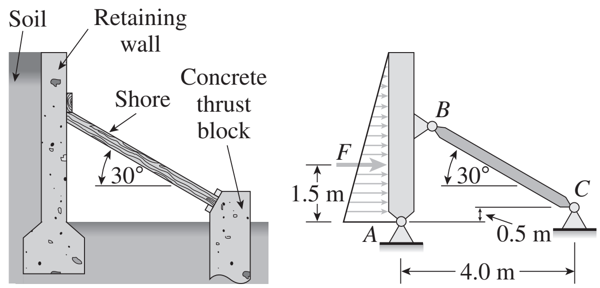
$$\sigma_{AB} = \frac{T_{AB}}{A} = 17{,}200 \text{ psi} \longleftarrow$$

$$\sigma_{BC} = \frac{T_{BC}}{A} = 21{,}300 \text{ psi} \longleftarrow$$

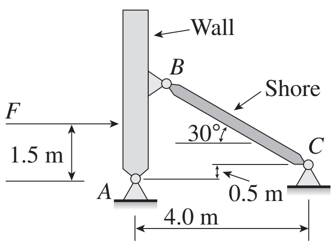
Problem 1.2-8 A long retaining wall is braced by wood shores set at an angle of 30° and supported by concrete thrust blocks, as shown in the first part of the figure. The shores are evenly spaced, 3 m apart.

For analysis purposes, the wall and shores are idealized as shown in the second part of the figure. Note that the base of the wall and both ends of the shores are assumed to be pinned. The pressure of the soil against the wall is assumed to be triangularly distributed, and the resultant force acting on a 3-meter length of the wall is $F = 190$ kN.

If each shore has a 150 mm × 150 mm square cross section, what is the compressive stress σ_c in the shores?



Solution 1.2-8 Retaining wall braced by wood shores



$F = 190$ kN

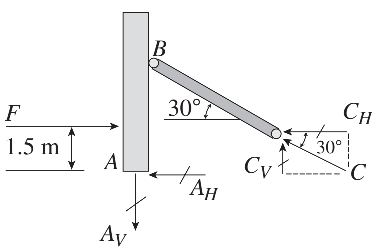
$A =$ area of one shore

$A = (150 \text{ mm})(150 \text{ mm})$

$= 22{,}500 \text{ mm}^2$

$= 0.0225 \text{ m}^2$

FREE-BODY DIAGRAM OF WALL AND SHORE



$C =$ compressive force in wood shore

$C_H =$ horizontal component of C

$C_V =$ vertical component of C

$C_H = C \cos 30°$

$C_V = C \sin 30°$

SUMMATION OF MOMENTS ABOUT POINT A

$\Sigma M_A = 0$ ↺+ ↻−

$-F(1.5 \text{ m}) + C_V(4.0 \text{ m}) + C_H(0.5 \text{ m}) = 0$

or

$-(190 \text{ kN})(1.5 \text{ m}) + C(\sin 30°)(4.0 \text{ m}) + C(\cos 30°)(0.5 \text{ m}) = 0$

$\therefore C = 117.14$ kN

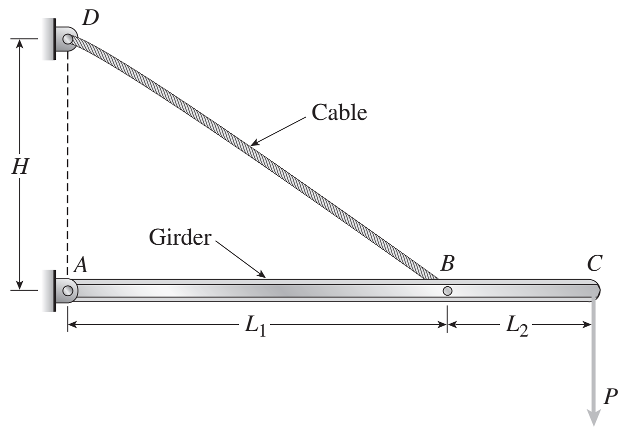
COMPRESSIVE STRESS IN THE SHORES

$$\sigma_c = \frac{C}{A} = \frac{117.14 \text{ kN}}{0.0225 \text{ m}^2}$$

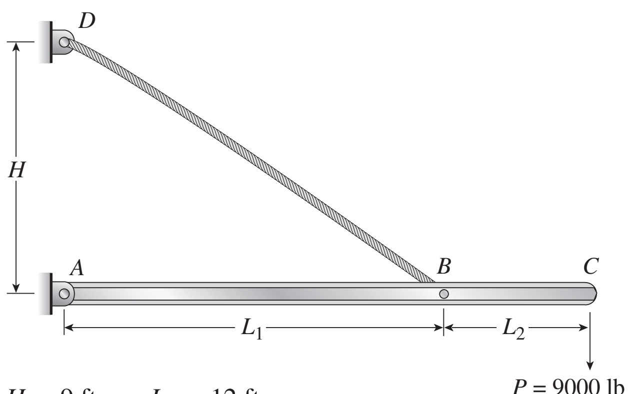
$$= 5.21 \text{ MPa} \quad \longleftarrow$$

Problem 1.2-9 A loading crane consisting of a steel girder *ABC* supported by a cable *BD* is subjected to a load *P* (see figure). The cable has an effective cross-sectional area $A = 0.471$ in^2. The dimensions of the crane are $H = 9$ ft, $L_1 = 12$ ft, and $L_2 = 4$ ft.

(a) If the load $P = 9000$ lb, what is the average tensile stress in the cable?
(b) If the cable stretches by 0.382 in., what is the average strain?



Solution 1.2-9 Loading crane with girder and cable



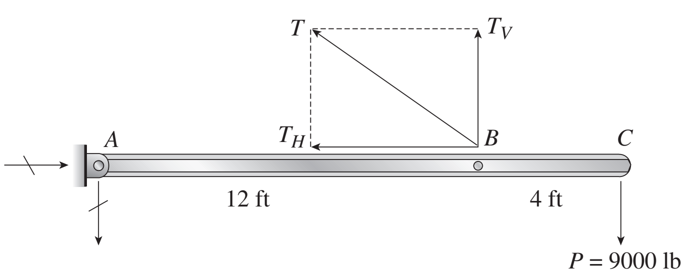
$H = 9$ ft $\quad L_1 = 12$ ft

$L_2 = 4$ ft $\quad A =$ effective area of cable

$A = 0.471$ in.^{2}

$P = 9000$ lb

FREE-BODY DIAGRAM OF GIRDER



$T =$ tensile force in cable

$P = 9000$ lb

EQUILIBRIUM

$$\Sigma M_A = 0$$

$$T_V\,(12\text{ ft}) - (9000\text{ lb})(16\text{ ft}) = 0$$

$$T_V = 12{,}000\text{ lb}$$

$$\frac{T_H}{T_V} = \frac{L_1}{H} = \frac{12\text{ ft}}{9\text{ ft}}$$

$$\therefore\ T_H = T_V\left(\frac{12}{9}\right)$$

$$T_H = (12{,}000\text{ lb})\left(\frac{12}{9}\right)$$

$$= 16{,}000\text{ lb}$$

TENSILE FORCE IN CABLE

$$T = \sqrt{T_H^2 + T_V^2} = \sqrt{(16{,}000\text{ lb})^2 + (12{,}000\text{ lb})^2}$$

$$= 20{,}000\text{ lb}$$

(a) AVERAGE TENSILE STRESS IN CABLE

$$\sigma = \frac{T}{A} = \frac{20{,}000\text{ lb}}{0.471\text{ in.}^2} = 42{,}500\text{ psi} \longleftarrow$$

(b) AVERAGE STRAIN IN CABLE

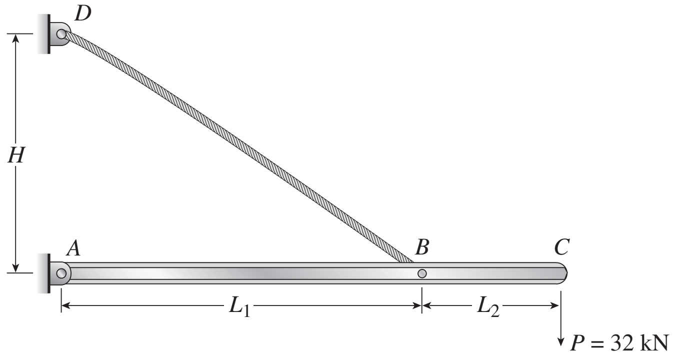
$L =$ length of cable $\quad L = \sqrt{H^2 + L_1^2} = 15$ ft

$\delta =$ stretch of cable $\quad \delta = 0.382$ in.

$$\varepsilon = \frac{\delta}{L} = \frac{0.382\text{ in.}}{(15\text{ ft})(12\text{ in./ft})} = 2120 \times 10^{-6} \longleftarrow$$

Problem 1.2-10 Solve the preceding problem if the load $P = 32$ kN; the cable has effective cross-sectional area $A = 481$ mm^2; the dimensions of the crane are $H = 1.6$ m, $L_1 = 3.0$ m, and $L_2 = 1.5$ m; and the cable stretches by 5.1 mm. Figure is with Prob. 1.2-9.

Solution 1.2-10 **Loading crane with girder and cable**

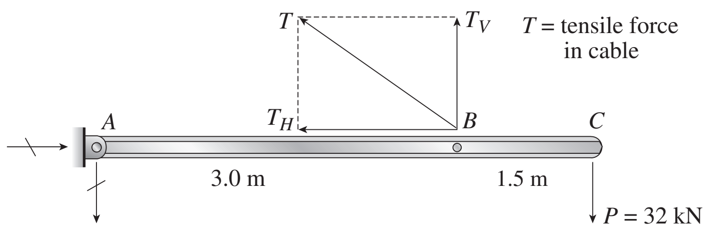


$H = 1.6$ m $\qquad L_1 = 3.0$ m

$L_2 = 1.5$ m $\qquad A =$ effective area of cable

$A = 481$ mm^2 $\qquad P = 32$ kN

FREE-BODY DIAGRAM OF GIRDER



EQUILIBRIUM

$\Sigma M_A = 0$

$T_V (3.0 \text{ m}) - (32 \text{ kN})(4.5 \text{ m}) = 0$

$T_V = 48$ kN

$$\frac{T_H}{T_V} = \frac{L_1}{H} = \frac{3.0 \text{ m}}{1.6 \text{ m}}$$

$$\therefore\ T_H = T_V\left(\frac{3.0}{1.6}\right)$$

$$T_H = (48 \text{ kN})\left(\frac{3.0}{1.6}\right)$$

$= 90$ kN

TENSILE FORCE IN CABLE

$$T = \sqrt{T_H^2 + T_V^2} = \sqrt{(90 \text{ kN})^2 + (48 \text{ kN})^2}$$

$= 102$ kN

(a) AVERAGE TENSILE STRESS IN CABLE

$$\sigma = \frac{T}{A} = \frac{102 \text{ kN}}{481 \text{ mm}^2} = 212 \text{ MPa} \longleftarrow$$

(b) AVERAGE STRAIN IN CABLE

$L =$ length of cable

$L = \sqrt{H^2 + L_1^2} = 3.4$ m

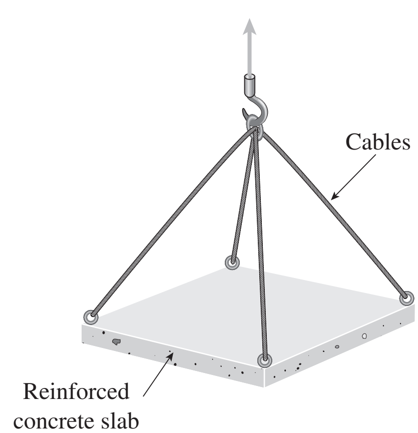
$\delta =$ stretch of cable

$\delta = 5.1$ mm

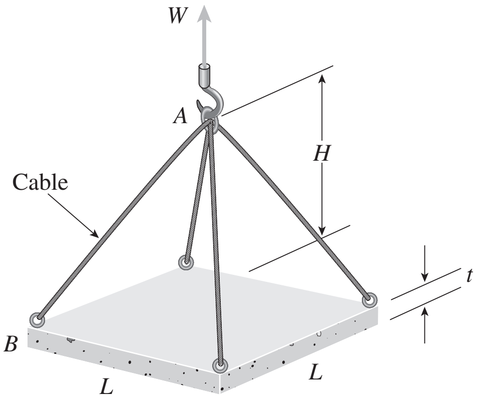
$$\varepsilon = \frac{\delta}{L} = \frac{5.1 \text{ mm}}{3.4 \text{ m}} = 1500 \times 10^{-6} \longleftarrow$$

Problem 1.2-11 A reinforced concrete slab 8.0 ft square and 9.0 in. thick is lifted by four cables attached to the corners, as shown in the figure. The cables are attached to a hook at a point 5.0 ft above the top of the slab. Each cable has an effective cross-sectional area $A = 0.12$ in^2.

Determine the tensile stress σ_t in the cables due to the weight of the concrete slab. (See Table H-1, Appendix H, for the weight density of reinforced concrete.)



Solution 1.2-11 Reinforced concrete slab supported by four cables



H = height of hook above slab

L = length of side of square slab

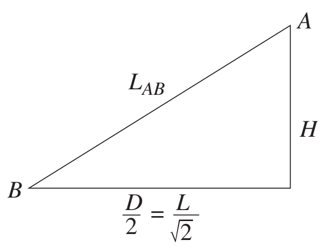
t = thickness of slab

γ = weight density of reinforced concrete

W = weight of slab = $\gamma L^2 t$

D = length of diagonal of slab = $L\sqrt{2}$

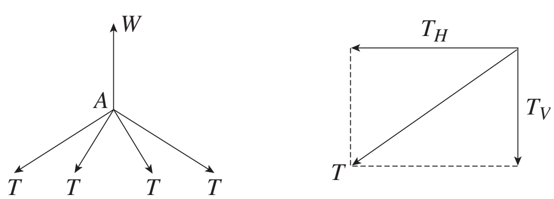
DIMENSIONS OF CABLE AB



L_{AB} = length of cable

$$= \sqrt{H^2 + \frac{L^2}{2}}$$

FREE-BODY DIAGRAM OF HOOK AT POINT A



T = tensile force in a cable

Cable AB:

$$\frac{T_V}{T} = \frac{H}{L_{AB}}$$

$$T_V = T\left(\frac{H}{\sqrt{H^2 + L^2/2}}\right) \qquad \text{(Eq. 1)}$$

EQUILIBRIUM

$$\Sigma F_{\text{vert}} = 0 \ \uparrow_+ \downarrow^-$$

$$W - 4T_V = 0$$

$$T_V = \frac{W}{4} \qquad \text{(Eq. 2)}$$

COMBINE EQS. (1) & (2):

$$T\left(\frac{H}{\sqrt{H^2 + L^2/2}}\right) = \frac{W}{4}$$

$$T = \frac{W}{4}\frac{\sqrt{H^2 + L^2/2}}{H} = \frac{W}{4}\sqrt{1 + L^2/2H^2}$$

TENSILE STRESS IN A CABLE

A = effective cross-sectional area of a cable

$$\sigma_t = \frac{T}{A} = \frac{W}{4A}\sqrt{1 + L^2/2H^2} \quad \longleftarrow$$

SUBSTITUTE NUMERICAL VALUES AND OBTAIN σ_t:

$H = 5.0$ ft $\quad L = 8.0$ ft $\quad t = 9.0$ in. $= 0.75$ ft

$\gamma = 150$ lb/ft^3 $\quad A = 0.12$ in.^{2}

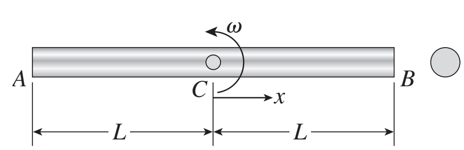
$W = \gamma L^2 t = 7200$ lb

$$\sigma_t = \frac{W}{4A}\sqrt{1 + L^2/2H^2} = 22{,}600 \text{ psi} \quad \longleftarrow$$

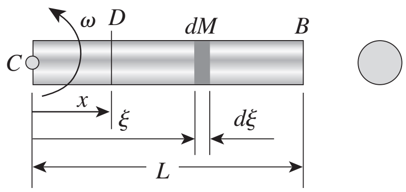
Problem 1.2-12 A round bar ACB of length $2L$ (see figure) rotates about an axis through the midpoint C with constant angular speed ω (radians per second). The material of the bar has weight density γ.

(a) Derive a formula for the tensile stress σ_x in the bar as a function of the distance x from the midpoint C.

(b) What is the maximum tensile stress σ_{max}?



Solution 1.2-12 **Rotating Bar**



ω = angular speed (rad/s)

A = cross-sectional area

γ = weight density

$\frac{\gamma}{g}$ = mass density

We wish to find the axial force F_x in the bar at Section D, distance x from the midpoint C.

The force F_x equals the inertia force of the part of the rotating bar from D to B.

Consider an element of mass dM at distance ξ from the midpoint C. The variable ξ ranges from x to L.

$$dM = \frac{\gamma}{g} A\, d\xi$$

dF = Inertia force (centrifugal force) of element of mass dM

$$dF = (dM)(\xi\omega^2) = \frac{\gamma}{g} A\omega^2 \xi d\xi$$

$$F_x = \int_D^B dF = \int_x^L \frac{\gamma}{g} A\omega^2 \xi d\xi = \frac{\gamma A\omega^2}{2g}(L^2 - x^2)$$

(a) Tensile stress in bar at distance x

$$\sigma_x = \frac{F_x}{A} = \frac{\gamma\omega^2}{2g}(L^2 - x^2) \longleftarrow$$

(b) Maximum tensile stress

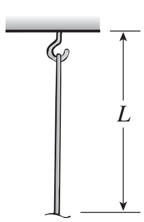
$$x = 0 \qquad \sigma_{max} = \frac{\gamma\omega^2 L^2}{2g} \longleftarrow$$

Mechanical Properties of Materials

Problem 1.3-1 Imagine that a long steel wire hangs vertically from a high-altitude balloon.

(a) What is the greatest length (feet) it can have without yielding if the steel yields at 40 ksi?
(b) If the same wire hangs from a ship at sea, what is the greatest length? (Obtain the weight densities of steel and sea water from Table H-1, Appendix H.)

Solution 1.3-1 **Hanging wire of length** L



W = total weight of steel wire

γ_S = weight density of steel

$= 490\ \text{lb/ft}^3$

γ_W = weight density of sea water

$= 63.8\ \text{lb/ft}^3$

A = cross-sectional area of wire

σ_{max} = 40 ksi (yield strength)

(a) Wire hanging in air

$$W = \gamma_S AL$$

$$\sigma_{max} = \frac{W}{A} = \gamma_S L$$

$$L_{max} = \frac{\sigma_{max}}{\gamma_S} = \frac{40{,}000\ \text{psi}}{490\ \text{lb/ft}^3}(144\ \text{in.}^2/\text{ft}^2)$$

$$= 11{,}800\ \text{ft} \longleftarrow$$

(b) Wire hanging in sea water

F = tensile force at top of wire

$$F = (\gamma_S - \gamma_W)AL \qquad \sigma_{max} = \frac{F}{A} = (\gamma_S - \gamma_W)L$$

$$L_{max} = \frac{\sigma_{max}}{\gamma_S - \gamma_W}$$

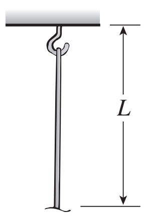
$$= \frac{40{,}000\ \text{psi}}{(490 - 63.8)\text{lb/ft}^3}(144\ \text{in.}^2/\text{ft}^2)$$

$$= 13{,}500\ \text{ft} \longleftarrow$$

Problem 1.3-2 Imagine that a long wire of tungsten hangs vertically from a high-altitude balloon.

(a) What is the greatest length (meters) it can have without breaking if the ultimate strength (or breaking strength) is 1500 MPa?
(b) If the same wire hangs from a ship at sea, what is the greatest length? (Obtain the weight densities of tungsten and sea water from Table H-1, Appendix H.)

Solution 1.3-2 Hanging wire of length L



W = total weight of tungsten wire

γ_T = weight density of tungsten = 190 kN/m^3

γ_W = weight density of sea water = 10.0 kN/m^3

A = cross-sectional area of wire

σ_{max} = 1500 MPa (breaking strength)

(a) Wire hanging in air

$$W = \gamma_T A L$$

$$\sigma_{max} = \frac{W}{A} = \gamma_T L$$

$$L_{max} = \frac{\sigma_{max}}{\gamma_T} = \frac{1500 \text{ MPa}}{190 \text{ kN/m}^3}$$

$$= 7900 \text{ m} \longleftarrow$$

(b) Wire hanging in sea water

F = tensile force at top of wire

$$F = (\gamma_T - \gamma_W)AL$$

$$\sigma_{max} = \frac{F}{A} = (\gamma_T - \gamma_W)L$$

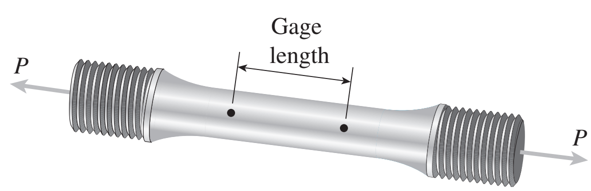
$$L_{max} = \frac{\sigma_{max}}{\gamma_T - \gamma_W}$$

$$= \frac{1500 \text{ MPa}}{(190 - 10.0) \text{ kN/m}^3}$$

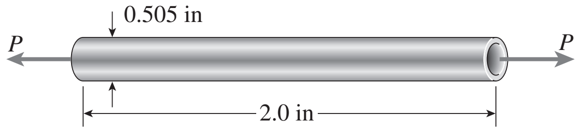
$$= 8300 \text{ m} \longleftarrow$$

Problem 1.3-3 Three different materials, designated *A, B,* and *C*, are tested in tension using test specimens having diameters of 0.505 in. and gage lengths of 2.0 in. (see figure). At failure, the distances between the gage marks are found to be 2.13, 2.48, and 2.78 in., respectively. Also, at the failure cross sections the diameters are found to be 0.484, 0.398, and 0.253 in., respectively.

Determine the percent elongation and percent reduction in area of each specimen, and then, using your own judgment, classify each material as brittle or ductile.



Solution 1.3-3 **Tensile tests of three materials**



$$\text{Percent elongation} = \frac{L_1 - L_0}{L_0}(100) = \left(\frac{L_1}{L_0} - 1\right)100$$

$L_0 = 2.0$ in.

$$\text{Percent elongation} = \left(\frac{L_1}{2.0} - 1\right)(100) \qquad \text{(Eq. 1)}$$

where L_1 is in inches.

$$\text{Percent reduction in area} = \frac{A_0 - A_1}{A_0}(100) = \left(1 - \frac{A_1}{A_0}\right)(100)$$

d_0 = initial diameter $\quad d_1$ = final diameter

$$\frac{A_1}{A_0} = \left(\frac{d_1}{d_0}\right)^2 \qquad d_0 = 0.505 \text{ in.}$$

Percent reduction in area

$$= \left[1 - \left(\frac{d_1}{0.505}\right)^2\right](100) \qquad \text{(Eq. 2)}$$

where d_1 is in inches.

Material	L_1 (in.)	d_1 (in.)	% Elongation (Eq. 1)	% Reduction (Eq. 2)	Brittle or Ductile?
A	2.13	0.484	6.5%	8.1%	Brittle
B	2.48	0.398	24.0%	37.9%	Ductile
C	2.78	0.253	39.0%	74.9%	Ductile

Problem 1.3-4 The *strength-to-weight ratio* of a structural material is defined as its load-carrying capacity divided by its weight. For materials in tension, we may use a characteristic tensile stress (as obtained from a stress-strain curve) as a measure of strength. For instance, either the yield stress or the ultimate stress could be used, depending upon the particular application. Thus, the strength-to-weight ratio $R_{S/W}$ for a material in tension is defined as

$$R_{S/W} = \frac{\sigma}{\gamma}$$

in which σ is the characteristic stress and γ is the weight density. Note that the ratio has units of length.

Using the ultimate stress σ_U as the strength parameter, calculate the strength-to-weight ratio (in units of meters) for each of the following materials: aluminum alloy 6061-T6, Douglas fir (in bending), nylon, structural steel ASTM-A572, and a titanium alloy. (Obtain the material properties from Tables H-1 and H-3 of Appendix H. When a range of values is given in a table, use the average value.)

Solution 1.3-4 **Strength-to-weight ratio**

The ultimate stress σ_U for each material is obtained from Table H-3, Appendix H, and the weight density γ is obtained from Table H-1.

The strength-to-weight ratio (meters) is

$$R_{S/W} = \frac{\sigma_U(\text{MPa})}{\gamma(\text{kN/m}^3)}(10^3)$$

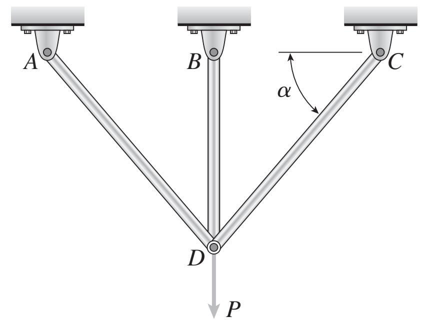
Values of σ_U, γ, and $R_{S/W}$ are listed in the table.

	σ_U (MPa)	γ (kN/m^3)	$R_{S/W}$ (m)
Aluminum alloy 6061-T6	310	26.0	11.9×10^3
Douglas fir	65	5.1	12.7×10^3
Nylon	60	9.8	6.1×10^3
Structural steel ASTM-A572	500	77.0	6.5×10^3
Titanium alloy	1050	44.0	23.9×10^3

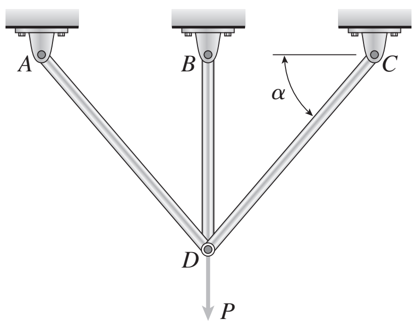
Titanium has a high strength-to-weight ratio, which is why it is used in space vehicles and high-performance airplanes. Aluminum is higher than steel, which makes it desirable for commercial aircraft. Some woods are also higher than steel, and nylon is about the same as steel.

Problem 1.3-5 A symmetrical framework consisting of three pin-connected bars is loaded by a force P (see figure). The angle between the inclined bars and the horizontal is $\alpha = 48°$. The axial strain in the middle bar is measured as 0.0713.

Determine the tensile stress in the outer bars if they are constructed of aluminum alloy having the stress-strain diagram shown in Fig. 1-13. (Express the stress in USCS units.)



Solution 1.3-5 **Symmetrical framework**

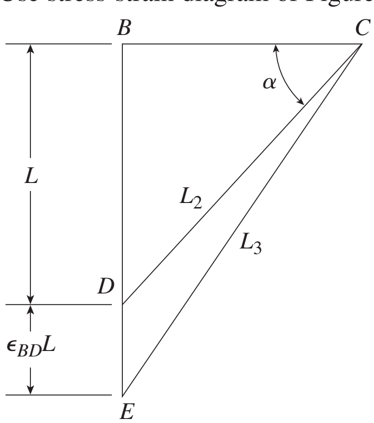


Aluminum alloy

$\alpha = 48°$

$\varepsilon_{BD} = 0.0713$

Use stress-strain diagram of Figure 1-13



L = length of bar BD

L_1 = distance BC

$= L \cot \alpha = L(\cot 48°) = 0.9004\,L$

L_2 = length of bar CD

$= L \csc \alpha = L(\csc 48°) = 1.3456\,L$

Elongation of bar BD = distance $DE = \varepsilon_{BD}L$

$\varepsilon_{BD}L = 0.0713\,L$

L_3 = distance CE

$$L_3 = \sqrt{L_1^2 + (L + \varepsilon_{BD}L)^2}$$

$$= \sqrt{(0.9004L)^2 + L^2(1 + 0.0713)^2}$$

$$= 1.3994\,L$$

δ = elongation of bar CD

$\delta = L_3 - L_2 = 0.0538L$

Strain in bar CD

$$= \frac{\delta}{L_2} = \frac{0.0538L}{1.3456L} = 0.0400$$

From the stress-strain diagram of Figure 1-13:

$\sigma \approx 31$ ksi ←

Problem 1.3-6 A specimen of a methacrylate plastic is tested in tension at room temperature (see figure), producing the stress-strain data listed in the accompanying table.

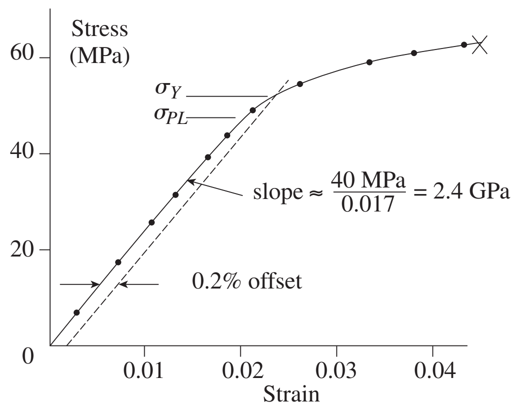
Plot the stress-strain curve and determine the proportional limit, modulus of elasticity (i.e., the slope of the initial part of the stress-strain curve), and yield stress at 0.2% offset. Is the material ductile or brittle?

STRESS-STRAIN DATA FOR PROBLEM 1.3-6

Stress (MPa)	Strain
8.0	0.0032
17.5	0.0073
25.6	0.0111
31.1	0.0129
39.8	0.0163
44.0	0.0184
48.2	0.0209
53.9	0.0260
58.1	0.0331
62.0	0.0429
62.1	Fracture

Solution 1.3-6 **Tensile test of a plastic**

Using the stress-strain data given in the problem statement, plot the stress-strain curve:



σ_{PL} = proportional limit $\sigma_{PL} \approx 47$ MPa ⟵

Modulus of elasticity (slope) ≈ 2.4 GPa ⟵

σ_Y = yield stress at 0.2% offset

$\sigma_Y \approx 53$ MPa ⟵

Material is *brittle*, because the strain after the proportional limit is exceeded is relatively small. ⟵

Problem 1.3-7 The data shown in the accompanying table were obtained from a tensile test of high-strength steel. The test specimen had a diameter of 0.505 in. and a gage length of 2.00 in. (see figure for Prob. 1.3-3). At fracture, the elongation between the gage marks was 0.12 in. and the minimum diameter was 0.42 in.

Plot the conventional stress-strain curve for the steel and determine the proportional limit, modulus of elasticity (i.e., the slope of the initial part of the stress-strain curve), yield stress at 0.1% offset, ultimate stress, percent elongation in 2.00 in., and percent reduction in area.

TENSILE-TEST DATA FOR PROBLEM 1.3-7

Load (lb)	Elongation (in.)
1,000	0.0002
2,000	0.0006
6,000	0.0019
10,000	0.0033
12,000	0.0039
12,900	0.0043
13,400	0.0047
13,600	0.0054
13,800	0.0063
14,000	0.0090
14,400	0.0102
15,200	0.0130
16,800	0.0230
18,400	0.0336
20,000	0.0507
22,400	0.1108
22,600	Fracture

Solution 1.3-7 Tensile test of high-strength steel

$d_0 = 0.505 \text{ in.} \qquad L_0 = 2.00 \text{ in.}$

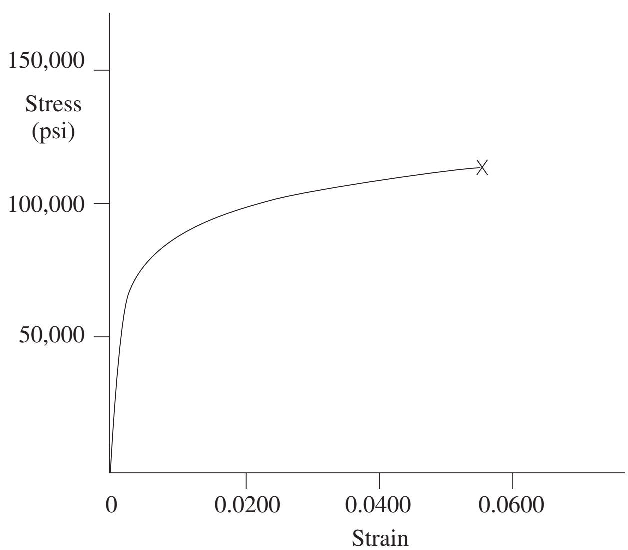
$$A_0 = \frac{\pi d_0^2}{4} = 0.200 \text{ in.}^2$$

CONVENTIONAL STRESS AND STRAIN

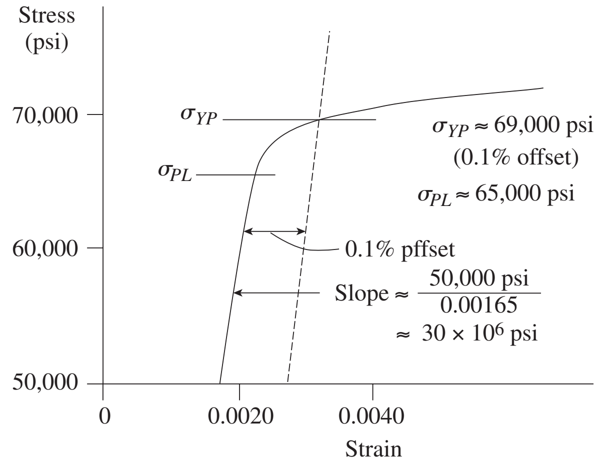
$$\sigma = \frac{P}{A_0} \qquad \varepsilon = \frac{\delta}{L_0}$$

Load P (lb)	Elongation δ (in.)	Stress σ (psi)	Strain ε
1,000	0.0002	5,000	0.00010
2,000	0.0006	10,000	0.00030
6,000	0.0019	30,000	0.00100
10,000	0.0033	50,000	0.00165
12,000	0.0039	60,000	0.00195
12,900	0.0043	64,500	0.00215
13,400	0.0047	67,000	0.00235
13,600	0.0054	68,000	0.00270
13,800	0.0063	69,000	0.00315
14,000	0.0090	70,000	0.00450
14,400	0.0102	72,000	0.00510
15,200	0.0130	76,000	0.00650
16,800	0.0230	84,000	0.01150
18,400	0.0336	92,000	0.01680
20,000	0.0507	100,000	0.02535
22,400	0.1108	112,000	0.05540
22,600	Fracture	113,000	

STRESS-STRAIN DIAGRAM



ENLARGEMENT OF PART OF THE STRESS-STRAIN CURVE



RESULTS

Proportional limit $\approx$ 65,000 psi ←

Modulus of elasticity (slope) $\approx 30 \times 10^6$ psi ←

Yield stress at 0.1% offset $\approx$ 69,000 psi ←

Ultimate stress (maximum stress)

$\approx$ 113,000 psi ←

Percent elongation in 2.00 in.

$$= \frac{L_1 - L_0}{L_0}(100)$$

$$= \frac{0.12 \text{ in.}}{2.00 \text{ in.}}(100) = 6\% \quad \longleftarrow$$

Percent reduction in area

$$= \frac{A_0 - A_1}{A_0}(100)$$

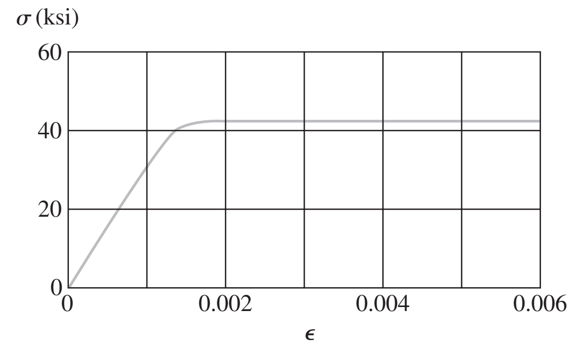
$$= \frac{0.200 \text{ in.}^2 - \frac{\pi}{4}(0.42 \text{ in.})^2}{0.200 \text{ in.}^2}(100)$$

$$= 31\% \quad \longleftarrow$$

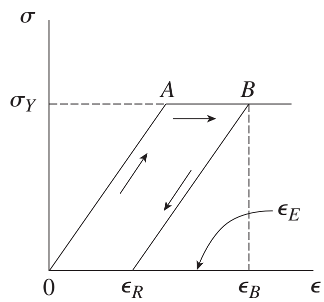
Elasticity, Plasticity, and Creep

Problem 1.4-1 A bar made of structural steel having the stress-strain diagram shown in the figure has a length of 48 in. The yield stress of the steel is 42 ksi and the slope of the initial linear part of the stress-strain curve (modulus of elasticity) is 30×10^3 ksi. The bar is loaded axially until it elongates 0.20 in., and then the load is removed.

How does the final length of the bar compare with its original length? (*Hint:* Use the concepts illustrated in Fig. 1-18b.)



Solution 1.4-1 **Steel bar in tension**



$L = 48$ in.

Yield stress $\sigma_Y = 42$ ksi

Slope $= 30 \times 10^3$ ksi

$\delta = 0.20$ in.

Stress and strain at point B

$$\sigma_B = \sigma_Y = 42 \text{ ksi}$$

$$\varepsilon_B = \frac{\delta}{L} = \frac{0.20 \text{ in.}}{48 \text{ in.}} = 0.00417$$

Elastic recovery ε_E

$$\varepsilon_E = \frac{\sigma_B}{\text{Slope}} = \frac{42 \text{ ksi}}{30 \times 10^3 \text{ ksi}} = 0.00140$$

Residual strain ε_R

$$\varepsilon_R = \varepsilon_B - \varepsilon_E = 0.00417 - 0.00140$$
$$= 0.00277$$

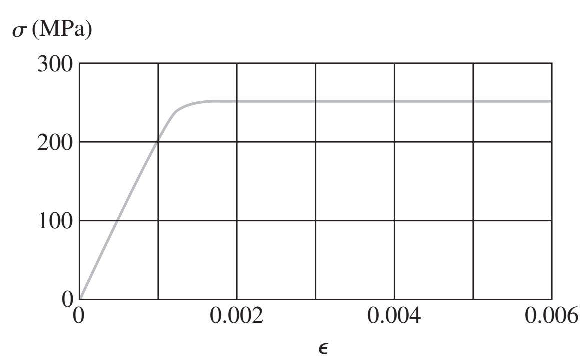
Permanent set

$$\varepsilon_R L = (0.00277)(48 \text{ in.})$$
$$= 0.13 \text{ in.}$$

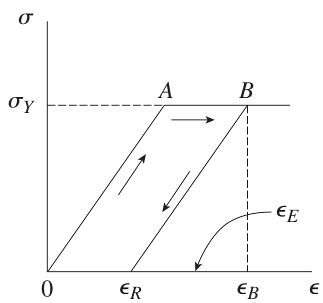
Final length of bar is 0.13 in. greater than its original length. ←

Problem 1.4-2 A bar of length 2.0 m is made of a structural steel having the stress-strain diagram shown in the figure. The yield stress of the steel is 250 MPa and the slope of the initial linear part of the stress-strain curve (modulus of elasticity) is 200 GPa. The bar is loaded axially until it elongates 6.5 mm, and then the load is removed.

How does the final length of the bar compare with its original length? (*Hint:* Use the concepts illustrated in Fig. 1-18b.)



Solution 1.4-2 **Steel bar in tension**



$L = 2.0\,\text{m} = 2000\text{ mm}$

Yield stress $\sigma_Y = 250\text{ MPa}$

Slope $= 200\text{ GPa}$

$\delta = 6.5\text{ mm}$

STRESS AND STRAIN AT POINT B

$\sigma_B = \sigma_Y = 250\text{ MPa}$

$$\varepsilon_B = \frac{\delta}{L} = \frac{6.5\text{ mm}}{2000\text{ mm}} = 0.00325$$

ELASTIC RECOVERY ε_E

$$\varepsilon_E = \frac{\sigma_B}{\text{Slope}} = \frac{250\text{ MPa}}{200\text{ GPa}} = 0.00125$$

RESIDUAL STRAIN ε_R

$$\varepsilon_R = \varepsilon_B - \varepsilon_E = 0.00325 - 0.00125 = 0.00200$$

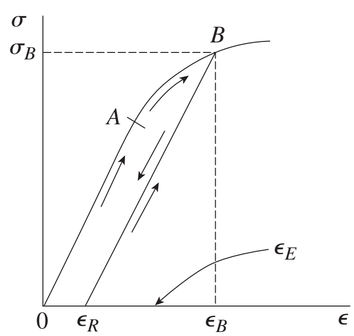
$$\text{Permanent set} = \varepsilon_R L = (0.00200)(2000\,\text{mm}) = 4.0\,\text{mm}$$

Final length of bar is 4.0 mm greater than its original length. ⟵

Problem 1.4-3 An aluminum bar has length $L = 4$ ft and diameter $d = 1.0$ in. The stress-strain curve for the aluminum is shown in Fig. 1-13 of Section 1.3. The initial straight-line part of the curve has a slope (modulus of elasticity) of 10×10^6 psi. The bar is loaded by tensile forces $P = 24$ k and then unloaded.

(a) What is the permanent set of the bar?
(b) If the bar is reloaded, what is the proportional limit? (*Hint:* Use the concepts illustrated in Figs. 1-18b and 1-19.)

Solution 1.4-3 **Aluminum bar in tension**



$L = 4\,\text{ft} = 48\,\text{in.}$

$d = 1.0\,\text{in.}$

$P = 24\,\text{k}$

See Fig. 1-13 for stress-strain diagram

Slope from O to A is 10×10^6 psi.

STRESS AND STRAIN AT POINT B

$$\sigma_B = \frac{P}{A} = \frac{24\text{ k}}{\frac{\pi}{4}(1.0\text{ in.})^2} = 31\text{ ksi}$$

From Fig. 1-13: $\varepsilon_B \approx 0.04$

ELASTIC RECOVERY ε_E

$$\varepsilon_E = \frac{\sigma_B}{\text{Slope}} = \frac{31\text{ ksi}}{10 \times 10^6\text{ psi}} = 0.0031$$

RESIDUAL STRAIN ε_R

$$\varepsilon_R = \varepsilon_B - \varepsilon_E = 0.04 - 0.0031 = 0.037$$

(Note: The accuracy in this result is very poor because ε_B is approximate.)

(a) PERMANENT SET

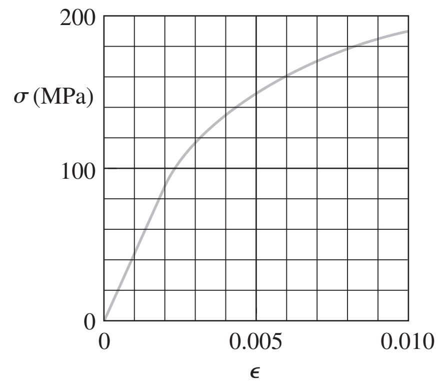
$$\varepsilon_R L = (0.037)(48\,\text{in.}) \approx 1.8\,\text{in.}$$ ⟵

(b) PROPORTIONAL LIMIT WHEN RELOADED

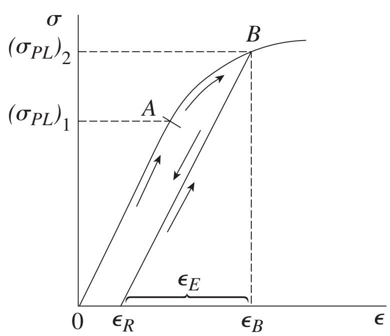
$\sigma_B = 31\,\text{ksi}$ ⟵

Problem 1.4-4 A circular bar of magnesium alloy is 800 mm long. The stress-strain diagram for the material is shown in the figure. The bar is loaded in tension to an elongation of 5.6 mm, and then the load is removed.

(a) What is the permanent set of the bar?
(b) If the bar is reloaded, what is the proportional limit?
(*Hint:* Use the concepts illustrated in Figs. 1-18b and 1-19.)



Solution 1.4-4 **Magnesium bar in tension**



$L = 800\,\text{mm}$

$\delta = 5.6\,\text{mm}$

$(\sigma_{PL})_1$ = initial proportional limit

= 88 MPa (from stress-strain diagram)

$(\sigma_{PL})_2$ = proportional limit when the bar is reloaded

INITIAL SLOPE OF STRESS-STRAIN CURVE

From σ-ε diagram:

At point A: $(\sigma_{PL})_1 = 88$ MPa

$\varepsilon_A = 0.002$

$$\text{Slope} = \frac{(\sigma_{PL})_1}{\varepsilon_A} = \frac{88\text{ MPa}}{0.002} = 44\text{ GPa}$$

STRESS AND STRAIN AT POINT B

$$\varepsilon_B = \frac{\delta}{L} = \frac{5.6\text{ mm}}{800\text{ mm}} = 0.007$$

From σ-ε diagram: $\sigma_B = (\sigma_{PL})_2 = 170$ MPa

ELASTIC RECOVERY ε_E

$$\varepsilon_E = \frac{\sigma_B}{\text{Slope}} = \frac{(\sigma_{PL})_2}{\text{Slope}} = \frac{170\text{ MPa}}{44\text{ GPa}} = 0.00386$$

RESIDUAL STRAIN ε_R

$$\varepsilon_R = \varepsilon_B - \varepsilon_E = 0.007 - 0.00386$$
$$= 0.00314$$

(a) PERMANENT SET

$$\varepsilon_R L = (0.00314)(800\text{ mm})$$
$$= 2.51\text{ mm} \longleftarrow$$

(b) PROPORTIONAL LIMIT WHEN RELOADED

$$(\sigma_{PL})_2 = \sigma_B = 170\text{ MPa} \longleftarrow$$

Problem 1.4-5 A wire of length $L = 4$ ft and diameter $d = 0.125$ in. is stretched by tensile forces $P = 600$ lb. The wire is made of a copper alloy having a stress-strain relationship that may be described mathematically by the following equation:

$$\sigma = \frac{18{,}000\epsilon}{1 + 300\epsilon} \qquad 0 \le \epsilon \le 0.03 \qquad (\sigma = \text{ksi})$$

in which ϵ is nondimensional and σ has units of kips per square inch (ksi).

(a) Construct a stress-strain diagram for the material.
(b) Determine the elongation of the wire due to the forces P.
(c) If the forces are removed, what is the permanent set of the bar?
(d) If the forces are applied again, what is the proportional limit?

Solution 1.4-5 **Wire stretched by forces P**

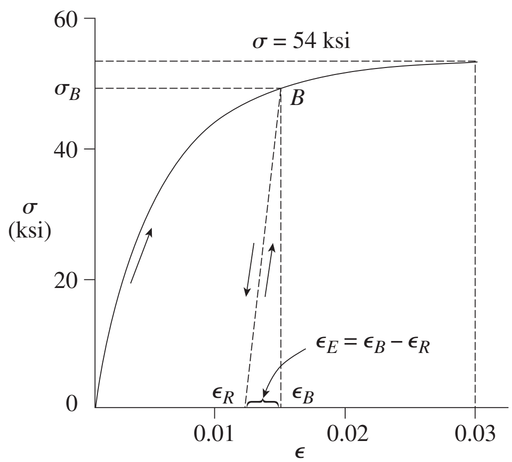
$L = 4 \text{ ft} = 48 \text{ in.} \quad d = 0.125 \text{ in.}$

$P = 600 \text{ lb}$

COPPER ALLOY

$$\sigma = \frac{18{,}000\varepsilon}{1 + 300\varepsilon} \qquad 0 \le \varepsilon \le 0.03 \ (\sigma = \text{ksi}) \qquad \text{(Eq. 1)}$$

(a) STRESS-STRAIN DIAGRAM (From Eq. 1)



INITIAL SLOPE OF STRESS-STRAIN CURVE

Take the derivative of σ with respect to ε:

$$\frac{d\sigma}{d\varepsilon} = \frac{(1 + 300\varepsilon)(18{,}000) - (18{,}000\varepsilon)(300)}{(1 + 300\varepsilon)^2}$$

$$= \frac{18{,}000}{(1 + 300\varepsilon)^2}$$

At $\varepsilon = 0$, $\dfrac{d\sigma}{d\varepsilon} = 18{,}000 \text{ ksi}$

$\therefore$ Initial slope $= 18{,}000$ ksi

ALTERNATIVE FORM OF THE STRESS-STRAIN RELATIONSHIP

Solve Eq. (1) for ε in terms of σ:

$$\varepsilon = \frac{\sigma}{18{,}000 - 300\sigma} \qquad 0 \le \sigma \le 54 \text{ ksi} \quad (\sigma = \text{ksi}) \quad \text{(Eq. 2)}$$

This equation may also be used when plotting the stress-strain diagram.

(b) ELONGATION δ OF THE WIRE

$$\sigma = \frac{P}{A} = \frac{600 \text{ lb}}{\frac{\pi}{4}(0.125 \text{ in.})^2} = 48{,}900 \text{ psi} = 48.9 \text{ ksi}$$

From Eq. (2) or from the stress-strain diagram:

$\varepsilon = 0.0147$

$\delta = \varepsilon L = (0.0147)(48 \text{ in.}) = 0.71 \text{ in.}$ ←

STRESS AND STRAIN AT POINT B (see diagram)

$\sigma_B = 48.9 \text{ ksi} \qquad \varepsilon_B = 0.0147$

ELASTIC RECOVERY ε_E

$$\varepsilon_E = \frac{\sigma_B}{\text{Slope}} = \frac{48.9 \text{ ksi}}{18{,}000 \text{ ksi}} = 0.00272$$

RESIDUAL STRAIN ε_R

$\varepsilon_R = \varepsilon_B - \varepsilon_E = 0.0147 - 0.0027 = 0.0120$

(c) Permanent set $= \varepsilon_R L = (0.0120)(48 \text{ in.})$

$= 0.58 \text{ in.}$ ←

(d) Proportional limit when reloaded $= \sigma_B$

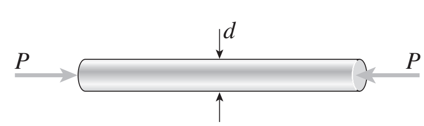
$\sigma_B = 49 \text{ ksi}$ ←

Linear Elasticity, Hooke's Law, and Poisson's Ratio

When solving the problems for Section 1.5, assume that the material behaves linearly elastically.

Problem 1.5-1 A high-strength steel bar used in a large crane has diameter $d = 2.00$ in. (see figure). The steel has modulus of elasticity $E = 29 \times 10^6$ psi and Poisson's ratio $\nu = 0.29$. Because of clearance requirements, the diameter of the bar is limited to 2.001 in. when it is compressed by axial forces.

What is the largest compressive load P_{max} that is permitted?



Solution 1.5-1 Steel bar in compression

STEEL BAR $d = 2.00\,\text{in.}$ Max. $\Delta d = 0.001\,\text{in.}$

$E = 29 \times 10^6\,\text{psi}$ $\nu = 0.29$

LATERAL STRAIN

$$\varepsilon' = \frac{\Delta d}{d} = \frac{0.001\ \text{in.}}{2.00\ \text{in.}} = 0.0005$$

AXIAL STRAIN

$$\varepsilon = -\frac{\varepsilon'}{\nu} = -\frac{0.0005}{0.29} = -0.001724$$

(shortening)

AXIAL STRESS

$$\sigma = E\varepsilon = (29 \times 10^6\,\text{psi})(-0.001724)$$
$$= -50.00\,\text{ksi (compression)}$$

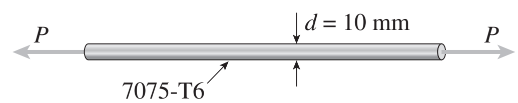
Assume that the yield stress for the high-strength steel is greater than 50 ksi. Therefore, Hooke's law is valid.

MAXIMUM COMPRESSIVE LOAD

$$P_{\text{max}} = \sigma A = (50.00\ \text{ksi})\left(\frac{\pi}{4}\right)(2.00\ \text{in.})^2$$
$$= 157\,\text{k} \longleftarrow$$

Problem 1.5-2 A round bar of 10 mm diameter is made of aluminum alloy 7075-T6 (see figure). When the bar is stretched by axial forces P, its diameter decreases by 0.016 mm.

Find the magnitude of the load P. (Obtain the material properties from Appendix H.)



Solution 1.5-2 Aluminum bar in tension

$d = 10\,\text{mm}$ $\Delta d = 0.016\,\text{mm}$

(Decrease in diameter)

7075-T6

From Table H-2: $E = 72\,\text{GPa}$ $\nu = 0.33$

From Table H-3: Yield stress $\sigma_Y = 480\ \text{MPa}$

LATERAL STRAIN

$$\varepsilon' = \frac{\Delta d}{d} = \frac{-0.016\ \text{mm}}{10\ \text{mm}} = -0.0016$$

AXIAL STRAIN

$$\varepsilon = -\frac{-\varepsilon'}{\nu} = \frac{0.0016}{0.33}$$
$$= 0.004848\ \text{(Elongation)}$$

AXIAL STRESS

$$\sigma = E\varepsilon = (72\,\text{GPa})(0.004848)$$
$$= 349.1\ \text{MPa}\ \text{(Tension)}$$

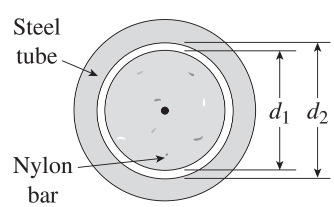
Because $\sigma < \sigma_Y$, Hooke's law is valid.

LOAD P (TENSILE FORCE)

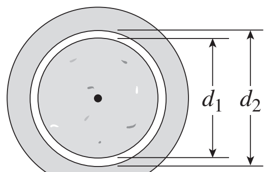
$$P = \sigma A = (349.1\ \text{MPa})\left(\frac{\pi}{4}\right)(10\ \text{mm})^2$$
$$= 27.4\,\text{kN} \longleftarrow$$

Problem 1.5-3 A nylon bar having diameter $d_1 = 3.50$ in. is placed inside a steel tube having inner diameter $d_2 = 3.51$ in. (see figure). The nylon bar is then compressed by an axial force P.

At what value of the force P will the space between the nylon bar and the steel tube be closed? (For nylon, assume $E = 400$ ksi and $\nu = 0.4$.)



Solution 1.5-3 **Nylon bar inside steel tube**



COMPRESSION

$d_1 = 3.50$ in. $\Delta d_1 = 0.01$ in.

$d_2 = 3.51$ in.

Nylon: $E = 400$ ksi $\nu = 0.4$

LATERAL STRAIN

$$\varepsilon' = \frac{\Delta d_1}{d_1} \text{ (Increase in diameter)}$$

$$\varepsilon' = \frac{0.01 \text{ in.}}{3.50 \text{ in.}} = 0.002857$$

AXIAL STRAIN

$$\varepsilon = -\frac{\varepsilon'}{\nu} = -\frac{0.002857}{0.4} = -0.007143$$

(Shortening)

AXIAL STRESS

$$\sigma = E\varepsilon = (400 \text{ ksi})(-0.007143)$$
$$= -2.857 \text{ ksi}$$

(Compressive stress)

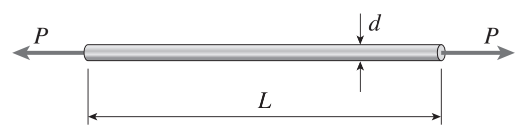
Assume that the yield stress is greater than σ and Hooke's law is valid.

FORCE P (COMPRESSION)

$$P = \sigma A = (2.857 \text{ ksi})\left(\frac{\pi}{4}\right)(3.50 \text{ in.})^2$$
$$= 27.5 \text{ k} \longleftarrow$$

Problem 1.5-4 A prismatic bar of circular cross section is loaded by tensile forces P (see figure). The bar has length $L = 1.5$ m and diameter $d = 30$ mm. It is made of aluminum alloy with modulus of elasticity $E = 75$ GPa and Poisson's ratio $\nu = ⅓$.

If the bar elongates by 3.6 mm, what is the decrease in diameter Δd? What is the magnitude of the load P?



Solution 1.5-4 **Aluminum bar in tension**

$L = 1.5$ m $d = 30$ mm

$E = 75$ GPa $\nu = ⅓$

$\delta = 3.6$ mm (elongation)

AXIAL STRAIN

$$\varepsilon = \frac{\delta}{L} = \frac{3.6 \text{ mm}}{1.5 \text{ m}} = 0.0024$$

LATERAL STRAIN

$$\varepsilon' = -\nu\varepsilon = -(\tfrac{1}{3})(0.0024)$$
$$= -0.0008$$

(Minus means decrease in diameter)

DECREASE IN DIAMETER

$$\Delta d = \varepsilon' d = (0.0008)(30 \text{ mm}) = 0.024 \text{ mm} \longleftarrow$$

AXIAL STRESS

$$\sigma = E\varepsilon = (75 \text{ GPa})(0.0024)$$
$$= 180 \text{ MPa}$$

(This stress is less than the yield stress, so Hooke's law is valid.)

LOAD P (TENSION)

$$P = \sigma A = (180 \text{ MPa})\left(\frac{\pi}{4}\right)(30 \text{ mm})^2$$
$$= 127 \text{ kN} \longleftarrow$$

Problem 1.5-5 A bar of monel metal (length $L = 8$ in., diameter $d = 0.25$ in.) is loaded axially by a tensile force $P = 1500$ lb (see figure from Prob. 1.5-4). Using the data in Table H-2, Appendix H, determine the increase in length of the bar and the percent decrease in its cross-sectional area.

Solution 1.5-5 Bar of monel metal in tension

$L = 8$ in. $\quad d = 0.25$ in. $\quad P = 1500$ lb

From Table H-2: $E = 25{,}000$ ksi $\quad \nu = 0.32$

Axial stress

$$\sigma = \frac{P}{A} = \frac{1500 \text{ lb}}{\frac{\pi}{4}(0.25 \text{ in.})^2} = 30{,}560 \text{ psi}$$

Assume σ is less than the proportional limit, so that Hooke's law is valid.

Axial strain

$$\varepsilon = \frac{\sigma}{E} = \frac{30{,}560 \text{ psi}}{25{,}000 \text{ ksi}} = 0.001222$$

Increase in length

$$\delta = \varepsilon L = (0.001222)(8 \text{ in.}) = 0.00978 \text{ in.} \quad \longleftarrow$$

Lateral strain

$$\varepsilon' = -\nu\varepsilon = -(0.32)(0.001222) = -0.0003910$$

Decrease in diameter

$$\Delta d = |\varepsilon' d| = (0.0003910)(0.25 \text{ in.}) = 0.0000978 \text{ in.}$$

Decrease in cross-sectional area

Original area: $A_0 = \dfrac{\pi d^2}{4}$

Final area:

$$A_1 = \frac{\pi}{4}(d - \Delta d)^2$$

$$A_1 = \frac{\pi}{4}[d^2 - 2d\Delta d + (\Delta d)^2]$$

Decrease in area:

$$\Delta A = A_0 - A_1$$

$$\Delta A = \frac{\pi}{4}(\Delta d)(2d - \Delta d)$$

Percent decrease in area

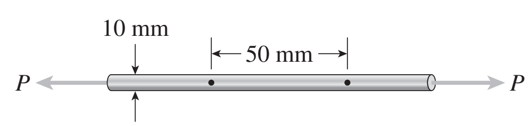
$$\text{Percent} = \frac{\Delta A}{A_0}(100) = \frac{(\Delta d)(2d - \Delta d)}{d^2}(100)$$

$$= \frac{(0.0000978)(0.4999)}{(0.25)^2}(100)$$

$$= 0.078\% \quad \longleftarrow$$

Problem 1.5-6 A tensile test is peformed on a brass specimen 10 mm in diameter using a gage length of 50 mm (see figure). When the tensile load P reaches a value of 20 kN, the distance between the gage marks has increased by 0.122 mm.

(a) What is the modulus of elasticity E of the brass?
(b) If the diameter decreases by 0.00830 mm, what is Poisson's ratio?



Solution 1.5-6 **Brass specimen in tension**

$d = 10\,\text{mm}$ Gage length $L = 50\,\text{mm}$

$P = 20\,\text{kN}$ $\delta = 0.122\,\text{mm}$ $\Delta d = 0.00830\,\text{mm}$

AXIAL STRESS

$$\sigma = \frac{P}{A} = \frac{20\text{ kN}}{\frac{\pi}{4}(10\text{ mm})^2} = 254.6\text{ MPa}$$

Assume σ is below the proportional limit so that Hooke's law is valid.

AXIAL STRAIN

$$\varepsilon = \frac{\delta}{L} = \frac{0.122\text{ mm}}{50\text{ mm}} = 0.002440$$

(a) MODULUS OF ELASTICITY

$$E = \frac{\sigma}{\varepsilon} = \frac{254.6\text{ MPa}}{0.002440} = 104\text{ GPa} \longleftarrow$$

(b) POISSON'S RATIO

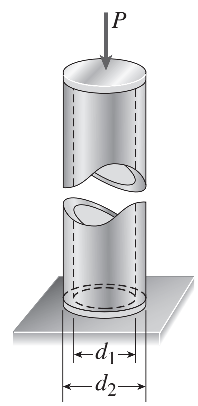
$$\varepsilon' = \nu\varepsilon$$

$$\Delta d = \varepsilon' d = \nu\varepsilon d$$

$$\nu = \frac{\Delta d}{\varepsilon d} = \frac{0.00830\text{ mm}}{(0.002440)(10\text{ mm})} = 0.34 \longleftarrow$$

Problem 1.5-7 A hollow steel cylinder is compressed by a force P (see figure). The cylinder has inner diameter $d_1 = 3.9$ in., outer diameter $d_2 = 4.5$ in., and modulus of elasticity $E = 30{,}000$ ksi. When the force P increases from zero to 40 k, the outer diameter of the cylinder increases by 455×10^{-6} in.

(a) Determine the increase in the inner diameter.
(b) Determine the increase in the wall thickness.
(c) Determine Poisson's ratio for the steel.



Solution 1.5-7 **Hollow steel cylinder**

$d_1 = 3.9\,\text{in.}$

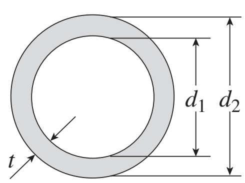
$d_2 = 4.5\,\text{in.}$

$t = 0.3\,\text{in.}$

$E = 30{,}000\,\text{ksi}$

$P = 40\,\text{k}$ (compression)

$\Delta d_2 = 455 \times 10^{-6}\,\text{in.}$ (increase)



LATERAL STRAIN

$$\varepsilon' = \frac{\Delta d_2}{d_2} = \frac{455 \times 10^{-6}\text{ in.}}{4.5\text{ in.}} = 0.0001011$$

(a) INCREASE IN INNER DIAMETER

$$\Delta d_1 = \varepsilon' d_1 = (0.0001011)(3.9\text{ in.})$$

$$= 394 \times 10^{-6}\text{ in.} \longleftarrow$$

(b) INCREASE IN WALL THICKNESS

$$\Delta t = \varepsilon' t = (0.0001011)(0.3\text{ in.})$$

$$= 30 \times 10^{-6}\text{ in.} \longleftarrow$$

(c) POISSON'S RATIO

Axial stress: $\sigma = \dfrac{P}{A}$

$$A = \frac{\pi}{4}[d_2^2 - d_1^2] = \frac{\pi}{4}[(4.5\text{ in.})^2 - (3.9\text{ in.})^2]$$

$$= 3.9584\text{ in.}^2$$

$$\sigma = \frac{P}{A} = \frac{40\text{ k}}{3.9584\text{ in.}^2}$$

$$= 10.105\text{ ksi (compression)}$$

($\sigma < \sigma_Y$; Hooke's law is valid)

Axial strain:

$$\varepsilon = \frac{\sigma}{E} = \frac{10.105\text{ ksi}}{30{,}000\text{ ksi}}$$

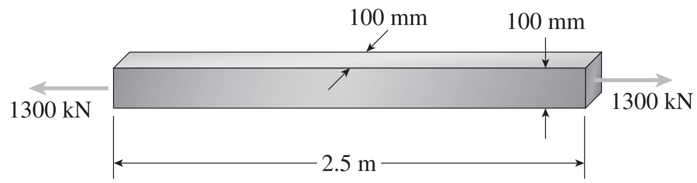
$$= 0.000337$$

$$\nu = \frac{\varepsilon'}{\varepsilon} = \frac{0.0001011}{0.000337}$$

$$= 0.30 \longleftarrow$$

Problem 1.5-8 A steel bar of length 2.5 m with a square cross section 100 mm on each side is subjected to an axial tensile force of 1300 kN (see figure). Assume that $E = 200$ GPa and $\nu = 0.3$.

Determine the increase in volume of the bar.



Solution 1.5-8 Square bar in tension

Find increase in volume.

Length: $L = 2.5\,\text{m} = 2500\,\text{mm}$

Side: $b = 100\,\text{mm}$

Force: $P = 1300\,\text{kN}$

$E = 200\,\text{GPa} \qquad \nu = 0.3$

AXIAL STRESS

$$\sigma = \frac{P}{A} = \frac{P}{b^2}$$

$$\sigma = \frac{1300\text{ kN}}{(100\text{ mm})^2} = 130\text{ MPa}$$

Stress σ is less than the yield stress, so Hooke's law is valid.

AXIAL STRAIN

$$\varepsilon = \frac{\sigma}{E} = \frac{130\text{ MPa}}{200\text{ GPa}}$$

$$= 650 \times 10^{-6}$$

INCREASE IN LENGTH

$$\Delta L = \varepsilon L = (650 \times 10^{-6})(2500\text{ mm})$$

$$= 1.625\text{ mm}$$

DECREASE IN SIDE DIMENSION

$$\varepsilon' = \nu\varepsilon = 195 \times 10^{-6}$$

$$\Delta b = \varepsilon' b = (195 \times 10^{-6})(100\text{ mm})$$

$$= 0.0195\text{ mm}$$

FINAL DIMENSIONS

$$L_1 = L + \Delta L = 2501.625\text{ mm}$$

$$b_1 = b - \Delta b = 99.9805\text{ mm}$$

FINAL VOLUME

$$V_1 = L_1 b_1^2 = 25{,}006{,}490\text{ mm}^3$$

INITIAL VOLUME

$$V = Lb^2 = 25{,}000{,}000\text{ mm}^3$$

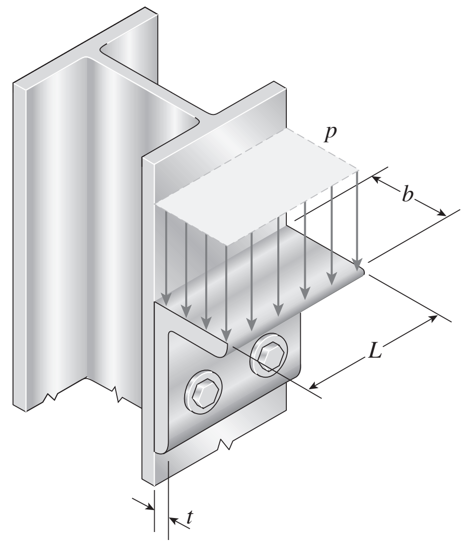
INCREASE IN VOLUME

$$\Delta V = V_1 - V = 6490\text{ mm}^3 \quad \longleftarrow$$

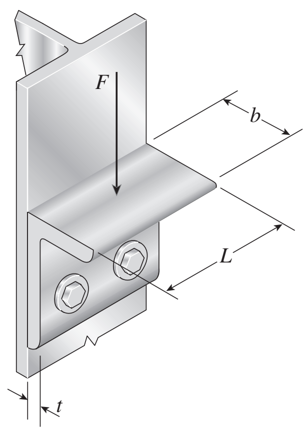
Shear Stress and Strain

Problem 1.6-1 An angle bracket having thickness $t = 0.5$ in. is attached to the flange of a column by two ⅝-inch diameter bolts (see figure). A uniformly distributed load acts on the top face of the bracket with a pressure $p = 300$ psi. The top face of the bracket has length $L = 6$ in. and width $b = 2.5$ in.

Determine the average bearing pressure σ_b between the angle bracket and the bolts and the average shear stress τ_{aver} in the bolts. (Disregard friction between the bracket and the column.)



Solution 1.6-1 **Angle bracket bolted to a column**



Two bolts

$d = 0.625$ in.

t = thickness of angle = 0.5 in.

$b = 2.5$ in.

$L = 6.0$ in.

p = pressure acting on top of the bracket

$= 300$ psi

F = resultant force acting on the bracket

$= pbL = (300 \text{ psi})(2.5 \text{ in.})(6.0 \text{ in.}) = 4.50 \text{ k}$

BEARING PRESSURE BETWEEN BRACKET AND BOLTS

A_b = bearing area of one bolt

$= dt = (0.625 \text{ in.})(0.5 \text{ in.}) = 0.3125 \text{ in.}^2$

$$\sigma_b = \frac{F}{2A_b} = \frac{4.50 \text{ k}}{2(0.3125 \text{ in.}^2)} = 7.20 \text{ ksi} \longleftarrow$$

AVERAGE SHEAR STRESS IN THE BOLTS

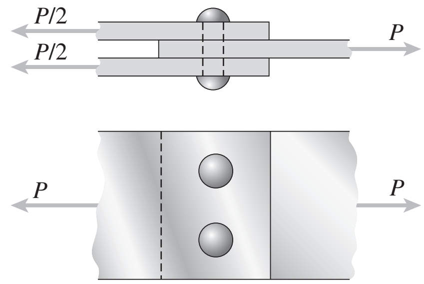
A_s = Shear area of one bolt

$$= \frac{\pi}{4}d^2 = \frac{\pi}{4}(0.625 \text{ in.})^2 = 0.3068 \text{ in.}^2$$

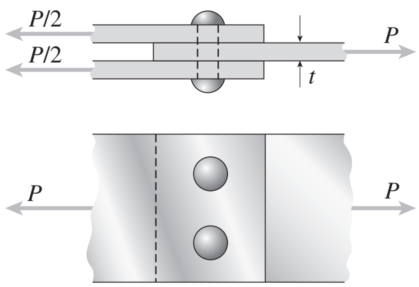
$$\tau_{aver} = \frac{F}{2A_s} = \frac{4.50 \text{ k}}{2(0.3068 \text{ in.}^2)} = 7.33 \text{ ksi} \longleftarrow$$

Problem 1.6-2 Three steel plates, each 16 mm thick, are joined by two 20-mm diameter rivets as shown in the figure.

(a) If the load $P = 50$ kN, what is the largest bearing stress acting on the rivets?
(b) If the ultimate shear stress for the rivets is 180 MPa, what force P_{ult} is required to cause the rivets to fail in shear? (Disregard friction between the plates.)



Solution 1.6-2 **Three plates joined by two rivets**



t = thickness of plates = 16 mm

d = diameter of rivets = 20 mm

P = 50 kN

τ_{ULT} = 180 MPa (for shear in the rivets)

(a) Maximum bearing stress on the rivets

Maximum stress occurs at the middle plate.

A_b = bearing area for one rivet

$= dt$

$$\sigma_b = \frac{P}{2A_b} = \frac{P}{2dt} = \frac{50\text{ kN}}{2(20\text{ mm})(16\text{ mm})}$$

$$= 78.1\text{ MPa} \longleftarrow$$

(b) Ultimate load in shear

Shear force on two rivets $= \dfrac{P}{2}$

Shear force on one rivet $= \dfrac{P}{4}$

Let A = cross-sectional area of one rivet

Shear stress $\tau = \dfrac{P/4}{A} = \dfrac{P}{4\left(\frac{\pi d^2}{4}\right)} = \dfrac{P}{\pi d^2}$

or, $P = \pi d^2 \tau$

At the ultimate load:

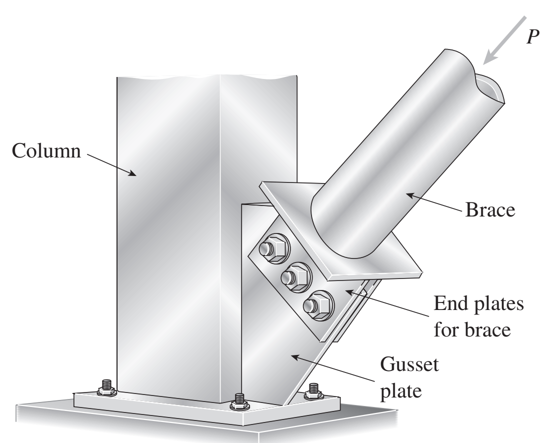
$$P_{ULT} = \pi d^2 \tau_{ULT} = \pi(20\text{ mm})^2(180\text{ MPa})$$

$$= 226\text{ kN} \longleftarrow$$

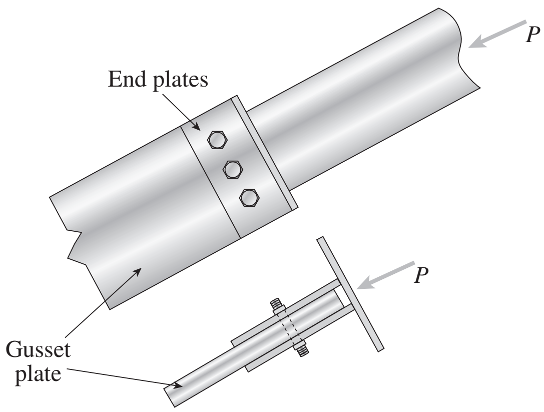
Problem 1.6-3 A bolted connection between a vertical column and a diagonal brace is shown in the figure. The connection consists of three ⅝-in. bolts that join two ¼-in. end plates welded to the brace and a ⅝-in. gusset plate welded to the column. The compressive load P carried by the brace equals 8.0 k.

Determine the following quantities:

(a) The average shear stress τ_{aver} in the bolts, and
(b) The average bearing stress σ_b between the gusset plate and the bolts. (Disregard friction between the plates.)



Solution 1.6-3 Diagonal brace



3 bolts in double shear

P = compressive force in brace = 8.0 k

d = diameter of bolts = ⅝ in. = 0.625 in.

t_1 = thickness of gusset plate

= ⅝ in. = 0.625 in.

t_2 = thickness of end plates

= ¼ in. = 0.25 in.

(a) AVERAGE SHEAR STRESS IN THE BOLTS

A = cross-sectional area of one bolt

$$= \frac{\pi d^2}{4} = 0.3068 \text{ in.}^2$$

V = shear force acting on one bolt

$$= \frac{1}{3}\left(\frac{P}{2}\right) = \frac{P}{6}$$

$$\tau_{\text{aver}} = \frac{V}{A} = \frac{P}{6A} = \frac{8.0 \text{ k}}{6(0.3068 \text{ in.}^2)}$$

$$= 4350 \text{ psi} \longleftarrow$$

(b) AVERAGE BEARING STRESS AGAINST GUSSET PLATE

A_b = bearing area of one bolt

$$= t_1 d = (0.625 \text{ in.})(0.625 \text{ in.}) = 0.3906 \text{ in.}^2$$

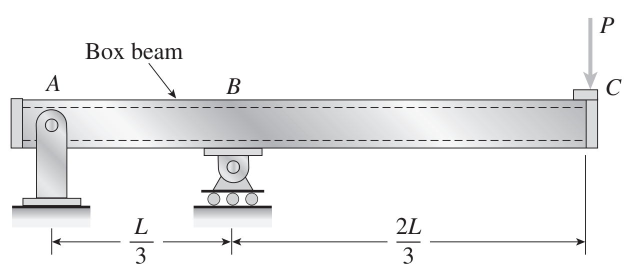
F = bearing force acting on gusset plate from one bolt

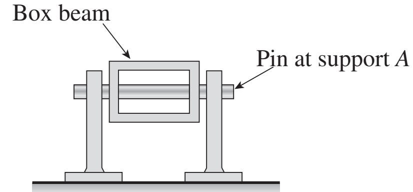
$$= \frac{P}{3}$$

$$\sigma_b = \frac{P}{3A_b} = \frac{8.0 \text{ k}}{3(0.3906 \text{ in.}^2)} = 6830 \text{ psi} \longleftarrow$$

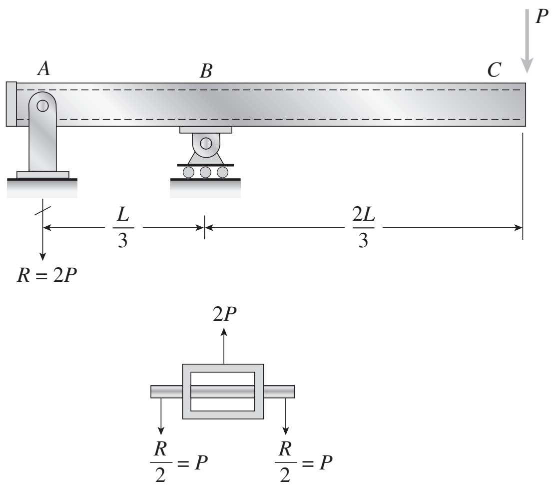
Problem 1.6-4 A hollow box beam ABC of length L is supported at end A by a 20-mm diameter pin that passes through the beam and its supporting pedestals (see figure). The roller support at B is located at distance $L/3$ from end A.

(a) Determine the average shear stress in the pin due to a load P equal to 10 kN.
(b) Determine the average bearing stress between the pin and the box beam if the wall thickness of the beam is equal to 12 mm.





Solution 1.6-4 Hollow box beam



$P = 10\,\text{kN}$

$d =$ diameter of pin $= 20\,\text{mm}$

$t =$ wall thickness of box beam $= 12\,\text{mm}$

(a) Average shear stress in pin

Double shear

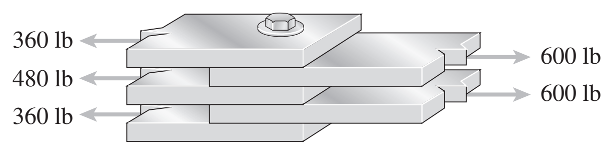
$$\tau_{\text{aver}} = \frac{2P}{2\left(\frac{\pi}{4}d^2\right)} = \frac{4P}{\pi d^2} = 31.8\text{ MPa} \quad \longleftarrow$$

(b) Average bearing stress on pin

$$\sigma_b = \frac{2P}{2(dt)} = \frac{P}{dt} = 41.7\text{ MPa} \quad \longleftarrow$$

Problem 1.6-5 The connection shown in the figure consists of five steel plates, each 3/16 in. thick, joined by a single 1/4-in. diameter bolt. The total load transferred between the plates is 1200 lb, distributed among the plates as shown.

(a) Calculate the largest shear stress in the bolt, disregarding friction between the plates.
(b) Calculate the largest bearing stress acting against the bolt.

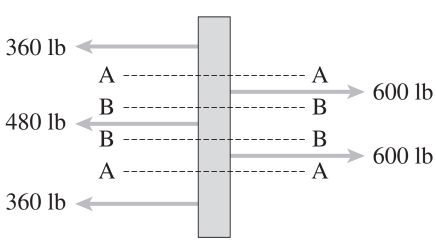


Solution 1.6-5 Plates joined by a bolt

$d =$ diameter of bolt $= 1/4$ in.

$t =$ thickness of plates $= 3/16$ in.

Free-body diagram of bolt



Section A − A: $V = 360\,\text{lb}$

Section B − B: $V = 240\,\text{lb}$

$V_{\text{max}} =$ max. shear force in bolt

$= 360\,\text{lb}$

(a) Maximum shear stress in bolt

$$\tau_{\text{max}} = \frac{V_{\text{max}}}{\pi \frac{d^2}{4}} = \frac{4V_{\text{max}}}{\pi d^2} = 7330\text{ psi} \quad \longleftarrow$$

(b) Maximum bearing stress

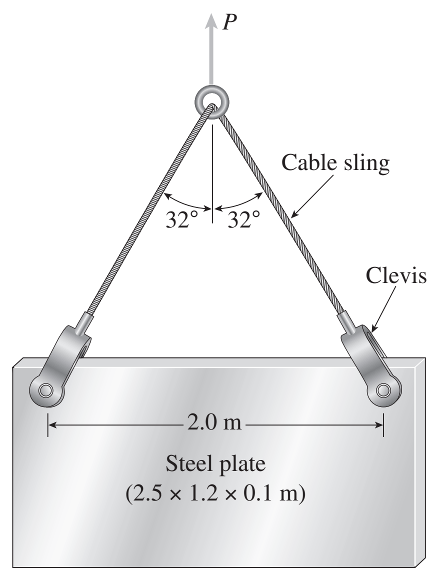
$F_{\text{max}} =$ maximum force applied by a plate against the bolt

$F_{\text{max}} = 600\,\text{lb}$

$$\sigma_b = \frac{F_{\text{max}}}{dt} = 12{,}800\text{ psi} \quad \longleftarrow$$

Problem 1.6-6 A steel plate of dimensions 2.5 × 1.2 × 0.1 m is hoisted by a cable sling that has a clevis at each end (see figure). The pins through the clevises are 18 mm in diameter and are located 2.0 m apart. Each half of the cable is at an angle of 32° to the vertical.

For these conditions, determine the average shear stress τ_{aver} in the pins and the average bearing stress σ_b between the steel plate and the pins.



Solution 1.6-6 **Steel plate hoisted by a sling**

Dimensions of plate: 2.5 × 1.2 × 0.1 m

Volume of plate: $V = (2.5)(1.2)(0.1)\text{ m} = 0.300\text{ m}^3$

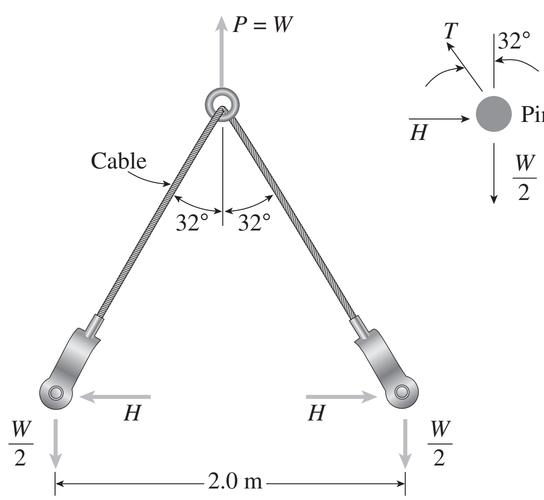
Weight density of steel: $\gamma = 77.0\text{ kN/m}^3$

Weight of plate: $W = \gamma V = 23.10\text{ kN}$

d = diameter of pin through clevis = 18 mm

t = thickness of plate = 0.1 m = 100 mm

FREE-BODY DIAGRAMS OF SLING AND PIN



TENSILE FORCE T IN CABLE

$\Sigma F_{vertical} = 0 \quad \uparrow_+ \downarrow^-$

$$T \cos 32° - \frac{W}{2} = 0$$

$$T = \frac{W}{2 \cos 32°} = \frac{23.10\text{ kN}}{2 \cos 32°} = 13.62\text{ kN}$$

SHEAR STRESS IN THE PINS (DOUBLE SHEAR)

$$\tau_{aver} = \frac{T}{2A_{pin}} = \frac{13.62\text{ kN}}{2(\frac{\pi}{4})(18\text{ mm})^2}$$

$$= 26.8\text{ MPa} \quad \longleftarrow$$

BEARING STRESS BETWEEN PLATE AND PINS

A_b = bearing area

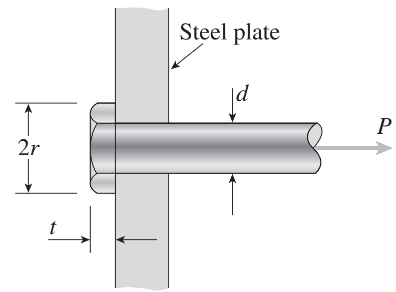
$= td$

$$\sigma_b = \frac{T}{td} = \frac{13.62\text{ kN}}{(100\text{ mm})(18\text{ mm})}$$

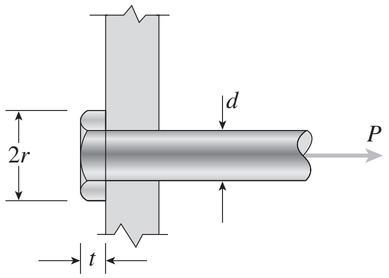
$$= 7.57\text{ MPa} \quad \longleftarrow$$

Problem 1.6-7 A special-purpose bolt of shank diameter $d = 0.50$ in. passes through a hole in a steel plate (see figure). The hexagonal head of the bolt bears directly against the steel plate. The radius of the circumscribed circle for the hexagon is $r = 0.40$ in. (which means that each side of the hexagon has length 0.40 in.). Also, the thickness t of the bolt head is 0.25 in. and the tensile force P in the bolt is 1000 lb.

(a) Determine the average bearing stress σ_b between the hexagonal head of the bolt and the plate.
(b) Determine the average shear stress τ_{aver} in the head of the bolt.



Solution 1.6-7 **Bolt in tension**

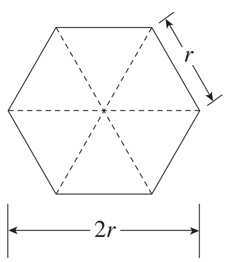


$d = 0.50$ in.

$r = 0.40$ in.

$t = 0.25$ in.

$P = 1000$ lb



Area of one equilateral triangle

$$= \frac{r^2\sqrt{3}}{4}$$

Area of hexagon

$$= \frac{3r^2\sqrt{3}}{2}$$

(a) Bearing stress between bolt head and plate

A_b = bearing area

A_b = area of hexagon minus area of bolt

$$= \frac{3r^2\sqrt{3}}{2} - \frac{\pi d^2}{4}$$

$$A_b = \frac{3}{2}(0.40\text{ in.})^2(\sqrt{3}) - \left(\frac{\pi}{4}\right)(0.50\text{ in.})^2$$

$$= 0.4157\text{ in.}^2 - 0.1963\text{ in.}^2$$

$$= 0.2194\text{ in.}^2$$

$$\sigma_b = \frac{P}{A_b} = \frac{1000\text{ lb}}{0.2194\text{ in.}^2} = 4560\text{ psi} \quad \longleftarrow$$

(b) Shear stress in head of bolt

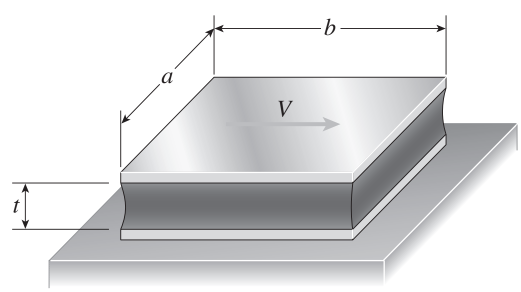
A_s = shear area $\quad A_s = \pi dt$

$$\tau_{\text{aver}} = \frac{P}{A_s} = \frac{P}{\pi dt} = \frac{1000\text{ lb}}{\pi(0.50\text{ in.})(0.25\text{ in.})}$$

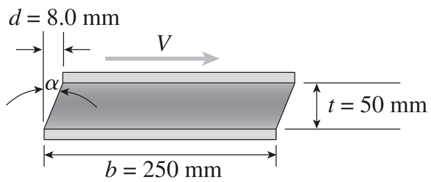
$$= 2550\text{ psi} \quad \longleftarrow$$

Problem 1.6-8 An elastomeric bearing pad consisting of two steel plates bonded to a chloroprene elastomer (an artificial rubber) is subjected to a shear force V during a static loading test (see figure). The pad has dimensions $a = 150$ mm and $b = 250$ mm, and the elastomer has thickness $t = 50$ mm. When the force V equals 12 kN, the top plate is found to have displaced laterally 8.0 mm with respect to the bottom plate.

What is the shear modulus of elasticity G of the chloroprene?



Solution 1.6-8 **Bearing pad subjected to shear**



$V = 12\,\text{kN}$

Width of pad: $a = 150\,\text{mm}$

Length of pad: $b = 250\,\text{mm}$

$d = 8.0\,\text{mm}$

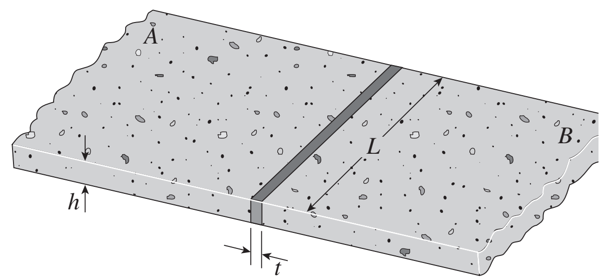
$$\tau_{\text{aver}} = \frac{V}{ab} = \frac{12\text{ kN}}{(150\text{ mm})(250\text{ mm})} = 0.32\text{ MPa}$$

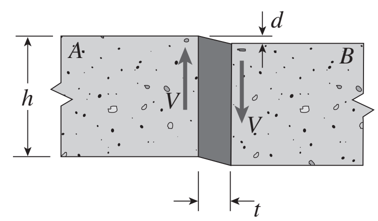
$$\gamma_{\text{aver}} = \frac{d}{t} = \frac{8.0\text{ mm}}{50\text{ mm}} = 0.16$$

$$G = \frac{\tau}{\gamma} = \frac{0.32\text{ MPa}}{0.16} = 2.0\text{ MPa} \longleftarrow$$

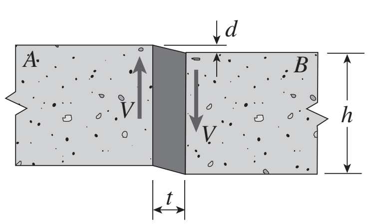
Problem 1.6-9 A joint between two concrete slabs A and B is filled with a flexible epoxy that bonds securely to the concrete (see figure). The height of the joint is $h = 4.0$ in., its length is $L = 40$ in., and its thickness is $t = 0.5$ in. Under the action of shear forces V, the slabs displace vertically through the distance $d = 0.002$ in. relative to each other.

(a) What is the average shear strain γ_{aver} in the epoxy?
(b) What is the magnitude of the forces V if the shear modulus of elasticity G for the epoxy is 140 ksi?





Solution 1.6-9 **Epoxy joint between concrete slabs**



$h = 4.0\,\text{in.}$

$t = 0.5\,\text{in.}$

$L = 40\,\text{in.}$

$d = 0.002\,\text{in.}$

$G = 140\,\text{ksi}$

(a) Average shear strain

$$\gamma_{\text{aver}} = \frac{d}{t} = 0.004 \longleftarrow$$

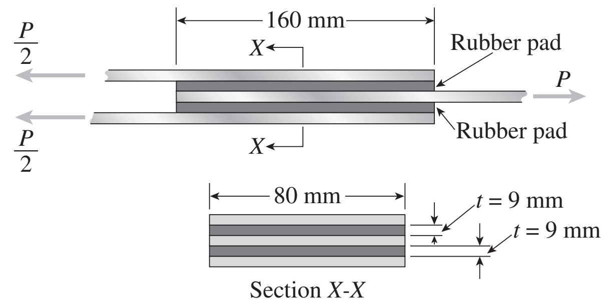
(b) Shear forces V

Average shear stress : $\tau_{\text{aver}} = G\gamma_{\text{aver}}$

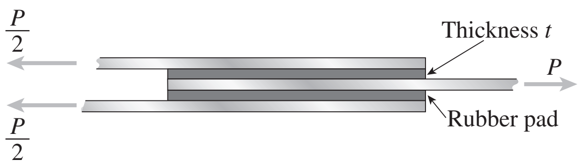
$$V = \tau_{\text{aver}}(hL) = G\gamma_{\text{aver}}(hL)$$
$$= (140\,\text{ksi})(0.004)(4.0\,\text{in.})(40\,\text{in.})$$
$$= 89.6\,\text{k} \longleftarrow$$

Problem 1.6-10 A flexible connection consisting of rubber pads (thickness $t = 9$ mm) bonded to steel plates is shown in the figure. The pads are 160 mm long and 80 mm wide.

(a) Find the average shear strain γ_{aver} in the rubber if the force $P = 16$ kN and the shear modulus for the rubber is $G = 1250$ kPa.
(b) Find the relative horizontal displacement δ between the interior plate and the outer plates.



Solution 1.6-10 **Rubber pads bonded to steel plates**



Rubber pads: $t = 9\,\text{mm}$

Length $L = 160\,\text{mm}$

Width $b = 80\,\text{mm}$

$G = 1250\,\text{kPa}$

$P = 16\,\text{kN}$

(a) Shear stress and strain in the rubber pads

$$\tau_{aver} = \frac{P/2}{bL} = \frac{8\text{ kN}}{(80\text{ mm})(160\text{ mm})} = 625\text{ kPa}$$

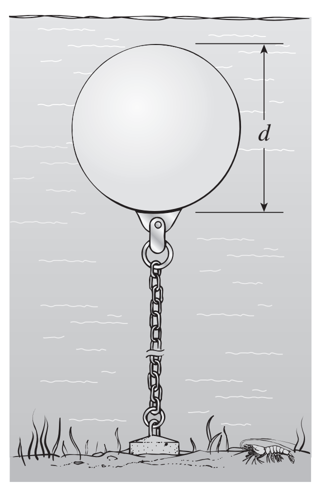
$$\gamma_{aver} = \frac{\tau_{aver}}{G} = \frac{625\text{ kPa}}{1250\text{ kPa}} = 0.50 \quad \longleftarrow$$

(b) Horizontal displacement

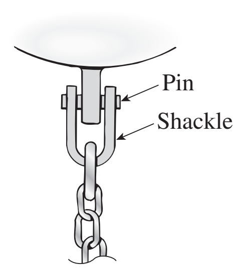
$$\delta = \gamma_{aver}t = (0.50)(9\text{ mm}) = 4.50\text{ mm} \quad \longleftarrow$$

Problem 1.6-11 A spherical fiberglass buoy used in an underwater experiment is anchored in shallow water by a chain [see part (a) of the figure]. Because the buoy is positioned just below the surface of the water, it is not expected to collapse from the water pressure. The chain is attached to the buoy by a shackle and pin [see part (b) of the figure]. The diameter of the pin is 0.5 in. and the thickness of the shackle is 0.25 in. The buoy has a diameter of 60 in. and weighs 1800 lb on land (not including the weight of the chain).

(a) Determine the average shear stress τ_{aver} in the pin.
(b) Determine the average bearing stress σ_b between the pin and the shackle.

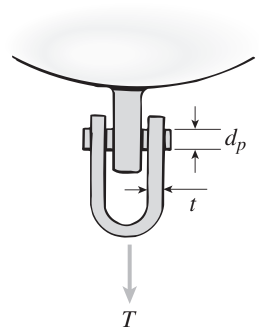


(a)



(b)

Solution 1.6-11 Submerged buoy



d = diameter of buoy

= 60 in.

T = tensile force in chain

d_p = diameter of pin

= 0.5 in.

t = thickness of shackle

= 0.25 in.

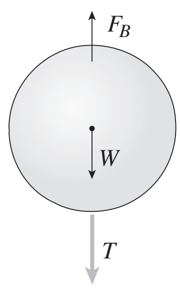
W = weight of buoy

= 1800 lb

γ_W = weight density of sea water

= 63.8 lb/ft^3

FREE-BODY DIAGRAM OF BUOY



F_B = buoyant force of water pressure (equals the weight of the displaced sea water)

V = volume of buoy

$$= \frac{\pi d^3}{6} = 65.45 \text{ ft}^3$$

$$F_B = \gamma_W V = 4176 \text{ lb}$$

EQUILIBRIUM

$$T = F_B - W = 2376 \text{ lb}$$

(a) AVERAGE SHEAR STRESS IN PIN

A_p = area of pin

$$A_p = \frac{\pi}{4} d_p^2 = 0.1963 \text{ in.}^2$$

$$\tau_{\text{aver}} = \frac{T}{2A_p} = 6050 \text{ psi} \longleftarrow$$

(b) BEARING STRESS BETWEEN PIN AND SHACKLE

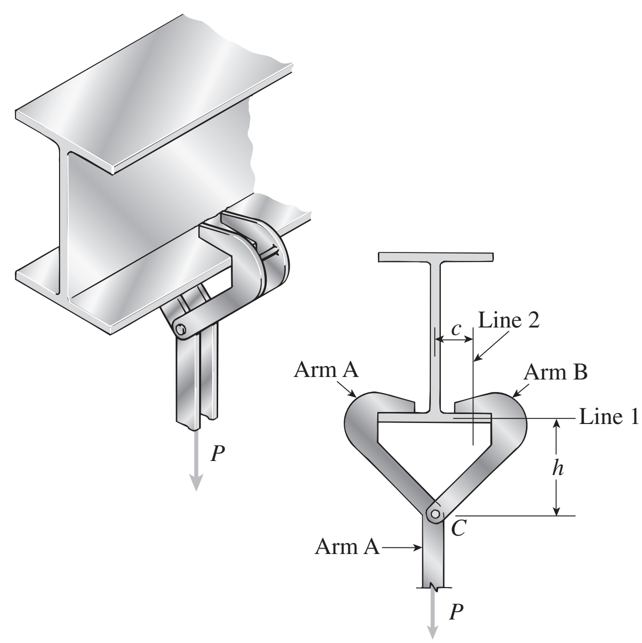
$$A_b = 2d_p t = 0.2500 \text{ in.}^2$$

$$\sigma_b = \frac{T}{A_b} = 9500 \text{ psi} \longleftarrow$$

Problem 1.6-12 The clamp shown in the figure is used to support a load hanging from the lower flange of a steel beam. The clamp consists of two arms (A and B) joined by a pin at C. The pin has diameter d = 12 mm. Because arm B straddles arm A, the pin is in double shear.

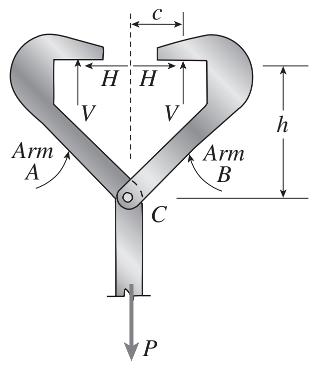
Line 1 in the figure defines the line of action of the resultant horizontal force H acting between the lower flange of the beam and arm B. The vertical distance from this line to the pin is h = 250 mm. Line 2 defines the line of action of the resultant vertical force V acting between the flange and arm B. The horizontal distance from this line to the centerline of the beam is c = 100 mm. The force conditions between arm A and the lower flange are symmetrical with those given for arm B.

Determine the average shear stress in the pin at C when the load P = 18 kN.



Solution 1.6-12 Clamp supporting a load P

FREE-BODY DIAGRAM OF CLAMP



$h = 250\,\text{mm}$

$c = 100\,\text{mm}$

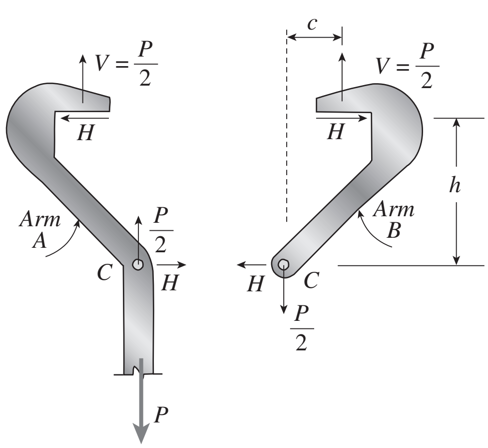
$P = 18\,\text{kN}$

From vertical equilibrium:

$$V = \frac{P}{2} = 9\text{ kN}$$

$d =$ diameter of pin at $C = 12\,\text{mm}$

FREE-BODY DIAGRAMS OF ARMS A AND B

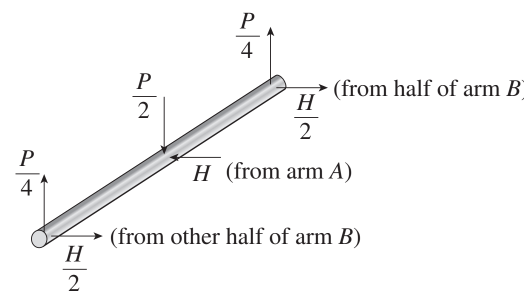


$\Sigma M_C = 0$

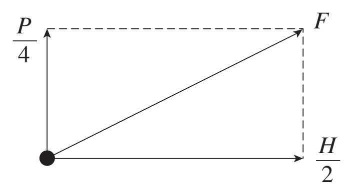
$Vc - Hh = 0$

$$H = \frac{Vc}{h} = \frac{Pc}{2h} = 3.6\text{ kN}$$

FREE-BODY DIAGRAM OF PIN



SHEAR FORCE F IN PIN



$$F = \sqrt{\left(\frac{P}{4}\right)^2 + \left(\frac{H}{2}\right)^2}$$

$$= 4.847\,\text{kN}$$

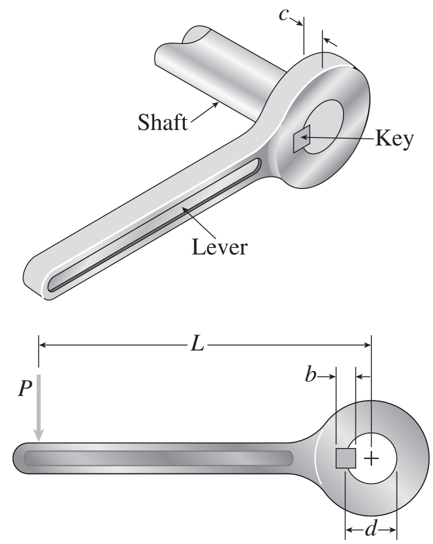
AVERAGE SHEAR STRESS IN THE PIN

$$\tau_{\text{aver}} = \frac{F}{A_{\text{pin}}} = \frac{F}{\frac{\pi d^2}{4}} = 42.9\text{ MPa} \quad \longleftarrow$$

Problem 1.6-13 A specially designed wrench is used to twist a circular shaft by means of a square key that fits into slots (or *keyways*) in the shaft and wrench, as shown in the figure. The shaft has diameter d, the key has a square cross section of dimensions $b \times b$, and the length of the key is c. The key fits half into the wrench and half into the shaft (i.e., the keyways have a depth equal to $b/2$).

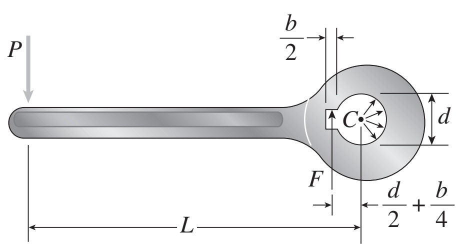
Derive a formula for the average shear stress τ_{aver} in the key when a load P is applied at distance L from the center of the shaft.

Hints: Disregard the effects of friction, assume that the bearing pressure between the key and the wrench is uniformly distributed, and be sure to draw free-body diagrams of the wrench and key.



Solution 1.6-13 Wrench with keyway

FREE-BODY DIAGRAM OF WRENCH



With friction disregarded, the bearing pressures between the wrench and the shaft are radial.

Because the bearing pressure between the wrench and the key is uniformly distributed, the force F acts at the midpoint of the keyway.

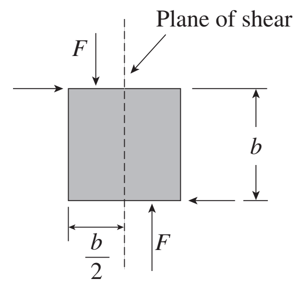
(Width of keyway = $b/2$)

$\Sigma M_C = 0$

$$PL - F\left(\frac{d}{2} + \frac{b}{4}\right) = 0$$

$$F = \frac{4PL}{2d + b}$$

FREE-BODY DIAGRAM OF KEY



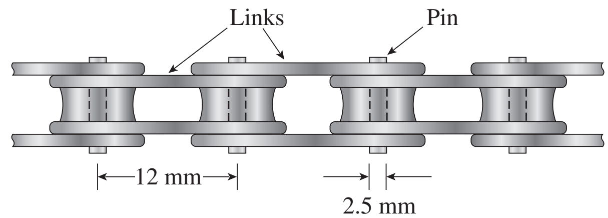
$$\tau_{aver} = \frac{F}{bc}$$

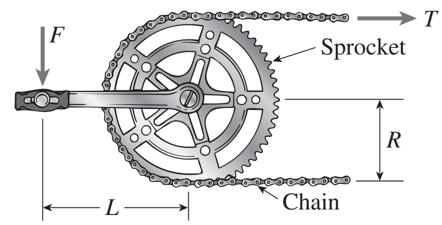
$$= \frac{4PL}{bc(2d + b)} \quad \leftarrow$$

Problem 1.6-14 A bicycle chain consists of a series of small links, each 12 mm long between the centers of the pins (see figure). You might wish to examine a bicycle chain and observe its construction. Note particularly the pins, which we will assume to have a diameter of 2.5 mm.

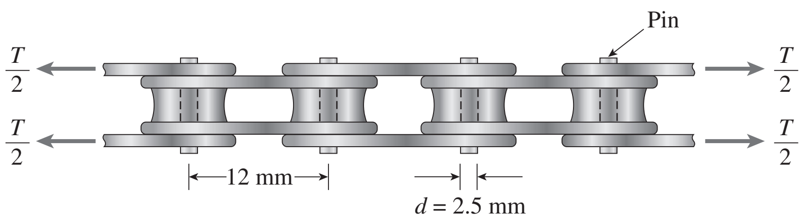
In order to solve this problem, you must now make two measurements on a bicycle (see figure): (1) the length L of the crank arm from main axle to pedal axle, and (2) the radius R of the sprocket (the toothed wheel, sometimes called the chainring).

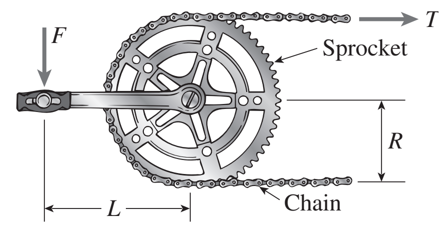
(a) Using your measured dimensions, calculate the tensile force T in the chain due to a force $F = 800$ N applied to one of the pedals.
(b) Calculate the average shear stress τ_{aver} in the pins.





Solution 1.6-14 **Bicycle chain**





F = force applied to pedal = 800 N

L = length of crank arm

R = radius of sprocket

MEASUREMENTS (FOR AUTHOR'S BICYCLE)

(1) $L = 162$ mm (2) $R = 90$ mm

(a) TENSILE FORCE T IN CHAIN

$$\Sigma M_{axle} = 0 \quad FL = TR \quad T = \frac{FL}{R}$$

Substitute numerical values:

$$T = \frac{(800\ N)(162\ \text{mm})}{90\ \text{mm}} = 1440\ \text{N} \quad \longleftarrow$$

(b) SHEAR STRESS IN PINS

$$\tau_{aver} = \frac{T/2}{A_{pin}} = \frac{T}{2\left(\frac{\pi d^2}{4}\right)} = \frac{2T}{\pi d^2}$$

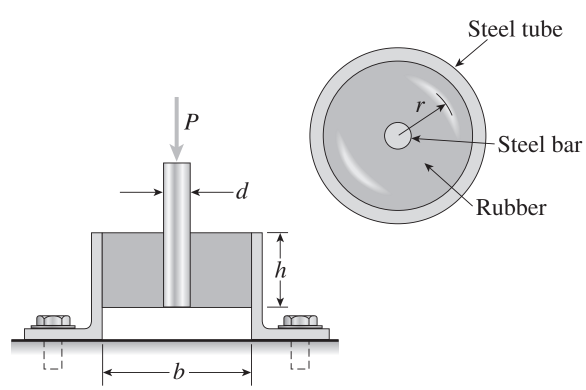
$$= \frac{2FL}{\pi d^2 R}$$

Substitute numerical values:

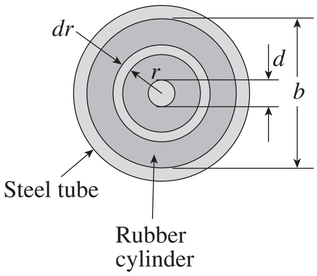
$$\tau_{aver} = \frac{2(800\ \text{N})(162\ \text{mm})}{\pi(2.5\ \text{mm})^2(90\ \text{mm})} = 147\ \text{MPa} \quad \longleftarrow$$

Problem 1.6-15 A shock mount constructed as shown in the figure is used to support a delicate instrument. The mount consists of an outer steel tube with inside diameter b, a central steel bar of diameter d that supports the load P, and a hollow rubber cylinder (height h) bonded to the tube and bar.

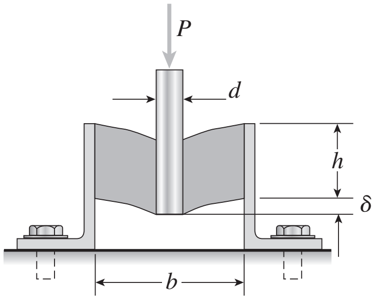
(a) Obtain a formula for the shear stress τ in the rubber at a radial distance r from the center of the shock mount.
(b) Obtain a formula for the downward displacement δ of the central bar due to the load P, assuming that G is the shear modulus of elasticity of the rubber and that the steel tube and bar are rigid.



Solution 1.6-15 Shock mount



r = radial distance from center of shock mount to element of thickness dr



(a) SHEAR STRESS τ AT RADIAL DISTANCE r

A_s = shear area at distance r

$= 2\pi rh$

$$\tau = \frac{P}{A_s} = \frac{P}{2\pi rh} \longleftarrow$$

(b) DOWNWARD DISPLACEMENT δ

γ = shear strain at distance r

$$\gamma = \frac{\tau}{G} = \frac{P}{2\pi rhG}$$

$d\delta$ = downward displacement for element dr

$$d\delta = \gamma dr = \frac{Pdr}{2\pi rhG}$$

$$\delta = \int d\delta = \int_{d/2}^{b/2} \frac{Pdr}{2\pi rhG}$$

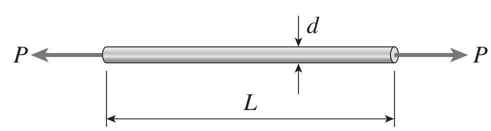
$$\delta = \frac{P}{2\pi hG}\int_{d/2}^{b/2}\frac{dr}{r} = \frac{P}{2\pi hG}[\ln r]_{d/2}^{b/2}$$

$$\delta = \frac{P}{2\pi hG}\ln\frac{b}{d} \longleftarrow$$

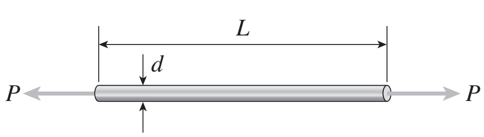
Allowable Stresses and Allowable Loads

Problem 1.7-1 A bar of solid circular cross section is loaded in tension by forces P (see figure). The bar has length $L = 16.0$ in. and diameter $d = 0.50$ in. The material is a magnesium alloy having modulus of elasticity $E = 6.4 \times 10^6$ psi. The allowable stress in tension is $\sigma_{allow} = 17{,}000$ psi, and the elongation of the bar must not exceed 0.04 in.

What is the allowable value of the forces P?



Solution 1.7-1 Magnesium bar in tension



$L = 16.0$ in.

$d = 0.50$ in.

$E = 6.4 \times 10^6$ psi

$\sigma_{allow} = 17{,}000$ psi $\quad \delta_{max} = 0.04$ in.

MAXIMUM LOAD BASED UPON ELONGATION

$$\varepsilon_{max} = \frac{\delta_{max}}{L} = \frac{0.04 \text{ in.}}{16 \text{ in.}} = 0.00250$$

$$\sigma_{max} = E\epsilon_{max} = (6.4 \times 10^6 \text{ psi})(0.00250)$$
$$= 16{,}000 \text{ psi}$$

$$P_{max} = \sigma_{max}A = (16{,}000 \text{ psi})\left(\frac{\pi}{4}\right)(0.50 \text{ in.})^2$$
$$= 3140 \text{ lb}$$

MAXIMUM LOAD BASED UPON TENSILE STRESS

$$P_{max} = \sigma_{allow}A = (17{,}000 \text{ psi})\left(\frac{\pi}{4}\right)(0.50 \text{ in.})^2$$
$$= 3340 \text{ lb}$$

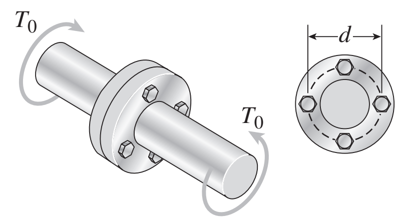
ALLOWABLE LOAD

Elongation governs.

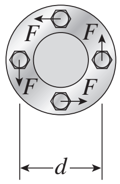
$P_{allow} = 3140$ lb $\longleftarrow$

Problem 1.7-2 A torque T_0 is transmitted between two flanged shafts by means of four 20-mm bolts (see figure). The diameter of the bolt circle is $d = 150$ mm.

If the allowable shear stress in the bolts is 90 MPa, what is the maximum permissible torque? (Disregard friction between the flanges.)



Solution 1.7-2 Shafts with flanges



T_0 = torque transmitted by bolts

d_B = bolt diameter = 20 mm

d = diameter of bolt circle

= 150 mm

$\tau_{allow} = 90$ MPa

F = shear force in one bolt

$$T_0 = 4F\left(\frac{d}{2}\right) = 2Fd$$

ALLOWABLE SHEAR FORCE IN ONE BOLT

$$F = \tau_{allow}A_{bolt} = (90 \text{ MPa})\left(\frac{\pi}{4}\right)(20 \text{ mm})^2$$

$$= 28.27 \text{ kN}$$

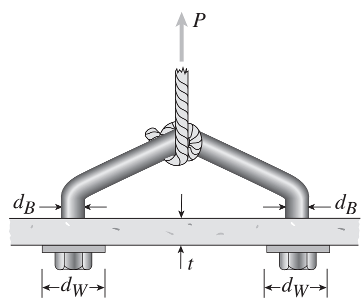
MAXIMUM TORQUE

$$T_0 = 2Fd = 2(28.27 \text{ kN})(150 \text{ mm})$$

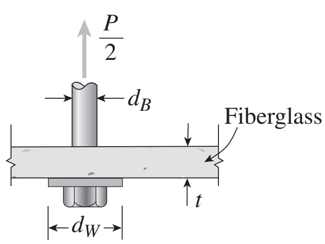
$$= 8.48 \text{ kN}\cdot\text{m} \quad \longleftarrow$$

Problem 1.7-3 A tie-down on the deck of a sailboat consists of a bent bar bolted at both ends, as shown in the figure. The diameter d_B of the bar is ¼ in., the diameter d_W of the washers is ⅞ in., and the thickness t of the fiberglass deck is ⅜ in.

If the allowable shear stress in the fiberglass is 300 psi, and the allowable bearing pressure between the washer and the fiberglass is 550 psi, what is the allowable load P_{allow} on the tie-down?



Solution 1.7-3 Bolts through fiberglass



$$d_B = \frac{1}{4} \text{ in.}$$

$$d_W = \frac{7}{8} \text{ in.}$$

$$t = \frac{3}{8} \text{ in.}$$

ALLOWABLE LOAD BASED UPON SHEAR STRESS IN FIBERGLASS

$\tau_{allow} = 300$ psi

Shear area $A_s = \pi d_W t$

$$\frac{P_1}{2} = \tau_{allow}A_s = \tau_{allow}(\pi d_W t)$$

$$= (300 \text{ psi})(\pi)\left(\frac{7}{8} \text{ in.}\right)\left(\frac{3}{8} \text{ in.}\right)$$

$$\frac{P_1}{2} = 309.3 \text{ lb}$$

$$P_1 = 619 \text{ lb}$$

ALLOWABLE LOAD BASED UPON BEARING PRESSURE

$\sigma_b = 550$ psi

Bearing area $A_b = \frac{\pi}{4}(d_W^2 - d_B^2)$

$$\frac{P_2}{2} = \sigma_b A_b = (550 \text{ psi})\left(\frac{\pi}{4}\right)\left[\left(\frac{7}{8} \text{ in.}\right)^2 - \left(\frac{1}{4} \text{ in.}\right)^2\right]$$

$$= 303.7 \text{ lb}$$

$$P_2 = 607 \text{ lb}$$

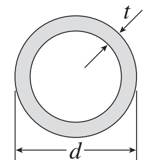
ALLOWABLE LOAD

Bearing pressure governs.

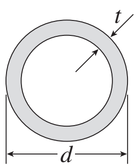
$$P_{allow} = 607 \text{ lb} \quad \longleftarrow$$

Problem 1.7-4 An aluminum tube serving as a compression brace in the fuselage of a small airplane has the cross section shown in the figure. The outer diameter of the tube is $d = 25$ mm and the wall thickness is $t = 2.5$ mm. The yield stress for the aluminum is $\sigma_Y = 270$ MPa and the ultimate stress is $\sigma_U = 310$ MPa.

Calculate the allowable compressive force P_{allow} if the factors of safety with respect to the yield stress and the ultimate stress are 4 and 5, respectively.



Solution 1.7-4 **Aluminum tube in compression**



$d = 25$ mm

$t = 2.5$ mm

$d_0 =$ inner diameter

$= 20$ mm

$$A_{\text{tube}} = \frac{\pi}{4}(d^2 - d_0^2) = 176.7 \text{ mm}^2$$

YIELD STRESS

$\sigma_Y = 270$ MPa

F.S. = 4

$$\sigma_{\text{allow}} = \frac{270 \text{ MPa}}{4} = 67.5 \text{ MPa}$$

ULTIMATE STRESS

$\sigma_U = 310$ MPa

F.S. = 5

$$\sigma_{\text{allow}} = \frac{310 \text{ MPa}}{5} = 62 \text{ MPa}$$

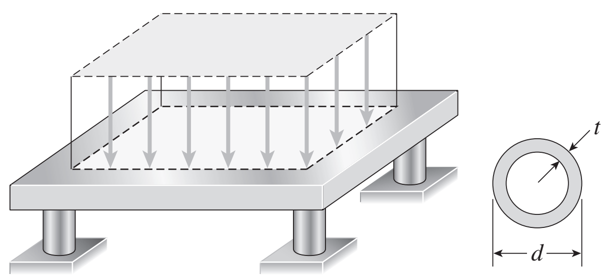
The ultimate stress governs.

ALLOWABLE COMPRESSIVE FORCE

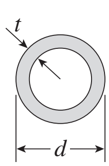
$$P_{\text{allow}} = \sigma_{\text{allow}} A_{\text{tube}} = (62 \text{ MPa})(176.7 \text{ mm}^2) = 11.0 \text{ kN} \longleftarrow$$

Problem 1.7-5 A steel pad supporting heavy machinery rests on four short, hollow, cast iron piers (see figure). The ultimate strength of the cast iron in compression is 50 ksi. The outer diameter of the piers is $d = 4.5$ in. and the wall thickness is $t = 0.40$ in.

Using a factor of safety of 3.5 with respect to the ultimate strength, determine the total load P that may be supported by the pad.



Solution 1.7-5 **Cast iron piers in compression**



Four piers

$\sigma_U = 50$ ksi

$n = 3.5$

$$\sigma_{\text{allow}} = \frac{\sigma_U}{n} = \frac{50 \text{ ksi}}{3.5} = 14.29 \text{ ksi}$$

$d = 4.5$ in.

$t = 0.4$ in.

$d_0 = d - 2t = 3.7$ in.

$$A = \frac{\pi}{4}(d^2 - d_o^2) = \frac{\pi}{4}[(4.5 \text{ in.})^2 - (3.7 \text{ in.})^2] = 5.152 \text{ in.}^2$$

$P_1 =$ allowable load on one pier

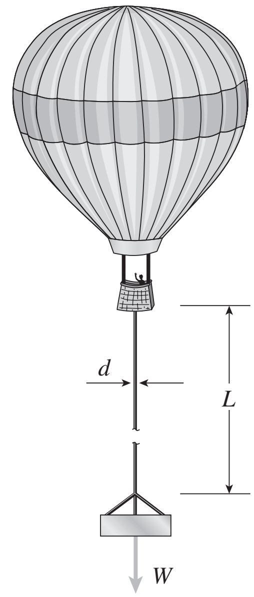
$= \sigma_{\text{allow}} A = (14.29 \text{ ksi})(5.152 \text{ in.}^2)$

$= 73.62$ k

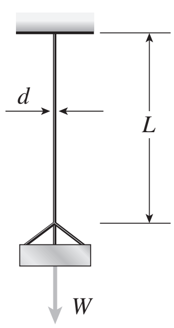
Total load $P = 4P_1 = 294 \text{ k} \longleftarrow$

Problem 1.7-6 A long steel wire hanging from a balloon carries a weight W at its lower end (see figure). The 4-mm diameter wire is 25 m long.

What is the maximum weight W_{max} that can safely be carried if the tensile yield stress for the wire is $\sigma_Y = 350$ MPa and a margin of safety against yielding of 1.5 is desired? (Include the weight of the wire in the calculations.)



Solution 1.7-6 **Wire hanging from a balloon**



$d = 4.0$ mm

$L = 25$ m

$\sigma_Y = 350$ MPa

Margin of safety = 1.5

Factor of safety = $n = 2.5$

$$\sigma_{allow} = \frac{\sigma_Y}{n} = 140 \text{ MPa}$$

Weight density of steel: $\gamma = 77.0$ kN/m^3

Weight of wire:

$$W_0 = \gamma A L = \gamma\left(\frac{\pi d^2}{4}\right)(L)$$

$$W_0 = (77.0 \text{ kN/m}^3)\left(\frac{\pi}{4}\right)(4.0 \text{ mm})^2(25 \text{ m})$$

$$= 24.19 \text{ N}$$

Total load $P = W_{max} + W_0 = \sigma_{allow} A$

$$W_{max} = \sigma_{allow} A - W_0$$

$$= (140 \text{ MPa})\left(\frac{\pi d^2}{4}\right) - 24.19 \text{ N}$$

$$= (140 \text{ MPa})\left(\frac{\pi}{4}\right)(4.0 \text{ mm})^2 - 24.19 \text{ N}$$

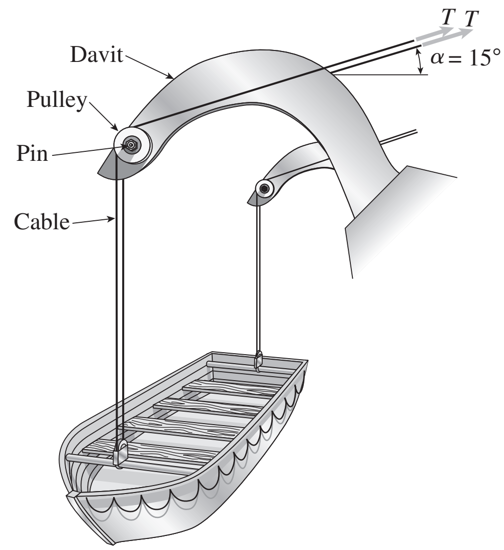
$$= 1759.3 \text{ N} - 24.2 \text{ N} = 1735.1 \text{ N}$$

$$W_{max} = 1740 \text{ N} \longleftarrow$$

Problem 1.7-7 A lifeboat hangs from two ship's davits, as shown in the figure. A pin of diameter $d = 0.80$ in. passes through each davit and supports two pulleys, one on each side of the davit.

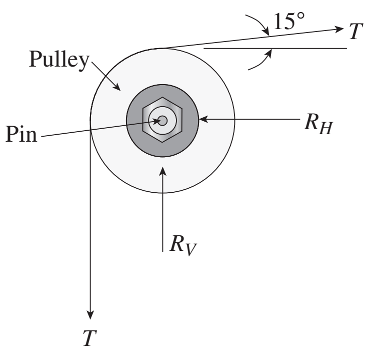
Cables attached to the lifeboat pass over the pulleys and wind around winches that raise and lower the lifeboat. The lower parts of the cables are vertical and the upper parts make an angle $\alpha = 15°$ with the horizontal. The allowable tensile force in each cable is 1800 lb, and the allowable shear stress in the pins is 4000 psi.

If the lifeboat weighs 1500 lb, what is the maximum weight that should be carried in the lifeboat?



Solution 1.7-7 Lifeboat supported by four cables

FREE-BODY DIAGRAM OF ONE PULLEY



Pin diameter $d = 0.80$ in.

T = tensile force in one cable

$T_{allow} = 1800$ lb

$\tau_{allow} = 4000$ psi

W = weight of lifeboat

$= 1500$ lb

$\Sigma F_{horiz} = 0 \qquad R_H = T \cos 15° = 0.9659T$

$\Sigma F_{vert} = 0 \qquad R_V = T - T \sin 15° = 0.7412T$

V = shear force in pin

$$V = \sqrt{(R_H)^2 + (R_V)^2} = 1.2175T$$

ALLOWABLE TENSILE FORCE IN ONE CABLE BASED UPON SHEAR IN THE PINS

$$V_{allow} = \tau_{allow} A_{pin} = (4000 \text{ psi})\left(\frac{\pi}{4}\right)(0.80 \text{ in.})^2$$

$$= 2011 \text{ lb}$$

$$V = 1.2175T \qquad T_1 = \frac{V_{allow}}{1.2175} = 1652 \text{ lb}$$

ALLOWABLE FORCE IN ONE CABLE BASED UPON TENSION IN THE CABLE

$T_2 = T_{allow} = 1800$ lb

MAXIMUM WEIGHT

Shear in the pins governs.

$T_{max} = T_1 = 1652$ lb

Total tensile force in four cables

$= 4T_{max} = 6608$ lb

$W_{max} = 4T_{max} - W$

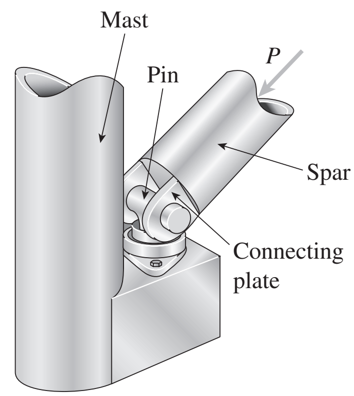
$= 6608 \text{ lb} - 1500 \text{ lb}$

$= 5110 \text{ lb} \longleftarrow$

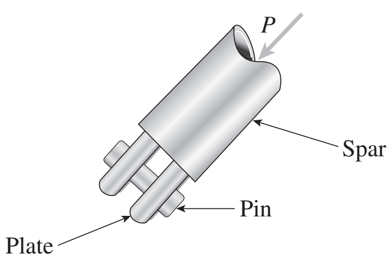
Problem 1.7-8 A ship's spar is attached at the base of a mast by a pin connection (see figure). The spar is a steel tube of outer diameter $d_2 = 80$ mm and inner diameter $d_1 = 70$ mm. The steel pin has diameter $d = 25$ mm, and the two plates connecting the spar to the pin have thickness $t = 12$ mm.

The allowable stresses are as follows: compressive stress in the spar, 70 MPa; shear stress in the pin, 45 MPa; and bearing stress between the pin and the connecting plates, 110 MPa.

Determine the allowable compressive force P_{allow} in the spar.



Solution 1.7-8 Pin connection for a ship's spar



Spar: $d_2 = 80\,\text{mm}$

$d_1 = 70\,\text{mm}$

Pin: $d = 25\,\text{mm}$

Plates: $t = 12\,\text{mm}$

Allowable load P based upon compression in the spar

$\sigma_c = 70$ MPa

$$A_c = \frac{\pi}{4}(d_2^2 - d_1^2) = \frac{\pi}{4}[(80\text{ mm})^2 - (70\text{ mm})^2]$$

$$= 1178.1\,\text{mm}^2$$

$$P_1 = \sigma_c A_c = (70\text{ MPa})(1178.1\,\text{mm}^2) = 82.5\,\text{kN}$$

Allowable load P based upon shear in the pin (double shear)

$\tau_{allow} = 45$ MPa

$$A_s = 2\left(\frac{\pi d^2}{4}\right) = \frac{\pi}{2}(25\text{ mm})^2 = 981.7\text{ mm}^2$$

$$P_2 = \tau_{allow} A_s = (45\text{ MPa})(981.7\,\text{mm}^2) = 44.2\text{ kN}$$

Allowable load P based upon bearing

$\sigma_b = 110$ MPa

$$A_b = 2dt = 2(25\,\text{mm})(12\,\text{mm}) = 600\,\text{mm}^2$$

$$P_3 = \sigma_b A_b = (110\,\text{MPa})(600\,\text{mm}^2) = 66.0\,\text{kN}$$

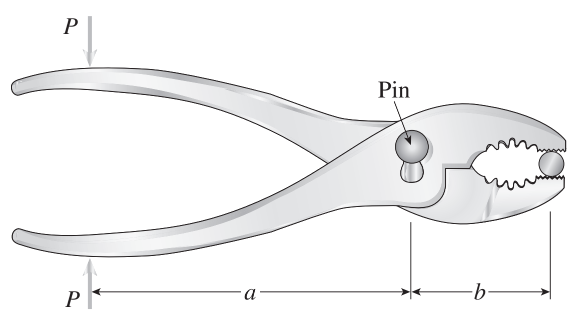
Allowable compressive load in the spar

Shear in the pin governs.

$P_{allow} = 44.2\,\text{kN}$ ←

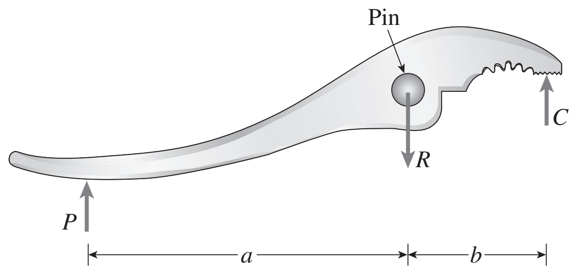
Problem 1.7-9 What is the maximum possible value of the clamping force C in the jaws of the pliers shown in the figure if $a = 3.75$ in., $b = 1.60$ in., and the ultimate shear stress in the 0.20-in. diameter pin is 50 ksi?

What is the maximum permissible value of the applied load P if a factor of safety of 3.0 with respect to failure of the pin is to be maintained?



Solution 1.7-9 **Forces in pliers**

FREE-BODY DIAGRAM OF ONE ARM



C = clamping force

R = reaction at pin

$a = 3.75$ in.

$b = 1.60$ in.

d = diameter of pin

$= 0.20$ in.

$\Sigma M_{pin} = 0 \quad Cb - Pa = 0$

$$C = \frac{Pa}{b} \quad P = \frac{Cb}{a} \quad \frac{C}{P} = \frac{a}{b}$$

$\Sigma F_{vert} = 0 \quad \uparrow^{+} \downarrow^{-} \quad P + C - R = 0$

$$R = P + C = P\left(1 + \frac{a}{b}\right) = C\left(1 + \frac{b}{a}\right)$$

V = shear force in pin (single shear)

$$V = R \qquad \therefore\ C = \frac{V}{1 + \frac{b}{a}} \quad \text{and} \quad P = \frac{V}{1 + \frac{a}{b}}$$

MAXIMUM CLAMPING FORCE C_{ult}

$\tau_{ult} = 50$ ksi

$$V_{ult} = \tau_{ult} A_{pin}$$

$$= (50\text{ ksi})\left(\frac{\pi}{4}\right)(0.20\text{ in.})^2$$

$$= 1571\text{ lb}$$

$$C_{ult} = \frac{V_{ult}}{1 + \frac{b}{a}} = \frac{1571\text{ lb}}{1 + \frac{1.60\text{ in.}}{3.75\text{ in.}}}$$

$$= 1100\text{ lb} \longleftarrow$$

MAXIMUM LOAD P_{ult}

$$P_{ult} = \frac{V_{ult}}{1 + \frac{a}{b}} = \frac{1571\text{ lb}}{1 + \frac{3.75\text{ in.}}{1.60\text{ in.}}} = 469.8\text{ lb}$$

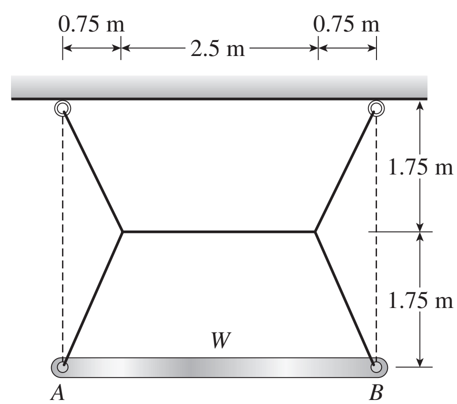
ALLOWABLE LOAD P_{allow}

$$P_{allow} = \frac{P_{ult}}{n} = \frac{469.8\text{ lb}}{3.0}$$

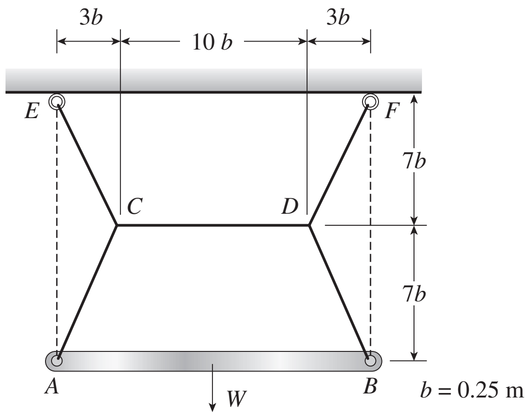
$$= 157\text{ lb} \longleftarrow$$

Problem 1.7-10 A metal bar AB of weight W is suspended by a system of steel wires arranged as shown in the figure. The diameter of the wires is 2 mm, and the yield stress of the steel is 450 MPa.

Determine the maximum permissible weight W_{max} for a factor of safety of 1.9 with respect to yielding.

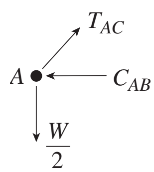


Solution 1.7-10 **Bar *AB* suspended by steel wires**



$$L_{AC} = L_{EC} = \sqrt{(3b)^2 + (7b)^2} = b\sqrt{58}$$

Free-body diagram of point *A*



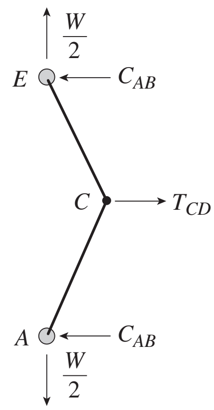
$$\Sigma F_{\text{vert}} = 0 \quad T_{AC}\left(\frac{7b}{b\sqrt{58}}\right) = \frac{W}{2}$$

$$T_{AC} = \frac{W\sqrt{58}}{14}$$

$$\Sigma F_{\text{horiz}} = 0 \quad T_{AC}\left(\frac{3b}{b\sqrt{58}}\right) = C_{AB}$$

$$C_{AB} = \frac{3W}{14}$$

Free-body diagram of wire *ACE*



$$\Sigma F_{\text{horiz}} = 0$$

$$T_{CD} = 2C_{AB} = \frac{3W}{7}$$

Allowable tensile force in a wire

$d = 2\,\text{mm} \qquad \sigma_Y = 450\,\text{MPa} \qquad \text{F.S.} = 1.9$

$$T_{\text{allow}} = \frac{\sigma_Y A}{n} = \frac{\sigma_Y\left(\frac{\pi d^2}{4}\right)}{n} = \left(\frac{450\text{ MPa}}{1.9}\right)\left(\frac{\pi}{4}\right)(2\text{ mm})^2 = 744.1\text{ N}$$

Maximum tensile forces in wires

$$T_{CD} = \frac{3W}{7} \qquad T_{AC} = \frac{W\sqrt{58}}{14}$$

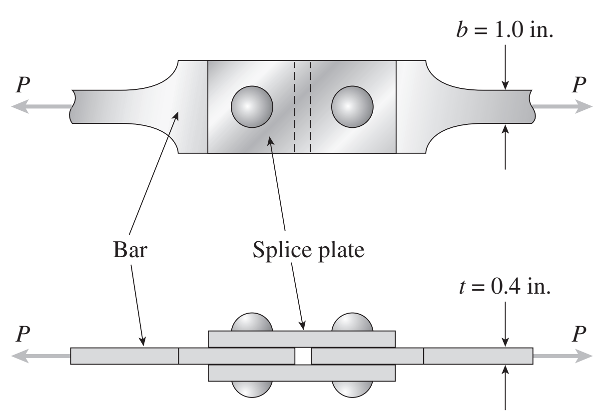
Force in wire *AC* is larger.

Maximum allowable weight *W*

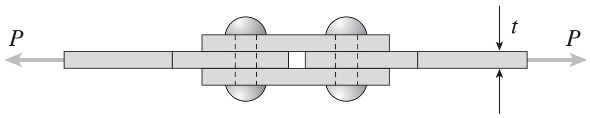
$$W_{\max} = \frac{14\,T_{AC}}{\sqrt{58}} = \frac{14\,T_{\text{allow}}}{\sqrt{58}} = \frac{14}{\sqrt{58}}(744.1\text{ N}) = 1370\text{ N} \longleftarrow$$

Problem 1.7-11 Two flat bars loaded in tension by forces P are spliced using two rectangular splice plates and two ⅝-in. diameter rivets (see figure). The bars have width $b = 1.0$ in. (except at the splice, where the bars are wider) and thickness $t = 0.4$ in. The bars are made of steel having an ultimate stress in tension equal to 60 ksi. The ultimate stresses in shear and bearing for the rivet steel are 25 ksi and 80 ksi, respectively.

Determine the allowable load P_{allow} if a safety factor of 2.5 is desired with respect to the ultimate load that can be carried. (Consider tension in the bars, shear in the rivets, and bearing between the rivets and the bars. Disregard friction between the plates.)



Solution 1.7-11 **Splice between two flat bars**



ULTIMATE LOAD BASED UPON TENSION IN THE BARS

Cross-sectional area of bars:

$A = bt \qquad b = 1.0\text{ in.} \qquad t = 0.4\text{ in.}$

$A = 0.40\text{ in.}^2$

$P_1 = \sigma_{ult}A = (60\text{ ksi})(0.40\text{ in.}^2) = 24.0\text{ k}$

ULTIMATE LOAD BASED UPON SHEAR IN THE RIVETS

Double shear $\qquad d$ = diameter of rivets

$d = 5/8$ in. $\qquad A_R$ = area of rivets

$$A_R = \frac{\pi d^2}{4} = \frac{\pi}{4}\left(\frac{5}{8}\text{ in.}\right)^2 = 0.3068\text{ in.}^2$$

$$P_2 = \tau_{ult}(2A_R) = 2(25\text{ ksi})(0.3068\text{ in.}^2)$$

$$= 15.34\text{ k}$$

ULTIMATE LOAD BASED UPON BEARING

A_b = bearing area = dt

$$P_3 = \sigma_b A_b = (80\text{ ksi})\left(\frac{5}{8}\text{ in.}\right)(0.4\text{ in.}) = 20.0\text{ k}$$

ULTIMATE LOAD

Shear governs. $P_{ult} = 15.34\text{ k}$

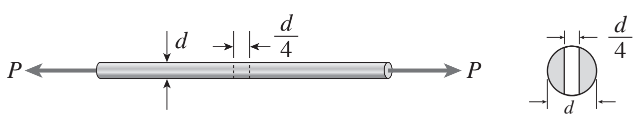
ALLOWABLE LOAD

$$P_{allow} = \frac{P_{ult}}{n} = \frac{15.34\text{ k}}{2.5} = 6.14\text{ k} \quad \longleftarrow$$

Problem 1.7-12 A solid bar of circular cross section (diameter d) has a hole of diameter $d/4$ drilled laterally through the center of the bar (see figure). The allowable average tensile stress on the net cross section of the bar is σ_{allow}.

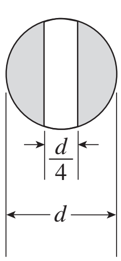
(a) Obtain a formula for the allowable load P_{allow} that the bar can carry in tension.
(b) Calculate the value of P_{allow} if the bar is made of brass with diameter $d = 40$ mm and $\sigma_{allow} = 80$ MPa.

(*Hint:* Use the formulas of Case 15, Appendix D.)



Solution 1.7-12 **Bar with a hole**

CROSS SECTION OF BAR



From Case 15, Appendix D:

$$A = 2r^2\left(\alpha - \frac{ab}{r^2}\right)$$

$$r = \frac{d}{2} \qquad a = \frac{d}{8}$$

$$b = \sqrt{r^2 - \left(\frac{d}{8}\right)^2} = d\sqrt{\frac{15}{64}} = \frac{d}{8}\sqrt{15}$$

$$\alpha = \arccos\frac{d/8}{r}$$

$$= \arccos\left(\frac{1}{4}\right)$$

$$A = 2\left(\frac{d}{2}\right)^2\left[\arccos\frac{1}{4} - \frac{\left(\frac{d}{8}\right)\left(\frac{d}{8}\sqrt{15}\right)}{(d/2)^2}\right]$$

$$= \frac{d^2}{2}\left(\arccos\frac{1}{4} - \frac{\sqrt{15}}{16}\right) = 0.5380\, d^2$$

(a) ALLOWABLE LOAD IN TENSION

$$P_{allow} = \sigma_{allow}A = 0.5380 d^2\,\sigma_{allow} \quad \longleftarrow$$

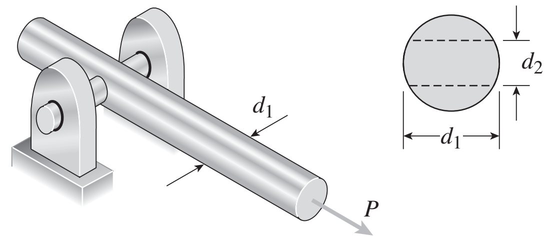
(b) SUBSTITUTE NUMERICAL VALUES

$\sigma_{allow} = 80\text{ MPa} \qquad d = 40\text{ mm}$

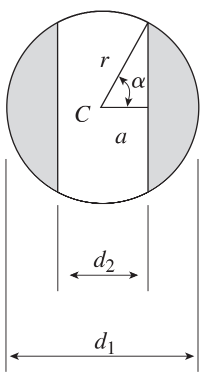
$P_{allow} = 68.9\text{ kN} \quad \longleftarrow$

Problem 1.7-13 A solid steel bar of diameter $d_1 = 2.25$ in. has a hole of diameter $d_2 = 1.125$ in. drilled through it (see figure). A steel pin of diameter d_2 passes through the hole and is attached to supports.

Determine the maximum permissible tensile load P_{allow} in the bar if the yield stress for shear in the pin is $\tau_Y = 17{,}000$ psi, the yield stress for tension in the bar is $\sigma_Y = 36{,}000$ psi, and a factor of safety of 2.0 with respect to yielding is required. (*Hint:* Use the formulas of Case 15, Appendix D.)



Solution 1.7-13 Bar with a hole



$d_1 = 2.25$ in.

$d_2 = 1.125$ in.

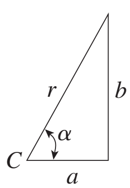
From Case 15, Appendix D:

$$A = 2r^2\left(\alpha - \frac{ab}{r^2}\right)$$

$$r = \frac{d_1}{2} = 1.125 \text{ in.}$$

$$\alpha = \arccos\frac{d_2/2}{d_1/2} = \arccos\frac{d_2}{d_1}$$

$$\frac{d_2}{d_1} = \frac{1.125 \text{ in.}}{2.25 \text{ in.}} = \frac{1}{2} \quad \alpha = \arccos\frac{1}{2} = 1.0472 \text{ rad}$$



$$a = \frac{d_2}{2} = 0.5625 \text{ in.}$$

$$b = \sqrt{r^2 - a^2} = 0.9743 \text{ in.}$$

$$A = 2r^2\left(\alpha - \frac{ab}{r^2}\right)$$

$$A = 2(1.125 \text{ in.})^2\left[1.0472 - \frac{(0.5625 \text{ in.})(0.9743 \text{ in.})}{(1.125 \text{ in.})^2}\right]$$

$$= 1.5546 \text{ in.}^2$$

ALLOWABLE LOAD BASED ON TENSION IN THE BAR

$$P_1 = \frac{\sigma_Y}{n}A = \frac{36{,}000 \text{ psi}}{2.0}(1.5546 \text{ in.}^2)$$

$$= 28.0 \text{ k}$$

ALLOWABLE LOAD BASED ON SHEAR IN THE PIN

Double shear

$$A_s = 2A_{pin} = 2\left(\frac{\pi d_2^2}{4}\right) = \frac{\pi}{2}(1.125 \text{ in.})^2$$

$$= 1.9880 \text{ in.}^2$$

$$P_2 = \frac{\tau_Y}{n}A_s = \frac{17{,}000 \text{ psi}}{2.0}(1.9880 \text{ in.})^2$$

$$= 16.9 \text{ k}$$

ALLOWABLE LOAD

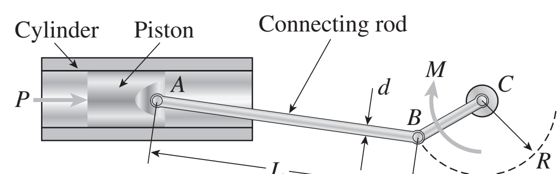
Shear in the pin governs.

$$P_{allow} = 16.9 \text{ k} \longleftarrow$$

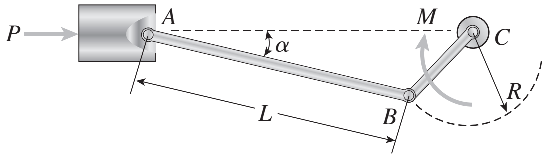
Problem 1.7-14 The piston in an engine is attached to a connecting rod *AB*, which in turn is connected to a crank arm *BC* (see figure). The piston slides without friction in a cylinder and is subjected to a force *P* (assumed to be constant) while moving to the right in the figure. The connecting rod, which has diameter *d* and length *L*, is attached at both ends by pins. The crank arm rotates about the axle at *C* with the pin at *B* moving in a circle of radius *R*. The axle at *C*, which is supported by bearings, exerts a resisting moment *M* against the crank arm.

(a) Obtain a formula for the maximum permissible force P_{allow} based upon an allowable compressive stress σ_c in the connecting rod.

(b) Calculate the force P_{allow} for the following data: $\sigma_c = 160$ MPa, $d = 9.00$ mm, and $R = 0.28L$.

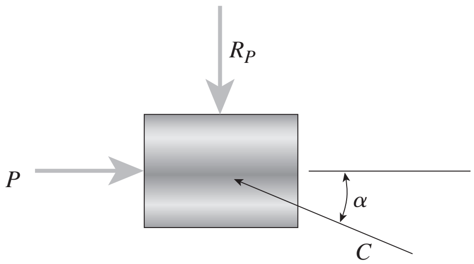


Solution 1.7-14 Piston and connecting rod



d = diameter of rod AB

FREE-BODY DIAGRAM OF PISTON



P = applied force (constant)

C = compressive force in connecting rod

R_P = resultant of reaction forces between cylinder and piston (no friction)

$\Sigma F_{\text{horiz}} = 0 \underset{+}{\rightarrow}\underset{-}{\leftarrow} \qquad P - C\cos\alpha = 0$

$P = C\cos\alpha$

MAXIMUM COMPRESSIVE FORCE C IN CONNECTING ROD

$$C_{\max} = \sigma_c A_c$$

in which A_c = area of connecting rod

$$A_c = \frac{\pi d^2}{4}$$

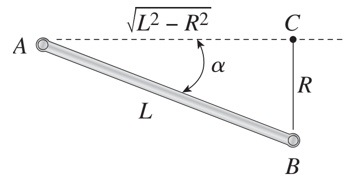
MAXIMUM ALLOWABLE FORCE P

$$P = C_{\max}\cos\alpha$$

$$= \sigma_c A_c \cos\alpha$$

The maximum allowable force P occurs when $\cos\alpha$ has its smallest value, which means that α has its largest value.

LARGEST VALUE OF α



The largest value of α occurs when point B is the farthest distance from line AC. The farthest distance is the radius R of the crank arm.

Therefore,

$\overline{BC} = R$

Also, $\overline{AC} = \sqrt{L^2 - R^2}$

$$\cos\alpha = \frac{\sqrt{L^2 - R^2}}{L} = \sqrt{1 - \left(\frac{R}{L}\right)^2}$$

(a) MAXIMUM ALLOWABLE FORCE P

$$P_{\text{allow}} = \sigma_c A_c \cos\alpha$$

$$= \sigma_c\left(\frac{\pi d^2}{4}\right)\sqrt{1 - \left(\frac{R}{L}\right)^2} \quad \leftarrow$$

(b) SUBSTITUTE NUMERICAL VALUES

$\sigma_c = 160$ MPa $\qquad d = 9.00$ mm

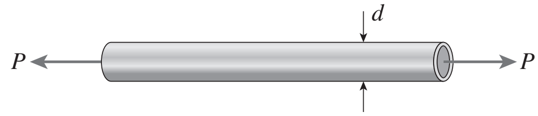
$R = 0.28L \qquad R/L = 0.28$

$P_{\text{allow}} = 9.77$ kN $\leftarrow$

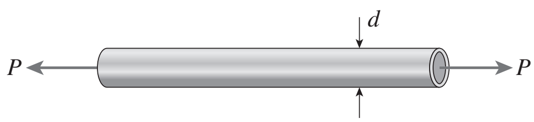
Design for Axial Loads and Direct Shear

Problem 1.8-1 An aluminum tube is required to transmit an axial tensile force $P = 34$ k (see figure). The thickness of the wall of the tube is to be 0.375 in.

What is the minimum required outer diameter d_{min} if the allowable tensile stress is 9000 psi?



Solution 1.8-1 Aluminum tube in tension



$$P = 34\,\text{k}$$

$$t = 0.375\,\text{in.}$$

$$\sigma_{allow} = 9000\,\text{psi}$$

$$A = \frac{\pi}{4}[d^2 - (d - 2t)^2] = \frac{\pi}{4}(4t)(d - t)$$

$$= \pi t(d - t)$$

$$P = \sigma_{allow}\, A = \pi t(d - t)\sigma_{allow}$$

SOLVE FOR d:

$$d = \frac{P}{\pi t \sigma_{allow}} + t$$

SUBSTITUTE NUMERICAL VALUES:

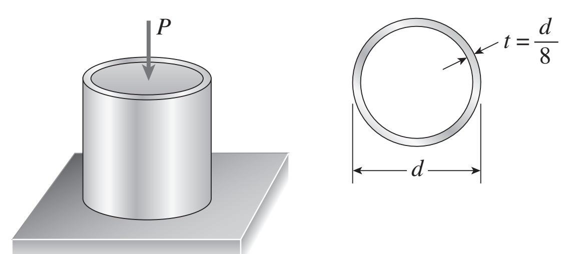
$$d_{min} = \frac{34\text{ k}}{\pi(0.375\text{ in.})(9000\text{ psi})} + 0.375\text{ in.}$$

$$= 3.207\text{ in.} + 0.375\text{ in.}$$

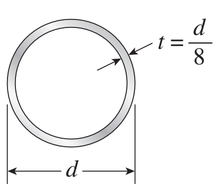
$$d_{min} = 3.58\text{ in.} \longleftarrow$$

Problem 1.8-2 A steel pipe having yield stress $\sigma_Y = 270$ MPa is to carry an axial compressive load $P = 1200$ kN (see figure). A factor of safety of 1.8 against yielding is to be used.

If the thickness t of the pipe is to be one-eighth of its outer diameter, what is the minimum required outer diameter d_{min}?



Solution 1.8-2 Steel pipe in compression



$$P = 1200\,\text{kN}$$

$$\sigma_Y = 270\,\text{MPa}$$

$$n = 1.8$$

$$\sigma_{allow} = 150\,\text{MPa}$$

$$A = \frac{\pi}{4}\left[d^2 - \left(d - \frac{d}{4}\right)^2\right] = \frac{7\pi d^2}{64}$$

$$P = \sigma_{allow}\, A = \frac{7\pi d^2}{64}\sigma_{allow}$$

SOLVE FOR d:

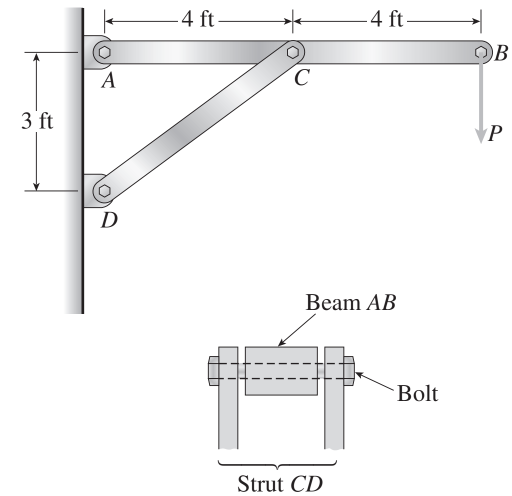
$$d^2 = \frac{64\,P}{7\pi\sigma_{allow}} \qquad d = 8\sqrt{\frac{P}{7\pi\sigma_{allow}}}$$

SUBSTITUTE NUMERICAL VALUES:

$$d_{min} = 8\sqrt{\frac{1200\text{ kN}}{7\pi\,(150\text{ MPa})}} = 153\text{ mm} \longleftarrow$$

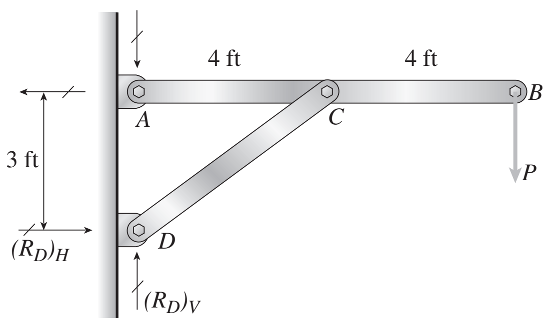
Problem 1.8-3 A horizontal beam *AB* supported by an inclined strut *CD* carries a load $P = 2500$ lb at the position shown in the figure. The strut, which consists of two bars, is connected to the beam by a bolt passing through the three bars meeting at joint *C*.

If the allowable shear stress in the bolt is 14,000 psi, what is the minimum required diameter d_{min} of the bolt?



Solution 1.8-3 **Beam *ACB* supported by a strut *CD***

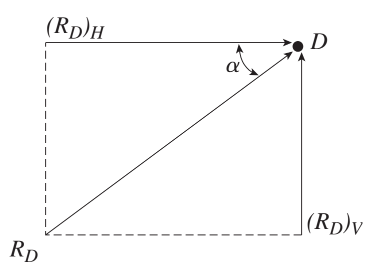
FREE-BODY DIAGRAM

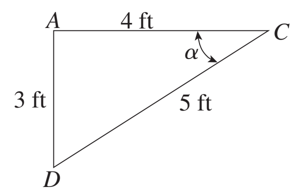


$$\Sigma M_A = 0 \qquad -P(8\text{ ft}) + (R_D)_H(3\text{ ft}) = 0$$

$$(R_D)_H = \frac{8}{3}P$$

REACTION AT JOINT *D*





F_{CD} = compressive force in strut = R_D

$$F_{CD} = (R_D)_H\left(\frac{5}{4}\right) = \left(\frac{5}{4}\right)\left(\frac{8P}{3}\right) = \frac{10P}{3}$$

SHEAR FORCE ACTING ON BOLT

$$V = \frac{F_{CD}}{2} = \frac{5P}{3}$$

REQUIRED AREA AND DIAMETER OF BOLT

$$A = \frac{V}{\tau_{allow}} = \frac{5P}{3\tau_{allow}} \qquad A = \frac{\pi d^2}{4} \qquad d^2 = \frac{20P}{3\pi\tau_{allow}}$$

SUBSTITUTE NUMERICAL VALUES:

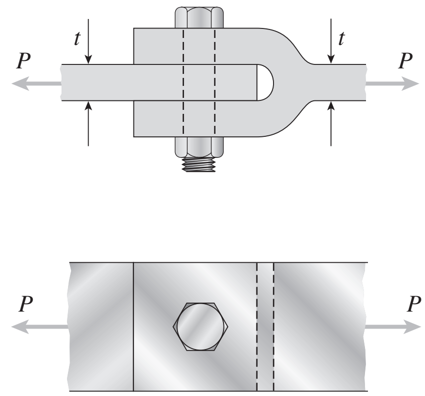
$P = 2500$ lb $\qquad \tau_{allow} = 14{,}000$ psi

$d^2 = 0.3789$ in.^{2}

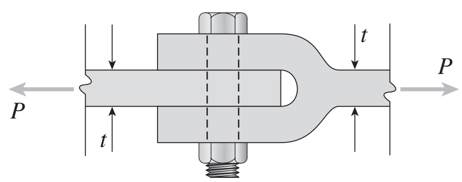
$d_{min} = 0.616$ in. ←

Problem 1.8-4 Two bars of rectangular cross section (thickness $t = 15$ mm) are connected by a bolt in the manner shown in the figure. The allowable shear stress in the bolt is 90 MPa and the allowable bearing stress between the bolt and the bars is 150 MPa.

If the tensile load $P = 31$ kN, what is the minimum required diameter d_{min} of the bolt?



Solution 1.8-4 **Bolted connection**



One bolt in double shear.

$P = 31\,\text{kN}$

$\tau_{allow} = 90\,\text{MPa}$

$\sigma_b = 150\,\text{MPa}$

$t = 15\,\text{mm}$

Find minimum diameter of bolt.

Based upon shear in the bolt

$$A_{bolt} = \frac{P}{2\tau_{allow}} \qquad \frac{\pi d^2}{4} = \frac{P}{2\tau_{allow}}$$

$$d^2 = \frac{2P}{\pi\tau_{allow}}$$

$$d_1 = \sqrt{\frac{2P}{\pi\tau_{allow}}} = \sqrt{\frac{2(31\text{ kN})}{\pi(90\text{ MPa})}}$$

$$= 14.8\text{ mm}$$

Based upon bearing between plate and bolt

$$A_{bearing} = \frac{P}{\sigma_b} \qquad dt = \frac{P}{\sigma_b}$$

$$d = \frac{P}{t\sigma_b} \qquad d_2 = \frac{31\text{ kN}}{(15\text{ mm})(150\text{ MPa})} = 13.8\text{ mm}$$

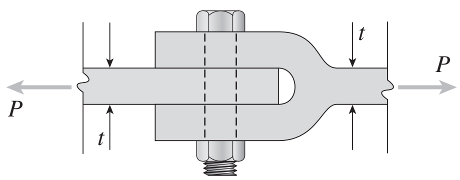
Minimum diameter of bolt

Shear governs.

$d_{min} = 14.8$ mm ⟵

Problem 1.8-5 Solve the preceding problem if the bars have thickness $t = 5/16$ in., the allowable shear stress is 12,000 psi, the allowable bearing stress is 20,000 psi, and the load $P = 1800$ lb.

Solution 1.8-5 Bolted connection



One bolt in double shear.

$P = 1800$ lb

$\tau_{allow} = 12{,}000$ psi

$\sigma_b = 20{,}000$ psi

$t = 5/16$ in.

Find minimum diameter of bolt.

BASED UPON SHEAR IN THE BOLT

$$A_{bolt} = \frac{P}{2\tau_{allow}} \qquad \frac{\pi d^2}{4} = \frac{P}{2\tau_{allow}}$$

$$d^2 = \frac{2P}{\pi\tau_{allow}}$$

$$d_1 = \sqrt{\frac{2P}{\pi\tau_{allow}}} = \sqrt{\frac{2(1800\text{ lb})}{\pi(12{,}000\text{ psi})}} = 0.309\text{ in.}$$

BASED UPON BEARING BETWEEN PLATE AND BOLT

$$A_{bearing} = \frac{P}{\sigma_b} \qquad dt = \frac{P}{\sigma_b}$$

$$d = \frac{P}{t\sigma_b} \qquad d_2 = \frac{1800\text{ lb}}{(\frac{5}{16}\text{ in.})(20{,}000\text{ psi})} = 0.288\text{ in.}$$

MINIMUM DIAMETER OF BOLT

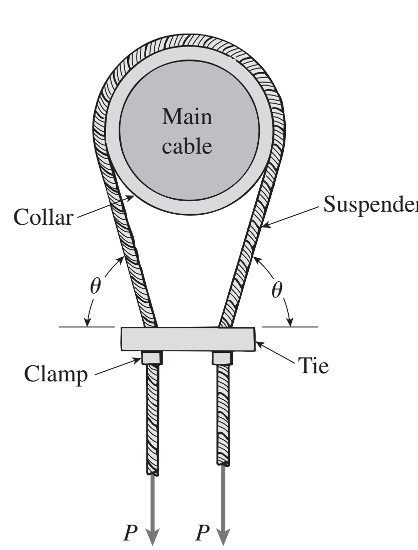
Shear governs.

$d_{min} = 0.309$ in. ⟵

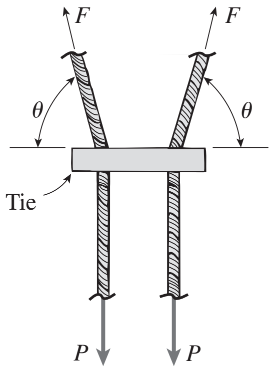
Problem 1.8-6 A suspender on a suspension bridge consists of a cable that passes over the main cable (see figure) and supports the bridge deck, which is far below. The suspender is held in position by a metal tie that is prevented from sliding downward by clamps around the suspender cable.

Let P represent the load in each part of the suspender cable, and let θ represent the angle of the suspender cable just above the tie. Finally, let σ_{allow} represent the allowable tensile stress in the metal tie.

(a) Obtain a formula for the minimum required cross-sectional area of the tie.
(b) Calculate the minimum area if $P = 130$ kN, $\theta = 75°$, and $\sigma_{allow} = 80$ MPa.



Solution 1.8-6 Suspender tie on a suspension bridge



F = tensile force in cable above tie

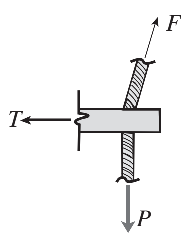
P = tensile force in cable below tie

σ_{allow} = allowable tensile stress in the tie

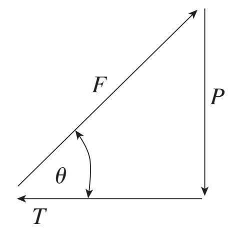
FREE-BODY DIAGRAM OF HALF THE TIE

Note: Include a small amount of the cable in the free-body diagram

T = tensile force in the tie



FORCE TRIANGLE



$$\cot\theta = \frac{T}{P}$$

$$T = P\cot\theta$$

(a) MINIMUM REQUIRED AREA OF TIE

$$A_{min} = \frac{T}{\sigma_{allow}} = \frac{P\cot\theta}{\sigma_{allow}} \quad \longleftarrow$$

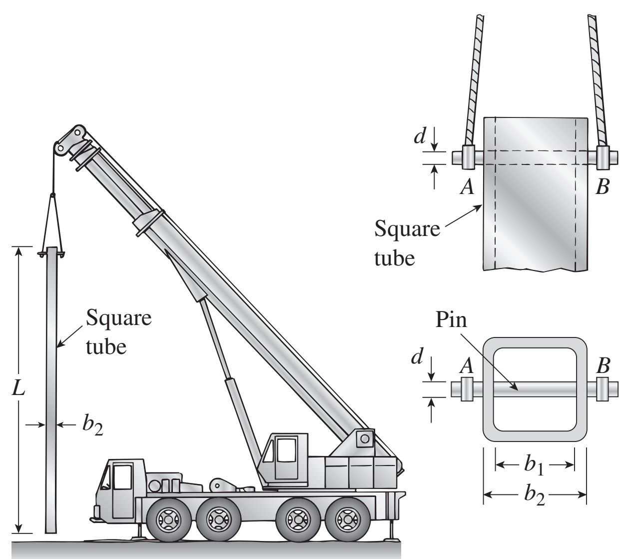
(b) SUBSTITUTE NUMERICAL VALUES:

$P = 130$ kN $\qquad \theta = 75°$ $\qquad \sigma_{allow} = 80$ MPa

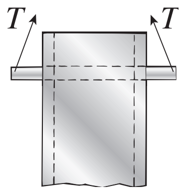
$A_{min} = 435\ \text{mm}^2 \quad \longleftarrow$

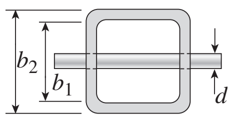
Problem 1.8-7 A square steel tube of length $L = 20$ ft and width $b_2 = 10.0$ in. is hoisted by a crane (see figure). The tube hangs from a pin of diameter d that is held by the cables at points A and B. The cross section is a hollow square with inner dimension $b_1 = 8.5$ in. and outer dimension $b_2 = 10.0$ in. The allowable shear stress in the pin is 8,700 psi, and the allowable bearing stress between the pin and the tube is 13,000 psi.

Determine the minimum diameter of the pin in order to support the weight of the tube. (*Note:* Disregard the rounded corners of the tube when calculating its weight.)



Solution 1.8-7 Tube hoisted by a crane





T = tensile force in cable

W = weight of steel tube

d = diameter of pin

b_1 = inner dimension of tube = 8.5 in.

b_2 = outer dimension of tube = 10.0 in.

L = length of tube = 20 ft

$\tau_{allow} = 8{,}700$ psi

$\sigma_b = 13{,}000$ psi

WEIGHT OF TUBE

γ_s = weight density of steel = 490 lb/ft^3

A = area of tube

$$= b_2^2 - b_1^2 = (10.0 \text{ in.})^2 - (8.5 \text{ in.})^2 = 27.75 \text{ in.}$$

$$W = \gamma_s AL$$

$$= (490 \text{ lb/ft}^3)(27.75 \text{ in.}^2)\left(\frac{1}{144}\ \frac{\text{ft}^2}{\text{in.}}\right)(20 \text{ ft})$$

$$= 1{,}889 \text{ lb}$$

DIAMETER OF PIN BASED UPON SHEAR

Double shear. $2\tau_{allow}A_{pin} = W$

$$2(8{,}700 \text{ psi})\left(\frac{\pi d^2}{4}\right) = 1889 \text{ lb}$$

$$d^2 = 0.1382 \text{ in.}^2 \qquad d_1 = 0.372 \text{ in.}$$

DIAMETER OF PIN BASED UPON BEARING

$$\sigma_b(b_2 - b_1)\, d = W$$

$$(13{,}000 \text{ psi})(10.0 \text{ in.} - 8.5 \text{ in.})\, d = 1{,}889 \text{ lb}$$

$$d_2 = 0.097 \text{ in.}$$

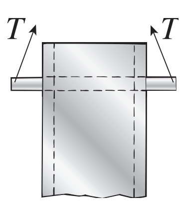
MINIMUM DIAMETER OF PIN

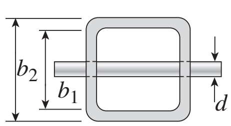
Shear governs.

$d_{min} = 0.372$ in. ⟵

Problem 1.8-8 Solve the preceding problem if the length L of the tube is 6.0 m, the outer width is $b_2 = 250$ mm, the inner dimension is $b_1 = 210$ mm, the allowable shear stress in the pin is 60 MPa, and the allowable bearing stress is 90 MPa.

Solution 1.8-8 Tube hoisted by a crane





T = tensile force in cable

W = weight of steel tube

d = diameter of pin

b_1 = inner dimension of tube = 210 mm

b_2 = outer dimension of tube = 250 mm

L = length of tube = 6.0 m

$\tau_{allow} = 60$ MPa

$\sigma_b = 90$ MPa

WEIGHT OF TUBE

γ_s = weight density of steel = 77.0 kN/m^3

A = area of tube

$$A = b_2^2 - b_1^2 = 18{,}400 \text{ mm}^2$$

$$W = \gamma_s AL = (77.0 \text{ kN/m}^3)(18{,}400 \text{ mm}^2)(6.0 \text{ m})$$

$$= 8.501 \text{ kN}$$

DIAMETER OF PIN BASED UPON SHEAR

Double shear. $2\tau_{allow}A_{pin} = W$

$$2(60 \text{ MPa})\left(\frac{\pi}{4}\right)d^2 = 8.501 \text{ kN} \qquad d^2 = 90.20 \text{ mm}^2$$

$$d_1 = 9.497 \text{ mm}$$

DIAMETER OF PIN BASED UPON BEARING

$$\sigma_b(b_2 - b_1)d = W$$

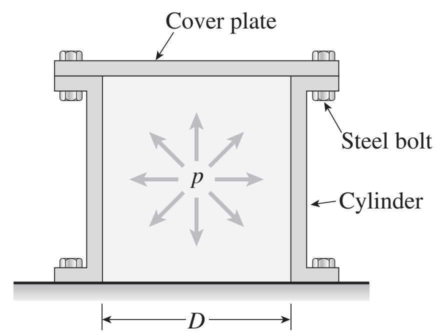
$$(90 \text{ MPa})(40 \text{ mm})d = 8.501 \text{ kN} \qquad d_2 = 2.361 \text{ mm}$$

MINIMUM DIAMETER OF PIN

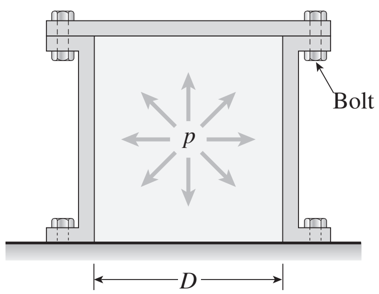
Shear governs. $d_{min} = 9.50$ mm ⟵

Problem 1.8-9 A pressurized circular cylinder has a sealed cover plate fastened with steel bolts (see figure). The pressure p of the gas in the cylinder is 290 psi, the inside diameter D of the cylinder is 10.0 in., and the diameter d_B of the bolts is 0.50 in.

If the allowable tensile stress in the bolts is 10,000 psi, find the number n of bolts needed to fasten the cover.



Solution 1.8-9 **Pressurized cylinder**



$$p = 290\,\text{psi}$$

$$D = 10.0\,\text{in.}$$

$$d_b = 0.50\,\text{in.}$$

$$\sigma_{\text{allow}} = 10{,}000\,\text{psi}$$

n = number of bolts

F = total force acting on the cover plate from the internal pressure

$$F = p\left(\frac{\pi D^2}{4}\right)$$

NUMBER OF BOLTS

P = tensile force in one bolt

$$P = \frac{F}{n} = \frac{\pi p D^2}{4n}$$

$$A_b = \text{area of one bolt} = \frac{\pi}{4} d_b^2$$

$$P = \sigma_{\text{allow}} A_b$$

$$\sigma_{\text{allow}} = \frac{P}{A_b} = \frac{\pi p D^2}{(4n)(\frac{\pi}{4})d_b^2} = \frac{pD^2}{n d_b^2}$$

$$n = \frac{pD^2}{d_b^2 \sigma_{\text{allow}}}$$

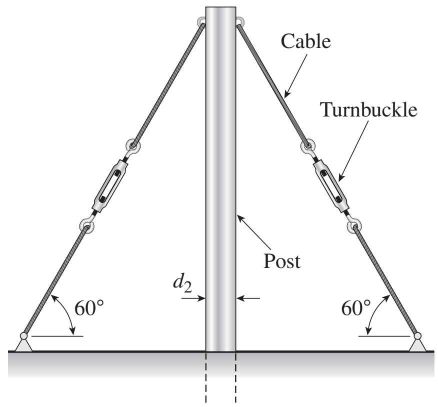
SUBSTITUTE NUMERICAL VALUES:

$$n = \frac{(290\text{ psi})(10\text{ in.})^2}{(0.5\text{ in.})^2(10{,}000\text{ psi})} = 11.6$$

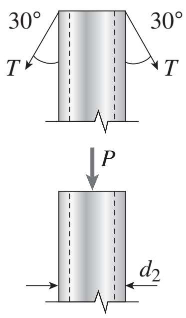
Use 12 bolts ⟵

Problem 1.8-10 A tubular post of outer diameter d_2 is guyed by two cables fitted with turnbuckles (see figure). The cables are tightened by rotating the turnbuckles, thus producing tension in the cables and compression in the post. Both cables are tightened to a tensile force of 110 kN. Also, the angle between the cables and the ground is 60°, and the allowable compressive stress in the post is σ_c = 35 MPa.

If the wall thickness of the post is 15 mm, what is the minimum permissible value of the outer diameter d_2?



Solution 1.8-10 **Tubular post with guy cables**



d_2 = outer diameter

d_1 = inner diameter

t = wall thickness

= 15 mm

T = tensile force in a cable

= 110 kN

σ_{allow} = 35 MPa

P = compressive force in post

$= 2T \cos 30°$

REQUIRED AREA OF POST

$$A = \frac{P}{\sigma_{\text{allow}}} = \frac{2T \cos 30°}{\sigma_{\text{allow}}}$$

AREA OF POST

$$A = \frac{\pi}{4}(d_2^2 - d_1^2) = \frac{\pi}{4}[d_2^2 - (d_2 - 2t)^2]$$

$$= \pi t(d_2 - t)$$

EQUATE AREAS AND SOLVE FOR d_2:

$$\frac{2T \cos 30°}{\sigma_{\text{allow}}} = \pi t(d_2 - t)$$

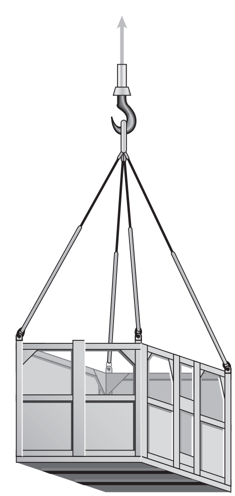
$$d_2 = \frac{2T \cos 30°}{\pi t \sigma_{\text{allow}}} + t \quad \longleftarrow$$

SUBSTITUTE NUMERICAL VALUES:

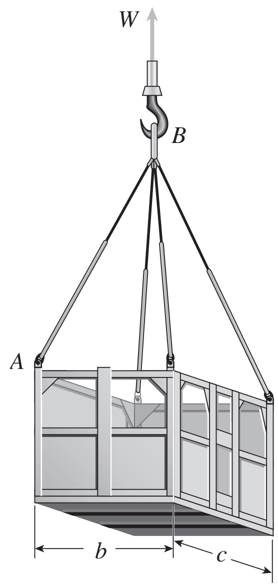
$$(d_2)_{\min} = 131 \text{ mm} \quad \longleftarrow$$

Problem 1.8-11 A cage for transporting workers and supplies on a construction site is hoisted by a crane (see figure). The floor of the cage is rectangular with dimensions 6 ft by 8 ft. Each of the four lifting cables is attached to a corner of the cage and is 13 ft long. The weight of the cage and its contents is limited by regulations to 9600 lb.

Determine the required cross-sectional area A_C of a cable if the breaking stress of a cable is 91 ksi and a factor of safety of 3.5 with respect to failure is desired.



Solution 1.8-11 Cage hoisted by a crane



Dimensions of cage:

$b = 6$ ft $\qquad\qquad c = 8$ ft

Length of a cable: $L = 13$ ft

Weight of cage and contents:

$W = 9600$ lb

Breaking stress of a cable:

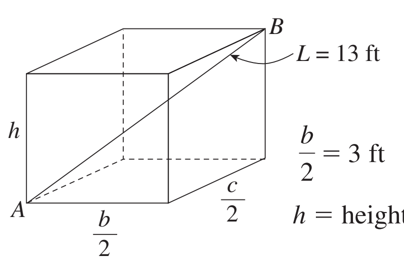
$\sigma_{\text{ult}} = 91$ ksi

Factor of safety: $n = 3.5$

$$\sigma_{\text{allow}} = \frac{\sigma_{\text{ult}}}{n} = \frac{91 \text{ ksi}}{3.5} = 26{,}000 \text{ psi}$$

Geometry of one cable (cable AB)

Point B is above the midpoint of the cage



$\frac{b}{2} = 3$ ft $\qquad \frac{c}{2} = 4$ ft

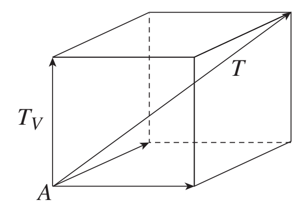
h = height from A to B

From geometry: $\quad L^2 = \left(\frac{b}{2}\right)^2 + \left(\frac{c}{2}\right)^2 + h^2$

$(13 \text{ ft})^2 = (3 \text{ ft})^2 + (4 \text{ ft})^2 + h^2$

Solving, $h = 12$ ft

Force in a cable



T = force in one cable (cable AB)

T_V = vertical component of T

(Each cable carries the same load.)

$$\therefore\ T_V = \frac{W}{4} = \frac{9600 \text{ lb}}{4} = 2400 \text{ lb}$$

$$\frac{T}{T_V} = \frac{L}{h} = \frac{13 \text{ ft}}{12 \text{ ft}}$$

$$T = \frac{13}{12} T_V = 2600 \text{ lb}$$

Required area of cable

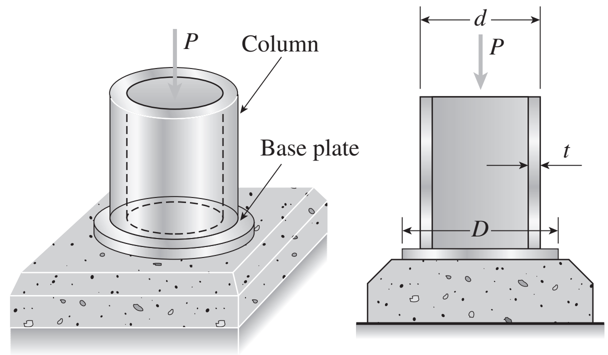
$$A_C = \frac{T}{\sigma_{\text{allow}}} = \frac{2{,}600 \text{ lb}}{26{,}000 \text{ psi}} = 0.100 \text{ in.}^2 \quad \longleftarrow$$

(Note: The diameter of the cable cannot be calculated from the area A_C, because a cable does not have a solid circular cross section. A cable consists of several strands wound together. For details, see Section 2.2.)

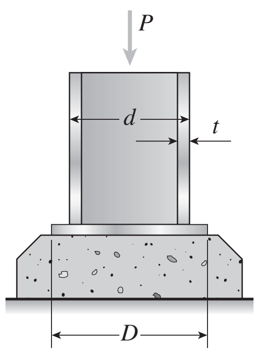
Problem 1.8-12 A steel column of hollow circular cross section is supported on a circular steel base plate and a concrete pedestal (see figure). The column has outside diameter d = 250 mm and supports a load P = 750 kN.

(a) If the allowable stress in the column is 55 MPa, what is the minimum required thickness t? Based upon your result, select a thickness for the column. (Select a thickness that is an even integer, such as 10, 12, 14, . . ., in units of millimeters.)

(b) If the allowable bearing stress on the concrete pedestal is 11.5 MPa, what is the minimum required diameter D of the base plate if it is designed for the allowable load P_{allow} that the column with the selected thickness can support?



Solution 1.8-12 **Hollow circular column**



$d = 250$ mm

$P = 750$ kN

$\sigma_{allow} = 55$ MPa (compression in column)

t = thickness of column

D = diameter of base plate

$\sigma_b = 11.5$ MPa (allowable pressure on concrete)

(a) THICKNESS t OF THE COLUMN

$$A = \frac{P}{\sigma_{allow}} \qquad A = \frac{\pi d^2}{4} - \frac{\pi}{4}(d - 2t)^2$$

$$= \frac{\pi}{4}(4t)(d - t) = \pi t(d - t)$$

$$\pi t(d - t) = \frac{P}{\sigma_{allow}}$$

$$\pi t^2 - \pi t d + \frac{P}{\sigma_{allow}} = 0$$

$$t^2 - dt + \frac{P}{\pi \sigma_{allow}} = 0 \qquad \text{(Eq. 1)}$$

SUBSTITUTE NUMERICAL VALUES IN EQ. (1):

$$t^2 - 250\,t + \frac{(750 \times 10^3\,\text{N})}{\pi(55\,\text{N/mm}^2)} = 0$$

(Note: In this eq., t has units of mm.)

$$t^2 - 250t + 4{,}340.6 = 0$$

Solve the quadratic eq. for t:

$t = 18.77$ mm $\quad t_{min} = 18.8$ mm ⟵

Use $t = 20$ mm ⟵

(b) DIAMETER D OF THE BASE PLATE

For the column, $P_{allow} = \sigma_{allow} A$

where A is the area of the column with $t = 20$ mm.

$A = \pi t(d - t) \quad P_{allow} = \sigma_{allow}\,\pi t(d - t)$

$$\text{Area of base plate} = \frac{\pi D^2}{4} = \frac{P_{allow}}{\sigma_b}$$

$$\frac{\pi D^2}{4} = \frac{\sigma_{allow}\pi t(d - t)}{\sigma_b}$$

$$D^2 = \frac{4\sigma_{allow} t(d - t)}{\sigma_b}$$

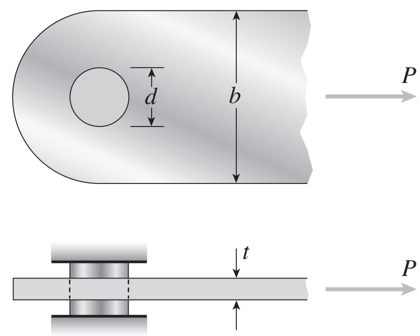
$$= \frac{4(55\text{ MPa})(20\text{ mm})(230\text{ mm})}{11.5\text{ MPa}}$$

$D^2 = 88{,}000\text{ mm}^2 \quad D = 296.6$ mm

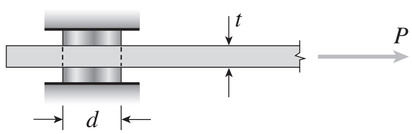
$D_{min} = 297$ mm ⟵

Problem 1.8-13 A bar of rectangular cross section is subjected to an axial load P (see figure). The bar has width $b = 2.0$ in. and thickness $t = 0.25$ in. A hole of diameter d is drilled through the bar to provide for a pin support. The allowable tensile stress on the net cross section of the bar is 20 ksi, and the allowable shear stress in the pin is 11.5 ksi.

(a) Determine the pin diameter d_m for which the load P will be a maximum.
(b) Determine the corresponding value P_{max} of the load.



Solution 1.8-13 **Bar with pin connection**



Width of bar $b = 2$ in.

Thickness $t = 0.25$ in.

$\sigma_{allow} = 20$ ksi

$\tau_{allow} = 11.5$ ksi

d = diameter of pin (inches)

P = axial load (pounds)

Allowable load based upon tension in bar

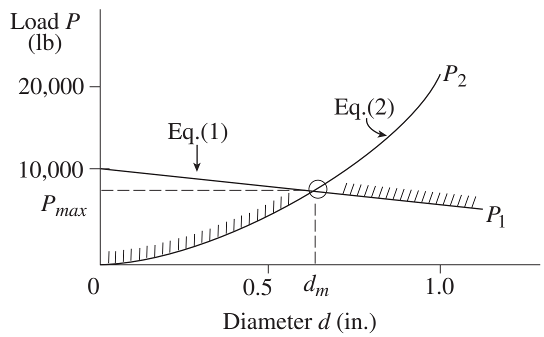
$$P_1 = \sigma_{allow} A_{net} = \sigma_{allow}(b - d)t$$
$$= (20{,}000\text{ psi})(2\text{ in.} - d)(0.25\text{ in.})$$
$$= 5{,}000(2 - d) = 10{,}000 - 5{,}000d \qquad \text{Eq. (1)}$$

Allowable load based upon shear in pin

Double shear

$$P_2 = 2\tau_{allow}\left(\frac{\pi d^2}{4}\right) = \tau_{allow}\left(\frac{\pi d^2}{2}\right)$$
$$= (11{,}500\text{ psi})\left(\frac{\pi d^2}{2}\right) = 18{,}064d^2 \qquad \text{Eq. (2)}$$

Graph of Eqs. (1) and (2)



(a) Maximum load occurs when $P_1 = P_2$

$$10{,}000 - 5{,}000d = 18{,}064d^2$$
$$\text{or } 18{,}064d^2 + 5{,}000d - 10{,}000 = 0$$

Solve quadratic equation:

$d = 0.6184$ in. $\quad d_m = 0.618$ in. ←

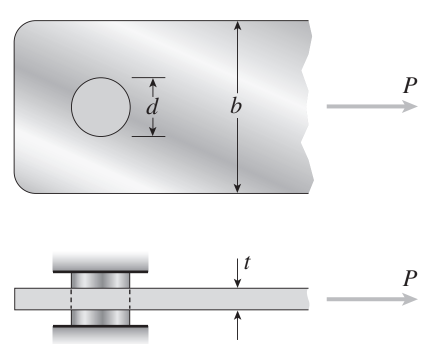
(b) Maximum load

Substitute $d = 0.6184$ in. into Eq. (1) or Eq. (2):

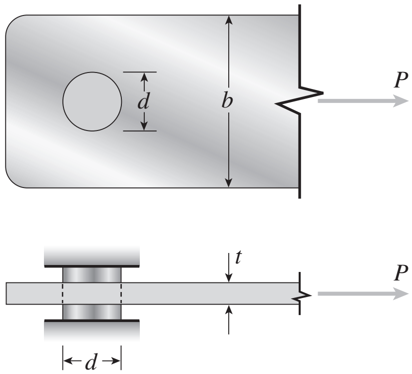
$P_{max} = 6910$ lb ←

Problem 1.8-14 A flat bar of width $b = 60$ mm and thickness $t = 10$ mm is loaded in tension by a force P (see figure). The bar is attached to a support by a pin of diameter d that passes through a hole of the same size in the bar. The allowable tensile stress on the net cross section of the bar is $\sigma_T = 140$ MPa, the allowable shear stress in the pin is $\tau_S = 80$ MPa, and the allowable bearing stress between the pin and the bar is $\sigma_B = 200$ MPa.

(a) Determine the pin diameter d_m for which the load P will be a maximum.
(b) Determine the corresponding value P_{max} of the load.



Solution 1.8-14 Bar with a pin connection



$b = 60\,\text{mm}$

$t = 10\,\text{mm}$

$d =$ diameter of hole and pin

$\sigma_T = 140\,\text{MPa}$

$\tau_S = 80\,\text{MPa}$

$\sigma_B = 200\,\text{MPa}$

UNITS USED IN THE FOLLOWING CALCULATIONS:

P is in kN

σ and τ are in N/mm² (same as MPa)

b, t, and d are in mm

TENSION IN THE BAR

$$P_T = \sigma_T(\text{Net area}) = \sigma_t(t)(b - d)$$

$$= (140\text{ MPa})(10\text{ mm})(60\text{ mm} - d)\left(\frac{1}{1000}\right)$$

$$= 1.40\,(60 - d) \qquad \text{(Eq. 1)}$$

SHEAR IN THE PIN

$$P_S = 2\tau_S A_{\text{pin}} = 2\tau_S\left(\frac{\pi d^2}{4}\right)$$

$$= 2(80\text{ MPa})\left(\frac{\pi}{4}\right)(d^2)\left(\frac{1}{1000}\right)$$

$$= 0.040\,\pi d^2 = 0.12566\,d^2 \qquad \text{(Eq. 2)}$$

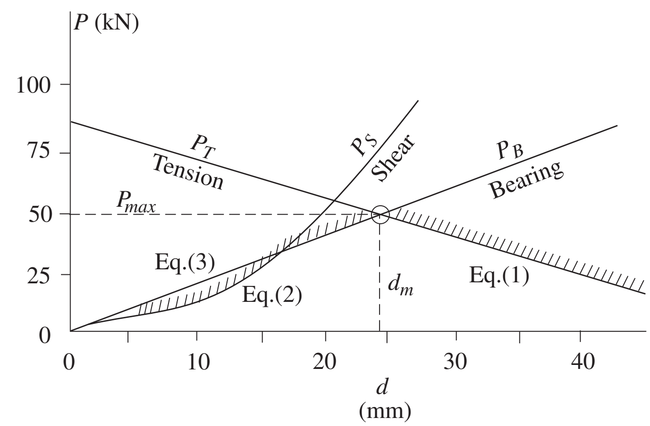
BEARING BETWEEN PIN AND BAR

$$P_B = \sigma_B\,td$$

$$= (200\text{ MPa})(10\text{ mm})(d)\left(\frac{1}{1000}\right)$$

$$= 2.0\,d \qquad \text{(Eq. 3)}$$

GRAPH OF EQS. (1), (2), AND (3)



(a) PIN DIAMETER d_m

$P_T = P_B$ or $1.40(60 - d) = 2.0\,d$

Solving, $d_m = \dfrac{84.0}{3.4}$ mm $= 24.7$ mm ←

(b) LOAD P_{max}

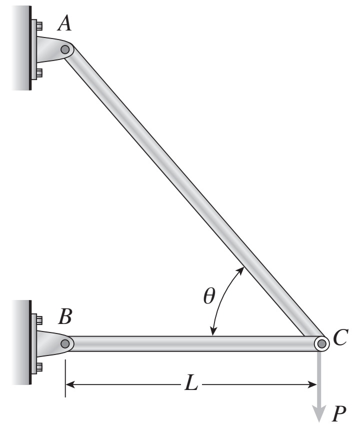
Substitute d_m into Eq. (1) or Eq. (3):

$P_{\text{max}} = 49.4$ kN ←

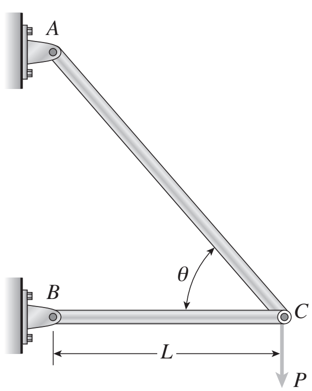
Problem 1.8-15 Two bars AC and BC of the same material support a vertical load P (see figure). The length L of the horizontal bar is fixed, but the angle θ can be varied by moving support A vertically and changing the length of bar AC to correspond with the new position of support A. The allowable stresses in the bars are the same in tension and compression.

We observe that when the angle θ is reduced, bar AC becomes shorter but the cross-sectional areas of both bars increase (because the axial forces are larger). The opposite effects occur if the angle θ is increased. Thus, we see that the weight of the structure (which is proportional to the volume) depends upon the angle θ.

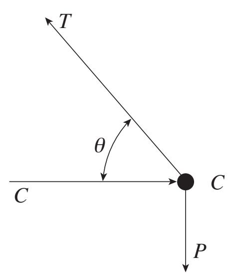
Determine the angle θ so that the structure has minimum weight without exceeding the allowable stresses in the bars. (*Note*: The weights of the bars are very small compared to the force P and may be disregarded.)



Solution 1.8-15 Two bars supporting a load P



Joint C



T = tensile force in bar AC

C = compressive force in bar BC

$$\Sigma F_{\text{vert}} = 0 \qquad T = \frac{P}{\sin\theta}$$

$$\Sigma F_{\text{horiz}} = 0 \qquad C = \frac{P}{\tan\theta}$$

Areas of bars

$$A_{AC} = \frac{T}{\sigma_{\text{allow}}} = \frac{P}{\sigma_{\text{allow}}\sin\theta}$$

$$A_{BC} = \frac{C}{\sigma_{\text{allow}}} = \frac{C}{\sigma_{\text{allow}}\tan\theta}$$

Lengths of bars

$$L_{AC} = \frac{L}{\cos\theta} \qquad L_{BC} = L$$

Weight of truss

γ = weight density of material

$$W = \gamma(A_{AC}L_{AC} + A_{BC}L_{BC})$$

$$= \frac{\gamma PL}{\sigma_{\text{allow}}}\left(\frac{1}{\sin\theta\,\cos\theta} + \frac{1}{\tan\theta}\right)$$

$$= \frac{\gamma PL}{\sigma_{\text{allow}}}\left(\frac{1+\cos^2\theta}{\sin\theta\,\cos\theta}\right) \qquad \text{Eq. (1)}$$

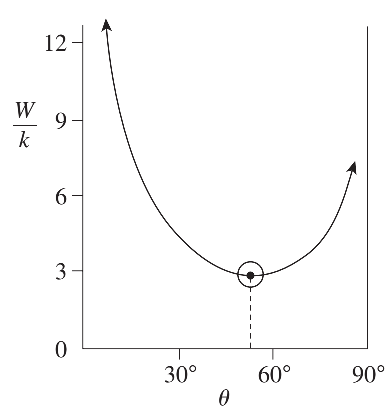
γ, P, L, and σ_{allow} are constants

W varies only with θ

Let $k = \dfrac{\gamma PL}{\sigma_{\text{allow}}}$ (k has units of force)

$$\frac{W}{k} = \frac{1+\cos^2\theta}{\sin\theta\cos\theta} \quad \text{(Nondimensional)} \qquad \text{Eq. (2)}$$

Graph of Eq. (2):



Angle θ that makes W a minimum

Use Eq. (2)

$$\text{Let } f = \frac{1 + \cos^2\theta}{\sin\theta\,\cos\theta}$$

$$\frac{df}{d\theta} = 0$$

$$\frac{df}{d\theta} = \frac{(\sin\theta\cos\theta)(2)(\cos\theta)(-\sin\theta) - (1 + \cos^2\theta)(-\sin^2\theta + \cos^2\theta)}{\sin^2\theta\cos^2\theta}$$

$$= \frac{-\sin^2\theta\cos^2\theta + \sin^2\theta - \cos^2\theta - \cos^4\theta}{\sin^2\theta\cos^2\theta}$$

Set the numerator $= 0$ and solve for θ:

$$-\sin^2\theta\cos^2\theta + \sin^2\theta - \cos^2\theta - \cos^4\theta = 0$$

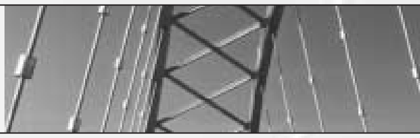
Replace $\sin^2\theta$ by $1 - \cos^2\theta$:

$$-(1 - \cos^2\theta)(\cos^2\theta) + 1 - \cos^2\theta - \cos^2\theta - \cos^4\theta = 0$$

Combine terms to simplify the equation:

$$1 - 3\cos^2\theta = 0 \qquad \cos\theta = \frac{1}{\sqrt{3}}$$

$$\theta = 54.7^\circ \quad \longleftarrow$$



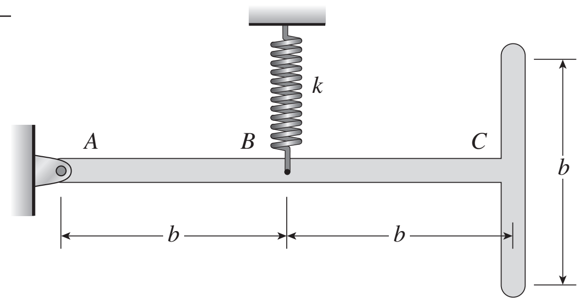
2

Axially Loaded Members

Changes in Lengths of Axially Loaded Members

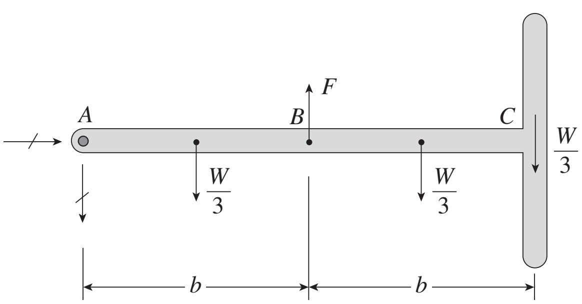
Problem 2.2-1 The T-shaped arm *ABC* shown in the figure lies in a vertical plane and pivots about a horizontal pin at *A*. The arm has constant cross-sectional area and total weight *W*. A vertical spring of stiffness *k* supports the arm at point *B*.

Obtain a formula for the elongation δ of the spring due to the weight of the arm.



Solution 2.2-1 **T-shaped arm**

FREE-BODY DIAGRAM OF ARM



F = tensile force in the spring

$\Sigma M_A = 0$

$$F(b) - \frac{W}{3}\left(\frac{b}{2}\right) - \frac{W}{3}\left(\frac{3b}{2}\right) - \frac{W}{3}(2b) = 0$$

$$F = \frac{4W}{3}$$

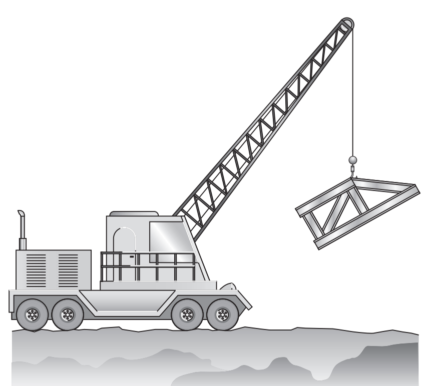
δ = elongation of the spring

$$\delta = \frac{F}{k} = \frac{4W}{3k} \quad \longleftarrow$$

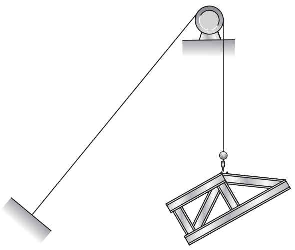
Problem 2.2-2 A steel cable with nominal diameter 25 mm (see Table 2-1) is used in a construction yard to lift a bridge section weighing 38 kN, as shown in the figure. The cable has an effective modulus of elasticity $E = 140$ GPa.

(a) If the cable is 14 m long, how much will it stretch when the load is picked up?

(b) If the cable is rated for a maximum load of 70 kN, what is the factor of safety with respect to failure of the cable?



Solution 2.2-2 Bridge section lifted by a cable



$A = 304 \text{ mm}^2$
(from Table 2-1)

$W = 38 \text{ kN}$

$E = 140 \text{ GPa}$

$L = 14 \text{ m}$

(a) Stretch of cable

$$\delta = \frac{WL}{EA} = \frac{(38 \text{ kN})(14 \text{ m})}{(140 \text{ GPa})(304 \text{ mm}^2)}$$

$$= 12.5 \text{ mm} \quad \longleftarrow$$

(b) Factor of safety

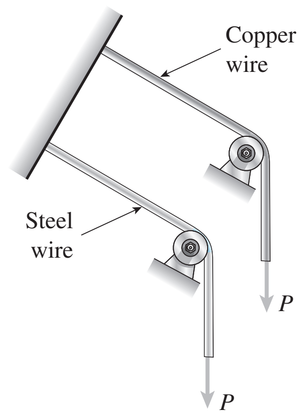
$P_{ULT} = 406 \text{ kN}$ (from Table 2-1)

$P_{\max} = 70 \text{ kN}$

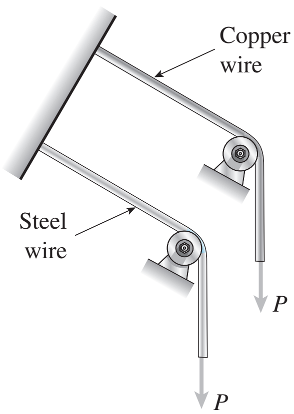
$$n = \frac{P_{ULT}}{P_{\max}} = \frac{406 \text{ kN}}{70 \text{ kN}} = 5.8 \quad \longleftarrow$$

Problem 2.2-3 A steel wire and a copper wire have equal lengths and support equal loads P (see figure). The moduli of elasticity for the steel and copper are $E_s = 30{,}000$ ksi and $E_c = 18{,}000$ ksi, respectively.

(a) If the wires have the same diameters, what is the ratio of the elongation of the copper wire to the elongation of the steel wire?
(b) If the wires stretch the same amount, what is the ratio of the diameter of the copper wire to the diameter of the steel wire?



Solution 2.2-3 Steel wire and copper wire



Equal lengths and equal loads

Steel: $E_s = 30{,}000$ ksi

Copper: $E_c = 18{,}000$ ksi

(a) Ratio of elongations (equal diameters)

$$\delta_c = \frac{PL}{E_c A} \qquad \delta_s = \frac{PL}{E_s A}$$

$$\frac{\delta_c}{\delta_s} = \frac{E_s}{E_c} = \frac{30}{18} = 1.67 \quad \longleftarrow$$

(b) Ratio of diameters (equal elongations)

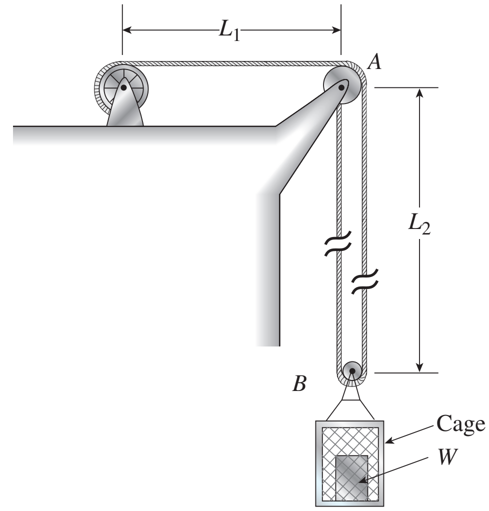
$$\delta_c = \delta_s \qquad \frac{PL}{E_c A_c} = \frac{PL}{E_s A_s} \quad \text{or} \quad E_c A_c = E_s A_s$$

$$E_c\left(\frac{\pi}{4}\right)d_c^2 = E_s\left(\frac{\pi}{4}\right)d_s^2$$

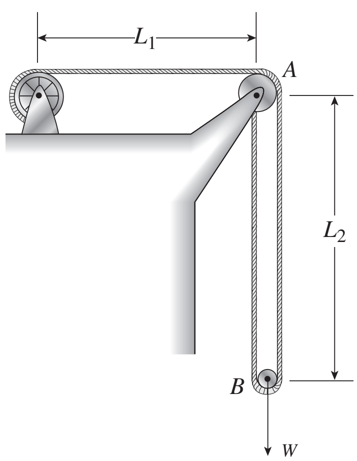
$$\frac{d_c^2}{d_s^2} = \frac{E_s}{E_c} \qquad \frac{d_c}{d_s} = \sqrt{\frac{E_s}{E_c}} = \sqrt{\frac{30}{18}} = 1.29 \quad \longleftarrow$$

Problem 2.2-4 By what distance h does the cage shown in the figure move downward when the weight W is placed inside it?

Consider only the effects of the stretching of the cable, which has axial rigidity $EA = 10{,}700$ kN. The pulley at A has diameter $d_A = 300$ mm and the pulley at B has diameter $d_B = 150$ mm. Also, the distance $L_1 = 4.6$ m, the distance $L_2 = 10.5$ m, and the weight $W = 22$ kN. (*Note:* When calculating the length of the cable, include the parts of the cable that go around the pulleys at A and B.)



Solution 2.2-4 **Cage supported by a cable**



$d_A = 300$ mm

$d_B = 150$ mm

$L_1 = 4.6$ m

$L_2 = 10.5$ m

$EA = 10{,}700$ kN

$W = 22$ kN

TENSILE FORCE IN CABLE

$$T = \frac{W}{2} = 11 \text{ kN}$$

LENGTH OF CABLE

$$L = L_1 + 2L_2 + \frac{1}{4}(\pi d_A) + \frac{1}{2}(\pi d_B)$$

$$= 4{,}600 \text{ mm} + 21{,}000 \text{ mm} + 236 \text{ mm} + 236 \text{ mm}$$

$$= 26{,}072 \text{ mm}$$

ELONGATION OF CABLE

$$\delta = \frac{TL}{EA} = \frac{(11 \text{ kN})(26{,}072 \text{ mm})}{(10{,}700 \text{ kN})} = 26.8 \text{ mm}$$

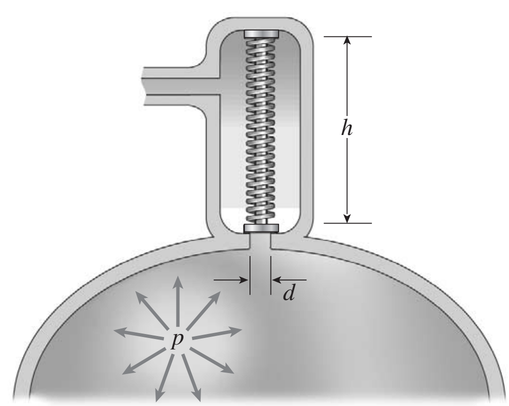
LOWERING OF THE CAGE

h = distance the cage moves downward

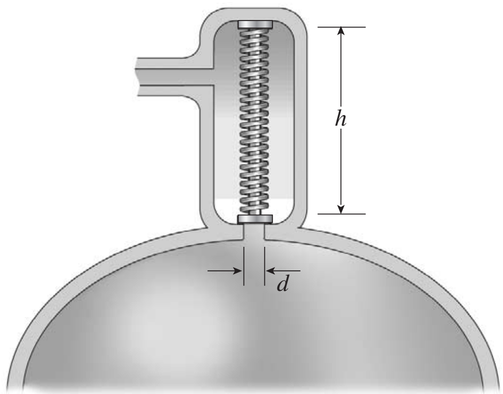
$$h = \frac{1}{2}\delta = 13.4 \text{ mm} \quad \longleftarrow$$

Problem 2.2-5 A safety valve on the top of a tank containing steam under pressure p has a discharge hole of diameter d (see figure). The valve is designed to release the steam when the pressure reaches the value p_{max}.

If the natural length of the spring is L and its stiffness is k, what should be the dimension h of the valve? (Express your result as a formula for h.)



Solution 2.2-5 Safety valve



h = height of valve (compressed length of the spring)

d = diameter of discharge hole

p = pressure in tank

p_{max} = pressure when valve opens

L = natural length of spring ($L > h$)

k = stiffness of spring

FORCE IN COMPRESSED SPRING

$F = k(L - h)$ (From Eq. 2-1a)

PRESSURE FORCE ON SPRING

$$P = p_{max}\left(\frac{\pi d^2}{4}\right)$$

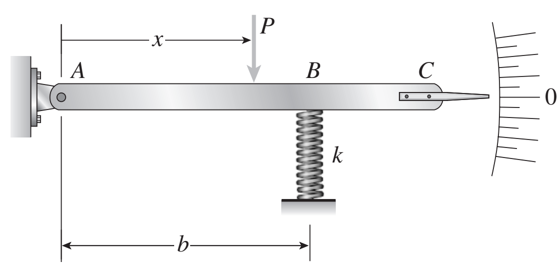
EQUATE FORCES AND SOLVE FOR h:

$$F = P \quad k(L - h) = \frac{\pi p_{max}\, d^2}{4}$$

$$h = L - \frac{\pi p_{max}\, d^2}{4k} \longleftarrow$$

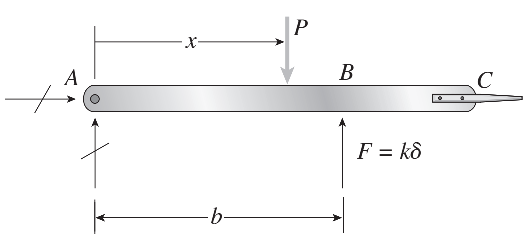
Problem 2.2-6 The device shown in the figure consists of a pointer ABC supported by a spring of stiffness $k = 800$ N/m. The spring is positioned at distance $b = 150$ mm from the pinned end A of the pointer. The device is adjusted so that when there is no load P, the pointer reads zero on the angular scale.

If the load $P = 8$ N, at what distance x should the load be placed so that the pointer will read 3° on the scale?



Solution 2.2-6 Pointer supported by a spring

FREE-BODY DIAGRAM OF POINTER



$P = 8$ N

$k = 800$ N/m

$b = 150$ mm

δ = displacement of spring

F = force in spring

$= k\delta$

$\Sigma M_A = 0$ ↺+

$$-Px + (k\delta)b = 0 \quad \text{or} \quad \delta = \frac{Px}{kb}$$

Let α = angle of rotation of pointer

$$\tan\alpha = \frac{\delta}{b} = \frac{Px}{kb^2} \qquad x = \frac{kb^2}{P}\tan\alpha \longleftarrow$$

SUBSTITUTE NUMERICAL VALUES:

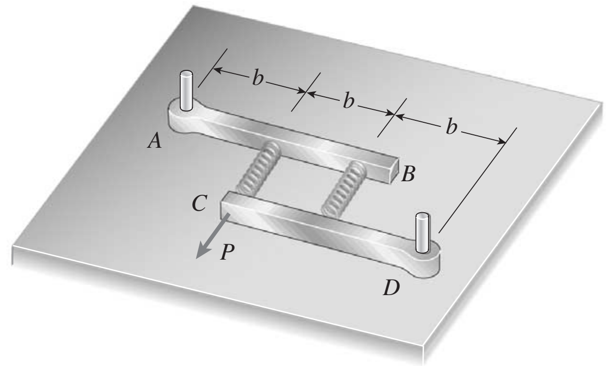
$\alpha = 3°$

$$x = \frac{(800 \text{ N/m})(150 \text{ mm})^2}{8 \text{ N}}\tan 3°$$

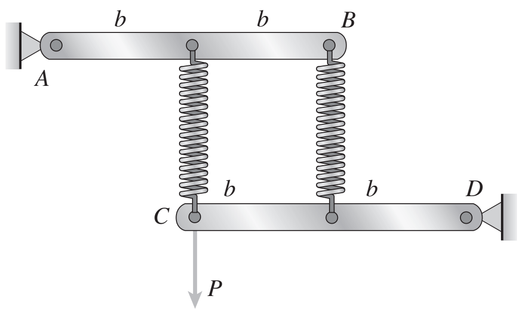
$$= 118 \text{ mm} \longleftarrow$$

Problem 2.2-7 Two rigid bars, AB and CD, rest on a smooth horizontal surface (see figure). Bar AB is pivoted end A and bar CD is pivoted at end D. The bars are connected to each other by two linearly elastic springs of stiffness k. Before the load P is applied, the lengths of the springs are such that the bars are parallel and the springs are without stress.

Derive a formula for the displacement δ_C at point C when the load P is acting. (Assume that the bars rotate through very small angles under the action of the load P.)



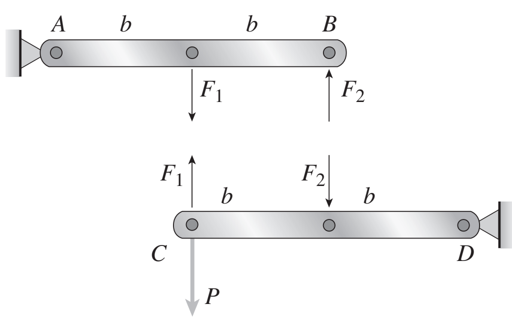
Solution 2.2-7 **Two bars connected by springs**



k = stiffness of springs

δ_C = displacement at point C due to load P

FREE-BODY DIAGRAMS



F_1 = tensile force in first spring

F_2 = compressive force in second spring

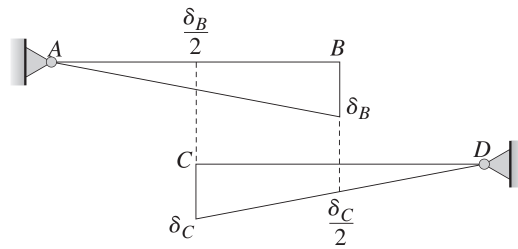
EQUILIBRIUM

$\Sigma M_A = 0 \qquad -bF_1 + 2bF_2 = 0 \quad F_1 = 2F_2$

$\Sigma M_D = 0 \quad 2bP - 2bF_1 + bF_2 = 0 \quad F_2 = 2F_1 - 2P$

Solving, $F_1 = \dfrac{4P}{3} \quad F_2 = \dfrac{2P}{3}$

DISPLACEMENT DIAGRAMS



δ_B = displacement of point B

δ_C = displacement of point C

Δ_1 = elongation of first spring

$= \delta_C - \dfrac{\delta_B}{2}$

Δ_2 = shortening of second spring

$= \delta_B - \dfrac{\delta_C}{2}$

Also, $\quad \Delta_1 = \dfrac{F_1}{k} = \dfrac{4P}{3k}; \quad \Delta_2 = \dfrac{F_2}{k} = \dfrac{2P}{3k}$

SOLVE THE EQUATIONS:

$\Delta_1 = \Delta_1 \quad \delta_C - \dfrac{\delta_B}{2} = \dfrac{4P}{3k}$

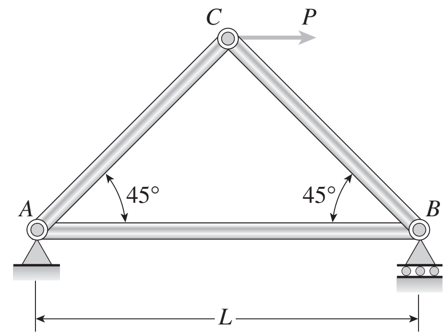
$\Delta_2 = \Delta_2 \quad \delta_B - \dfrac{\delta_C}{2} = \dfrac{2P}{3k}$

Eliminate δ_B and obtain δ_C:

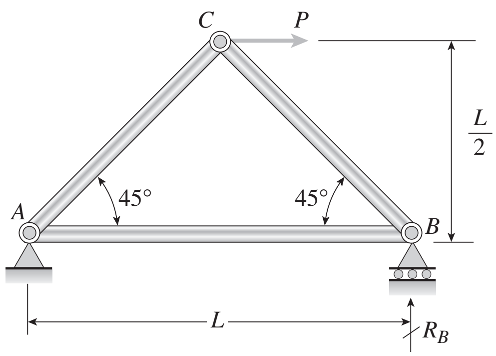
$$\delta_C = \frac{20P}{9k} \quad \longleftarrow$$

Problem 2.2-8 The three-bar truss ABC shown in the figure has a span $L = 3$ m and is constructed of steel pipes having cross-sectional area $A = 3900$ mm^2 and modulus of elasticity $E = 200$ GPa. A load P acts horizontally to the right at joint C.

(a) If $P = 650$ kN, what is the horizontal displacement of joint B?
(b) What is the maximum permissible load P_{max} if the displacement of joint B is limited to 1.5 mm?



Solution 2.2-8 **Truss with horizontal load**



$L = 3$ m

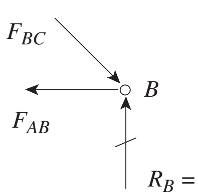
$A = 3900$ mm^2

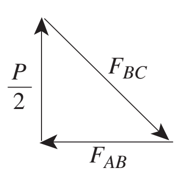
$E = 200$ GPa

$\Sigma M_A = 0$ gives $R_B = \dfrac{P}{2}$

FREE-BODY DIAGRAM OF JOINT B

Force triangle:





From force triangle,

$$F_{AB} = \frac{P}{2} \text{ (tension)}$$

(a) HORIZONTAL DISPLACEMENT δ_B

$P = 650$ kN

$$\delta_B = \frac{F_{AB}\,L_{AB}}{EA} = \frac{PL}{2EA}$$

$$= \frac{(650 \text{ kN})(3 \text{ m})}{2(200 \text{ GPa})(3900 \text{ mm}^2)}$$

$$= 1.25 \text{ mm} \longleftarrow$$

(b) MAXIMUM LOAD P_{max}

$\delta_{max} = 1.5$ mm

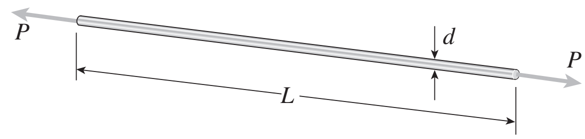
$$\frac{P_{max}}{\delta_{max}} = \frac{P}{\delta} \qquad P_{max} = P\left(\frac{\delta_{max}}{\delta}\right)$$

$$P_{max} = (650 \text{ kN})\left(\frac{1.5 \text{ mm}}{1.25 \text{ mm}}\right)$$

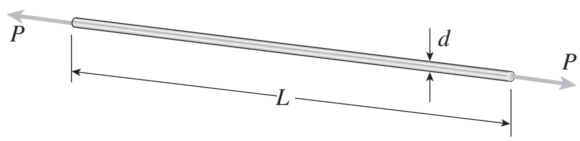
$$= 780 \text{ kN} \longleftarrow$$

Problem 2.2-9 An aluminum wire having a diameter $d = 2$ mm and length $L = 3.8$ m is subjected to a tensile load P (see figure). The aluminum has modulus of elasticity $E = 75$ GPa.

If the maximum permissible elongation of the wire is 3.0 mm and the allowable stress in tension is 60 MPa, what is the allowable load P_{max}?



Solution 2.2-9 **Aluminum wire in tension**



$d = 2$ mm

$L = 3.8$ m

$E = 75$ GPa

$$A = \frac{\pi d^2}{4} = 3.142 \text{ mm}^2$$

Maximum load based upon elongation

$$\delta_{max} = 3.0 \text{ mm} \qquad \delta = \frac{PL}{EA}$$

$$P_{max} = \frac{EA}{L}\delta_{max}$$

$$= \frac{(75 \text{ GPa})(3.142 \text{ mm}^2)}{3.8 \text{ m}}(3.0 \text{ mm})$$

$$= 186 \text{ N}$$

Maximum load based upon stress

$$\sigma_{allow} = 60 \text{ MPa} \qquad \sigma = \frac{P}{A}$$

$$P_{max} = A\sigma_{allow} = (3.142 \text{ mm}^2)(60 \text{ MPa})$$

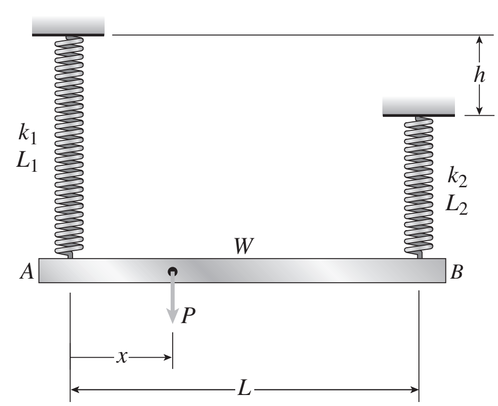
$$= 189 \text{ N}$$

Allowable load

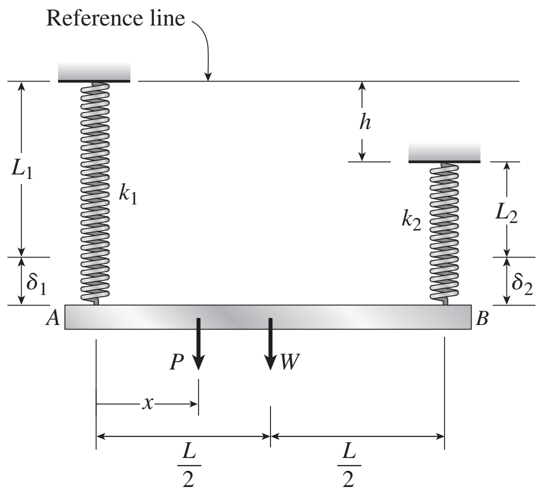
Elongation governs. $P_{max} = 186$ N ←

Problem 2.2-10 A uniform bar AB of weight $W = 25$ N is supported by two springs, as shown in the figure. The spring on the left has stiffness $k_1 = 300$ N/m and natural length $L_1 = 250$ mm. The corresponding quantities for the spring on the right are $k_2 = 400$ N/m and $L_2 = 200$ mm. The distance between the springs is $L = 350$ mm, and the spring on the right is suspended from a support that is distance $h = 80$ mm below the point of support for the spring on the left.

At what distance x from the left-hand spring should a load $P = 18$ N be placed in order to bring the bar to a horizontal position?



Solution 2.2-10 Bar supported by two springs



$W = 25$ N

$k_1 = 300$ N/m

$k_2 = 400$ N/m

$L = 350$ mm

$h = 80$ mm

$P = 18$ N

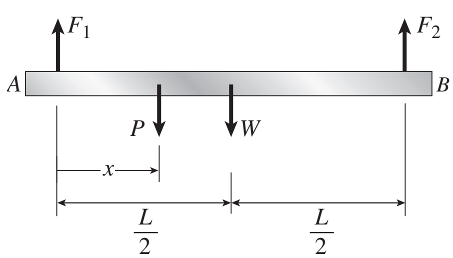
Natural lengths of springs

$L_1 = 250$ mm $\quad L_2 = 200$ mm

Objective

Find distance x for bar AB to be horizontal.

Free-body diagram of bar AB



$\Sigma M_A = 0$

$$F_2 L - P_X - \frac{WL}{2} = 0 \qquad \text{(Eq. 1)}$$

$\Sigma F_{\text{vert}} = 0 \quad \uparrow_{+} \quad \downarrow^{-}$

$$F_1 + F_2 - P - W = 0 \qquad \text{(Eq. 2)}$$

Solve Eqs. (1) and (2):

$$F_1 = P\left(1 - \frac{x}{L}\right) + \frac{W}{2} \qquad F_2 = \frac{P_x}{L} + \frac{W}{2}$$

Substitute numerical values:

Units: Newtons and meters

$$F_1 = (18)\left(1 - \frac{x}{0.350}\right) + 12.5 = 30.5 - 51.429x$$

$$F_2 = (18)\left(\frac{x}{0.350}\right) + 12.5 = 51.429x + 12.5$$

Elongations of the springs

$$\delta_1 = \frac{F_1}{k_1} = \frac{F_1}{300} = 0.10167 - 0.17143x$$

$$\delta_2 = \frac{F_2}{k_2} = \frac{F_2}{400} = 0.12857x + 0.031250$$

Bar AB remains horizontal

Points A and B are the same distance below the reference line (see figure above).

$$\therefore\ L_1 + \delta_1 = h + L_2 + \delta_2$$

or $\quad 0.250 + 0.10167 - 0.17143\,x$
$\quad = 0.080 + 0.200 + 0.12857\,x + 0.031250$

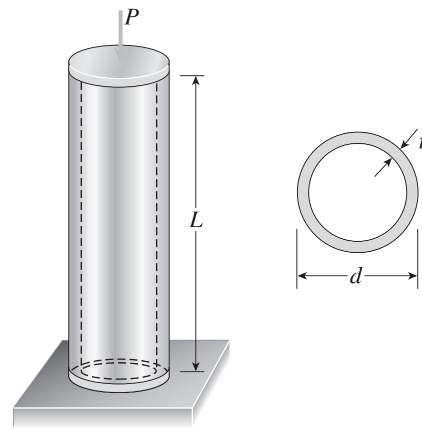
Solve for x:

$0.300\,x = 0.040420 \qquad x = 0.1347$ m

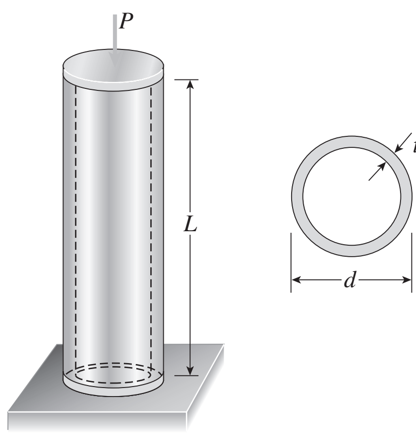
$x = 135$ mm $\leftarrow$

Problem 2.2-11 A hollow, circular, steel column ($E = 30{,}000$ ksi) is subjected to a compressive load P, as shown in the figure. The column has length $L = 8.0$ ft and outside diameter $d = 7.5$ in. The load $P = 85$ k.

If the allowable compressive stress is 7000 psi and the allowable shortening of the column is 0.02 in., what is the minimum required wall thickness t_{min}?



Solution 2.2-11 **Column in compression**



$P = 85$ k

$E = 30{,}000$ ksi

$L = 8.0$ ft

$d = 7.5$ in.

$\sigma_{allow} = 7{,}000$ psi

$\delta_{allow} = 0.02$ in.

Required area based upon allowable stress

$$\sigma = \frac{P}{A} \quad A = \frac{P}{\sigma_{allow}} = \frac{85 \text{ k}}{7{,}000 \text{ psi}} = 12.14 \text{ in.}^2$$

Required area based upon allowable shortening

$$\delta = \frac{PL}{EA} \quad A = \frac{PL}{E\delta_{allow}} = \frac{(85 \text{ k})(96 \text{ in.})}{(30{,}000 \text{ ksi})(0.02 \text{ in.})}$$

$$= 13.60 \text{ in.}^2$$

Shortening governs

$$A_{min} = 13.60 \text{ in.}^2$$

Minimum thickness t_{min}

$$A = \frac{\pi}{4}[d^2 - (d - 2t)^2] \quad \text{or}$$

$$\frac{4A}{\pi} - d^2 = -(d - 2t)^2$$

$$(d - 2t)^2 = d^2 - \frac{4A}{\pi} \quad \text{or} \quad d - 2t = \sqrt{d^2 - \frac{4A}{\pi}}$$

$$t = \frac{d}{2} - \sqrt{\left(\frac{d}{2}\right)^2 - \frac{A}{\pi}} \quad \text{or}$$

$$t_{min} = \frac{d}{2} - \sqrt{\left(\frac{d}{2}\right)^2 - \frac{A_{min}}{\pi}}$$

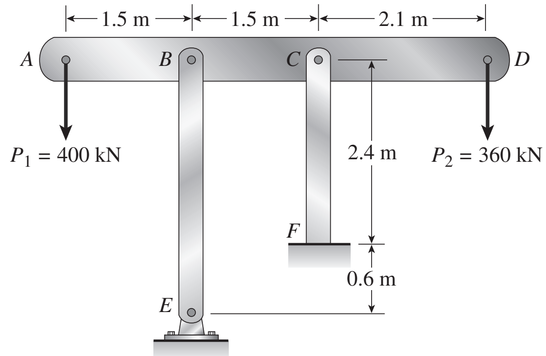
Substitute numerical values

$$t_{min} = \frac{7.5 \text{ in.}}{2} - \sqrt{\left(\frac{7.5 \text{ in.}}{2}\right)^2 - \frac{13.60 \text{ in.}^2}{\pi}}$$

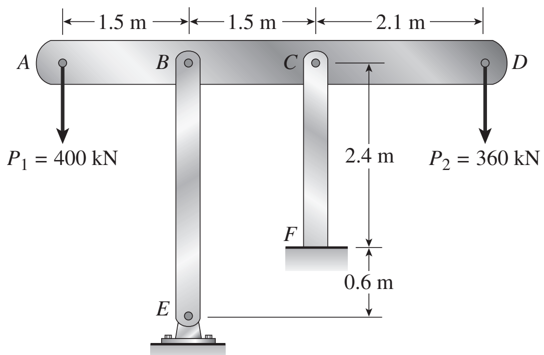
$$t_{min} = 0.63 \text{ in.} \longleftarrow$$

Problem 2.2-12 The horizontal rigid beam *ABCD* is supported by vertical bars *BE* and *CF* and is loaded by vertical forces $P_1 = 400$ kN and $P_2 = 360$ kN acting at points *A* and *D*, respectively (see figure). Bars *BE* and *CF* are made of steel ($E = 200$ GPa) and have cross-sectional areas $A_{BE} = 11{,}100\ \text{mm}^2$ and $A_{CF} = 9{,}280\ \text{mm}^2$. The distances between various points on the bars are shown in the figure.

Determine the vertical displacements δ_A and δ_D of points *A* and *D*, respectively.



Solution 2.2-12 Rigid beam supported by vertical bars



$A_{BE} = 11{,}100\ \text{mm}^2$

$A_{CF} = 9{,}280\ \text{mm}^2$

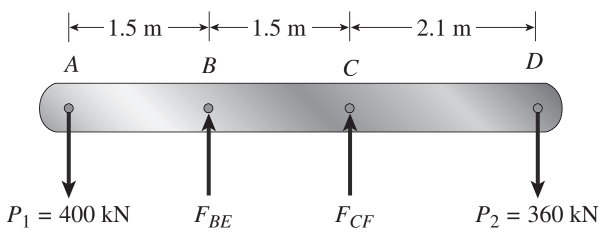
$E = 200$ GPa

$L_{BE} = 3.0$ m

$L_{CF} = 2.4$ m

$P_1 = 400$ kN; $P_2 = 360$ kN

FREE-BODY DIAGRAM OF BAR *ABCD*



$\Sigma M_B = 0$

$(400\ \text{kN})(1.5\ \text{m}) + F_{CF}(1.5\ \text{m}) - (360\ \text{kN})(3.6\ \text{m}) = 0$

$F_{CF} = 464$ kN

$\Sigma M_C = 0$

$(400\ \text{kN})(3.0\ \text{m}) - F_{BE}(1.5\ \text{m}) - (360\ \text{kN})(2.1\ \text{m}) = 0$

$F_{BE} = 296$ kN

SHORTENING OF BAR *BE*

$$\delta_{BE} = \frac{F_{BE}L_{BE}}{EA_{BE}} = \frac{(296\ \text{kN})(3.0\ \text{m})}{(200\ \text{GPa})(11{,}100\ \text{mm}^2)}$$

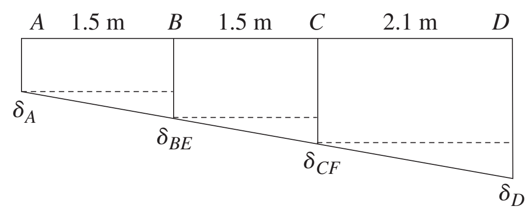
$$= 0.400\ \text{mm}$$

SHORTENING OF BAR *CF*

$$\delta_{CF} = \frac{F_{CF}L_{CF}}{EA_{CF}} = \frac{(464\ \text{kN})(2.4\ \text{m})}{(200\ \text{GPa})(9{,}280\ \text{mm}^2)}$$

$$= 0.600\ \text{mm}$$

DISPLACEMENT DIAGRAM



$\delta_{BE} - \delta_A = \delta_{CF} - \delta_{BE}$ or $\delta_A = 2\delta_{BE} - \delta_{CF}$

$\delta_A = 2(0.400\ \text{mm}) - 0.600\ \text{m}$

$= 0.200\ \text{mm}$

(Downward)

$$\delta_D - \delta_{CF} = \frac{2.1}{1.5}(\delta_{CF} - \delta_{BE})$$

$$\text{or} \quad \delta_D = \frac{12}{5}\delta_{CF} - \frac{7}{5}\delta_{BE}$$

$$= \frac{12}{5}(0.600\ \text{mm}) - \frac{7}{5}(0.400\ \text{mm})$$

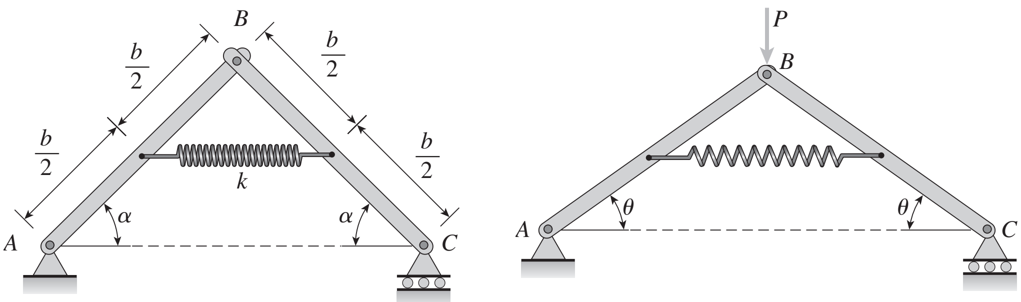
$= 0.880\ \text{mm}$

(Downward)

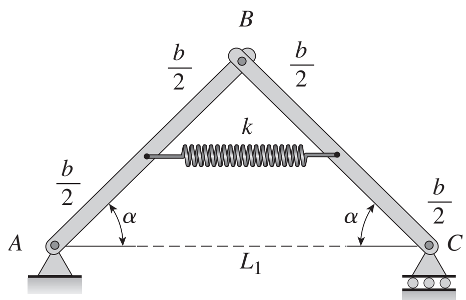
Problem 2.2-13 A framework ABC consists of two rigid bars AB and BC, each having length b (see the first part of the figure). The bars have pin connections at A, B, and C and are joined by a spring of stiffness k. The spring is attached at the midpoints of the bars. The framework has a pin support at A and a roller support at C, and the bars are at an angle α to the hoizontal.

When a vertical load P is applied at joint B (see the second part of the figure) the roller support C moves to the right, the spring is stretched, and the angle of the bars decreases from α to the angle θ.

Determine the angle θ and the increase δ in the distance between points A and C. (Use the following data; $b = 8.0$ in., $k = 16$ lb/in., $\alpha = 45°$, and $P = 10$ lb.)



Solution 2.2-13 **Framework with rigid bars and a spring**



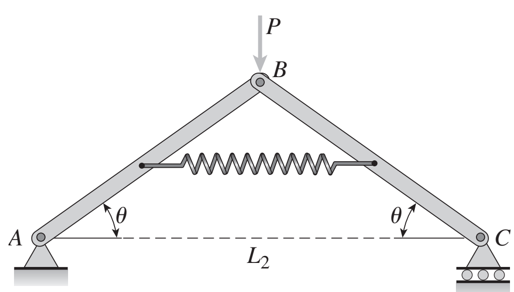
With no load

L_1 = span from A to C

$= 2b \cos \alpha$

S_1 = length of spring

$$= \frac{L_1}{2} = b \cos \alpha$$



With load P

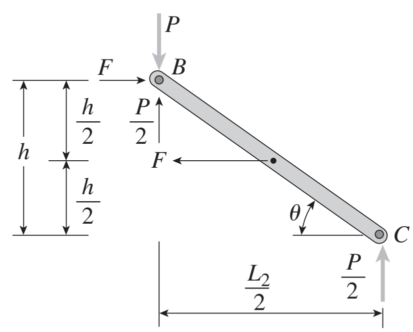
L_2 = span from A to C

$= 2b \cos \theta$

S_2 = length of spring

$$= \frac{L_2}{2} = b \cos \theta$$

Free-body diagram of BC



h = height from C to $B = b \sin \theta$

$$\frac{L_2}{2} = b \cos \theta$$

F = force in spring due to load P

$\Sigma M_B = 0$

$$\frac{P}{2}\left(\frac{L_2}{2}\right) - F\left(\frac{h}{2}\right) = 0 \text{ or } P \cos \theta = F \sin \theta \qquad \text{(Eq. 1)}$$

(Continued)

DETERMINE THE ANGLE θ

ΔS = elongation of spring

$= S_2 - S_1 = b(\cos\theta - \cos\alpha)$

For the spring: $F = k(\Delta S)$

$F = bk(\cos\theta - \cos\alpha)$

Substitute F into Eq. (1):

$P\cos\theta = bk(\cos\theta - \cos\alpha)(\sin\theta)$

or $\quad \frac{\mathrm{P}}{\mathrm{bk}}\cot\theta - \cos\theta + \cos\alpha = 0 \quad \longleftarrow \quad$ (Eq. 2)

This equation must be solved numerically for the angle θ.

DETERMINE THE DISTANCE δ

$\delta = L_2 - L_1 = 2b\cos\theta - 2b\cos\alpha$

$= 2b(\cos\theta - \cos\alpha)$

From Eq. (2): $\cos\alpha = \cos\theta - \frac{P\cot\theta}{bk}$

Therefore,

$$\delta = 2b\left(\cos\theta - \cos\theta + \frac{P\cot\theta}{bk}\right)$$

$$= \frac{2P}{b}\cot\theta \quad \longleftarrow \qquad \text{(Eq. 3)}$$

NUMERICAL RESULTS

$b = 8.0$ in. $\quad k = 16$ lb/in. $\quad \alpha = 45°\quad P = 10$ lb

Substitute into Eq. (2):

$0.078125\cot\theta - \cos\theta + 0.707107 = 0$ (Eq. 4)

Solve Eq. (4) numerically:

$\theta = 35.1° \quad \longleftarrow$

Substitute into Eq. (3):

$\delta = 1.78$ in. $\quad \longleftarrow$

Problem 2.2-14 Solve the preceding problem for the following data: $b = 200$ mm, $k = 3.2$ kN/m, $\alpha = 45°$, and $P = 50$ N.

Solution 2.2-14 Framework with rigid bars and a spring

See the solution to the preceding problem.

Eq. (2): $\quad \frac{P}{bk}\cot\theta - \cos\theta + \cos\alpha = 0$

Eq. (3): $\quad \delta = \frac{2P}{k}\cot\theta$

NUMERICAL RESULTS

$b = 200$ mm $\quad k = 3.2$ kN/m $\quad \alpha = 45° \quad P = 50$ N

Substitute into Eq. (2):

$0.078125\cot\theta - \cos\theta + 0.707107 = 0$ (Eq. 4)

Solve Eq. (4) numerically:

$\theta = 35.1° \quad \longleftarrow$

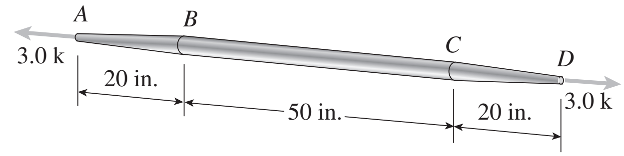
Substitute into Eq. (3):

$\delta = 44.5$ mm $\quad \longleftarrow$

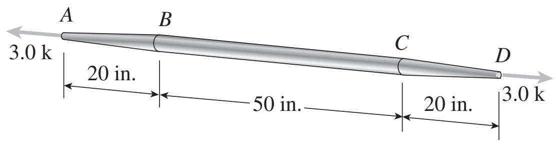
Changes in Lengths under Nonuniform Conditions

Problem 2.3-1 Calculate the elongation of a copper bar of solid circular cross section with tapered ends when it is stretched by axial loads of magnitude 3.0 k (see figure).

The length of the end segments is 20 in. and the length of the prismatic middle segment is 50 in. Also, the diameters at cross sections A, B, C, and D are 0.5, 1.0, 1.0, and 0.5 in., respectively, and the modulus of elasticity is 18,000 ksi. (*Hint:* Use the result of Example 2-4.)



Solution 2.3-1 **Bar with tapered ends**



$d_A = d_D = 0.5 \text{ in.} \quad P = 3.0 \text{ k}$

$d_B = d_C = 1.0 \text{ in.} \quad E = 18{,}000 \text{ ksi}$

END SEGMENT ($L = 20$ in.)

From Example 2-4:

$$\delta = \frac{4PL}{\pi E\, d_A\, d_B}$$

$$\delta_1 = \frac{4(3.0 \text{ k})(20 \text{ in.})}{\pi(18{,}000 \text{ ksi})(0.5 \text{ in.})(1.0 \text{ in.})} = 0.008488 \text{ in.}$$

MIDDLE SEGMENT ($L = 50$ in.)

$$\delta_2 = \frac{PL}{EA} = \frac{(3.0 \text{ k})(50 \text{ in.})}{(18{,}000 \text{ ksi})(\frac{\pi}{4})(1.0 \text{ in.})^2}$$

$$= 0.01061 \text{in.}$$

ELONGATION OF BAR

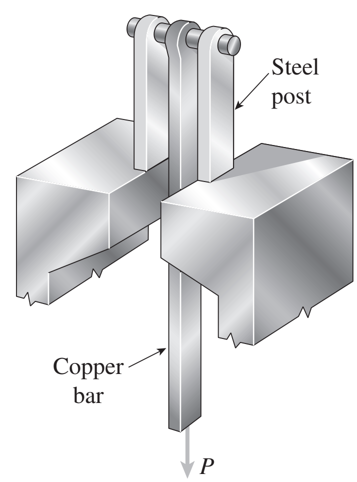
$$\delta = \sum \frac{NL}{EA} = 2\delta_1 + \delta_2$$

$$= 2(0.008488 \text{ in.}) + (0.01061 \text{ in.})$$

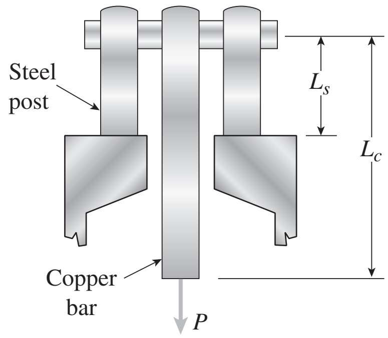
$$= 0.0276 \text{ in.} \longleftarrow$$

Problem 2.3-2 A long, rectangular copper bar under a tensile load P hangs from a pin that is supported by two steel posts (see figure). The copper bar has a length of 2.0 m, a cross-sectional area of 4800 mm^2, and a modulus of elasticity $E_c = 120$ GPa. Each steel post has a height of 0.5 m, a cross-sectional area of 4500 mm^2, and a modulus of elasticity $E_s = 200$ GPa.

(a) Determine the downward displacement δ of the lower end of the copper bar due to a load $P = 180$ kN.
(b) What is the maximum permissible load P_{max} if the displacement δ is limited to 1.0 mm?



Solution 2.3-2 **Copper bar with a tensile load**



$L_c = 2.0$ m

$A_c = 4800$ mm^2

$E_c = 120$ GPa

$L_s = 0.5$ m

$A_s = 4500$ mm^2

$E_s = 200$ GPa

(a) DOWNWARD DISPLACEMENT δ ($P = 180$ kN)

$$\delta_c = \frac{PL_c}{E_c A_c} = \frac{(180\text{ kN})(2.0\text{ m})}{(120\text{ GPa})(4800\text{ mm}^2)}$$

$$= 0.625\text{ mm}$$

$$\delta_s = \frac{(P/2)L_s}{E_s A_s} = \frac{(90\text{ kN})(0.5\text{ m})}{(200\text{ GPa})(4500\text{ mm}^2)}$$

$$= 0.050\text{ mm}$$

$$\delta = \delta_c + \delta_s = 0.625\text{ mm} + 0.050\text{ mm}$$

$$= 0.675\text{ mm} \longleftarrow$$

(b) MAXIMUM LOAD P_{max} ($\delta_{max} = 1.0$ mm)

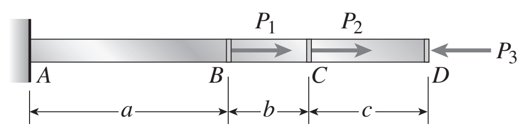
$$\frac{P_{max}}{P} = \frac{\delta_{max}}{\delta} \qquad P_{max} = P\left(\frac{\delta_{max}}{\delta}\right)$$

$$P_{max} = (180\text{ kN})\left(\frac{1.0\text{ mm}}{0.675\text{ mm}}\right) = 267\text{ kN} \longleftarrow$$

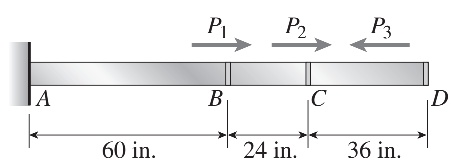
Problem 2.3-3 A steel bar *AD* (see figure) has a cross-sectional area of 0.40 in.^{2} and is loaded by forces $P_1 = 2700$ lb, $P_2 = 1800$ lb, and $P_3 = 1300$ lb. The lengths of the segments of the bar are $a = 60$ in., $b = 24$ in., and $c = 36$ in.

(a) Assuming that the modulus of elasticity $E = 30 \times 10^6$ psi, calculate the change in length δ of the bar. Does the bar elongate or shorten?

(b) By what amount P should the load P_3 be increased so that the bar does not change in length when the three loads are applied?



Solution 2.3-3 **Steel bar loaded by three forces**



$A = 0.40$ in.^{2} $\quad P_1 = 2700$ lb $\quad P_2 = 1800$ lb

$P_3 = 1300$ lb $\quad E = 30 \times 10^6$ psi

AXIAL FORCES

$N_{AB} = P_1 + P_2 - P_3 = 3200$ lb

$N_{BC} = P_2 - P_3 = 500$ lb

$N_{CD} = -P_3 = -1300$ lb

(a) CHANGE IN LENGTH

$$\delta = \sum \frac{N_i L_i}{E_i A_i}$$

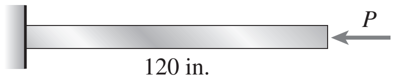
$$= \frac{1}{EA}(N_{AB}L_{AB} + N_{BC}L_{BC} + N_{CD}L_{CD})$$

$$= \frac{1}{(30 \times 10^6\text{ psi})(0.40\text{ in.}^2)}[(3200\text{ lb})(60\text{ in.})$$

$$+ (500\text{ lb})(24\text{ in.}) - (1300\text{ lb})(36\text{ in.})]$$

$$= 0.0131\text{ in. (elongation)} \longleftarrow$$

(b) INCREASE IN P_3 FOR NO CHANGE IN LENGTH



P = increase in force P_3

The force P must produce a shortening equal to 0.0131 in. in order to have no change in length.

$$\therefore\ 0.0131 \text{ in.} = \delta = \frac{PL}{EA}$$

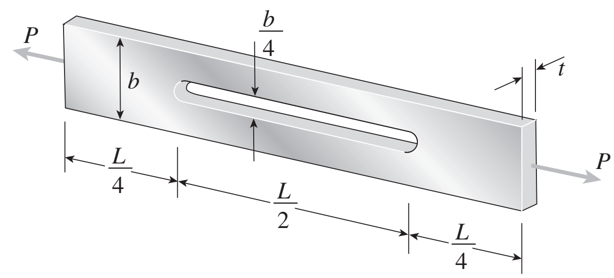
$$= \frac{P(120 \text{ in.})}{(30 \times 10^6 \text{ psi})(0.40 \text{ in.}^2)}$$

$$P = 1310 \text{ lb} \quad \longleftarrow$$

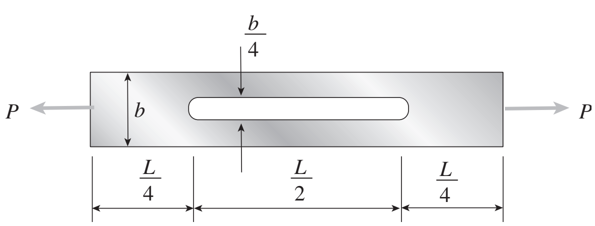
Problem 2.3-4 A rectangular bar of length L has a slot in the middle half of its length (see figure). The bar has width b, thickness t, and modulus of elasticity E. The slot has width $b/4$.

(a) Obtain a formula for the elongation δ of the bar due to the axial loads P.

(b) Calculate the elongation of the bar if the material is high-strength steel, the axial stress in the middle region is 160 MPa, the length is 750 mm, and the modulus of elasticity is 210 GPa.



Solution 2.3-4 **Bar with a slot**



t = thickness $\quad L$ = length of bar

(a) ELONGATION OF BAR

$$\delta = \sum \frac{N_i L_i}{EA_i} = \frac{P(L/4)}{E(bt)} + \frac{P(L/2)}{E(\frac{3}{4}bt)} + \frac{P(L/4)}{E(bt)}$$

$$= \frac{PL}{Ebt}\left(\frac{1}{4} + \frac{4}{6} + \frac{1}{4}\right) = \frac{7PL}{6Ebt} \quad \longleftarrow$$

STRESS IN MIDDLE REGION

$$\sigma = \frac{P}{A} = \frac{P}{(\frac{3}{4}bt)} = \frac{4P}{3bt} \quad \text{or} \quad \frac{P}{bt} = \frac{3\sigma}{4}$$

Substitute into the equation for δ:

$$\delta = \frac{7PL}{6Ebt} = \frac{7L}{6E}\left(\frac{P}{bt}\right) = \frac{7L}{6E}\left(\frac{3\sigma}{4}\right)$$

$$= \frac{7\sigma L}{8E}$$

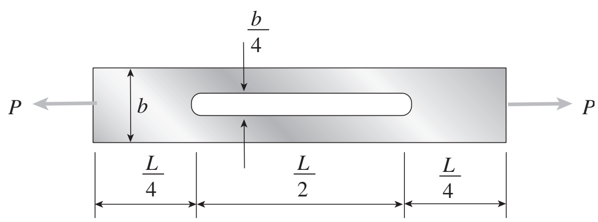
(b) SUBSTITUTE NUMERICAL VALUES:

$\sigma = 160$ MPa $\quad L = 750$ mm $\quad E = 210$ GPa

$$\delta = \frac{7(160 \text{ MPa})(750 \text{ mm})}{8\,(210 \text{ GPa})} = 0.500 \text{ mm} \quad \longleftarrow$$

Problem 2.3-5 Solve the preceding problem if the axial stress in the middle region is 24,000 psi, the length is 30 in., and the modulus of elasticity is 30×10^6 psi.

Solution 2.3-5 Bar with a slot



t = thickness $\quad L$ = length of bar

(a) Elongation of bar

$$\delta = \sum \frac{N_i L_i}{EA_i} = \frac{P(L/4)}{E(bt)} + \frac{P(L/2)}{E\left(\frac{3}{4}bt\right)} + \frac{P(L/4)}{E(bt)}$$

$$= \frac{PL}{Ebt}\left(\frac{1}{4} + \frac{4}{6} + \frac{1}{4}\right) = \frac{7PL}{6Ebt} \quad \longleftarrow$$

Stress in middle region

$$\sigma = \frac{P}{A} = \frac{P}{\left(\frac{3}{4}bt\right)} = \frac{4P}{3bt} \quad \text{or} \quad \frac{P}{bt} = \frac{3\sigma}{4}$$

Substitute into the equation for δ:

$$\delta = \frac{7PL}{6Ebt} = \frac{7L}{6E}\left(\frac{P}{bt}\right) = \frac{7L}{6E}\left(\frac{3\sigma}{4}\right)$$

$$= \frac{7\sigma L}{8E}$$

(b) Substitute numerical values:

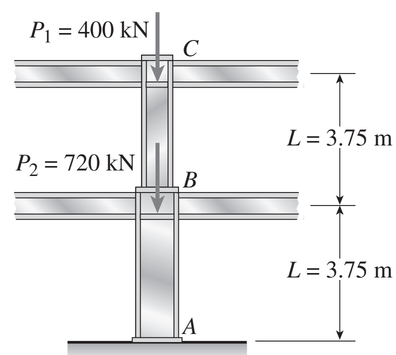
$\sigma = 24{,}000$ psi $\quad L = 30$ in.

$E = 30 \times 10^6$ psi

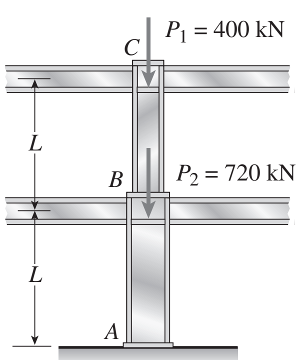
$$\delta = \frac{7(24{,}000 \text{ psi})(30 \text{ in.})}{8(30 \times 10^6 \text{ psi})} = 0.0210 \text{ in.} \quad \longleftarrow$$

Problem 2.3-6 A two-story building has steel columns AB in the first floor and BC in the second floor, as shown in the figure. The roof load P_1 equals 400 kN and the second-floor load P_2 equals 720 kN. Each column has length $L = 3.75$ m. The cross-sectional areas of the first- and second-floor columns are 11,000 mm² and 3,900 mm², respectively.

(a) Assuming that $E = 206$ GPa, determine the total shortening δ_{AC} of the two columns due to the combined action of the loads P_1 and P_2.
(b) How much additional load P_0 can be placed at the top of the column (point C) if the total shortening δ_{AC} is not to exceed 4.0 mm?



Solution 2.3-6 Steel columns in a building



L = length of each column

$= 3.75$ m

$E = 206$ GPa

$A_{AB} = 11{,}000$ mm²

$A_{BC} = 3{,}900$ mm²

(a) Shortening δ_{AC} of the two columns

$$\delta_{AC} = \sum \frac{N_i L_i}{E_i A_i} = \frac{N_{AB} L}{EA_{AB}} + \frac{N_{BC} L}{EA_{BC}}$$

$$= \frac{(1120 \text{ kN})(3.75 \text{ m})}{(206 \text{ GPa})(11{,}000 \text{ mm}^2)}$$

$$+ \frac{(400 \text{ kN})(3.75 \text{ m})}{(206 \text{ GPa})(3{,}900 \text{ mm}^2)}$$

$$= 1.8535 \text{ mm} + 1.8671 \text{ mm} = 3.7206 \text{ mm}$$

$$\delta_{AC} = 3.72 \text{ mm} \quad \longleftarrow$$

(b) ADDITIONAL LOAD P_0 AT POINT C

$(\delta_{AC})_{max} = 4.0$ mm

δ_0 = additional shortening of the two columns due to the load P_0

$$\delta_0 = (\delta_{AC})_{max} - \delta_{AC} = 4.0 \text{ mm} - 3.7206 \text{ mm} = 0.2794 \text{ mm}$$

Also, $$\delta_0 = \frac{P_0 L}{EA_{AB}} + \frac{P_0 L}{EA_{BC}} = \frac{P_0 L}{E}\left(\frac{1}{A_{AB}} + \frac{1}{A_{BC}}\right)$$

Solve for P_0:

$$P_0 = \frac{E\delta_0}{L}\left(\frac{A_{AB}A_{BC}}{A_{AB} + A_{BC}}\right)$$

SUBSTITUTE NUMERICAL VALUES:

$E = 206 \times 10^9$ N/m² $\quad \delta_0 = 0.2794 \times 10^{-3}$ m

$L = 3.75$ m $\quad A_{AB} = 11{,}000 \times 10^{-6}$ m²

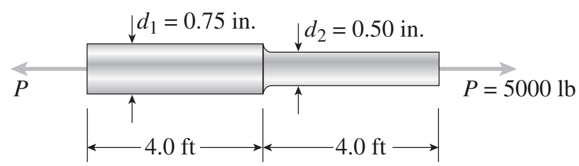
$A_{BC} = 3{,}900 \times 10^{-6}$ m²

$P_0 = 44{,}200$ N $= 44.2$ kN ←

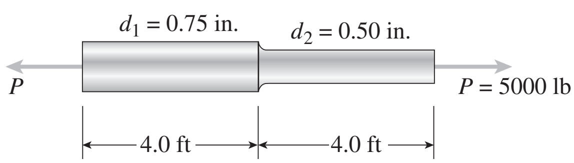
Problem 2.3-7 A steel bar 8.0 ft long has a circular cross section of diameter $d_1 = 0.75$ in. over one-half of its length and diameter $d_2 = 0.5$ in. over the other half (see figure). The modulus of elasticity $E = 30 \times 10^6$ psi.

(a) How much will the bar elongate under a tensile load $P = 5000$ lb?

(b) If the same volume of material is made into a bar of constant diameter d and length 8.0 ft, what will be the elongation under the same load P?



Solution 2.3-7 **Bar in tension**



$P = 5000$ lb

$E = 30 \times 10^6$ psi

$L = 4$ ft $= 48$ in.

(a) ELONGATION OF NONPRISMATIC BAR

$$\delta = \sum \frac{N_i L_i}{E_i A_i} = \frac{PL}{E}\sum \frac{1}{A_i}$$

$$\delta = \frac{(5000 \text{ lb})(48 \text{ in.})}{30 \times 10^6 \text{ psi}} \times \left[\frac{1}{\frac{\pi}{4}(0.75 \text{ in})^2} + \frac{1}{\frac{\pi}{4}(0.50 \text{ in.})^2}\right]$$

$= 0.0589$ in. ←

(b) ELONGATION OF PRISMATIC BAR OF SAME VOLUME

Original bar: $V_o = A_1 L + A_2 L = L(A_1 + A_2)$

Prismatic bar: $V_p = A_p(2L)$

Equate volumes and solve for A_p:

$V_o = V_p \qquad L(A_1 + A_2) = A_p(2L)$

$$A_p = \frac{A_1 + A_2}{2} = \frac{1}{2}\left(\frac{\pi}{4}\right)(d_1^2 + d_2^2)$$

$$= \frac{\pi}{8}[(0.75 \text{ in.})^2 + (0.50 \text{ in.})^2] = 0.3191 \text{ in.}^2$$

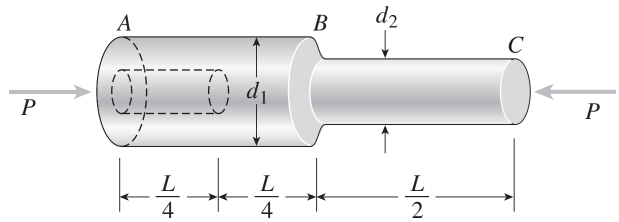
$$\delta = \frac{P(2L)}{EA_p} = \frac{(5000 \text{ lb})(2)(48 \text{ in.})}{(30 \times 10^6 \text{ psi})(0.3191 \text{ in.}^2)}$$

$= 0.0501$ in. ←

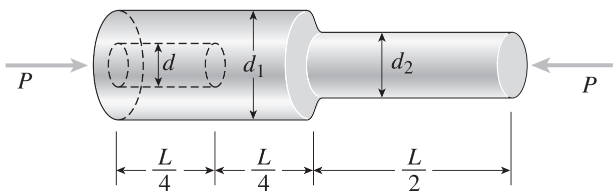
NOTE: A prismatic bar of the same volume will *always* have a smaller change in length than will a nonprismatic bar, provided the constant axial load P, modulus E, and total length L are the same.

Problem 2.3-8 A bar *ABC* of length *L* consists of two parts of equal lengths but different diameters (see figure). Segment *AB* has diameter $d_1 = 100$ mm and segment *BC* has diameter $d_2 = 60$ mm. Both segments have length $L/2 = 0.6$ m. A longitudinal hole of diameter *d* is drilled through segment *AB* for one-half of its length (distance $L/4 = 0.3$ m). The bar is made of plastic having modulus of elasticity $E = 4.0$ GPa. Compressive loads $P = 110$ kN act at the ends of the bar.

If the shortening of the bar is limited to 8.0 mm, what is the maximum allowable diameter d_{max} of the hole?



Solution 2.3-8 **Bar with a hole**



d = diameter of hole

SHORTENING δ OF THE BAR

$$\delta = \sum \frac{N_i L_i}{E_i A_i} = \frac{P}{E} \sum \frac{L_i}{A_i}$$

$$= \frac{P}{E}\left[\frac{L/4}{\frac{\pi}{4}(d_1^2 - d^2)} + \frac{L/4}{\frac{\pi}{4}d_1^2} + \frac{L/2}{\frac{\pi}{4}d_2^2}\right]$$

$$= \frac{PL}{\pi E}\left(\frac{1}{d_1^2 - d^2} + \frac{1}{d_1^2} + \frac{2}{d_2^2}\right) \qquad \text{(Eq. 1)}$$

NUMERICAL VALUES (DATA):

δ = maximum allowable shortening of the bar = 8.0 mm

$P = 110$ kN $\quad L = 1.2$ m $\quad E = 4.0$ GPa

$d_1 = 100$ mm

d_{max} = maximum allowable diameter of the hole

$d_2 = 60$ mm

SUBSTITUTE NUMERICAL VALUES INTO EQ. (1) FOR δ AND SOLVE FOR $d = d_{max}$:

UNITS: Newtons and meters

$$0.008 = \frac{(110{,}000)(1.2)}{\pi(4.0 \times 10^9)} \times \left[\frac{1}{(0.1)^2 - d^2} + \frac{1}{(0.1)^2} + \frac{2}{(0.06)^2}\right]$$

$$761.598 = \frac{1}{0.01 - d^2} + \frac{1}{0.01} + \frac{2}{0.0036}$$

$$\frac{1}{0.01 - d^2} = 761.598 - 100 - 555.556 = 106.042$$

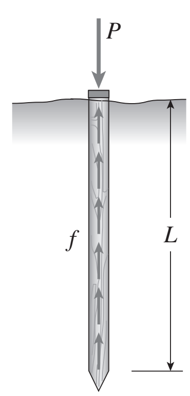
$$d^2 = 569.81 \times 10^{-6}\ \text{m}^2$$

$$d = 0.02387\ \text{m}$$

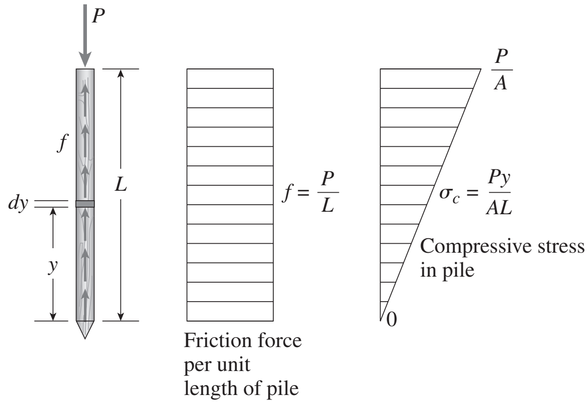
$$d_{max} = 23.9\ \text{mm} \quad \longleftarrow$$

Problem 2.3-9 A wood pile, driven into the earth, supports a load *P* entirely by friction along its sides (see figure). The friction force *f* per unit length of pile is assumed to be uniformly distributed over the surface of the pile. The pile has length *L*, cross-sectional area *A*, and modulus of elasticity *E*.

(a) Derive a formula for the shortening δ of the pile in terms of *P*, *L*, *E*, and *A*.
(b) Draw a diagram showing how the compressive stress σ_c varies throughout the length of the pile.



Solution 2.3-9 **Wood pile with friction**



FROM FREE-BODY DIAGRAM OF PILE:

$$\Sigma F_{\text{vert}} = 0 \quad \uparrow_+ \ \downarrow^- \quad fL - P = 0 \quad f = \frac{P}{L} \qquad \text{(Eq. 1)}$$

(a) SHORTENING δ OF PILE:

At distance y from the base:

$$N(y) = \text{axial force} \qquad N(y) = fy \qquad \text{(Eq. 2)}$$

$$d\delta = \frac{N(y)\,dy}{EA} = \frac{fy\,dy}{EA}$$

$$\delta = \int_0^L d\delta = \frac{f}{EA}\int_0^L ydy = \frac{fL^2}{2EA} = \frac{PL}{2EA}$$

$$\delta = \frac{PL}{2EA} \longleftarrow$$

(b) COMPRESSIVE STRESS σ_c IN PILE

$$\sigma_c = \frac{N(y)}{A} = \frac{fy}{A} = \frac{Py}{AL} \longleftarrow$$

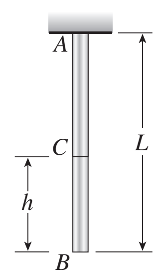
At the base ($y = 0$): $\sigma_c = 0$

At the top($y = L$): $\sigma_c = \dfrac{P}{A}$

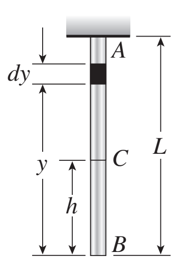
See the diagram above.

Problem 2.3-10 A prismatic bar AB of length L, cross-sectional area A, modulus of elasticity E, and weight W hangs vertically under its own weight (see figure).

(a) Derive a formula for the downward displacement δ_C of point C, located at distance h from the lower end of the bar.
(b) What is the elongation δ_B of the entire bar?
(c) What is the ratio β of the elongation of the upper half of the bar to the elongation of the lower half of the bar?



Solution 2.3-10 **Prismatic bar hanging vertically**



W = Weight of bar

(a) DOWNWARD DISPLACEMENT δ_C

Consider an element at distance y from the lower end.

$$N(y) = \frac{Wy}{L} \qquad d\delta = \frac{N(y)dy}{EA} = \frac{Wydy}{EAL}$$

$$\delta_C = \int_h^L d\delta = \int_h^L \frac{Wydy}{EAL} = \frac{W}{2EAL}(L^2 - h^2)$$

$$\delta_C = \frac{W}{2EAL}(L^2 - h^2) \longleftarrow$$

(b) ELONGATION OF BAR ($h = 0$)

$$\delta_B = \frac{WL}{2EA} \longleftarrow$$

(c) RATIO OF ELONGATIONS

Elongation of upper half of bar $\left(h = \dfrac{L}{2}\right)$:

$$\delta_{\text{upper}} = \frac{3WL}{8EA}$$

Elongation of lower half of bar:

$$\delta_{\text{lower}} = \delta_B - \delta_{\text{upper}} = \frac{WL}{2EA} - \frac{3WL}{8EA} = \frac{WL}{8EA}$$

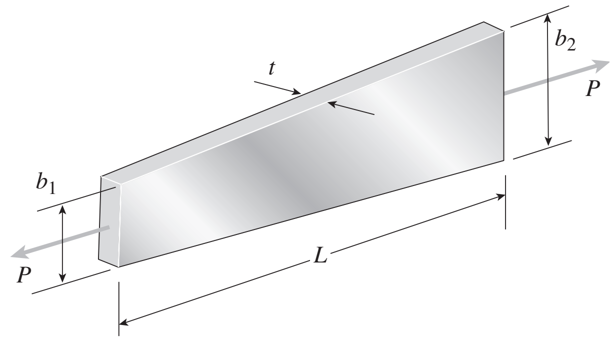
$$\beta = \frac{\delta_{\text{upper}}}{\delta_{\text{lower}}} = \frac{3/8}{1/8} = 3 \longleftarrow$$

Problem 2.3-11 A flat bar of rectangular cross section, length L, and constant thickness t is subjected to tension by forces P (see figure). The width of the bar varies linearly from b_1 at the smaller end to b_2 at the larger end. Assume that the angle of taper is small.

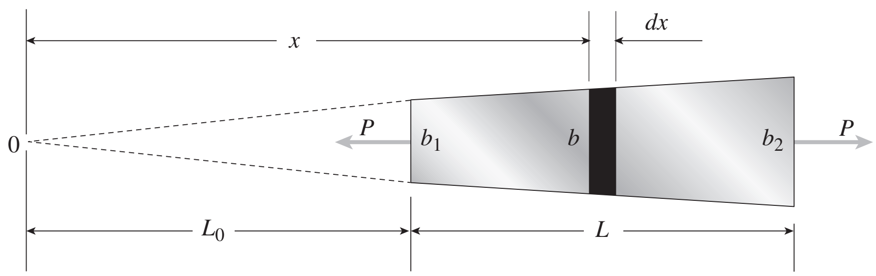
(a) Derive the following formula for the elongation of the bar:

$$\delta = \frac{PL}{Et(b_2 - b_1)} \ln \frac{b_2}{b_1}$$

(b) Calculate the elongation, assuming $L = 5$ ft, $t = 1.0$ in., $P = 25$ k, $b_1 = 4.0$ in., $b_2 = 6.0$ in., and $E = 30 \times 10^6$ psi.



Solution 2.3-11 **Tapered bar (rectangular cross section)**



t = thickness (constant)

$$b = b_1\left(\frac{x}{L_0}\right) \quad b_2 = b_1\left(\frac{L_0 + L}{L_0}\right) \qquad \text{(Eq. 1)}$$

$$A(x) = bt = b_1 t\left(\frac{x}{L_0}\right)$$

(a) Elongation of the bar

$$d\delta = \frac{P\,dx}{EA(x)} = \frac{PL_0\,dx}{Eb_1 t x}$$

$$\delta = \int_{L_0}^{L_0+L} d\delta = \frac{PL_0}{Eb_1 t}\int_{L_0}^{L_0+L}\frac{dx}{x}$$

$$= \frac{PL_0}{Eb_1 t}\ln x\Big|_{L_0}^{L_0+L} = \frac{PL_0}{Eb_1 t}\ln\frac{L_0+L}{L_0} \qquad \text{(Eq. 2)}$$

From Eq. (1): $$\frac{L_0 + L}{L_0} = \frac{b_2}{b_1} \qquad \text{(Eq. 3)}$$

Solve Eq. (3) for L_0: $$L_0 = L\left(\frac{b_1}{b_2 - b_1}\right) \qquad \text{(Eq. 4)}$$

Substitute Eqs. (3) and (4) into Eq. (2):

$$\delta = \frac{PL}{Et(b_2 - b_1)}\ln\frac{b_2}{b_1} \longleftarrow \qquad \text{(Eq. 5)}$$

(b) Substitute numerical values:

$L = 5$ ft $= 60$ in. $\quad t = 10$ in.

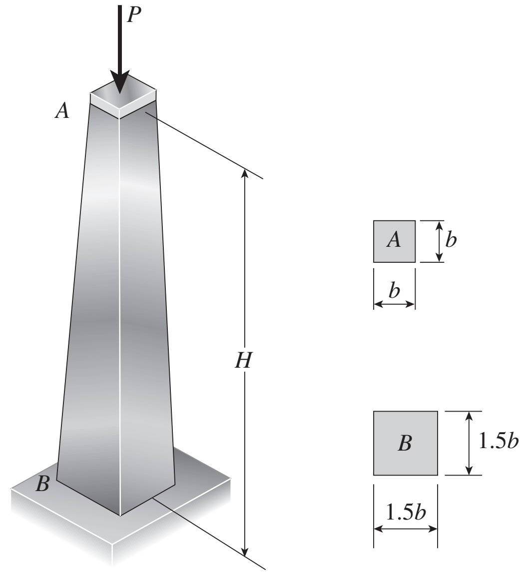
$P = 25$ k $\quad b_1 = 4.0$ in.

$b_2 = 6.0$ in. $\quad E = 30 \times 10^6$ psi

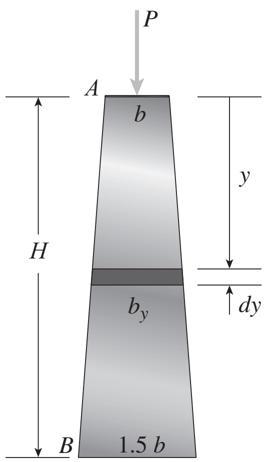
From Eq. (5): $\delta = 0.010$ in. $\longleftarrow$

Problem 2.3-12 A post AB supporting equipment in a laboratory is tapered uniformly throughout its height H (see figure). The cross sections of the post are square, with dimensions $b \times b$ at the top and $1.5b \times 1.5b$ at the base.

Derive a formula for the shortening δ of the post due to the compressive load P acting at the top. (Assume that the angle of taper is small and disregard the weight of the post itself.)



Solution 2.3-12 **Tapered post**



Square cross sections

b = width at A

$1.5b$ = width at B

b_y = width at distance y

$$= b + (1.5b - b)\frac{y}{H}$$

$$= \frac{b}{H}(H + 0.5y)$$

A_y = cross-sectional area at distance y

$$= (b_y)^2 = \frac{b^2}{H^2}(H + 0.5y)^2$$

SHORTENING OF ELEMENT dy

$$d\delta = \frac{Pdy}{EA_y} = \frac{Pdy}{E\left(\dfrac{b^2}{H^2}\right)(H + 0.5y)^2}$$

SHORTENING OF ENTIRE POST

$$\delta = \int d\delta = \frac{PH^2}{Eb^2}\int_0^H \frac{dy}{(H + 0.5y)^2}$$

From Appendix C: $\displaystyle\int \frac{dx}{(a + bx)^2} = -\frac{1}{b(a + bx)}$

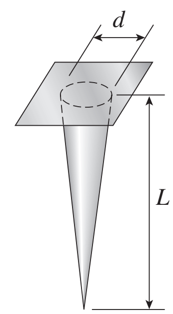
$$\delta = \frac{PH^2}{Eb^2}\left[-\frac{1}{(0.5)(H + 0.5y)}\right]_0^H$$

$$= \frac{PH^2}{Eb^2}\left[-\frac{1}{(0.5)(1.5H)} + \frac{1}{0.5H}\right]$$

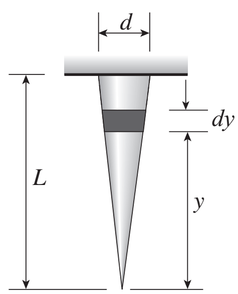
$$= \frac{2PH}{3Eb^2} \quad \longleftarrow$$

Problem 2.3-13 A long, slender bar in the shape of a right circular cone with length L and base diameter d hangs vertically under the action of its own weight (see figure). The weight of the cone is W and the modulus of elasticity of the material is E.

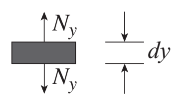
Derive a formula for the increase δ in the length of the bar due to its own weight. (Assume that the angle of taper of the cone is small.)



Solution 2.3-13 Conical bar hanging vertically



ELEMENT OF BAR



W = weight of cone

TERMINOLOGY

N_y = axial force acting on element dy

A_y = cross-sectional area at element dy

A_B = cross-sectional area at base of cone

$$= \frac{\pi d^2}{4}$$

V = volume of cone

$$= \frac{1}{3} A_B L$$

V_y = volume of cone below element dy

$$= \frac{1}{3} A_y y$$

W_y = weight of cone below element dy

$$= \frac{V_y}{V}(W) = \frac{A_y y W}{A_B L}$$

$N_y = W_y$

ELONGATION OF ELEMENT dy

$$d\delta = \frac{N_y\,dy}{E\,A_y} = \frac{Wy\,dy}{E\,A_B L} = \frac{4W}{\pi\,d^2\,EL}\,y\,dy$$

ELONGATION OF CONICAL BAR

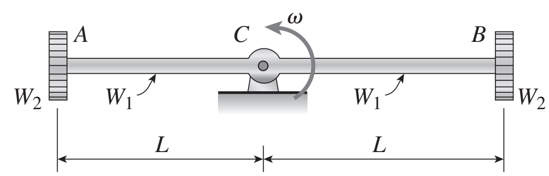
$$\delta = \int d\delta = \frac{4W}{\pi d^2 EL}\int_0^L y\,dy = \frac{2WL}{\pi\,d^2 E} \quad \longleftarrow$$

Problem 2.3-14 A bar ABC revolves in a horizontal plane about a vertical axis at the midpoint C (see figure). The bar, which has length $2L$ and cross-sectional area A, revolves at constant angular speed ω. Each half of the bar (AC and BC) has weight W_1 and supports a weight W_2 at its end.

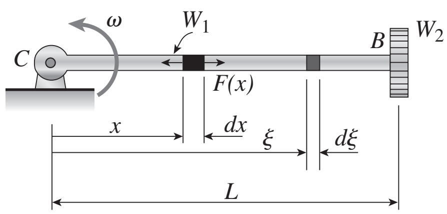
Derive the following formula for the elongation of one-half of the bar (that is, the elongation of either AC or BC):

$$\delta = \frac{L^2\omega^2}{3gEA}(W_1 + 3W_2)$$

in which E is the modulus of elasticity of the material of the bar and g is the acceleration of gravity.



Solution 2.3-14 **Rotating bar**



ω = angular speed

A = cross-sectional area

E = modulus of elasticity

g = acceleration of gravity

$F(x)$ = axial force in bar at distance x from point C

Consider an element of length dx at distance x from point C.

To find the force $F(x)$ acting on this element, we must find the inertia force of the part of the bar from distance x to distance L, plus the inertia force of the weight W_2.

Since the inertia force varies with distance from point C, we now must consider an element of length $d\xi$ at distance ξ, where ξ varies from x to L.

Mass of element $d\xi = \dfrac{d\xi}{L}\left(\dfrac{W_1}{g}\right)$

Acceleration of element $= \xi\omega^2$

Centrifugal force produced by element

$$= (\text{mass})(\text{acceleration}) = \frac{W_1\omega^2}{gL}\xi d\xi$$

Centrifugal force produced by weight W_2

$$= \left(\frac{W_2}{g}\right)(L\omega^2)$$

AXIAL FORCE $F(x)$

$$F(x) = \int_{\xi=x}^{\xi=L}\frac{W_1\omega^2}{gL}\xi d\xi + \frac{W_2L\omega^2}{g}$$

$$= \frac{W_1\omega^2}{2gL}(L^2 - x^2) + \frac{W_2L\omega^2}{g}$$

ELONGATION OF BAR BC

$$\delta = \int_0^L \frac{F(x)\,dx}{EA}$$

$$= \int_0^L \frac{W_1\omega^2}{2gLEA}(L^2 - x^2)dx + \int_0^L \frac{W_2L\omega^2 dx}{gEA}$$

$$= \frac{W_1\omega^2}{2gLEA}\left[\int_0^L L^2\,dx - \int_0^L x^2\,dx\right] + \frac{W_2L\omega^2}{gEA}\int_0^L dx$$

$$= \frac{W_1L^2\omega^2}{3gEA} + \frac{W_2L^2\omega^2}{gEA}$$

$$= \frac{L^2\omega^2}{3gEA}(W_1 + 3W_2) \longleftarrow$$

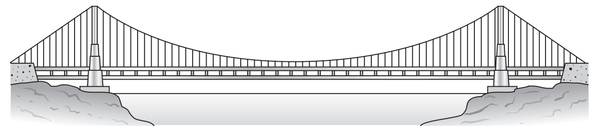
Problem 2.3-15 The main cables of a suspension bridge [see part (a) of the figure] follow a curve that is nearly parabolic because the primary load on the cables is the weight of the bridge deck, which is uniform in intensity along the horizontal. Therefore, let us represent the central region *AOB* of one of the main cables [see part (b) of the figure] as a parabolic cable supported at points *A* and *B* and carrying a uniform load of intensity *q* along the horizontal. The span of the cable is *L*, the sag is *h*, the axial rigidity is *EA*, and the origin of coordinates is at midspan.

(a) Derive the following formula for the elongation of cable *AOB* shown in part (b) of the figure:

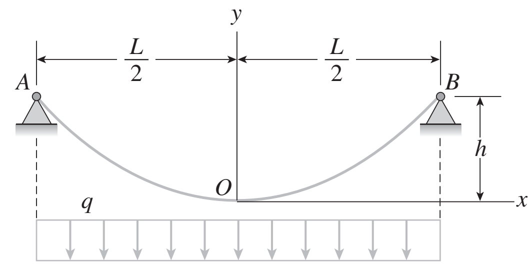
$$\delta = \frac{qL^3}{8hEA}\left(1 + \frac{16h^2}{3L^2}\right)$$

(b) Calculate the elongation δ of the central span of one of the main cables of the Golden Gate Bridge, for which the dimensions and properties are $L = 4200$ ft, $h = 470$ ft, $q = 12{,}700$ lb/ft, and $E = 28{,}800{,}000$ psi. The cable consists of 27,572 parallel wires of diameter 0.196 in.

Hint: Determine the tensile force *T* at any point in the cable from a free-body diagram of part of the cable; then determine the elongation of an element of the cable of length *ds*; finally, integrate along the curve of the cable to obtain an equation for the elongation δ.

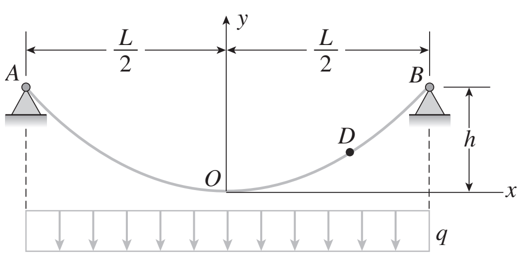


(a)



(b)

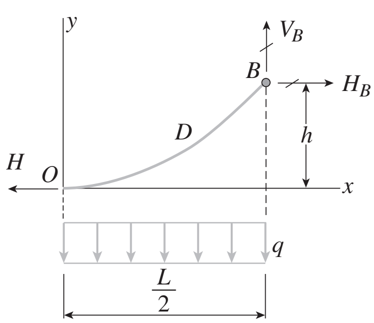
Solution 2.3-15 Cable of a suspension bridge



Equation of parabolic curve:

$$y = \frac{4hx^2}{L^2}$$

$$\frac{dy}{dx} = \frac{8hx}{L^2}$$



FREE-BODY DIAGRAM OF HALF OF CABLE

$$\Sigma M_B = 0$$

$$-Hh + \frac{qL}{2}\left(\frac{L}{4}\right) = 0$$

$$H = \frac{qL^2}{8h}$$

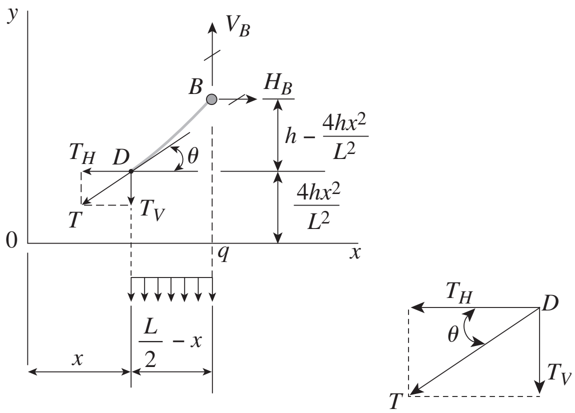
$$\Sigma F_{\text{horizontal}} = 0$$

$$H_B = H = \frac{qL^2}{8h} \qquad \text{(Eq. 1)}$$

$$\Sigma F_{\text{vertical}} = 0$$

$$V_B = \frac{qL}{2} \qquad \text{(Eq. 2)}$$

FREE-BODY DIAGRAM OF SEGMENT *DB* OF CABLE



$$\Sigma F_{\text{horiz}} = 0 \qquad T_H = H_B = \frac{qL^2}{8h} \qquad \text{(Eq. 3)}$$

$$\Sigma F_{\text{vert}} = 0 \qquad V_B - T_V - q\left(\frac{L}{2} - x\right) = 0$$

$$T_V = V_B - q\left(\frac{L}{2} - x\right) = \frac{qL}{2} - \frac{qL}{2} + qx$$

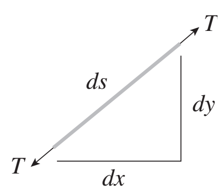
$$= qx \qquad \text{(Eq. 4)}$$

TENSILE FORCE *T* IN CABLE

$$T = \sqrt{T_H^2 + T_V^2} = \sqrt{\left(\frac{qL^2}{8h}\right)^2 + (qx)^2}$$

$$= \frac{qL^2}{8h}\sqrt{1 + \frac{64h^2x^2}{L^4}} \qquad \text{(Eq. 5)}$$

ELONGATION $d\delta$ OF AN ELEMENT OF LENGTH *ds*



$$d\delta = \frac{T ds}{EA}$$

$$ds = \sqrt{(dx)^2 + (dy)^2} = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$= dx\sqrt{1 + \left(\frac{8hx}{L^2}\right)^2}$$

$$= dx\sqrt{1 + \frac{64h^2x^2}{L^4}} \qquad \text{(Eq. 6)}$$

(a) ELONGATION δ OF CABLE *AOB*

$$\delta = \int d\delta = \int \frac{T\,ds}{EA}$$

Substitute for *T* from Eq. (5) and for *ds* from Eq. (6):

$$\delta = \frac{1}{EA}\int \frac{qL^2}{8h}\left(1 + \frac{64h^2x^2}{L^4}\right) dx$$

For both halves of cable:

$$\delta = \frac{2}{EA}\int_0^{L/2} \frac{qL^2}{8h}\left(1 + \frac{64h^2x^2}{L^4}\right) dx$$

$$\delta = \frac{qL^3}{8hEA}\left(1 + \frac{16h^2}{3L^2}\right) \longleftarrow \qquad \text{(Eq. 7)}$$

(b) GOLDEN GATE BRIDGE CABLE

$L = 4200$ ft $\qquad h = 470$ ft

$q = 12{,}700$ lb/ft $\qquad E = 28{,}800{,}000$ psi

27,572 wires of diameter $d = 0.196$ in.

$$A = (27{,}572)\left(\frac{\pi}{4}\right)(0.196 \text{ in.})^2 = 831.90 \text{ in.}^2$$

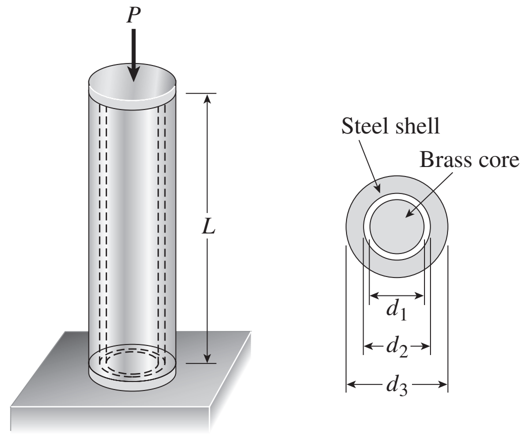
Substitute into Eq. (7):

$\delta = 133.7$ in $= 11.14$ ft $\longleftarrow$

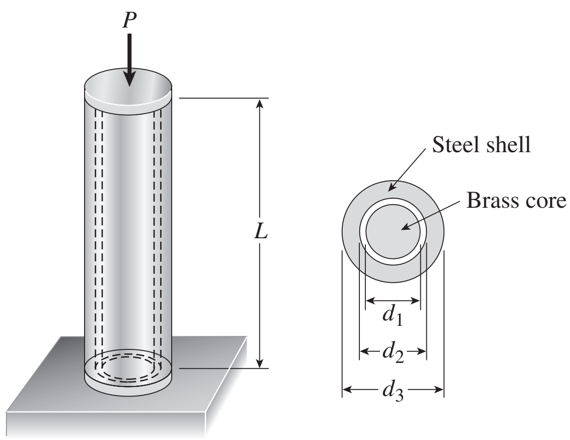
Statically Indeterminate Structures

Problem 2.4-1 The assembly shown in the figure consists of a brass core (diameter $d_1 = 0.25$ in.) surrounded by a steel shell (inner diameter $d_2 = 0.28$ in., outer diameter $d_3 = 0.35$ in.). A load P compresses the core and shell, which have length $L = 4.0$ in. The moduli of elasticity of the brass and steel are $E_b = 15 \times 10^6$ psi and $E_s = 30 \times 10^6$ psi, respectively.

(a) What load P will compress the assembly by 0.003 in.?
(b) If the allowable stress in the steel is 22 ksi and the allowable stress in the brass is 16 ksi, what is the allowable compressive load P_{allow}? (*Suggestion:* Use the equations derived in Example 2-5.)



Solution 2.4-1 **Cylindrical assembly in compression**



$d_1 = 0.25$ in. $\quad E_b = 15 \times 10^6$ psi

$d_2 = 0.28$ in. $\quad E_s = 30 \times 10^6$ psi

$d_3 = 0.35$ in. $\quad A_s = \dfrac{\pi}{4}(d_3^2 - d_2^2) = 0.03464 \text{ in.}^2$

$L = 4.0$ in. $\quad A_b = \dfrac{\pi}{4} d_1^2 = 0.04909 \text{ in.}^2$

(a) DECREASE IN LENGTH ($\delta = 0.003$ in.)

Use Eq. (2-13) of Example 2-5.

$$\delta = \frac{PL}{E_s A_s + E_b A_b} \quad \text{or}$$

$$P = (E_s A_s + E_b A_b)\left(\frac{\delta}{L}\right)$$

Substitute numerical values:

$$E_s A_s + E_b A_b = (30 \times 10^6 \text{ psi})(0.03464 \text{ in.}^2) + (15 \times 10^6 \text{ psi})(0.04909 \text{ in.}^2) = 1.776 \times 10^6 \text{ lb}$$

$$P = (1.776 \times 10^6 \text{ lb})\left(\frac{0.003 \text{ in.}}{4.0 \text{ in.}}\right) = 1330 \text{ lb} \quad \longleftarrow$$

(b) ALLOWABLE LOAD

$\sigma_s = 22$ ksi $\qquad \sigma_b = 16$ ksi

Use Eqs. (2-12a and b) of Example 2-5.

For steel:

$$\sigma_s = \frac{PE_s}{E_s A_s + E_b A_b} \qquad P_s = (E_s A_s + E_b A_b)\frac{\sigma_s}{E_s}$$

$$P_s = (1.776 \times 10^6 \text{ lb})\left(\frac{22 \text{ ksi}}{30 \times 10^6 \text{ psi}}\right) = 1300 \text{ lb}$$

For brass:

$$\sigma_b = \frac{PE_b}{E_s A_s + E_b A_b} \qquad P_s = (E_s A_s + E_b A_b)\frac{\sigma_b}{E_b}$$

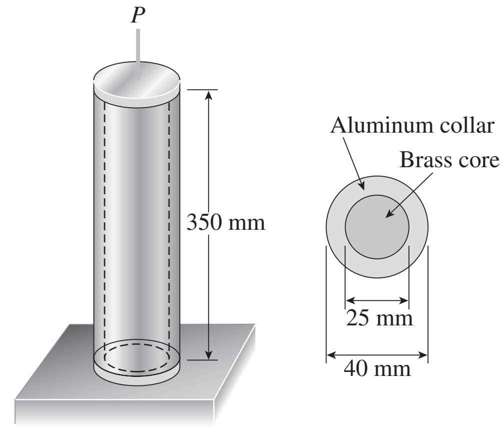
$$P_s = (1.776 \times 10^6 \text{ lb})\left(\frac{16 \text{ ksi}}{15 \times 10^6 \text{ psi}}\right) = 1890 \text{ lb}$$

Steel governs. $\quad P_{allow} = 1300$ lb $\quad \longleftarrow$

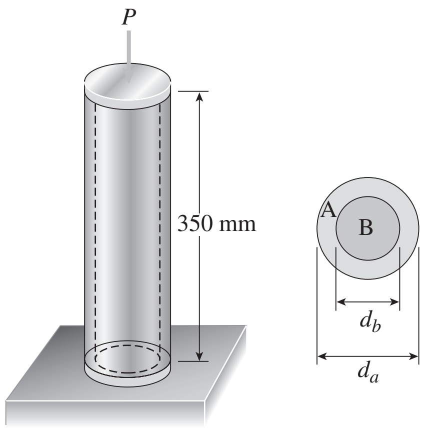
Problem 2.4-2 A cylindrical assembly consisting of a brass core and an aluminum collar is compressed by a load P (see figure). The length of the aluminum collar and brass core is 350 mm, the diameter of the core is 25 mm, and the outside diameter of the collar is 40 mm. Also, the moduli of elasticity of the aluminum and brass are 72 GPa and 100 GPa, respectively.

(a) If the length of the assembly decreases by 0.1% when the load P is applied, what is the magnitude of the load?

(b) What is the maximum permissible load P_{max} if the allowable stresses in the aluminum and brass are 80 MPa and 120 MPa, respectively? (*Suggestion:* Use the equations derived in Example 2-5.)



Solution 2.4-2 **Cylindrical assembly in compression**



A = aluminum

B = brass

L = 350 mm

d_a = 40 mm

d_b = 25 mm

$$A_a = \frac{\pi}{4}(d_a^2 - d_b^2)$$

$$= 765.8 \text{ mm}^2$$

$$E_a = 72 \text{ GPa} \quad E_b = 100 \text{ GPa} \quad A_b = \frac{\pi}{4} d_b^2$$

$$= 490.9 \text{ mm}^2$$

(a) Decrease in length

(δ = 0.1% of L = 0.350 mm)

Use Eq. (2-13) of Example 2-5.

$$\delta = \frac{PL}{E_a A_a + E_b A_b} \quad \text{or}$$

$$P = (E_a A_a + E_b A_b)\left(\frac{\delta}{L}\right)$$

Substitute numerical values:

$$E_a A_a + E_b A_b = (72 \text{ GPa})(765.8 \text{ mm}^2) + (100 \text{ GPa})(490.9 \text{ mm}^2)$$

$$= 55.135 \text{ MN} + 49.090 \text{ MN}$$

$$= 104.23 \text{ MN}$$

$$P = (104.23 \text{ MN})\left(\frac{0.350 \text{ mm}}{350 \text{ mm}}\right)$$

$$= 104.2 \text{ kN} \quad \longleftarrow$$

(b) Allowable load

σ_a = 80 MPa $\qquad$ σ_b = 120 MPa

Use Eqs. (2-12a and b) of Example 2-5.

For aluminum:

$$\sigma_a = \frac{PE_a}{E_a A_a + E_b A_b} \quad P_a = (E_a A_a + E_b A_b)\left(\frac{\sigma_a}{E_a}\right)$$

$$P_a = (104.23 \text{ MN})\left(\frac{80 \text{ MPa}}{72 \text{ GPa}}\right) = 115.8 \text{ kN}$$

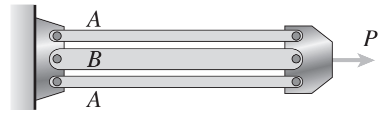
For brass:

$$\sigma_b = \frac{PE_b}{E_a A_a + E_b A_b} \quad P_b = (E_a A_a + E_b A_b)\left(\frac{\sigma_b}{E_b}\right)$$

$$P_b = (104.23 \text{ MN})\left(\frac{120 \text{ MPa}}{100 \text{ GPa}}\right) = 125.1 \text{ kN}$$

Aluminum governs. $\quad P_{max} = 116 \text{ kN} \quad \longleftarrow$

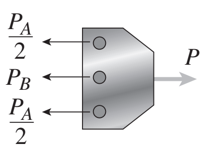
Problem 2.4-3 Three prismatic bars, two of material A and one of material B, transmit a tensile load P (see figure). The two outer bars (material A) are identical. The cross-sectional area of the middle bar (material B) is 50% larger than the cross-sectional area of one of the outer bars. Also, the modulus of elasticity of material A is twice that of material B.



(a) What fraction of the load P is transmitted by the middle bar?
(b) What is the ratio of the stress in the middle bar to the stress in the outer bars?
(c) What is the ratio of the strain in the middle bar to the strain in the outer bars?

Solution 2.4-3 Prismatic bars in tension

FREE-BODY DIAGRAM OF END PLATE



EQUATION OF EQUILIBRIUM

$$\Sigma F_{\text{horiz}} = 0 \qquad P_A + P_B - P = 0 \tag{1}$$

EQUATION OF COMPATIBILITY

$$\delta_A = \delta_B \tag{2}$$

FORCE-DISPLACEMENT RELATIONS

A_A = total area of both outer bars

$$\delta_A = \frac{P_A L}{E_A A_A} \qquad \delta_B = \frac{P_B L}{E_B A_B} \tag{3}$$

Substitute into Eq. (2):

$$\frac{P_A L}{E_A A_A} = \frac{P_B L}{E_B A_B} \tag{4}$$

SOLUTION OF THE EQUATIONS

Solve simultaneously Eqs. (1) and (4):

$$P_A = \frac{E_A A_A P}{E_A A_A + E_B A_B} \qquad P_B = \frac{E_B A_B P}{E_A A_A + E_B A_B} \tag{5}$$

Substitute into Eq. (3):

$$\delta = \delta_A = \delta_B = \frac{PL}{E_A A_A + E_B A_B} \tag{6}$$

STRESSES:

$$\sigma_A = \frac{P_A}{A_A} = \frac{E_A P}{E_A A_A + E_B A_B}$$
$$\sigma_B = \frac{P_B}{A_B} = \frac{E_B P}{E_A A_A + E_B A_B} \tag{7}$$

(a) LOAD IN MIDDLE BAR

$$\frac{P_B}{P} = \frac{E_B A_B}{E_A A_A + E_B A_B} = \frac{1}{\frac{E_A A_A}{E_B A_B} + 1}$$

Given: $\frac{E_A}{E_B} = 2 \qquad \frac{A_A}{A_B} = \frac{1 + 1}{1.5} = \frac{4}{3}$

$$\therefore \frac{P_B}{P} = \frac{1}{\left(\frac{E_A}{E_B}\right)\left(\frac{A_A}{A_B}\right) + 1} = \frac{1}{\frac{8}{3} + 1} = \frac{3}{11} \quad \longleftarrow$$

(b) RATIO OF STRESSES

$$\frac{\sigma_B}{\sigma_A} = \frac{E_B}{E_A} = \frac{1}{2} \quad \longleftarrow$$

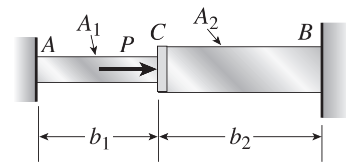
(c) RATIO OF STRAINS

All bars have the same strain

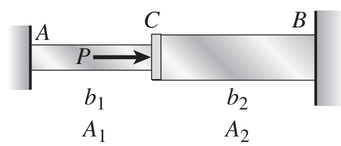
Ratio = 1 $\longleftarrow$

Problem 2.4-4 A bar *ACB* having two different cross-sectional areas A_1 and A_2 is held between rigid supports at *A* and *B* (see figure). A load *P* acts at point *C*, which is distance b_1 from end *A* and distance b_2 from end *B*.

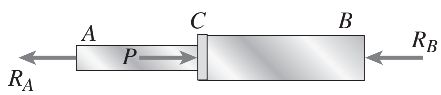
(a) Obtain formulas for the reactions R_A and R_B at supports *A* and *B*, respectively, due to the load *P*.
(b) Obtain a formula for the displacement δ_C of point *C*.
(c) What is the ratio of the stress σ_1 in region *AC* to the stress σ_2 in region *CB*?



Solution 2.4-4 **Bar with intermediate load**



FREE-BODY DIAGRAM



EQUATION OF EQUILIBRIUM

$\Sigma F_{\text{horiz}} = 0 \qquad R_A + R_B = P \qquad$ (Eq. 1)

EQUATION OF COMPATIBILITY

δ_{AC} = elongation of *AC*

δ_{CB} = shortening of *CB*

$\delta_{AC} = \delta_{CB}$ (Eq. 2)

FORCE DISPLACEMENT RELATIONS

$$\delta_{AC} = \frac{R_A b_1}{EA_1} \qquad \delta_{CB} = \frac{R_B b_2}{EA_2} \qquad \text{(Eqs. 3\&4)}$$

(a) SOLUTION OF EQUATIONS

Substitute Eq. (3) and Eq. (4) into Eq. (2):

$$\frac{R_A b_1}{EA_1} = \frac{R_B b_2}{EA_2} \qquad \text{(Eq. 5)}$$

Solve Eq. (1) and Eq. (5) simultaneously:

$$R_A = \frac{b_2 A_1 P}{b_1 A_2 + b_2 A_1} \qquad R_B = \frac{b_1 A_2 P}{b_1 A_2 + b_2 A_1} \quad \longleftarrow$$

(b) DISPLACEMENT OF POINT *C*

$$\delta_C = \delta_{AC} = \frac{R_A b_1}{EA_1} = \frac{b_1 b_2 P}{E(b_1 A_2 + b_2 A_1)} \quad \longleftarrow$$

(c) RATIO OF STRESSES

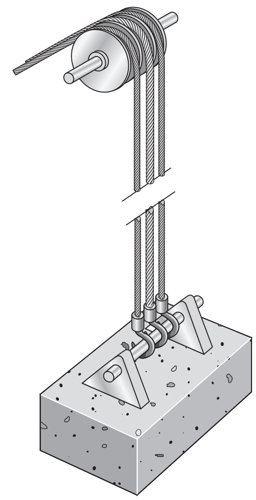
$$\sigma_1 = \frac{R_A}{A_1} \text{ (tension)} \qquad \sigma_2 = \frac{R_B}{A_2} \text{ (compression)}$$

$$\frac{\sigma_1}{\sigma_2} = \frac{b_2}{b_1} \quad \longleftarrow$$

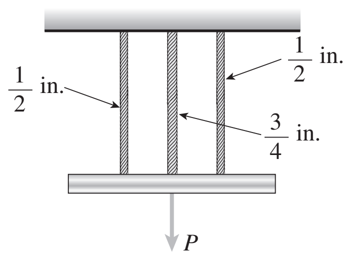
(Note that if $b_1 = b_2$, the stresses are numerically equal regardless of the areas A_1 and A_2.)

Problem 2.4-5 Three steel cables jointly support a load of 12 k (see figure). The diameter of the middle cable is ¾ in. and the diameter of each outer cable is ½ in. The tensions in the cables are adjusted so that each cable carries one-third of the load (i.e., 4 k). Later, the load is increased by 9 k to a total load of 21 k.

(a) What percent of the total load is now carried by the middle cable?
(b) What are the stresses σ_M and σ_O in the middle and outer cables, respectively? (*Note:* See Table 2-1 in Section 2.2 for properties of cables.)



Solution 2.4-5 **Three cables in tension**



AREAS OF CABLES (from Table 2-1)

Middle cable: $A_M = 0.268 \text{ in.}^2$

Outer cables: $A_O = 0.119 \text{ in.}^2$

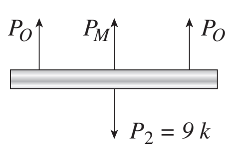
(for each cable)

FIRST LOADING

$$P_1 = 12\text{ k}\left(\text{Each cable carries } \frac{P_1}{3} \text{ or } 4\text{ k.}\right)$$

SECOND LOADING

$P_2 = 9$ k (additional load)



EQUATION OF EQUILIBRIUM

$$\Sigma F_{vert} = 0 \qquad 2P_O + P_M - P_2 = 0 \tag{1}$$

EQUATION OF COMPATIBILITY

$$\delta_M = \delta_O \tag{2}$$

FORCE-DISPLACEMENT RELATIONS

$$\delta_M = \frac{P_M L}{EA_M} \qquad \delta_O = \frac{P_o L}{EA_o} \tag{3, 4}$$

SUBSTITUTE INTO COMPATIBILITY EQUATION:

$$\frac{P_M L}{EA_M} = \frac{P_O L}{EA_O} \qquad \frac{P_M}{A_M} = \frac{P_O}{A_O} \tag{5}$$

SOLVE SIMULTANEOUSLY EQS. (1) AND (5):

$$P_M = P_2\left(\frac{A_M}{A_M + 2A_O}\right) = (9\text{ k})\left(\frac{0.268 \text{ in.}^2}{0.506 \text{ in.}^2}\right)$$

$$= 4.767\text{ k}$$

$$P_o = P_2\left(\frac{A_o}{A_M + 2A_O}\right) = (9\text{ k})\left(\frac{0.119 \text{ in.}^2}{0.506 \text{ in.}^2}\right)$$

$$= 2.117\text{ k}$$

FORCES IN CABLES

Middle cable: Force = 4 k + 4.767 k = 8.767 k

Outer cables: Force = 4 k + 2.117 k = 6.117 k

(for each cable)

(a) PERCENT OF TOTAL LOAD CARRIED BY MIDDLE CABLE

$$\text{Percent} = \frac{8.767\text{ k}}{21\text{ k}}(100\%) = 41.7\% \quad \longleftarrow$$

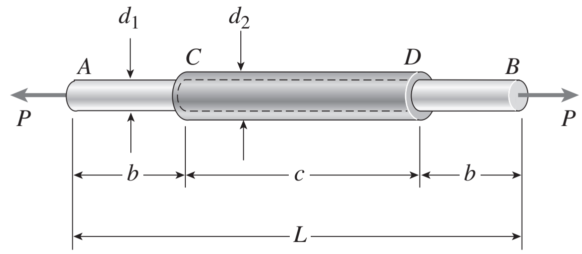
(b) STRESSES IN CABLES ($\sigma = P/A$)

$$\text{Middle cable: } \sigma_M = \frac{8.767\text{ k}}{0.268 \text{ in.}^2} = 32.7\text{ ksi} \quad \longleftarrow$$

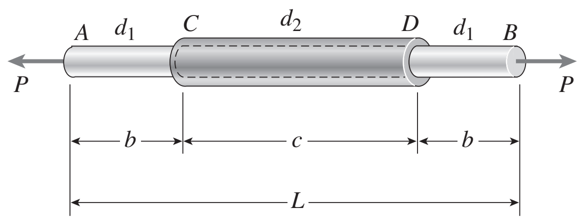
$$\text{Outer cables: } \sigma_O = \frac{6.117\text{ k}}{0.119 \text{ in.}^2} = 51.4\text{ ksi} \quad \longleftarrow$$

Problem 2.4-6 A plastic rod AB of length $L = 0.5$ m has a diameter $d_1 = 30$ mm (see figure). A plastic sleeve CD of length $c = 0.3$ m and outer diameter $d_2 = 45$ mm is securely bonded to the rod so that no slippage can occur between the rod and the sleeve. The rod is made of an acrylic with modulus of elasticity $E_1 = 3.1$ GPa and the sleeve is made of a polyamide with $E_2 = 2.5$ GPa.

(a) Calculate the elongation δ of the rod when it is pulled by axial forces $P = 12$ kN.
(b) If the sleeve is extended for the full length of the rod, what is the elongation?
(c) If the sleeve is removed, what is the elongation?



Solution 2.4-6 Plastic rod with sleeve



$P = 12$ kN $\quad d_1 = 30$ mm $\quad b = 100$ mm

$L = 500$ mm $\quad d_2 = 45$ mm $\quad c = 300$ mm

Rod: $E_1 = 3.1$ GPa

Sleeve: $E_2 = 2.5$ GPa

$$\text{Rod: } A_1 = \frac{\pi d_1^2}{4} = 706.86 \text{ mm}^2$$

$$\text{Sleeve: } A_2 = \frac{\pi}{4}(d_2^2 - d_1^2) = 883.57 \text{ mm}^2$$

$E_1A_1 + E_2A_2 = 4.400$ MN

(a) Elongation of rod

$$\text{Part } AC\text{: } \delta_{AC} = \frac{Pb}{E_1A_1} = 0.5476 \text{ mm}$$

$$\text{Part } CD\text{: } \delta_{CD} = \frac{Pc}{E_1A_1E_2A_2}$$

$$= 0.81815 \text{ mm}$$

(From Eq. 2-13 of Example 2-5)

$$\delta = 2\delta_{AC} + \delta_{CD} = 1.91 \text{ mm} \longleftarrow$$

(b) Sleeve at full length

$$\delta = \delta_{CD}\left(\frac{L}{c}\right) = (0.81815 \text{ mm})\left(\frac{500 \text{ mm}}{300 \text{ mm}}\right)$$

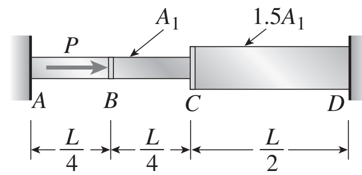
$$= 1.36 \text{ mm} \longleftarrow$$

(c) Sleeve removed

$$\delta = \frac{PL}{E_1A_1} = 2.74 \text{ mm} \longleftarrow$$

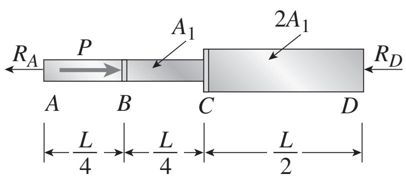
Problem 2.4-7 The axially loaded bar *ABCD* shown in the figure is held between rigid supports. The bar has cross-sectional area A_1 from *A* to *C* and $2A_1$ from *C* to *D*.

(a) Derive formulas for the reactions R_A and R_D at the ends of the bar.
(b) Determine the displacements δ_B and δ_C at points *B* and *C*, respectively.
(c) Draw a diagram in which the abscissa is the distance from the left-hand support to any point in the bar and the ordinate is the horizontal displacement δ at that point.



Solution 2.4-7 Bar with fixed ends

FREE-BODY DIAGRAM OF BAR



EQUATION OF EQUILIBRIUM

$\Sigma F_{\text{horiz}} = 0 \qquad R_A + R_D = P$ (Eq. 1)

EQUATION OF COMPATIBILITY

$\delta_{AB} + \delta_{BC} + \delta_{CD} = 0$ (Eq. 2)

Positive means elongation.

FORCE-DISPLACEMENT EQUATIONS

$$\delta_{AB} = \frac{R_A(L/4)}{EA_1} \qquad \delta_{BC} = \frac{(R_A - P)(L/4)}{EA_1} \qquad \text{(Eqs. 3, 4)}$$

$$\delta_{CD} = -\frac{R_D(L/2)}{E(2A_1)} \qquad \text{(Eq. 5)}$$

SOLUTION OF EQUATIONS

Substitute Eqs. (3), (4), and (5) into Eq. (2):

$$\frac{R_A L}{4EA_1} + \frac{(R_A - P)(L)}{4EA_1} - \frac{R_D L}{4EA_1} = 0 \quad \text{(Eq. 6)}$$

(a) REACTIONS

Solve simultaneously Eqs. (1) and (6):

$$R_A = \frac{2P}{3} \qquad R_D = \frac{P}{3} \quad \longleftarrow$$

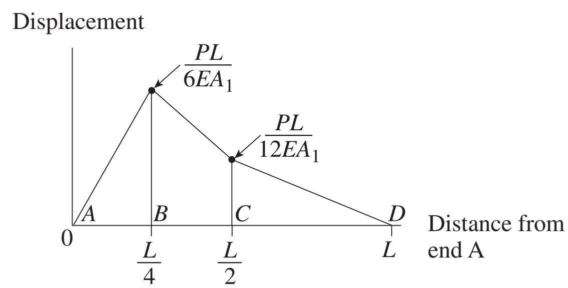
(b) DISPLACEMENTS AT POINTS *B* AND *C*

$$\delta_B = \delta_{AB} = \frac{R_A L}{4EA_1} = \frac{PL}{6EA_1} \text{ (To the right)} \quad \longleftarrow$$

$$\delta_C = |\delta_{CD}| = \frac{R_D L}{4EA_1}$$

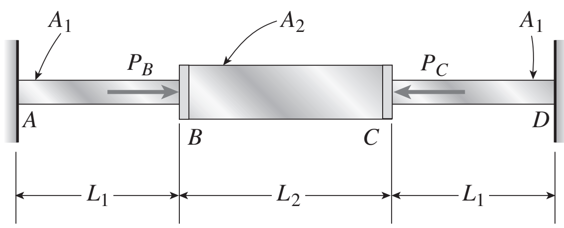
$$= \frac{PL}{12EA_1} \text{ (To the right)} \quad \longleftarrow$$

(c) DISPLACEMENT DIAGRAM

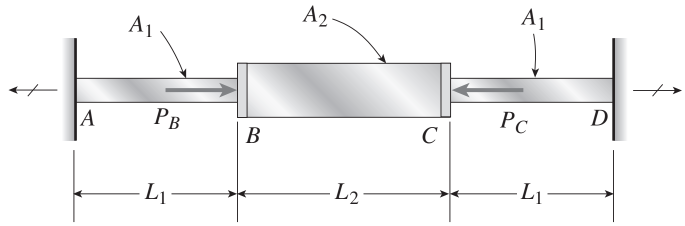


Problem 2.4-8 The fixed-end bar *ABCD* consists of three prismatic segments, as shown in the figure. The end segments have cross-sectional area $A_1 = 840\ \text{mm}^2$ and length $L_1 = 200$ mm. The middle segment has cross-sectional area $A_2 = 1260\ \text{mm}^2$ and length $L_2 = 250$ mm. Loads P_B and P_C are equal to 25.5 kN and 17.0 kN, respectively.

(a) Determine the reactions R_A and R_D at the fixed supports.
(b) Determine the compressive axial force F_{BC} in the middle segment of the bar.



Solution 2.4-8 Bar with three segments



$P_B = 25.5$ kN $\quad P_C = 17.0$ kN

$L_1 = 200$ mm $\quad L_2 = 250$ mm

$A_1 = 840\ \text{mm}^2 \quad A_2 = 1260\ \text{mm}^2$

m = meter

FREE-BODY DIAGRAM

EQUATION OF EQUILIBRIUM

$\Sigma F_{\text{horiz}} = 0 \quad \underset{+}{\rightarrow} \quad \underset{-}{\leftarrow}$

$P_B + R_D - P_C - R_A = 0$ or

$R_A - R_D = P_B - P_C = 8.5\ \text{kN} \qquad \text{(Eq. 1)}$

EQUATION OF COMPATIBILITY

δ_{AD} = elongation of entire bar

$\delta_{AD} = \delta_{AB} + \delta_{BC} + \delta_{CD} = 0 \qquad \text{(Eq. 2)}$

FORCE-DISPLACEMENT RELATIONS

$$\delta_{AB} = \frac{R_A L_1}{EA_1} = \frac{R_A}{E}\left(238.095\ \frac{1}{\text{m}}\right) \qquad \text{(Eq. 3)}$$

$$\delta_{BC} = \frac{(R_A - P_B)L_2}{EA_2}$$

$$= \frac{R_A}{E}\left(198.413\ \frac{1}{\text{m}}\right) - \frac{P_B}{E}\left(198.413\ \frac{1}{\text{m}}\right) \qquad \text{(Eq. 4)}$$

$$\delta_{CD} = \frac{R_D L_1}{EA_1} = \frac{R_D}{E}\left(238.095\ \frac{1}{\text{m}}\right) \qquad \text{(Eq. 5)}$$

SOLUTION OF EQUATIONS

Substitute Eqs. (3), (4), and (5) into Eq. (2):

$$\frac{R_A}{E}\left(238.095\ \frac{1}{\text{m}}\right) + \frac{R_A}{E}\left(198.413\ \frac{1}{\text{m}}\right)$$

$$-\frac{P_B}{E}\left(198.413\ \frac{1}{\text{m}}\right) + \frac{R_D}{E}\left(238.095\ \frac{1}{\text{m}}\right) = 0$$

Simplify and substitute $P_B = 25.5$ kN:

$$R_A\left(436.508\ \frac{1}{\text{m}}\right) + R_D\left(238.095\ \frac{1}{\text{m}}\right)$$

$$= 5{,}059.53\ \frac{\text{kN}}{\text{m}} \qquad \text{(Eq. 6)}$$

(a) REACTIONS R_A AND R_D

Solve simultaneously Eqs. (1) and (6).

From (1): $R_D = R_A - 8.5$ kN

Substitute into (6) and solve for R_A:

$$R_A\left(674.603\ \frac{1}{\text{m}}\right) = 7083.34\ \frac{\text{kN}}{\text{m}}$$

$R_A = 10.5\ \text{kN} \ \longleftarrow$

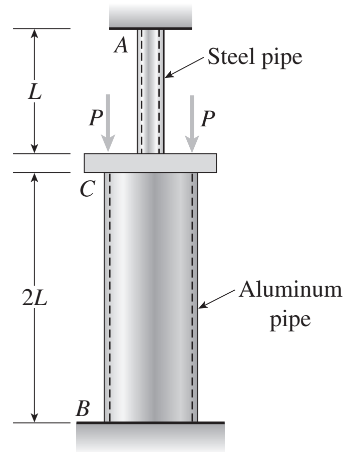
$R_D = R_A - 8.5\ \text{kN} = 2.0\ \text{kN} \ \longleftarrow$

(b) COMPRESSIVE AXIAL FORCE F_{BC}

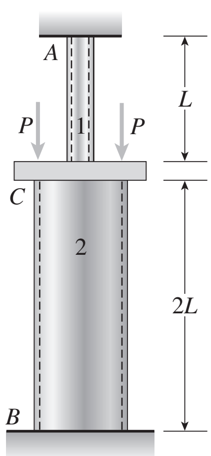
$F_{BC} = P_B - R_A = P_C - R_D = 15.0\ \text{kN} \ \longleftarrow$

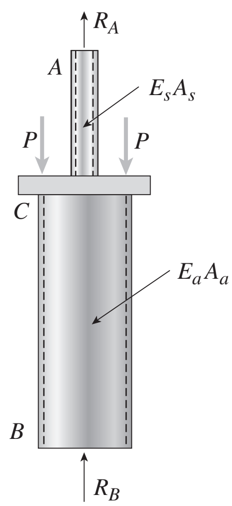
Problem 2.4-9 The aluminum and steel pipes shown in the figure are fastened to rigid supports at ends A and B and to a rigid plate C at their junction. The aluminum pipe is twice as long as the steel pipe. Two equal and symmetrically placed loads P act on the plate at C.

(a) Obtain formulas for the axial stresses σ_a and σ_s in the aluminum and steel pipes, respectively.
(b) Calculate the stresses for the following data: $P = 12$ k, cross-sectional area of aluminum pipe $A_a = 8.92$ in.^{2}, cross-sectional area of steel pipe $A_s = 1.03$ in.^{2}, modulus of elasticity of aluminum $E_a = 10 \times 10^6$ psi, and modulus of elasticity of steel $E_s = 29 \times 10^6$ psi.



Solution 2.4-9 Pipes with intermediate loads





Pipe 1 is steel.
Pipe 2 is aluminum.

EQUATION OF EQUILIBRIUM

$\Sigma F_{\text{vert}} = 0 \qquad R_A + R_B = 2P \qquad$ (Eq. 1)

EQUATION OF COMPATIBILITY

$\delta_{AB} = \delta_{AC} + \delta_{CB} = 0 \qquad$ (Eq. 2)

(A positive value of δ means elongation.)

FORCE-DISPLACEMENT RELATIONS

$$\delta_{AC} = \frac{R_A L}{E_s A_s} \qquad \delta_{BC} = -\frac{R_B(2L)}{E_a A_a} \qquad \text{(Eqs. 3, 4))}$$

SOLUTION OF EQUATIONS

Substitute Eqs. (3) and (4) into Eq. (2):

$$\frac{R_A L}{E_s A_s} - \frac{R_B(2L)}{E_a A_a} = 0 \qquad \text{(Eq. 5)}$$

Solve simultaneously Eqs. (1) and (5):

$$R_A = \frac{4E_s A_s P}{E_a A_a + 2E_s A_s} \qquad R_B = \frac{2E_a A_a P}{E_a A_a + 2E_s A_s} \qquad \text{(Eqs. 6, 7)}$$

(a) AXIAL STRESSES

$$\text{Aluminum: } \sigma_a = \frac{R_B}{A_a} = \frac{2E_a P}{E_a A_a + 2E_s A_s} \longleftarrow \qquad \text{(Eq. 8)}$$

(compression)

$$\text{Steel: } \sigma_s = \frac{R_A}{A_s} = \frac{4E_s P}{E_a A_a + 2E_s A_s} \longleftarrow \qquad \text{(Eq. 9)}$$

(tension)

(b) NUMERICAL RESULTS

$P = 12$ k $\qquad A_a = 8.92$ in.^{2} $\qquad A_s = 1.03$ in.^{2}

$E_a = 10 \times 10^6$ psi $\qquad E_s = 29 \times 10^6$ psi

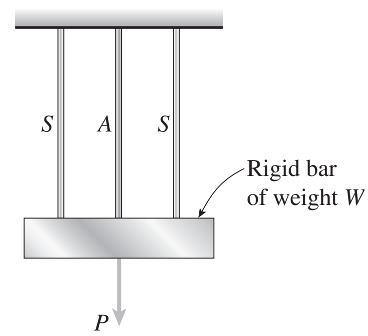
Substitute into Eqs. (8) and (9):

$\sigma_a = 1{,}610$ psi (compression) $\longleftarrow$

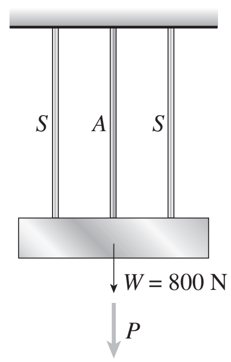
$\sigma_s = 9{,}350$ psi (tension) $\longleftarrow$

Problem 2.4-10 A rigid bar of weight $W = 800$ N hangs from three equally spaced vertical wires, two of steel and one of aluminum (see figure). The wires also support a load P acting at the midpoint of the bar. The diameter of the steel wires is 2 mm, and the diameter of the aluminum wire is 4 mm.

What load P_{allow} can be supported if the allowable stress in the steel wires is 220 MPa and in the aluminum wire is 80 MPa? (Assume $E_s = 210$ GPa and $E_a = 70$ GPa.)



Solution 2.4-10 **Rigid bar hanging from three wires**



STEEL WIRES

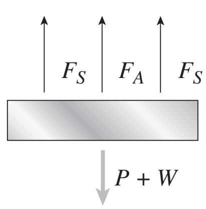
$d_s = 2$ mm $\quad \sigma_s = 220$ MPa $\quad E_s = 210$ GPa

ALUMINUM WIRES

$d_A = 4$ mm $\quad \sigma_A = 80$ MPa

$E_A = 70$ GPa

FREE-BODY DIAGRAM OF RIGID BAR



EQUATION OF EQUILIBRIUM

$\Sigma F_{vert} = 0$

$$2F_s + F_A - P - W = 0 \qquad \text{(Eq. 1)}$$

EQUATION OF COMPATIBILITY

$$\delta_s = \delta_A \qquad \text{(Eq. 2)}$$

FORCE DISPLACEMENT RELATIONS

$$\delta_s = \frac{F_s L}{E_s A_s} \qquad \delta_A = \frac{F_A L}{E_A A_A} \qquad \text{(Eqs. 3, 4)}$$

SOLUTION OF EQUATIONS

Substitute (3) and (4) into Eq. (2):

$$\frac{F_s L}{E_s A_s} = \frac{F_A L}{E_A A_A} \qquad \text{(Eq. 5)}$$

Solve simultaneously Eqs. (1) and (5):

$$F_A = (P + W)\left(\frac{E_A A_A}{E_A A_A + 2E_s A_s}\right) \qquad \text{(Eq. 6)}$$

$$F_s = (P + W)\left(\frac{E_s A_s}{E_A A_A + 2E_s A_s}\right) \qquad \text{(Eq. 7)}$$

STRESSES IN THE WIRES

$$\sigma_A = \frac{F_A}{A_A} = \frac{(P + W)E_A}{E_A A_A + 2E_s A_s} \qquad \text{(Eq. 8)}$$

$$\sigma_s = \frac{F_s}{A_s} = \frac{(P + W)E_s}{E_A A_A + 2E_s A_s} \qquad \text{(Eq. 9)}$$

ALLOWABLE LOADS (FROM EQS. (8) AND (9))

$$P_A = \frac{\sigma_A}{E_A}(E_A A_A + 2E_s A_s) - W \qquad \text{(Eq. 10)}$$

$$P_s = \frac{\sigma_s}{E_s}(E_A A_A + 2E_s A_s) - W \qquad \text{(Eq. 11)}$$

SUBSTITUTE NUMERICAL VALUES INTO EQS. (10) AND (11):

$$A_s = \frac{\pi}{4}(2\text{ mm})^2 = 3.1416\text{ mm}^2$$

$$A_A = \frac{\pi}{4}(4\text{ mm})^2 = 12.5664\text{ mm}^2$$

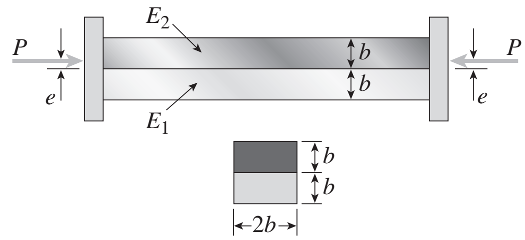
$P_A = 1713$ N

$P_s = 1504$ N

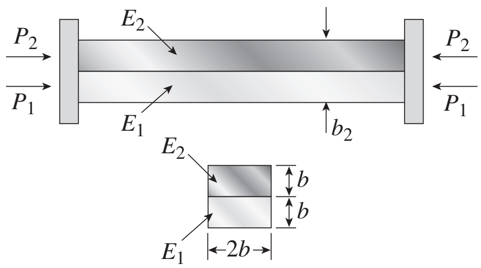
Steel governs. $\quad P_{allow} = 1500$ N $\longleftarrow$

Problem 2.4-11 A *bimetallic* bar (or composite bar) of square cross section with dimensions $2b \times 2b$ is constructed of two different metals having moduli of elasticity E_1 and E_2 (see figure). The two parts of the bar have the same cross-sectional dimensions. The bar is compressed by forces P acting through rigid end plates. The line of action of the loads has an eccentricity e of such magnitude that each part of the bar is stressed uniformly in compression.

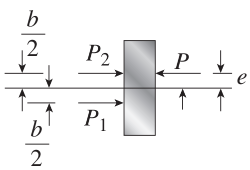
(a) Determine the axial forces P_1 and P_2 in the two parts of the bar.
(b) Determine the eccentricity e of the loads.
(c) Determine the ratio σ_1/σ_2 of the stresses in the two parts of the bar.



Solution 2.4-11 **Bimetallic bar in compression**



FREE-BODY DIAGRAM
(Plate at right-hand end)



EQUATIONS OF EQUILIBRIUM

$$\Sigma F = 0 \qquad P_1 + P_2 = P \qquad \text{(Eq. 1)}$$

$$\Sigma M = 0 \qquad Pe + P_1\left(\frac{b}{2}\right) - P_2\left(\frac{b}{2}\right) = 0 \qquad \text{(Eq. 2)}$$

EQUATION OF COMPATIBILITY

$$\delta_2 = \delta_1$$

$$\frac{P_2 L}{E_2 A} = \frac{P_1 L}{E_1 A} \quad \text{or} \quad \frac{P_2}{E_2} = \frac{P_1}{E_1} \qquad \text{(Eq. 3)}$$

(a) AXIAL FORCES

Solve simultaneously Eqs. (1) and (3):

$$P_1 = \frac{PE_1}{E_1 + E_2} \qquad P_2 = \frac{PE_2}{E_1 + E_2} \quad \longleftarrow$$

(b ECCENTRICITY OF LOAD P

Substitute P_1 and P_2 into Eq. (2) and solve for e:

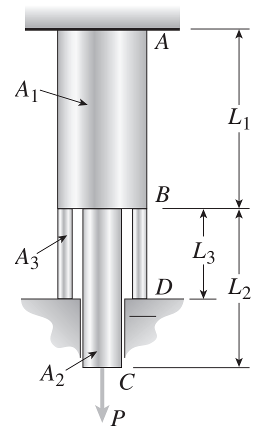
$$e = \frac{b(E_2 - E_1)}{2(E_2 + E_1)} \quad \longleftarrow$$

(c) RATIO OF STRESSES

$$\sigma_1 = \frac{P_1}{A} \qquad \sigma_2 = \frac{P_2}{A} \qquad \frac{\sigma_1}{\sigma_2} = \frac{P_1}{P_2} = \frac{E_1}{E_2} \quad \longleftarrow$$

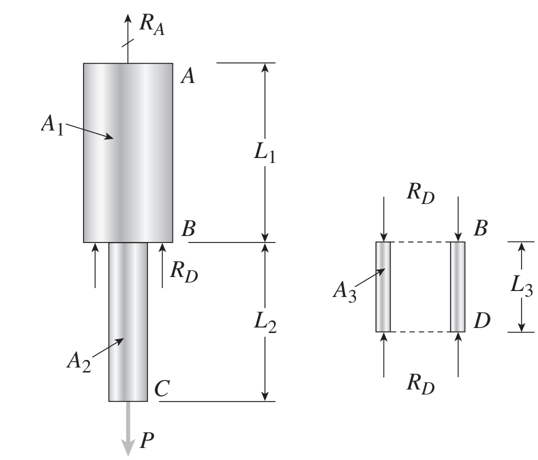
Problem 2.4-12 A circular steel bar ABC ($E = 200$ GPa) has cross-sectional area A_1 from A to B and cross-sectional area A_2 from B to C (see figure). The bar is supported rigidly at end A and is subjected to a load P equal to 40 kN at end C. A circular steel collar BD having cross-sectional area A_3 supports the bar at B. The collar fits snugly at B and D when there is no load.

Determine the elongation δ_{AC} of the bar due to the load P. (Assume $L_1 = 2L_3 = 250$ mm, $L_2 = 225$ mm, $A_1 = 2A_3 = 960$ mm^2, and $A_2 = 300$ mm^2.)



Solution 2.4-12 Bar supported by a collar

FREE-BODY DIAGRAM OF BAR ABC AND COLLAR BD



EQUILIBRIUM OF BAR ABC

$\Sigma F_{\text{vert}} = 0 \qquad R_A + R_D - P = 0$ (Eq. 1)

COMPATIBILITY (distance AD does not change)

$\delta_{AB}(\text{bar}) + \delta_{BD}(\text{collar}) = 0$ (Eq. 2)

(Elongation is positive.)

FORCE-DISPLACEMENT RELATIONS

$$\delta_{AB} = \frac{R_A L_1}{EA_1} \qquad \delta_{BD} = -\frac{R_D L_3}{EA_3}$$

Substitute into Eq. (2):

$$\frac{R_A L_1}{EA_1} - \frac{R_D L_3}{EA_3} = 0 \qquad \text{(Eq. 3)}$$

SOLVE SIMULTANEOUSLY EQS. (1) AND (3):

$$R_A = \frac{PL_3A_1}{L_1A_3 + L_3A_1} \qquad R_D = \frac{PL_1A_3}{L_1A_3 + L_3A_1}$$

CHANGES IN LENGTHS (Elongation is positive)

$$\delta_{AB} = \frac{R_AL_1}{EA_1} = \frac{PL_1L_3}{E(L_1A_3 + L_3A_1)} \qquad \delta_{BC} = \frac{PL_2}{EA_2}$$

ELONGATION OF BAR ABC

$$\delta_{AC} = \delta_{AB} + \delta_{AC}$$

SUBSTITUTE NUMERICAL VALUES:

$P = 40$ kN $\qquad E = 200$ GPa

$L_1 = 250$ mm

$L_2 = 225$ mm

$L_3 = 125$ mm

$A_1 = 960$ mm^2

$A_2 = 300$ mm^2

$A_3 = 480$ mm^2

RESULTS:

$R_A = R_D = 20$ kN

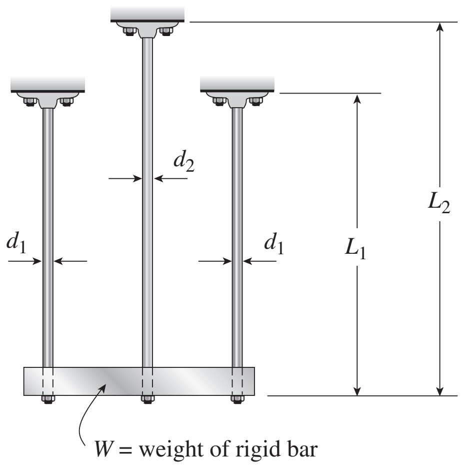
$\delta_{AB} = 0.02604$ mm

$\delta_{BC} = 0.15000$ mm

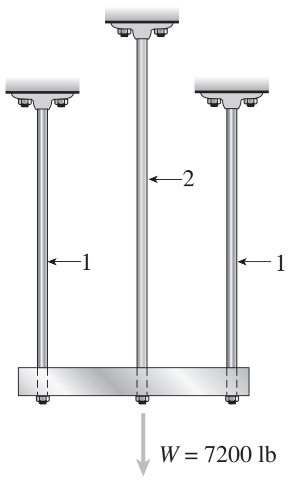
$\delta_{AC} = \delta_{AB} + \delta_{AC} = 0.176$ mm ←

Problem 2.4-13 A horizontal rigid bar of weight $W = 7200$ lb is supported by three slender circular rods that are equally spaced (see figure). The two outer rods are made of aluminum ($E_1 = 10 \times 10^6$ psi) with diameter $d_1 = 0.4$ in. and length $L_1 = 40$ in. The inner rod is magnesium ($E_2 = 6.5 \times 10^6$ psi) with diameter d_2 and length L_2. The allowable stresses in the aluminum and magnesium are 24,000 psi and 13,000 psi, respectively.

If it is desired to have all three rods loaded to their maximum allowable values, what should be the diameter d_2 and length L_2 of the middle rod?



Solution 2.4-13 **Bar supported by three rods**



BAR 1 ALUMINUM

$E_1 = 10 \times 10^6$ psi

$d_1 = 0.4$ in.

$L_1 = 40$ in.

$\sigma_1 = 24{,}000$ psi

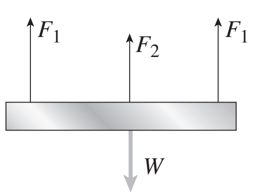
BAR 2 MAGNESIUM

$E_2 = 6.5 \times 10^6$ psi

$d_2 = ?$ $\quad L_2 = ?$

$\sigma_2 = 13{,}000$ psi

FREE-BODY DIAGRAM OF RIGID BAR
EQUATION OF EQUILIBRIUM



$$\Sigma F_{\text{vert}} = 0$$

$$2F_1 + F_2 - W = 0 \qquad \text{(Eq. 1)}$$

FULLY STRESSED RODS

$$F_1 = \sigma_1 A_1 \qquad F_2 = \sigma_2 A_2$$

$$A_1 = \frac{\pi d_1^2}{4} \qquad A_2 = \frac{\pi d_2^2}{4}$$

Substitute into Eq. (1):

$$2\sigma_1\left(\frac{\pi d_1^2}{4}\right) + \sigma_2\left(\frac{\pi d_2^2}{4}\right) = W$$

Diameter d_1 is known; solve for d_2:

$$d_2^2 = \frac{4W}{\pi \sigma_2} - \frac{2\sigma_1 d_1^2}{\sigma_2} \quad \longleftarrow \qquad \text{(Eq. 2)}$$

SUBSTITUTE NUMERICAL VALUES:

$$d_2^2 = \frac{4(7200 \text{ lb})}{\pi(13{,}000 \text{ psi})} - \frac{2(24{,}000 \text{ psi})\ (0.4 \text{ in.})^2}{13{,}000 \text{ psi}}$$

$$= 0.70518 \text{ in.}^2 - 0.59077 \text{ in.}^2 = 0.11441 \text{ in.}^2$$

$$d_2 = 0.338 \text{ in.} \quad \longleftarrow$$

EQUATION OF COMPATIBILITY

$$\delta_1 = \delta_2 \qquad \text{(Eq. 3)}$$

FORCE-DISPLACEMENT RELATIONS

$$\delta_1 = \frac{F_1 L_1}{E_1 A_1} = \sigma_1\left(\frac{L_1}{E_1}\right) \qquad \text{(Eq. 4)}$$

$$\delta_2 = \frac{F_2 L_2}{E_2 A_2} = \sigma_2\left(\frac{L_2}{E_2}\right) \qquad \text{(Eq. 5)}$$

Substitute (4) and (5) into Eq. (3):

$$\sigma_1\left(\frac{L_1}{E_1}\right) = \sigma_2\left(\frac{L_2}{E_2}\right)$$

Length L_1 is known; solve for L_2:

$$L_2 = L_1\left(\frac{\sigma_1 E_2}{\sigma_2 E_1}\right) \quad \longleftarrow \qquad \text{(Eq. 6)}$$

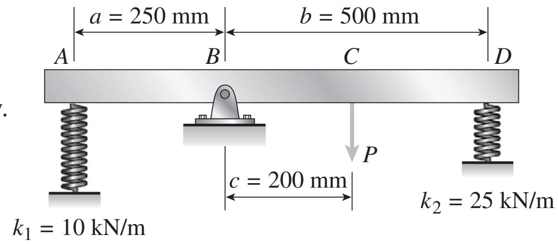
SUBSTITUTE NUMERICAL VALUES:

$$L_2 = (40 \text{ in.})\left(\frac{24{,}000 \text{ psi}}{13{,}000 \text{ psi}}\right)\left(\frac{6.5 \times 10^6 \text{ psi}}{10 \times 10^6 \text{ psi}}\right)$$

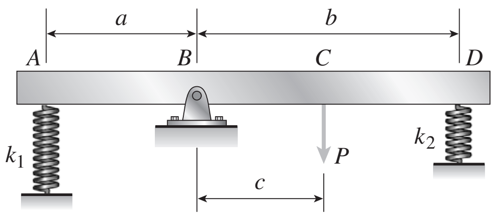
$$= 48.0 \text{ in.} \quad \longleftarrow$$

Problem 2.4-14 A rigid bar *ABCD* is pinned at point *B* and supported by springs at *A* and *D* (see figure). The springs at *A* and *D* have stiffnesses $k_1 = 10$ kN/m and $k_2 = 25$ kN/m, respectively, and the dimensions *a*, *b*, and *c* are 250 mm, 500 mm, and 200 mm, respectively. A load *P* acts at point *C*.

If the angle of rotation of the bar due to the action of the load *P* is limited to 3°, what is the maximum permissible load P_{max}?



Solution 2.4-14 **Rigid bar supported by springs**



NUMERICAL DATA

$a = 250$ mm

$b = 500$ mm

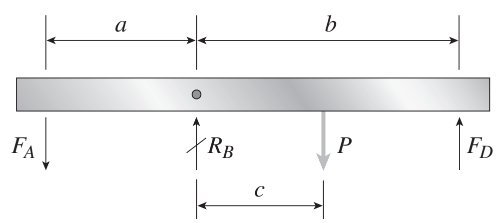
$c = 200$ mm

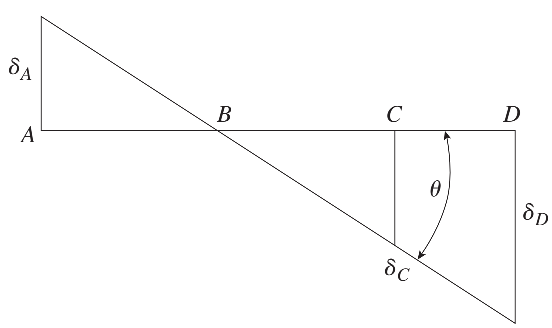
$k_1 = 10$ kN/m

$k_2 = 25$ kN/m

$$\theta_{max} = 3° = \frac{\pi}{60} \text{ rad}$$

FREE-BODY DIAGRAM AND DISPLACEMENT DIAGRAM





EQUATION OF EQUILIBRIUM

$$\Sigma M_B = 0 \quad F_A(a) - P(c) + F_D(b) = 0 \qquad \text{(Eq. 1)}$$

EQUATION OF COMPATIBILITY

$$\frac{\delta_A}{a} = \frac{\delta_D}{b} \qquad \text{(Eq. 2)}$$

FORCE-DISPLACEMENT RELATIONS

$$\delta_A = \frac{F_A}{k_1} \qquad \delta_D = \frac{F_D}{k_2} \qquad \text{(Eqs. 3, 4)}$$

SOLUTION OF EQUATIONS

Substitute (3) and (4) into Eq. (2):

$$\frac{F_A}{ak_1} = \frac{F_D}{bk_2} \qquad \text{(Eq. 5)}$$

SOLVE SIMULTANEOUSLY EQS. (1) AND (5):

$$F_A = \frac{ack_1P}{a^2k_1 + b^2k_2} \qquad F_D = \frac{bck_2P}{a^2k_1 + b^2k_2}$$

ANGLE OF ROTATION

$$\delta_D = \frac{F_D}{k_2} = \frac{bcP}{a^2k_1 + b^2k_2} \qquad \theta = \frac{\delta_D}{b} = \frac{cP}{a^2k_1 + b^2k_2}$$

MAXIMUM LOAD

$$P = \frac{\theta}{c}(a^2k_1 + b^2k_2)$$

$$P_{max} = \frac{\theta_{max}}{c}(a^2k_1 + b^2k_2) \quad \longleftarrow$$

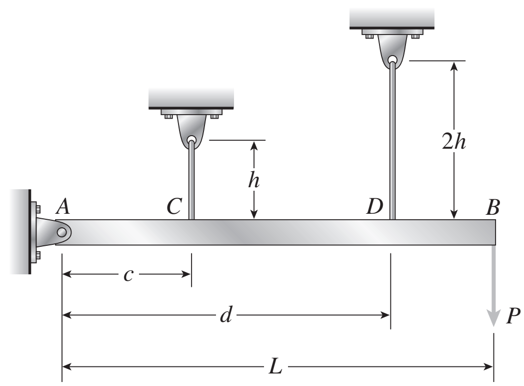
SUBSTITUTE NUMERICAL VALUES:

$$P_{max} = \frac{\pi/60 \text{ rad}}{200 \text{ mm}}[(250 \text{ mm})^2(10 \text{ kN/m}) + (500 \text{ mm})^2(25 \text{ kN/m})]$$

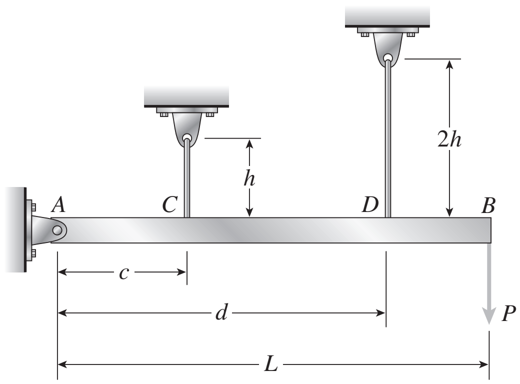
$$= 1800 \text{ N} \quad \longleftarrow$$

Problem 2.4-15 A rigid bar *AB* of length $L = 66$ in. is hinged to a support at *A* and supported by two vertical wires attached at points *C* and *D* (see figure). Both wires have the same cross-sectional area ($A = 0.0272$ in.^{2}) and are made of the same material (modulus $E = 30 \times 10^6$ psi). The wire at *C* has length $h = 18$ in. and the wire at *D* has length twice that amount. The horizontal distances are $c = 20$ in. and $d = 50$ in.

(a) Determine the tensile stresses σ_C and σ_D in the wires due to the load $P = 340$ lb acting at end *B* of the bar.
(b) Find the downward displacement δ_B at end *B* of the bar.



Solution 2.4-15 Bar supported by two wires



$h = 18$ in.

$2h = 36$ in.

$c = 20$ in.

$d = 50$ in.

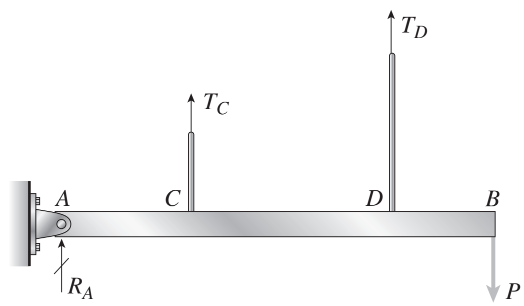
$L = 66$ in.

$E = 30 \times 10^6$ psi

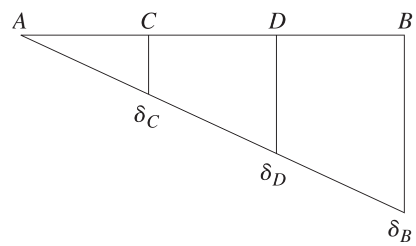
$A = 0.0272$ in.^{2}

$P = 340$ lb

FREE-BODY DIAGRAM



DISPLACEMENT DIAGRAM



EQUATION OF EQUILIBRIUM

$\Sigma M_A = 0$ $\quad T_C(c) + T_D(d) = PL$ (Eq. 1)

EQUATION OF COMPATIBILITY

$$\frac{\delta_C}{c} = \frac{\delta_D}{d} \qquad \text{(Eq. 2)}$$

FORCE-DISPLACEMENT RELATIONS

$$\delta_C = \frac{T_C h}{EA} \qquad \delta_D = \frac{T_D(2h)}{EA} \qquad \text{(Eqs. 3, 4)}$$

SOLUTION OF EQUATIONS

Substitute (3) and (4) into Eq. (2):

$$\frac{T_C h}{cEA} = \frac{T_D(2h)}{dEA} \quad \text{or} \quad \frac{T_C}{c} = \frac{2T_D}{d} \qquad \text{(Eq. 5)}$$

TENSILE FORCES IN THE WIRES

Solve simultaneously Eqs. (1) and (5):

$$T_C = \frac{2cPL}{2c^2 + d^2} \qquad T_D = \frac{dPL}{2c^2 + d^2} \qquad \text{(Eqs. 6, 7)}$$

TENSILE STRESSES IN THE WIRES

$$\sigma_C = \frac{T_C}{A} = \frac{2cPL}{A(2c^2 + d^2)} \qquad \text{(Eq. 8)}$$

$$\sigma_D = \frac{T_D}{A} = \frac{dPL}{A(2c^2 + d^2)} \qquad \text{(Eq. 9)}$$

DISPLACEMENT AT END OF BAR

$$\delta_B = \delta_D\left(\frac{L}{d}\right) = \frac{2hT_D}{EA}\left(\frac{L}{d}\right) = \frac{2hPL^2}{EA(2c^2 + d^2)} \qquad \text{(Eq. 10)}$$

SUBSTITUTE NUMERICAL VALUES

$$2c^2 + d^2 = 2(20 \text{ in.})^2 + (50 \text{ in.})^2 = 3300 \text{ in.}^2$$

(a) $$\sigma_C = \frac{2cPL}{A(2c^2 + d^2)} = \frac{2(20 \text{ in.})(340 \text{ lb})(66 \text{ in.})}{(0.0272 \text{ in.}^2)(3300 \text{ in.}^2)}$$

$$= 10{,}000 \text{ psi} \longleftarrow$$

$$\sigma_D = \frac{dPL}{A(2c^2 + d^2)} = \frac{(50 \text{ in.})(340 \text{ lb})(66 \text{ in.})}{(0.0272 \text{ in.}^2)(3300 \text{ in.}^2)}$$

$$= 12{,}500 \text{ psi} \longleftarrow$$

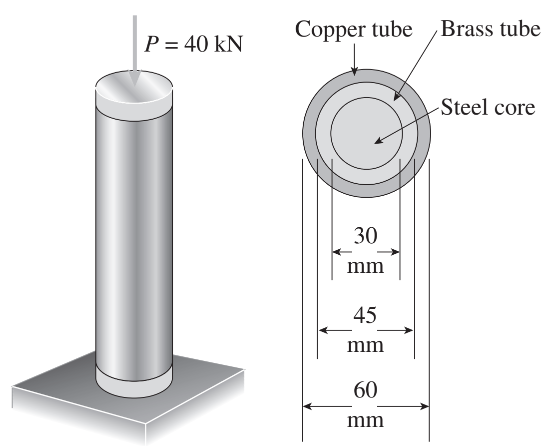
(b) $$\delta_B = \frac{2hPL^2}{EA(2c^2 + d^2)}$$

$$= \frac{2(18 \text{ in.})(340 \text{ lb})(66 \text{ in.})^2}{(30 \times 10^6 \text{ psi})(0.0272 \text{ in.}^2)(3300 \text{ in.}^2)}$$

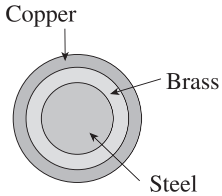
$$= 0.0198 \text{ in.} \longleftarrow$$

Problem 2.4-16 A trimetallic bar is uniformly compressed by an axial force $P = 40$ kN applied through a rigid end plate (see figure). The bar consists of a circular steel core surrounded by brass and copper tubes. The steel core has diameter 30 mm, the brass tube has outer diameter 45 mm, and the copper tube has outer diameter 60 mm. The corresponding moduli of elasticity are $E_s = 210$ GPa, $E_b = 100$ GPa, and $E_c = 120$ GPa.

Calculate the compressive stresses σ_s, σ_b, and σ_c in the steel, brass, and copper, respectively, due to the force P.



Solution 2.4-16 Trimetallic bar in compression

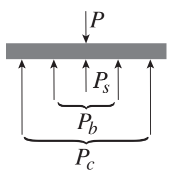


P_s = compressive force in steel core

P_b = compressive force in brass tube

P_c = compressive force in copper tube

FREE-BODY DIAGRAM OF RIGID END PLATE



EQUATION OF EQUILIBRIUM

$\Sigma F_{\text{vert}} = 0 \qquad P_s + P_b + P_c = P \qquad$ (Eq. 1)

EQUATIONS OF COMPATIBILITY

$\delta_s = \delta_b \qquad \delta_c = \delta_s \qquad$ (Eqs. 2)

FORCE-DISPLACEMENT RELATIONS

$$\delta_s = \frac{P_s L}{E_s A_s} \qquad \delta_b = \frac{P_b L}{E_b A_b} \qquad \delta_c = \frac{P_c L}{E_c A_c} \qquad \text{(Eqs. 3, 4, 5)}$$

SOLUTION OF EQUATIONS

Substitute (3), (4), and (5) into Eqs. (2):

$$P_b = P_s \frac{E_b A_b}{E_s A_s} \qquad P_c = P_s \frac{E_c A_c}{E_s A_s} \qquad \text{(Eqs. 6, 7)}$$

SOLVE SIMULTANEOUSLY EQS. (1), (6), AND (7):

$$P_s = P \frac{E_s A_s}{E_s A_s + E_b A_b + E_c A_c}$$

$$P_b = P \frac{E_b A_b}{E_s A_s + E_b A_b + E_c A_c}$$

$$P_c = P \frac{E_c A_c}{E_s A_s + E_b A_b + E_c + A_c}$$

COMPRESSIVE STRESSES

Let $\Sigma EA = E_s A_s + E_b A_b + E_c A_c$

$$\sigma_s = \frac{P_s}{A_s} = \frac{PE_s}{\Sigma EA} \qquad \sigma_b = \frac{P_b}{A_b} = \frac{PE_b}{\Sigma EA}$$

$$\sigma_c = \frac{P_c}{A_c} = \frac{PE_c}{\Sigma EA}$$

SUBSTITUTE NUMERICAL VALUES:

$P = 40$ kN $\qquad E_s = 210$ GPa

$E_b = 100$ GPa $\qquad E_c = 120$ GPa

$d_1 = 30$ mm $\qquad d_2 = 45$ mm $\qquad d_3 = 60$ mm

$$A_s = \frac{\pi}{4} d_1^2 = 706.86 \text{ mm}^2$$

$$A_b = \frac{\pi}{4}(d_2^2 - d_1^2) = 883.57 \text{ mm}^2$$

$$A_c = \frac{\pi}{4}(d_3^2 - d_2^2) = 1237.00 \text{ mm}^2$$

$\Sigma EA = 385.238 \times 10^6$ N

$$\sigma_s = \frac{PE_s}{\Sigma EA} = 21.8 \text{ MPa} \longleftarrow$$

$$\sigma_b = \frac{PE_b}{\Sigma EA} = 10.4 \text{ MPa} \longleftarrow$$

$$\sigma_c = \frac{PE_c}{\Sigma EA} = 12.5 \text{ MPa} \longleftarrow$$

Thermal Effects

Problem 2.5-1 The rails of a railroad track are welded together at their ends (to form continuous rails and thus eliminate the clacking sound of the wheels) when the temperature is 60°F.

What compressive stress σ is produced in the rails when they are heated by the sun to 120°F if the coefficient of thermal expansion $\alpha = 6.5 \times 10^{-6}$/°F and the modulus of elasticity $E = 30 \times 10^6$ psi?

Solution 2.5-1 Expansion of railroad rails

The rails are prevented from expanding because of their great length and lack of expansion joints.

Therefore, each rail is in the same condition as a bar with fixed ends (see Example 2-7).

The compressive stress in the rails may be calculated from Eq. (2-18).

$\Delta T = 120°\text{F} - 60°\text{F} = 60°\text{F}$

$$\sigma = E\alpha(\Delta T)$$
$$= (30 \times 10^6 \text{ psi})(6.5 \times 10^{-6}/°\text{F})(60°\text{F})$$
$$\sigma = 11{,}700 \text{ psi} \longleftarrow$$

Problem 2.5-2 An aluminum pipe has a length of 60 m at a temperature of 10°C. An adjacent steel pipe at the same temperature is 5 mm longer than the aluminum pipe.

At what temperature (degrees Celsius) will the aluminum pipe be 15 mm longer than the steel pipe? (Assume that the coefficients of thermal expansion of aluminum and steel are $\alpha_a = 23 \times 10^{-6}$/°C and $\alpha_s = 12 \times 10^{-6}$/°C, respectively.)

Solution 2.5-2 Aluminum and steel pipes

Initial conditions

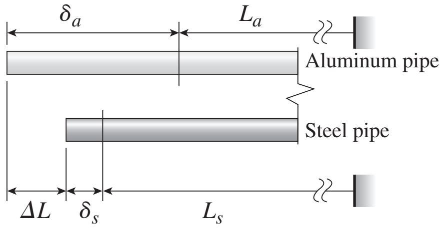
$L_a = 60$ m	$T_0 = 10°$C
$L_s = 60.005$ m	$T_0 = 10°$C
$\alpha_a = 23 \times 10^{-6}$/°C	$\alpha_s = 12 \times 10^{-6}$/°C

Final conditions

Aluminum pipe is longer than the steel pipe by the amount $\Delta L = 15$ mm.

ΔT = increase in temperature

$\delta_a = \alpha_a(\Delta T)L_a \qquad \delta_s = \alpha_s(\Delta T)L_s$



From the figure above:

$\delta_a + L_a = \Delta L + \delta_s + L_s$

or, $\alpha_a(\Delta T)L_a + L_a = \Delta L + \alpha_s(\Delta T)L_s + L_s$

Solve for ΔT:

$$\Delta T = \frac{\Delta L + (L_s - L_a)}{\alpha_a L_a - \alpha_s L_s} \longleftarrow$$

Substitute numerical values:

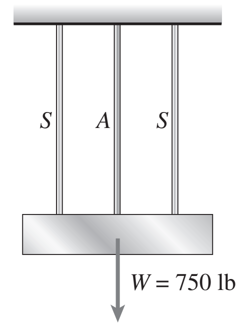
$\alpha_a L_a - \alpha_s L_s = 659.9 \times 10^{-6}$ m/°C

$$\Delta T = \frac{15 \text{ mm} + 5 \text{ mm}}{659.9 \times 10^{-6} \text{ m/°C}} = 30.31°\text{C}$$

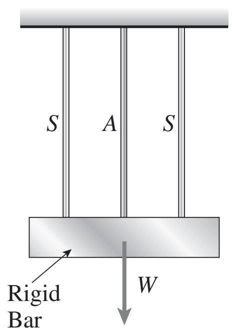
$$T = T_0 + \Delta T = 10°\text{C} + 30.31°\text{C}$$
$$= 40.3°\text{C} \longleftarrow$$

Problem 2.5-3 A rigid bar of weight $W = 750$ lb hangs from three equally spaced wires, two of steel and one of aluminum (see figure). The diameter of the wires is ⅛ in. Before they were loaded, all three wires had the same length.

What temperature increase ΔT in all three wires will result in the entire load being carried by the steel wires? (Assume $E_s = 30 \times 10^6$ psi, $\alpha_s = 6.5 \times 10^{-6}$/°F, and $\alpha_a = 12 \times 10^{-6}$/°F.)



Solution 2.5-3 Bar supported by three wires



S = steel A = aluminum

$W = 750$ lb

$$d = \frac{1}{8} \text{ in.}$$

$$A_s = \frac{\pi d^2}{4} = 0.012272 \text{ in.}^2$$

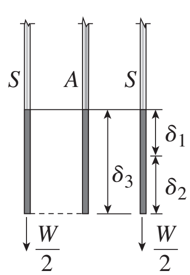
$E_s = 30 \times 10^6$ psi

$E_s A_s = 368{,}155$ lb

$\alpha_s = 6.5 \times 10^{-6}$/°F

$\alpha_a = 12 \times 10^{-6}$/°F

L = Initial length of wires



δ_1 = increase in length of a steel wire due to temperature increase ΔT

$= \alpha_s (\Delta T) L$

δ_2 = increase in length of a steel wire due to load $W/2$

$$= \frac{WL}{2E_s A_s}$$

δ_3 = increase in length of aluminum wire due to temperature increase ΔT

$= \alpha_a (\Delta T) L$

For no load in the aluminum wire:

$$\delta_1 + \delta_2 = \delta_3$$

$$\alpha_s(\Delta T)L + \frac{WL}{2E_s A_s} = \alpha_a(\Delta T)L$$

or

$$\Delta T = \frac{W}{2E_s A_s(\alpha_a - \alpha_s)} \quad \longleftarrow$$

Substitute numerical values:

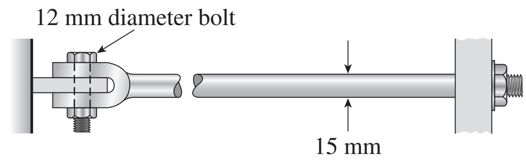
$$\Delta T = \frac{750 \text{ lb}}{(2)(368{,}155 \text{ lb})(5.5 \times 10^{-6}/°\text{F})}$$

$$= 185°\text{F} \quad \longleftarrow$$

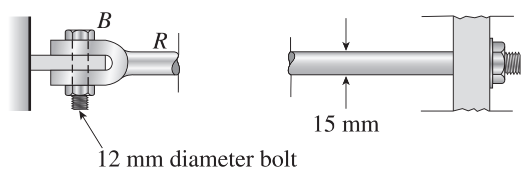
Note: If the temperature increase is larger than ΔT, the aluminum wire would be in compression, which is not possible. Therefore, the steel wires continue to carry all of the load. If the temperature increase is less than ΔT, the aluminum wire will be in tension and carry part of the load.

Problem 2.5-4 A steel rod of diameter 15 mm is held snugly (but without any initial stresses) between rigid walls by the arrangement shown in the figure.

Calculate the temperature drop ΔT (degrees Celsius) at which the average shear stress in the 12-mm diameter bolt becomes 45 MPa. (For the steel rod, use $\alpha = 12 \times 10^{-6}$/°C and $E = 200$ GPa.)



Solution 2.5-4 **Steel rod with bolted connection**



R = rod

B = bolt

P = tensile force in steel rod due to temperature drop ΔT

A_R = cross-sectional area of steel rod

From Eq. (2-17) of Example 2-7: $P = EA_R\alpha(\Delta T)$

Bolt is in double shear.

V = shear force acting over one cross section of the bolt

$$V = P/2 = \frac{1}{2}EA_R\alpha(\Delta T)$$

τ = average shear stress on cross section of the bolt

A_B = cross-sectional area of bolt

$$\tau = \frac{V}{A_B} = \frac{EA_R\,\alpha(\Delta T)}{2A_B}$$

Solve for ΔT: $\Delta T = \dfrac{2\tau A_B}{EA_R\alpha}$

$A_B = \dfrac{\pi d_B^2}{4}$ where d_B = diameter of bolt

$A_R = \dfrac{\pi d_R^2}{4}$ where d_R = diameter of steel rod

$$\Delta T = \frac{2\tau d_B^2}{E\alpha d_R^2} \quad \longleftarrow$$

SUBSTITUTE NUMERICAL VALUES:

$\tau = 45$ MPa $\quad d_B = 12$ mm $\quad d_R = 15$ mm

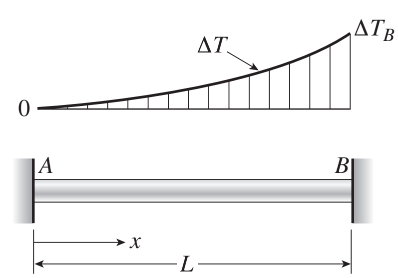
$\alpha = 12 \times 10^{-6}$/°C $\quad E = 200$ GPa

$$\Delta T = \frac{2(45\text{ MPa})(12\text{ mm})^2}{(200\text{ GPa})(12 \times 10^{-6}/°\text{C})(15\text{ mm})^2}$$

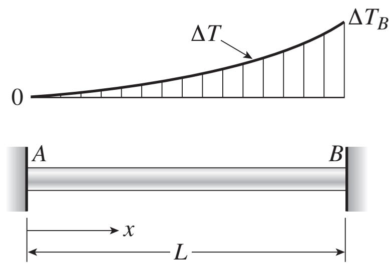
$$\Delta T = 24°\text{C} \quad \longleftarrow$$

Problem 2.5-5 A bar AB of length L is held between rigid supports and heated nonuniformly in such a manner that the temperature increase ΔT at distance x from end A is given by the expression $\Delta T = \Delta T_B x^3/L^3$, where ΔT_B is the increase in temperature at end B of the bar (see figure).

Derive a formula for the compressive stress σ_c in the bar. (Assume that the material has modulus of elasticity E and coefficient of thermal expansion α.)



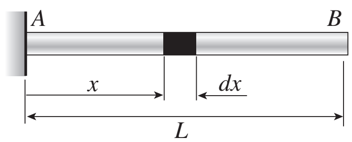
Solution 2.5-5 Bar with nonuniform temperature change



At distance x:

$$\Delta T = \Delta T_B\left(\frac{x^3}{L^3}\right)$$

Remove the support at end B of the bar:



Consider an element dx at a distance x from end A.

$d\delta$ = Elongation of element dx

$$d\delta = \alpha(\Delta T)dx = \alpha(\Delta T_B)\left(\frac{x^3}{L^3}\right)dx$$

δ = elongation of bar

$$\delta = \int_0^L d\delta = \int_0^L \alpha(\Delta T_B)\left(\frac{x^3}{L^3}\right)dx = \frac{1}{4}\alpha(\Delta T_B)L$$

Compressive force P required to shorten the bar by the amount δ

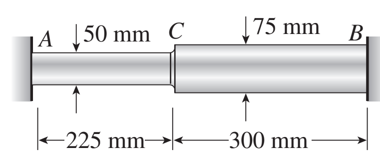
$$P = \frac{EA\delta}{L} = \frac{1}{4}EA\alpha(\Delta T_B)$$

Compressive stress in the bar

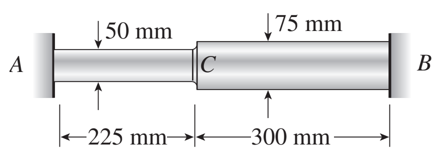
$$\sigma_c = \frac{P}{A} = \frac{E\alpha(\Delta T_B)}{4} \quad \longleftarrow$$

Problem 2.5-6 A plastic bar ACB having two different solid circular cross sections is held between rigid supports as shown in the figure. The diameters in the left- and right-hand parts are 50 mm and 75 mm, respectively. The corresponding lengths are 225 mm and 300 mm. Also, the modulus of elasticity E is 6.0 GPa, and the coefficient of thermal expansion α is 100×10^{-6}/°C. The bar is subjected to a uniform temperature increase of 30°C.

Calculate the following quantities: (a) the compressive force P in the bar; (b) the maximum compressive stress σ_c; and (c) the displacement δ_C of point C.



Solution 2.5-6 Bar with rigid supports



$E = 6.0$ GPa $\qquad \alpha = 100 \times 10^{-6}$/°C

Left-hand part:

$L_1 = 225$ mm $\qquad d_1 = 50$ mm

$$A_1 = \frac{\pi}{4}d_1^2 = \frac{\pi}{4}(50\text{ mm})^2$$

$$= 1963.5\text{ mm}^2$$

$\Delta T = 30$°C

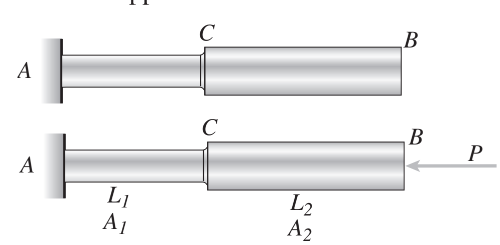
Right-hand part:

$L_2 = 300$ mm $\qquad d_2 = 75$ mm

$$A_2 = \frac{\pi}{4}d_2^2 = \frac{\pi}{4}(75\text{ mm})^2 = 4417.9\text{ mm}^2$$

(a) Compressive force P

Remove the support at end B.



δ_T = elongation due to temperature

$P = \alpha(\Delta T)(L_1+L_2)$

$= 1.5750$ mm

δ_P = shortening due to P

$$= \frac{PL_1}{EA_1} + \frac{PL_2}{EA_2}$$

$= P(19.0986 \times 10^{-9}$ m/N$+11.3177 \times 10^{-9}$ m/N)

$= (30.4163 \times 10^{-9}$ m/N$)P$

(P = newtons)

Compatibility: $\delta_T = \delta_P$

1.5750×10^{-3} m $= (30.4163 \times 10^{-9}$ m/N$)P$

$P = 51{,}781$ N or $P = 51.8$ kN ←

(b) Maximum compressive stress

$$\sigma_c = \frac{P}{A_1} = \frac{51.78 \text{ kN}}{1963.5 \text{ mm}^2} = 26.4 \text{ MPa} \quad \leftarrow$$

(c) Displacement of point C

δ_C = Shortening of AC

$$\delta_C = \frac{PL_1}{EA_1} - \alpha(\Delta T)L_1$$

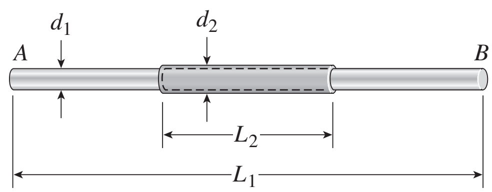
$= 0.9890$ mm $- 0.6750$ mm

$\delta_C = 0.314$ mm ←

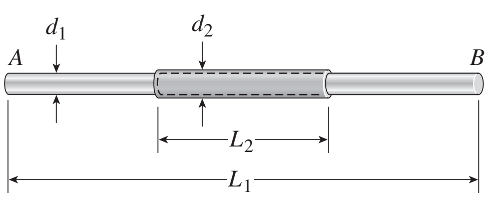
(Positive means AC shortens and point C displaces to the left.)

Problem 2.5-7 A circular steel rod AB (diameter $d_1 = 1.0$ in., length $L_1 = 3.0$ ft) has a bronze sleeve (outer diameter $d_2 = 1.25$ in., length $L_2 = 1.0$ ft) shrunk onto it so that the two parts are securely bonded (see figure).

Calculate the total elongation δ of the steel bar due to a temperature rise $\Delta T = 500°$F. (Material properties are as follows: for steel, $E_s = 30 \times 10^6$ psi and $\alpha_s = 6.5 \times 10^{-6}$/°F; for bronze, $E_b = 15 \times 10^6$ psi and $\alpha_b = 11 \times 10^{-6}$/°F.)



Solution 2.5-7 **Steel rod with bronze sleeve**



$L_1 = 36$ in. $L_2 = 12$ in.

Elongation of the two outer parts of the bar

$\delta_1 = \alpha_s(\Delta T)(L_1 - L_2)$

$= (6.5 \times 10^{-6}$/°F)(500°F)(36 in. $-$ 12 in.)

$= 0.07800$ in.

Elongation of the middle part of the bar

The steel rod and bronze sleeve lengthen the same amount, so they are in the same condition as the bolt and sleeve of Example 2-8. Thus, we can calculate the elongation from Eq. (2-21):

$$\delta_2 = \frac{(\alpha_s E_s A_s + \alpha_b E_b A_b)(\Delta T)L_2}{E_s A_s + E_b A_b}$$

Substitute numerical values:

$\alpha_s = 6.5 \times 10^{-6}$/°F $\alpha_b = 11 \times 10^{-6}$/°F

$E_s = 30 \times 10^6$ psi $E_b = 15 \times 10^6$ psi

$d_1 = 1.0$ in.

$$A_s = \frac{\pi}{4} d_1^2 = 0.78540 \text{ in.}^2$$

$d_2 = 1.25$ in.

$$A_b = \frac{\pi}{4}(d_2^2 - d_1^2) = 0.44179 \text{ in.}^2$$

$\Delta T = 500°$F $L_2 = 12.0$ in.

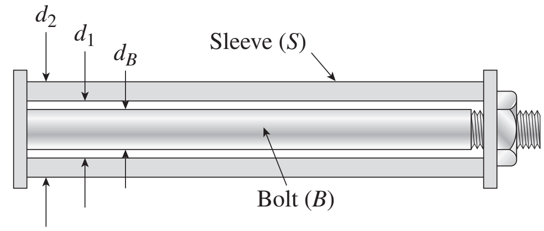
$\delta_2 = 0.04493$ in.

Total elongation

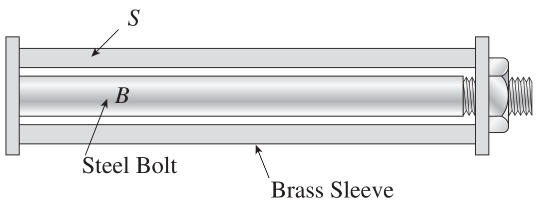
$\delta = \delta_1 + \delta_2 = 0.123$ in. ←

Problem 2.5-8 A brass sleeve S is fitted over a steel bolt B (see figure), and the nut is tightened until it is just snug. The bolt has a diameter $d_B = 25$ mm, and the sleeve has inside and outside diameters $d_1 = 26$ mm and $d_2 = 36$ mm, respectively.

Calculate the temperature rise ΔT that is required to produce a compressive stress of 25 MPa in the sleeve. (Use material properties as follows: for the sleeve, $\alpha_S = 21 \times 10^{-6}/°C$ and $E_S = 100$ GPa; for the bolt, $\alpha_B = 10 \times 10^{-6}/°C$ and $E_B = 200$ GPa.) (*Suggestion:* Use the results of Example 2-8.)



Solution 2.5-8 Brass sleeve fitted over a Steel bolt



Subscript S means "sleeve".

Subscript B means "bolt".

Use the results of Example 2-8.

σ_S = compressive force in sleeve

EQUATION (2-20a):

$$\sigma_S = \frac{(\alpha_S - \alpha_B)(\Delta T)E_S E_B A_B}{E_S A_S + E_B A_B} \text{ (Compression)}$$

SOLVE FOR ΔT:

$$\Delta T = \frac{\sigma_S(E_S A_S + E_B A_B)}{(\alpha_S - \alpha_B)E_S E_B A_B}$$

or

$$\Delta T = \frac{\sigma_S}{E_S(\alpha_S - \alpha_B)}\left(1 + \frac{E_S A_S}{E_B A_B}\right) \longleftarrow$$

SUBSTITUTE NUMERICAL VALUES:

$\sigma_S = 25$ MPa

$d_2 = 36$ mm $\quad d_1 = 26$ mm $\quad d_B = 25$ mm

$E_S = 100$ GPa $\quad E_B = 200$ GPa

$\alpha_S = 21 \times 10^{-6}/°C \quad \alpha_B = 10 \times 10^{-6}/°C$

$$A_S = \frac{\pi}{4}(d_2^2 - d_1^2) = \frac{\pi}{4}(620 \text{ mm}^2)$$

$$A_B = \frac{\pi}{4}(d_B)^2 = \frac{\pi}{4}(625 \text{ mm}^2)$$

$$1 + \frac{E_S A_S}{E_B A_B} = 1.496$$

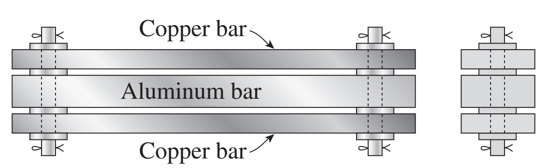
$$\Delta T = \frac{25 \text{ MPa } (1.496)}{(100 \text{ GPa})(11 \times 10^{-6}/°C)}$$

$$\Delta T = 34°C \longleftarrow$$

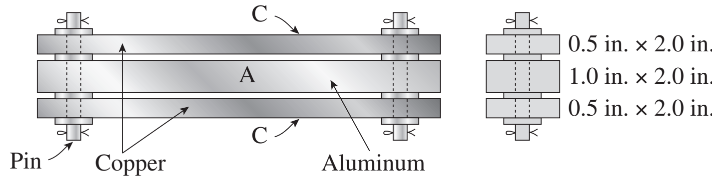
(Increase in temperature)

Problem 2.5-9 Rectangular bars of copper and aluminum are held by pins at their ends, as shown in the figure. Thin spacers provide a separation between the bars. The copper bars have cross-sectional dimensions 0.5 in. $\times$ 2.0 in., and the aluminum bar has dimensions 1.0 in. $\times$ 2.0 in.

Determine the shear stress in the 7/16 in. diameter pins if the temperature is raised by 100°F. (For copper, $E_c = 18{,}000$ ksi and $\alpha_c = 9.5 \times 10^{-6}/°F$; for aluminum, $E_a = 10{,}000$ ksi and $\alpha_a = 13 \times 10^{-6}/°F$.) *Suggestion:* Use the results of Example 2-8.



Solution 2.5-9 **Rectangular bars held by pins**



Diameter of pin: $d_P = \frac{7}{16}$ in. = 0.4375 in.

Area of pin: $A_P = \frac{\pi}{4} d_P^2 = 0.15033 \text{ in.}^2$

Area of two copper bars: $A_c = 2.0 \text{ in.}^2$

Area of aluminum bar: $A_a = 2.0 \text{ in.}^2$

$\Delta T = 100°\text{F}$

Copper: $E_c = 18{,}000$ ksi $\qquad \alpha_c = 9.5 \times 10^{-6}/°\text{F}$

Aluminum: $E_a = 10{,}000$ ksi $\qquad \alpha_a = 13 \times 10^{-6}/°\text{F}$

Use the results of Example 2-8.

Find the forces P_a and P_c in the aluminum bar and copper bar, respectively, from Eq. (2-19).

Replace the subscript "*S*" in that equation by "*a*" (for aluminum) and replace the subscript "*B*" by "*c*" (for copper), because α for aluminum is larger than α for copper.

$$P_a = P_c = \frac{(\alpha_a - \alpha_c)(\Delta T)E_a A_a E_c A_c}{E_a A_a + E_c A_c}$$

Note that P_a is the compressive force in the aluminum bar and P_c is the combined tensile force in the two copper bars.

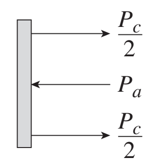
$$P_a = P_c = \frac{(\alpha_a - \alpha_c)(\Delta T)E_c A_c}{1 + \dfrac{E_c A_c}{E_a A_a}}$$

SUBSTITUTE NUMERICAL VALUES:

$$P_a = P_c = \frac{(3.5 \times 10^{-6}/°\text{F})(100°\text{F})(18{,}000 \text{ ksi})(2 \text{ in.}^2)}{1 + \left(\dfrac{18}{10}\right)\left(\dfrac{2.0}{2.0}\right)}$$

$$= 4{,}500 \text{ lb}$$

FREE-BODY DIAGRAM OF PIN AT THE LEFT END



V = shear force in pin

$= P_c/2$

$= 2{,}250$ lb

τ = average shear stress on cross section of pin

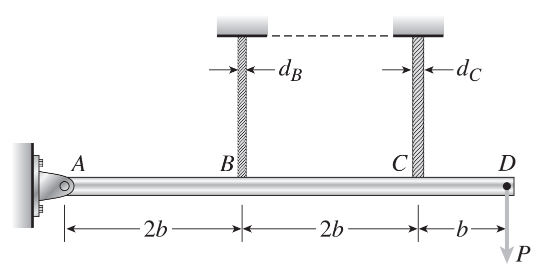
$$\tau = \frac{V}{A_P} = \frac{2{,}250 \text{ lb}}{0.15033 \text{ in.}^2}$$

$$\tau = 15.0 \text{ ksi} \quad \longleftarrow$$

Problem 2.5-10 A rigid bar *ABCD* is pinned at end *A* and supported by two cables at points *B* and *C* (see figure). The cable at *B* has nominal diameter $d_B = 12$ mm and the cable at *C* has nominal diameter $d_C = 20$ mm. A load *P* acts at end *D* of the bar.

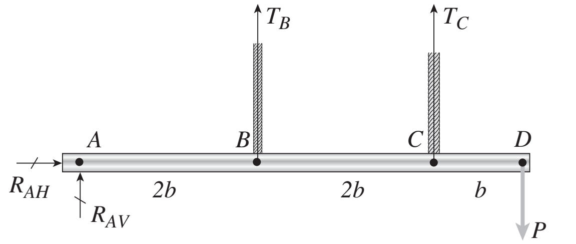
What is the allowable load *P* if the temperature rises by 60°C and each cable is required to have a factor of safety of at least 5 against its ultimate load?

(*Note:* The cables have effective modulus of elasticity $E = 140$ GPa and coefficient of thermal expansion $\alpha = 12 \times 10^{-6}/°\text{C}$. Other properties of the cables can be found in Table 2-1, Section 2.2.)



Solution 2.5-10 **Rigid bar supported by two cables**

FREE-BODY DIAGRAM OF BAR *ABCD*



T_B = force in cable B $\qquad$ T_C = force in cable C

$d_B = 12$ mm $\qquad d_C = 20$ mm

From Table 2-1:

$A_B = 76.7\text{ mm}^2$ $\qquad E = 140$ GPa

$\Delta T = 60°\text{C}$ $\qquad A_C = 173\text{ mm}^2$

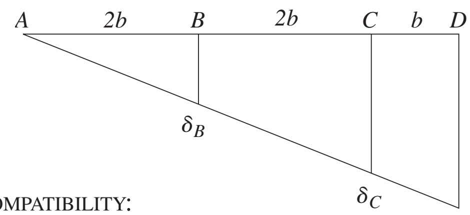
$\alpha = 12 \times 10^{-6}/°\text{C}$

EQUATION OF EQUILIBRIUM

$\Sigma M_A = 0 \quad T_B(2b) + T_C(4b) - P(5b) = 0$

or $\quad 2T_B + 4T_C = 5P \qquad$ (Eq. 1)

DISPLACEMENT DIAGRAM



COMPATIBILITY:

$\delta_C = 2\delta_B \qquad$ (Eq. 2)

FORCE-DISPLACEMENT AND TEMPERATURE-DISPLACEMENT RELATIONS

$$\delta_B = \frac{T_B L}{EA_B} + \alpha(\Delta T)L \qquad \text{(Eq. 3)}$$

$$\delta_C = \frac{T_C L}{EA_C} + \alpha(\Delta T)L \qquad \text{(Eq. 4)}$$

SUBSTITUTE EQS. (3) AND (4) INTO EQ. (2):

$$\frac{T_C L}{EA_C} + \alpha(\Delta T)L = \frac{2T_B L}{EA_B} + 2\alpha(\Delta T)L$$

or

$$2T_B A_C - T_C A_B = -E\alpha(\Delta T)A_B A_C \qquad \text{(Eq. 5)}$$

SUBSTITUTE NUMERICAL VALUES INTO EQ. (5):

$T_B(346) - T_C(76.7) = -1{,}338{,}000 \qquad$ (Eq. 6)

in which T_B and T_C have units of newtons.

SOLVE SIMULTANEOUSLY EQS. (1) AND (6):

$T_B = 0.2494\,P - 3{,}480 \qquad$ (Eq. 7)

$T_C = 1.1253\,P + 1{,}740 \qquad$ (Eq. 8)

in which P has units of newtons.

SOLVE EQS. (7) AND (8) FOR THE LOAD P:

$P_B = 4.0096\,T_B + 13{,}953 \qquad$ (Eq. 9)

$P_C = 0.8887\,T_C - 1{,}546 \qquad$ (Eq. 10)

ALLOWABLE LOADS

From Table 2-1:

$(T_B)_{\text{ULT}} = 102{,}000$ N $\qquad (T_C)_{\text{ULT}} = 231{,}000$ N

Factor of safety = 5

$(T_B)_{\text{allow}} = 20{,}400$ N $\qquad (T_C)_{\text{allow}} = 46{,}200$ N

From Eq. (9): $P_B = (4.0096)(20{,}400\text{ N}) + 13{,}953\text{ N}$
$= 95{,}700$ N

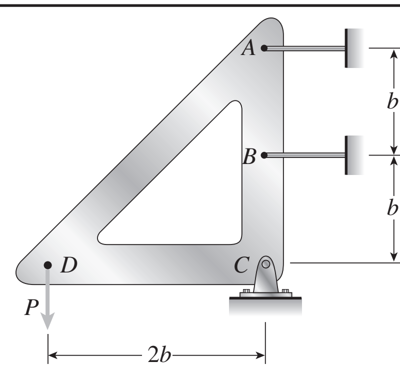
From Eq. (10): $P_C = (0.8887)(46{,}200\text{ N}) - 1546\text{ N}$
$= 39{,}500$ N

Cable C governs.

$P_{\text{allow}} = 39.5$ kN ⟵

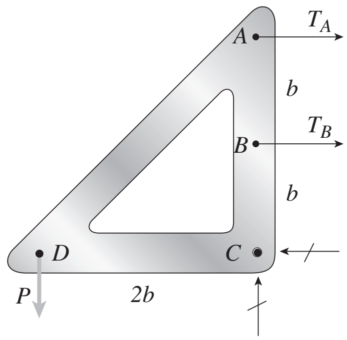
Problem 2.5-11 A rigid triangular frame is pivoted at C and held by two identical horizontal wires at points A and B (see figure). Each wire has axial rigidity $EA = 120$ k and coefficient of thermal expansion $\alpha = 12.5 \times 10^{-6}/°\text{F}$.

(a) If a vertical load $P = 500$ lb acts at point D, what are the tensile forces T_A and T_B in the wires at A and B, respectively?
(b) If, while the load P is acting, both wires have their temperatures raised by 180°F, what are the forces T_A and T_B?
(c) What further increase in temperature will cause the wire at B to become slack?



Solution 2.5-11 Triangular frame held by two wires

FREE-BODY DIAGRAM OF FRAME

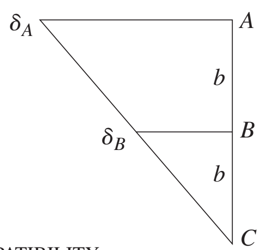


EQUATION OF EQUILIBRIUM

$\Sigma M_C = 0$

$P(2b) - T_A(2b) - T_B(b) = 0$ or $2T_A + T_B = 2P$ (Eq. 1)

DISPLACEMENT DIAGRAM



EQUATION OF COMPATIBILITY

$\delta_A = 2\delta_B$ (Eq. 2)

(a) LOAD P ONLY

Force-displacement relations:

$$\delta_A = \frac{T_A L}{EA} \qquad \delta_B = \frac{T_B L}{EA} \qquad \text{(Eq. 3, 4)}$$

(L = length of wires at A and B.)

Substitute (3) and (4) into Eq. (2):

$$\frac{T_A L}{EA} = \frac{2T_B L}{EA}$$

or $T_A = 2T_B$ (Eq. 5)

Solve simultaneously Eqs. (1) and (5):

$$T_A = \frac{4P}{5} \qquad T_B = \frac{2P}{5} \qquad \text{(Eqs. 6, 7)}$$

Numerical values:

$P = 500$ lb

$\therefore T_A = 400$ lb $\quad T_B = 200$ lb ←

(b) LOAD P AND TEMPERATURE INCREASE ΔT

Force-displacement and temperature-displacement relations:

$$\delta_A = \frac{T_A L}{EA} + \alpha(\Delta T)L \qquad \text{(Eq. 8)}$$

$$\delta_B = \frac{T_B L}{EA} + \alpha(\Delta T)L \qquad \text{(Eq. 9)}$$

Substitute (8) and (9) into Eq. (2):

$$\frac{T_A L}{EA} + \alpha(\Delta T)L = \frac{2T_B L}{EA} + 2\alpha(\Delta T)L$$

or $T_A - 2T_B = EA\alpha(\Delta T)$ (Eq. 10)

Solve simultaneously Eqs. (1) and (10):

$$T_A = \frac{1}{5}[4P + EA\alpha(\Delta T)] \qquad \text{(Eq. 11)}$$

$$T_B = \frac{2}{5}[P - EA\alpha(\Delta T)] \qquad \text{(Eq. 12)}$$

Substitute numerical values:

$P = 500$ lb $\qquad EA = 120{,}000$ lb

$\Delta T = 180°F$

$\alpha = 12.5 \times 10^{-6}/°F$

$$T_A = \frac{1}{5}(2000 \text{ lb} + 270 \text{ lb}) = 454 \text{ lb} \quad \longleftarrow$$

$$T_B = \frac{2}{5}(500 \text{ lb} - 270 \text{ lb}) = 92 \text{ lb} \quad \longleftarrow$$

(c) WIRE B BECOMES SLACK

Set $T_B = 0$ in Eq. (12):

$P = EA\alpha(\Delta T)$

or

$$\Delta T = \frac{P}{EA\alpha} = \frac{500 \text{ lb}}{(120{,}000 \text{ lb})(12.5 \times 10^{-6}/°F)}$$

$= 333.3°F$

Further increase in temperature:

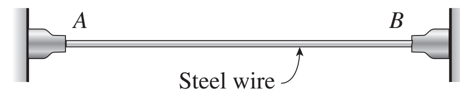
$\Delta T = 333.3°F - 180°F$

$= 153°F$ ←

Misfits and Prestrains

Problem 2.5-12 A steel wire AB is stretched between rigid supports (see figure). The initial prestress in the wire is 42 MPa when the temperature is 20°C.

(a) What is the stress σ in the wire when the temperature drops to 0°C?
(b) At what temperature T will the stress in the wire become zero? (Assume $\alpha = 14 \times 10^{-6}$/°C and $E = 200$ GPa.)



Solution 2.5-12 **Steel wire with initial prestress**

Initial prestress: $\sigma_1 = 42$ MPa

Initial temperature: $T_1 = 20$°C

$E = 200$ GPa

$\alpha = 14 \times 10^{-6}$/°C

(a) Stress σ when temperature drops to 0°C

$T_2 = 0$°C $\qquad \Delta T = 20$°C

Note: Positive ΔT means a *decrease* in temperature and an *increase* in the stress in the wire.

Negative ΔT means an *increase* in temperature and a *decrease* in the stress.

Stress σ equals the initial stress σ_1 plus the additional stress σ_2 due to the temperature drop.

From Eq. (2-18): $\sigma_2 = E\alpha(\Delta T)$

$$\sigma = \sigma_1 + \sigma_2 = \sigma_1 + E\alpha(\Delta T)$$
$$= 42 \text{ MPa} + (200 \text{ GPa})(14 \times 10^{-6}/°\text{C})(20°\text{C})$$
$$= 42 \text{ MPa} + 56 \text{ MPa} = 98 \text{ MPa} \longleftarrow$$

(b) Temperature when stress equals zero

$$\sigma = \sigma_1 + \sigma_2 = 0 \qquad \sigma_1 + E\alpha(\Delta T) = 0$$

$$\Delta T = -\frac{\sigma_1}{E\alpha}$$

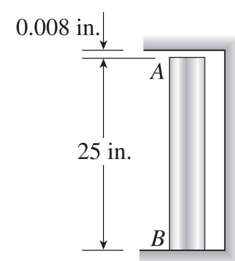
(Negative means increase in temp.)

$$\Delta T = -\frac{42 \text{ MPa}}{(200 \text{ GPa})(14 \times 10^{-6}/°\text{C})} = -15°\text{C}$$

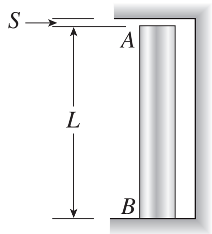
$$T = 20°\text{C} + 15°\text{C} = 35°\text{C} \longleftarrow$$

Problem 2.5-13 A copper bar AB of length 25 in. is placed in position at room temperature with a gap of 0.008 in. between end A and a rigid restraint (see figure).

Calculate the axial compressive stress σ_c in the bar if the temperature rises 50°F. (For copper, use $\alpha = 9.6 \times 10^{-6}$/°F and $E = 16 \times 10^6$ psi.)



Solution 2.5-13 Bar with a gap



$L = 25$ in.

$S = 0.008$ in.

$\Delta T = 50°\text{F}$ (increase)

$\alpha = 9.6 \times 10^{-6}/°\text{F}$

$E = 16 \times 10^{6}$ psi

δ = elongation of the bar if it is free to expand

$= \alpha(\Delta T)L$

δ_C = elongation that is prevented by the support

$= \alpha(\Delta T)L - S$

ε_C = strain in the bar due to the restraint

$= \delta_C/L$

σ_c = stress in the bar

$$= E\varepsilon_C = \frac{E\delta_C}{L} = \frac{E}{L}[\alpha(\Delta T)L - S] \quad \longleftarrow$$

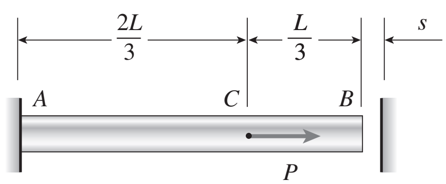
Note: This result is valid only if $\alpha(\Delta T)L \geq S$. (Otherwise, the gap is not closed).

Substitute numerical values:

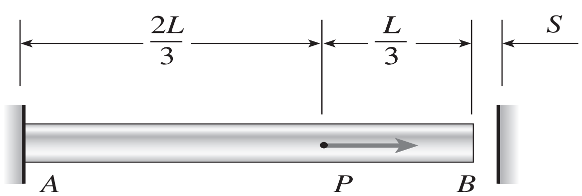
$$\sigma_c = \frac{16 \times 10^6 \text{ psi}}{25 \text{ in.}}[(9.6 \times 10^{-6}/°\text{F})(50°\text{F})(25 \text{ in.}) - 0.008 \text{ in.}] = 2{,}560 \text{ psi} \quad \longleftarrow$$

Problem 2.5-14 A bar *AB* having length *L* and axial rigidity *EA* is fixed at end *A* (see figure). At the other end a small gap of dimension *s* exists between the end of the bar and a rigid surface. A load *P* acts on the bar at point *C*, which is two-thirds of the length from the fixed end.

If the support reactions produced by the load *P* are to be equal in magnitude, what should be the size *s* of the gap?



Solution 2.5-14 Bar with a gap (load *P*)



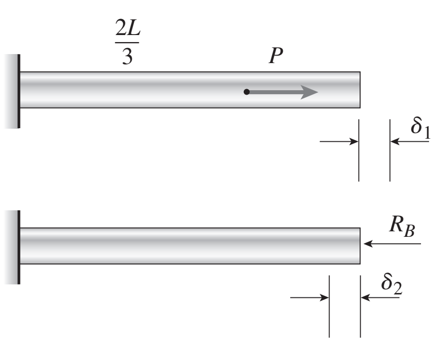
L = length of bar

S = size of gap

EA = axial rigidity

Reactions must be equal; find S.

FORCE-DISPLACEMENT RELATIONS



$$\delta_1 = \frac{P(\frac{2L}{3})}{EA}$$

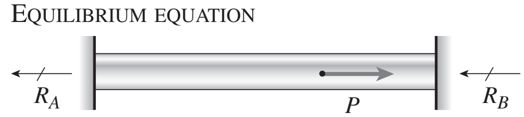
$$\delta_2 = \frac{R_B L}{EA}$$

COMPATIBILITY EQUATION

$\delta_1 - \delta_2 = S$ or

$$\frac{2PL}{3EA} - \frac{R_B L}{EA} = S \qquad \text{(Eq. 1)}$$

EQUILIBRIUM EQUATION



R_A = reaction at end A (to the left)

R_B = reaction at end B (to the left)

$P = R_A + R_B$

Reactions must be equal.

$$\therefore R_A = R_B \qquad P = 2R_B \qquad R_B = \frac{P}{2}$$

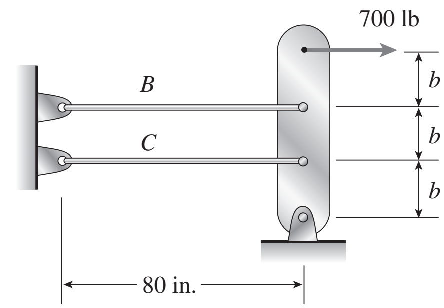
Substitute for R_B in Eq. (1):

$$\frac{2PL}{3EA} - \frac{PL}{2EA} = S \quad \text{or} \quad S = \frac{PL}{6EA} \quad \longleftarrow$$

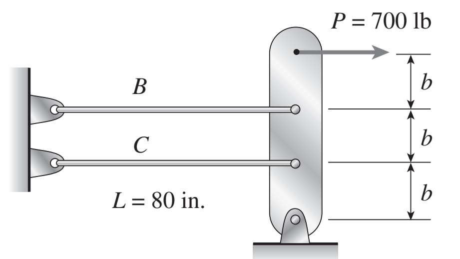
NOTE: The gap closes when the load reaches the value $P/4$. When the load reaches the value P, equal to $6EAs/L$, the reactions are equal ($R_A = R_B = P/2$). When the load is between $P/4$ and P, R_A is greater than R_B. If the load exceeds P, R_B is greater than R_A.

Problem 2.5-15 Wires B and C are attached to a support at the left-hand end and to a pin-supported rigid bar at the right-hand end (see figure). Each wire has cross-sectional area $A = 0.03$ in.^{2} and modulus of elasticity $E = 30 \times 10^6$ psi. When the bar is in a vertical position, the length of each wire is $L = 80$ in. However, before being attached to the bar, the length of wire B was 79.98 in. and of wire C was 79.95 in.

Find the tensile forces T_B and T_C in the wires under the action of a force $P = 700$ lb acting at the upper end of the bar.



Solution 2.5-15 Wires B and C attached to a bar



$P = 700$ lb

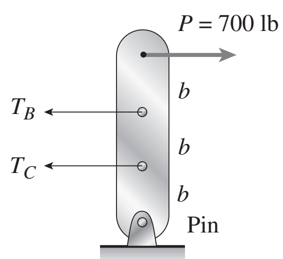
$A = 0.03$ in.^{2}

$E = 30 \times 10^6$ psi

$L_B = 79.98$ in.

$L_C = 79.95$ in.

Equilibrium Equation



$\Sigma M_{\text{pin}} = 0$

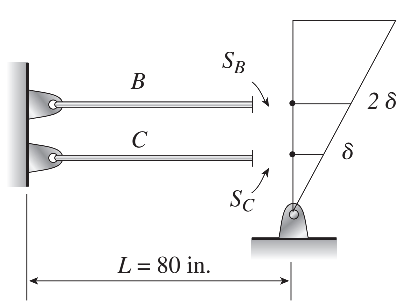
$T_C(b) + T_B(2b) = P(3b)$

$2T_B + T_C = 3P$ (Eq. 1)

Displacement diagram

$S_B = 80 \text{ in.} - L_B = 0.02$ in.

$S_C = 80 \text{ in.} - L_C = 0.05$ in.



Elongation of wires:

$\delta_B = S_B + 2\delta$ (Eq. 2)

$\delta_C = S_C + \delta$ (Eq. 3)

Force-displacement relations

$$\delta_B = \frac{T_B L}{EA} \qquad \delta_C = \frac{T_C L}{EA} \qquad \text{(Eqs. 4, 5)}$$

Solution of equations

Combine Eqs. (2) and (4):

$$\frac{T_B L}{EA} = S_B + 2\delta \qquad \text{(Eq. 6)}$$

Combine Eqs. (3) and (5):

$$\frac{T_C L}{EA} = S_C + \delta \qquad \text{(Eq. 7)}$$

Eliminate δ between Eqs. (6) and (7):

$$T_B - 2T_C = \frac{EAS_B}{L} - \frac{2EAS_C}{L} \qquad \text{(Eq. 8)}$$

Solve simultaneously Eqs. (1) and (8):

$$T_B = \frac{6P}{5} + \frac{EAS_B}{5L} - \frac{2EAS_C}{5L} \quad \longleftarrow$$

$$T_C = \frac{3P}{5} - \frac{2EAS_B}{5L} + \frac{4EAS_C}{5L} \quad \longleftarrow$$

Substitute numerical values:

$$\frac{EA}{5L} = 2250 \text{ lb/in.}$$

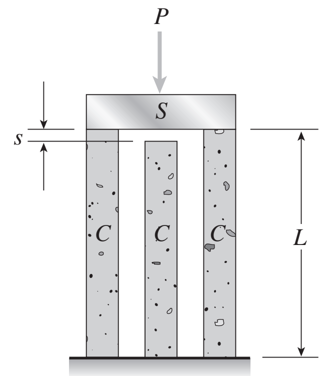
$T_B = 840 \text{ lb} + 45 \text{ lb} - 225 \text{ lb} = 660 \text{ lb}$ ⟵

$T_C = 420 \text{ lb} - 90 \text{ lb} + 450 \text{ lb} = 780 \text{ lb}$ ⟵

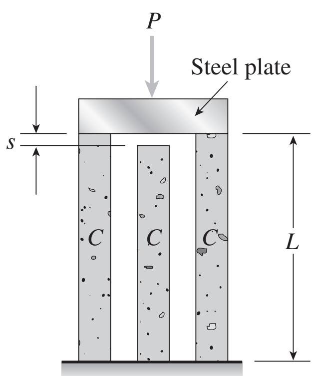
(Both forces are positive, which means tension, as required for wires.)

Problem 2.5-16 A rigid steel plate is supported by three posts of high-strength concrete each having an effective cross-sectional area $A = 40{,}000\ \text{mm}^2$ and length $L = 2$ m (see figure). Before the load P is applied, the middle post is shorter than the others by an amount $s = 1.0$ mm.

Determine the maximum allowable load P_{allow} if the allowable compressive stress in the concrete is $\sigma_{\text{allow}} = 20$ MPa. (Use $E = 30$ GPa for concrete.)



Solution 2.5-16 **Plate supported by three posts**



s = size of gap = 1.0 mm

L = length of posts = 2.0 m

$A = 40{,}000\ \text{mm}^2$

$\sigma_{\text{allow}} = 20$ MPa

$E = 30$ GPa

C = concrete post

DOES THE GAP CLOSE?

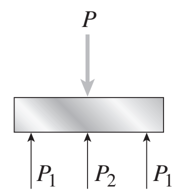
Stress in the two outer posts when the gap is just closed:

$$\sigma = E\varepsilon = E\left(\frac{s}{L}\right) = (30\ \text{GPa})\left(\frac{1.0\ \text{mm}}{2.0\ \text{m}}\right)$$

$$= 15\ \text{MPa}$$

Since this stress is less than the allowable stress, the allowable force P will close the gap.

EQUILIBRIUM EQUATION



$$2P_1 + P_2 = P \qquad \text{(Eq. 1)}$$

COMPATIBILITY EQUATION

δ_1 = shortening of outer posts

δ_2 = shortening of inner post

$$\delta_1 = \delta_2 + s \qquad \text{(Eq. 2)}$$

FORCE-DISPLACEMENT RELATIONS

$$\delta_1 = \frac{P_1 L}{EA} \qquad \delta_2 = \frac{P_2 L}{EA} \qquad \text{(Eqs. 3, 4)}$$

SOLUTION OF EQUATIONS

Substitute (3) and (4) into Eq. (2):

$$\frac{P_1 L}{EA} = \frac{P_2 L}{EA} + s \quad \text{or} \quad P_1 - P_2 = \frac{EAs}{L} \qquad \text{(Eq. 5)}$$

Solve simultaneously Eqs. (1) and (5):

$$P = 3P_1 - \frac{EAs}{L}$$

By inspection, we know that P_1 is larger than P_2. Therefore, P_1 will control and will be equal to $\sigma_{\text{allow}} A$.

$$P_{\text{allow}} = 3\sigma_{\text{allow}} A - \frac{EAs}{L}$$

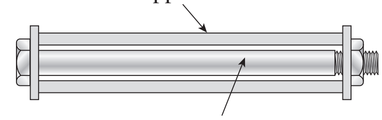
$$= 2400\ \text{kN} - 600\ \text{kN} = 1800\ \text{kN}$$

$$= 1.8\ \text{MN} \longleftarrow$$

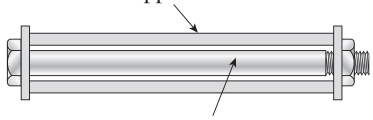
Problem 2.5-17 A copper tube is fitted around a steel bolt and the nut is turned until it is just snug (see figure). What stresses σ_s and σ_c will be produced in the steel and copper, respectively, if the bolt is now tightened by a quarter turn of the nut?

The copper tube has length $L = 16$ in. and cross-sectional area $A_c = 0.6$ in.^{2}, and the steel bolt has cross-sectional area $A_s = 0.2$ in.^{2} The pitch of the threads of the bolt is $p = 52$ mils (a mil is one-thousandth of an inch). Also, the moduli of elasticity of the steel and copper are $E_s = 30 \times 10^6$ psi and $E_c = 16 \times 10^6$ psi, respectively.

Note: The pitch of the threads is the distance advanced by the nut in one complete turn (see Eq. 2-22).



Solution 2.5-17 Steel bolt and copper tube



$L = 16$ in.

$p = 52$ mils $= 0.052$ in.

$n = \frac{1}{4}$ (See Eq. 2-22)

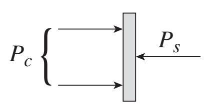
Steel bolt: $A_s = 0.2$ in.^{2}

$E_s = 30 \times 10^6$ psi

Copper tube: $A_c = 0.6$ in.^{2}

$E_c = 16 \times 10^6$ psi

EQUILIBRIUM EQUATION

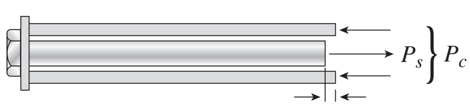


P_s = tensile force in steel bolt

P_c = compressive force in copper tube

$$P_c = P_s \qquad \text{(Eq. 1)}$$

COMPATIBILITY EQUATION



δ_c = shortening of copper tube

δ_s = elongation of steel bolt

$$\delta_c + \delta_s = np \qquad \text{(Eq. 2)}$$

FORCE-DISPLACEMENT RELATIONS

$$\delta_c = \frac{P_c L}{E_c A_c} \qquad \delta_s = \frac{P_s L}{E_s A_s} \qquad \text{(Eq. 3, Eq. 4)}$$

SOLUTION OF EQUATIONS

Substitute (3) and (4) into Eq. (2):

$$\frac{P_c L}{E_c A_c} + \frac{P_s L}{E_s A_s} = np \qquad \text{(Eq. 5)}$$

Solve simultaneously Eqs. (1) and (5):

$$P_s = P_c = \frac{np E_s A_s E_c A_c}{L(E_s A_s + E_c A_c)} \qquad \text{(Eq. 6)}$$

Substitute numerical values:

$P_s = P_c = 3{,}000$ lb

STRESSES

Steel bolt:

$$\sigma_s = \frac{P_s}{A_s} = \frac{3{,}000 \text{ lb}}{0.2 \text{ in.}^2} = 15 \text{ ksi (tension)} \quad \longleftarrow$$

Copper tube:

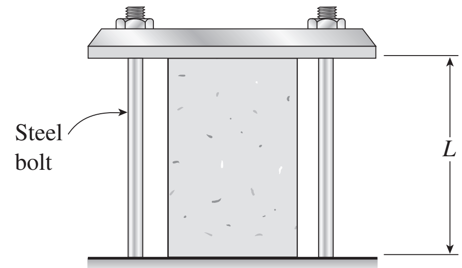
$$\sigma_c = \frac{P_c}{A_c} = \frac{3{,}000 \text{ lb}}{0.6 \text{ in.}^2}$$

$$= 5 \text{ ksi (compression)} \quad \longleftarrow$$

Problem 2.5-18 A plastic cylinder is held snugly between a rigid plate and a foundation by two steel bolts (see figure).

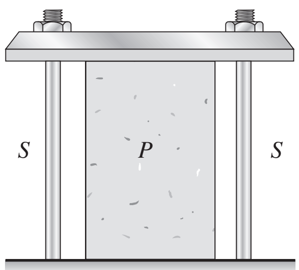
Determine the compressive stress σ_p in the plastic when the nuts on the steel bolts are tightened by one complete turn.

Data for the assembly are as follows: length $L = 200$ mm, pitch of the bolt threads $p = 1.0$ mm, modulus of elasticity for steel $E_s = 200$ GPa, modulus of elasticity for the plastic $E_p = 7.5$ GPa, cross-sectional area of one bolt $A_s = 36.0\text{ mm}^2$, and cross-sectional area of the plastic cylinder $A_p = 960\text{ mm}^2$.



Probs. 2.5-18 and 2.5-19

Solution 2.5-18 Plastic cylinder and two steel bolts



$L = 200$ mm

$P = 1.0$ mm

$E_s = 200$ GPa

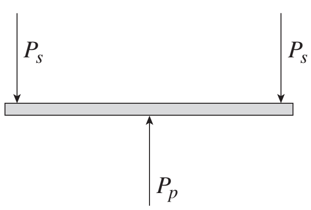
$A_s = 36.0\text{ mm}^2$ (for one bolt)

$E_p = 7.5$ GPa

$A_p = 960\text{ mm}^2$

$n = 1$ (See Eq. 2-22)

EQUILIBRIUM EQUATION

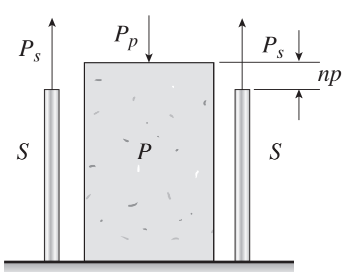


P_s = tensile force in one steel bolt

P_p = compressive force in plastic cylinder

$$P_p = 2P_s \qquad \text{(Eq. 1)}$$

COMPATIBILITY EQUATION



δ_s = elongation of steel bolt

δ_p = shortening of plastic cylinder

$$\delta_s + \delta_p = np \qquad \text{(Eq. 2)}$$

FORCE-DISPLACEMENT RELATIONS

$$\delta_s = \frac{P_s L}{E_s A_s} \quad \delta_p = \frac{P_p L}{E_p A_p} \qquad \text{(Eq. 3, Eq. 4)}$$

SOLUTION OF EQUATIONS

Substitute (3) and (4) into Eq. (2):

$$\frac{P_s L}{E_s A_s} + \frac{P_p L}{E_p A_p} = np \qquad \text{(Eq. 5)}$$

Solve simultaneously Eqs. (1) and (5):

$$P_p = \frac{2npE_s A_s E_p A_p}{L(E_p A_p + 2E_s A_s)}$$

STRESS IN THE PLASTIC CYLINDER

$$\sigma_p = \frac{P_p}{A_p} = \frac{2\,np\,E_s A_s E_p}{L(E_p A_p + 2E_s A_s)} \longleftarrow$$

SUBSTITUTE NUMERICAL VALUES:

$$N = E_s A_s E_p = 54.0 \times 10^{15}\ \text{N}^2/\text{m}^2$$

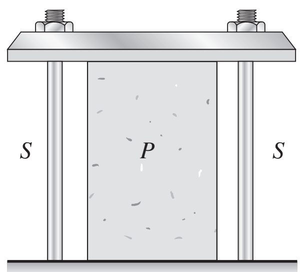
$$D = E_p A_p + 2E_s A_s = 21.6 \times 10^6\ \text{N}$$

$$\sigma_p = \frac{2np}{L}\left(\frac{N}{D}\right) = \frac{2(1)(1.0\text{ mm})}{200\text{ mm}}\left(\frac{N}{D}\right)$$

$$= 25.0\text{ MPa} \longleftarrow$$

Problem 2.5-19 Solve the preceding problem if the data for the assembly are as follows: length $L = 10$ in., pitch of the bolt threads $p = 0.058$ in., modulus of elasticity for steel $E_s = 30 \times 10^6$ psi, modulus of elasticity for the plastic $E_p = 500$ ksi, cross-sectional area of one bolt $A_s = 0.06$ in.^{2}, and cross-sectional area of the plastic cylinder $A_p = 1.5$ in.^{2}

Solution 2.5-19 Plastic cylinder and two steel bolts



$L = 10$ in.

$p = 0.058$ in.

$E_s = 30 \times 10^6$ psi

$A_s = 0.06$ in.^{2} (for one bolt)

$E_p = 500$ ksi

$A_p = 1.5$ in.^{2}

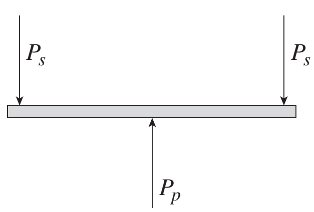
$n = 1$ (see Eq. 2-22)

EQUILIBRIUM EQUATION

P_s = tensile force in one steel bolt

P_p = compressive force in plastic cylinder

$$P_p = 2P_s \qquad \text{(Eq. 1)}$$

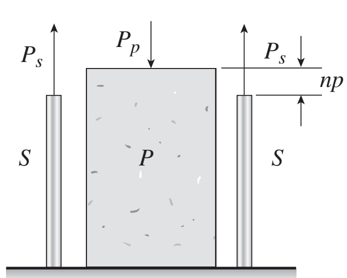


COMPATIBILITY EQUATION

δ_s = elongation of steel bolt

δ_p = shortening of plastic cylinder

$$\delta_s + \delta_p = np \qquad \text{(Eq. 2)}$$



FORCE-DISPLACEMENT RELATIONS

$$\delta_s = \frac{P_s L}{E_s A_s} \qquad \delta_p = \frac{P_p L}{E_p A_p} \qquad \text{(Eq. 3, Eq. 4)}$$

SOLUTION OF EQUATIONS

Substitute (3) and (4) into Eq. (2):

$$\frac{P_s L}{E_s A_s} + \frac{P_p L}{E_p A_p} = np \qquad \text{(Eq. 5)}$$

Solve simultaneously Eqs. (1) and (5):

$$P_p = \frac{2\,np\,E_s A_s E_p A_p}{L(E_p A_p + 2E_s A_s)}$$

STRESS IN THE PLASTIC CYLINDER

$$\sigma_p = \frac{P_p}{A_p} = \frac{2\,np\,E_s A_s E_p}{L(E_p A_p + 2E_s A_s)} \longleftarrow$$

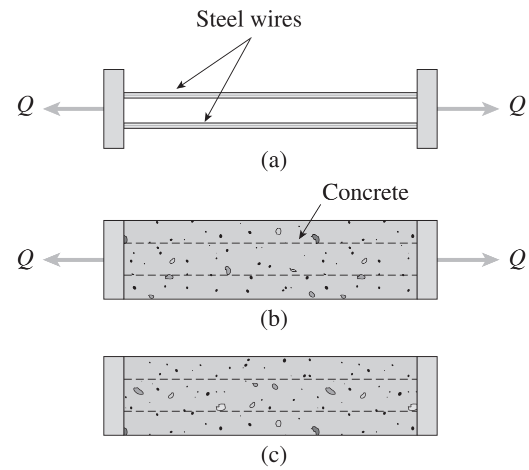
SUBSTITUTE NUMERICAL VALUES:

$$N = E_s A_s E_p = 900 \times 10^9 \text{ lb}^2/\text{in.}^2$$

$$D = E_p A_p + 2E_s A_s = 4350 \times 10^3 \text{ lb}$$

$$\sigma_P = \frac{2np}{L}\left(\frac{N}{D}\right) = \frac{2(1)(0.058 \text{ in.})}{10 \text{ in.}}\left(\frac{N}{D}\right)$$

$$= 2400 \text{ psi} \longleftarrow$$

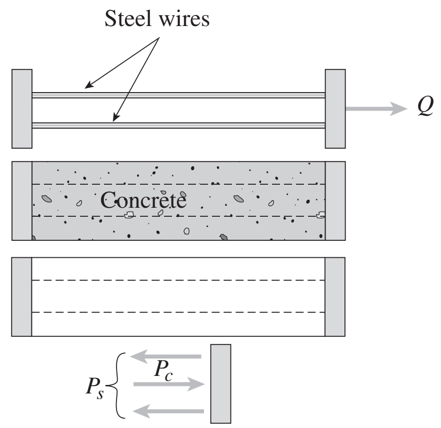


Problem 2.5-20 Prestressed concrete beams are sometimes manufactured in the following manner. High-strength steel wires are stretched by a jacking mechanism that applies a force Q, as represented schematically in part (a) of the figure. Concrete is then poured around the wires to form a beam, as shown in part (b).

After the concrete sets properly, the jacks are released and the force Q is removed [see part (c) of the figure]. Thus, the beam is left in a prestressed condition, with the wires in tension and the concrete in compression.

Let us assume that the prestressing force Q produces in the steel wires an initial stress $\sigma_0 = 620$ MPa. If the moduli of elasticity of the steel and concrete are in the ratio 12:1 and the cross-sectional areas are in the ratio 1:50, what are the final stresses σ_s and σ_c in the two materials?

Solution 2.5-20 Prestressed concrete beam



L = length

σ_0 = initial stress in wires

$$= \frac{Q}{A_s} = 620 \text{ MPa}$$

A_s = total area of steel wires

A_c = area of concrete

$$= 50\, A_s$$

$E_s = 12\, E_c$

P_s = final tensile force in steel wires

P_c = final compressive force in concrete

EQUILIBRIUM EQUATION

$$P_s = P_c \qquad \text{(Eq. 1)}$$

COMPATIBILITY EQUATION AND FORCE-DISPLACEMENT RELATIONS

δ_1 = initial elongation of steel wires

$$= \frac{QL}{E_s A_s} = \frac{\sigma_0 L}{E_s}$$

δ_2 = final elongation of steel wires

$$= \frac{P_s L}{E_s A_s}$$

δ_3 = shortening of concrete

$$= \frac{P_c L}{E_c A_c}$$

$$\delta_1 - \delta_2 = \delta_3 \quad \text{or} \quad \frac{\sigma_0 L}{E_s} - \frac{P_s L}{E_s A_s} = \frac{P_c L}{E_c A_c} \quad \text{(Eq. 2, Eq. 3)}$$

Solve simultaneously Eqs. (1) and (3):

$$P_s = P_c = \frac{\sigma_0 A_s}{1 + \dfrac{E_s A_s}{E_c A_c}}$$

STRESSES

$$\sigma_s = \frac{P_s}{A_s} = \frac{\sigma_0}{1 + \dfrac{E_s A_s}{E_c A_c}} \quad \longleftarrow$$

$$\sigma_c = \frac{P_c}{A_c} = \frac{\sigma_0}{\dfrac{A_c}{A_s} + \dfrac{E_s}{E_c}} \quad \longleftarrow$$

SUBSTITUTE NUMERICAL VALUES:

$$\sigma_0 = 620 \text{ MPa} \qquad \frac{E_s}{E_c} = 12 \qquad \frac{A_s}{A_c} = \frac{1}{50}$$

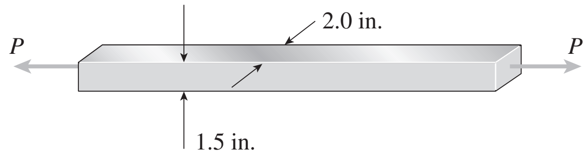
$$\sigma_s = \frac{620 \text{ MPa}}{1 + \dfrac{12}{50}} = 500 \text{ MPa (Tension)} \quad \longleftarrow$$

$$\sigma_c = \frac{620 \text{ MPa}}{50 + 12} = 10 \text{ MPa (Compression)} \quad \longleftarrow$$

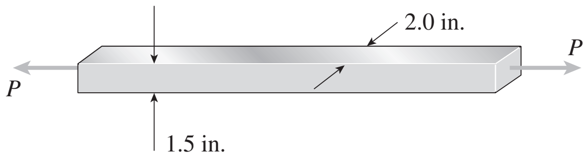
Stresses on Inclined Sections

Problem 2.6-1 A steel bar of rectangular cross section (1.5 in. × 2.0 in.) carries a tensile load P (see figure). The allowable stresses in tension and shear are 15,000 psi and 7,000 psi, respectively.

Determine the maximum permissible load P_{max}.



Solution 2.6-1 **Rectangular bar in tension**



$A = 1.5 \text{ in.} \times 2.0 \text{ in.}$

$= 3.0 \text{ in.}^2$

Maximum Normal Stress:

$$\sigma_x = \frac{P}{A}$$

Maximum shear stress: $\tau_{max} = \dfrac{\sigma_x}{2} = \dfrac{P}{2A}$

$\sigma_{allow} = 15{,}000 \text{ psi} \quad \tau_{allow} = 7{,}000 \text{ psi}$

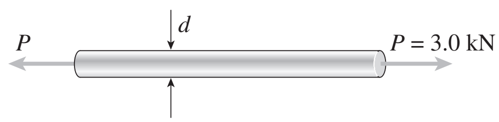
Because τ_{allow} is less than one-half of σ_{allow}, the shear stress governs.

$P_{max} = 2\tau_{allow} A = 2(7{,}000 \text{ psi})\,(3.0 \text{ in.}^2)$

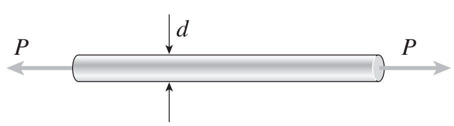
$= 42{,}000 \text{ lb}$ ←

Problem 2.6-2 A circular steel rod of diameter d is subjected to a tensile force $P = 3.0$ kN (see figure). The allowable stresses in tension and shear are 120 MPa and 50 MPa, respectively.

What is the minimum permissible diameter d_{min} of the rod?



Solution 2.6-2 **Steel rod in tension**



$P = 3.0 \text{ kN} \quad A = \dfrac{\pi d^2}{4}$

Maximum normal stress: $\sigma_x = \dfrac{P}{A}$

Maximum shear stress: $\tau_{max} = \dfrac{\sigma_x}{2} = \dfrac{P}{2A}$

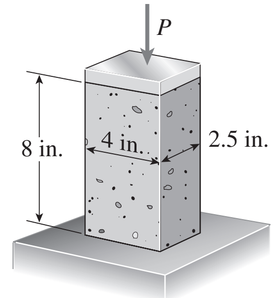
$\sigma_{allow} = 120 \text{ MPa} \qquad \tau_{allow} = 50 \text{ MPa}$

Because τ_{allow} is less than one-half of σ_{allow}, the shear stress governs.

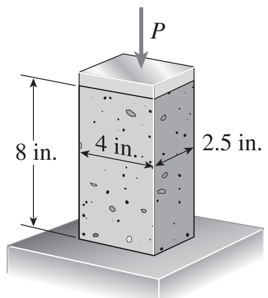
$$\tau_{max} = \frac{P}{2A} \quad \text{or} \quad 50 \text{ MPa} = \frac{3.0 \text{ kN}}{(2)\left(\dfrac{\pi d^2}{4}\right)}$$

Solve for d: $d_{min} = 6.18$ mm ←

Problem 2.6-3 A standard brick (dimensions 8 in. $\times$ 4 in. $\times$ 2.5 in.) is compressed lengthwise by a force P, as shown in the figure. If the ultimate shear stress for brick is 1200 psi and the ultimate compressive stress is 3600 psi, what force P_{max} is required to break the brick?



Solution 2.6-3 **Standard brick in compression**



$A = 2.5 \text{ in.} \times 4.0 \text{ in.} = 10.0 \text{ in.}^2$

Maximum normal stress:

$$\sigma_x = \frac{P}{A}$$

Maximum shear stress:

$$\tau_{max} = \frac{\sigma_x}{2} = \frac{P}{2A}$$

$$\sigma_{ult} = 3600 \text{ psi} \qquad \tau_{ult} = 1200 \text{ psi}$$

Because τ_{ult} is less than one-half of σ_{ult}, the shear stress governs.

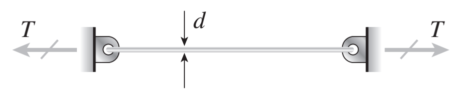
$$\tau_{max} = \frac{P}{2A} \quad \text{or} \quad P_{max} = 2A\tau_{ult}$$

$$P_{max} = 2(10.0 \text{ in.}^2)(1200 \text{ psi})$$

$$= 24{,}000 \text{ lb} \longleftarrow$$

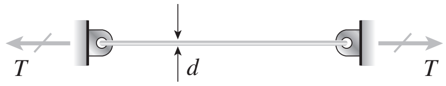
Problem 2.6-4 A brass wire of diameter $d = 2.42$ mm is stretched tightly between rigid supports so that the tensile force is $T = 92$ N (see figure).

What is the maximum permissible temperature drop ΔT if the allowable shear stress in the wire is 60 MPa? (The coefficient of thermal expansion for the wire is 20×10^{-6}/°C and the modulus of elasticity is 100 GPa.)



Probs. 2.6-4 and 2.6-5

Solution 2.6-4 **Brass wire in tension**



$d = 2.42$ mm

$$A = \frac{\pi d^2}{4} = 4.60 \text{ mm}^2$$

$\alpha = 20 \times 10^{-6}/°\text{C} \quad E = 100 \text{ GPa} \quad \tau_{allow} = 60 \text{ MPa}$

Initial tensile force: $T = 92$ N

Stress due to initial tension: $\sigma_x = \dfrac{T}{A}$

Stress due to temperature drop: $\sigma_x = E\alpha(\Delta T)$

(see Eq. 2-18 of Section 2.5)

Total stress: $\sigma_x = \dfrac{T}{A} + E\alpha(\Delta T)$

MAXIMUM SHEAR STRESS

$$\tau_{max} = \frac{\sigma_x}{2} = \frac{1}{2}\left[\frac{T}{A} + E\alpha(\Delta T)\right]$$

Solve for temperature drop ΔT:

$$\Delta T = \frac{2\tau_{max} - T/A}{E\alpha} \qquad \tau_{max} = \tau_{allow}$$

SUBSTITUTE NUMERICAL VALUES:

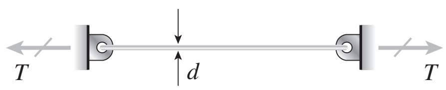
$$\Delta T = \frac{2(60 \text{ MPa}) - (92 \text{ N})/(4.60 \text{ mm}^2)}{(100 \text{ GPa})(20 \times 10^{-6}/°\text{C})}$$

$$= \frac{120 \text{ MPa} - 20 \text{ MPa}}{2 \text{ MPa}/°\text{C}} = 50°\text{C} \longleftarrow$$

Problem 2.6-5 A brass wire of diameter d = 1/16 in. is stretched between rigid supports with an initial tension T of 32 lb (see figure).

(a) If the temperature is lowered by 50°F, what is the maximum shear stress τ_{max} in the wire?
(b) If the allowable shear stress is 10,000 psi, what is the maximum permissible temperature drop? (Assume that the coefficient of thermal expansion is 10.6×10^{-6}/°F and the modulus of elasticity is 15×10^6 psi.)

Solution 2.6-5 **Brass wire in tension**



$$d = \frac{1}{16} \text{ in.}$$

$$A = \frac{\pi d^2}{4}$$

$$= 0.003068 \text{ in.}^2$$

$$\alpha = 10.6 \times 10^{-6}/°\text{F}$$

$$E = 15 \times 10^6 \text{ psi}$$

Initial tensile force: $T = 32$ lb

Stress due to initial tension: $\sigma_x = \dfrac{T}{A}$

Stress due to temperature drop: $\sigma_x = E\alpha(\Delta T)$

(see Eq. 2-18 of Section 2.5)

Total stress: $\sigma_x = \dfrac{T}{A} + E\alpha(\Delta T)$

(a) Maximum shear stress when temperature drops 50°F

$$\tau_{max} = \frac{\sigma_x}{2} = \frac{1}{2}\left[\frac{T}{A} + E\alpha(\Delta T)\right] \qquad \text{(Eq. 1)}$$

Substitute numerical values:

$\tau_{max} = 9{,}190$ psi ←

(b) Maximum permissible temperature drop if $\tau_{allow} = 10{,}000$ psi ←

Solve Eq. (1) for ΔT:

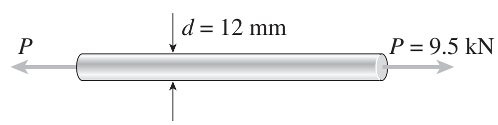
$$\Delta T = \frac{2\tau_{max} - T/A}{E\alpha} \qquad \tau_{max} = \tau_{allow}$$

Substitute numerical values:

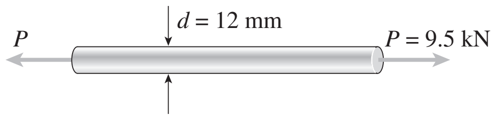
$\Delta T = 60.2$°F ←

Problem 2.6-6 A steel bar with diameter d = 12 mm is subjected to a tensile load P = 9.5 kN (see figure).

(a) What is the maximum normal stress σ_{max} in the bar?
(b) What is the maximum shear stress τ_{max}?
(c) Draw a stress element oriented at 45° to the axis of the bar and show all stresses acting on the faces of this element.



Solution 2.6-6 **Steel bar in tension**



$P = 9.5$ kN

(a) MAXIMUM NORMAL STRESS

$$\sigma_x = \frac{P}{A} = \frac{9.5 \text{ kN}}{\frac{\pi}{4}(12 \text{ mm})^2} = 84.0 \text{ MPa}$$

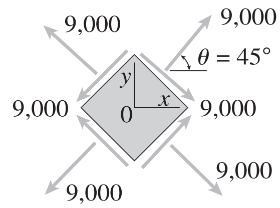
$\sigma_{max} = 84.0$ MPa ⟵

(b) MAXIMUM SHEAR STRESS

The maximum shear stress is on a 45° plane and equals $\sigma_x/2$.

$$\tau_{max} = \frac{\sigma_x}{2} = 42.0 \text{ MPa}$$ ⟵

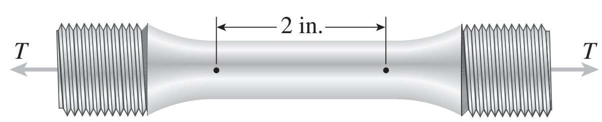
(c) STRESS ELEMENT AT $\theta = 45°$



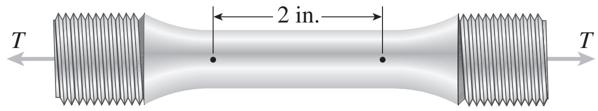
NOTE: All stresses have units of MPa.

Problem 2.6-7 During a tension test of a mild-steel specimen (see figure), the extensometer shows an elongation of 0.00120 in. with a gage length of 2 in. Assume that the steel is stressed below the proportional limit and that the modulus of elasticity $E = 30 \times 10^6$ psi.

(a) What is the maximum normal stress σ_{max} in the specimen?
(b) What is the maximum shear stress τ_{max}?
(c) Draw a stress element oriented at an angle of 45° to the axis of the bar and show all stresses acting on the faces of this element.



Solution 2.6-7 Tension test



Elongation: $\delta = 0.00120$ in.

(2 in. gage length)

Strain: $\varepsilon = \dfrac{\delta}{L} = \dfrac{0.00120 \text{ in.}}{2 \text{ in.}} = 0.00060$

Hooke's law : $\sigma_x = E\varepsilon = (30 \times 10^6 \text{ psi})(0.00060)$

$= 18{,}000$ psi

(a) MAXIMUM NORMAL STRESS

σ_x is the maximum normal stress.

$\sigma_{max} = 18{,}000$ psi ⟵

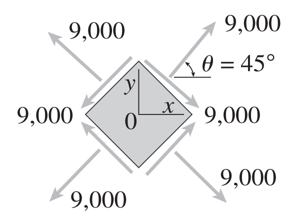
(b) MAXIMUM SHEAR STRESS

The maximum shear stress is on a 45° plane and equals $\sigma_x/2$.

$\tau_{max} = \dfrac{\sigma_x}{2} = 9{,}000$ psi ⟵

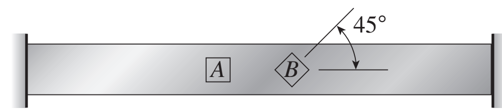
(c) STRESS ELEMENT AT $\theta = 45°$

NOTE: All stresses have units of psi.

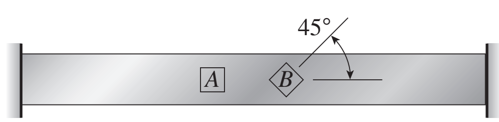


Problem 2.6-8 A copper bar with a rectangular cross section is held without stress between rigid supports (see figure). Subsequently, the temperature of the bar is raised 50°C.

Determine the stresses on all faces of the elements A and B, and show these stresses on sketches of the elements. (Assume $\alpha = 17.5 \times 10^{-6}$/°C and $E = 120$ GPa.)



Solution 2.6-8 Copper bar with rigid supports



$\Delta T = 50°C$ (Increase)

$\alpha = 17.5 \times 10^{-6}$/°C

$E = 120$ GPa

STRESS DUE TO TEMPERATURE INCREASE

$\sigma_x = E\alpha\,(\Delta T)$ (See Eq. 2-18 of Section 2.5)

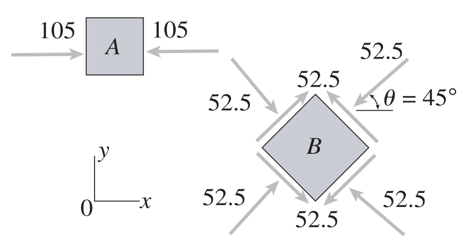
$= 105$ MPa (Compression)

MAXIMUM SHEAR STRESS

$\tau_{max} = \dfrac{\sigma_x}{2}$

$= 52.5$ MPa

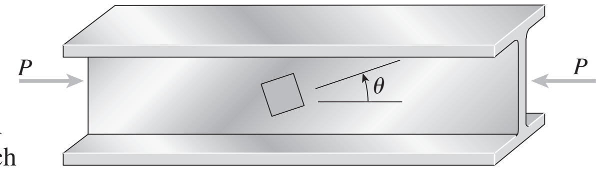
STRESSES ON ELEMENTS A AND B



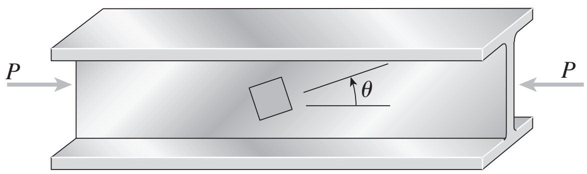
NOTE: All stresses have units of MPa.

Problem 2.6-9 A compression member in a bridge truss is fabricated from a wide-flange steel section (see figure). The cross-sectional area $A = 7.5 \text{ in.}^2$ and the axial load $P = 90$ k.

Determine the normal and shear stresses acting on all faces of stress elements located in the web of the beam and oriented at (a) an angle $\theta = 0°$, (b) an angle $\theta = 30°$, and (c) an angle $\theta = 45°$. In each case, show the stresses on a sketch of a properly oriented element.



Solution 2.6-9 Truss member in compression



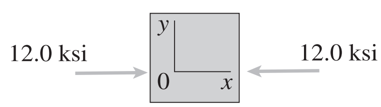
$P = 90$ k

$A = 7.5 \text{ in.}^2$

$$\sigma_x = -\frac{P}{A} = -\frac{90 \text{ k}}{7.5 \text{ in.}^2}$$

$$= -12.0 \text{ ksi (Compression)}$$

(a) $\theta = 0°$



(b) $\theta = 30°$

Use Eqs. (2-29a) and (2-29b):

$$\sigma_\theta = \sigma_x \cos^2\theta = (-12.0 \text{ ksi})(\cos 30°)^2$$

$$= -9.0 \text{ ksi}$$

$$\tau_\theta = -\sigma_x \sin\theta \cos\theta = -(-12.0 \text{ ksi})(\sin 30°)(\cos 30°)$$

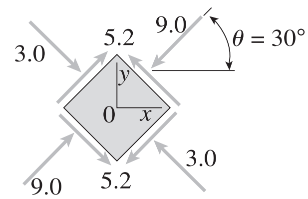
$$= 5.2 \text{ ksi}$$

$$\theta = 30° + 90° = 120°$$

$$\sigma_\theta = \sigma_x \cos^2\theta = (-12.0 \text{ ksi})(\cos 120°)^2 = -3.0 \text{ ksi}$$

$$\tau_\theta = -\sigma_x \sin\theta \cos\theta = -(-12.0 \text{ ksi})(\sin 120°)(\cos 120°)$$

$$= -5.2 \text{ ksi}$$



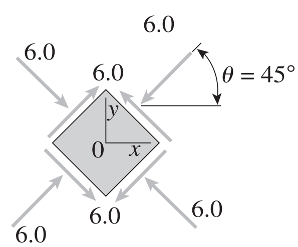
NOTE: All stresses have units of ksi.

(c) $\theta = 45°$

$$\sigma_\theta = \sigma_x \cos^2\theta = (-12.0 \text{ ksi})(\cos 45°)^2 = -6.0 \text{ ksi}$$

$$\tau_\theta = -\sigma_x \sin\theta \cos\theta = -(-12.0 \text{ ksi})(\sin 45°)(\cos 45°)$$

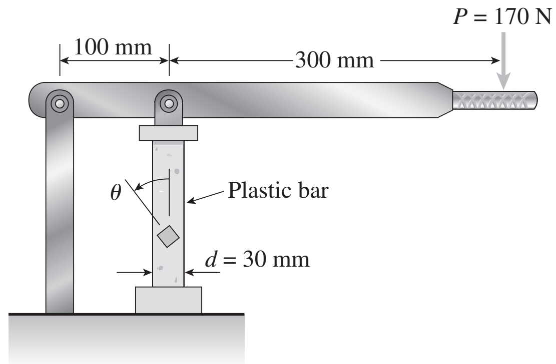
$$= 6.0 \text{ ksi}$$



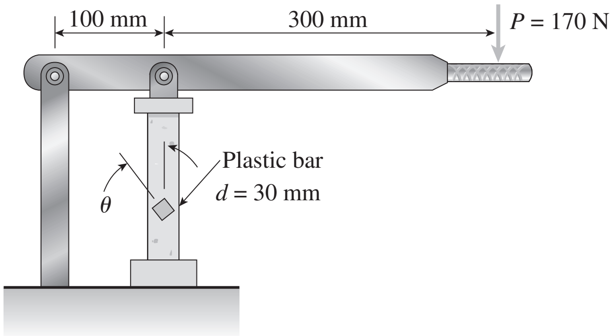
NOTE: All stresses have units of ksi.

Problem 2.6-10 A plastic bar of diameter $d = 30$ mm is compressed in a testing device by a force $P = 170$ N applied as shown in the figure.

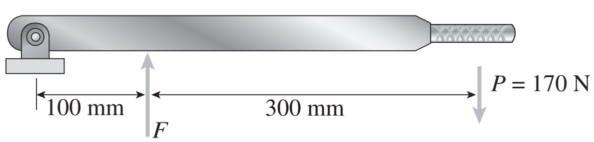
Determine the normal and shear stresses acting on all faces of stress elements oriented at (a) an angle $\theta = 0°$, (b) an angle $\theta = 22.5°$, and (c) an angle $\theta = 45°$. In each case, show the stresses on a sketch of a properly oriented element.



Solution 2.6-10 Plastic bar in compression



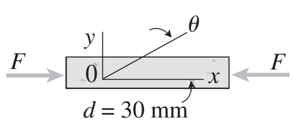
FREE-BODY DIAGRAM



F = Compressive force in plastic bar

$F = 4P = 4(170 \text{ N}) = 680 \text{ N}$

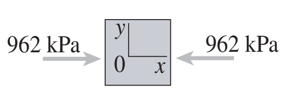
PLASTIC BAR (ROTATED TO THE HORIZONTAL)



$$\sigma_x = -\frac{F}{A} = -\frac{680 \text{ N}}{\frac{\pi}{4}(30 \text{ mm})^2}$$

$$= -962.0 \text{ kPa (Compression)}$$

(a) $\theta = 0°$



(b) $\theta = 22.5°$

Use Eqs. (2-29a) and (2-29b)

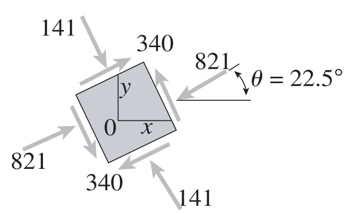
$$\sigma_\theta = \sigma_x \cos^2\theta = (-962.0 \text{ kPa})(\cos 22.5°)^2$$
$$= -821 \text{ kPa}$$

$$\tau_\theta = -\sigma_x \sin\theta\cos\theta$$
$$= -(-962.0 \text{ kPa})(\sin 22.5°)(\cos 22.5°)$$
$$= 340 \text{ kPa}$$

$$\theta = 22.5° + 90° = 112.5°$$

$$\sigma_\theta = \sigma_x \cos^2\theta = (-962.0 \text{ kPa})(\cos 112.5°)^2$$
$$= -141 \text{ kPa}$$

$$\tau_\theta = -\sigma_x \sin\theta\cos\theta$$
$$= -(-962.0 \text{ kPa})(\sin 112.5°)(\cos 112.5°)$$
$$= -340 \text{ kPa}$$

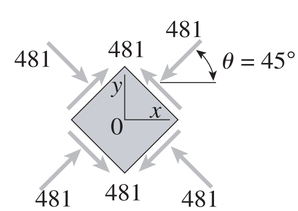


NOTE: All stresses have units of kPa.

(c) $\theta = 45°$

$$\sigma_\theta = \sigma_x \cos^2\theta = (-962.0 \text{ kPa})(\cos 45°)^2$$
$$= -481 \text{ kPa}$$

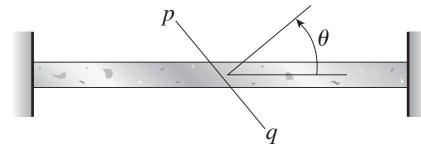
$$\tau_\theta = -\sigma_x \sin\theta\cos\theta$$
$$= -(-962.0 \text{ kPa})(\sin 45°)(\cos 45°) = 481 \text{ kPa}$$



NOTE: All stresses have units of kPa.

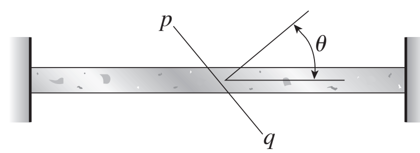
Problem 2.6-11 A plastic bar fits snugly between rigid supports at room temperature (68°F) but with no initial stress (see figure). When the temperature of the bar is raised to 160°F, the compressive stress on an inclined plane *pq* becomes 1700 psi.

(a) What is the shear stress on plane *pq*? (Assume $\alpha = 60 \times 10^{-6}$/°F and $E = 450 \times 10^3$ psi.)
(b) Draw a stress element oriented to plane *pq* and show the stresses acting on all faces of this element.



Probs. 2.6-11 and 2.6-12

Solution 2.6-11 **Plastic bar between rigid supports**



$\alpha = 60 \times 10^{-6}$/°F $\quad E = 450 \times 10^3$ psi

Temperature increase:

$\Delta T = 160°\text{F} - 68°\text{F} = 92°\text{F}$

NORMAL STRESS σ_x IN THE BAR

$\sigma_x = -E\alpha(\Delta T)$ (See Eq. 2-18 in Section 2.5)

$\sigma_x = -(450 \times 10^3 \text{ psi})(60 \times 10^{-6}/°\text{F})(92°\text{F})$

$= -2484$ psi (Compression)

ANGLE θ TO PLANE *pq*

$\sigma_\theta = \sigma_x \cos^2\theta$ For plane *pq*: $\sigma_\theta = -1700$ psi

Therefore, $-1700 \text{ psi} = (-2484 \text{ psi})(\cos^2\theta)$

$$\cos^2\theta = \frac{-1700 \text{ psi}}{-2484 \text{ psi}} = 0.6844$$

$\cos\theta = 0.8273 \quad \theta = 34.18°$

(a) SHEAR STRESS ON PLANE *pq*

$\tau_\theta = -\sigma_x \sin\theta\cos\theta$

$= -(-2484 \text{ psi})(\sin 34.18°)(\cos 34.18°)$

$= 1150$ psi (Counter clockwise) ←

(b) STRESS ELEMENT ORIENTED TO PLANE *pq*

$\theta = 34.18° \quad \sigma_\theta = -1700 \text{ psi} \quad \tau_\theta = 1150$ psi

$\theta = 34.18° + 90° = 124.18°$

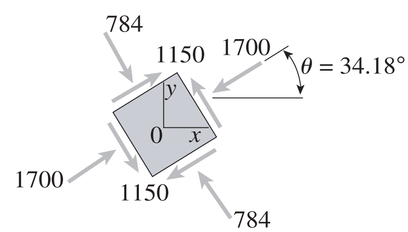
$\sigma_\theta = \sigma_x \cos^2\theta = (-2484 \text{ psi})(\cos 124.18°)^2$

$= -784$ psi

$\tau_\theta = -\sigma_x \sin\theta\cos\theta$

$= -(-2484 \text{ psi})(\sin 124.18°)(\cos 124.18°)$

$= -1150$ psi

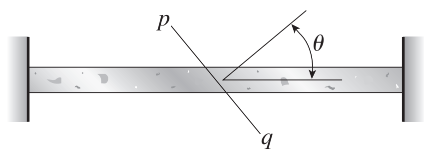


NOTE: All stresses have units of psi.

Problem 2.6-12 A copper bar is held snugly (but without any initial stress) between rigid supports (see figure). The allowable stresses on the inclined plane *pq*, for which $\theta = 55°$, are specified as 60 MPa in compression and 30 MPa in shear.

(a) What is the maximum permissible temperature rise ΔT if the allowable stresses on plane *pq* are not to be exceeded? (Assume $\alpha = 17 \times 10^{-6}$/°C and $E = 120$ GPa.)
(b) If the temperature increases by the maximum permissible amount, what are the stresses on plane *pq*?

Solution 2.6-12 **Copper bar between rigid supports**



$\alpha = 17 \times 10^{-6}$/°C

$E = 120$ GPa

Plane *pq*: $\theta = 55°$

Allowable stresses on plane *pq*:

$\sigma_{allow} = 60$ MPa (Compression)

$\tau_{allow} = 30$ MPa (Shear)

(a) Maximum permissible temperature rise ΔT

$\sigma_\theta = \sigma_x \cos^2\theta \quad -60 \text{ MPa} = \sigma_x (\cos 55°)^2$

$\sigma_x = -182.4$ MPa

$\tau_\theta = -\sigma_x \sin\theta \cos\theta$

$30 \text{ MPa} = -\sigma_x (\sin 55°)(\cos 55°)$

$\sigma_x = -63.85$ MPa

Shear stress governs. $\sigma_x = -63.85$ MPa

Due to temperature increase ΔT:

$\sigma_x = -E\alpha(\Delta T)$ (See Eq. 2-18 in Section 2.5)

$-63.85 \text{ MPa} = -(120 \text{ GPa})(17 \times 10^{-6}/°\text{C})(\Delta T)$

$\Delta T = 31.3°\text{C}$ ←

(b) Stresses on plane *pq*

$\sigma_x = -63.85$ MPa

$\sigma_\theta = \sigma_x \cos^2\theta = (-63.85 \text{ MPa})(\cos 55°)^2$

$= -21.0$ MPa (Compression) ←

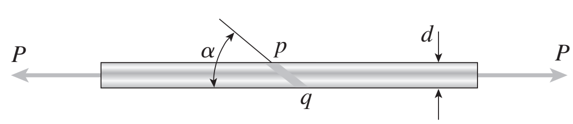
$\tau_\theta = -\sigma_x \sin\theta \cos\theta$

$= -(-63.85 \text{ MPa})(\sin 55°)(\cos 55°)$

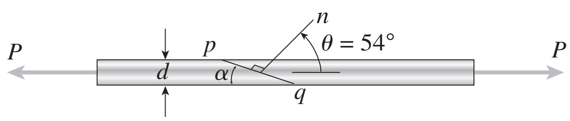
$= 30.0$ MPa (Counter clockwise) ←

Problem 2.6-13 A circular brass bar of diameter *d* is composed of two segments brazed together on a plane *pq* making an angle $\alpha = 36°$ with the axis of the bar (see figure). The allowable stresses in the brass are 13,500 psi in tension and 6500 psi in shear. On the brazed joint, the allowable stresses are 6000 psi in tension and 3000 psi in shear.

If the bar must resist a tensile force $P = 6000$ lb, what is the minimum required diameter d_{min} of the bar?



Solution 2.6-13 **Brass bar in tension**



$\alpha = 36°$

$\theta = 90° - \alpha = 54°$

$P = 6000$ lb

$$A = \frac{\pi d^2}{4}$$

STRESS σ_x BASED UPON ALLOWABLE STRESSES IN THE BRASS

Tensile stress ($\theta = 0°$): $\sigma_{allow} = 13{,}500$ psi

$$\sigma_x = 13{,}500 \text{ psi} \qquad (1)$$

Shear stress ($\theta = 45°$): $\tau_{allow} = 6500$ psi

$$\tau_{max} = \frac{\sigma_x}{2}$$

$$\sigma_x = 2\,\tau_{allow}$$
$$= 13{,}000 \text{ psi} \qquad (2)$$

STRESS σ_x BASED UPON ALLOWABLE STRESSES ON THE BRAZED JOINT ($\theta = 54°$)

$\sigma_{allow} = 6000$ psi (tension)

$\tau_{allow} = 3000$ psi (shear)

Tensile stress: $\sigma_\theta = \sigma_x \cos^2\theta$

$$\sigma_x = \frac{\sigma_{allow}}{\cos^2\theta} = \frac{6000 \text{ psi}}{(\cos 54°)^2}$$
$$= 17{,}370 \text{ psi} \qquad (3)$$

Shear stress: $\tau_\theta = -\sigma_x \sin\theta\cos\theta$

$$\sigma_x = \left|\frac{\tau_{allow}}{\sin\theta\cos\theta}\right| = \frac{3{,}000 \text{ psi}}{(\sin 54°)(\cos 54°)}$$
$$= 6{,}310 \text{ psi} \qquad (4)$$

ALLOWABLE STRESS

Compare (1), (2), (3), and (4).

Shear stress on the brazed joint governs.

$\sigma_x = 6310$ psi

DIAMETER OF BAR

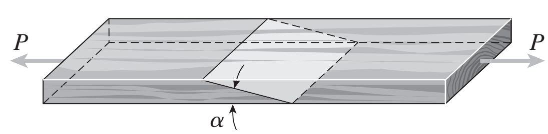
$$A = \frac{P}{\sigma_x} = \frac{6000 \text{ lb}}{6310 \text{ psi}} = 0.951 \text{ in.}^2$$

$$A = \frac{\pi d^2}{4} \quad d^2 = \frac{4A}{\pi} \quad d_{min} = \sqrt{\frac{4A}{\pi}}$$

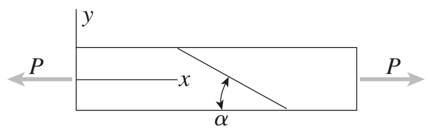
$$d_{min} = 1.10 \text{ in.} \longleftarrow$$

Problem 2.6-14 Two boards are joined by gluing along a scarf joint, as shown in the figure. For purposes of cutting and gluing, the angle α between the plane of the joint and the faces of the boards must be between 10° and 40°. Under a tensile load P, the normal stress in the boards is 4.9 MPa.

(a) What are the normal and shear stresses acting on the glued joint if $\alpha = 20°$?
(b) If the allowable shear stress on the joint is 2.25 MPa, what is the largest permissible value of the angle α?
(c) For what angle α will the shear stress on the glued joint be numerically equal to twice the normal stress on the joint?



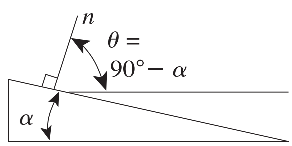
Solution 2.6-14 **Two boards joined by a scarf joint**



$10° \leq \alpha \leq 40°$

Due to load P: $\sigma_x = 4.9$ MPa

(a) STRESSES ON JOINT WHEN $\alpha = 20°$



$\theta = 90° - \alpha = 70°$

$\sigma_\theta = \sigma_x \cos^2\theta = (4.9 \text{ MPa})(\cos 70°)^2$

$= 0.57$ MPa ⟵

$\tau_\theta = -\sigma_x \sin\theta \cos\theta$

$= (-4.9 \text{ MPa})(\sin 70°)(\cos 70°)$

$= -1.58$ MPa ⟵

(b) LARGEST ANGLE α IF $\tau_{allow} = 2.25$ MPa

$\tau_{allow} = -\sigma_x \sin\theta \cos\theta$

The shear stress on the joint has a negative sign. Its numerical value cannot exceed $\tau_{allow} = 2.25$ MPa. Therefore,

$-2.25 \text{ MPa} = -(4.9 \text{ MPa})(\sin\theta)(\cos\theta)$ or
$\sin\theta \cos\theta = 0.4592$

From trigonometry: $\sin\theta\cos\theta = \dfrac{1}{2}\sin 2\theta$

Therefore: $\sin 2\theta = 2(0.4592) = 0.9184$

Solving : $2\theta = 66.69°$ or $113.31°$

$\theta = 33.34°$ or $56.66°$

$\alpha = 90° - \theta \quad \therefore \alpha = 56.66°$ or $33.34°$

Since α must be between 10° and 40°, we select

$\alpha = 33.3°$ ⟵

Note: If α is between 10° and 33.3°,

$|\tau_\theta| < 2.25$ MPa.

If α is between 33.3° and 40°,

$|\tau_\theta| > 2.25$ MPa.

(c) WHAT IS α if $\tau_\theta = 2\sigma_\theta$?

Numerical values only:

$|\tau_\theta| = \sigma_x \sin\theta\cos\theta \qquad |\sigma_\theta| = \sigma_x \cos^2\theta$

$$\left|\frac{\tau_\theta}{\sigma_\theta}\right| = 2$$

$\sigma_x \sin\theta\cos\theta = 2\sigma_x \cos^2\theta$

$\sin\theta = 2\cos\theta$ or $\tan\theta = 2$

$\theta = 63.43° \quad \alpha = 90° - \theta$

$\alpha = 26.6°$ ⟵

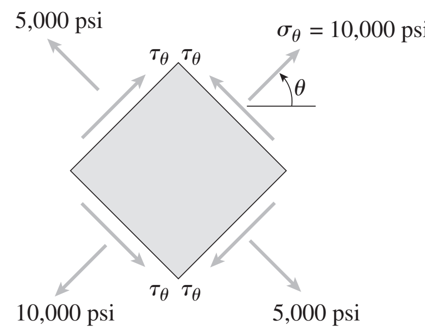
NOTE: For $\alpha = 26.6°$ and $\theta = 63.4°$, we find $\sigma_\theta =$ 0.98 MPa and $\tau_\theta = -1.96$ MPa.

Thus, $\left|\dfrac{\tau_\theta}{\sigma_\theta}\right| = 2$ as required.

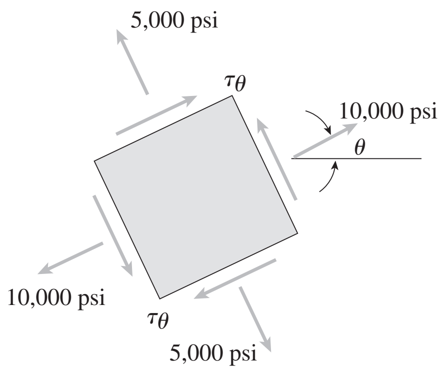
Problem 2.6-15 Acting on the sides of a stress element cut from a bar in uniaxial stress are tensile stresses of 10,000 psi and 5,000 psi, as shown in the figure.

(a) Determine the angle θ and the shear stress τ_θ and show all stresses on a sketch of the element.

(b) Determine the maximum normal stress σ_{max} and the maximum shear stress τ_{max} in the material.



Solution 2.6-15 **Bar in uniaxial stress**



(a) ANGLE θ AND SHEAR STRESS τ_θ

$$\sigma_\theta = \sigma_x \cos^2\theta$$

$$\sigma_\theta = 10{,}000 \text{ psi}$$

$$\sigma_x = \frac{\sigma_\theta}{\cos^2\theta} = \frac{10{,}000 \text{ psi}}{\cos^2\theta} \qquad (1)$$

PLANE AT ANGLE $\theta + 90°$

$$\sigma_{\theta+90°} = \sigma_x[\cos(\theta + 90°)]^2 = \sigma_x[-\sin\theta]^2$$

$$= \sigma_x \sin^2\theta$$

$$\sigma_{\theta+90°} = 5{,}000 \text{ psi}$$

$$\sigma_x = \frac{\sigma_{\theta+90°}}{\sin^2\theta} = \frac{5{,}000 \text{ psi}}{\sin^2\theta} \qquad (2)$$

Equate (1) and (2):

$$\frac{10{,}000 \text{ psi}}{\cos^2\theta} = \frac{5{,}000 \text{ psi}}{\sin^2\theta}$$

$$\tan^2\theta = \frac{1}{2} \qquad \tan\theta = \frac{1}{\sqrt{2}} \qquad \theta = 35.26° \quad \longleftarrow$$

From Eq. (1) or (2):

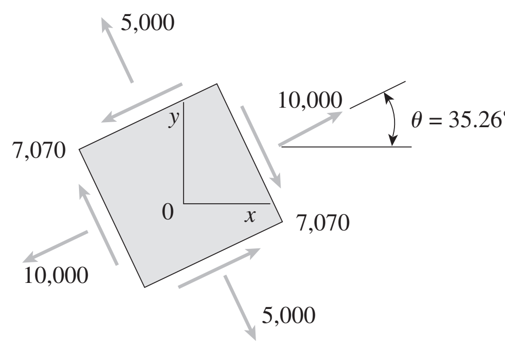
$$\sigma_x = 15{,}000 \text{ psi}$$

$$\tau_\theta = -\sigma_x \sin\theta \cos\theta$$

$$= (-15{,}000 \text{ psi})(\sin 35.26°)(\cos 35.26°)$$

$$= -7{,}070 \text{ psi} \quad \longleftarrow$$

Minus sign means that τ_θ acts clockwise on the plane for which $\theta = 35.26°$.



NOTE: All stresses have units of psi.

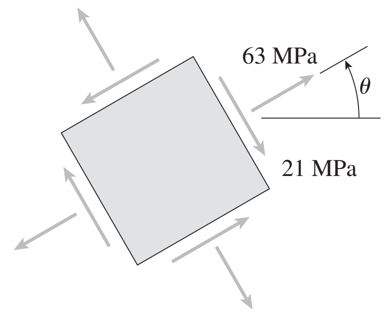
(b) MAXIMUM NORMAL AND SHEAR STRESSES

$$\sigma_{max} = \sigma_x = 15{,}000 \text{ psi} \quad \longleftarrow$$

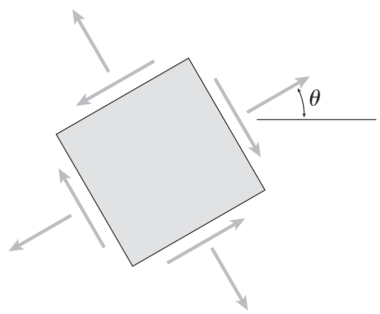
$$\tau_{max} = \frac{\sigma_x}{2} = 7{,}500 \text{ psi} \quad \longleftarrow$$

Problem 2.6-16 A prismatic bar is subjected to an axial force that produces a tensile stress $\sigma_\theta = 63$ MPa and a shear stress $\tau_\theta = -21$ MPa on a certain inclined plane (see figure).

Determine the stresses acting on all faces of a stress element oriented at $\theta = 30°$ and show the stresses on a sketch of the element.



Solution 2.6-16 **Bar in uniaxial stress**



$\sigma_\theta = 63$ MPa $\quad \tau_\theta = -21$ MPa

INCLINED PLANE AT ANGLE θ

$\sigma_\theta = \sigma_x \cos^2\theta$

$63 \text{ MPa} = \sigma_x \cos^2\theta$

$$\sigma_x = \frac{63 \text{ MPa}}{\cos^2\theta} \tag{1}$$

$\tau_\theta = -\sigma_x \sin\theta\cos\theta$

$-21 \text{ MPa} = -\sigma_x \sin\theta\cos\theta$

$$\sigma_x = \frac{21 \text{ MPa}}{\sin\theta\cos\theta} \tag{2}$$

Equate (1) and (2):

$$\frac{63 \text{ MPa}}{\cos^2\theta} = \frac{21 \text{ MPa}}{\sin\theta\cos\theta}$$

or

$$\tan\theta = \frac{21}{63} = \frac{1}{3} \qquad \theta = 18.43°$$

From (1) or (2): $\sigma_x = 70.0$ MPa (tension)

STRESS ELEMENT AT $\theta = 30°$

$\sigma_\theta = \sigma_x \cos^2\theta = (70 \text{ MPa})(\cos 30°)^2$

$= 52.5$ MPa

$\tau_\theta = -\sigma_x \sin\theta\cos\theta$

$= (-70 \text{ MPa})(\sin 30°)(\cos 30°)$

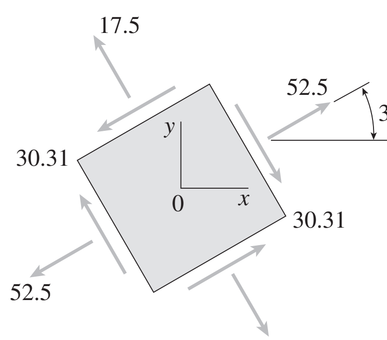
$= -30.31$ MPa

Plane at $\theta = 30° + 90° = 120°$

$\sigma_\theta = (70 \text{ MPa})(\cos 120°)^2 = 17.5$ MPa

$\tau_\theta = (-70 \text{ MPa})(\sin 120°)(\cos 120°)$

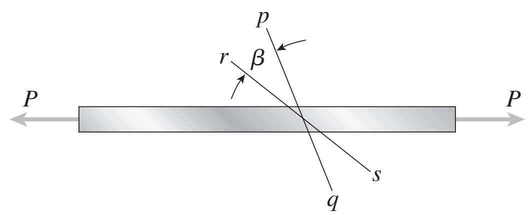
$= 30.31$ MPa



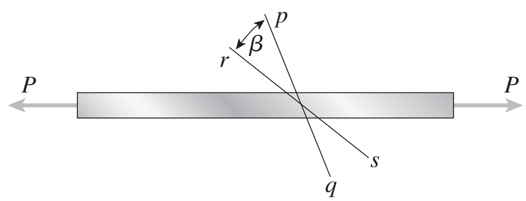
NOTE: All stresses have units of MPa.

Problem 2.6-17 The normal stress on plane *pq* of a prismatic bar in tension (see figure) is found to be 7500 psi. On plane *rs*, which makes an angle $\beta = 30°$ with plane *pq*, the stress is found to be 2500 psi.

Determine the maximum normal stress σ_{max} and maximum shear stress τ_{max} in the bar.



Solution 2.6-17 Bar in tension



Eq. (2-29a):

$\sigma_\theta = \sigma_x \cos^2\theta$

$\beta = 30°$

PLANE *pq*: $\sigma_1 = \sigma_x \cos^2\theta_1$ $\quad \sigma_1 = 7500$ psi

PLANE *rs*: $\sigma_2 = \sigma_x \cos^2(\theta_1 + \beta)$ $\quad \sigma_2 = 2500$ psi

Equate σ_x from σ_1 and σ_2:

$$\sigma_x = \frac{\sigma_1}{\cos^2\theta_1} = \frac{\sigma_2}{\cos^2(\theta_1 + \beta)} \qquad \text{(Eq. 1)}$$

or

$$\frac{\cos^2\theta_1}{\cos^2(\theta_1 + \beta)} = \frac{\sigma_1}{\sigma_2} \quad \frac{\cos\theta_1}{\cos(\theta_1 + \beta)} = \sqrt{\frac{\sigma_1}{\sigma_2}} \qquad \text{(Eq. 2)}$$

SUBSTITUTE NUMERICAL VALUES INTO EQ. (2):

$$\frac{\cos\theta_1}{\cos(\theta_1 + 30°)} = \sqrt{\frac{7500 \text{ psi}}{2500 \text{ psi}}} = \sqrt{3} = 1.7321$$

Solve by iteration or a computer program:

$\theta_1 = 30°$

MAXIMUM NORMAL STRESS (FROM EQ. 1)

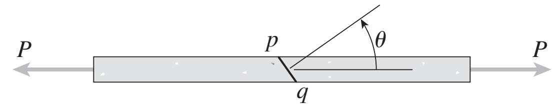
$$\sigma_{max} = \sigma_x = \frac{\sigma_1}{\cos^2\theta_1} = \frac{7500 \text{ psi}}{\cos^2 30°}$$

$$= 10{,}000 \text{ psi} \quad \longleftarrow$$

MAXIMUM SHEAR STRESS

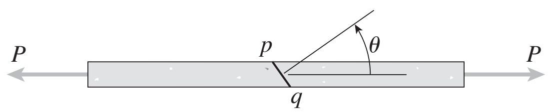
$$\tau_{max} = \frac{\sigma_x}{2} = 5{,}000 \text{ psi} \quad \longleftarrow$$

Problem 2.6-18 A tension member is to be constructed of two pieces of plastic glued along plane *pq* (see figure). For purposes of cutting and gluing, the angle θ must be between 25° and 45°. The allowable stresses on the glued joint in tension and shear are 5.0 MPa and 3.0 MPa, respectively.



(a) Determine the angle θ so that the bar will carry the largest load P. (Assume that the strength of the glued joint controls the design.)
(b) Determine the maximum allowable load P_{max} if the cross-sectional area of the bar is 225 mm^2.

Solution 2.6-18 **Bar in tension with glued joint**



$25° < \theta < 45°$

$A = 225 \text{ mm}^2$

On glued joint: $\sigma_{allow} = 5.0$ MPa

$\tau_{allow} = 3.0$ MPa

ALLOWABLE STRESS σ_x IN TENSION

$$\sigma_\theta = \sigma_x \cos^2\theta \qquad \sigma_x = \frac{\sigma_\theta}{\cos^2\theta} = \frac{5.0 \text{ MPa}}{\cos^2\theta} \qquad (1)$$

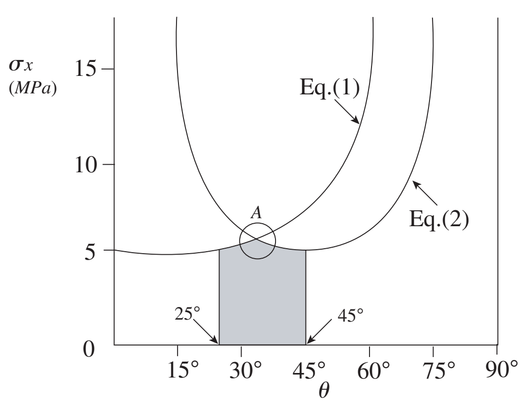
$$\tau_\theta = -\sigma_x \sin\theta\cos\theta$$

Since the direction of τ_θ is immaterial, we can write:
$|\tau_\theta| = \sigma_x \sin\theta\cos\theta$

or

$$\sigma_x = \frac{|\tau_\theta|}{\sin\theta\cos\theta} = \frac{3.0 \text{ MPa}}{\sin\theta\cos\theta} \qquad (2)$$

GRAPH OF EQS. (1) AND (2)



(a) DETERMINE ANGLE θ FOR LARGEST LOAD

Point A gives the largest value of σ_x and hence the largest load. To determine the angle θ corresponding to point A, we equate Eqs. (1) and (2).

$$\frac{5.0 \text{ MPa}}{\cos^2\theta} = \frac{3.0 \text{ MPa}}{\sin\theta\cos\theta}$$

$$\tan\theta = \frac{3.0}{5.0} \qquad \theta = 30.96° \quad \longleftarrow$$

(b) DETERMINE THE MAXIMUM LOAD

From Eq. (1) or Eq. (2):

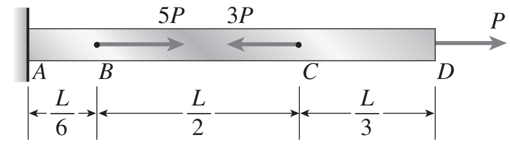
$$\sigma_x = \frac{5.0 \text{ MPa}}{\cos^2\theta} = \frac{3.0 \text{ MPa}}{\sin\theta\cos\theta} = 6.80 \text{ MPa}$$

$$P_{max} = \sigma_x A = (6.80 \text{ MPa})(225 \text{ mm}^2)$$
$$= 1.53 \text{ kN} \quad \longleftarrow$$

Strain Energy

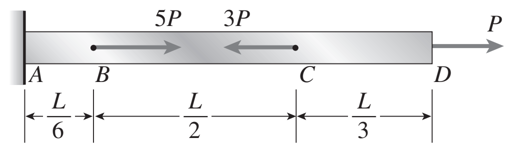
When solving the problems for Section 2.7, assume that the material behaves linearly elastically.

Problem 2.7-1 A prismatic bar AD of length L, cross-sectional area A, and modulus of elasticity E is subjected to loads $5P$, $3P$, and P acting at points B, C, and D, respectively (see figure). Segments AB, BC, and CD have lengths $L/6$, $L/2$, and $L/3$, respectively.



(a) Obtain a formula for the strain energy U of the bar.
(b) Calculate the strain energy if $P = 6$ k, $L = 52$ in., $A = 2.76$ in.^{2}, and the material is aluminum with $E = 10.4 \times 10^6$ psi.

Solution 2.7-1 Bar with three loads



$P = 6$ k

$L = 52$ in.

$E = 10.4 \times 10^6$ psi

$A = 2.76$ in.^{2}

INTERNAL AXIAL FORCES

$N_{AB} = 3P \qquad N_{BC} = -2P \qquad N_{CD} = P$

LENGTHS

$$L_{AB} = \frac{L}{6} \qquad L_{BC} = \frac{L}{2} \qquad L_{CD} = \frac{L}{3}$$

(a) STRAIN ENERGY OF THE BAR (EQ. 2-40)

$$U = \sum \frac{N_i^2 L_i}{2E_i A_i}$$

$$= \frac{1}{2EA}\left[(3P)^2\left(\frac{L}{6}\right) + (-2P)^2\left(\frac{L}{2}\right) + (P)^2\left(\frac{L}{3}\right)\right]$$

$$= \frac{P^2 L}{2EA}\left(\frac{23}{6}\right) = \frac{23P^2L}{12EA} \quad \longleftarrow$$

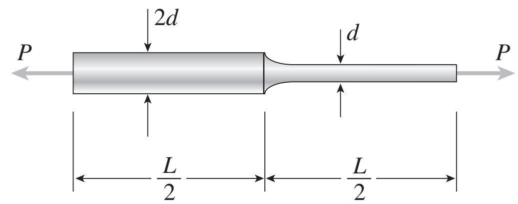
(b) SUBSTITUTE NUMERICAL VALUES:

$$U = \frac{23(6\text{ k})^2(52\text{ in.})}{12(10.4 \times 10^6\text{ psi})(2.76\text{ in.}^2)}$$

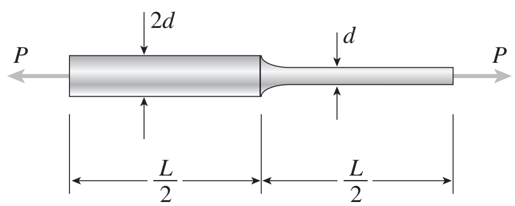
$$= 125\text{ in.-lb} \quad \longleftarrow$$

Problem 2.7-2 A bar of circular cross section having two different diameters d and $2d$ is shown in the figure. The length of each segment of the bar is $L/2$ and the modulus of elasticity of the material is E.

(a) Obtain a formula for the strain energy U of the bar due to the load P.
(b) Calculate the strain energy if the load $P = 27$ kN, the length $L = 600$ mm, the diameter $d = 40$ mm, and the material is brass with $E = 105$ GPa.



Solution 2.7-2 **Bar with two segments**



(a) STRAIN ENERGY OF THE BAR

Add the strain energies of the two segments of the bar (see Eq. 2-40).

$$U = \sum_{i=1}^{2} \frac{N_i^2 L_i}{2\,E_i A_i} = \frac{P^2(L/2)}{2E}\left[\frac{1}{\frac{\pi}{4}(2d)^2} + \frac{1}{\frac{\pi}{4}(d^2)}\right]$$

$$= \frac{P^2L}{\pi E}\left(\frac{1}{4d^2} + \frac{1}{d^2}\right) = \frac{5P^2L}{4\pi E d^2} \longleftarrow$$

(b) SUBSTITUTE NUMERICAL VALUES:

$P = 27$ kN $\qquad L = 600$ mm

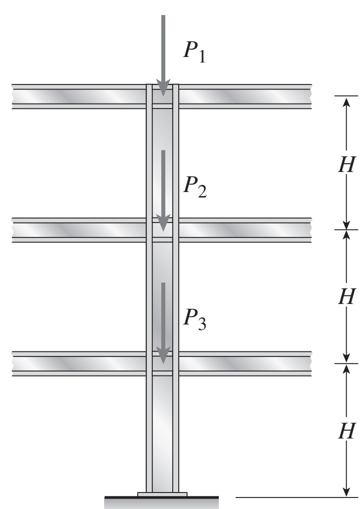
$d = 40$ mm $\qquad E = 105$ GPa

$$U = \frac{5(27\text{ kN})^2(600\text{ mm})}{4\pi(105\text{ GPa})(40\text{ mm})^2}$$

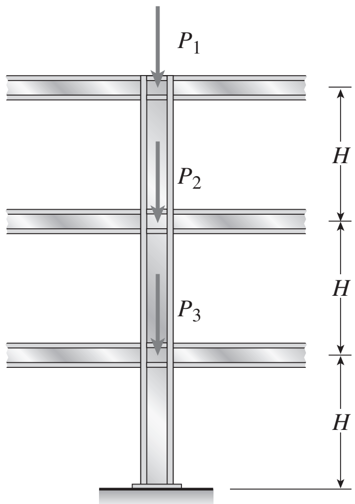
$$= 1.036\text{ N}\cdot\text{m} = 1.036\text{ J} \longleftarrow$$

Problem 2.7-3 A three-story steel column in a building supports roof and floor loads as shown in the figure. The story height H is 10.5 ft, the cross-sectional area A of the column is 15.5 in.^{2}, and the modulus of elasticity E of the steel is 30×10^6 psi.

Calculate the strain energy U of the column assuming $P_1 = 40$ k and $P_2 = P_3 = 60$ k.



Solution 2.7-3 **Three-story column**



$H = 10.5$ ft $\qquad E = 30 \times 10^6$ psi

$A = 15.5$ in.^{2} $\qquad P_1 = 40$ k

$P_2 = P_3 = 60$ k

To find the strain energy of the column, add the strain energies of the three segments (see Eq. 2-40).

Upper segment: $N_1 = -P_1$

Middle segment: $N_2 = -(P_1 + P_2)$

Lower segment: $N_3 = -(P_1 + P_2 + P_3)$

STRAIN ENERGY

$$U = \sum \frac{N_i^2 L_i}{2E_i A_i}$$

$$= \frac{H}{2EA}[P_1^2 + (P_1 + P_2)^2 + (P_1 + P_2 + P_3)^2]$$

$$= \frac{H}{2EA}[Q]$$

$$[Q] = (40\text{ k})^2 + (100\text{ k})^2 + (160\text{ k})^2 = 37{,}200\text{ k}^2$$

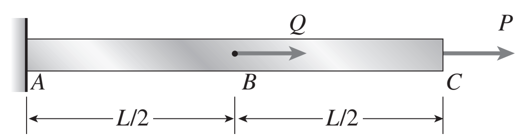
$$2EA = 2(30 \times 10^6\text{ psi})(15.5\text{ in.}^2) = 930 \times 10^6\text{ lb}$$

$$U = \frac{(10.5\text{ ft})(12\text{ in./ft})}{930 \times 10^6\text{ lb}}[37{,}200\text{ k}^2]$$

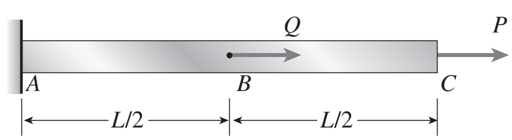
$$= 5040\text{ in.-lb} \quad \longleftarrow$$

Problem 2.7-4 The bar ABC shown in the figure is loaded by a force P acting at end C and by a force Q acting at the midpoint B. The bar has constant axial rigidity EA.

(a) Determine the strain energy U_1 of the bar when the force P acts alone ($Q = 0$).
(b) Determine the strain energy U_2 when the force Q acts alone ($P = 0$).
(c) Determine the strain energy U_3 when the forces P and Q act simultaneously upon the bar.



Solution 2.7-4 **Bar with two loads**



(a) FORCE P ACTS ALONE ($Q = 0$)

$$U_1 = \frac{P^2 L}{2EA} \quad \longleftarrow$$

(b) FORCE Q ACTS ALONE ($P = 0$)

$$U_2 = \frac{Q^2(L/2)}{2EA} = \frac{Q^2 L}{4EA} \quad \longleftarrow$$

(c) FORCES P AND Q ACT SIMULTANEOUSLY

$$\text{Segment } BC\text{: } U_{BC} = \frac{P^2(L/2)}{2EA} = \frac{P^2 L}{4EA}$$

$$\text{Segment } AB\text{: } U_{AB} = \frac{(P + Q)^2(L/2)}{2EA}$$

$$= \frac{P^2 L}{4EA} + \frac{PQL}{2EA} + \frac{Q^2 L}{4EA}$$

$$U_3 = U_{BC} + U_{AB} = \frac{P^2 L}{2EA} + \frac{PQL}{2EA} + \frac{Q^2 L}{4EA} \quad \longleftarrow$$

(Note that U_3 is *not* equal to $U_1 + U_2$. In this case, $U_3 > U_1 + U_2$. However, if Q is reversed in direction, $U_3 < U_1 + U_2$. Thus, U_3 may be larger or smaller than $U_1 + U_2$.)

Problem 2.7-5 Determine the strain energy per unit volume (units of psi) and the strain energy per unit weight (units of in.) that can be stored in each of the materials listed in the accompanying table, assuming that the material is stressed to the proportional limit.

DATA FOR PROBLEM 2.7-5

Material	Weight density (lb/in.^{3})	Modulus of elasticity (ksi)	Proportional limit (psi)
Mild steel	0.284	30,000	36,000
Tool steel	0.284	30,000	75,000
Aluminum	0.0984	10,500	60,000
Rubber (soft)	0.0405	0.300	300

Solution 2.7-5 **Strain-energy density**

DATA:

Material	Weight density (lb/in.^{3})	Modulus of elasticity (ksi)	Proportional limit (psi)
Mild steel	0.284	30,000	36,000
Tool steel	0.284	30,000	75,000
Aluminum	0.0984	10,500	60,000
Rubber (soft)	0.0405	0.300	300

STRAIN ENERGY PER UNIT VOLUME

$$U = \frac{P^2L}{2EA} \qquad \text{Volume } V = AL$$

$$\text{Stress } \sigma = \frac{P}{A}$$

$$u = \frac{U}{V} = \frac{\sigma^2}{2E}$$

At the proportional limit:

$u = u_R$ = modulus of resistance

$$u_R = \frac{\sigma_{PL}^2}{2E} \qquad \text{(Eq. 1)}$$

STRAIN ENERGY PER UNIT WEIGHT

$$U = \frac{P^2L}{2EA} \qquad \text{Weight } W = \gamma AL$$

γ = weight density

$$u_W = \frac{U}{W} = \frac{\sigma^2}{2\gamma E}$$

At the proportional limit:

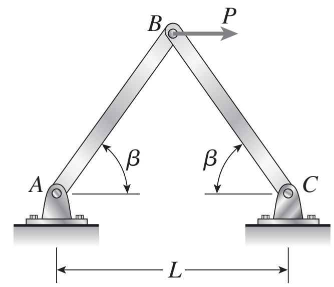
$$u_W = \frac{\sigma_{PL}^2}{2\gamma E} \qquad \text{(Eq. 2)}$$

RESULTS

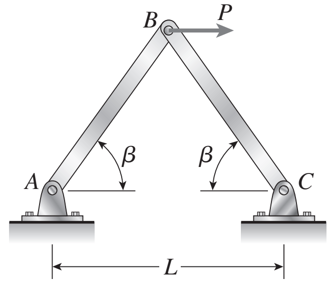
	u_R (psi)	u_W (in.)
Mild steel	22	76
Tool steel	94	330
Aluminum	171	1740
Rubber (soft)	150	3700

Problem 2.7-6 The truss *ABC* shown in the figure is subjected to a horizontal load *P* at joint *B*. The two bars are identical with cross-sectional area *A* and modulus of elasticity *E*.

(a) Determine the strain energy *U* of the truss if the angle $\beta = 60°$.
(b) Determine the horizontal displacement δ_B of joint *B* by equating the strain energy of the truss to the work done by the load.



Solution 2.7-6 Truss subjected to a load *P*



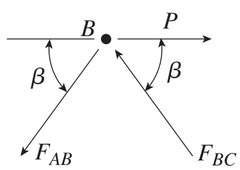
$\beta = 60°$

$L_{AB} = L_{BC} = L$

$\sin \beta = \sqrt{3}/2$

$\cos \beta = 1/2$

FREE-BODY DIAGRAM OF JOINT *B*



$\Sigma F_{\text{vert}} = 0 \quad \uparrow_{+} \quad \downarrow^{-}$

$-F_{AB} \sin \beta + F_{BC} \sin \beta = 0$

$F_{AB} = F_{BC}$ (Eq. 1)

$\Sigma F_{\text{horiz}} = 0 \quad \xrightarrow{+} \quad \xleftarrow{-}$

$-F_{AB} \cos \beta - F_{BC} \cos \beta + P = 0$

$$F_{AB} = F_{BC} = \frac{P}{2 \cos \beta} = \frac{P}{2(1/2)} = P \qquad \text{(Eq. 2)}$$

Axial forces: $N_{AB} = P$ (tension)

$N_{BC} = -P$ (compression)

(a) STRAIN ENERGY OF TRUSS (EQ. 2-40)

$$U = \sum \frac{N_i^2 L_i}{2E_i A_i} = \frac{(N_{AB})^2 L}{2EA} + \frac{(N_{BC})^2 L}{2EA}$$

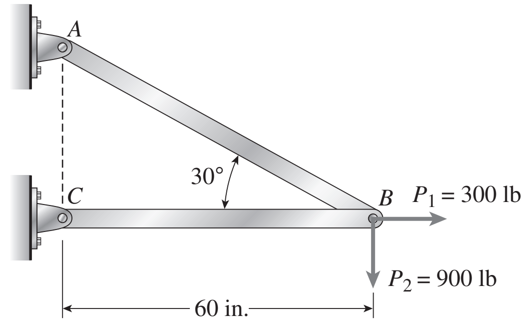
$$= \frac{P^2 L}{EA} \quad \longleftarrow$$

(b) HORIZONTAL DISPLACEMENT OF JOINT *B* (EQ. 2-42)

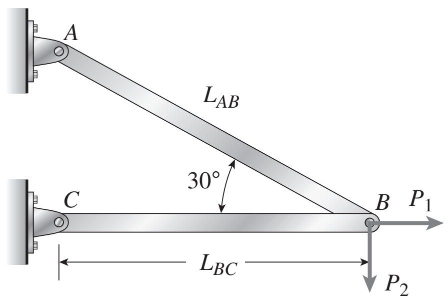
$$\delta_B = \frac{2U}{P} = \frac{2}{P}\left(\frac{P^2 L}{EA}\right) = \frac{2PL}{EA} \quad \longleftarrow$$

Problem 2.7-7 The truss *ABC* shown in the figure supports a horizontal load $P_1 = 300$ lb and a vertical load $P_2 = 900$ lb. Both bars have cross-sectional area $A = 2.4$ in.^{2} and are made of steel with $E = 30 \times 10^6$ psi.

(a) Determine the strain energy U_1 of the truss when the load P_1 acts alone ($P_2 = 0$).
(b) Determine the strain energy U_2 when the load P_2 acts alone ($P_1 = 0$).
(c) Determine the strain energy U_3 when both loads act simultaneously.



Solution 2.7-7 Truss with two loads



$P_1 = 300$ lb

$P_2 = 900$ lb

$A = 2.4$ in.^{2}

$E = 30 \times 10^6$ psi

$L_{BC} = 60$ in.

$\beta = 30°$

$$\sin\beta = \sin 30° = \frac{1}{2}$$

$$\cos\beta = \cos 30° = \frac{\sqrt{3}}{2}$$

$$L_{AB} = \frac{L_{BC}}{\cos 30°} = \frac{120}{\sqrt{3}}\text{ in.} = 69.282\text{ in.}$$

$$2EA = 2(30 \times 10^6\text{ psi})(2.4\text{ in.}^2) = 144 \times 10^6\text{ lb}$$

Forces F_{AB} and F_{BC} in the bars

From equilibrium of joint *B*:

$$F_{AB} = 2P_2 = 1800\text{ lb}$$

$$F_{BC} = P_1 - P_2\sqrt{3} = 300\text{ lb} - 1558.8\text{ lb}$$

Force	P_1 alone	P_2 alone	P_1 and P_2
F_{AB}	0	1800 lb	1800 lb
F_{BC}	300 lb	-1558.8 lb	-1258.8 lb

(a) Load P_1 acts alone

$$U_1 = \frac{(F_{BC})^2 L_{BC}}{2EA} = \frac{(300\text{ lb})^2(60\text{ in.})}{144 \times 10^6\text{ lb}}$$

$$= 0.0375\text{ in.-lb} \longleftarrow$$

(b) Load P_2 acts alone

$$U_2 = \frac{1}{2EA}\left[(F_{AB})^2 L_{AB} + (F_{BC})^2 L_{BC}\right]$$

$$= \frac{1}{2EA}\left[(1800\text{ lb})^2(69.282\text{ in.}) + (-1558.8\text{ lb})^2(60\text{ in.})\right]$$

$$= \frac{370.265 \times 10^6\text{ lb}^2\text{-in.}}{144 \times 10^6\text{ lb}} = 2.57\text{ in.-lb} \longleftarrow$$

(c) Loads P_1 and P_2 act simultaneously

$$U_3 = \frac{1}{2EA}\left[(F_{AB})^2 L_{AB} + (F_{BC})^2 L_{BC}\right]$$

$$= \frac{1}{2EA}\left[(1800\text{ lb})^2(69.282\text{ in.}) + (-1258.8\text{ lb})^2(60\text{ in.})\right]$$

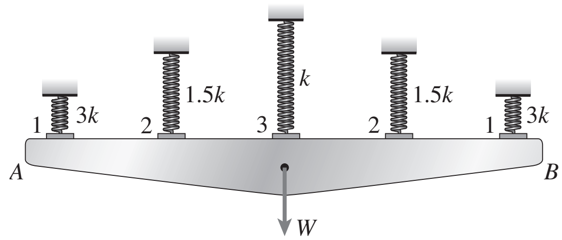
$$= \frac{319.548 \times 10^6\text{ lb}^2\text{-in.}}{144 \times 10^6\text{ lb}}$$

$$= 2.22\text{ in.-lb} \longleftarrow$$

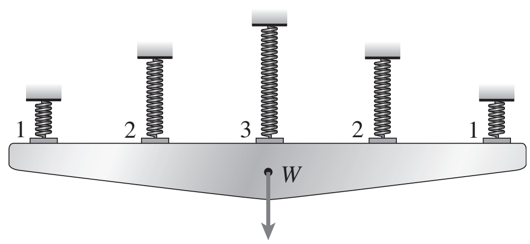
Note: The strain energy U_3 is *not* equal to $U_1 + U_2$.

Problem 2.7-8 The statically indeterminate structure shown in the figure consists of a horizontal rigid bar AB supported by five equally spaced springs. Springs 1, 2, and 3 have stiffnesses $3k$, $1.5k$, and k, respectively. When unstressed, the lower ends of all five springs lie along a horizontal line. Bar AB, which has weight W, causes the springs to elongate by an amount δ.

(a) Obtain a formula for the total strain energy U of the springs in terms of the downward displacement δ of the bar.
(b) Obtain a formula for the displacement δ by equating the strain energy of the springs to the work done by the weight W.
(c) Determine the forces F_1, F_2, and F_3 in the springs.
(d) Evaluate the strain energy U, the displacement δ, and the forces in the springs if $W = 600$ N and $k = 7.5$ N/mm.



Solution 2.7-8 **Rigid bar supported by springs**



$k_1 = 3k$

$k_2 = 1.5k$

$k_3 = k$

δ = downward displacement of rigid bar

For a spring: $U = \dfrac{k\delta^2}{2}$ Eq. (2-38b)

(a) STRAIN ENERGY U OF ALL SPRINGS

$$U = 2\left(\frac{3k\delta^2}{2}\right) + 2\left(\frac{1.5k\delta^2}{2}\right) + \frac{k\delta^2}{2}$$

$$= 5k\delta^2 \longleftarrow$$

(b) DISPLACEMENT δ

Work done by the weight W equals $\dfrac{W\delta}{2}$

Strain energy of the springs equals $5k\delta^2$

$$\therefore \frac{W\delta}{2} = 5k\delta^2 \quad \text{and} \quad \delta = \frac{W}{10k} \longleftarrow$$

(c) FORCES IN THE SPRINGS

$$F_1 = 3k\delta = \frac{3W}{10} \quad F_2 = 1.5k\delta = \frac{3W}{20} \longleftarrow$$

$$F_3 = k\delta = \frac{W}{10} \longleftarrow$$

(d) NUMERICAL VALUES

$W = 600\,\text{N} \quad k = 7.5\,\text{N/mm} = 7500\,\text{N/mm}$

$$U = 5k\delta^2 = 5k\left(\frac{W}{10k}\right)^2 = \frac{W^2}{20k}$$

$$= 2.4\ \text{N} \cdot \text{m} = 2.4\ \text{J} \longleftarrow$$

$$\delta = \frac{W}{10k} = 8.0\ \text{mm} \longleftarrow$$

$$F_1 = \frac{3W}{10} = 180\ \text{N} \longleftarrow$$

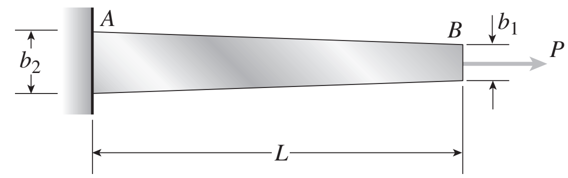
$$F_2 = \frac{3W}{20} = 90\ \text{N} \longleftarrow$$

$$F_3 = \frac{W}{10} = 60\ \text{N} \longleftarrow$$

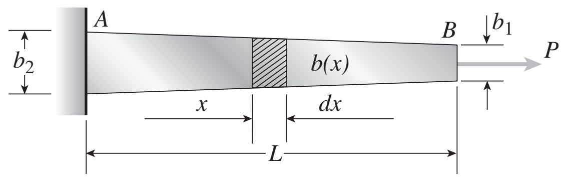
NOTE: $W = 2F_1 + 2F_2 + F_3 = 600\,\text{N}$ (Check)

Problem 2.7-9 A slightly tapered bar AB of rectangular cross section and length L is acted upon by a force P (see figure). The width of the bar varies uniformly from b_2 at end A to b_1 at end B. The thickness t is constant.

(a) Determine the strain energy U of the bar.
(b) Determine the elongation δ of the bar by equating the strain energy to the work done by the force P.



Solution 2.7-9 Tapered bar of rectangular cross section



$$b(x) = b_2 - \frac{(b_2 - b_1)x}{L}$$

$$A(x) = tb(x)$$

$$= t\left[b_2 - \frac{(b_2 - b_1)x}{L}\right]$$

(a) Strain energy of the bar

$$U = \int \frac{[N(x)]^2 dx}{2EA(x)} \qquad \text{(Eq. 2-41)}$$

$$= \int_0^L \frac{P^2 dx}{2Etb(x)} = \frac{P^2}{2Et}\int_0^L \frac{dx}{b_2 - (b_2 - b_1)\frac{x}{L}} \qquad (1)$$

From Appendix C: $\displaystyle\int \frac{dx}{a + bx} = \frac{1}{b}\ln(a + bx)$

Apply this integration formula to Eq. (1):

$$U = \frac{P^2}{2Et}\left[\frac{1}{-(b_2 - b_1)(\frac{1}{L})}\ln\left[b_2 - \frac{(b_2 - b_1)x}{L}\right]\right]_0^L$$

$$= \frac{P^2}{2Et}\left[\frac{-L}{(b_2 - b_1)}\ln b_1 - \frac{-L}{(b_2 - b_1)}\ln b_2\right]$$

$$U = \frac{P^2 L}{2Et(b_2 - b_1)}\ln\frac{b_2}{b_1} \quad \longleftarrow$$

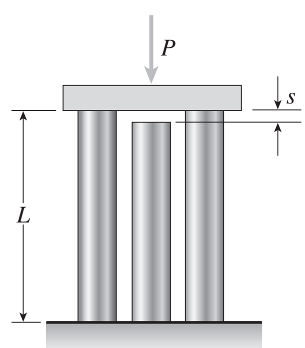
(b) Elongation of the bar (Eq. 2-42)

$$\delta = \frac{2U}{P} = \frac{PL}{Et(b_2 - b_1)}\ln\frac{b_2}{b_1} \quad \longleftarrow$$

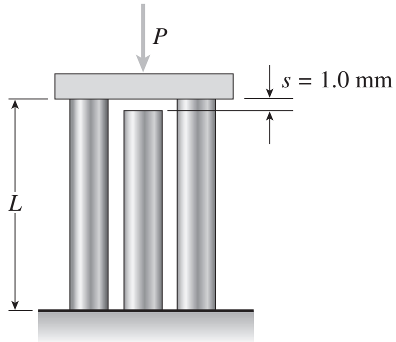
Note: This result agrees with the formula derived in Prob. 2.3-11.

Problem 2.7-10 A compressive load P is transmitted through a rigid plate to three magnesium-alloy bars that are identical except that initially the middle bar is slightly shorter than the other bars (see figure). The dimensions and properties of the assembly are as follows: length $L = 1.0$ m, cross-sectional area of each bar $A = 3000\ \text{mm}^2$, modulus of elasticity $E = 45$ GPa, and the gap $s = 1.0$ mm.

(a) Calculate the load P_1 required to close the gap.
(b) Calculate the downward displacement δ of the rigid plate when $P = 400$ kN.
(c) Calculate the total strain energy U of the three bars when $P = 400$ kN.
(d) Explain why the strain energy U is *not* equal to $P\delta/2$. (*Hint:* Draw a load-displacement diagram.)



Solution 2.7-10 **Three bars in compression**



$s = 1.0\,\text{mm}$

$L = 1.0\,\text{m}$

For each bar:

$A = 3000\,\text{mm}^2$

$E = 45\,\text{GPa}$

$$\frac{EA}{L} = 135 \times 10^6\ \text{N/m}$$

(a) LOAD P_1 REQUIRED TO CLOSE THE GAP

In general, $\delta = \dfrac{PL}{EA}$ and $P = \dfrac{EA\delta}{L}$

For two bars, we obtain:

$$P_1 = 2\left(\frac{EAs}{L}\right) = 2(135 \times 10^6\ \text{N/m})(1.0\ \text{mm})$$

$$P_1 = 270\ \text{kN} \longleftarrow$$

(b) DISPLACEMENT δ FOR $P = 400\,\text{kN}$

Since $P > P_1$, all three bars are compressed. The force P equals P_1 plus the additional force required to compress all three bars by the amount $\delta - s$.

$$P = P_1 + 3\left(\frac{EA}{L}\right)(\delta - s)$$

or $400\ \text{kN} = 270\ \text{kN} + 3(135 \times 10^6\ \text{N/m})(\delta - 0.001\,\text{m})$

Solving, we get $\delta = 1.321\ \text{mm} \longleftarrow$

(c) STRAIN ENERGY U FOR $P = 400\,\text{kN}$

$$U = \sum \frac{EA\delta^2}{2L}$$

Outer bars: $\delta = 1.321\,\text{mm}$

Middle bar: $\delta = 1.321\,\text{mm} - s$

$= 0.321\,\text{mm}$

$$U = \frac{EA}{2L}[2(1.321\ \text{mm})^2 + (0.321\ \text{mm})^2]$$

$$= \frac{1}{2}(135 \times 10^6\ \text{N/m})(3.593\ \text{mm}^2)$$

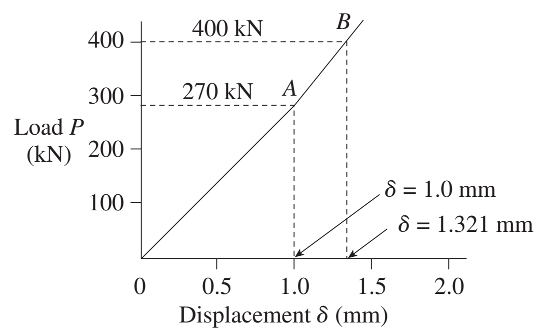
$$= 243\ \text{N} \cdot \text{m} = 243\ \text{J} \longleftarrow$$

(d) LOAD-DISPLACEMENT DIAGRAM

$U = 243\ \text{J} = 243\,\text{N} \cdot \text{m}$

$$\frac{P\delta}{2} = \frac{1}{2}(400\ \text{kN})(1.321\ \text{mm}) = 264\ \text{N} \cdot \text{m}$$

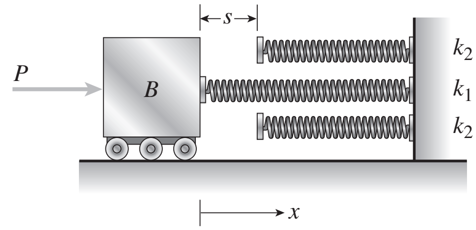
The strain energy U is *not* equal to $\dfrac{P\delta}{2}$ because the load-displacement relation is not linear.



U = area under line OAB.

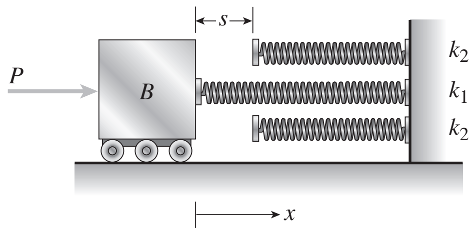
$\dfrac{P\delta}{2}$ = area under a straight line from O to B, which is larger than U.

Problem 2.7-11 A block B is pushed against three springs by a force P (see figure). The middle spring has stiffness k_1 and the outer springs each have stiffness k_2. Initially, the springs are unstressed and the middle spring
is longer than the outer springs (the difference in length is denoted s).



(a) Draw a force-displacement diagram with the force P as ordinate and the displacement x of the block as abscissa.
(b) From the diagram, determine the strain energy U_1 of the springs when $x = 2s$.
(c) Explain why the strain energy U_1 is not equal to $P\delta/2$, where $\delta = 2s$.

Solution 2.7-11 **Block pushed against three springs**



Force P_0 required to close the gap:

$$P_0 = k_1 s \qquad (1)$$

FORCE-DISPLACEMENT RELATION BEFORE GAP IS CLOSED

$$P = k_1 x \qquad (0 \le x \le s)(0 \le P \le P_0) \qquad (2)$$

FORCE-DISPLACEMENT RELATION AFTER GAP IS CLOSED

All three springs are compressed. Total stiffness equals $k_1 + 2k_2$. Additional displacement equals $x - s$. Force P equals P_0 plus the force required to compress all three springs by the amount $x - s$.

$$P = P_0 + (k_1 + 2k_2)(x - s)$$

$$= k_1 s + (k_1 + 2k_2)x - k_1 s - 2k_2 s$$

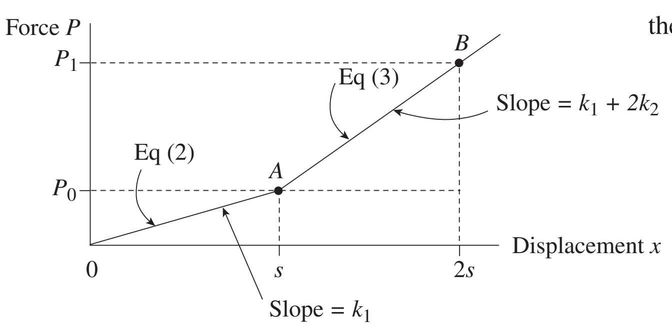
$$P = (k_1 + 2k_2)x - 2k_2 s \quad (x \ge s); (P \ge P_0) \qquad (3)$$

P_1 = force P when $x = 2s$

Substitute $x = 2s$ into Eq. (3):

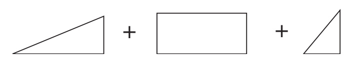
$$P_1 = 2(k_1 + k_2)s \qquad (4)$$

(a) FORCE-DISPLACEMENT DIAGRAM



(b) STRAIN ENERGY U_1 WHEN $x = 2s$

U_1 = Area below force-displacement curve

= 

$$= \frac{1}{2}P_0 s + P_0 s + \frac{1}{2}(P_1 - P_0)s = P_0 s + \frac{1}{2}P_1 s$$

$$= k_1 s^2 + (k_1 + k_2)s^2$$

$$U_1 = (2k_1 + k_2)s^2 \quad \leftarrow \qquad (5)$$

(c) STRAIN ENERGY U_1 IS NOT EQUAL TO $\frac{P\delta}{2}$

For $\delta = 2s$: $\frac{P\delta}{2} = \frac{1}{2}P_1(2s) = P_1 s = 2(k_1 + k_2)s^2$

(This quantity is greater than U_1.)

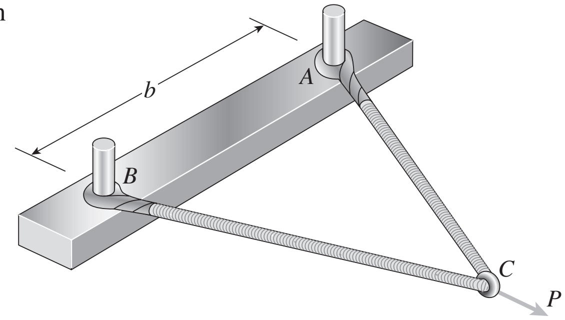
U_1 = area under line OAB.

$\frac{P\delta}{2}$ = area under a straight line from O to B, which is larger than U_1.

Thus, $\frac{P\delta}{2}$ is *not* equal to the strain energy because the force-displacement relation is not linear.

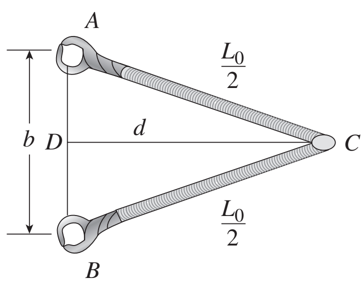
Problem 2.7-12 A bungee cord that behaves linearly elastically has an unstressed length $L_0 = 760$ mm and a stiffness $k = 140$ N/m. The cord is attached to two pegs, distance $b = 380$ mm apart, and pulled at its midpoint by a force $P = 80$ N (see figure).

(a) How much strain energy U is stored in the cord?
(b) What is the displacement δ_C of the point where the load is applied?
(c) Compare the strain energy U with the quantity $P\delta_C/2$.
(*Note:* The elongation of the cord is *not* small compared to its original length.)



Solution 2.7-12 Bungee cord subjected to a load *P*.

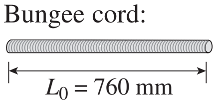
DIMENSIONS BEFORE THE LOAD P IS APPLIED



$L_0 = 760$ mm $\quad \frac{L_0}{2} = 380$ mm

$b = 380$ mm

Bungee cord:

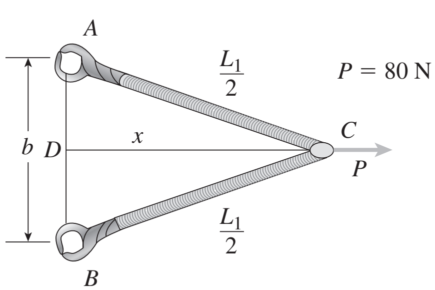


$k = 140$ N/m

From triangle ACD:

$$d = \frac{1}{2}\sqrt{L_0^2 - b^2} = 329.09 \text{ mm} \qquad (1)$$

DIMENSIONS AFTER THE LOAD P IS APPLIED



Let x = distance CD

Let L_1 = stretched length of bungee cord

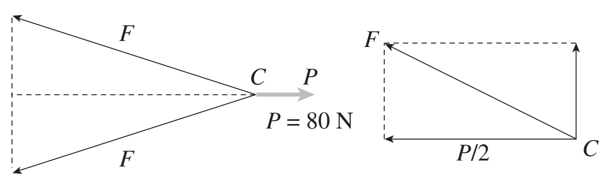
From triangle ACD:

$$\frac{L_1}{2} = \sqrt{\left(\frac{b}{2}\right)^2 + x^2} \qquad (2)$$

$$L_1 = \sqrt{b^2 + 4x^2} \qquad (3)$$

EQUILIBRIUM AT POINT C

Let F = tensile force in bungee cord



$$\frac{F}{P/2} = \frac{L_1/2}{x} \qquad F = \left(\frac{P}{2}\right)\left(\frac{L_1}{2}\right)\left(\frac{1}{x}\right)$$

$$= \frac{P}{2}\sqrt{1 + \left(\frac{b}{2x}\right)^2} \qquad (4)$$

ELONGATION OF BUNGEE CORD

Let δ = elongation of the entire bungee cord

$$\delta = \frac{F}{k} = \frac{P}{2k}\sqrt{1 + \frac{b^2}{4x^2}} \qquad (5)$$

Final length of bungee cord = original length + δ

$$L_1 = L_0 + \delta = L_0 + \frac{P}{2k}\sqrt{1 + \frac{b^2}{4x^2}} \qquad (6)$$

(Continued)

SOLUTION OF EQUATIONS

Combine Eqs. (6) and (3):

$$L_1 = L_0 + \frac{P}{2k}\sqrt{1 + \frac{b^2}{4x^2}} = \sqrt{b^2 + 4x^2}$$

or $$L_1 = L_0 + \frac{P}{4kx}\sqrt{b^2 + 4x^2} = \sqrt{b^2 + 4x^2}$$

$$L_0 = \left(1 - \frac{P}{4kx}\right)\sqrt{b^2 + 4x^2} \qquad (7)$$

This equation can be solved for x.

SUBSTITUTE NUMERICAL VALUES INTO EQ. (7):

$$760 \text{ mm} = \left[1 - \frac{(80 \text{ N})(1000 \text{ mm/m})}{4(140 \text{ N/m})x}\right] \times \sqrt{(380 \text{ mm})^2 + 4x^2} \qquad (8)$$

$$760 = \left(1 - \frac{142.857}{x}\right)\sqrt{144{,}400 + 4x^2} \quad (9) \qquad (9)$$

Units: x is in millimeters

Solve for x (Use trial & error or a computer program):

$x = 497.88$ mm

(a) STRAIN ENERGY U OF THE BUNGEE CORD

$$U = \frac{k\delta^2}{2} \qquad k = 140 \text{ N/m} \qquad P = 80 \text{ N}$$

From Eq. (5):

$$\delta = \frac{P}{2k}\sqrt{1 + \frac{b^2}{4x^2}} = 305.81 \text{ mm}$$

$$U = \frac{1}{2}(140 \text{ N/m})(305.81 \text{ mm})^2 = 6.55 \text{ N}\cdot\text{m}$$

$U = 6.55$ J ←

(b) DISPLACEMENT δ_C OF POINT C

$$\delta_C = x - d = 497.88 \text{ mm} - 329.09 \text{ mm}$$
$$= 168.8 \text{ mm}$$ ←

(c) COMPARISON OF STRAIN ENERGY U WITH THE QUANTITY $P\delta_C/2$

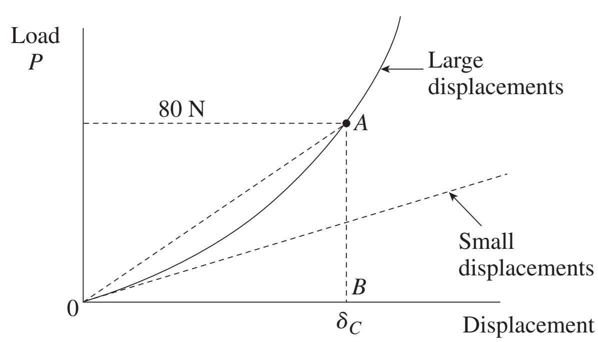
$U = 6.55$ J

$$\frac{P\delta_C}{2} = \frac{1}{2}(80 \text{ N})(168.8 \text{ mm}) = 6.75 \text{ J}$$

The two quantities are not the same. The work done by the load P is *not* equal to $P\delta_C/2$ because the load-displacement relation (see below) is non-linear when the displacements are large. (The *work* done by the load P *is* equal to the strain energy because the bungee cord behaves elastically and there are no energy losses.)

U = area OAB under the curve OA.

$\frac{P\delta_C}{2}$ = area of triangle OAB, which is greater than U.

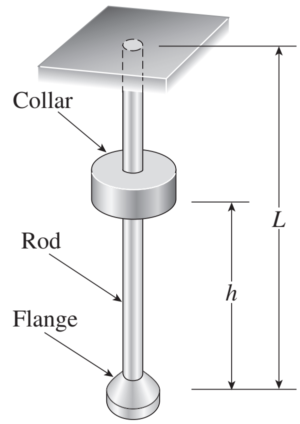


Impact Loading

The problems for Section 2.8 are to be solved on the basis of the assumptions and idealizations described in the text. In particular, assume that the material behaves linearly elastically and no energy is lost during the impact.

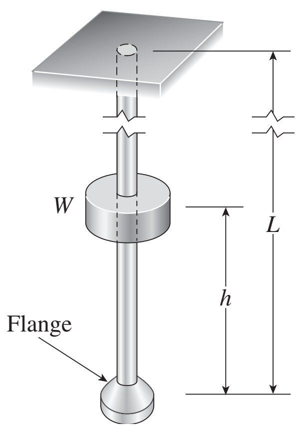
Problem 2.8-1 A sliding collar of weight $W = 150$ lb falls from a height $h = 2.0$ in. onto a flange at the bottom of a slender vertical rod (see figure). The rod has length $L = 4.0$ ft, cross-sectional area $A = 0.75$ in.^{2}, and modulus of elasticity $E = 30 \times 10^6$ psi.

Calculate the following quantities: (a) the maximum downward displacement of the flange, (b) the maximum tensile stress in the rod, and (c) the impact factor.



Probs. 2.8-1, 2.8-2, and 2.8-3

Solution 2.8-1 Collar falling onto a flange



$W = 150$ lb

$h = 2.0$ in. $\qquad L = 4.0$ ft $= 48$ in.

$E = 30 \times 10^6$ psi $\qquad A = 0.75$ in.^{2}

(a) Downward displacement of flange

$$\delta_{st} = \frac{WL}{EA} = 0.00032 \text{ in.}$$

Eq. of (2-53):

$$\delta_{max} = \delta_{st}\left[1 + \left(1 + \frac{2h}{\delta_{st}}\right)^{1/2}\right]$$

$$= 0.0361 \text{ in.} \longleftarrow$$

(b) Maximum tensile stress (Eq. 2-55)

$$\sigma_{max} = \frac{E\delta_{max}}{L} = 22{,}600 \text{ psi} \longleftarrow$$

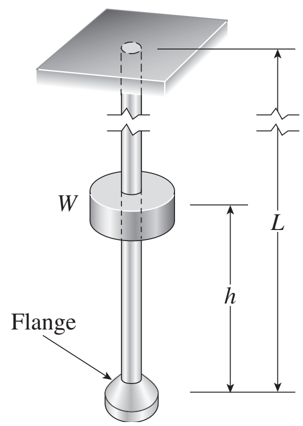
(c) Impact factor (Eq. 2-61)

$$\text{Impact factor} = \frac{\delta_{max}}{\delta_{st}} = \frac{0.0361 \text{ in.}}{0.00032 \text{ in.}}$$

$$= 113 \longleftarrow$$

Problem 2.8-2 Solve the preceding problem if the collar has mass $M = 80$ kg, the height $h = 0.5$ m, the length $L = 3.0$ m, the cross-sectional area $A = 350\ \text{mm}^2$, and the modulus of elasticity $E = 170$ GPa.

Solution 2.8-2 Collar falling onto a flange



$M = 80\,\text{kg}$

$W = Mg = (80\,\text{kg})(9.81\,\text{m/s}^2)$

$= 784.8\,\text{N}$

$h = 0.5\,\text{m}$ $\quad L = 3.0\,\text{m}$

$E = 170\,\text{GPa}$ $\quad A = 350\,\text{mm}^2$

(a) DOWNWARD DISPLACEMENT OF FLANGE

$$\delta_{st} = \frac{WL}{EA} = 0.03957\text{ mm}$$

Eq. (2-53): $$\delta_{max} = \delta_{st}\left[1 + \left(1 + \frac{2h}{\delta_{st}}\right)^{1/2}\right]$$

$$= 6.33\text{ mm} \longleftarrow$$

(b) MAXIMUM TENSILE STRESS (EQ. 2-55)

$$\sigma_{max} = \frac{E\delta_{max}}{L} = 359\text{ MPa} \longleftarrow$$

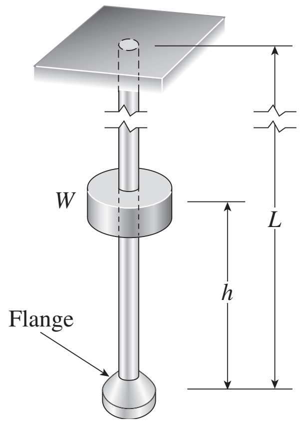
(c) IMPACT FACTOR (EQ. 2-61)

$$\text{Impact factor} = \frac{\delta_{max}}{\delta_{st}} = \frac{6.33\text{ mm}}{0.03957\text{ mm}}$$

$$= 160 \longleftarrow$$

Problem 2.8-3 Solve Problem 2.8-1 if the collar has weight $W = 50$ lb, the height $h = 2.0$ in., the length $L = 3.0$ ft, the cross-sectional area $A = 0.25\ \text{in.}^2$, and the modulus of elasticity $E = 30{,}000$ ksi.

Solution 2.8-3 Collar falling onto a flange



$W = 50\,\text{lb}$ $\quad h = 2.0\,\text{in.}$

$L = 3.0\,\text{ft} = 36\,\text{in.}$

$E = 30{,}000\,\text{psi}$ $\quad A = 0.25\,\text{in.}^2$

(a) DOWNWARD DISPLACEMENT OF FLANGE

$$\delta_{st} = \frac{WL}{EA} = 0.00024\text{ in.}$$

Eq. (2-53): $$\delta_{max} = \delta_{st}\left[1 + \left(1 + \frac{2h}{\delta_{st}}\right)^{1/2}\right]$$

$$= 0.0312\text{ in.} \longleftarrow$$

(b) MAXIMUM TENSILE STRESS (EQ. 2-55)

$$\sigma_{max} = \frac{E\delta_{max}}{L} = 26{,}000\text{ psi} \longleftarrow$$

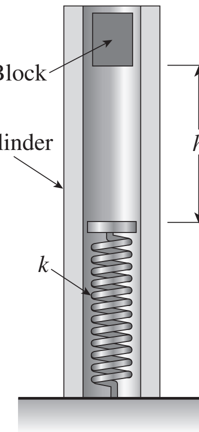
(c) IMPACT FACTOR (EQ. 2-61)

$$\text{Impact factor} = \frac{\delta_{max}}{\delta_{st}} = \frac{0.0312\text{ in.}}{0.00024\text{ in.}}$$

$$= 130 \longleftarrow$$

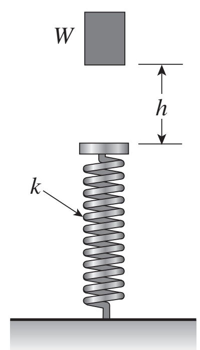
Problem 2.8-4 A block weighing $W = 5.0$ N drops inside a cylinder from a height $h = 200$ mm onto a spring having stiffness $k = 90$ N/m (see figure).

(a) Determine the maximum shortening of the spring due to the impact, and (b) determine the impact factor.



Prob. 2.8-4 and 2.8-5

Solution 2.8-4 **Block dropping onto a spring**



$W = 5.0\,\text{N} \qquad h = 200\,\text{mm} \qquad k = 90\,\text{N/m}$

(a) MAXIMUM SHORTENING OF THE SPRING

$$\delta_{st} = \frac{W}{k} = \frac{5.0\text{ N}}{90\text{ N/m}} = 55.56\text{ mm}$$

Eq. (2-53): $$\delta_{max} = \delta_{st}\left[1 + \left(1 + \frac{2h}{\delta_{st}}\right)^{1/2}\right]$$

$$= 215\text{ mm} \longleftarrow$$

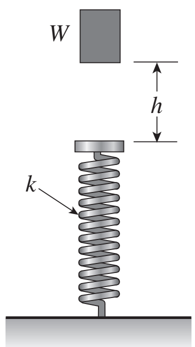
(b) IMPACT FACTOR (EQ. 2-61)

$$\text{Impact factor} = \frac{\delta_{max}}{\delta_{st}} = \frac{215\text{ mm}}{55.56\text{ mm}}$$

$$= 3.9 \longleftarrow$$

Problem 2.8-5 Solve the preceding problem if the block weighs $W = 1.0$ lb, $h = 12$ in., and $k = 0.5$ lb/in.

Solution 2.8-5 Block dropping onto a spring



$W = 1.0\,\text{lb}$ $\quad h = 12\,\text{in.}$ $\quad k = 0.5\,\text{lb/in.}$

(a) MAXIMUM SHORTENING OF THE SPRING

$$\delta_{st} = \frac{W}{k} = \frac{1.0\text{ lb}}{0.5\text{ lb/in.}} = 2.0\text{ in.}$$

Eq. (2-53): $\quad \delta_{max} = \delta_{st}\left[1 + \left(1 + \frac{2h}{\delta_{st}}\right)^{1/2}\right]$

$$= 9.21\text{ in.} \longleftarrow$$

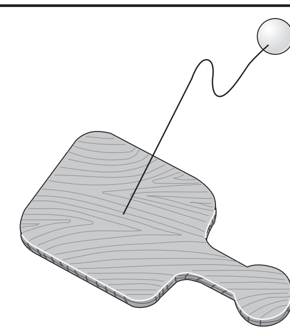
(b) IMPACT FACTOR (EQ. 2-61)

$$\text{Impact factor} = \frac{\delta_{max}}{\delta_{st}} = \frac{9.21\text{ in.}}{2.0\text{ in.}}$$

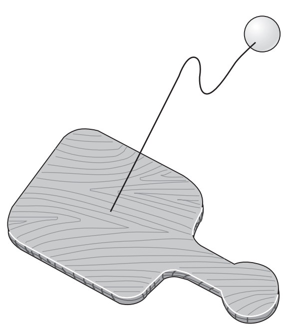
$$= 4.6 \longleftarrow$$

Problem 2.8-6 A small rubber ball (weight $W = 450$ mN) is attached by a rubber cord to a wood paddle (see figure). The natural length of the cord is $L_0 = 200$ mm, its cross-sectional area is $A = 1.6\text{ mm}^2$, and its modulus of elasticity is $E = 2.0$ MPa. After being struck by the paddle, the ball stretches the cord to a total length $L_1 = 900$ mm.

What was the velocity v of the ball when it left the paddle? (Assume linearly elastic behavior of the rubber cord, and disregard the potential energy due to any change in elevation of the ball.)



Solution 2.8-6 Rubber ball attached to a paddle



$g = 9.81\,\text{m/s}^2 \quad E = 2.0\,\text{MPa}$

$A = 1.6\,\text{mm}^2 \quad L_0 = 200\,\text{mm}$

$L_1 = 900\,\text{mm} \quad W = 450\,\text{mN}$

WHEN THE BALL LEAVES THE PADDLE

$$KE = \frac{Wv^2}{2g}$$

WHEN THE RUBBER CORD IS FULLY STRETCHED:

$$U = \frac{EA\delta^2}{2L_0} = \frac{EA}{2L_0}(L_1 - L_0)^2$$

CONSERVATION OF ENERGY

$$KE = U \qquad \frac{Wv^2}{2g} = \frac{EA}{2L_0}(L_1 - L_0)^2$$

$$v^2 = \frac{gEA}{WL_0}(L_1 - L_0)^2$$

$$v = (L_1 - L_0)\sqrt{\frac{gEA}{WL_0}} \longleftarrow$$

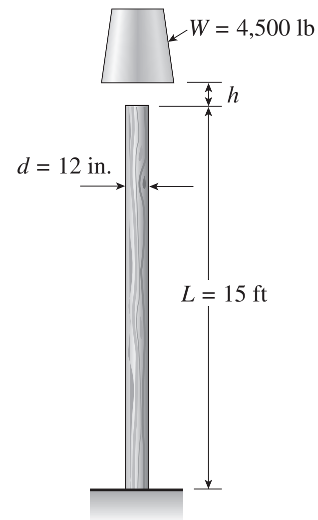
SUBSTITUTE NUMERICAL VALUES:

$$v = (700\text{ mm})\sqrt{\frac{(9.81\text{ m/s}^2)(2.0\text{ MPa})(1.6\text{ mm}^2)}{(450\text{ mN})(200\text{ mm})}}$$

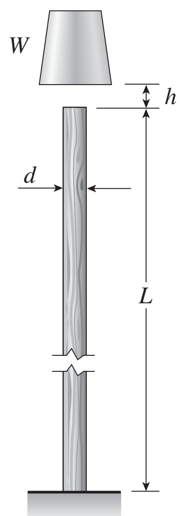
$$= 13.1\text{ m/s} \longleftarrow$$

Problem 2.8-7 A weight $W = 4500$ lb falls from a height h onto a vertical wood pole having length $L = 15$ ft, diameter $d = 12$ in., and modulus of elasticity $E = 1.6 \times 10^6$ psi (see figure).

If the allowable stress in the wood under an impact load is 2500 psi, what is the maximum permissible height h?



Solution 2.8-7 **Weight falling on a wood pole**



$W = 4500\,\text{lb} \qquad d = 12\,\text{in.}$

$L = 15\,\text{ft} = 180\,\text{in.}$

$$A = \frac{\pi d^2}{4} = 113.10\ \text{in.}^2$$

$E = 1.6 \times 10^6\,\text{psi}$

$\sigma_{allow} = 2500\,\text{psi}\ (= \sigma_{max})$

Find h_{max}

STATIC STRESS

$$\sigma_{st} = \frac{W}{A} = \frac{4500\ \text{lb}}{113.10\ \text{in.}^2} = 39.79\ \text{psi}$$

MAXIMUM HEIGHT h_{max}

$$\text{Eq. (2-59):} \quad \sigma_{max} = \sigma_{st}\left[1 + \left(1 + \frac{2hE}{L\sigma_{st}}\right)^{1/2}\right]$$

or

$$\frac{\sigma_{max}}{\sigma_{st}} - 1 = \left(1 + \frac{2hE}{L\sigma_{st}}\right)^{1/2}$$

Square both sides and solve for h:

$$h = h_{max} = \frac{L\sigma_{max}}{2E}\left(\frac{\sigma_{max}}{\sigma_{st}} - 2\right) \longleftarrow$$

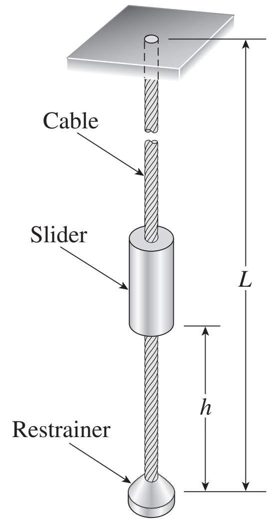
SUBSTITUTE NUMERICAL VALUES:

$$h_{max} = \frac{(180\ \text{in.})(2500\ \text{psi})}{2(1.6 \times 10^6\ \text{psi})}\left(\frac{2500\ \text{psi}}{39.79\ \text{psi}} - 2\right)$$

$$= 8.55\ \text{in.} \longleftarrow$$

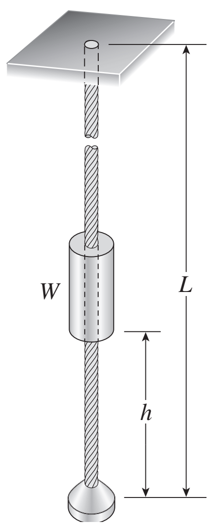
Problem 2.8-8 A cable with a restrainer at the bottom hangs vertically from its upper end (see figure). The cable has an effective cross-sectional area $A = 40 \text{ mm}^2$ and an effective modulus of elasticity $E = 130$ GPa. A slider of mass $M = 35$ kg drops from a height $h = 1.0$ m onto the restrainer.

If the allowable stress in the cable under an impact load is 500 MPa, what is the minimum permissible length L of the cable?



Probs. 2.8-8 and 2.8-9

Solution 2.8-8 Slider on a cable



$W = Mg = (35\,\text{kg})(9.81\,\text{m/s}^2) = 343.4\,\text{N}$

$A = 40\,\text{mm}^2 \qquad E = 130\,\text{GPa}$

$h = 1.0\,\text{m} \qquad \sigma_{\text{allow}} = \sigma_{\text{max}} = 500\,\text{MPa}$

Find minimum length L_{min}

STATIC STRESS

$$\sigma_{st} = \frac{W}{A} = \frac{343.4 \text{ N}}{40 \text{ mm}^2} = 8.585 \text{ MPa}$$

MINIMUM LENGTH L_{min}

$$\text{Eq. (2–59):} \quad \sigma_{\text{max}} = \sigma_{st}\left[1 + \left(1 + \frac{2hE}{L\sigma_{st}}\right)^{1/2}\right]$$

or

$$\frac{\sigma_{\text{max}}}{\sigma_{st}} - 1 = \left(1 + \frac{2hE}{L\sigma_{st}}\right)^{1/2}$$

Square both sides and solve for L:

$$L = L_{\text{min}} = \frac{2Eh\sigma_{st}}{\sigma_{\text{max}}(\sigma_{\text{max}} - 2\sigma_{st})} \quad \longleftarrow$$

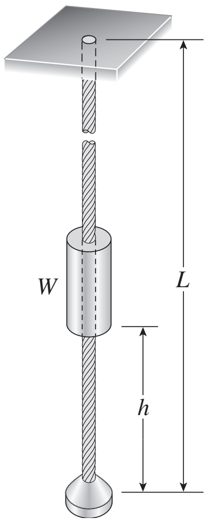
SUBSTITUTE NUMERICAL VALUES:

$$L_{\text{min}} = \frac{2(130 \text{ GPa})(1.0 \text{ m})(8.585 \text{ MPa})}{(500 \text{ MPa})[500 \text{ MPa} - 2(8.585 \text{ MPa})]}$$

$$= 9.25 \text{ mm} \quad \longleftarrow$$

Problem 2.8-9 Solve the preceding problem if the slider has weight $W = 100$ lb, $h = 45$ in., $A = 0.080 \text{ in.}^2$, $E = 21 \times 10^6$ psi, and the allowable stress is 70 ksi.

Solution 2.8-9 **Slider on a cable**



$W = 100 \text{ lb}$

$A = 0.080 \text{ in.}^2 \quad E = 21 \times 10^6 \text{ psi}$

$h = 45 \text{ in} \quad \sigma_{allow} = \sigma_{max} = 70 \text{ ksi}$

Find minimum length L_{min}

STATIC STRESS

$$\sigma_{st} = \frac{W}{A} = \frac{100 \text{ lb}}{0.080 \text{ in.}^2} = 1250 \text{ psi}$$

MINIMUM LENGTH L_{min}

Eq. (2-59): $$\sigma_{max} = \sigma_{st}\left[1 + \left(1 + \frac{2hE}{L\sigma_{st}}\right)^{1/2}\right]$$

or

$$\frac{\sigma_{max}}{\sigma_{st}} - 1 = \left(1 + \frac{2hE}{L\sigma_{st}}\right)^{1/2}$$

Square both sides and solve for L:

$$L = L_{min} = \frac{2Eh\sigma_{st}}{\sigma_{max}(\sigma_{max} - 2\sigma_{st})} \longleftarrow$$

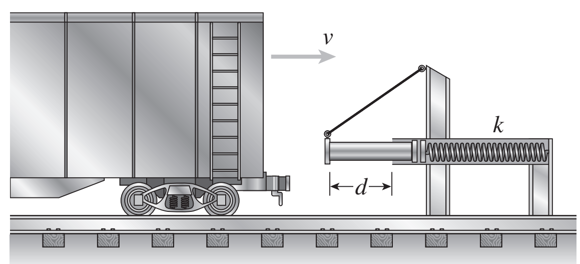
SUBSTITUTE NUMERICAL VALUES:

$$L_{min} = \frac{2(21 \times 10^6 \text{ psi})(45 \text{ in.})(1250 \text{ psi})}{(70{,}000 \text{ psi})[70{,}000 \text{ psi} - 2(1250 \text{ psi})]}$$

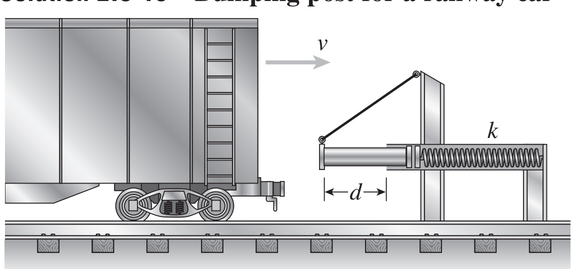
$$= 500 \text{ in.} \longleftarrow$$

Problem 2.8-10 A bumping post at the end of a track in a railway yard has a spring constant $k = 8.0$ MN/m (see figure). The maximum possible displacement d of the end of the striking plate is 450 mm.

What is the maximum velocity v_{max} that a railway car of weight $W = 545$ kN can have without damaging the bumping post when it strikes it?



Solution 2.8-10 **Bumping post for a railway car**



$k = 8.0 \text{ MN/m} \qquad W = 545 \text{ kN}$

d = maximum displacement of spring

$d = \delta_{max} = 450 \text{ mm}$

Find v_{max}

KINETIC ENERGY BEFORE IMPACT

$$KE = \frac{Mv^2}{2} = \frac{Wv^2}{2g}$$

STRAIN ENERGY WHEN SPRING IS COMPRESSED TO THE MAXIMUM ALLOWABLE AMOUNT

$$U = \frac{k\delta_{max}^2}{2} = \frac{kd^2}{2}$$

CONSERVATION OF ENERGY

$$KE = U \qquad \frac{Wv^2}{2g} = \frac{kd^2}{2} \qquad v^2 = \frac{kd^2}{W/g}$$

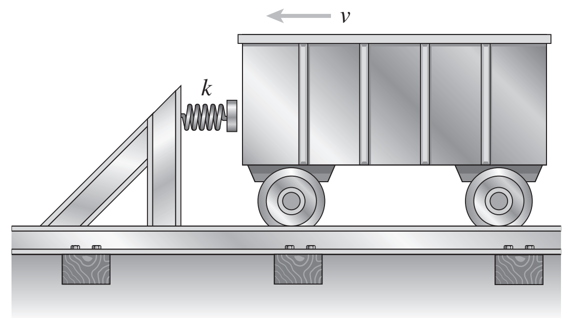
$$v = v_{max} = d\sqrt{\frac{k}{W/g}} \longleftarrow$$

SUBSTITUTE NUMERICAL VALUES:

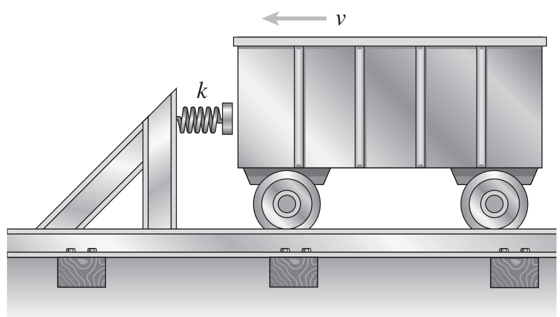
$$v_{max} = (450 \text{ mm})\sqrt{\frac{8.0 \text{ MN/m}}{(545 \text{ kN})/(9.81 \text{ m/s}^2)}}$$

$$= 5400 \text{ mm/s} = 5.4 \text{ m/s} \longleftarrow$$

Problem 2.8-11 A bumper for a mine car is constructed with a spring of stiffness $k = 1120$ lb/in. (see figure). If a car weighing 3450 lb is traveling at velocity $v = 7$ mph when it strikes the spring, what is the maximum shortening of the spring?



Solution 2.8-11 Bumper for a mine car



$k = 1120\,\text{lb/in.} \quad W = 3450\,\text{lb}$

$v = 7\,\text{mph} = 123.2\,\text{in./sec}$

$g = 32.2\,\text{ft/sec}^2 = 386.4\,\text{in./sec}^2$

Find the shortening δ_{max} of the spring.

KINETIC ENERGY JUST BEFORE IMPACT

$$KE = \frac{Mv^2}{2} = \frac{Wv^2}{2g}$$

STRAIN ENERGY WHEN SPRING IS FULLY COMPRESSED

$$U = \frac{k\delta_{max}^2}{2}$$

Conservation of energy

$$KE = U \qquad \frac{Wv^2}{2g} = \frac{k\delta_{max}^2}{2}$$

Solve for δ_{max}: $\quad \delta_{max} = \sqrt{\frac{Wv^2}{gk}}$ ⟵

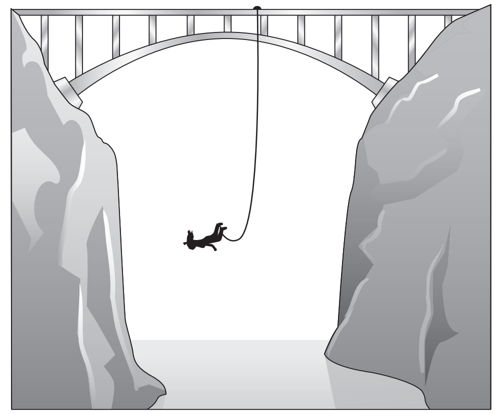
SUBSTITUTE NUMERICAL VALUES:

$$\delta_{max} = \sqrt{\frac{(3450\text{ lb})(123.2\text{ in./sec})^2}{(386.4\text{ in./sec}^2)(1120\text{ lb/in.})}}$$

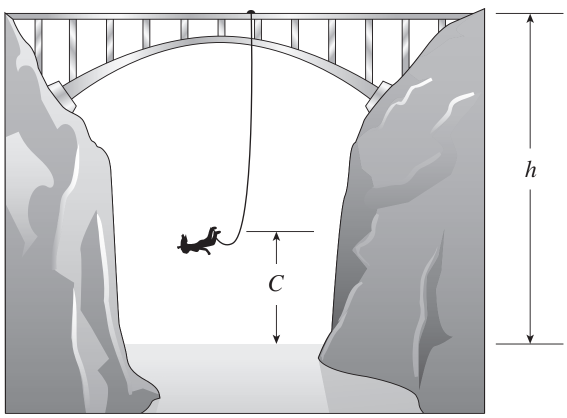
$$= 11.0\text{ in.}$$ ⟵

Problem 2.8-12 A bungee jumper having a mass of 55 kg leaps from a bridge, braking her fall with a long elastic shock cord having axial rigidity $EA = 2.3$ kN (see figure).

If the jumpoff point is 60 m above the water, and if it is desired to maintain a clearance of 10 m between the jumper and the water, what length L of cord should be used?



Solution 2.8-12 Bungee jumper



$$W = Mg = (55 \text{ kg})(9.81 \text{ m/s}^2)$$
$$= 539.55 \text{ N}$$

$EA = 2.3$ kN

Height: $h = 60$ m

Clearance: $C = 10$ m

Find length L of the bungee cord.

$P.E.$ = Potential energy of the jumper at the top of bridge (with respect to lowest position)

$$= W(L + \delta_{max})$$

U = strain energy of cord at lowest position

$$= \frac{EA\delta_{max}^2}{2L}$$

CONSERVATION OF ENERGY

$$P.E. = U \quad W(L + \delta_{max}) = \frac{EA\delta_{max}^2}{2L}$$

or $\quad \delta_{max}^2 - \dfrac{2WL}{EA}\delta_{max} - \dfrac{2WL^2}{EA} = 0$

SOLVE QUADRATIC EQUATION FOR δ_{max}:

$$\delta_{max} = \frac{WL}{EA} + \left[\left(\frac{WL}{EA}\right)^2 + 2L\left(\frac{WL}{EA}\right)\right]^{1/2}$$
$$= \frac{WL}{EA}\left[1 + \left(1 + \frac{2EA}{W}\right)^{1/2}\right]$$

VERTICAL HEIGHT

$$h = C + L + \delta_{max}$$
$$h - C = L + \frac{WL}{EA}\left[1 + \left(1 + \frac{2EA}{W}\right)^{1/2}\right]$$

SOLVE FOR L:

$$L = \frac{h - C}{1 + \dfrac{W}{EA}\left[1 + \left(1 + \dfrac{2EA}{W}\right)^{1/2}\right]} \quad \longleftarrow$$

SUBSTITUTE NUMERICAL VALUES:

$$\frac{W}{EA} = \frac{539.55 \text{ N}}{2.3 \text{ kN}} = 0.234587$$

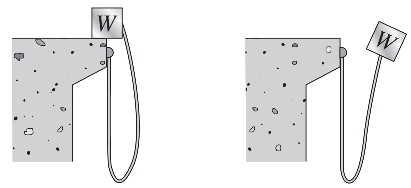
Numerator $= h - C = 60 \text{ m} - 10 \text{ m} = 50 \text{ m}$

$$\text{Denominator} = 1 + (0.234587) \times \left[1 + \left(1 + \frac{2}{0.234587}\right)^{1/2}\right]$$
$$= 1.9586$$

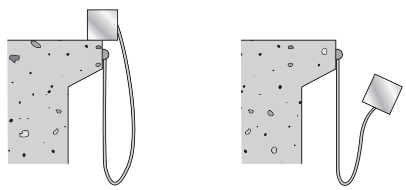
$$L = \frac{50 \text{ m}}{1.9586} = 25.5 \text{ m} \quad \longleftarrow$$

Problem 2.8-13 A weight W rests on top of a wall and is attached to one end of a very flexible cord having cross-sectional area A and modulus of elasticity E (see figure). The other end of the cord is attached securely to the wall. The weight is then pushed off the wall and falls freely the full length of the cord.

(a) Derive a formula for the impact factor.
(b) Evaluate the impact factor if the weight, when hanging statically, elongates the band by 2.5% of its original length.



Solution 2.8-13 Weight falling off a wall



W = Weight

Properties of elastic cord:

E = modulus of elasticity

A = cross-sectional area

L = original length

δ_{max} = elongation of elastic cord

$P.E.$ = potential energy of weight before fall (with respect to lowest position)

$P.E. = W(L + \delta_{max})$

Let U = strain energy of cord at lowest position

$$U = \frac{EA\delta_{max}^2}{2L}$$

CONSERVATION OF ENERGY

$$P.E. = U \qquad W(L + \delta_{max}) = \frac{EA\delta_{max}^2}{2\,L}$$

$$\text{or} \quad \delta_{max}^2 - \frac{2WL}{EA}\delta_{max} - \frac{2WL^2}{EA} = 0$$

SOLVE QUADRATIC EQUATION FOR δ_{max}:

$$\delta_{max} = \frac{WL}{EA} + \left[\left(\frac{WL}{EA}\right)^2 + 2L\left(\frac{WL}{EA}\right)\right]^{1/2}$$

STATIC ELONGATION

$$\delta_{st} = \frac{WL}{EA}$$

IMPACT FACTOR

$$\frac{\delta_{max}}{\delta_{st}} = 1 + \left[1 + \frac{2EA}{W}\right]^{1/2} \longleftarrow$$

NUMERICAL VALUES

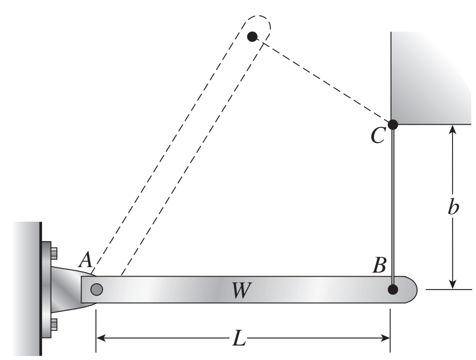
$$\delta_{st} = (2.5\%)(L) = 0.025L$$

$$\delta_{st} = \frac{WL}{EA} \qquad \frac{W}{EA} = 0.025 \qquad \frac{EA}{W} = 40$$

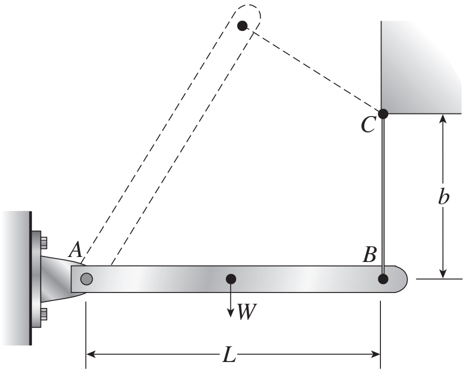
$$\text{Impact factor} = 1 + [1 + 2(40)]^{1/2} = 10 \longleftarrow$$

Problem 2.8-14 A rigid bar AB having mass $M = 1.0$ kg and length $L = 0.5$ m is hinged at end A and supported at end B by a nylon cord BC (see figure). The cord has cross-sectional area $A = 30$ mm^2, length $b = 0.25$ m, and modulus of elasticity $E = 2.1$ GPa.

If the bar is raised to its maximum height and then released, what is the maximum stress in the cord?



Solution 2.8-14 Falling bar AB



RIGID BAR:

$W = Mg = (1.0\,\text{kg})(9.81\,\text{m/s}^2)$

$= 9.81\text{ N}$

$L = 0.5\,\text{m}$

NYLON CORD:

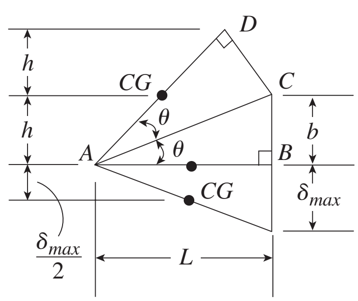
$A = 30\,\text{mm}^2$

$b = 0.25\,\text{m}$

$E = 2.1\,\text{GPa}$

Find maximum stress $\sigma_{\max}$ in cord BC.

GEOMETRY OF BAR AB AND CORD BC



$\overline{CD} = \overline{CB} = b$

$\overline{AD} = \overline{AB} = L$

h = height of center of gravity of raised bar AD

$\delta_{\max}$ = elongation of cord

From triangle ABC: $\sin\theta = \dfrac{b}{\sqrt{b^2 + L^2}}$

$$\cos\theta = \frac{L}{\sqrt{b^2 + L^2}}$$

From line AD : $\sin 2\theta = \dfrac{2h}{AD} = \dfrac{2h}{L}$

From Appendix C: $\sin 2\theta = 2\sin\theta\cos\theta$

$$\therefore \frac{2h}{L} = 2\left(\frac{b}{\sqrt{b^2 + L^2}}\right)\left(\frac{L}{\sqrt{b^2 + L^2}}\right) = \frac{2bL}{b^2 + L^2}$$

and $\quad h = \dfrac{bL^2}{b^2 + L^2}$ (Eq. 1)

CONSERVATION OF ENERGY

$P.E.$ = potential energy of raised bar AD

$$= W\left(h + \frac{\delta_{\max}}{2}\right)$$

U = strain energy of stretched cord $= \dfrac{EA\delta_{\max}^2}{2b}$

$P.E. = U \quad W\left(h + \dfrac{\delta_{\max}}{2}\right) = \dfrac{EA\delta_{\max}^2}{2b}$ (Eq. 2)

For the cord: $\delta_{\max} = \dfrac{\sigma_{\max} b}{E}$

Substitute into Eq. (2) and rearrange:

$$\sigma_{\max}^2 - \frac{W}{A}\sigma_{\max} - \frac{2WhE}{bA} = 0 \qquad \text{(Eq. 3)}$$

Substitute from Eq. (1) into Eq. (3):

$$\sigma_{\max}^2 - \frac{W}{A}\sigma_{\max} - \frac{2WL^2E}{A(b^2 + L^2)} = 0 \qquad \text{(Eq. 4)}$$

SOLVE FOR $\sigma_{\max}$:

$$\sigma_{\max} = \frac{W}{2A}\left[1 + \sqrt{1 + \frac{8L^2EA}{W(b^2 + L^2)}}\right] \longleftarrow$$

SUBSTITUTE NUMERICAL VALUES:

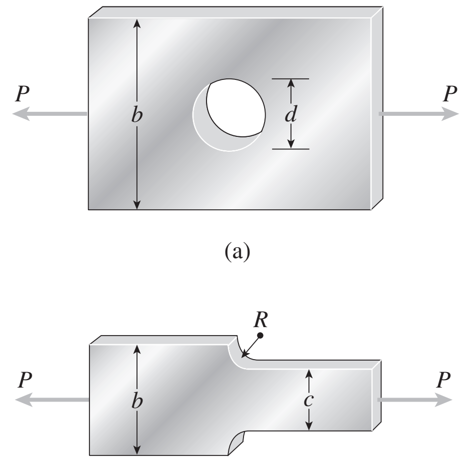
$\sigma_{\max} = 33.3\text{ MPa} \longleftarrow$

Stress Concentrations

The problems for Section 2.10 are to be solved by considering the stress-concentration factors and assuming linearly elastic behavior.

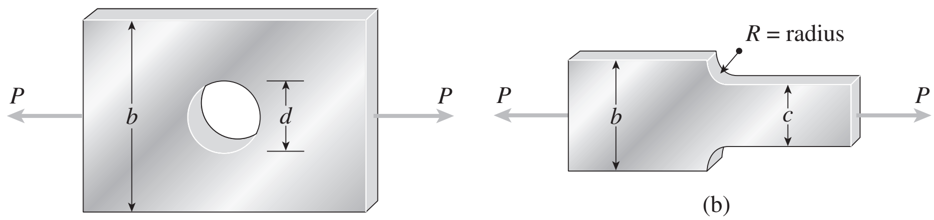
Problem 2.10-1 The flat bars shown in parts (a) and (b) of the figure are subjected to tensile forces $P = 3.0$ k. Each bar has thickness $t = 0.25$ in.

(a) For the bar with a circular hole, determine the maximum stresses for hole diameters $d = 1$ in. and $d = 2$ in. if the width $b = 6.0$ in.
(b) For the stepped bar with shoulder fillets, determine the maximum stresses for fillet radii $R = 0.25$ in. and $R = 0.5$ in. if the bar widths are $b = 4.0$ in. and $c = 2.5$ in.



Probs. 2.10-1 and 2.10-2 (b)

Solution 2.10-1 Flat bars in tension



$P = 3.0\,\text{k} \quad t = 0.25\,\text{in.}$

(a) Bar with circular hole ($b = 6$ in.)

Obtain K from Fig. 2-63

For $d = 1\,\text{in.}$: $\quad c = b - d = 5\,\text{in.}$

$$\sigma_{nom} = \frac{P}{ct} = \frac{3.0\text{ k}}{(5\text{ in.})(0.25\text{ in.})} = 2.40\text{ ksi}$$

$$d/b = \frac{1}{6} \qquad K \approx 2.60$$

$$\sigma_{max} = k\sigma_{nom} \approx 6.2\text{ ksi} \quad \longleftarrow$$

For $d = 2$ in.: $c = b - d = 4\,\text{in.}$

$$\sigma_{nom} = \frac{P}{ct} = \frac{3.0\text{ k}}{(4\text{ in.})(0.25\text{ in.})} = 3.00\text{ ksi}$$

$$d/b = \frac{1}{3} \qquad K \approx 2.31$$

$$\sigma_{max} = K\sigma_{nom} \approx 6.9\text{ ksi} \quad \longleftarrow$$

(b) Stepped bar with shoulder fillets

$b = 4.0\,\text{in.} \quad c = 2.5\,\text{in.}$; Obtain k from Fig. 2-64

$$\sigma_{nom} = \frac{P}{ct} = \frac{3.0\text{ k}}{(2.5\text{ in.})(0.25\text{ in.})} = 4.80\text{ ksi}$$

For $R = 0.25\,\text{in.}$: $R/c = 0.1 \qquad b/c = 1.60$

$$k \approx 2.30 \quad \sigma_{max} = K\sigma_{nom} \approx 11.0\text{ ksi} \quad \longleftarrow$$

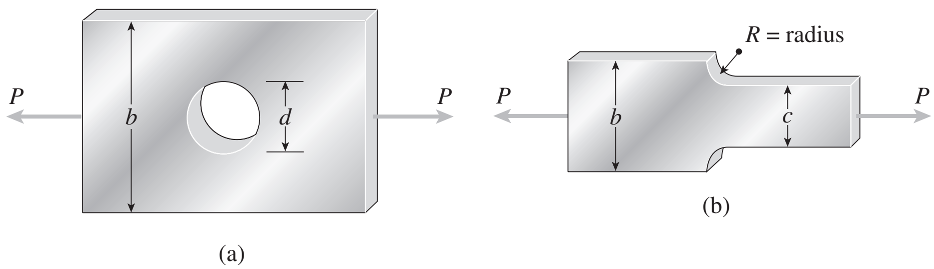
For $R = 0.5\,\text{in.}$: $R/c = 0.2 \qquad b/c = 1.60$

$$K \approx 1.87 \quad \sigma_{max} = K\sigma_{nom} \approx 9.0\text{ ksi} \quad \longleftarrow$$

Problem 2.10-2 The flat bars shown in parts (a) and (b) of the figure are subjected to tensile forces $P = 2.5$ kN. Each bar has thickness $t = 5.0$ mm.

(a) For the bar with a circular hole, determine the maximum stresses for hole diameters $d = 12$ mm and $d = 20$ mm if the width $b = 60$ mm.
(b) For the stepped bar with shoulder fillets, determine the maximum stresses for fillet radii $R = 6$ mm and $R = 10$ mm if the bar widths are $b = 60$ mm and $c = 40$ mm.

Solution 2.10-2 **Flat bars in tension**



$P = 2.5\,\text{kN} \quad t = 5.0\,\text{mm}$

(a) BAR WITH CIRCULAR HOLE ($b = 60\,\text{mm}$)

Obtain K from Fig. 2-63

FOR $d = 12\,\text{mm}$: $c = b - d = 48\,\text{mm}$

$$\sigma_{nom} = \frac{P}{ct} = \frac{2.5\text{ kN}}{(48\text{ mm})(5\text{ mm})} = 10.42\text{ MPa}$$

$$d/b = \frac{1}{5} \qquad K \approx 2.51$$

$$\sigma_{max} = K\sigma_{nom} \approx 26\text{ MPa} \longleftarrow$$

FOR $d = 20\,\text{mm}$: $c = b - d = 40\,\text{mm}$

$$\sigma_{nom} = \frac{P}{ct} = \frac{2.5\text{ kN}}{(40\text{ mm})(5\text{ mm})} = 12.50\text{ MPa}$$

$$d/b = \frac{1}{3} \qquad K \approx 2.31$$

$$\sigma_{max} = K\sigma_{nom} \approx 29\text{ MPa} \longleftarrow$$

(b) STEPPED BAR WITH SHOULDER FILLETS

$b = 60\,\text{mm} \qquad c = 40\,\text{mm};$

Obtain K from Fig. 2-64

$$\sigma_{nom} = \frac{P}{ct} = \frac{2.5\text{ kN}}{(40\text{ mm})(5\text{ mm})} = 12.50\text{ MPa}$$

FOR $R = 6\,\text{mm}$: $R/c = 0.15 \qquad b/c = 1.5$

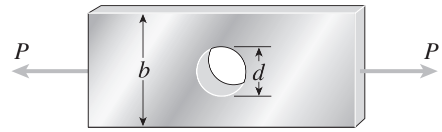
$$K \approx 2.00 \qquad \sigma_{max} = K\sigma_{nom} \approx 25\text{ MPa} \longleftarrow$$

FOR $R = 10\,\text{mm}$: $R/c = 0.25 \qquad b/c = 1.5$

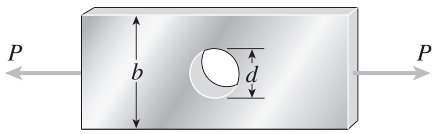
$$K \approx 1.75 \qquad \sigma_{max} = K\sigma_{nom} \approx 22\text{ MPa} \longleftarrow$$

Problem 2.10-3 A flat bar of width b and thickness t has a hole of diameter d drilled through it (see figure). The hole may have any diameter that will fit within the bar.

What is the maximum permissible tensile load P_{max} if the allowable tensile stress in the material is σ_t?



Solution 2.10-3 **Flat bar in tension**



t = thickness

σ_t = allowable tensile stress

Find P_{max}

Find K from Fig. 2-64

$$P_{max} = \sigma_{nom} ct = \frac{\sigma_{max}}{K} ct = \frac{\sigma_t}{K}(b - d)t$$

$$= \frac{\sigma_t}{K} bt\left(1 - \frac{d}{b}\right)$$

Because σ_t, b, and t are constants, we write:

$$P^* = \frac{P_{max}}{\sigma_t bt} = \frac{1}{K}\left(1 - \frac{d}{b}\right)$$

$\frac{d}{b}$	K	P^*
0	3.00	0.333
0.1	2.73	0.330
0.2	2.50	0.320
0.3	2.35	0.298
0.4	2.24	0.268

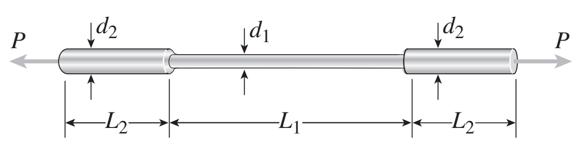
We observe that P_{max} decreases as d/b increases. Therefore, the maximum load occurs when the hole becomes very small.

$$\left(\frac{d}{b} \rightarrow 0 \quad \text{and} \quad K \rightarrow 3\right)$$

$$P_{max} = \frac{\sigma_t bt}{3} \quad \longleftarrow$$

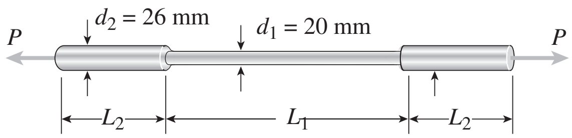
Problem 2.10-4 A round brass bar of diameter $d_1 = 20$ mm has upset ends of diameter $d_2 = 26$ mm (see figure). The lengths of the segments of the bar are $L_1 = 0.3$ m and $L_2 = 0.1$ m. Quarter-circular fillets are used at the shoulders of the bar, and the modulus of elasticity of the brass is $E = 100$ GPa.

If the bar lengthens by 0.12 mm under a tensile load P, what is the maximum stress σ_{max} in the bar?



Probs. 2.10-4 and 2.10-5

Solution 2.10-4 Round brass bar with upset ends



$E = 100\,\text{GPa}$

$\delta = 0.12\,\text{mm}$

$L_2 = 0.1\,\text{m}$

$L_1 = 0.3\,\text{m}$

$$R = \text{radius of fillets} = \frac{26\text{ mm} - 20\text{ mm}}{2} = 3\text{ mm}$$

$$\delta = 2\left(\frac{PL_2}{EA_2}\right) + \frac{PL_1}{EA_1}$$

Solve for P: $\quad P = \dfrac{\delta EA_1 A_2}{2L_2 A_1 + L_1 A_2}$

Use Fig. 2-65 for the stress-concentration factor:

$$\sigma_{\text{nom}} = \frac{P}{A_1} = \frac{\delta EA_2}{2L_2 A_1 + L_1 A_2} = \frac{\delta E}{2L_2(\frac{A_1}{A_2}) + L_1}$$

$$= \frac{\delta E}{2L_2(\frac{d_1}{d_2})^2 + L_1}$$

SUBSTITUTE NUMERICAL VALUES:

$$\sigma_{\text{nom}} = \frac{(0.12\text{ mm})(100\text{ GPa})}{2(0.1\text{ m})(\frac{20}{26})^2 + 0.3\text{ m}} = 28.68\text{ MPa}$$

$$\frac{R}{D_1} = \frac{3\text{ mm}}{20\text{ mm}} = 0.15$$

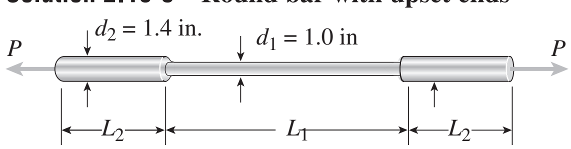
Use the dashed curve in Fig. 2-65. $K \approx 1.6$

$$\sigma_{\text{max}} = K\sigma_{\text{nom}} \approx (1.6)(28.68\text{ MPa})$$

$$\approx 46\text{ MPa} \longleftarrow$$

Problem 2.10-5 Solve the preceding problem for a bar of monel metal having the following properties: $d_1 = 1.0$ in., $d_2 = 1.4$ in., $L_1 = 20.0$ in., $L_2 = 5.0$ in., and $E = 25 \times 10^6$ psi. Also, the bar lengthens by 0.0040 in. when the tensile load is applied.

Solution 2.10-5 Round bar with upset ends



$E = 25 \times 10^6$ psi

$\delta = 0.0040\,\text{in.}$

$L_1 = 20\,\text{in.}$

$L_2 = 5\,\text{in.}$

R = radius of fillets $\quad R = \dfrac{1.4\text{ in.} - 1.0\text{ in.}}{2}$

$= 0.2$ in.

$$\delta = 2\left(\frac{PL_2}{EA_2}\right) + \frac{PL_1}{EA_1}$$

Solve for P: $P = \dfrac{\delta EA_1A_2}{2L_2A_1 + L_1A_2}$

Use Fig. 2-65 for the stress-concentration factor.

$$\sigma_{\text{nom}} = \frac{P}{A_1} = \frac{\delta EA_2}{2L_2A_1 + L_1A_2} = \frac{\delta E}{2L_2(\frac{A_1}{A_2}) + L_1}$$

$$= \frac{\delta E}{2L_2(\frac{d_1}{d_2})^2 + L_1}$$

SUBSTITUTE NUMERICAL VALUES:

$$\sigma_{\text{nom}} = \frac{(0.0040\text{ in.})(25 \times 10^6\text{ psi})}{2(5\text{ in.})(\frac{1.0}{1.4})^2 + 20\text{ in.}} = 3{,}984\text{ psi}$$

$$\frac{R}{D_1} = \frac{0.2\text{ in.}}{1.0\text{ in.}} = 0.2$$

Use the dashed curve in Fig. 2-65. $K \approx 1.53$

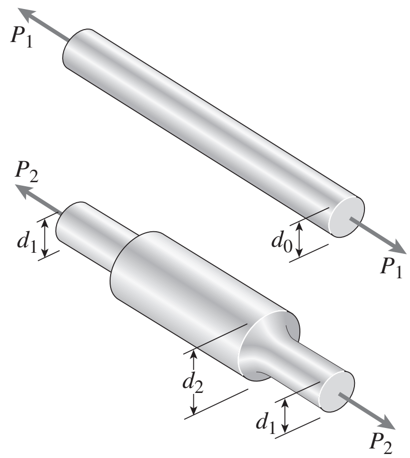
$$\sigma_{\text{max}} = K\sigma_{\text{nom}} \approx (1.53)(3984\text{ psi})$$

$$\approx 6100\text{ psi} \longleftarrow$$

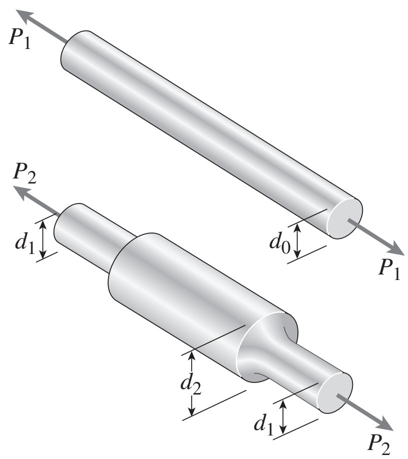
Problem 2.10-6 A prismatic bar of diameter $d_0 = 20$ mm is being compared with a stepped bar of the same diameter ($d_1 = 20$ mm) that is enlarged in the middle region to a diameter $d_2 = 25$ mm (see figure). The radius of the fillets in the stepped bar is 2.0 mm.

(a) Does enlarging the bar in the middle region make it stronger than the prismatic bar? Demonstrate your answer by determining the maximum permissible load P_1 for the prismatic bar and the maximum permissible load P_2 for the enlarged bar, assuming that the allowable stress for the material is 80 MPa.

(b) What should be the diameter d_0 of the prismatic bar if it is to have the same maximum permissible load as does the stepped bar?



Soluton 2.10-6 **Prismatic bar and stepped bar**



$d_0 = 20\,\text{mm}$

$d_1 = 20\,\text{mm}$

$d_2 = 25\,\text{mm}$

Fillet radius: $R = 2\,\text{mm}$

Allowable stress: $\sigma_t = 80\,\text{MPa}$

(a) Comparison of bars

$$\text{Prismatic bar: } P_1 = \sigma_t A_0 = \sigma_t\left(\frac{\pi d_0^2}{4}\right)$$

$$= (80\text{ MPa})\left(\frac{\pi}{4}\right)(20\text{ mm})^2 = 25.1\text{ kN} \longleftarrow$$

Stepped bar: See Fig. 2-65 for the stress-concentration factor.

$R = 2.0\,\text{mm} \quad D_1 = 20\,\text{mm} \quad D_2 = 25\,\text{mm}$

$R/D_1 = 0.10 \quad D_2/D_1 = 1.25 \quad K \approx 1.75$

$$\sigma_{\text{nom}} = \frac{P_2}{\frac{\pi}{4}d_1^2} = \frac{P_2}{A_1} \qquad \sigma_{\text{nom}} = \frac{\sigma_{\text{max}}}{K}$$

$$P_2 = \sigma_{\text{nom}} A_1 = \frac{\sigma_{\text{max}}}{K} A_1 = \frac{\sigma_t}{K} A_1$$

$$= \left(\frac{80\text{ MPa}}{1.75}\right)\left(\frac{\pi}{4}\right)(20\text{ mm})^2$$

$$\approx 14.4\text{ kN} \longleftarrow$$

Enlarging the bar makes it *weaker*, not stronger. The ratio of loads is $P_1/P_2 = K = 1.75$

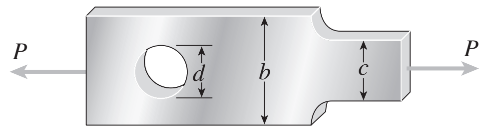
(b) Diameter of prismatic bar for the same allowable load

$$P_1 = P_2 \quad \sigma_t\left(\frac{\pi d_0^2}{4}\right) = \frac{\sigma_t}{K}\left(\frac{\pi d_1^2}{4}\right) \quad d_0^2 = \frac{d_1^2}{K}$$

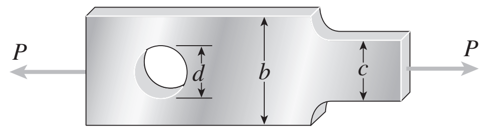
$$d_0 = \frac{d_1}{\sqrt{K}} \approx \frac{20\text{ mm}}{\sqrt{1.75}} \approx 15.1\text{ mm} \longleftarrow$$

Problem 2.10-7 A stepped bar with a hole (see figure) has widths $b = 2.4$ in. and $c = 1.6$ in. The fillets have radii equal to 0.2 in.

What is the diameter d_{max} of the largest hole that can be drilled through the bar without reducing the load-carrying capacity?



Solution 10-7 **Stepped bar with a hole**



$b = 2.4$ in.

$c = 1.6$ in.

Fillet radius: $R = 0.2$ in.

Find d_{max}

BASED UPON FILLETS (Use Fig. 2-64)

$b = 2.4$ in. $c = 1.6$ in. $R = 0.2$ in. $R/c = 0.125$

$b/c = 1.5$ $K \approx 2.10$

$$P_{max} = \sigma_{nom} ct = \frac{\sigma_{max}}{K} ct = \frac{\sigma_{max}}{K}\left(\frac{c}{b}\right)(bt)$$

$$\approx 0.317\, bt\, \sigma_{max}$$

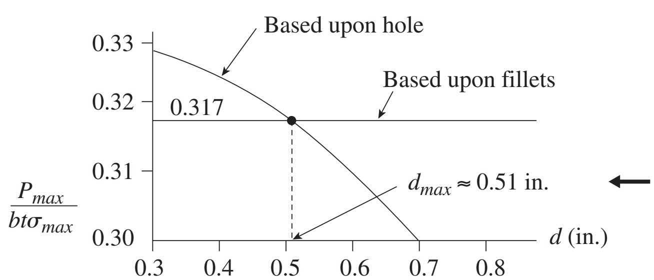
BASED UPON HOLE (Use Fig. 2-63)

$b = 2.4$ in. d = diameter of the hole (in.) $c_1 = b - d$

$$P_{max} = \sigma_{nom}\, c_1 t = \frac{\sigma_{max}}{K}(b - d)t$$

$$= \frac{1}{K}\left(1 - \frac{d}{b}\right) bt\sigma_{max}$$

d (in.)	d/b	K	$P_{max}/bt\sigma_{max}$
0.3	0.125	2.66	0.329
0.4	0.167	2.57	0.324
0.5	0.208	2.49	0.318
0.6	0.250	2.41	0.311
0.7	0.292	2.37	0.299



Nonlinear Behavior (Changes in Lengths of Bars)

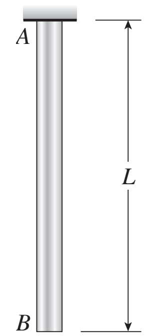
Problem 2.11-1 A bar AB of length L and weight density γ hangs vertically under its own weight (see figure). The stress-strain relation for the material is given by the Ramberg-Osgood equation (Eq. 2-71):

$$\epsilon = \frac{\sigma}{E} + \frac{\sigma_0 \alpha}{E}\left(\frac{\sigma}{\sigma_0}\right)^m$$

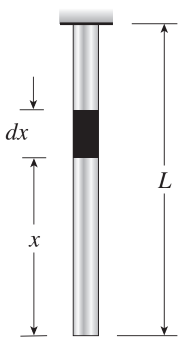
Derive the following formula

$$\delta = \frac{\gamma L^2}{2E} + \frac{\sigma_0 \alpha L}{(m+1)E}\left(\frac{\gamma L}{\sigma_0}\right)^m$$

for the elongation of the bar.



Solution 2.11-1 **Bar hanging under its own weight**



Let A = cross-sectional area

Let N = axial force at distance x

$N = \gamma A x$

$$\sigma = \frac{N}{A} = \gamma x$$

STRAIN AT DISTANCE x

$$\varepsilon = \frac{\sigma}{E} + \frac{\sigma_0 \alpha}{E}\left(\frac{\sigma}{\sigma_0}\right)^m = \frac{\gamma x}{E} + \frac{\sigma_0 \alpha}{E}\left(\frac{\gamma x}{\sigma_0}\right)^m$$

ELONGATION OF BAR

$$\delta = \int_0^L \varepsilon\, dx = \int_0^L \frac{\gamma x}{E}\, dx + \frac{\sigma_0 \alpha}{E}\int_0^L \left(\frac{\gamma x}{\sigma_0}\right)^m dx$$

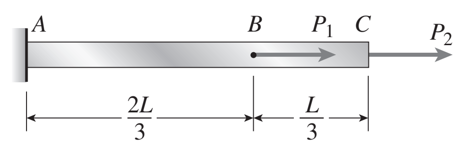
$$= \frac{\gamma L^2}{2E} + \frac{\sigma_0 \alpha L}{(m+1)E}\left(\frac{\gamma L}{\sigma_0}\right)^m \qquad \text{Q.E.D.} \leftarrow$$

Problem 2.11-2 A prismatic bar of length $L = 1.8$ m and cross-sectional area $A = 480\ \text{mm}^2$ is loaded by forces $P_1 = 30$ kN and $P_2 = 60$ kN (see figure). The bar is constructed of magnesium alloy having a stress-strain curve described by the following Ramberg-Osgood equation:

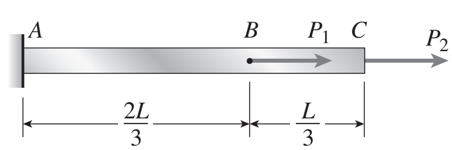
$$\epsilon = \frac{\sigma}{45{,}000} + \frac{1}{618}\left(\frac{\sigma}{170}\right)^{10} \qquad (\sigma = \text{MPa})$$

in which σ has units of megapascals.

(a) Calculate the displacement δ_C of the end of the bar when the load P_1 acts alone.
(b) Calculate the displacement when the load P_2 acts alone.
(c) Calculate the displacement when both loads act simultaneously.



Solution 2.11-2 **Axially loaded bar**



$L = 1.8\ \text{m} \qquad A = 480\ \text{mm}^2$

$P_1 = 30\ \text{kN} \quad P_2 = 60\ \text{kN}$

Ramberg–Osgood Equation:

$$\varepsilon = \frac{\sigma}{45{,}000} + \frac{1}{618}\left(\frac{\sigma}{170}\right)^{10} (\sigma = \text{MPa})$$

Find displacement at end of bar.

(a) P_1 ACTS ALONE

$$AB: \quad \sigma = \frac{P_1}{A} = \frac{30 \text{ kN}}{480 \text{ mm}^2} = 62.5 \text{ MPa}$$

$$\varepsilon = 0.001389$$

$$\delta_c = \varepsilon\left(\frac{2L}{3}\right) = 1.67 \text{ mm} \quad \longleftarrow$$

(b) P_2 ACTS ALONE

$$ABC: \quad \sigma = \frac{P_2}{A} = \frac{60 \text{ kN}}{480 \text{ mm}^2} = 125 \text{ MPa}$$

$$\varepsilon = 0.002853$$

$$\delta_c = \varepsilon L = 5.13 \text{ mm} \quad \longleftarrow$$

(c) BOTH P_1 AND P_2 ARE ACTING

$$AB: \quad \sigma = \frac{P_1 + P_2}{A} = \frac{90 \text{ kN}}{480 \text{ mm}^2} = 187.5 \text{ MPa}$$

$$\varepsilon = 0.008477$$

$$\delta_{AB} = \varepsilon\left(\frac{2L}{3}\right) = 10.17 \text{ mm}$$

$$BC: \quad \sigma = \frac{P_2}{A} = \frac{60 \text{ kN}}{480 \text{ mm}^2} = 125 \text{ MPa}$$

$$\varepsilon = 0.002853$$

$$\delta_{BC} = \varepsilon\left(\frac{L}{3}\right) = 1.71 \text{ mm}$$

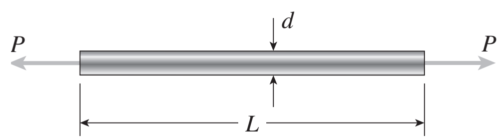
$$\delta_C = \delta_{AB} + \delta_{BC} = 11.88 \text{ mm} \quad \longleftarrow$$

(Note that the displacement when both loads act simultaneously is *not* equal to the sum of the displacements when the loads act separately.)

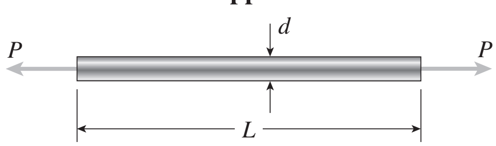
Problem 2.11-3 A circular bar of length $L = 32$ in. and diameter $d = 0.75$ in. is subjected to tension by forces P (see figure). The wire is made of a copper alloy having the following *hyperbolic stress-strain relationship*:

$$\sigma = \frac{18{,}000\epsilon}{1 + 300\epsilon} \qquad 0 \le \epsilon \le 0.03 \qquad (\sigma = \text{ksi})$$

(a) Draw a stress-strain diagram for the material.
(b) If the elongation of the wire is limited to 0.25 in. and the maximum stress is limited to 40 ksi, what is the allowable load P?



Solution 2.11-3 **Copper bar in tension**

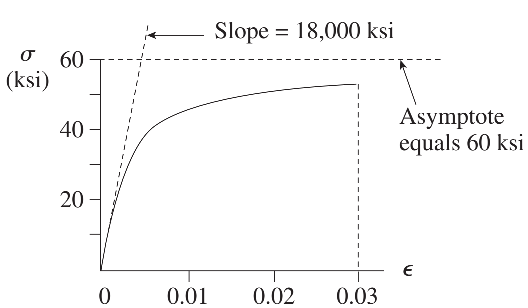


$L = 32$ in. $\qquad d = 0.75$ in.

$$A = \frac{\pi d^2}{4} = 0.4418 \text{ in.}^2$$

(a) STRESS-STRAIN DIAGRAM

$$\sigma = \frac{18{,}000\varepsilon}{1 + 300\varepsilon} \qquad 0 \le \varepsilon \le 0.03 \qquad (\sigma = \text{ksi})$$



(b) ALLOWABLE LOAD P

Max. elongation $\delta_{max} = 0.25$ in.

Max. stress $\sigma_{max} = 40$ ksi

Based upon elongation:

$$\varepsilon_{max} = \frac{\delta_{max}}{L} = \frac{0.25 \text{ in.}}{32 \text{ in.}} = 0.007813$$

$$\sigma_{max} = \frac{18{,}000\, \varepsilon_{max}}{1 + 300\, \varepsilon_{max}} = 42.06 \text{ ksi}$$

BASED UPON STRESS:

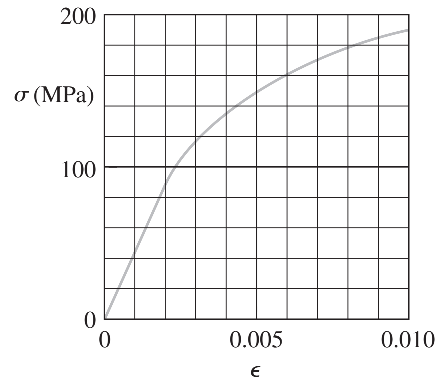
$\sigma_{max} = 40$ ksi

Stress governs. $P = \sigma_{max} A = (40 \text{ ksi})(0.4418 \text{ in.}^2)$

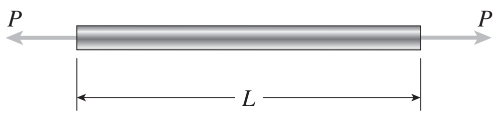
$= 17.7$ k $\quad \longleftarrow$

Problem 2.11-4 A prismatic bar in tension has length $L = 2.0$ m and cross-sectional area $A = 249$ mm^2. The material of the bar has the stress-strain curve shown in the figure.

Determine the elongation δ of the bar for each of the following axial loads: $P = 10$ kN, 20 kN, 30 kN, 40 kN, and 45 kN. From these results, plot a diagram of load P versus elongation δ (load-displacement diagram).



Solution 2.11-4 **Bar in tension**



$L = 2.0$ m

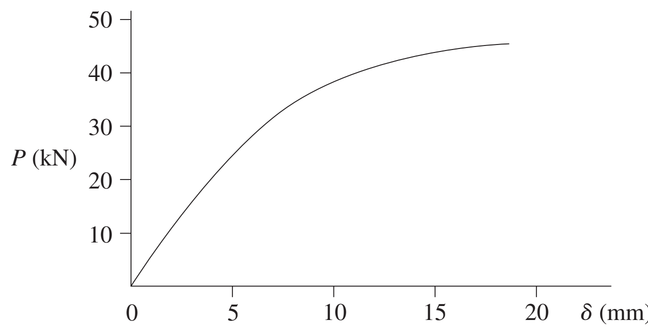
$A = 249$ mm^2

STRESS-STRAIN DIAGRAM

(See the problem statement for the diagram)

LOAD-DISPLACEMENT DIAGRAM

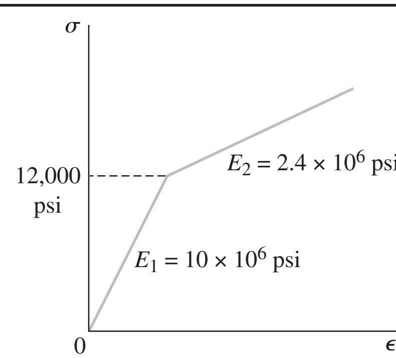
P (kN)	$\sigma = P/A$ (MPa)	ε (from diagram)	$\delta = \varepsilon L$ (mm)
10	40	0.0009	1.8
20	80	0.0018	3.6
30	120	0.0031	6.2
40	161	0.0060	12.0
45	181	0.0081	16.2



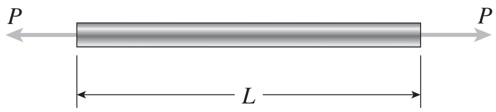
NOTE: The load-displacement curve has the same shape as the stress-strain curve.

Problem 2.11-5 An aluminum bar subjected to tensile forces P has length $L = 150$ in. and cross-sectional area $A = 2.0$ in.^{2} The stress-strain behavior of the aluminum may be represented approximately by the bilinear stress-strain diagram shown in the figure.

Calculate the elongation δ of the bar for each of the following axial loads: $P = 8$ k, 16 k, 24 k, 32 k, and 40 k. From these results, plot a diagram of load P versus elongation δ (load-displacement diagram).



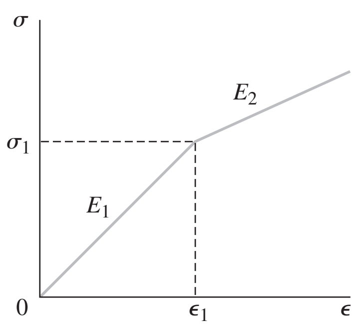
Solution 2.11-5 **Aluminum bar in tension**



$L = 150$ in.

$A = 2.0$ in.^{2}

STRESS-STRAIN DIAGRAM



$E_1 = 10 \times 10^6$ psi

$E_2 = 2.4 \times 10^6$ psi

$\sigma_1 = 12{,}000$ psi

$$\varepsilon_1 = \frac{\sigma_1}{E_1} = \frac{12{,}000 \text{ psi}}{10 \times 10^6 \text{ psi}}$$

$$= 0.0012$$

For $0 \leq \sigma \leq \sigma_1$:

$$\varepsilon = \frac{\sigma}{E_2} = \frac{\sigma}{10 \times 10^6 \text{ psi}} \quad (\sigma = \text{psi}) \qquad \text{Eq. (1)}$$

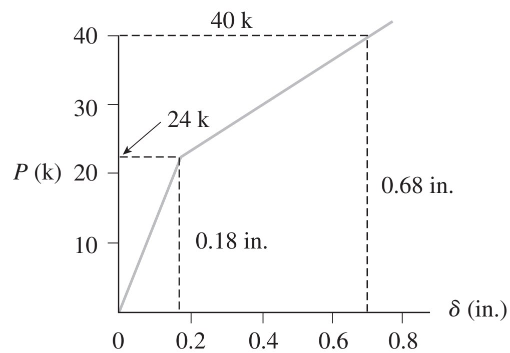
For $\sigma \geq \sigma_1$:

$$\varepsilon = \varepsilon_1 + \frac{\sigma - \sigma_1}{E_2} = 0.0012 + \frac{\sigma - 12{,}000}{2.4 \times 10^6}$$

$$= \frac{\sigma}{2.4 \times 10^6} - 0.0038 \quad (\sigma = \text{psi}) \qquad \text{Eq. (2)}$$

LOAD-DISPLACEMENT DIAGRAM

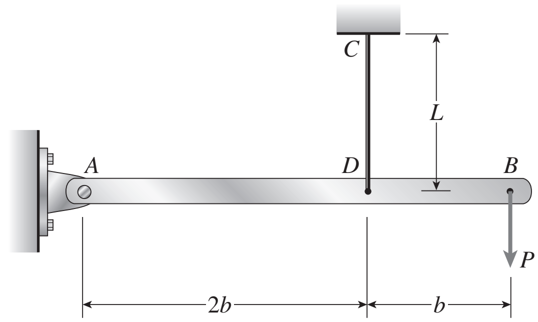
P (k)	$\sigma = P/A$ (psi)	ε (from Eq. 1 or Eq. 2)	$\delta = \varepsilon L$ (in.)
8	4,000	0.00040	0.060
16	8,000	0.00080	0.120
24	12,000	0.00120	0.180
32	16,000	0.00287	0.430
40	20,000	0.00453	0.680



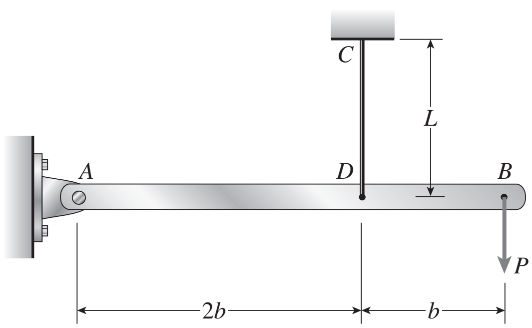
Problem 2.11-6 A rigid bar AB, pinned at end A, is supported by a wire CD and loaded by a force P at end B (see figure). The wire is made of high-strength steel having modulus of elasticity $E = 210$ GPa and yield stress $\sigma_Y = 820$ MPa. The length of the wire is $L = 1.0$ m and its diameter is $d = 3$ mm. The stress-strain diagram for the steel is defined by the *modified power law*, as follows:

$$\sigma = E\epsilon \qquad 0 \le \sigma \le \sigma_Y$$
$$\sigma = \sigma_Y\left(\frac{E\epsilon}{\sigma_Y}\right)^n \qquad \sigma \ge \sigma_Y$$

(a) Assuming $n = 0.2$, calculate the displacement δ_B at the end of the bar due to the load P. Take values of P from 2.4 kN to 5.6 kN in increments of 0.8 kN.
(b) Plot a load-displacement diagram showing P versus δ_B.



Solution 2.11-6 **Rigid bar supported by a wire**



Wire: $E = 210\,\text{GPa}$

$\sigma_Y = 820\,\text{MPa}$

$L = 1.0\,\text{m}$

$d = 3\,\text{mm}$

$$A = \frac{\pi d^2}{4} = 7.0686\ \text{mm}^2$$

STRESS-STRAIN DIAGRAM

$$\sigma = E\varepsilon \quad (0 \le \sigma \le \sigma_Y) \tag{1}$$

$$\sigma = \sigma_Y\left(\frac{E\varepsilon}{\sigma_Y}\right)^n \quad (\sigma \ge \sigma_Y) \quad (n = 0.2) \tag{2}$$

(a) DISPLACEMENT δ_B AT END OF BAR

$$\delta = \text{elongation of wire} \quad \delta_B = \frac{3}{2}\delta = \frac{3}{2}\varepsilon L \tag{3}$$

Obtain ε from stress-strain equations:

$$\text{From Eq. (1):}\ \ \varepsilon = \frac{\sigma}{E} \quad (0 \le \sigma \le \sigma_Y) \tag{4}$$

$$\text{From Eq. (2):}\ \ \varepsilon = \frac{\sigma_Y}{E}\left(\frac{\sigma}{\sigma_Y}\right)^{1/n} \tag{5}$$

$$\text{Axial force in wire: } F = \frac{3P}{2}$$

$$\text{Stress in wire: } \sigma = \frac{F}{A} = \frac{3P}{2A} \tag{6}$$

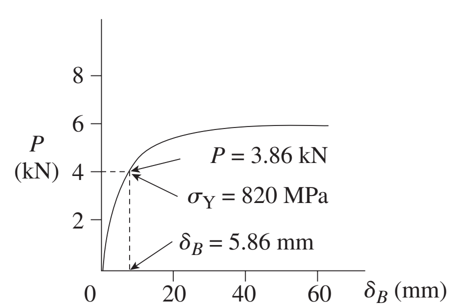
PROCEDURE: Assume a value of P
Calculate σ from Eq. (6)
Calculate ε from Eq. (4) or (5)
Calculate δ_B from Eq. (3)

P (kN)	σ (MPa) Eq. (6)	ε Eq. (4) or (5)	δ_B (mm) Eq. (3)
2.4	509.3	0.002425	3.64
3.2	679.1	0.003234	4.85
4.0	848.8	0.004640	6.96
4.8	1018.6	0.01155	17.3
5.6	1188.4	0.02497	37.5

For $\sigma = \sigma_Y = 820\,\text{MPa}$:

$\varepsilon = 0.0039048 \quad P = 3.864\,\text{kN} \quad \delta_B = 5.86\,\text{mm}$

(b) LOAD-DISPLACEMENT DIAGRAM

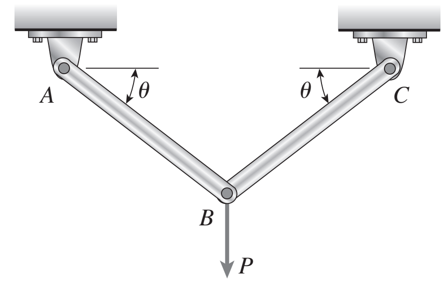


Elastoplastic Analysis

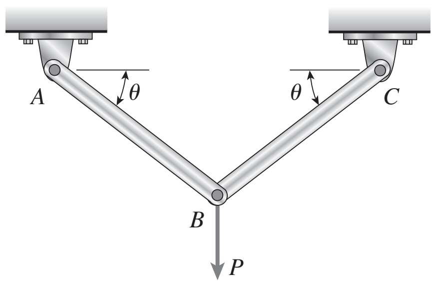
The problems for Section 2.12 are to be solved assuming that the material is elastoplastic with yield stress σ_Y, yield strain ϵ_Y, and modulus of elasticity E in the linearly elastic region (see Fig. 2-70).

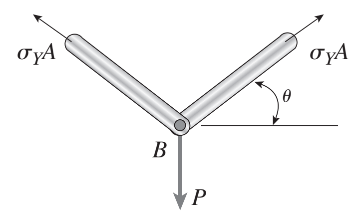
Problem 2.12-1 Two identical bars AB and BC support a vertical load P (see figure). The bars are made of steel having a stress-strain curve that may be idealized as elastoplastic with yield stress σ_Y. Each bar has cross-sectional area A.

Determine the yield load P_Y and the plastic load P_P.



Solution 2.12-1 Two bars supporting a load P





Structure is statically determinate. The yield load P_Y and the plastic lead P_P occur at the same time, namely, when both bars reach the yield stress.

JOINT B

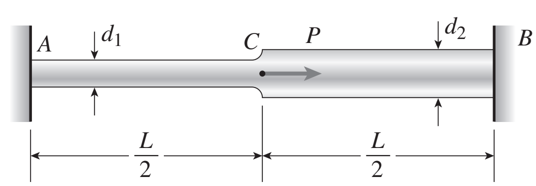
$\Sigma F_{\text{vert}} = 0$

$(2\sigma_Y A)\sin\theta = P$

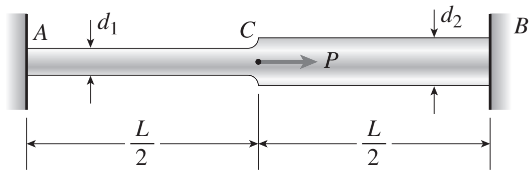
$P_Y = P_P = 2\sigma_Y A \sin\theta$ ←

Problem 2.12-2 A stepped bar ACB with circular cross sections is held between rigid supports and loaded by an axial force P at midlength (see figure). The diameters for the two parts of the bar are d_1 = 20 mm and d_2 = 25 mm, and the material is elastoplastic with yield stress σ_Y = 250 MPa.

Determine the plastic load P_P.



Solution 2.12-2 **Bar between rigid supports**

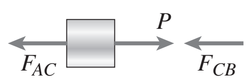


$d_1 = 20\,\text{mm} \qquad d_2 = 25\,\text{mm} \qquad \sigma_Y = 250\,\text{MPa}$

DETERMINE THE PLASTIC LOAD P_P:

At the plastic load, all parts of the bar are stressed to the yield stress.

Point C:



$F_{AC} = \sigma_Y A_1 \qquad F_{CB} = \sigma_Y A_2$

$P = F_{AC} + F_{CB}$

$P_P = \sigma_Y A_1 + \sigma_Y A_2 = \sigma_Y(A_1 + A_2)$ ←

SUBSTITUTE NUMERICAL VALUES:

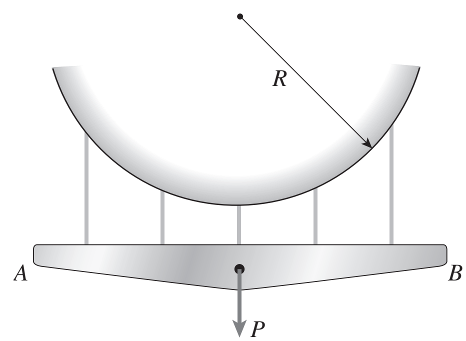
$$P_P = (250\text{ MPa})\left(\frac{\pi}{4}\right)(d_1^2 + d_2^2)$$

$$= (250\text{ MPa})\left(\frac{\pi}{4}\right)[(20\text{ mm})^2 + (25\text{ mm})^2]$$

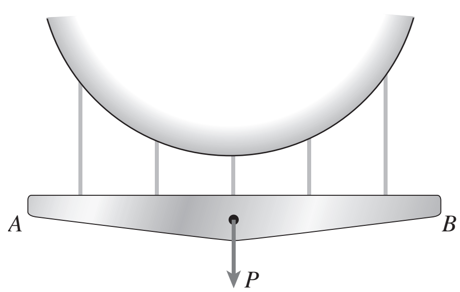
$$= 201\text{ kN}$$ ←

Problem 2.12-3 A horizontal rigid bar AB supporting a load P is hung from five symmetrically placed wires, each of cross-sectional area A (see figure). The wires are fastened to a curved surface of radius R.

(a) Determine the plastic load P_P if the material of the wires is elastoplastic with yield stress σ_Y.
(b) How is P_P changed if bar AB is flexible instead of rigid?
(c) How is P_P changed if the radius R is increased?

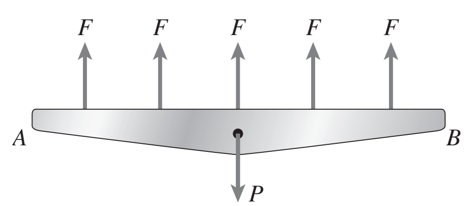


Solution 2.12-3 **Rigid bar supported by five wires**



(a) PLASTIC LOAD P_P

At the plastic load, each wire is stressed to the yield stress. $\therefore\ P_P = 5\sigma_Y A$ ←



$F = \sigma_Y A$

(b) BAR AB IS FLEXIBLE

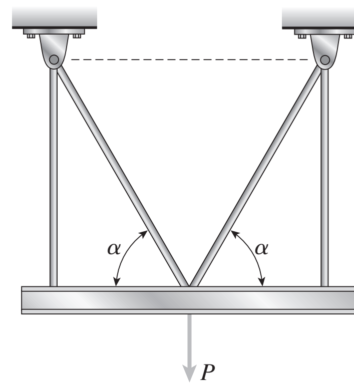
At the plastic load, each wire is stressed to the yield stress, so the plastic load is not changed. ←

(c) RADIUS R IS INCREASED

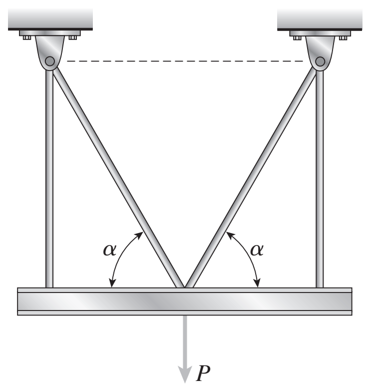
Again, the forces in the wires are not changed, so the plastic load is not changed. ←

Problem 2.12-4 A load P acts on a horizontal beam that is supported by four rods arranged in the symmetrical pattern shown in the figure. Each rod has cross-sectional area A and the material is elastoplastic with yield stress σ_Y.

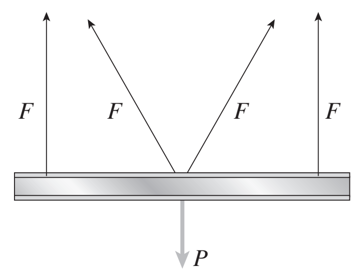
Determine the plastic load P_P.



Solution 2.12-4 **Beam supported by four rods**



At the plastic load, all four rods are stressed to the yield stress.



$F = \sigma_Y A$

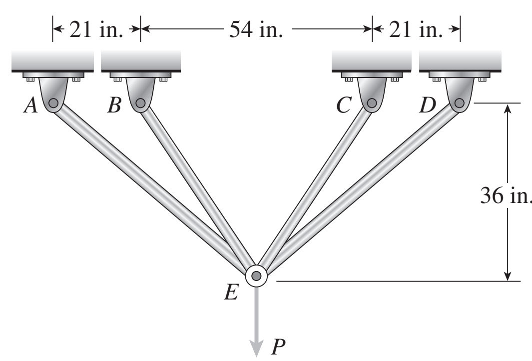
Sum forces in the vertical direction and solve for the load:

$P_P = 2F + 2F \sin \alpha$

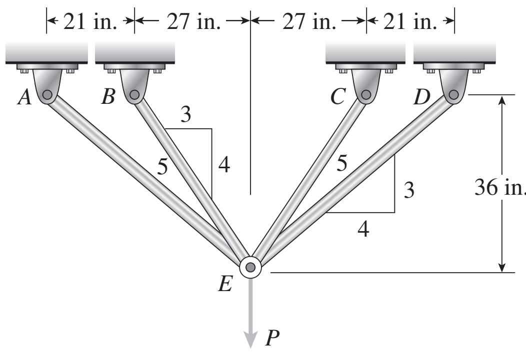
$P_P = 2\sigma_Y A\ (1 + \sin \alpha)$ ⟵

Problem 2.12-5 The symmetric truss $ABCDE$ shown in the figure is constructed of four bars and supports a load P at joint E. Each of the two outer bars has a cross-sectional area of 0.307 in.^{2}, and each of the two inner bars has an area of 0.601 in.^{2} The material is elastoplastic with yield stress σ_Y = 36 ksi.

Determine the plastic load P_P.



Solution 2.12-5 **Truss with four bars**



PLASTIC LOAD P_P

At the plastic load, all bars are stressed to the yield stress.

$F_{AE} = \sigma_Y A_{AE}$ $\qquad F_{BE} = \sigma_Y A_{BE}$

$$P_P = \frac{6}{5}\sigma_Y A_{AE} + \frac{8}{5}\sigma_Y A_{BE} \quad \longleftarrow$$

SUBSTITUTE NUMERICAL VALUES:

$A_{AE} = 0.307\text{ in.}^2$ $\;A_{BE} = 0.601\text{ in.}^2$

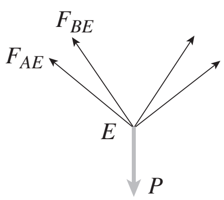
$\sigma_Y = 36\text{ ksi}$

$$P_P = \frac{6}{5}(36\text{ ksi})(0.307\text{ in.}^2) + \frac{8}{5}(36\text{ ksi})(0.601\text{ in.}^2)$$

$$= 13.26\text{ k} + 34.62\text{ k} = 47.9\text{ k} \quad \longleftarrow$$

$L_{AE} = 60\text{ in.}$ $\qquad L_{BE} = 45\text{ in.}$

JOINT E



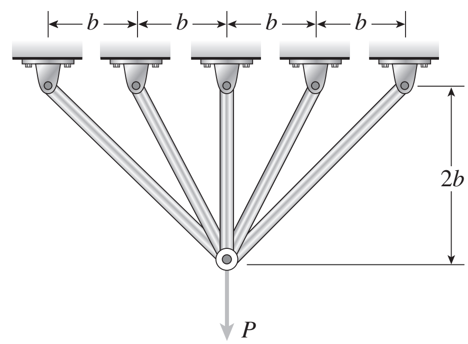
Equilibrium:

$$2F_{AE}\left(\frac{3}{5}\right) + 2F_{BE}\left(\frac{4}{5}\right) = P$$

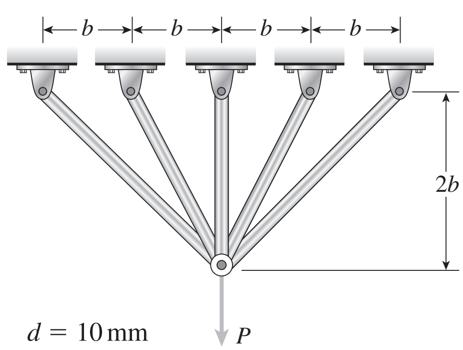
or

$$P = \frac{6}{5}F_{AE} + \frac{8}{5}F_{BE}$$

Problem 2.12-6 Five bars, each having a diameter of 10 mm, support a load P as shown in the figure. Determine the plastic load P_P if the material is elastoplastic with yield stress $\sigma_Y = 250$ MPa.



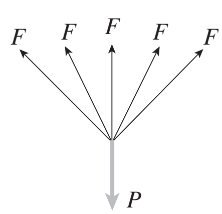
Solution 2.12-6 **Truss consisting of five bars**



$d = 10\text{ mm}$

$$A = \frac{\pi d^2}{4} = 78.54\text{ mm}^2$$

$\sigma_Y = 250\text{ MPa}$



At the plastic load, all five bars are stressed to the yield stress

$$F = \sigma_Y A$$

Sum forces in the vertical direction and solve for the load:

$$P_P = 2F\left(\frac{1}{\sqrt{2}}\right) + 2F\left(\frac{2}{\sqrt{5}}\right) + F$$

$$= \frac{\sigma_Y A}{5}(5\sqrt{2} + 4\sqrt{5} + 5)$$

$$= 4.2031\,\sigma_Y A \quad \longleftarrow$$

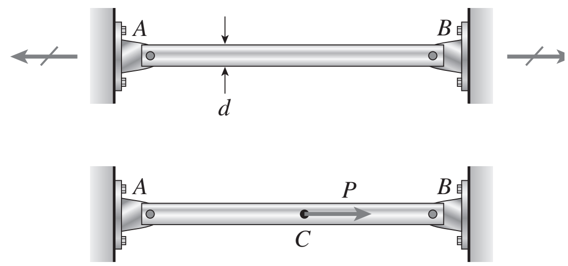
Substitute numerical values:

$$P_P = (4.2031)(250\text{ MPa})(78.54\text{ mm}^2)$$

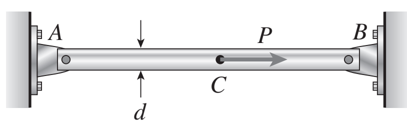
$$= 82.5\text{ kN} \quad \longleftarrow$$

Problem 2.12-7 A circular steel rod AB of diameter $d = 0.60$ in. is stretched tightly between two supports so that initially the tensile stress in the rod is 10 ksi (see figure). An axial force P is then applied to the rod at an intermediate location C.

(a) Determine the plastic load P_P if the material is elastoplastic with yield stress $\sigma_Y = 36$ ksi.
(b) How is P_P changed if the initial tensile stress is doubled to 20 ksi?



Solution 2.12-7 Bar held between rigid supports



$d = 0.6$ in.

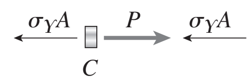
$\sigma_Y = 36$ ksi

Initial tensile stress = 10 ksi

(a) Plastic load P_P

The presence of the initial tensile stress does not affect the plastic load. Both parts of the bar must yield in order to reach the plastic load.

Point C:



$$P_P = 2\sigma_Y A = (2)(36 \text{ ksi})\left(\frac{\pi}{4}\right)(0.60 \text{ in.})^2$$

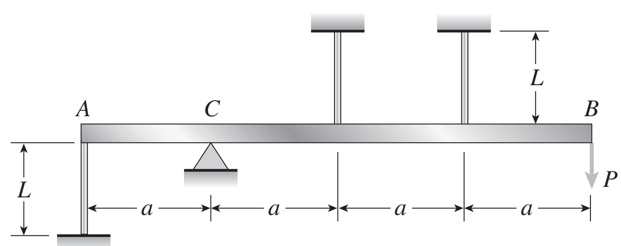
$$= 20.4 \text{ k} \longleftarrow$$

(b) Initial tensile stress is doubled

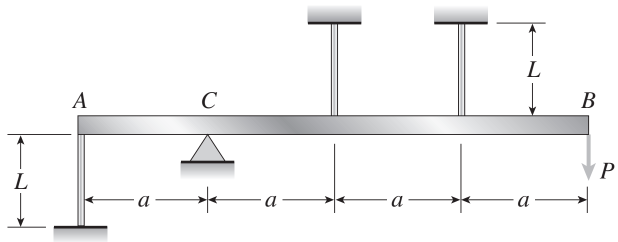
P_P is not changed. $\longleftarrow$

Problem 2.12-8 A rigid bar ACB is supported on a fulcrum at C and loaded by a force P at end B (see figure). Three identical wires made of an elastoplastic material (yield stress σ_Y and modulus of elasticity E) resist the load P. Each wire has cross-sectional area A and length L.

(a) Determine the yield load P_Y and the corresponding yield displacement δ_Y at point B.
(b) Determine the plastic load P_P and the corresponding displacement δ_P at point B when the load just reaches the value P_P.
(c) Draw a load-displacement diagram with the load P as ordinate and the displacement δ_B of point B as abscissa.

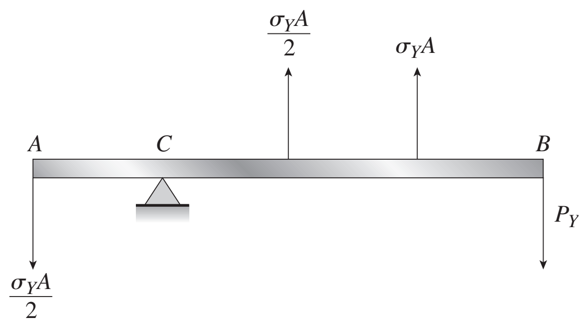


Solution 2.12-8 Rigid bar supported by wires



(a) YIELD LOAD P_Y

Yielding occurs when the most highly stressed wire reaches the yield stress σ_Y.



$\Sigma M_C = 0$

$$P_Y = \sigma_Y A \quad \leftarrow$$

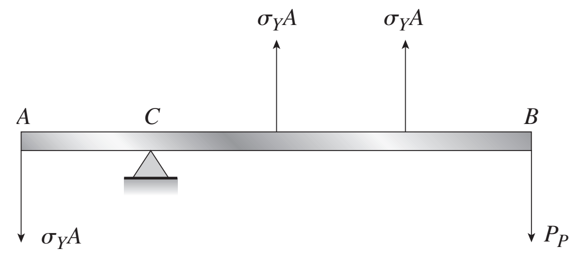
At point A:

$$\delta_A = \left(\frac{\sigma_Y A}{2}\right)\left(\frac{L}{EA}\right) = \frac{\sigma_Y L}{2E}$$

At point B:

$$\delta_B = 3\delta_A = \delta_Y = \frac{3\sigma_Y L}{2E} \quad \leftarrow$$

(b) PLASTIC LOAD P_P



At the plastic load, all wires reach the yield stress.

$\Sigma M_C = 0$

$$P_P = \frac{4\sigma_Y A}{3} \quad \leftarrow$$

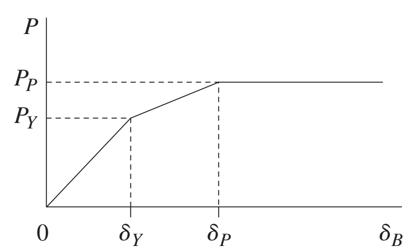
At point A:

$$\delta_A = (\sigma_Y A)\left(\frac{L}{EA}\right) = \frac{\sigma_Y L}{E}$$

At point B:

$$\delta_B = 3\delta_A = \delta_P = \frac{3\sigma_Y L}{E} \quad \leftarrow$$

(c) LOAD-DISPLACEMENT DIAGRAM

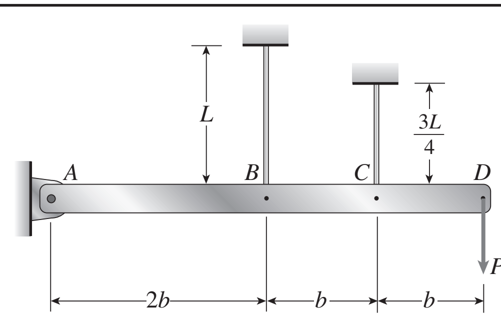


$$P_P = \frac{4}{3} P_Y$$

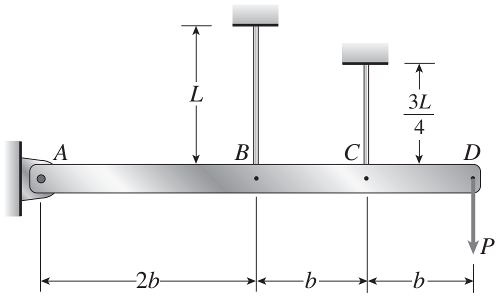
$$\delta_P = 2\delta_Y$$

Problem 2.12-9 The structure shown in the figure consists of a horizontal rigid bar $ABCD$ supported by two steel wires, one of length L and the other of length $3L/4$. Both wires have cross-sectional area A and are made of elastoplastic material with yield stress σ_Y and modulus of elasticity E. A vertical load P acts at end D of the bar.

(a) Determine the yield load P_Y and the corresponding yield displacement δ_Y at point D.
(b) Determine the plastic load P_P and the corresponding displacement δ_P at point D when the load just reaches the value P_P.
(c) Draw a load-displacement diagram with the load P as ordinate and the displacement δ_D of point D as abscissa.



Solution 2.12-9 Rigid bar supported by two wires

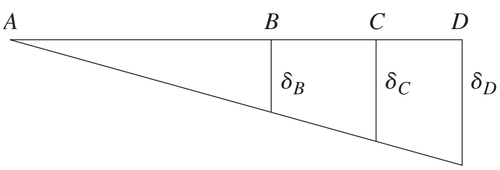


A = cross-sectional area

σ_Y = yield stress

E = modulus of elasticity

DISPLACEMENT DIAGRAM

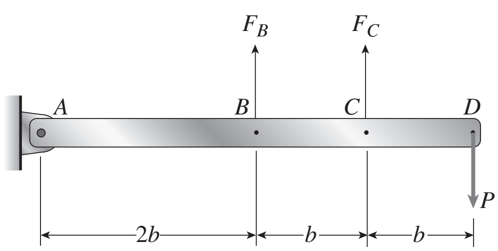


COMPATIBILITY:

$$\delta_C = \frac{3}{2}\delta_B \qquad (1)$$

$$\delta_D = 2\delta_B \qquad (2)$$

FREE-BODY DIAGRAM



EQUILIBRIUM:

$$\Sigma M_A = 0 \qquad F_B(2b) + F_C(3b) = P(4b)$$

$$2F_B + 3F_C = 4P \qquad (3)$$

FORCE-DISPLACEMENT RELATIONS

$$\delta_B = \frac{F_BL}{EA} \qquad \delta_C = \frac{F_C\left(\frac{3}{4}L\right)}{EA} \qquad (4, 5)$$

Substitute into Eq. (1):

$$\frac{3F_CL}{4EA} = \frac{3F_BL}{2EA}$$

$$F_C = 2F_B \qquad (6)$$

STRESSES

$$\sigma_B = \frac{F_B}{A} \quad \sigma_C = \frac{F_C}{A} \quad \therefore \ \sigma_C = 2\sigma_B \qquad (7)$$

Wire C has the larger stress. Therefore, it will yield first.

(a) YIELD LOAD

$$\sigma_C = \sigma_Y \quad \sigma_B = \frac{\sigma_C}{2} = \frac{\sigma_Y}{2} \quad \text{(From Eq. 7)}$$

$$F_C = \sigma_Y A \quad F_B = \frac{1}{2}\sigma_Y A$$

From Eq. (3):

$$2\left(\frac{1}{2}\sigma_Y A\right) + 3(\sigma_Y A) = 4P$$

$$P = P_Y = \sigma_Y A \longleftarrow$$

From Eq. (4):

$$\delta_B = \frac{F_BL}{EA} = \frac{\sigma_Y L}{2E}$$

From Eq. (2):

$$\delta_D = \delta_Y = 2\delta_B = \frac{\sigma_Y L}{E} \longleftarrow$$

(b) PLASTIC LOAD

At the plastic load, both wires yield.

$$\sigma_B = \sigma_Y = \sigma_C \qquad F_B = F_C = \sigma_Y A$$

From Eq. (3):

$$2(\sigma_Y A) + 3(\sigma_Y A) = 4P$$

$$P = P_P = \frac{5}{4}\sigma_Y A \longleftarrow$$

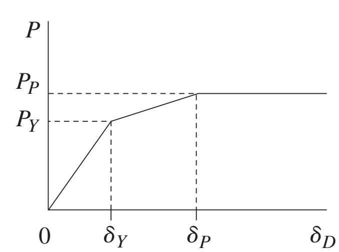
From Eq. (4):

$$\delta_B = \frac{F_BL}{EA} = \frac{\sigma_Y L}{E}$$

From Eq. (2):

$$\delta_D = \delta_P = 2\delta_B = \frac{2\sigma_Y L}{E} \longleftarrow$$

(c) LOAD-DISPLACEMENT DIAGRAM

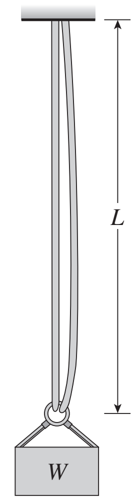


$$P_P = \frac{5}{4}P_Y$$

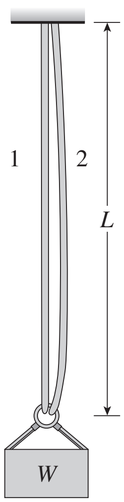
$$\delta_P = 2\delta_Y$$

Problem 2.12-10 Two cables, each having a length L of approximately 40 m, support a loaded container of weight W (see figure). The cables, which have effective cross-sectional area $A = 48.0\ \text{mm}^2$ and effective modulus of elasticity $E = 160$ GPa, are identical except that one cable is longer than the other when they are hanging separately and unloaded. The difference in lengths is $d = 100$ mm. The cables are made of steel having an elastoplastic stress-strain diagram with $\sigma_Y = 500$ MPa. Assume that the weight W is initially zero and is slowly increased by the addition of material to the container.

(a) Determine the weight W_Y that first produces yielding of the shorter cable. Also, determine the corresponding elongation δ_Y of the shorter cable.
(b) Determine the weight W_P that produces yielding of both cables. Also, determine the elongation δ_P of the shorter cable when the weight W just reaches the value W_P.
(c) Construct a load-displacement diagram showing the weight W as ordinate and the elongation δ of the shorter cable as abscissa. (*Hint*: The load displacement diagram is not a single straight line in the region $0 \le W \le W_Y$.)



Solution 2.12-10 **Two cables supporting a load**



$L = 40\ \text{m} \quad A = 48.0\ \text{mm}^2$

$E = 160\ \text{GPa}$

d = difference in length = 100 mm

$\sigma_Y = 500\ \text{MPa}$

INITIAL STRETCHING OF CABLE 1

Initially, cable 1 supports all of the load.

Let W_1 = load required to stretch cable 1 to the same length as cable 2

$$W_1 = \frac{EA}{L}d = 19.2\ \text{kN}$$

$\delta_1 = 100$ mm (elongation of cable 1)

$$\sigma_1 = \frac{W_1}{A} = \frac{Ed}{L} = 400\ \text{MPa}\ (\sigma_1 < \sigma_Y \ \therefore\ \text{OK})$$

(a) YIELD LOAD W_Y

Cable 1 yields first. $F_1 = \sigma_Y A = 24\ \text{kN}$

δ_{1Y} = total elongation of cable 1

$$\delta_{1Y} = \frac{F_1 L}{EA} = \frac{\sigma_Y L}{E} = 0.125\ \text{m} = 125\ \text{mm}$$

$\delta_Y = \delta_{1Y} = 125\ \text{mm}$ ←

δ_{2Y} = elongation of cable 2

$= \delta_{1Y} - d = 25\ \text{mm}$

$$F_2 = \frac{EA}{L}\delta_{2Y} = 4.8\ \text{kN}$$

$W_Y = F_1 + F_2 = 24\ \text{kN} + 4.8\ \text{kN}$

$= 28.8\ \text{kN}$ ←

(b) PLASTIC LOAD W_P

$F_1 = \sigma_Y A \quad F_2 = \sigma_Y A$

$W_P = 2\sigma_Y A = 48\ \text{kN}$ ←

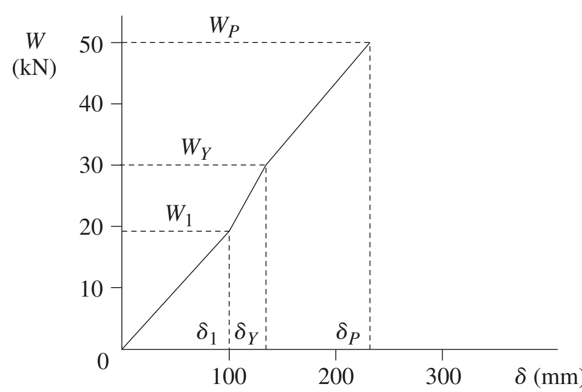
δ_{2P} = elongation of cable 2

$$= F_2\left(\frac{L}{EA}\right) = \frac{\sigma_Y L}{E} = 0.125\ \text{mm} = 125\ \text{mm}$$

$\delta_{1P} = \delta_{2P} + d = 225\ \text{mm}$

$\delta_P = \delta_{1P} = 225\ \text{mm}$ ←

(c) LOAD-DISPLACEMENT DIAGRAM



$$\frac{W_Y}{W_1} = 1.5 \qquad \frac{\delta_Y}{\delta_1} = 1.25$$

$$\frac{W_P}{W_Y} = 1.667 \qquad \frac{\delta_P}{\delta_Y} = 1.8$$

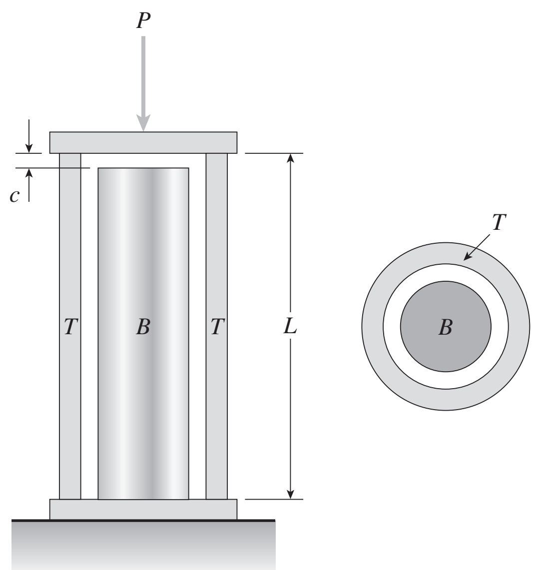
$0 < W < W_1$: slope = 192,000 N/m

$W_1 < W < W_Y$: slope = 384,000 N/m

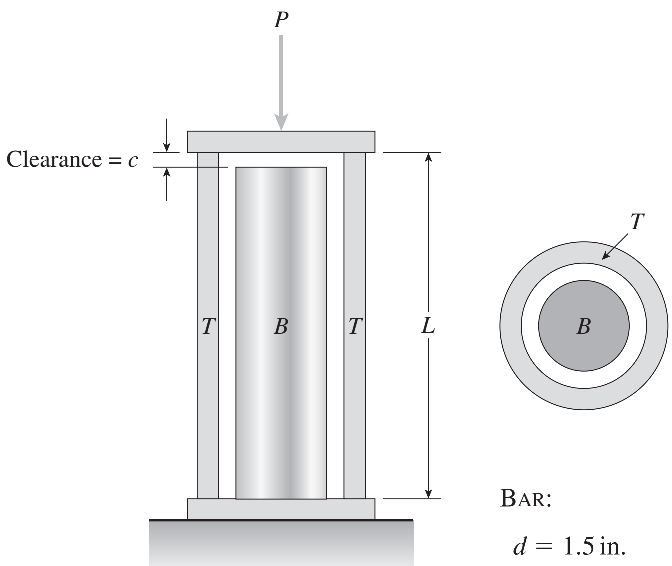
$W_Y < W < W_P$: slope = 192,000 N/m

Problem 2.12-11 A hollow circular tube T of length $L = 15$ in. is uniformly compressed by a force P acting through a rigid plate (see figure). The outside and inside diameters of the tube are 3.0 and 2.75 in., respectively. A concentric solid circular bar B of 1.5 in. diameter is mounted inside the tube. When no load is present, there is a clearance $c = 0.010$ in. between the bar B and the rigid plate. Both bar and tube are made of steel having an elastoplastic stress-strain diagram with $E = 29 \times 10^3$ ksi and $\sigma_Y = 36$ ksi.

(a) Determine the yield load P_Y and the corresponding shortening δ_Y of the tube.
(b) Determine the plastic load P_P and the corresponding shortening δ_P of the tube.
(c) Construct a load-displacement diagram showing the load P as ordinate and the shortening δ of the tube as abscissa. (*Hint:* The load-displacement diagram is not a single straight line in the region $0 \leq P \leq P_Y$.)



Solution 2.12-11 Tube and bar supporting a load



$L = 15$ in.

$c = 0.010$ in.

$E = 29 \times 10^3$ ksi

$\sigma_Y = 36$ ksi

TUBE:

$d_2 = 3.0$ in.

$d_1 = 2.75$ in.

$$A_T = \frac{\pi}{4}(d_2^2 - d_1^2) = 1.1290 \text{ in.}^2$$

BAR:

$d = 1.5$ in.

$$A_B = \frac{\pi d^2}{4} = 1.7671 \text{ in.}^2$$

INITIAL SHORTENING OF TUBE T

Initially, the tube supports all of the load.

Let P_1 = load required to close the clearance

$$P_1 = \frac{EA_T}{L}c = 21{,}827 \text{ lb}$$

Let δ_1 = shortening of tube $\quad \delta_1 = c = 0.010$ in.

$$\sigma_1 = \frac{P_1}{A_T} = 19{,}330 \text{ psi} \qquad (\sigma_1 < \sigma_Y \;\therefore\; \text{OK})$$

(Continued)

(a) YIELD LOAD P_Y

Because the tube and bar are made of the same material, and because the strain in the tube is larger than the strain in the bar, the tube will yield first.

$F_T = \sigma_Y A_T = 40{,}644 \text{ lb}$

δ_{TY} = shortening of tube at the yield stress

$$\delta_{TY} = \frac{F_T L}{EA_T} = \frac{\sigma_Y L}{E} = 0.018621 \text{ in.}$$

$\delta_Y = \delta_{TY} = 0.01862 \text{ in.}$ ⟵

δ_{BY} = shortening of bar

$= \delta_{TY} - c = 0.008621 \text{ in.}$

$$F_B = \frac{EA_B}{L}\delta_{BY} = 29{,}453 \text{ lb}$$

$P_Y = F_T + F_B = 40{,}644 \text{ lb} + 29{,}453 \text{ lb}$
$= 70{,}097 \text{ lb}$

$P_Y = 70{,}100 \text{ lb}$ ⟵

(b) PLASTIC LOAD P_P

$F_T = \sigma_Y A_T \qquad F_B = \sigma_Y A_B$

$P_P = F_T + F_B = \sigma_Y(A_T + A_B)$

$= 104{,}300 \text{ lb}$ ⟵

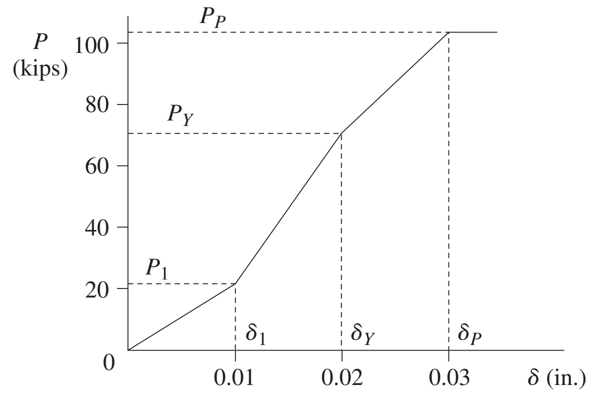
δ_{BP} = shortening of bar

$$= F_B\left(\frac{L}{EA_B}\right) = \frac{\sigma_Y L}{E} = 0.018621 \text{ in.}$$

$\delta_{TP} = \delta_{BP} + c = 0.028621 \text{ in.}$

$\delta_P = \delta_{TP} = 0.02862 \text{ in.}$ ⟵

(c) LOAD-DISPLACEMENT DIAGRAM



$$\frac{P_Y}{P_1} = 3.21 \qquad \frac{\delta_Y}{\delta_1} = 1.86$$

$$\frac{P_P}{P_Y} = 1.49 \qquad \frac{\delta_P}{\delta_Y} = 1.54$$

$0 < P < P_1$: slope = 2180 k/in.

$P_1 < P < P_Y$: slope = 5600 k/in.

$P_Y < P < P_P$: slope = 3420 k/in.

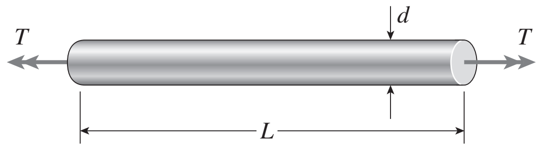
3

Torsion

Torsional Deformations

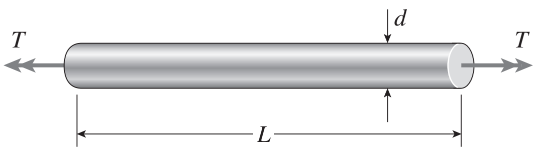
Problem 3.2-1 A copper rod of length $L = 18.0$ in. is to be twisted by torques T (see figure) until the angle of rotation between the ends of the rod is 3.0°.

If the allowable shear strain in the copper is 0.0006 rad, what is the maximum permissible diameter of the rod?



Probs. 3.2-1 and 3.2-2

Solution 3.2-1 Copper rod in torsion



$L = 18.0$ in.

$$\phi = 3.0° = (3.0)\left(\frac{\pi}{180}\right) \text{rad}$$

$$= 0.05236 \text{ rad}$$

$\gamma_{allow} = 0.0006$ rad

Find d_{max}

From Eq. (3-3):

$$\gamma_{max} = \frac{r\phi}{L} = \frac{d\phi}{2L}$$

$$d_{max} = \frac{2L\gamma_{allow}}{\phi} = \frac{(2)(18.0 \text{ in.})(0.0006 \text{ rad})}{0.05236 \text{ rad}}$$

$$d_{max} = 0.413 \text{ in.} \longleftarrow$$

Problem 3.2-2 A plastic bar of diameter $d = 50$ mm is to be twisted by torques T (see figure) until the angle of rotation between the ends of the bar is 5.0°.

If the allowable shear strain in the plastic is 0.012 rad, what is the minimum permissible length of the bar?

Solution 3.2-2 Plastic bar in torsion

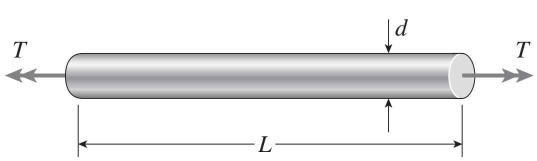
$d = 50$ mm

$$\phi = 5.0° = (5.0)\left(\frac{\pi}{180}\right) \text{rad} = 0.08727 \text{ rad}$$

$\gamma_{allow} = 0.012$ rad

Find L_{min}

From Eq. (3-3): $\gamma_{max} = \dfrac{r\phi}{L} = \dfrac{d\phi}{2L}$

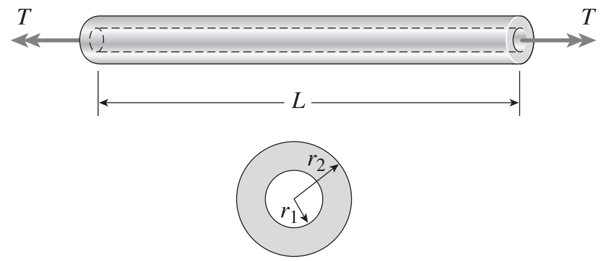


$$L_{min} = \frac{d\phi}{2\gamma_{allow}} = \frac{(50 \text{ mm})(0.08727 \text{ rad})}{(2)(0.012 \text{ rad})}$$

$$L_{min} = 182 \text{ mm} \longleftarrow$$

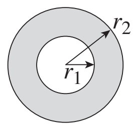
Problem 3.2-3 A circular aluminum tube subjected to pure torsion by torques T (see figure) has an outer radius r_2 equal to twice the inner radius r_1.

(a) If the maximum shear strain in the tube is measured as 400×10^{-6} rad, what is the shear strain γ_1 at the inner surface?
(b) If the maximum allowable rate of twist is 0.15 degrees per foot and the maximum shear strain is to be kept at 400×10^{-6} rad by adjusting the torque T, what is the minimum required outer radius $(r_2)_{min}$?



Problems 3.2-3, 3.2-4, and 3.2-5

Solution 3.2-3 **Circular aluminum tube**



$$r_2 = 2r_1$$

$$\gamma_{max} = 400 \times 10^{-6} \text{ rad}$$

$$\theta_{allow} = 0.15°/\text{ft}$$

$$= (0.15°/\text{ft})\left(\frac{\pi}{180}\,\frac{\text{rad}}{\text{degree}}\right)\left(\frac{1}{12}\,\frac{\text{ft}}{\text{in.}}\right)$$

$$= 218.2 \times 10^{-6} \text{ rad/in.}$$

(a) Shear strain at inner surface

From Eq. (3-5b):

$$\gamma_1 = \frac{1}{2}\gamma_2 = \frac{1}{2}(400 \times 10^{-6} \text{ rad})$$

$$\gamma_1 = 200 \times 10^{-6} \text{ rad} \longleftarrow$$

(b) Minimum outer radius

From Eq. (3-5a):

$$\gamma_{max} = r_2 \frac{\phi}{L} = r_2\theta$$

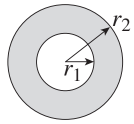
$$(r_2)_{min} = \frac{\gamma_{max}}{\theta_{allow}} = \frac{400 \times 10^{-6} \text{ rad}}{218.2 \times 10^{-6} \text{ rad/in.}}$$

$$(r_2)_{min} = 1.83 \text{ in.} \longleftarrow$$

Problem 3.2-4 A circular steel tube of length $L = 0.90$ m is loaded in torsion by torques T (see figure).

(a) If the inner radius of the tube is $r_1 = 40$ mm and the measured angle of twist between the ends is 0.5°, what is the shear strain γ_1 (in radians) at the inner surface?
(b) If the maximum allowable shear strain is 0.0005 rad and the angle of twist is to be kept at 0.5° by adjusting the torque T, what is the maximum permissible outer radius $(r_2)_{max}$?

Solution 3.2-4 **Circular steel tube**



$L = 0.90$ m

$r_1 = 40$ mm

$$\phi = 0.5° = (0.5°)\left(\frac{\pi}{180}\ \frac{\text{rad}}{\text{degree}}\right)$$

$$= 0.008727 \text{ rad}$$

$$\gamma_{max} = 0.0005 \text{ rad}$$

(a) Shear strain at inner surface

From Eq. (3-5b):

$$\gamma_{min} = \gamma_1 = r_1\frac{\phi}{L} = \frac{(40 \text{ mm})(0.008727 \text{ rad})}{900 \text{ mm}}$$

$$\gamma_1 = 388 \times 10^{-6} \text{ rad} \longleftarrow$$

(b) Maximum outer radius

From Eq. (3-5a):

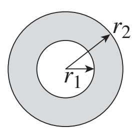
$$\gamma_{max} = \gamma_2 = r_2\frac{\phi}{L}; \quad r_2 = \frac{\gamma_{max}L}{\phi}$$

$$(r_2)_{max} = \frac{(0.0005 \text{ rad})(900 \text{ mm})}{0.008727 \text{ rad}}$$

$$(r_2)_{max} = 51.6 \text{ mm} \longleftarrow$$

Problem 3.2-5 Solve the preceding problem if the length $L = 50$ in., the inner radius $r_1 = 1.5$ in., the angle of twist is 0.6°, and the allowable shear strain is 0.0004 rad.

Solution 3.2-5 **Circular steel tube**



$L = 50$ in.

$r_1 = 1.5$ in.

$$\phi = 0.6° = (0.6°)\left(\frac{\pi}{180}\ \frac{\text{rad}}{\text{degree}}\right)$$

$$= 0.010472 \text{ rad}$$

$$\gamma_{max} = 0.0004 \text{ rad}$$

(a) Shear strain at inner surface

From Eq. (3-5b):

$$\gamma_{min} = \gamma_1 = r_1\frac{\phi}{L} = \frac{(1.5 \text{ in.})(0.010472 \text{ rad})}{50 \text{ in.}}$$

$$\gamma_1 = 314 \times 10^{-6} \text{ rad} \longleftarrow$$

(b) Maximum outer radius

From Eq. (3-5a):

$$\gamma_{max} = \gamma_2 = r_2\frac{\phi}{L}; \; r_2 = \frac{\gamma_{max}L}{\phi}$$

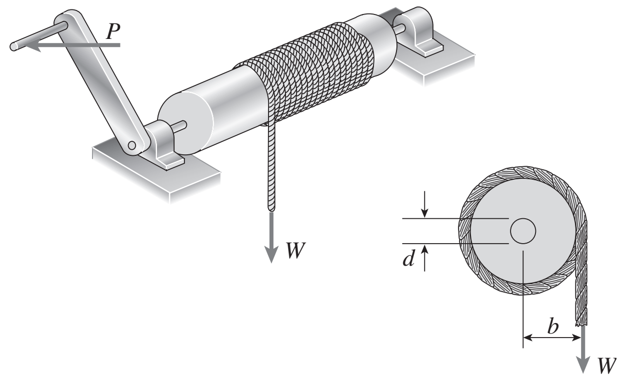
$$(r_2)_{max} = \frac{(0.0004 \text{ rad})(50 \text{ in.})}{0.010472 \text{ rad}}$$

$$(r_2)_{max} = 1.91 \text{ in.} \longleftarrow$$

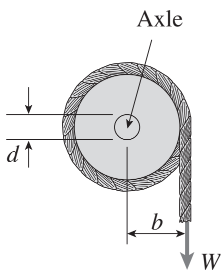
Circular Bars and Tubes

Problem 3.3-1 A prospector uses a hand-powered winch (see figure) to raise a bucket of ore in his mine shaft. The axle of the winch is a steel rod of diameter $d = 0.625$ in. Also, the distance from the center of the axle to the center of the lifting rope is $b = 4.0$ in.

If the weight of the loaded bucket is $W = 100$ lb, what is the maximum shear stress in the axle due to torsion?



Solution 3.3-1 **Hand-powered winch**



$d = 0.625$ in.

$b = 4.0$ in.

$W = 100$ lb

Torque T applied to the axle:

$T = Wb = 400$ lb-in.

MAXIMUM SHEAR STRESS IN THE AXLE

From Eq. (3-12):

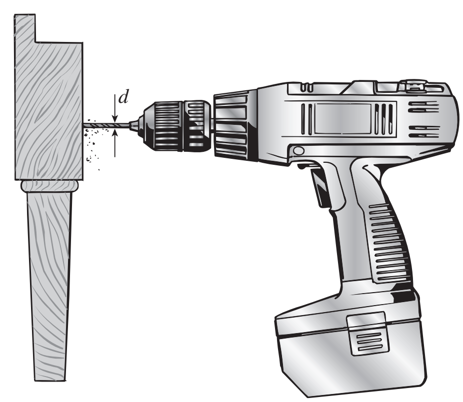
$$\tau_{max} = \frac{16T}{\pi d^3}$$

$$\tau_{max} = \frac{(16)(400 \text{ lb-in})}{\pi(0.625 \text{ in.})^3}$$

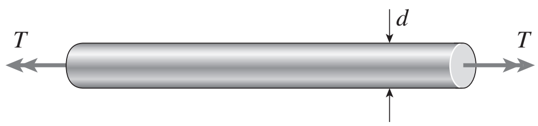
$$\tau_{max} = 8{,}340 \text{ psi} \longleftarrow$$

Problem 3.3-2 When drilling a hole in a table leg, a furniture maker uses a hand-operated drill (see figure) with a bit of diameter $d = 4.0$ mm.

(a) If the resisting torque supplied by the table leg is equal to 0.3 N·m, what is the maximum shear stress in the drill bit?
(b) If the shear modulus of elasticity of the steel is $G = 75$ GPa, what is the rate of twist of the drill bit (degrees per meter)?



Solution 3.3-2 **Torsion of a drill bit**



$d = 4.0$ mm $\quad T = 0.3$ N·m $\quad G = 75$ GPa

(a) MAXIMUM SHEAR STRESS

From Eq. (3-12):

$$\tau_{max} = \frac{16T}{\pi d^3}$$

$$\tau_{max} = \frac{16(0.3 \text{ N} \cdot \text{m})}{\pi (4.0 \text{ mm})^3}$$

$$\tau_{max} = 23.8 \text{ MPa} \quad \longleftarrow$$

(b) RATE OF TWIST

From Eq. (3-14):

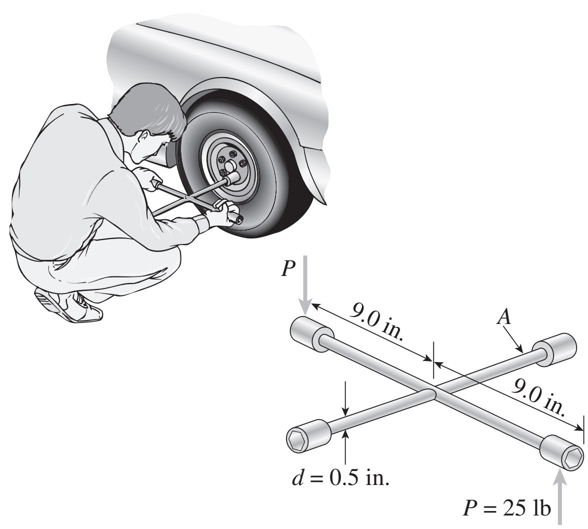
$$\theta = \frac{T}{GI_P}$$

$$\theta = \frac{0.3 \text{ N} \cdot \text{m}}{(75 \text{ GPa})\left(\frac{\pi}{32}\right)(4.0 \text{ mm})^4}$$

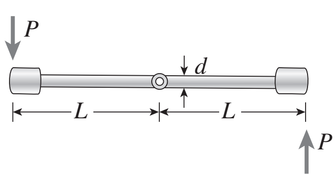
$$\theta = 0.1592 \text{ rad/m} = 9.12°/\text{m} \quad \longleftarrow$$

Problem 3.3-3 While removing a wheel to change a tire, a driver applies forces $P = 25$ lb at the ends of two of the arms of a lug wrench (see figure). The wrench is made of steel with shear modulus of elasticity $G = 11.4 \times 10^6$ psi. Each arm of the wrench is 9.0 in. long and has a solid circular cross section of diameter $d = 0.5$ in.

(a) Determine the maximum shear stress in the arm that is turning the lug nut (arm A).
(b) Determine the angle of twist (in degrees) of this same arm.



Solution 3.3-3 **Lug wrench**

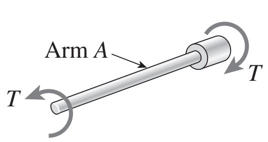


$P = 25$ lb

$L = 9.0$ in.

$d = 0.5$ in.

$G = 11.4 \times 10^6$ psi



$T =$ torque acting on arm A

$T = P(2L) = 2(25 \text{ lb})(9.0 \text{ in.})$

$= 450$ lb-in.

(a) MAXIMUM SHEAR STRESS

From Eq. (3-12):

$$\tau_{max} = \frac{16T}{\pi d^3} = \frac{(16)(450 \text{ lb-in.})}{\pi (0.5 \text{ in.})^3}$$

$$\tau_{max} = 18{,}300 \text{ psi} \quad \longleftarrow$$

(b) ANGLE OF TWIST

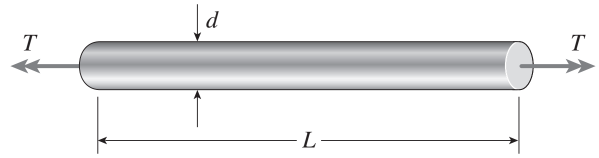
From Eq. (3-15):

$$\phi = \frac{TL}{GI_P} = \frac{(450 \text{ lb-in.})(9.0 \text{ in.})}{(11.4 \times 10^6 \text{ psi})\left(\frac{\pi}{32}\right)(0.5 \text{ in.})^4}$$

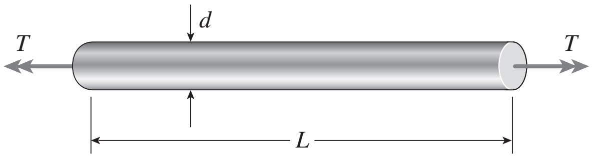
$$\phi = 0.05790 \text{ rad} = 3.32° \quad \longleftarrow$$

Problem 3.3-4 An aluminum bar of solid circular cross section is twisted by torques T acting at the ends (see figure). The dimensions and shear modulus of elasticity are as follows: $L = 1.2$ m, $d = 30$ mm, and $G = 28$ GPa.

(a) Determine the torsional stiffness of the bar.
(b) If the angle of twist of the bar is 4°, what is the maximum shear stress? What is the maximum shear strain (in radians)?



Solution 3.3-4 **Aluminum bar in torsion**



$L = 1.2$ m $\quad d = 30$ mm

$G = 28$ GPa $\quad \phi = 4°$

(a) Torsional stiffness

$$k_T = \frac{GI_P}{L} = \frac{G\pi d^4}{32L} = \frac{(28\text{ GPa})(\pi)(30\text{ mm})^4}{32(1.2\text{ m})}$$

$$k_T = 1860 \text{ N} \cdot \text{m} \longleftarrow$$

(b) Maximum shear stress

$$\phi = 4° = (4°)(\pi/180)rad = 0.069813 \text{ rad}$$

From Eq. (3-15):

$$\phi = \frac{TL}{GI_P} \quad T = \frac{GI_P\phi}{L}$$

From Eq. (3-11):

$$\tau_{max} = \frac{Tr}{I_P} = \frac{Td}{2I_P} = \left(\frac{GI_P\phi}{L}\right)\left(\frac{d}{2I_P}\right)$$

$$\tau_{max} = \frac{Gd\phi}{2L}$$

$$= \frac{(28\text{ GPa})(30\text{ mm})(0.069813\text{ rad})}{2(1.2\text{ m})}$$

$$= 24.43 \text{ MPa}$$

$$\tau_{max} = 24.4 \text{ MPa} \longleftarrow$$

Maximum shear strain

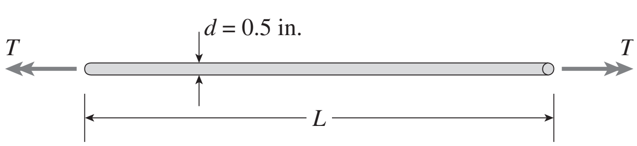
Hooke's Law:

$$\gamma_{max} = \frac{\tau_{max}}{G} = \frac{24.43\text{ MPa}}{28\text{ GPa}}$$

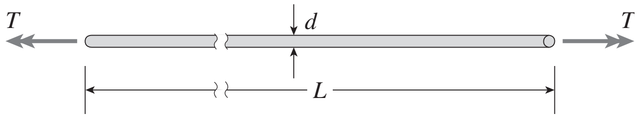
$$\gamma_{max} = 873 \times 10^{-6} \text{ rad} \longleftarrow$$

Problem 3.3-5 A high-strength steel drill rod used for boring a hole in the earth has a diameter of 0.5 in. (see figure).The allowable shear stress in the steel is 40 ksi and the shear modulus of elasticity is 11,600 ksi.

What is the minimum required length of the rod so that one end of the rod can be twisted 30° with respect to the other end without exceeding the allowable stress?



Solution 3.3-5 **Steel drill rod**



$G = 11{,}600$ psi

$d = 0.5$ in.

$\phi = 30° = (30°)\left(\frac{\pi}{180}\right)\text{rad} = 0.52360$ rad

$\tau_{\text{allow}} = 40$ ksi

MINIMUM LENGTH

From Eq. (3-12): $\tau_{\max} = \frac{16T}{\pi d^3}$ (1)

From Eq. (3-15): $\phi = \frac{TL}{GI_P} = \frac{32TL}{G\pi d^4}$

$T = \frac{G\pi d^4\phi}{32L}$; substitute T into Eq. (1):

$$\tau_{\max} = \left(\frac{16}{\pi d^3}\right)\left(\frac{G\pi d^4\phi}{32L}\right) = \frac{Gd\phi}{2L}$$

$$L_{\min} = \frac{Gd\phi}{2\tau_{\text{allow}}}$$

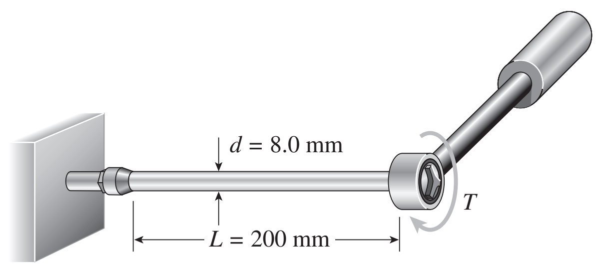
$$= \frac{(11{,}600 \text{ ksi})(0.5 \text{ in.})(0.52360 \text{ rad})}{2(40 \text{ ksi})}$$

$L_{\min} = 38.0$ in. ←

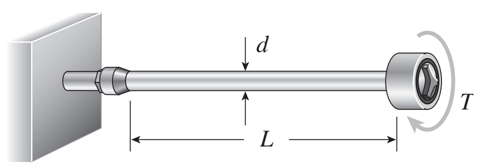
Problem 3.3-6 The steel shaft of a socket wrench has a diameter of 8.0 mm. and a length of 200 mm (see figure).

If the allowable stress in shear is 60 MPa, what is the maximum permissible torque $T_{\max}$ that may be exerted with the wrench?

Through what angle ϕ (in degrees) will the shaft twist under the action of the maximum torque? (Assume $G = 78$ GPa and disregard any bending of the shaft.)



Solution 3.3-6 **Socket wrench**



$d = 8.0$ mm $\quad L = 200$ mm

$\tau_{\text{allow}} = 60$ MPa $\quad G = 78$ GPa

MAXIMUM PERMISSIBLE TORQUE

From Eq. (3-12): $\tau_{\max} = \frac{16T}{\pi d^3}$

$$T_{\max} = \frac{\pi d^3 \tau_{\max}}{16}$$

$$T_{\max} = \frac{\pi(8.0 \text{ mm})^3(60 \text{ MPa})}{16}$$

$T_{\max} = 6.03$ N · m ←

ANGLE OF TWIST

From Eq. (3-15): $\phi = \frac{T_{\max}L}{GI_P}$

From Eq. (3-12): $T_{\max} = \frac{\pi d^3 \tau_{\max}}{16}$

$$\phi = \left(\frac{\pi d^3\tau_{\max}}{16}\right)\left(\frac{L}{GI_P}\right) \qquad I_P = \frac{\pi d^4}{32}$$

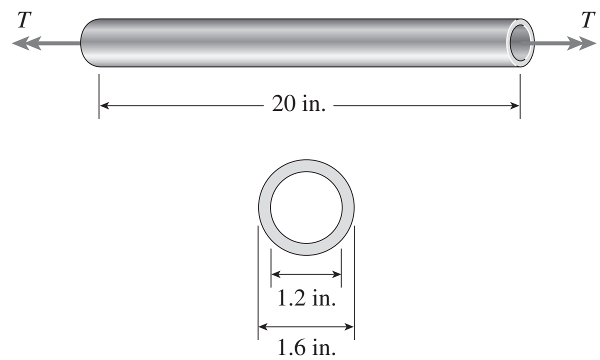
$$\phi = \frac{\pi d^3 \tau_{\max} L(32)}{16G(\pi d^4)} = \frac{2\tau_{\max}L}{Gd}$$

$$\phi = \frac{2(60 \text{ MPa})(200 \text{ mm})}{(78 \text{ GPa})(8.0 \text{ mm})} = 0.03846 \text{ rad}$$

$\phi = (0.03846 \text{ rad})\left(\frac{180}{\pi}\text{deg/rad}\right) = 2.20°$ ←

Problem 3.3-7 A circular tube of aluminum is subjected to torsion by torques T applied at the ends (see figure). The bar is 20 in. long, and the inside and outside diameters are 1.2 in. and 1.6 in., respectively. It is determined by measurement that the angle of twist is 3.63° when the torque is 5800 lb-in.

Calculate the maximum shear stress τ_{max} in the tube, the shear modulus of elasticity G, and the maximum shear strain γ_{max} (in radians).

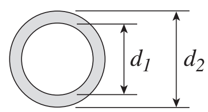


Solution 3.3-7 **Aluminum tube in torsion**

$L = 20$ in.

$d_1 = 1.2$ in.

$d_2 = 1.6$ in.



$T = 5800$ lb-in.

$\phi = 3.63° = 0.063355$ rad

$$I_P = \frac{\pi}{32}(d_2^4 - d_1^4) = 0.43982 \text{ in.}^4$$

MAXIMUM SHEAR STRESS

$$\tau_{max} = \frac{Tr}{I_P} = \frac{(5800 \text{ lb-in.})(0.8 \text{ in.})}{0.43982 \text{ in.}^4}$$

$$\tau_{max} = 10{,}550 \text{ psi} \longleftarrow$$

SHEAR MODULUS OF ELASTICITY

$$\phi = \frac{TL}{GI_P} \qquad G = \frac{TL}{\phi I_P}$$

$$G = \frac{(5800 \text{ lb-in.})(20 \text{ in.})}{(0.063355 \text{ rad})(0.43982 \text{ in.}^4)}$$

$$G = 4.16 \times 10^6 \text{ psi} \longleftarrow$$

MAXIMUM SHEAR STRAIN

$$\gamma_{max} = \frac{\tau_{max}}{G}$$

$$\gamma_{max} = \left(\frac{T_r}{I_P}\right)\left(\frac{\phi I_P}{TL}\right) = \frac{r\phi}{L}$$

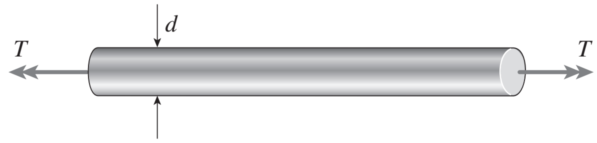
$$\gamma_{max} = \frac{(0.8 \text{ in.})(0.063355 \text{ rad})}{20 \text{ in.}}$$

$$\gamma_{max} = 0.00253 \text{ rad} \longleftarrow$$

Problem 3.3-8 A propeller shaft for a small yacht is made of a solid steel bar 100 mm in diameter. The allowable stress in shear is 50 MPa, and the allowable rate of twist is 2.0° in 3 meters.

Assuming that the shear modulus of elasticity is $G = 80$ GPa, determine the maximum torque T_{max} that can be applied to the shaft.

Solution 3.3-8 **Propeller shaft**



$d = 100$ mm

$G = 80$ GPa $\qquad \tau_{allow} = 50$ MPa

$$\theta = 2° \text{ in } 3 \text{ m} = \frac{1}{3}(2°)\left(\frac{\pi}{180}\right) \text{rad/m}$$

$$= 0.011636 \text{ rad/m}$$

MAX. TORQUE BASED UPON SHEAR STRESS

$$\tau = \frac{16T}{\pi d^3} \qquad T_1 = \frac{\pi d^3 \tau_{allow}}{16}$$

$$= \frac{\pi(100 \text{ mm})^3(50 \text{ MPa})}{16}$$

$$T_1 = 9820 \text{ N} \cdot \text{m} \longleftarrow$$

MAX. TORQUE BASED UPON RATE OF TWIST

$$\theta = \frac{T}{GI_P} \qquad T_2 = GI_P\theta = G\left(\frac{\pi d^4}{32}\right)\theta$$

$$= (80 \text{ GPa})\left(\frac{\pi}{32}\right)(100 \text{ mm})^4(0.011636 \text{ rad/m})$$

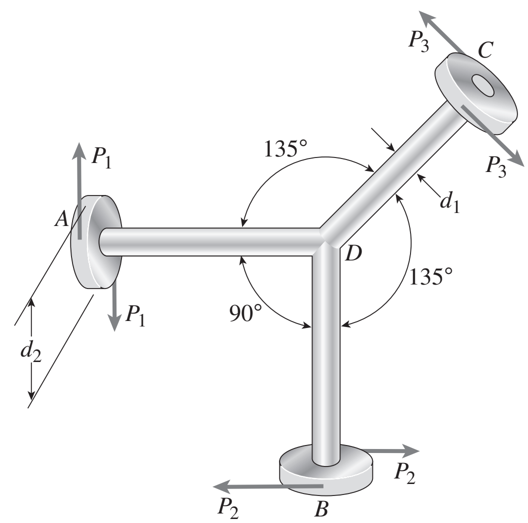
$$T_2 = 9140 \text{ N} \cdot \text{m} \longleftarrow$$

RATE OF TWIST GOVERNS

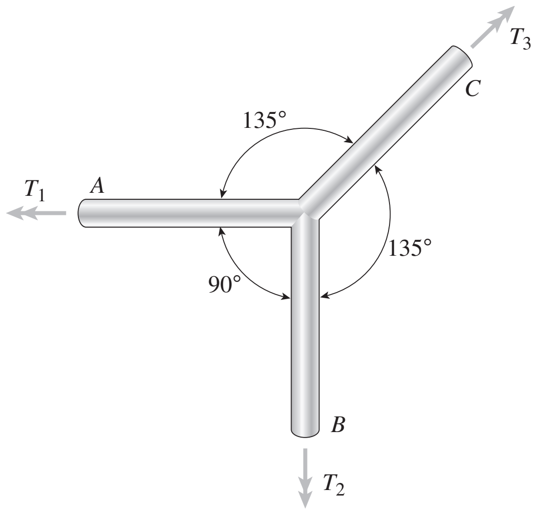
$$T_{max} = 9140 \text{ N} \cdot \text{m} \longleftarrow$$

Problem 3.3-9 Three identical circular disks A, B, and C are welded to the ends of three identical solid circular bars (see figure). The bars lie in a common plane and the disks lie in planes perpendicular to the axes of the bars. The bars are welded at their intersection D to form a rigid connection. Each bar has diameter $d_1 = 0.5$ in. and each disk has diameter $d_2 = 3.0$ in.

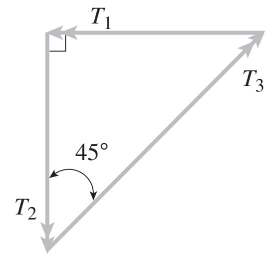
Forces P_1, P_2, and P_3 act on disks A, B, and C, respectively, thus subjecting the bars to torsion. If $P_1 = 28$ lb, what is the maximum shear stress τ_{max} in any of the three bars?



Solution 3.3-9 Three circular bars



THE THREE TORQUES MUST BE IN EQUILIBRIUM



d_1 = diameter of bars

= 0.5 in.

d_2 = diameter of disks

= 3.0 in.

$P_1 = 28$ lb

$T_1 = P_1 d_2 \qquad T_2 = P_2 d_2 \qquad T_3 = P_3 d_2$

T_3 is the largest torque

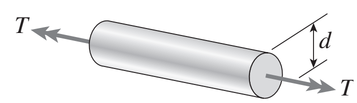
$T_3 = T_1\sqrt{2} = P_1 d_2 \sqrt{2}$

MAXIMUM SHEAR STRESS (Eq. 3-12)

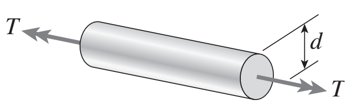
$$\tau_{\max} = \frac{16T}{\pi d^3} = \frac{16T_3}{\pi d_1^3} = \frac{16P_1 d_2\sqrt{2}}{\pi d_1^3}$$

$$\tau_{\max} = \frac{16(28\text{ lb})(3.0\text{ in.})\sqrt{2}}{\pi(0.5\text{ in.})^3} = 4840\text{ psi} \longleftarrow$$

Problem 3.3-10 The steel axle of a large winch on an ocean liner is subjected to a torque of 1.5 kN·m (see figure). What is the minimum required diameter $d_{\min}$ if the allowable shear stress is 50 MPa and the allowable rate of twist is 0.8°/m? (Assume that the shear modulus of elasticity is 80 GPa.)



Solution 3.3-10 Axle of a large winch



$T = 1.5\text{ kN}\cdot\text{m}$

$G = 80\text{ GPa}$

$\tau_{\text{allow}} = 50\text{ MPa}$

$$\theta_{\text{allow}} = 0.8°/\text{m} = (0.8°)\left(\frac{\pi}{180}\right)\text{rad/m}$$

$$= 0.013963\text{ rad/m}$$

MIN. DIAMETER BASED UPON SHEAR STRESS

$$\tau = \frac{16T}{\pi d^3} \qquad d^3 = \frac{16T}{\pi \tau_{\text{allow}}}$$

$$d^3 = \frac{16(1.5\text{ kN}\cdot\text{m})}{\pi(50\text{ MPa})} = 152.789 \times 10^{-6}\text{ m}^3$$

$d = 0.05346$ m $\quad d_{\min} = 53.5$ mm

MIN. DIAMETER BASED UPON RATE OF TWIST

$$\theta = \frac{T}{GI_p} = \frac{32T}{G\pi d^4} \qquad d^4 = \frac{32T}{\pi G\theta_{\text{allow}}}$$

$$d^4 = \frac{32(1.5\text{ kN}\cdot\text{m})}{\pi(80\text{ GPa})(0.013963\text{ rad/m})}$$

$$= 0.00001368\text{ m}^4$$

$d = 0.0608$ m $\quad d_{\min} = 60.8$ mm

RATE OF TWIST GOVERNS

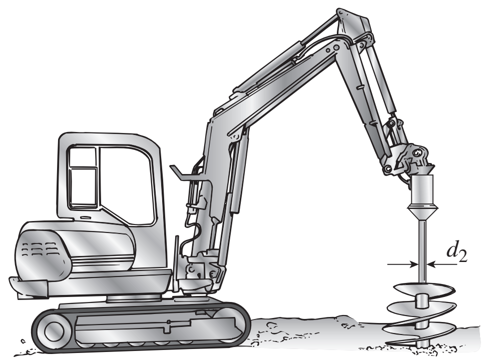
$d_{\min} = 60.8$ mm $\longleftarrow$

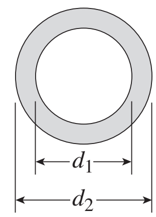
Problem 3.3-11 A hollow steel shaft used in a construction auger has outer diameter $d_2 = 6.0$ in. and inner diameter $d_1 = 4.5$ in. (see figure). The steel has shear modulus of elasticity $G = 11.0 \times 10^6$ psi.

For an applied torque of 150 k-in., determine the following quantities:

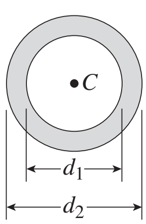
(a) shear stress τ_2 at the outer surface of the shaft,
(b) shear stress τ_1 at the inner surface, and
(c) rate of twist θ (degrees per unit of length).

Also, draw a diagram showing how the shear stresses vary in magnitude along a radial line in the cross section.





Solution 3.3-11 **Construction auger**



$d_2 = 6.0$ in. $\quad r_2 = 3.0$ in.

$d_1 = 4.5$ in. $\quad r_1 = 2.25$ in.

$G = 11 \times 10^6$ psi

$T = 150$ k-in.

$$I_P = \frac{\pi}{32}(d_2^4 - d_1^4) = 86.98 \text{ in.}^4$$

(a) Shear stress at outer surface

$$\tau_2 = \frac{Tr_2}{I_P} = \frac{(150 \text{ k-in.})(3.0 \text{ in.})}{86.98 \text{ in.}^4}$$

$$= 5170 \text{ psi} \quad \longleftarrow$$

(b) Shear stress at inner surface

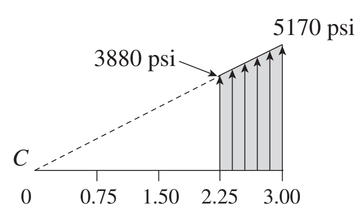
$$\tau_1 = \frac{Tr_1}{I_P} = \frac{r_1}{r_2}\tau_2 = 3880 \text{ psi} \quad \longleftarrow$$

(c) Rate of twist

$$\theta = \frac{T}{GI_P} = \frac{(150 \text{ k-in.})}{(11 \times 10^6 \text{ psi})(86.98 \text{ in.}^4)}$$

$$\theta = 157 \times 10^{-6} \text{ rad/in.} = 0.00898°/\text{in.} \quad \longleftarrow$$

(d) Shear stress diagram



Problem 3.3-12 Solve the preceding problem if the shaft has outer diameter $d_2 = 150$ mm and inner diameter $d_1 = 100$ mm. Also, the steel has shear modulus of elasticity $G = 75$ GPa and the applied torque is 16 kN·m.

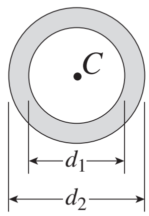
Solution 3.3-12 Construction auger

$d_2 = 150$ mm $\quad r_2 = 75$ mm

$d_1 = 100$ mm $\quad r_1 = 50$ mm

$G = 75$ GPa

$T = 16$ kN · m



$$I_P = \frac{\pi}{32}(d_2^4 - d_1^4) = 39.88 \times 10^6 \text{ mm}^4$$

(a) Shear stress at outer surface

$$\tau_2 = \frac{Tr_2}{I_P} = \frac{(16 \text{ kN} \cdot \text{m})(75 \text{ mm})}{39.88 \times 10^6 \text{ mm}^4}$$

$$= 30.1 \text{ MPa} \longleftarrow$$

(b) Shear stress at inner surface

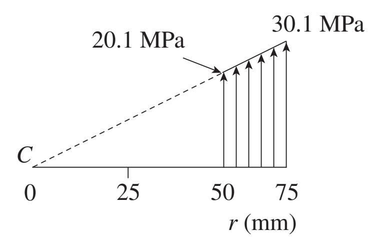
$$\tau_1 = \frac{Tr_1}{I_P} = \frac{r_1}{r_2}\tau_2 = 20.1 \text{ MPa} \longleftarrow$$

(c) Rate of twist

$$\theta = \frac{T}{GI_P} = \frac{16 \text{ kN} \cdot \text{m}}{(75 \text{ GPa})(39.88 \times 10^6 \text{ mm}^4)}$$

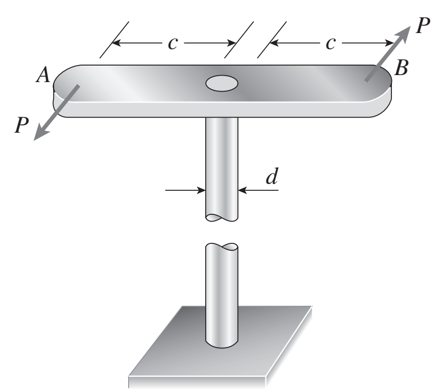
$$\theta = 0.005349 \text{ rad/m} = 0.306°/\text{m} \longleftarrow$$

(d) Shear stress diagram

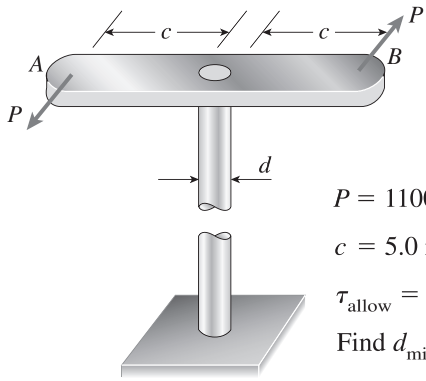


Problem 3.3-13 A vertical pole of solid circular cross section is twisted by horizontal forces $P = 1100$ lb acting at the ends of a horizontal arm AB (see figure). The distance from the outside of the pole to the line of action of each force is $c = 5.0$ in.

If the allowable shear stress in the pole is 4500 psi, what is the minimum required diameter d_{min} of the pole?



Solution 3.3-13 **Vertical pole**



$P = 1100$ lb

$c = 5.0$ in.

$\tau_{allow} = 4500$ psi

Find d_{min}

TORSION FORMULA

$$\tau_{max} = \frac{Tr}{I_P} = \frac{Td}{2I_P}$$

$$T = P(2c + d) \quad I_P = \frac{\pi d^4}{32}$$

$$\tau_{max} = \frac{P(2c + d)d}{\pi d^4/16} = \frac{16P(2c + d)}{\pi d^3}$$

$$(\pi\tau_{max})d^3 - (16P)d - 32Pc = 0$$

SUBSTITUTE NUMERICAL VALUES:

UNITS: Pounds, Inches

$$(\pi)(4500)d^3 - (16)(1100)d - 32(1100)(5.0) = 0$$

or

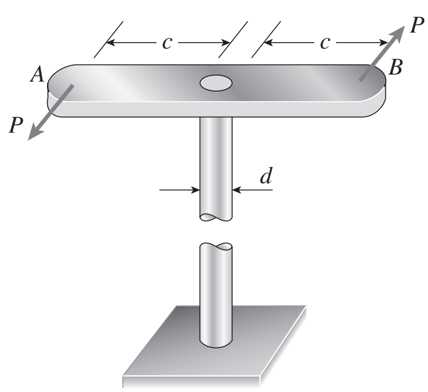
$$d^3 - 1.24495d - 12.4495 = 0$$

Solve numerically: $d = 2.496$ in.

$d_{min} = 2.50$ in. ⟵

Problem 3.3-14 Solve the preceding problem if the horizontal forces have magnitude $P = 5.0$ kN, the distance $c = 125$ mm, and the allowable shear stress is 30 MPa.

Solution 3.3-14 **Vertical pole**



$P = 5.0$ kN

$c = 125$ mm

$\tau_{allow} = 30$ MPa

Find d_{min}

TORSION FORMULA

$$\tau_{max} = \frac{Tr}{I_P} = \frac{Td}{2I_P}$$

$$T = P(2c + d) \quad I_P = \frac{\pi d^4}{32}$$

$$\tau_{max} = \frac{P(2c + d)d}{\pi d^4/16} = \frac{16P(2c + d)}{\pi d^3}$$

$$(\pi\tau_{max})d^3 - (16P)d - 32Pc = 0$$

SUBSTITUTE NUMERICAL VALUES:

UNITS: Newtons, Meters

$$(\pi)(30 \times 10^6)d^3 - (16)(5000)d - 32(5000)(0.125) = 0$$

or

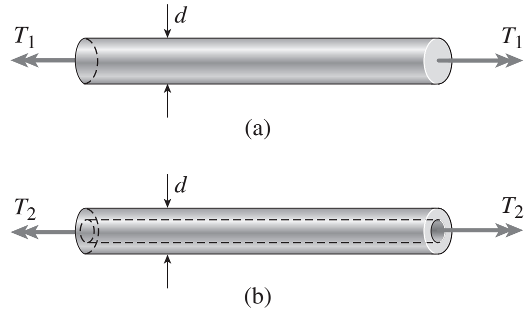
$$d^3 - 848.826 \times 10^{-6}d - 212.207 \times 10^{-6} = 0$$

Solve numerically: $d = 0.06438$ m

$d_{min} = 64.4$ mm ⟵

Problem 3.3-15 A solid brass bar of diameter $d = 1.2$ in. is subjected to torques T_1, as shown in part (a) of the figure. The allowable shear stress in the brass is 12 ksi.

(a) What is the maximum permissible value of the torques T_1?
(b) If a hole of diameter 0.6 in. is drilled longitudinally through the bar, as shown in part (b) of the figure, what is the maximum permissible value of the torques T_2?
(c) What is the percent decrease in torque and the percent decrease in weight due to the hole?

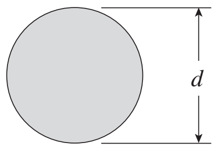


Solution 3.3-15 Brass bar in torsion

(a) SOLID BAR

$d = 1.2$ in.

$\tau_{allow} = 12$ ksi

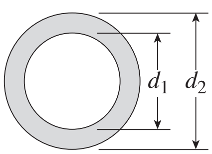


Find max. torque T_1

$$\tau_{max} = \frac{16T}{\pi d^3} \qquad T_1 = \frac{\pi d^3 \tau_{allow}}{16}$$

$$T_1 = \frac{\pi(1.2 \text{ in.})^3(12 \text{ ksi})}{16}$$

$$= 4072 \text{ lb-in.} \longleftarrow$$



(b) BAR WITH A HOLE

$d_2 = d = 1.2$ in.

$d_1 = 0.6$ in.

$$\tau_{max} = \frac{Tr}{I_P} = \frac{Td/2}{\frac{\pi}{32}(d_2^4 - d_1^4)} = \frac{16Td_2}{\pi(d_2^4 - d_1^4)}$$

$$T_2 = \frac{\pi(d_2^4 - d_1^4)\tau_{allow}}{16d_2}$$

$$T_2 = \frac{\pi[(1.2 \text{ in.})^4 - (0.6 \text{ in.})^4](12 \text{ ksi})}{16(1.2 \text{ in.})}$$

$$T_2 = 3817 \text{ lb-in.} \longleftarrow$$

(c) PERCENT DECREASE IN TORQUE

$$\frac{T_2}{T_1} = \frac{\pi(d_2^4 - d_1^4)\tau_{allow}}{16d_2} \cdot \frac{16}{\pi d_2^3 \tau_{allow}} = 1 - \left(\frac{d_1}{d_2}\right)^4$$

$$\frac{d_1}{d_2} = \frac{1}{2} \qquad \frac{T_2}{T_1} = 0.9375$$

% decrease = 6.25% $\longleftarrow$

PERCENT DECREASE IN WEIGHT

$$\frac{W_2}{W_1} = \frac{A_2}{A_1} = \frac{d_2^2 - d_1^2}{d_2^2} = 1 - \left(\frac{d_1}{d_2}\right)^2$$

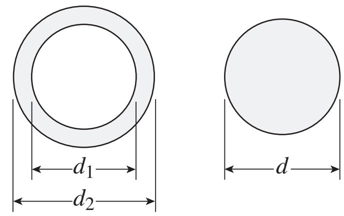
$$\frac{d_1}{d_2} = \frac{1}{2} \qquad \frac{W_2}{W_1} = \frac{3}{4}$$

% decrease = 25% $\longleftarrow$

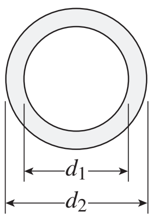
NOTE: The hollow bar weighs 25% less than the solid bar with only a 6.25% decrease in strength.

Problem 3.3-16 A hollow aluminum tube used in a roof structure has an outside diameter $d_2 = 100$ mm and an inside diameter $d_1 = 80$ mm (see figure). The tube is 2.5 m long, and the aluminum has shear modulus $G = 28$ GPa.

(a) If the tube is twisted in pure torsion by torques acting at the ends, what is the angle of twist ϕ (in degrees) when the maximum shear stress is 50 MPa?
(b) What diameter d is required for a solid shaft (see figure) to resist the same torque with the same maximum stress?
(c) What is the ratio of the weight of the hollow tube to the weight of the solid shaft?



Solution 3.3-16 **Hollow aluminum tube**



$d_2 = 100$ mm

$d_1 = 80$ mm

$L = 2.5$ m

$G = 28$ GPa

$\tau_{max} = 50$ MPa

(a) ANGLE OF TWIST FOR THE TUBE

$$\tau_{max} = \frac{Tr}{I_p} = \frac{Td_2}{2I_p}, \qquad T = \frac{2I_p\tau_{max}}{d_2}$$

$$\phi = \frac{TL}{GI_p} = \left(\frac{2I_p\tau_{max}}{d_2}\right)\left(\frac{L}{GI_p}\right)$$

$$\phi = \frac{2\tau_{max}L}{Gd_2}$$

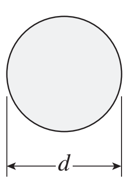
$$\phi = \frac{2(50\text{ MPa})(2.5\text{ m})}{(28\text{ GPa})(100\text{ mm})} = 0.08929\text{ rad}$$

$$\phi = 5.12° \longleftarrow$$

(b) DIAMETER OF A SOLID SHAFT

τ_{max} is the same as for tube.

Torque is the same.



For the tube: $T = \dfrac{2I_P\tau_{max}}{d_2}$

$$T = \frac{2\tau_{max}}{d_2}\left(\frac{\pi}{32}\right)(d_2^4 - d_1^4)$$

FOR THE SOLID SHAFT:

$$\tau_{max} = \frac{16T}{\pi d^3} = \frac{16}{\pi d^3}\left(\frac{2\tau_{max}}{d_2}\right)\left(\frac{\pi}{32}\right)(d_2^4 - d_1^4)$$

Solve for d^3: $d^3 = \dfrac{d_2^4 - d_1^4}{d_2}$

$$d^3 = \frac{(100\text{ mm})^4 - (80\text{ mm})^4}{100\text{ mm}} = 590{,}400\text{ mm}^3$$

$$d = 83.9\text{ mm} \longleftarrow$$

(c) RATIO OF WEIGHTS

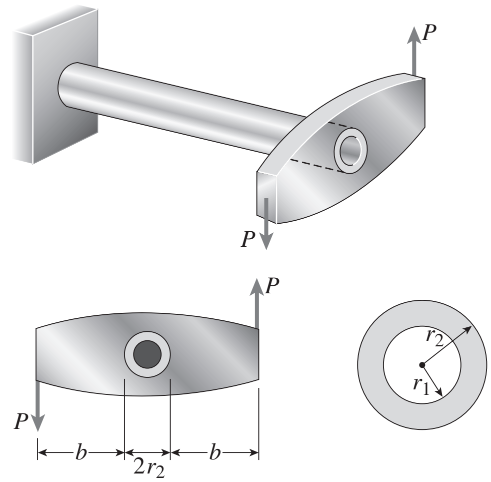
$$\frac{W_{tube}}{W_{solid}} = \frac{A_{tube}}{A_{solid}} = \frac{d_2^2 - d_1^2}{d^2}$$

$$\frac{W_{tube}}{W_{solid}} = \frac{(100\text{ mm})^2 - (80\text{ mm})^2}{(83.9\text{ mm})^2} = 0.51 \longleftarrow$$

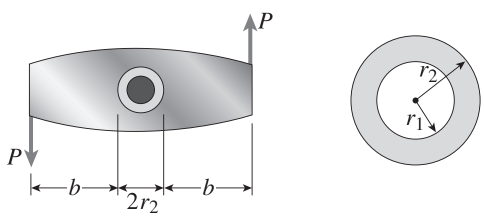
The weight of the tube is 51% of the weight of the solid shaft, but they resist the same torque.

Problem 3.3-17 A circular tube of inner radius r_1 and outer radius r_2 is subjected to a torque produced by forces $P = 900$ lb (see figure). The forces have their lines of action at a distance $b = 5.5$ in. from the outside of the tube.

If the allowable shear stress in the tube is 6300 psi and the inner radius $r_1 = 1.2$ in., what is the minimum permissible outer radius r_2?



Solution 3.3-17 Circular tube in torsion



$P = 900$ lb

$b = 5.5$ in.

$\tau_{allow} = 6300$ psi

$r_1 = 1.2$ in.

Find minimum permissible radius r_2

TORSION FORMULA

$T = 2P(b + r_2)$

$$I_P = \frac{\pi}{2}(r_2^4 - r_1^4)$$

$$\tau_{max} = \frac{Tr_2}{I_P} = \frac{2P(b + r_2)r_2}{\frac{\pi}{2}(r_2^4 - r_1^4)} = \frac{4P(b + r_2)r_2}{\pi(r_2^4 - r_1^4)}$$

All terms in this equation are known except r_2.

SOLUTION OF EQUATION

UNITS: Pounds, Inches

Substitute numerical values:

$$6300 \text{ psi} = \frac{4(900 \text{ lb})(5.5 \text{ in.} + r_2)(r_2)}{\pi[(r_2^4) - (1.2 \text{ in.})^4]}$$

or

$$\frac{r_2^4 - 2.07360}{r_2(r_2 + 5.5)} - 0.181891 = 0$$

or

$$r_2^4 - 0.181891\, r_2^2 - 1.000402\, r_2 - 2.07360 = 0$$

Solve numerically:

$r_2 = 1.3988$ in.

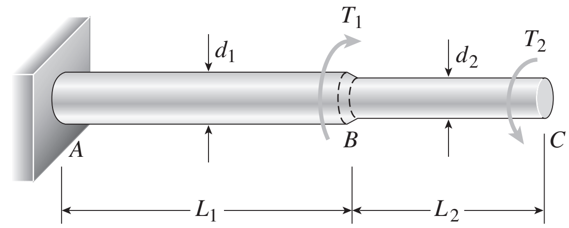
MINIMUM PERMISSIBLE RADIUS

$r_2 = 1.40$ in. ←

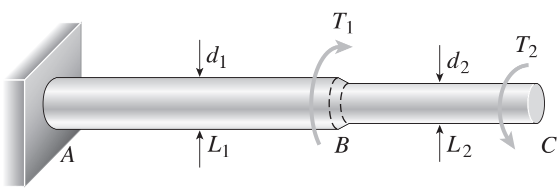
Nonuniform Torsion

Problem 3.4-1 A stepped shaft *ABC* consisting of two solid circular segments is subjected to torques T_1 and T_2 acting in opposite directions, as shown in the figure. The larger segment of the shaft has diameter $d_1 = 2.25$ in. and length $L_1 = 30$ in.; the smaller segment has diameter $d_2 = 1.75$ in. and length $L_2 = 20$ in. The material is steel with shear modulus $G = 11 \times 10^6$ psi, and the torques are $T_1 = 20{,}000$ lb-in. and $T_2 = 8{,}000$ lb-in.

Calculate the following quantities: (a) the maximum shear stress τ_{max} in the shaft, and (b) the angle of twist ϕ_C (in degrees) at end *C*.



Solution 3.4-1 **Stepped shaft**



$d_1 = 2.25$ in. $\quad L_1 = 30$ in.

$d_2 = 1.75$ in. $\quad L_2 = 20$ in.

$G = 11 \times 10^6$ psi

$T_1 = 20{,}000$ lb-in.

$T_2 = 8{,}000$ lb-in.

SEGMENT *AB*

$T_{AB} = T_2 - T_1 = -12{,}000$ lb-in.

$$\tau_{AB} = \left|\frac{16\,T_{AB}}{\pi d_1^3}\right| = \frac{16(12{,}000\text{ lb-in.})}{\pi(2.25\text{ in.})^3} = 5365\text{ psi}$$

$$\phi_{AB} = \frac{T_{AB}L_1}{G(I_p)_{AB}} = \frac{(-12{,}000\text{ lb-in.})(30\text{ in.})}{(11 \times 10^6\text{ psi})\left(\frac{\pi}{32}\right)(2.25\text{ in.})^4}$$

$$= -0.013007\text{ rad}$$

SEGMENT *BC*

$T_{BC} = +T_2 = 8{,}000$ lb-in.

$$\tau_{BC} = \frac{16\,T_{BC}}{\pi d_2^3} = \frac{16(8{,}000\text{ lb-in.})}{\pi(1.75\text{ in.})^3} = 7602\text{ psi}$$

$$\phi_{BC} = \frac{T_{BC}L_2}{G(I_p)_{BC}} = \frac{(8{,}000\text{ lb-in.})(20\text{ in.})}{(11 \times 10^6\text{ psi})\left(\frac{\pi}{32}\right)(1.75\text{ in.})^4}$$

$$= +0.015797\text{ rad}$$

(a) MAXIMUM SHEAR STRESS

Segment *BC* has the maximum stress

$\tau_{max} = 7600$ psi ←

(b) ANGLE OF TWIST AT END *C*

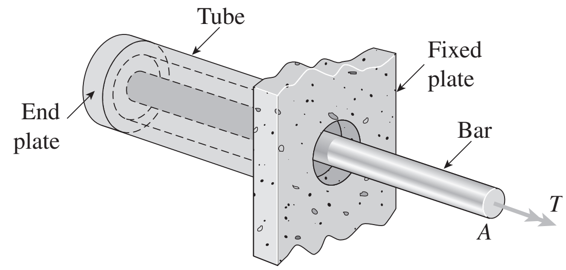
$\phi_C = \phi_{AB} + \phi_{BC} = (-0.013007 + 0.015797)$ rad

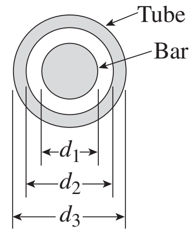
$\phi_C = 0.002790$ rad $= 0.16°$ ←

Problem 3.4-2 A circular tube of outer diameter $d_3 = 70$ mm and inner diameter $d_2 = 60$ mm is welded at the right-hand end to a fixed plate and at the left-hand end to a rigid end plate (see figure). A solid circular bar of diameter $d_1 = 40$ mm is inside of, and concentric with, the tube. The bar passes through a hole in the fixed plate and is welded to the rigid end plate.

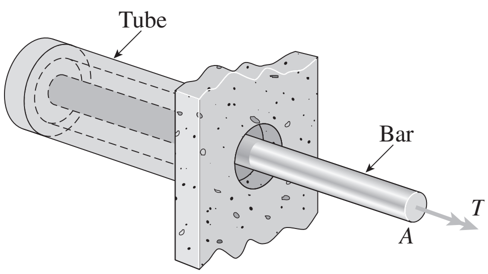
The bar is 1.0 m long and the tube is half as long as the bar. A torque $T = 1000$ N · m acts at end A of the bar. Also, both the bar and tube are made of an aluminum alloy with shear modulus of elasticity $G = 27$ GPa.

(a) Determine the maximum shear stresses in both the bar and tube.
(b) Determine the angle of twist (in degrees) at end A of the bar.





Solution 3.4-2 Bar and tube



TUBE

$d_3 = 70$ mm $\quad d_2 = 60$ mm

$L_{\text{tube}} = 0.5$ m $\quad G = 27$ GPa

$$(I_p)_{\text{tube}} = \frac{\pi}{32}(d_3^4 - d_2^4)$$

$$= 1.0848 \times 10^6 \text{ mm}^4$$

BAR

$d_1 = 40$ mm $\quad L_{\text{bar}} = 1.0$ m $\quad G = 27$ GPa

$$(I_p)_{\text{bar}} = \frac{\pi d_1^4}{32} = 251.3 \times 10^3 \text{ mm}^4$$

TORQUE

$T = 1000$ N · m

(a) MAXIMUM SHEAR STRESSES

$$\text{Bar: } \tau_{\text{bar}} = \frac{16T}{\pi d_1^3} = 79.6 \text{ MPa} \quad \longleftarrow$$

$$\text{Tube: } \tau_{\text{tube}} = \frac{T(d_3/2)}{(I_p)_{\text{tube}}} = 32.3 \text{ MPa} \quad \longleftarrow$$

(b) ANGLE OF TWIST AT END A

$$\text{Bar: } \phi_{\text{bar}} = \frac{TL_{\text{bar}}}{G(I_p)_{\text{bar}}} = 0.1474 \text{ rad}$$

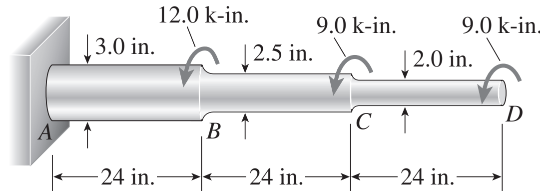
$$\text{Tube: } \phi_{\text{tube}} = \frac{TL_{\text{tube}}}{G(I_p)_{\text{tube}}} = 0.0171 \text{ rad}$$

$$\phi_A = \phi_{\text{bar}} + \phi_{\text{tube}} = 0.1474 + 0.0171 = 0.1645 \text{ rad}$$

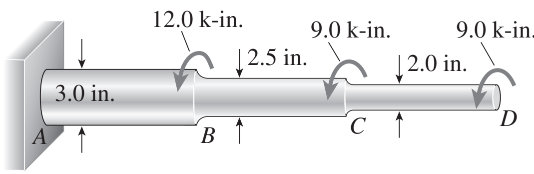
$$\phi_A = 9.43° \quad \longleftarrow$$

Problem 3.4-3 A stepped shaft *ABCD* consisting of solid circular segments is subjected to three torques, as shown in the figure. The torques have magnitudes 12.0 k-in., 9.0 k-in., and 9.0 k-in. The length of each segment is 24 in. and the diameters of the segments are 3.0 in., 2.5 in., and 2.0 in. The material is steel with shear modulus of elasticity $G = 11.6 \times 10^3$ ksi.

(a) Calculate the maximum shear stress τ_{max} in the shaft.
(b) Calculate the angle of twist ϕ_D (in degrees) at end *D*.



Solution 3.4-3 **Stepped shaft**



$G = 11.6 \times 10^3$ ksi

$r_{AB} = 1.5$ in.

$r_{BC} = 1.25$ in. $\quad r_{CD} = 1.0$ in.

$L_{AB} = L_{BC} = L_{CD} = 24$ in.

TORQUES

$T_{AB} = 12.0 + 9.0 + 9.0 = 30$ k-in.

$T_{BC} = 9.0 + 9.0 = 18$ k-in.

$T_{CD} = 9.0$ k-in.

POLAR MOMENTS OF INERTIA

$$(I_p)_{AB} = \frac{\pi}{32}(3.0 \text{ in.})^4 = 7.952 \text{ in.}^4$$

$$(I_p)_{BC} = \frac{\pi}{32}(2.5 \text{ in.})^4 = 3.835 \text{ in.}^4$$

$$(I_p)_{CD} = \frac{\pi}{32}(2.0 \text{ in.})^4 = 1.571 \text{ in.}^4$$

(a) SHEAR STRESSES

$$\tau_{AB} = \frac{T_{AB}\, r_{AB}}{(I_p)_{AB}} = 5660 \text{ psi}$$

$$\tau_{BC} = \frac{T_{BC}\, r_{BC}}{(I_p)_{BC}} = 5870 \text{ psi}$$

$$\tau_{CD} = \frac{T_{CD}\, r_{CD}}{(I_p)_{CD}} = 5730 \text{ psi}$$

$\tau_{max} = 5870$ psi $\longleftarrow$

(b) ANGLE OF TWIST AT END *D*

$$\phi_{AB} = \frac{T_{AB} L_{AB}}{G(I_p)_{AB}} = 0.007805 \text{ rad}$$

$$\phi_{BC} = \frac{T_{BC} L_{BC}}{G(I_p)_{BC}} = 0.009711 \text{ rad}$$

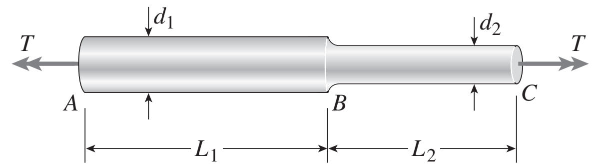
$$\phi_{CD} = \frac{T_{CD} L_{CD}}{G(I_p)_{CD}} = 0.011853 \text{ rad}$$

$\phi_D = \phi_{AB} + \phi_{BC} + \phi_{CD} = 0.02937$ rad

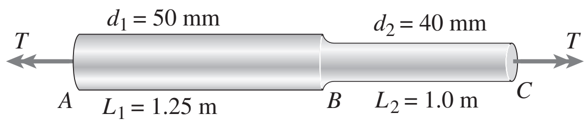
$\phi_D = 1.68°$ $\longleftarrow$

Problem 3.4-4 A solid circular bar *ABC* consists of two segments, as shown in the figure. One segment has diameter $d_1 = 50$ mm and length $L_1 = 1.25$ m; the other segment has diameter $d_2 = 40$ mm and length $L_2 = 1.0$ m.

What is the allowable torque T_{allow} if the shear stress is not to exceed 30 MPa and the angle of twist between the ends of the bar is not to exceed 1.5°? (Assume $G = 80$ GPa.)



Solution 3.4-4 **Bar consisting of two segments**



$\tau_{allow} = 30$ MPa

$\phi_{allow} = 1.5° = 0.02618$ rad

$G = 80$ GPa

ALLOWABLE TORQUE BASED UPON SHEAR STRESS

Segment *BC* has the smaller diameter and hence the larger stress.

$$\tau_{max} = \frac{16T}{\pi d^3} \qquad T_{allow} = \frac{\pi d_2^3 \tau_{allow}}{16} = 3.77 \text{ N} \cdot \text{m}$$

ALLOWABLE TORQUE BASED UPON ANGLE OF TWIST

$$\phi = \sum \frac{T_i L_i}{G I_{Pi}} = \frac{TL_1}{GI_{P1}} + \frac{TL_2}{GI_{P2}} = \frac{T}{G}\left(\frac{L_1}{I_{P1}} + \frac{L_2}{I_{P2}}\right)$$

$$\phi = \frac{32T}{\pi G}\left(\frac{L_1}{d_1^4} + \frac{L_2}{d_2^4}\right)$$

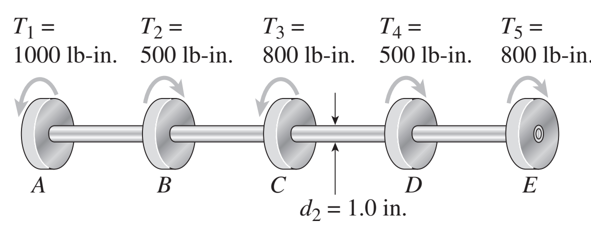
$$T_{allow} = \frac{\pi \phi_{allow} G}{32\left(\frac{L_1}{d_1^4} + \frac{L_2}{d_2^4}\right)} = 348 \text{ N} \cdot \text{m}$$

ANGLE OF TWIST GOVERNS

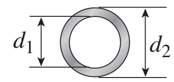
$T_{allow} = 348$ N · m $\longleftarrow$

Problem 3.4-5 A hollow tube *ABCDE* constructed of monel metal is subjected to five torques acting in the directions shown in the figure. The magnitudes of the torques are $T_1 = 1000$ lb-in., $T_2 = T_4 = 500$ lb-in., and $T_3 = T_5 = 800$ lb-in. The tube has an outside diameter $d_2 = 1.0$ in. The allowable shear stress is 12,000 psi and the allowable rate of twist is 2.0°/ft.

Determine the maximum permissible inside diameter d_1 of the tube.



Solution 3.4-5 Hollow tube of monel metal



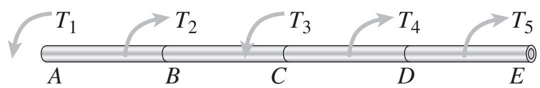
$d_2 = 1.0$ in. $\tau_{allow} = 12{,}000$ psi

$\theta_{allow} = 2°/\text{ft} = 0.16667°/\text{in.}$

$= 0.002909$ rad/in.

From Table H-2, Appendix H: $G = 9500$ ksi

TORQUES



$T_1 = 1000$ lb-in. $T_2 = 500$ lb-in. $T_3 = 800$ lb-in.

$T_4 = 500$ lb-in. $T_5 = 800$ lb-in.

INTERNAL TORQUES

$T_{AB} = -T_1 = -1000$ lb-in.

$T_{BC} = -T_1 + T_2 = -500$ lb-in.

$T_{CD} = -T_1 + T_2 - T_3 = -1300$ lb-in.

$T_{DE} = -T_1 + T_2 - T_3 + T_4 = -800$ lb-in.

Largest torque (absolute value only):

$T_{max} = 1300$ lb-in.

REQUIRED POLAR MOMENT OF INERTIA BASED UPON ALLOWABLE SHEAR STRESS

$$\tau_{max} = \frac{T_{max}r}{I_P} \qquad I_P = \frac{T_{max}(d_2/2)}{\tau_{allow}} = 0.05417 \text{ in.}^4$$

REQUIRED POLAR MOMENT OF INERTIA BASED UPON ALLOWABLE ANGLE OF TWIST

$$\theta = \frac{T_{max}}{GI_P} \qquad I_P = \frac{T_{max}}{G\theta_{allow}} = 0.04704 \text{ in.}^4$$

SHEAR STRESS GOVERNS

Required $I_P = 0.05417$ in.^{4}

$$I_P = \frac{\pi}{32}(d_2^4 - d_1^4)$$

$$d_1^4 = d_2^4 - \frac{32I_P}{\pi} = (1.0 \text{ in.})^4 - \frac{32(0.05417 \text{ in.}^4)}{\pi}$$

$$= 0.4482 \text{ in.}^4$$

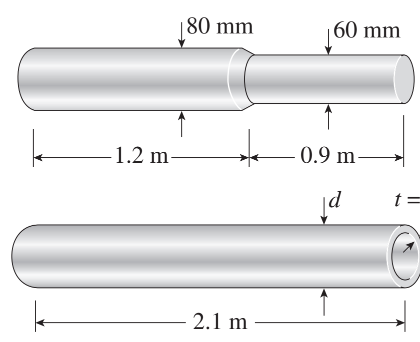
$d_1 = 0.818$ in. ⟵

(Maximum permissible inside diameter)

Problem 3.4-6 A shaft of solid circular cross section consisting of two segments is shown in the first part of the figure. The left-hand segment has diameter 80 mm and length 1.2 m; the right-hand segment has diameter 60 mm and length 0.9 m.

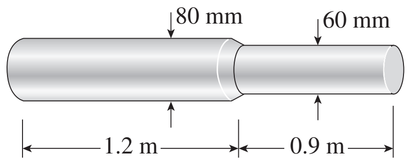
Shown in the second part of the figure is a hollow shaft made of the same material and having the same length. The thickness t of the hollow shaft is $d/10$, where d is the outer diameter. Both shafts are subjected to the same torque.

If the hollow shaft is to have the same torsional stiffness as the solid shaft, what should be its outer diameter d?



Solution 3.4-6 Solid and hollow shafts

SOLID SHAFT CONSISTING OF TWO SEGMENTS

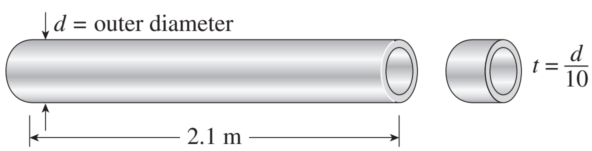


$$\phi_1 = \Sigma \frac{TL_i}{GI_{Pi}} = \frac{T(1.2\text{ m})}{G\left(\frac{\pi}{32}\right)(80\text{ mm})^4} + \frac{T(0.9\text{ m})}{G\left(\frac{\pi}{32}\right)(60\text{ mm})^4}$$

$$= \frac{32T}{\pi G}(29{,}297\text{ m}^{-3} + 69{,}444\text{ m}^{-3})$$

$$= \frac{32T}{\pi G}(98{,}741\text{ m}^{-3})$$

HOLLOW SHAFT



d_0 = inner diameter = $0.8d$

$$\phi_2 = \frac{TL}{GI_p} = \frac{T(2.1\text{ m})}{G\left(\frac{\pi}{32}\right)[d^4 - (0.8d)^4]}$$

$$= \frac{32T}{\pi G}\left(\frac{2.1\text{ m}}{0.5904\,d^4}\right) = \frac{32T}{\pi G}\left(\frac{3.5569\text{ m}}{d^4}\right)$$

UNITS: d = meters

TORSIONAL STIFFNESS

$k_T = \dfrac{T}{\phi}$ Torque T is the same for both shafts.

$\therefore$ For equal stiffnesses, $\phi_1 = \phi_2$

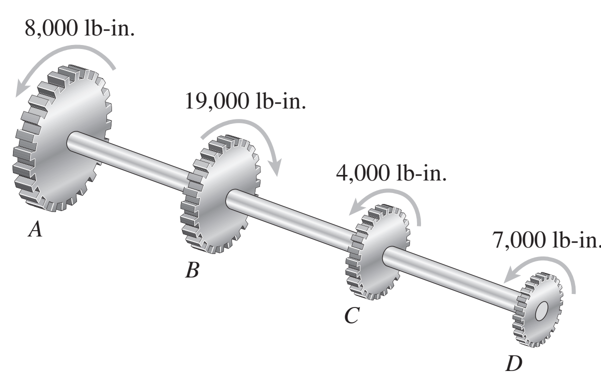
$$98{,}741\text{ m}^{-3} = \frac{3.5569\text{ m}}{d^4}$$

$$d^4 = \frac{3.5569}{98{,}741} = 36.023 \times 10^{-6}\text{ m}^4$$

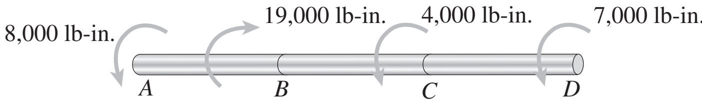
$$d = 0.0775\text{ m} = 77.5\text{ mm} \quad \longleftarrow$$

Problem 3.4-7 Four gears are attached to a circular shaft and transmit the torques shown in the figure. The allowable shear stress in the shaft is 10,000 psi.

(a) What is the required diameter d of the shaft if it has a solid cross section?
(b) What is the required outside diameter d if the shaft is hollow with an inside diameter of 1.0 in.?



Solution 3.4-7 **Shaft with four gears**



$\tau_{\text{allow}} = 10{,}000 \text{ psi}$ $\quad T_{BC} = +11{,}000 \text{ lb-in.}$

$T_{AB} = -8000 \text{ lb-in.}$ $\quad T_{CD} = +7000 \text{ lb-in.}$

(a) SOLID SHAFT

$$\tau_{\max} = \frac{16T}{\pi d^3}$$

$$d^3 = \frac{16T_{\max}}{\pi\tau_{\text{allow}}} = \frac{16(11{,}000 \text{ lb-in.})}{\pi(10{,}000 \text{ psi})} = 5.602 \text{ in.}^3$$

Required $d = 1.78$ in. ←

(b) HOLLOW SHAFT

Inside diameter $d_0 = 1.0$ in.

$$\tau_{\max} = \frac{Tr}{I_p} \qquad \tau_{\text{allow}} = \frac{T_{\max}\left(\frac{d}{2}\right)}{I_p}$$

$$10{,}000 \text{ psi} = \frac{(11{,}000 \text{ lb-in.})\left(\frac{d}{2}\right)}{\left(\frac{\pi}{32}\right)[d^4 - (1.0 \text{ in.})^4]}$$

UNITS: d = inches

$$10{,}000 = \frac{56{,}023\, d}{d^4 - 1}$$

or

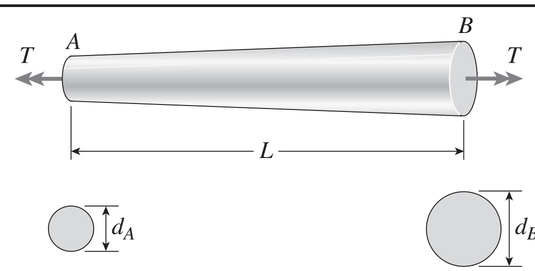
$$d_4 - 5.6023\, d - 1 = 0$$

Solving, $d = 1.832$

Required $d = 1.83$ in. ←

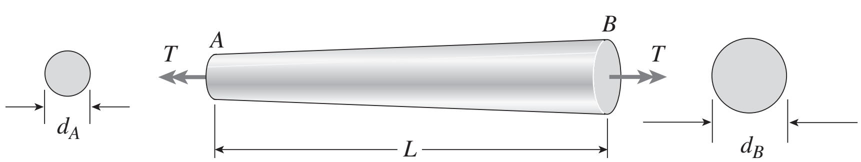
Problem 3.4-8 A tapered bar AB of solid circular cross section is twisted by torques T (see figure). The diameter of the bar varies linearly from d_A at the left-hand end to d_B at the right-hand end.

For what ratio d_B/d_A will the angle of twist of the tapered bar be one-half the angle of twist of a prismatic bar of diameter d_A? (The prismatic bar is made of the same material, has the same length, and is subjected to the same torque as the tapered bar.) *Hint*: Use the results of Example 3-5.



Problems 3.4-8, 3.4-9 and 3.4-10

Solution 3.4-8 **Tapered bar** ***AB***



TAPERED BAR (From Eq. 3-27)

$$\phi_1 = \frac{TL}{G(I_P)_A}\left(\frac{\beta^2 + \beta + 1}{3\beta^3}\right) \qquad \beta = \frac{d_B}{d_A}$$

PRISMATIC BAR

$$\phi_2 = \frac{TL}{G(I_P)_A}$$

ANGLE OF TWIST

$$\phi_1 = \frac{1}{2}\phi_2 \qquad \frac{\beta^2 + \beta + 1}{3\beta^3} = \frac{1}{2}$$

or $\quad 3\beta^3 - 2\beta^2 - 2\beta - 2 = 0$

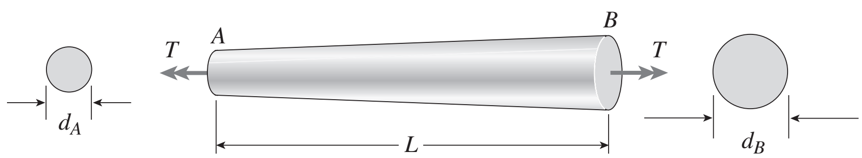
SOLVE NUMERICALLY:

$$\beta = \frac{d_B}{d_A} = 1.45$$ ←

Problem 3.4-9 A tapered bar AB of solid circular cross section is twisted by torques $T = 36{,}000$ lb-in. (see figure). The diameter of the bar varies linearly from d_A at the left-hand end to d_B at the right-hand end. The bar has length $L = 4.0$ ft and is made of an aluminum alloy having shear modulus of elasticity $G = 3.9 \times 10^6$ psi. The allowable shear stress in the bar is 15,000 psi and the allowable angle of twist is 3.0°.

If the diameter at end B is 1.5 times the diameter at end A, what is the minimum required diameter d_A at end A? (*Hint:* Use the results of Example 3-5).

Solution 3.4-9 **Tapered bar**



$d_B = 1.5\, d_A$

$T = 36{,}000$ lb-in.

$L = 4.0$ ft $= 48$ in.

$G = 3.9 \times 10^6$ psi

$\tau_{\text{allow}} = 15{,}000$ psi

$\phi_{\text{allow}} = 3.0°$

$= 0.0523599$ rad

MINIMUM DIAMETER BASED UPON ALLOWABLE SHEAR STRESS

$$\tau_{\max} = \frac{16T}{\pi d_A^3} \qquad d_A^3 = \frac{16T}{\pi \tau_{\text{allow}}} = \frac{16(36{,}000 \text{ lb-in.})}{\pi(15{,}000 \text{ psi})}$$

$$= 12.2231 \text{ in.}^3$$

$$d_A = 2.30 \text{ in.}$$

MINIMUM DIAMETER BASED UPON ALLOWABLE ANGLE OF TWIST (From Eq. 3-27)

$$\beta = d_B/d_A = 1.5$$

$$\phi = \frac{TL}{G(I_P)_A}\left(\frac{\beta^2 + \beta + 1}{3\beta^3}\right) = \frac{TL}{G(I_P)_A}(0.469136)$$

$$= \frac{(36{,}000 \text{ lb-in.})(48 \text{ in.})}{(3.9 \times 10^6 \text{ psi})\left(\frac{\pi}{32}\right) d_A^4}(0.469136)$$

$$= \frac{2.11728 \text{ in.}^4}{d_A^4}$$

$$d_A^4 = \frac{2.11728 \text{ in.}^4}{\phi_{\text{allow}}} = \frac{2.11728 \text{ in.}^4}{0.0523599 \text{ rad}}$$

$$= 40.4370 \text{ in.}^4$$

$$d_A = 2.52 \text{ in.}$$

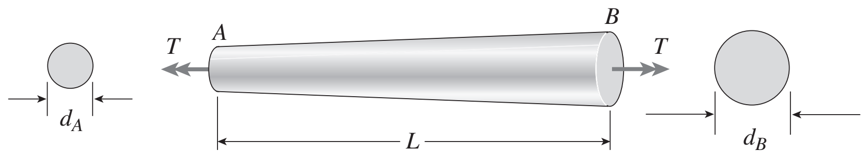
ANGLE OF TWIST GOVERNS

Min. $d_A = 2.52$ in. ←

Problem 3.4-10 The bar shown in the figure is tapered linearly from end A to end B and has a solid circular cross section. The diameter at the smaller end of the bar is $d_A = 25$ mm and the length is $L = 300$ mm. The bar is made of steel with shear modulus of elasticity $G = 82$ GPa.

If the torque $T = 180\ \text{N} \cdot \text{m}$ and the allowable angle of twist is 0.3°, what is the minimum allowable diameter d_B at the larger end of the bar? (*Hint:* Use the results of Example 3-5.)

Solution 3.4-10 **Tapered bar**



$d_A = 25$ mm

$L = 300$ mm

$G = 82$ GPa

$T = 180\ \text{N} \cdot \text{m}$

$\phi_{\text{allow}} = 0.3°$

Find d_B

DIAMETER BASED UPON ALLOWABLE ANGLE OF TWIST

(From Eq. 3-27)

$$\beta = \frac{d_B}{d_A}$$

$$\phi = \frac{TL}{G(I_P)_A}\left(\frac{\beta^2 + \beta + 1}{3\beta^3}\right) \qquad (I_P)_A = \frac{\pi}{32}d_A^4$$

$$(0.3°)\left(\frac{\pi}{180}\,\frac{\text{rad}}{\text{degrees}}\right) = \frac{(180\ \text{N}\cdot\text{m})(0.3\ \text{m})}{(82\ \text{GPa})\left(\frac{\pi}{32}\right)(25\ \text{mm})^4}\left(\frac{\beta^2 + \beta + 1}{3\beta^3}\right)$$

$$0.304915 = \frac{\beta^2 + \beta + 1}{3\beta^3}$$

$$0.914745\beta^3 - \beta^2 - 1 = 0$$

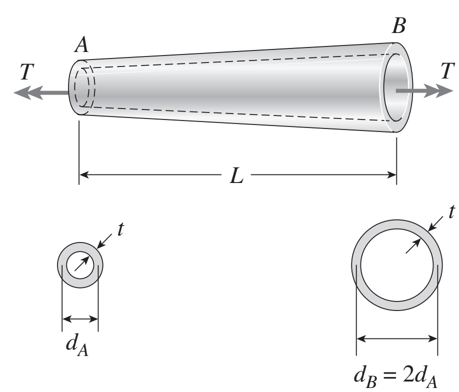
SOLVE NUMERICALLY:

$\beta = 1.94452$

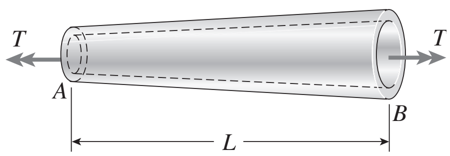
Min. $d_B = \beta d_A = 48.6$ mm ←

Problem 3.4-11 A uniformly tapered tube AB of hollow circular cross section is shown in the figure. The tube has constant wall thickness t and length L. The average diameters at the ends are d_A and $d_B = 2d_A$. The polar moment of inertia may be represented by the approximate formula $I_P \approx \pi d^3 t/4$ (see Eq. 3-18).

Derive a formula for the angle of twist ϕ of the tube when it is subjected to torques T acting at the ends.



Solution 3.4-11 **Tapered tube**

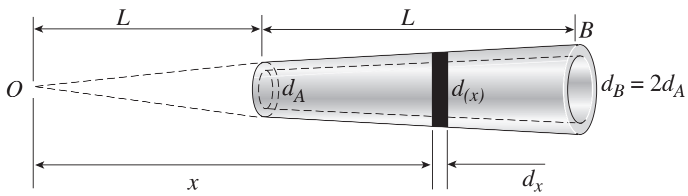


t = thickness (constant)

d_A, d_B = average diameters at the ends

$d_B = 2d_A \quad I_P = \dfrac{\pi d^3 t}{4}$ (approximate formula)

ANGLE OF TWIST



Take the origin of coordinates at point O.

$$d(x) = \frac{x}{2L}(d_B) = \frac{x}{L}d_A$$

$$I_P(x) = \frac{\pi[d(x)]^3 t}{4} = \frac{\pi t d_A^3}{4L^3}x^3$$

For element of length dx:

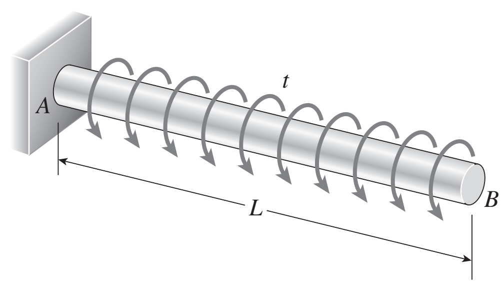
$$d\phi = \frac{T dx}{GI_P(x)} = \frac{T dx}{G\left(\dfrac{\pi t d_A^3}{4L^3}\right)x^3} = \frac{4TL^3}{\pi G t d_A^3}\cdot\frac{dx}{x^3}$$

For entire bar:

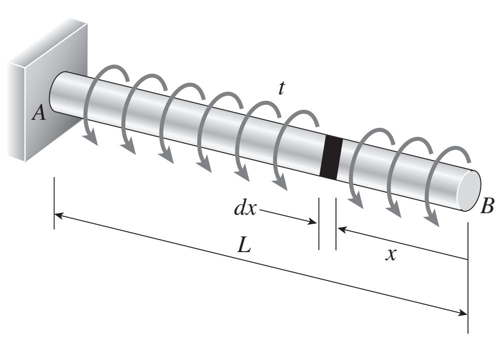
$$\phi = \int_L^{2L} d\phi = \frac{4TL^3}{\pi G t d_A^3}\int_L^{2L}\frac{dx}{x^3} = \frac{3TL}{2\pi G t d_A^3} \longleftarrow$$

Problem 3.4-12 A prismatic bar AB of length L and solid circular cross section (diameter d) is loaded by a distributed torque of constant intensity t per unit distance (see figure).

(a) Determine the maximum shear stress τ_{max} in the bar.
(b) Determine the angle of twist ϕ between the ends of the bar.



Solution 3.4-12 **Bar with distributed torque**



t = intensity of distributed torque

d = diameter

G = shear modulus of elasticity

(a) MAXIMUM SHEAR STRESS

$$T_{max} = tL \qquad \tau_{max} = \frac{16T_{max}}{\pi d^3} = \frac{16tL}{\pi d^3} \longleftarrow$$

(b) ANGLE OF TWIST

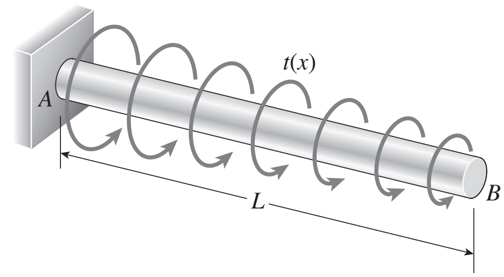
$$T(x) = tx \qquad I_P = \frac{\pi d^4}{32}$$

$$d\phi = \frac{T(x)dx}{GI_p} = \frac{32\,tx\,dx}{\pi G d^4}$$

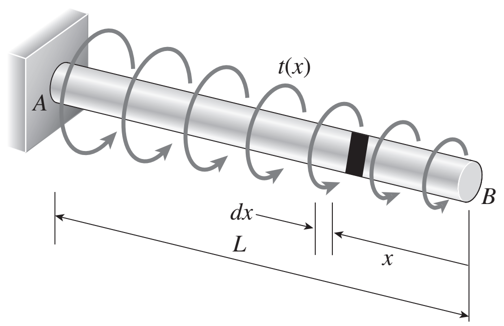
$$\phi = \int_0^L d\phi = \frac{32t}{\pi G d^4}\int_0^L x\,dx = \frac{16tL^2}{\pi G d^4} \longleftarrow$$

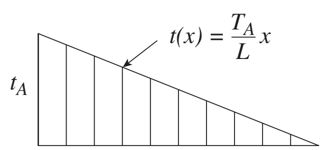
Problem 3.4-13 A prismatic bar AB of solid circular cross section (diameter d) is loaded by a distributed torque (see figure). The intensity of the torque, that is, the torque per unit distance, is denoted $t(x)$ and varies linearly from a maximum value t_A at end A to zero at end B. Also, the length of the bar is L and the shear modulus of elasticity of the material is G.

(a) Determine the maximum shear stress τ_{max} in the bar.
(b) Determine the angle of twist ϕ between the ends of the bar.



Solution 3.4-13 **Bar with linearly varying torque**





$t(x)$ = intensity of distributed torque

t_A = maximum intensity of torque

d = diameter

G = shear modulus

T_A = maximum torque

$= \frac{1}{2} t_A L$

(a) Maximum shear stress

$$\tau_{max} = \frac{16T_{max}}{\pi d^3} = \frac{16T_A}{\pi d^3} = \frac{8t_A L}{\pi d^3} \longleftarrow$$

(b) Angle of twist

$T(x)$ = torque at distance x from end B

$$T(x) = \frac{t(x)x}{2} = \frac{t_A x^2}{2L} \qquad I_P = \frac{\pi d^4}{32}$$

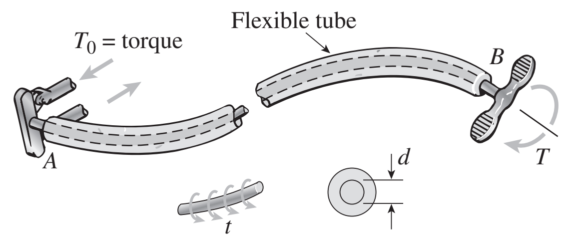
$$d\phi = \frac{T(x)\,dx}{GI_P} = \frac{16t_A x^2\,dx}{\pi G L d^4}$$

$$\phi = \int_0^L d\phi = \frac{16t_A}{\pi G L d^4}\int_0^L x^2 dx = \frac{16t_A L^2}{3\pi G d^4} \longleftarrow$$

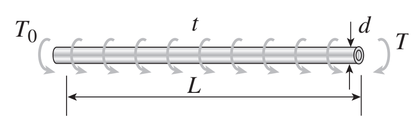
Problem 3.4-14 A magnesium-alloy wire of diameter $d = 4$ mm and length L rotates inside a flexible tube in order to open or close a switch from a remote location (see figure). A torque T is applied manually (either clockwise or counterclockwise) at end B, thus twisting the wire inside the tube. At the other end A, the rotation of the wire operates a handle that opens or closes the switch.

A torque $T_0 = 0.2$ N · m is required to operate the switch. The torsional stiffness of the tube, combined with friction between the tube and the wire, induces a distributed torque of constant intensity $t = 0.04$ N·m/m (torque per unit distance) acting along the entire length of the wire.

(a) If the allowable shear stress in the wire is $\tau_{allow} = 30$ MPa, what is the longest permissible length L_{max} of the wire?
(b) If the wire has length $L = 4.0$ m and the shear modulus of elasticity for the wire is $G = 15$ GPa, what is the angle of twist ϕ (in degrees) between the ends of the wire?



Solution 3.4-14 Wire inside a flexible tube



$d = 4$ mm

$T_0 = 0.2$ N · m

$t = 0.04$ N · m/m

(a) MAXIMUM LENGTH L_{max}

$\tau_{allow} = 30$ MPa

Equilibrium: $T = tL + T_0$

From Eq. (3-12): $\tau_{max} = \dfrac{16T}{\pi d^3}$ $\quad T = \dfrac{\pi d^3 \tau_{max}}{16}$

$$tL + T_0 = \frac{\pi d^3 \tau_{max}}{16}$$

$$L = \frac{1}{16t}(\pi d^3 \tau_{max} - 16T_0)$$

$$L_{max} = \frac{1}{16t}(\pi d^3 \tau_{allow} - 16T_0) \longleftarrow$$

Substitute numerical values: $L_{max} = 4.42$ m $\longleftarrow$

(b) ANGLE OF TWIST ϕ

$L = 4$ m $\qquad G = 15$ GPa

ϕ_1 = angle of twist due to distributed torque t

$$= \frac{16tL^2}{\pi G d^4} \text{ (from problem 3.4-12)}$$

ϕ_2 = angle of twist due to torque T_0

$$= \frac{T_0 L}{G I_P} = \frac{32\, T_0 L}{\pi G d^4} \text{ (from Eq. 3-15)}$$

ϕ = total angle of twist

$= \phi_1 + \phi_2$

$$\phi = \frac{16L}{\pi G d^4}(tL + 2T_0) \longleftarrow$$

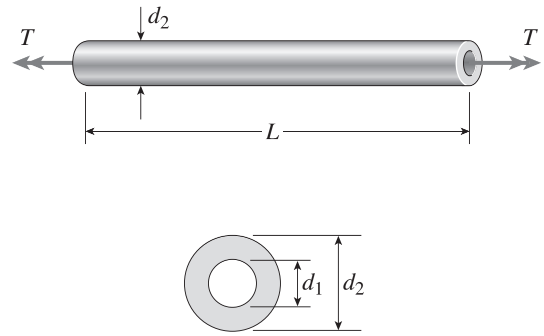
Substitute numerical values:

$\phi = 2.971$ rad $= 170°$ $\longleftarrow$

Pure Shear

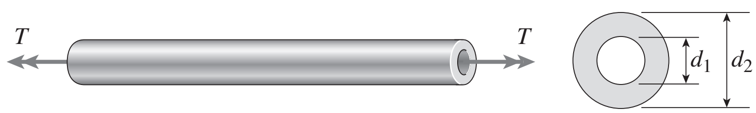
Problem 3.5-1 A hollow aluminum shaft (see figure) has outside diameter $d_2 = 4.0$ in. and inside diameter $d_1 = 2.0$ in. When twisted by torques T, the shaft has an angle of twist per unit distance equal to 0.54°/ft. The shear modulus of elasticity of the aluminum is $G = 4.0 \times 10^6$ psi.

(a) Determine the maximum tensile stress σ_{max} in the shaft.
(b) Determine the magnitude of the applied torques T.



Problems 3.5-1, 3.5-2, and 3.5-3

Solution 3.5-1 Hollow aluminum shaft



$d_2 = 4.0$ in. $\quad d_1 = 2.0$ in. $\quad \theta = 0.54°/\text{ft}$

$G = 4.0 \times 10^6$ psi

MAXIMUM SHEAR STRESS

$\tau_{max} = Gr\theta$ (from Eq. 3-7a)

$r = d_2/2 = 2.0$ in.

$$\theta = (0.54°/\text{ft})\left(\frac{1}{12}\ \frac{\text{ft}}{\text{in.}}\right)\left(\frac{\pi}{180}\ \frac{\text{rad}}{\text{degree}}\right)$$

$$= 785.40 \times 10^{-6}\ \text{rad/in.}$$

$$\tau_{max} = (4.0 \times 10^6\ \text{psi})(2.0\ \text{in.})(785.40 \times 10^{-6}\ \text{rad/in.})$$

$$= 6283.2\ \text{psi}$$

(a) MAXIMUM TENSILE STRESS

σ_{max} occurs on a 45° plane and is equal to τ_{max}.

$\sigma_{max} = \tau_{max} = 6280$ psi ⟵

(b) APPLIED TORQUE

Use the torsion formula $\tau_{max} = \dfrac{Tr}{I_P}$

$$T = \frac{\tau_{max} I_P}{r} \qquad I_P = \frac{\pi}{32}[(4.0\ \text{in.})^4 - (2.0\ \text{in.})^4]$$

$$= 23.562\ \text{in.}^4$$

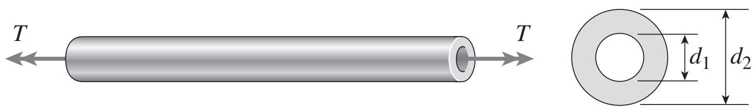
$$T = \frac{(6283.2\ \text{psi})(23.562\ \text{in.}^4)}{2.0\ \text{in.}}$$

$$= 74{,}000\ \text{lb-in.}$$ ⟵

Problem 3.5-2 A hollow steel bar ($G = 80$ GPa) is twisted by torques T (see figure). The twisting of the bar produces a maximum shear strain $\gamma_{max} = 640 \times 10^{-6}$ rad. The bar has outside and inside diameters of 150 mm and 120 mm, respectively.

(a) Determine the maximum tensile strain in the bar.
(b) Determine the maximum tensile stress in the bar.
(c) What is the magnitude of the applied torques T?

Solution 3.5-2 Hollow steel bar



$G = 80$ GPa $\quad \gamma_{max} = 640 \times 10^{-6}$ rad

$d_2 = 150$ mm $\quad d_1 = 120$ mm

$$I_P = \frac{\pi}{32}(d_2^4 - d_1^4)$$

$$= \frac{\pi}{32}[(150\text{ mm})^4 - (120\text{ mm})^4]$$

$$= 29.343 \times 10^6\text{ mm}^4$$

(a) Maximum tensile strain

$$\varepsilon_{max} = \frac{\gamma_{max}}{2} = 320 \times 10^{-6} \longleftarrow$$

(b) Maximum tensile stress

$$\tau_{max} = G\gamma_{max} = (80\text{ GPa})(640 \times 10^{-6})$$

$$= 51.2\text{ MPa}$$

$$\sigma_{max} = \tau_{max} = 51.2\text{ MPa} \longleftarrow$$

(c) Applied torques

Torsion formula: $\tau_{max} = \dfrac{Tr}{I_P} = \dfrac{Td_2}{2I_P}$

$$T = \frac{2I_P\tau_{max}}{d_2} = \frac{2(29.343 \times 10^6\text{ mm}^4)(51.2\text{ MPa})}{150\text{ mm}}$$

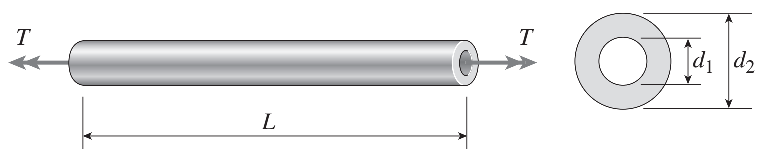
$$= 20{,}030\text{ N}\cdot\text{m}$$

$$= 20.0\text{ kN}\cdot\text{m} \longleftarrow$$

Problem 3.5-3 A tubular bar with outside diameter $d_2 = 4.0$ in. is twisted by torques $T = 70.0$ k-in. (see figure). Under the action of these torques, the maximum tensile stress in the bar is found to be 6400 psi.

(a) Determine the inside diameter d_1 of the bar.
(b) If the bar has length $L = 48.0$ in. and is made of aluminum with shear modulus $G = 4.0 \times 10^6$ psi, what is the angle of twist ϕ (in degrees) between the ends of the bar?
(c) Determine the maximum shear strain γ_{max} (in radians)?

Solution 3.5-3 **Tubular bar**



$d_2 = 4.0$ in. $\quad T = 70.0$ k-in. $= 70{,}000$ lb-in.

$\sigma_{max} = 6400$ psi $\quad \tau_{max} = \sigma_{max} = 6400$ psi

(a) INSIDE DIAMETER d_1

Torsion formula: $\tau_{max} = \dfrac{Tr}{I_P} = \dfrac{Td_2}{2I_P}$

$$I_P = \frac{Td_2}{2\tau_{max}} = \frac{(70.0 \text{ k-in.})(4.0 \text{ in.})}{2(6400 \text{ psi})}$$

$$= 21.875 \text{ in.}^4$$

Also, $I_p = \dfrac{\pi}{32}(d_2^4 - d_1^4) = \dfrac{\pi}{32}[(4.0 \text{ in.})^4 - d_1^4]$

Equate formulas:

$$\frac{\pi}{32}[256 \text{ in.}^4 - d_1^4] = 21.875 \text{ in.}^4$$

Solve for d_1: $\quad d_1 = 2.40$ in. ⟵

(b) ANGLE OF TWIST ϕ

$L = 48$ in. $\quad G = 4.0 \times 10^6$ psi

$$\phi = \frac{TL}{GI_p}$$

From torsion formula, $T = \dfrac{2I_P\tau_{max}}{d_2}$

$$\therefore \phi = \frac{2I_P\tau_{max}}{d_2}\left(\frac{L}{GI_P}\right) = \frac{2L\tau_{max}}{Gd_2}$$

$$= \frac{2(48 \text{ in.})(6400 \text{ psi})}{(4.0 \times 10^6 \text{ psi})(4.0 \text{ in.})} = 0.03840 \text{ rad}$$

$\phi = 2.20°$ ⟵

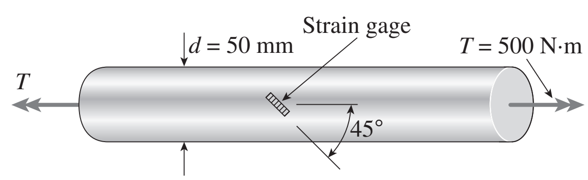
(c) MAXIMUM SHEAR STRAIN

$$\gamma_{max} = \frac{\tau_{max}}{G} = \frac{6400 \text{ psi}}{4.0 \times 10^6 \text{ psi}}$$

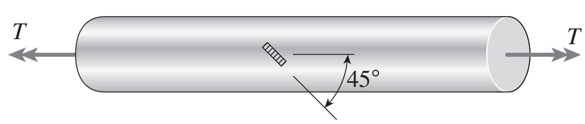
$= 1600 \times 10^{-6}$ rad ⟵

Problem 3.5-4 A solid circular bar of diameter $d = 50$ mm (see figure) is twisted in a testing machine until the applied torque reaches the value $T = 500$ N · m. At this value of torque, a strain gage oriented at 45° to the axis of the bar gives a reading $\epsilon = 339 \times 10^{-6}$.

What is the shear modulus G of the material?



Solution 3.5-4 **Bar in a testing machine**



Strain gage at 45°:

$\varepsilon_{max} = 339 \times 10^{-6}$

$d = 50$ mm

$T = 500$ N · m

SHEAR STRAIN (FROM EQ. 3-33)

$\gamma_{max} = 2\varepsilon_{max} = 678 \times 10^{-6}$

SHEAR STRESS (FROM EQ. 3-12)

$$\tau_{max} = \frac{16T}{\pi d^3} = \frac{16(500 \text{ N} \cdot \text{m})}{\pi(0.050 \text{ m})^3} = 20.372 \text{ MPa}$$

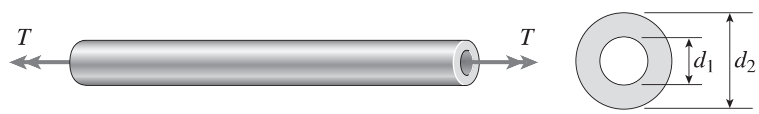
SHEAR MODULUS

$$G = \frac{\tau_{max}}{\gamma_{max}} = \frac{20.372 \text{ MPa}}{678 \times 10^{-6}} = 30.0 \text{ GPa}$$ ⟵

Problem 3.5-5 A steel tube ($G = 11.5 \times 10^6$ psi) has an outer diameter $d_2 = 2.0$ in. and an inner diameter $d_1 = 1.5$ in. When twisted by a torque T, the tube develops a maximum normal strain of 170×10^{-6}.

What is the magnitude of the applied torque T?

Solution 3.5-5 **Steel tube**



$G = 11.5 \times 10^6$ psi $\quad d_2 = 2.0$ in. $\quad d_1 = 1.5$ in.

$\varepsilon_{max} = 170 \times 10^{-6}$

$$I_P = \frac{\pi}{32}(d_2^4 - d_1^4) = \frac{\pi}{32}[(2.0 \text{ in.})^4 - (1.5 \text{ in.})^4]$$

$$= 1.07379 \text{ in.}^4$$

Shear strain (from Eq. 3-33)

$\gamma_{max} = 2\varepsilon_{max} = 340 \times 10^{-6}$

Shear Stress (from torsion formula)

$$\tau_{max} = \frac{Tr}{I_P} = \frac{Td_2}{2I_P}$$

Also, $\tau_{max} = G\gamma_{max}$

Equate expressions:

$$\frac{Td_2}{2I_P} = G\gamma_{max}$$

Solve for torque

$$T = \frac{2GI_P\gamma_{max}}{d_2}$$

$$= \frac{2(11.5 \times 10^6 \text{ psi})(1.07379 \text{ in.}^4)(340 \times 10^{-6})}{2.0 \text{ in.}}$$

$$= 4200 \text{ lb-in.} \longleftarrow$$

Problem 3.5-6 A solid circular bar of steel ($G = 78$ GPa) transmits a torque $T = 360$ N·m. The allowable stresses in tension, compression, and shear are 90 MPa, 70 MPa, and 40 MPa, respectively. Also, the allowable tensile strain is 220×10^{-6}.

Determine the minimum required diameter d of the bar.

Solution 3.5-6 **Solid circular bar of steel**

$T = 360\ \text{N}\cdot\text{m} \quad G = 78\ \text{GPa}$

ALLOWABLE STRESSES

Tension: 90 MPa Compression: 70 MPa

Shear: 40 MPa

Allowable tensile strain: $\varepsilon_{\max} = 220 \times 10^{-6}$

DIAMETER BASED UPON ALLOWABLE STRESS

The maximum tensile, compressive, and shear stresses in a bar in pure torsion are numerically equal. Therefore, the lowest allowable stress (shear stress) governs.

$\tau_{\text{allow}} = 40\ \text{MPa}$

$$\tau_{\max} = \frac{16T}{\pi d^3} \qquad d^3 = \frac{16T}{\pi\tau_{\text{allow}}} = \frac{16(360\ \text{N}\cdot\text{m})}{\pi(40\ \text{MPa})}$$

$$d^3 = 45.837 \times 10^{-6}\ \text{m}^3$$

$$d = 0.0358\ \text{m} = 35.8\ \text{mm}$$

DIAMETER BASED UPON ALLOWABLE TENSILE STRAIN

$$\gamma_{\max} = 2\varepsilon_{\max};\ \tau_{\max} = G\gamma_{\max} = 2G\varepsilon_{\max}$$

$$\tau_{\max} = \frac{16T}{\pi d^3} \qquad d^3 = \frac{16T}{\pi\tau_{\max}} = \frac{16T}{2\pi G\varepsilon_{\max}}$$

$$d^3 = \frac{16(360\ \text{N}\cdot\text{m})}{2\pi(78\ \text{GPa})(220 \times 10^{-6})}$$

$$= 53.423 \times 10^{-6}\ \text{m}^3$$

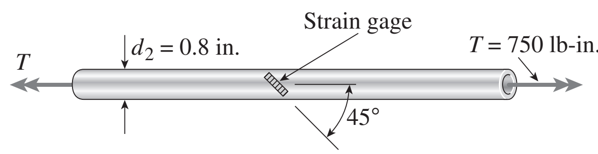
$$d = 0.0377\ \text{m} = 37.7\ \text{mm}$$

TENSILE STRAIN GOVERNS

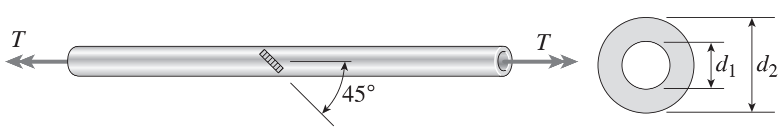
$d_{\min} = 37.7\ \text{mm}$ ⟵

Problem 3.5-7 The normal strain in the 45° direction on the surface of a circular tube (see figure) is 880×10^{-6} when the torque $T = 750$ lb-in. The tube is made of copper alloy with $G = 6.2 \times 10^6$ psi.

If the outside diameter d_2 of the tube is 0.8 in., what is the inside diameter d_1?



Solution 3.5-7 **Circular tube with strain gage**



$d_2 = 0.80$ in. $\quad T = 750$ lb-in. $\quad G = 6.2 \times 10^6$ psi

Strain gage at 45°: $\varepsilon_{\max} = 880 \times 10^{-6}$

MAXIMUM SHEAR STRAIN

$$\gamma_{\max} = 2\varepsilon_{\max}$$

MAXIMUM SHEAR STRESS

$$\tau_{\max} = G\gamma_{\max} = 2G\varepsilon_{\max}$$

$$\tau_{\max} = \frac{T(d_2/2)}{I_P} \qquad I_P = \frac{Td_2}{2\tau_{\max}} = \frac{Td_2}{4G\varepsilon_{\max}}$$

$$I_P = \frac{\pi}{32}(d_2^4 - d_1^4) = \frac{Td_2}{4G\varepsilon_{\max}}$$

$$d_2^4 - d_1^4 = \frac{8Td_2}{\pi G\varepsilon_{\max}} \qquad d_1^4 = d_2^4 - \frac{8Td_2}{\pi G\varepsilon_{\max}}$$

INSIDE DIAMETER

Substitute numerical values:

$$d_1^4 = (0.8\ \text{in.})^4 - \frac{8(750\ \text{lb-in.})(0.80\ \text{in.})}{\pi(6.2 \times 10^6\ \text{psi})(880 \times 10^{-6})}$$

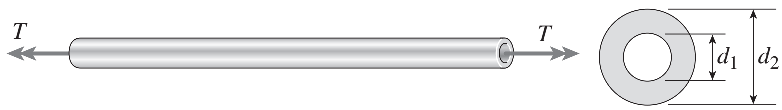
$$= 0.4096\ \text{in.}^4 - 0.2800\ \text{in.}^4 = 0.12956\ \text{in.}^4$$

$d_1 = 0.60$ in. ⟵

Problem 3.5-8 An aluminum tube has inside diameter $d_1 = 50$ mm, shear modulus of elasticity $G = 27$ GPa, and torque $T = 4.0$ kN · m. The allowable shear stress in the aluminum is 50 MPa and the allowable normal strain is 900×10^{-6}.

Determine the required outside diameter d_2.

Solution 3.5-8 Aluminum tube



$d_1 = 50$ mm $\quad G = 27$ GPa

$T = 4.0$ kN · m $\quad \tau_{allow} = 50$ MPa $\quad \varepsilon_{allow} = 900 \times 10^{-6}$

Determine the required diameter d_2.

ALLOWABLE SHEAR STRESS

$(\tau_{allow})_1 = 50$ MPa

ALLOWABLE SHEAR STRESS BASED ON NORMAL STRAIN

$$\varepsilon_{max} = \frac{\gamma}{2} = \frac{\tau}{2G} \qquad \tau = 2G\varepsilon_{max}$$

$$(\tau_{allow})_2 = 2G\varepsilon_{allow} = 2(27 \text{ GPa})(900 \times 10^{-6})$$
$$= 48.6 \text{ MPa}$$

NORMAL STRAIN GOVERNS

$\tau_{allow} = 48.60$ MPa

REQUIRED DIAMETER

$$\tau = \frac{Tr}{I_P} \qquad 48.6 \text{ MPa} = \frac{(4000 \text{ N}\cdot\text{m})(d_2/2)}{\frac{\pi}{32}[d_2^4 - (0.050 \text{ m})^4]}$$

Rearrange and simplify:

$$d_2^4 - (419.174 \times 10^{-6})d_2 - 6.25 \times 10^{-6} = 0$$

Solve numerically:

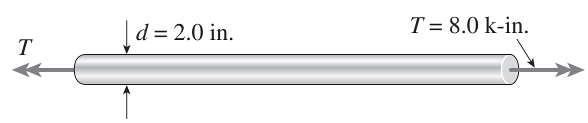
$d_2 = 0.07927$ m

$d_2 = 79.3$ mm ⟵

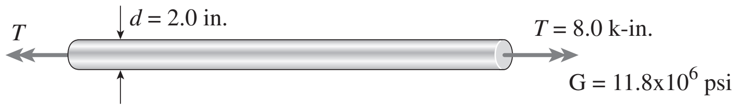
Problem 3.5-9 A solid steel bar ($G = 11.8 \times 10^6$ psi) of diameter $d = 2.0$ in. is subjected to torques $T = 8.0$ k-in. acting in the directions shown in the figure.

(a) Determine the maximum shear, tensile, and compressive stresses in the bar and show these stresses on sketches of properly oriented stress elements.

(b) Determine the corresponding maximum strains (shear, tensile, and compressive) in the bar and show these strains on sketches of the deformed elements.



Solution 3.5-9 **Solid steel bar**



$T = 8.0$ k-in.

$G = 11.8 \times 10^6$ psi

(a) MAXIMUM STRESSES

$$\tau_{max} = \frac{16T}{\pi d^3} = \frac{16(8000 \text{ lb-in.})}{\pi(2.0 \text{ in.})^3}$$

$$= 5093 \text{ psi} \quad \longleftarrow$$

$$\sigma_t = 5090 \text{ psi} \qquad \sigma_c = -5090 \text{ psi} \quad \longleftarrow$$

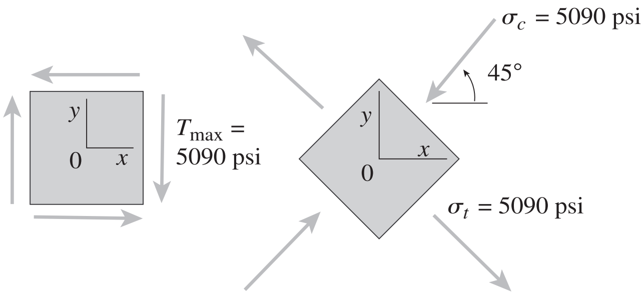
(b) MAXIMUM STRAINS

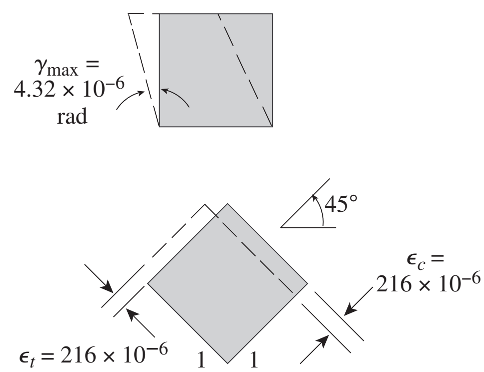
$$\gamma_{max} = \frac{\tau_{max}}{G} = \frac{5093 \text{ psi}}{11.8 \times 10^6 \text{ psi}}$$

$$= 432 \times 10^{-6} \text{ rad} \quad \longleftarrow$$

$$\varepsilon_{max} = \frac{\gamma_{max}}{2} = 216 \times 10^{-6}$$

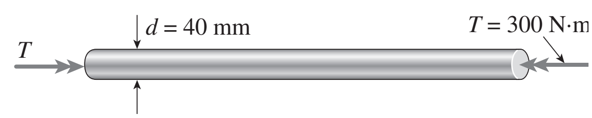
$$\varepsilon_t = 216 \times 10^{-6} \qquad \varepsilon_c = -216 \times 10^{-6} \quad \longleftarrow$$



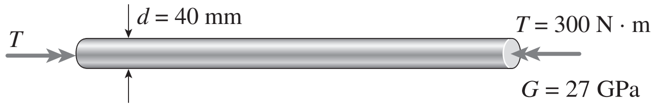


Problem 3.5-10 A solid aluminum bar ($G = 27$ GPa) of diameter $d = 40$ mm is subjected to torques $T = 300$ N · m acting in the directions shown in the figure.

(a) Determine the maximum shear, tensile, and compressive stresses in the bar and show these stresses on sketches of properly oriented stress elements.
(b) Determine the corresponding maximum strains (shear, tensile, and compressive) in the bar and show these strains on sketches of the deformed elements.



Solution 3.5-10 **Solid aluminum bar**



(a) MAXIMUM STRESSES

$$\tau_{max} = \frac{16T}{\pi d^3} = \frac{16(300 \text{ N} \cdot \text{m})}{\pi(0.040 \text{ m})^3}$$

$$= 23.87 \text{ MPa} \quad \longleftarrow$$

$$\sigma_t = 23.9 \text{ MPa} \qquad \sigma_c = -23.9 \text{ MPa} \quad \longleftarrow$$

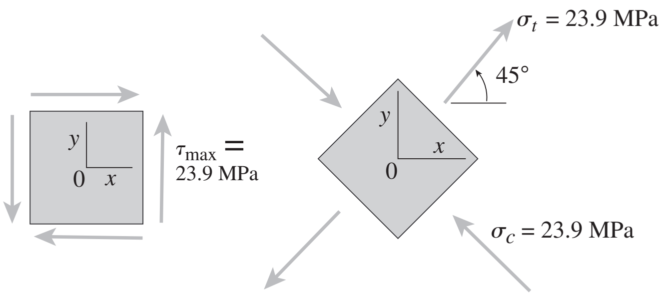
(b) MAXIMUM STRAINS

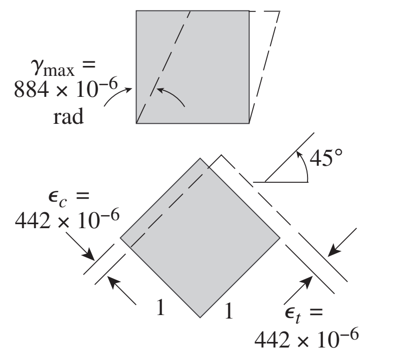
$$\gamma_{max} = \frac{\tau_{max}}{G} = \frac{23.87 \text{ MPa}}{27 \text{ GPa}}$$

$$= 884 \times 10^{-6} \text{ rad} \quad \longleftarrow$$

$$\varepsilon_{max} = \frac{\gamma_{max}}{2} = 442 \times 10^{-6}$$

$$\varepsilon_t = 442 \times 10^{-6} \qquad \varepsilon_c = -442 \times 10^{-6} \quad \longleftarrow$$

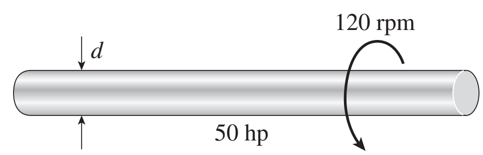




Transmission of Power

Problem 3.7-1 A generator shaft in a small hydroelectric plant turns at 120 rpm and delivers 50 hp (see figure).

(a) If the diameter of the shaft is $d = 3.0$ in., what is the maximum shear stress τ_{max} in the shaft?
(b) If the shear stress is limited to 4000 psi, what is the minimum permissible diameter d_{min} of the shaft?



Solution 3.7-1 **Generator shaft**

$n = 120$ rpm $\qquad H = 50$ hp $\qquad d$ = diameter

TORQUE

$$H = \frac{2\pi n T}{33{,}000} \qquad H = \text{hp} \quad n = \text{rpm} \quad T = \text{lb-ft}$$

$$T = \frac{33{,}000\, H}{2\pi n} = \frac{(33{,}000)(50 \text{ hp})}{2\pi(120 \text{ rpm})}$$

$$= 2188 \text{ lb-ft} = 26{,}260 \text{ lb-in.}$$

(a) MAXIMUM SHEAR STRESS τ_{max}

$d = 3.0$ in.

$$\tau_{max} = \frac{16T}{\pi d^3} = \frac{16(26{,}260 \text{ lb-in.})}{\pi(3.0 \text{ in.})^3}$$

$$\tau_{max} = 4950 \text{ psi} \quad \longleftarrow$$

(b) MINIMUM DIAMETER d_{min}

$$\tau_{allow} = 4000 \text{ psi}$$

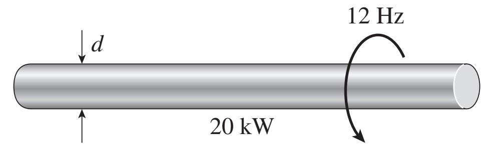
$$d^3 = \frac{16T}{\pi \tau_{allow}} = \frac{16(26{,}260 \text{ lb-in.})}{\pi(4000 \text{ psi})} = 334.44 \text{ in.}^3$$

$$d_{min} = 3.22 \text{ in.} \quad \longleftarrow$$

Problem 3.7-2 A motor drives a shaft at 12 Hz and delivers 20 kW of power (see figure).

(a) If the shaft has a diameter of 30 mm, what is the maximum shear stress τ_{max} in the shaft?

(b) If the maximum allowable shear stress is 40 MPa, what is the minimum permissible diameter d_{min} of the shaft?



Solution 3.7-2 **Motor-driven shaft**

$f = 12$ Hz $\quad P = 20$ kW $= 20{,}000$ N · m/s

TORQUE

$P = 2\pi f T \quad P = \text{watts} \qquad f = \text{Hz} = \text{s}^{-1}$

T = Newton meters

$$T = \frac{P}{2\pi f} = \frac{20{,}000 \text{ W}}{2\pi(12 \text{ Hz})} = 265.3 \text{ N}\cdot\text{m}$$

(a) MAXIMUM SHEAR STRESS τ_{max}

$d = 30$ mm

$$\tau_{max} = \frac{16T}{\pi d^3} = \frac{16(265.3 \text{ N}\cdot\text{m})}{\pi(0.030 \text{ m})^3}$$

$$= 50.0 \text{ MPa} \quad \longleftarrow$$

(b) MINIMUM DIAMETER d_{min}

$\tau_{allow} = 40$ MPa

$$d^3 = \frac{16T}{\pi\tau_{allow}} = \frac{16(265.3 \text{ N}\cdot\text{m})}{\pi(40 \text{ MPa})}$$

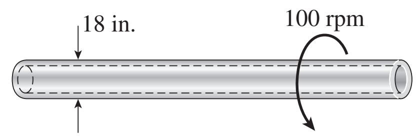
$$= 33.78 \times 10^{-6} \text{ m}^3$$

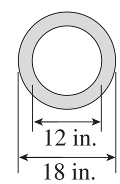
$$d_{min} = 0.0323 \text{ m} = 32.3 \text{ mm} \quad \longleftarrow$$

Problem 3.7-3 The propeller shaft of a large ship has outside diameter 18 in. and inside diameter 12 in., as shown in the figure. The shaft is rated for a maximum shear stress of 4500 psi.

(a) If the shaft is turning at 100 rpm, what is the maximum horsepower that can be transmitted without exceeding the allowable stress?

(b) If the rotational speed of the shaft is doubled but the power requirements remain unchanged, what happens to the shear stress in the shaft?





Solution 3.7-3 **Hollow propeller shaft**

$d_2 = 18$ in. $\qquad d_1 = 12$ in. $\quad \tau_{allow} = 4500$ psi

$$I_P = \frac{\pi}{32}(d_2^4 - d_1^4) = 8270.2 \text{ in.}^4$$

TORQUE

$$\tau_{max} = \frac{T(d_2/2)}{I_P} \qquad T = \frac{2\tau_{allow} I_P}{d_2}$$

$$T = \frac{2(4500 \text{ psi})(8270.2 \text{ in.}^4)}{18 \text{ in.}}$$

$$= 4.1351 \times 10^6 \text{ lb-in.}$$

$$= 344{,}590 \text{ lb-ft.}$$

(a) HORSEPOWER

$$n = 100 \text{ rpm} \quad H = \frac{2\pi nT}{33{,}000}$$

n = rpm $\qquad T$ = lb-ft $\qquad H$ = hp

$$H = \frac{2\pi(100 \text{ rpm})(344{,}590 \text{ lb-ft})}{33{,}000}$$

$$= 6560 \text{ hp} \quad \longleftarrow$$

(b) ROTATIONAL SPEED IS DOUBLED

$$H = \frac{2\pi nT}{33{,}000}$$

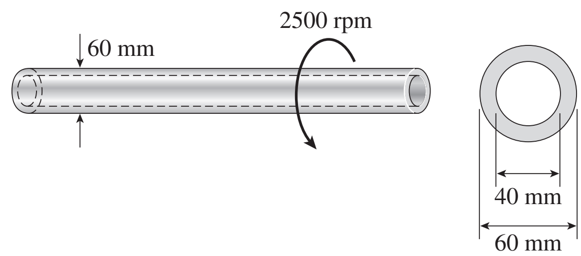
If n is doubled but H remains the same, then T is halved.

If T is halved, so is the maximum shear stress.

$\therefore$ Shear stress is halved $\longleftarrow$

Problem 3.7-4 The drive shaft for a truck (outer diameter 60 mm and inner diameter 40 mm) is running at 2500 rpm (see figure).

(a) If the shaft transmits 150 kW, what is the maximum shear stress in the shaft?
(b) If the allowable shear stress is 30 MPa, what is the maximum power that can be transmitted?



Solution 3.7-4 Drive shaft for a truck

$d_2 = 60$ mm $\quad d_1 = 40$ mm $\quad n = 2500$ rpm

$$I_P = \frac{\pi}{32}(d_2^4 - d_1^4) = 1.0210 \times 10^{-6}\ \text{m}^4$$

(a) Maximum shear stress τ_{max}

P = power (watts) $\quad P = 150$ kW $= 150{,}000$ W

T = torque (newton meters) $\quad n$ = rpm

$$P = \frac{2\pi n T}{60} \qquad T = \frac{60P}{2\pi n}$$

$$T = \frac{60(150{,}000\ \text{W})}{2\pi(2500\ \text{rpm})} = 572.96\ \text{N}\cdot\text{m}$$

$$\tau_{max} = \frac{Td_2}{2\,I_P} = \frac{(572.96\ \text{N}\cdot\text{m})(0.060\ \text{m})}{2(1.0210 \times 10^{-6}\ \text{m}^4)}$$

$$= 16.835\ \text{MPa}$$

$$\tau_{max} = 16.8\ \text{MPa} \longleftarrow$$

(b) Maximum power P_{max}

$\tau_{allow} = 30$ MPa

$$P_{max} = P\frac{\tau_{allow}}{\tau_{max}} = (150\ \text{kW})\left(\frac{30\ \text{MPa}}{16.835\ \text{MPa}}\right)$$

$$= 267\ \text{kW} \longleftarrow$$

Problem 3.7-5 A hollow circular shaft for use in a pumping station is being designed with an inside diameter equal to 0.75 times the outside diameter. The shaft must transmit 400 hp at 400 rpm without exceeding the allowable shear stress of 6000 psi.

Determine the minimum required outside diameter d.

Solution 3.7-5 Hollow shaft

d = outside diameter

d_0 = inside diameter
$= 0.75\, d$

$H = 400$ hp $\quad n = 400$ rpm

$\tau_{allow} = 6000$ psi

$$I_P = \frac{\pi}{32}[d^4 - (0.75\,d)^4] = 0.067112\,d^4$$

Torque

$$H = \frac{2\pi n T}{33{,}000}$$

H = hp $\quad n$ = rpm $\quad T$ = lb-ft

$$T = \frac{33{,}000\ \text{H}}{2\pi n} = \frac{(33{,}000)(400\ \text{hp})}{2\pi(400\ \text{rpm})}$$

$$= 5252.1\ \text{lb-ft} = 63{,}025\ \text{lb-in.}$$

Minimum outside diameter

$$\tau_{max} = \frac{Td}{2I_P} \qquad I_P = \frac{Td}{2\tau_{max}} = \frac{Td}{2\tau_{allow}}$$

$$0.067112\,d^4 = \frac{(63{,}025\ \text{lb-in.})(d)}{2(6000\ \text{psi})}$$

$$d^3 = 78.259\ \text{in.}^3 \qquad d_{min} = 4.28\ \text{in.} \longleftarrow$$

Problem 3.7-6 A tubular shaft being designed for use on a construction site must transmit 120 kW at 1.75 Hz. The inside diameter of the shaft is to be one-half of the outside diameter.

If the allowable shear stress in the shaft is 45 MPa, what is the minimum required outside diameter d?

Solution 3.7-6 Tubular shaft

d = outside diameter

d_0 = inside diameter

$= 0.5\,d$

$P = 120$ kW $= 120{,}000$ W $\quad f = 1.75$ Hz

$\tau_{allow} = 45$ MPa

$$I_P = \frac{\pi}{32}[d^4 - (0.5\,d)^4] = 0.092039\,d^4$$

TORQUE

$P = 2\pi f T \qquad P$ = watts $\qquad f$ = Hz

T = newton meters

$$T = \frac{P}{2\pi f} = \frac{120{,}000\ \text{W}}{2\pi(1.75\ \text{Hz})} = 10{,}913.5\ \text{N}\cdot\text{m}$$

MINIMUM OUTSIDE DIAMETER

$$\tau_{max} = \frac{Td}{2I_P} \qquad I_P = \frac{Td}{2\tau_{max}} = \frac{Td}{2\tau_{allow}}$$

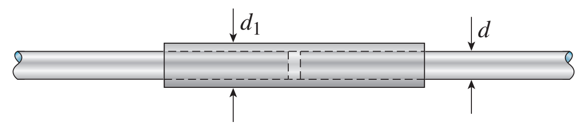
$$0.092039\,d^4 = \frac{(10{,}913.5\ \text{N.m})(d)}{2(45\ \text{MPa})}$$

$d^3 = 0.0013175\ \text{m}^3 \qquad d = 0.1096$ m

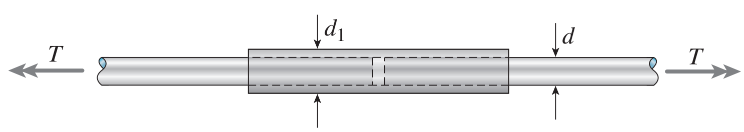
$d_{min} = 110$ mm ⟵

Problem 3.7-7 A propeller shaft of solid circular cross section and diameter d is spliced by a collar of the same material (see figure). The collar is securely bonded to both parts of the shaft.

What should be the minimum outer diameter d_1 of the collar in order that the splice can transmit the same power as the solid shaft?



Solution 3.7-7 Splice in a propeller shaft



SOLID SHAFT

$$\tau_{max} = \frac{16T_1}{\pi d^3} \qquad T_1 = \frac{\pi d^3 \tau_{max}}{16}$$

HOLLOW COLLAR

$$I_P = \frac{\pi}{32}(d_1^4 - d^4) \qquad \tau_{max} = \frac{T_2 r}{I_P} = \frac{T_2(d_1/2)}{I_P}$$

$$T_2 = \frac{2\tau_{max} I_P}{d_1} = \frac{2\tau_{max}}{d_1}\left(\frac{\pi}{32}\right)(d_1^4 - d^4)$$

$$= \frac{\pi\tau_{max}}{16\,d_1}(d_1^4 - d^4)$$

EQUATE TORQUES

For the same power, the torques must be the same. For the same material, both parts can be stressed to the same maximum stress.

$$\therefore\ T_1 = T_2 \qquad \frac{\pi d^3 \tau_{max}}{16} = \frac{\pi\tau_{max}}{16 d_1}(d_1^4 - d^4)$$

$$\text{or}\ \left(\frac{d_1}{d}\right)^4 - \frac{d_1}{d} - 1 = 0 \qquad \text{(Eq. 1)}$$

MINIMUM OUTER DIAMETER

Solve Eq. (1) numerically:

Min. $d_1 = 1.221\,d$ ⟵

Problem 3.7-8 What is the maximum power that can be delivered by a hollow propeller shaft (outside diameter 50 mm, inside diameter 40 mm, and shear modulus of elasticity 80 GPa) turning at 600 rpm if the allowable shear stress is 100 MPa and the allowable rate of twist is 3.0°/m?

Solution 3.7-8 Hollow propeller shaft

$d_2 = 50$ mm $\quad d_1 = 40$ mm
$G = 80$ GPa $\quad n = 600$ rpm

$\tau_{\text{allow}} = 100$ MPa $\quad \theta_{\text{allow}} = 3.0°/\text{m}$

$$I_P = \frac{\pi}{32}(d_2^4 - d_1^4) = 362.3 \times 10^{-9}\ \text{m}^4$$

BASED UPON ALLOWABLE SHEAR STRESS

$$\tau_{\max} = \frac{T_1(d_2/2)}{I_P} \qquad T_1 = \frac{2\tau_{\text{allow}} I_P}{d_2}$$

$$T_1 = \frac{2(100\ \text{MPa})(362.3 \times 10^{-9}\ \text{m}^4)}{0.050\ \text{m}}$$

$$= 1449\ \text{N}\cdot\text{m}$$

BASED UPON ALLOWABLE RATE OF TWIST

$$\theta = \frac{T_2}{GI_P} \qquad T_2 = GI_P\theta_{\text{allow}}$$

$$T_2 = (80\ \text{GPa})(362.3 \times 10^{-9}\ \text{m}^4)(3.0°/\text{m}) \times \left(\frac{\pi}{180}\ \text{rad/degree}\right)$$

$$T_2 = 1517\ \text{N}\cdot\text{m}$$

SHEAR STRESS GOVERNS

$$T_{\text{allow}} = T_1 = 1449\ \text{N}\cdot\text{m}$$

MAXIMUM POWER

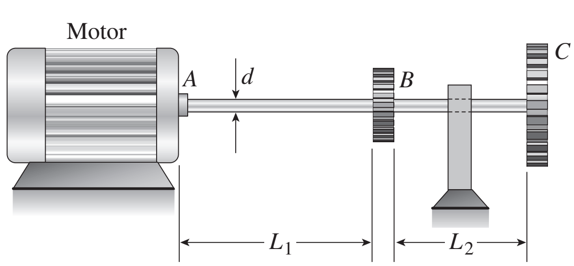
$$P = \frac{2\pi nT}{60} = \frac{2\pi(600\ \text{rpm})(1449\ \text{N}\cdot\text{m})}{60}$$

$$P = 91{,}047\ \text{W}$$

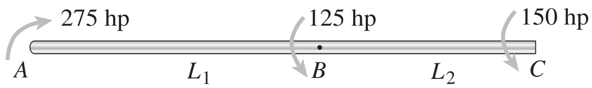
$$P_{\max} = 91.0\ \text{kW} \quad \longleftarrow$$

Problem 3.7-9 A motor delivers 275 hp at 1000 rpm to the end of a shaft (see figure). The gears at B and C take out 125 and 150 hp, respectively.

Determine the required diameter d of the shaft if the allowable shear stress is 7500 psi and the angle of twist between the motor and gear C is limited to 1.5°. (Assume $G = 11.5 \times 10^6$ psi, $L_1 = 6$ ft, and $L_2 = 4$ ft.)



Solution 3.7-9 Motor-driven shaft



$L_1 = 6$ ft

$L_2 = 4$ ft

d = diameter

$n = 1000$ rpm

$\tau_{allow} = 7500$ psi

$(\phi_{AC})_{allow} = 1.5° = 0.02618$ rad

$G = 11.5 \times 10^6$ psi

TORQUES ACTING ON THE SHAFT

$$H = \frac{2\pi nT}{33{,}000} \quad H = \text{hp} \quad n = \text{rpm} \quad T = \text{lb-ft}$$

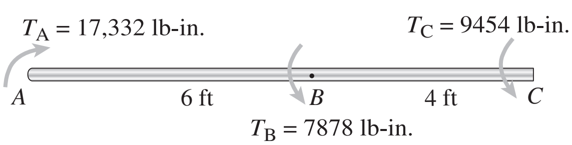
$$T = \frac{33{,}000\ H}{2\pi n}$$

$$\text{At point } A\text{: } T_A = \frac{33{,}000(275\text{ hp})}{2\pi(1000\text{ rpm})} = 1444\text{ lb-ft} = 17{,}332\text{ lb-in.}$$

$$\text{At point } B\text{: } T_B = \frac{125}{275} \quad T_A = 7878\text{ lb-in.}$$

$$\text{At point } C\text{: } T_C = \frac{150}{275} \quad T_A = 9454\text{ lb-in.}$$

FREE-BODY DIAGRAM



$T_A = 17{,}332$ lb-in.

$T_C = 9454$ lb-in.

d = diameter

$T_B = 7878$ lb-in.

INTERNAL TORQUES

$T_{AB} = 17{,}332$ lb-in.

$T_{BC} = 9454$ lb-in.

DIAMETER BASED UPON ALLOWABLE SHEAR STRESS

The larger torque occurs in segment AB

$$\tau_{max} = \frac{16T_{AB}}{\pi d^3} \qquad d^3 = \frac{16T_{AB}}{\pi\tau_{allow}}$$

$$= \frac{16(17{,}332\text{ lb-in.})}{\pi(7500\text{ psi})} = 11.77\text{ in.}^3$$

$d = 2.27$ in.

DIAMETER BASED UPON ALLOWABLE ANGLE OF TWIST

$$I_P = \frac{\pi d^4}{32} \qquad \phi = \frac{TL}{GI_P} = \frac{32TL}{\pi Gd^4}$$

Segment AB:

$$\phi_{AB} = \frac{32\ T_{AB}L_{AB}}{\pi Gd^4}$$

$$= \frac{32(17{,}330\text{ lb-in.})(6\text{ ft})(12\text{ in./ft})}{\pi(11.5 \times 10^6\text{ psi})d^4}$$

$$\phi_{AB} = \frac{1.1052}{d^4}$$

Segment BC:

$$\phi_{BC} = \frac{32\ T_{BC}L_{BC}}{\pi Gd^4}$$

$$= \frac{32(9450\text{ lb-in.})(4\text{ ft})(12\text{ in./ft})}{\pi(11.5 \times 10^6\text{ psi})d^4}$$

$$\phi_{BC} = \frac{0.4018}{d^4}$$

$$\text{From } A \text{ to } C\text{: } \phi_{AC} = \phi_{AB} + \phi_{BC} = \frac{1.5070}{d^4}$$

$$(\phi_{AC})_{allow} = 0.02618\text{ rad}$$

$$\therefore\ 0.02618 = \frac{1.5070}{d^4} \quad \text{and} \quad d = 2.75\text{ in.}$$

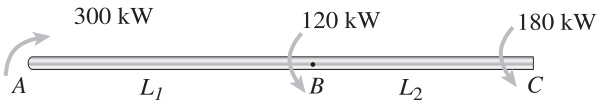
Angle of twist governs

$d = 2.75$ in. ←

Problem 3.7-10 The shaft *ABC* shown in the figure is driven by a motor that delivers 300 kW at a rotational speed of 32 Hz. The gears at *B* and *C* take out 120 and 180 kW, respectively. The lengths of the two parts of the shaft are $L_1 = 1.5$ m and $L_2 = 0.9$ m.

Determine the required diameter d of the shaft if the allowable shear stress is 50 MPa, the allowable angle of twist between points A and C is 4.0°, and $G = 75$ GPa.

Solution 3.7-10 Motor-driven shaft



$L_1 = 1.5$ m

$L_2 = 0.9$ m

$d =$ diameter

$f = 32$ Hz

$\tau_{\text{allow}} = 50$ MPa

$G = 75$ GPa

$(\phi_{AC})_{\text{allow}} = 4° = 0.06981$ rad

TORQUES ACTING ON THE SHAFT

$P = 2\pi f T \qquad P = \text{watts} \qquad f = \text{Hz}$

$T =$ newton meters

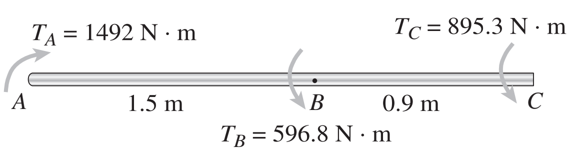
$$T = \frac{P}{2\pi f}$$

At point A: $T_A = \dfrac{300{,}000 \text{ W}}{2\pi(32 \text{ Hz})} = 1492 \text{ N}\cdot\text{m}$

At point B: $T_B = \dfrac{120}{300}\, T_A = 596.8 \text{ N}\cdot\text{m}$

At point C: $T_C = \dfrac{180}{300}\, T_A = 895.3 \text{ N}\cdot\text{m}$

FREE-BODY DIAGRAM



$T_A = 1492$ N · m

$T_B = 596.8$ N · m

$T_C = 895.3$ N · m

$d =$ diameter

INTERNAL TORQUES

$T_{AB} = 1492$ N · m

$T_{BC} = 895.3$ N · m

DIAMETER BASED UPON ALLOWABLE SHEAR STRESS

The larger torque occurs in segment *AB*

$$\tau_{\max} = \frac{16\, T_{AB}}{\pi\, d^3} \qquad d^3 = \frac{16\, T_{AB}}{\pi \tau_{\text{allow}}} = \frac{16(1492 \text{ N}\cdot\text{m})}{\pi(50 \text{ MPa})}$$

$d^3 = 0.0001520 \text{ m}^3 \qquad d = 0.0534 \text{ m} = 53.4 \text{ mm}$

DIAMETER BASED UPON ALLOWABLE ANGLE OF TWIST

$$I_P = \frac{\pi d^4}{32} \qquad \phi = \frac{TL}{GI_P} = \frac{32TL}{\pi G d^4}$$

Segment *AB*:

$$\phi_{AB} = \frac{32\, T_{AB} L_{AB}}{\pi G d^4} = \frac{32(1492 \text{ N}\cdot\text{m})(1.5 \text{ m})}{\pi(75 \text{ GPa}) d^4}$$

$$\phi_{AB} = \frac{0.3039 \times 10^{-6}}{d^4}$$

Segment *BC*:

$$\phi_{BC} = \frac{32\, T_{BC} L_{BC}}{\pi G d^4} = \frac{32(895.3 \text{ N}\cdot\text{m})(0.9 \text{ m})}{\pi(75 \text{ GPa}) d^4}$$

$$\phi_{BC} = \frac{0.1094 \times 10^{-6}}{d^4}$$

From A to C: $\phi_{AC} = \phi_{AB} + \phi_{BC} = \dfrac{0.4133 \times 10^{-6}}{d^4}$

$(\phi_{AC})_{\text{allow}} = 0.06981$ rad

$$\therefore \quad 0.06981 = \frac{0.4133 \times 10^{-6}}{d^4}$$

and $d = 0.04933$ m

$= 49.3$ mm

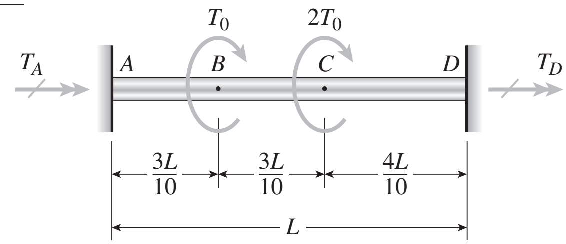
SHEAR STRESS GOVERNS

$d = 53.4$ mm ←

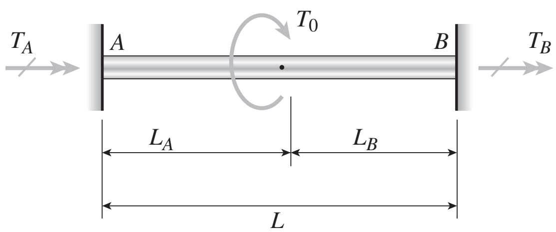
Statically Indeterminate Torsional Members

Problem 3.8-1 A solid circular bar *ABCD* with fixed supports is acted upon by torques T_0 and $2T_0$ at the locations shown in the figure.

Obtain a formula for the maximum angle of twist ϕ_{max} of the bar. (*Hint:* Use Eqs. 3-46a and b of Example 3-9 to obtain the reactive torques.)



Solution 3.8-1 **Circular bar with fixed ends**



From Eqs. (3-46a and b):

$$T_A = \frac{T_0 L_B}{L}$$

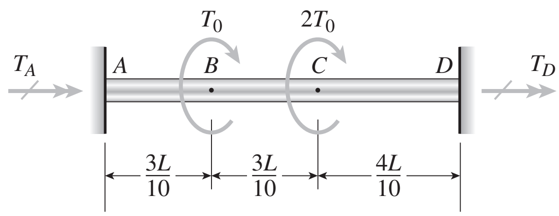
$$T_B = \frac{T_0 L_A}{L}$$

APPLY THE ABOVE FORMULAS TO THE GIVEN BAR:

$$T_A = T_0\left(\frac{7}{10}\right) + 2T_0\left(\frac{4}{10}\right) = \frac{15T_0}{10}$$

$$T_D = T_0\left(\frac{3}{10}\right) + 2T_0\left(\frac{6}{10}\right) = \frac{15T_0}{10}$$

ANGLE OF TWIST AT SECTION B



$$\phi_B = \phi_{AB} = \frac{T_A(3L/10)}{GI_P} = \frac{9T_0L}{20GI_P}$$

ANGLE OF TWIST AT SECTION C

$$\phi_C = \phi_{CD} = \frac{T_D(4L/10)}{GI_P} = \frac{3T_0L}{5GI_P}$$

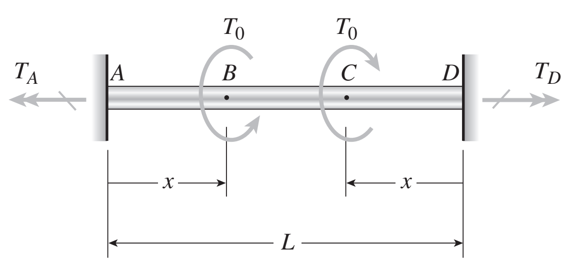
MAXIMUM ANGLE OF TWIST

$$\phi_{max} = \phi_C = \frac{3T_0L}{5GI_P} \quad \longleftarrow$$

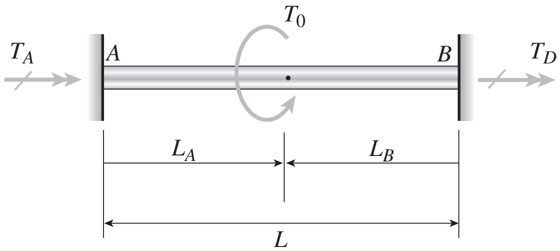
Problem 3.8-2 A solid circular bar *ABCD* with fixed supports at ends *A* and *D* is acted upon by two equal and oppositely directed torques T_0, as shown in the figure. The torques are applied at points *B* and *C*, each of which is located at distance *x* from one end of the bar. (The distance *x* may vary from zero to $L/2$.)

(a) For what distance *x* will the angle of twist at points *B* and *C* be a maximum?

(b) What is the corresponding angle of twist ϕ_{max}?
(*Hint:* Use Eqs. 3-46a and b of Example 3-9 to obtain the reactive torques.)



Solution 3.8-2 **Circular bar with fixed ends**

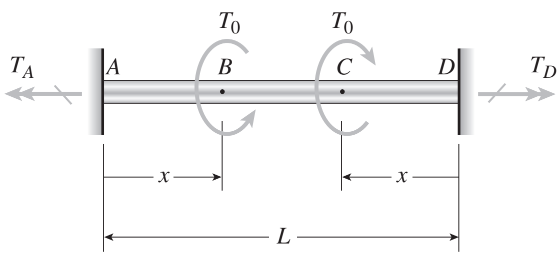


From Eqs. (3-46a and b):

$$T_A = \frac{T_0 L_B}{L}$$

$$T_B = \frac{T_0 L_A}{L}$$

APPLY THE ABOVE FORMULAS TO THE GIVEN BAR:



$$T_A = \frac{T_0(L - x)}{L} - \frac{T_0 x}{L} = \frac{T_0}{L}(L - 2x) \qquad T_D = T_A$$

(a) ANGLE OF TWIST AT SECTIONS B AND C

$$\phi_B = \phi_{AB} = \frac{T_A x}{GI_P} = \frac{T_0}{GI_P L}(L - 2x)(x)$$

$$\frac{d\phi_B}{dx} = \frac{T_0}{GI_P L}(L - 4x)$$

$$\frac{d\phi_B}{dx} = 0; \ L - 4x = 0$$

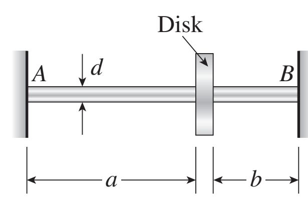
or $\quad x = \dfrac{L}{4}$ ←

(b) MAXIMUM ANGLE OF TWIST

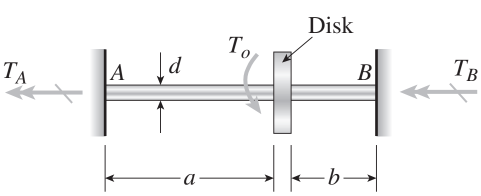
$$\phi_{\max} = (\phi_B)_{\max} = (\phi_B)_{x=\frac{L}{4}} = \frac{T_0 L}{8GI_P}$$ ←

Problem 3.8-3 A solid circular shaft AB of diameter d is fixed against rotation at both ends (see figure). A circular disk is attached to the shaft at the location shown.

What is the largest permissible angle of rotation $\phi_{\max}$ of the disk if the allowable shear stress in the shaft is τ_{allow}? (Assume that $a > b$. Also, use Eqs. 3-46a and b of Example 3-9 to obtain the reactive torques.)



Solution 3.8-3 **Shaft fixed at both ends**



$L = a + b$

$a > b$

Assume that a torque T_0 acts at the disk.

The reactive torques can be obtained from Eqs. (3-46a and b):

$$T_A = \frac{T_0 b}{L} \qquad T_B = \frac{T_0 a}{L}$$

Since $a > b$, the larger torque (and hence the larger stress) is in the right hand segment.

$$\tau_{\max} = \frac{T_B(d/2)}{I_P} = \frac{T_0 ad}{2LI_P}$$

$$T_0 = \frac{2LI_P \tau_{\max}}{ad} \qquad (T_0)_{\max} = \frac{2LI_P \tau_{\text{allow}}}{ad}$$

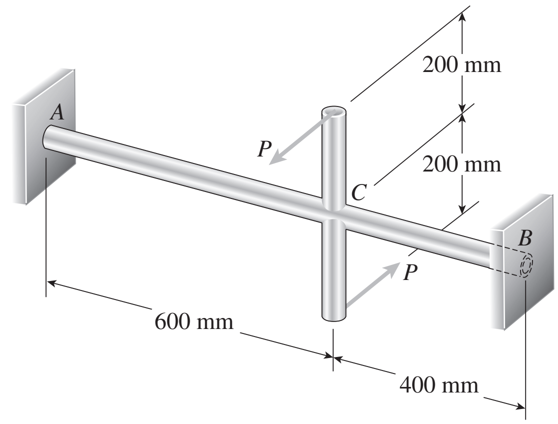
ANGLE OF ROTATION OF THE DISK (FROM Eq. 3-49)

$$\phi = \frac{T_0 ab}{GLI_P}$$

$$\phi_{\max} = \frac{(T_0)_{\max} ab}{GLI_P} = \frac{2b\tau_{\text{allow}}}{Gd}$$ ←

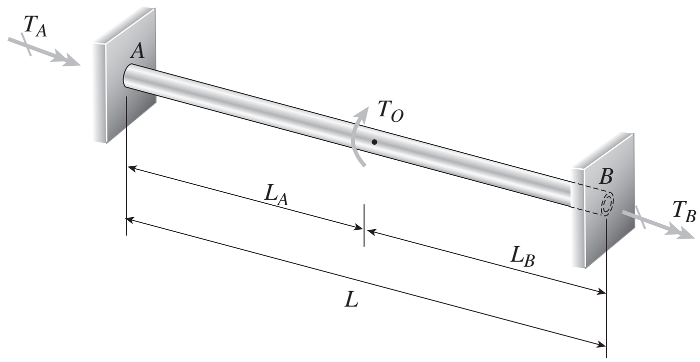
Problem 3.8-4 A hollow steel shaft *ACB* of outside diameter 50 mm and inside diameter 40 mm is held against rotation at ends *A* and *B* (see figure). Horizontal forces *P* are applied at the ends of a vertical arm that is welded to the shaft at point *C*.

Determine the allowable value of the forces *P* if the maximum permissible shear stress in the shaft is 45 MPa. (*Hint:* Use Eqs. 3-46a and b of Example 3-9 to obtain the reactive torques.)



Solution 3.8-4 **Hollow shaft with fixed ends**

GENERAL FORMULAS:

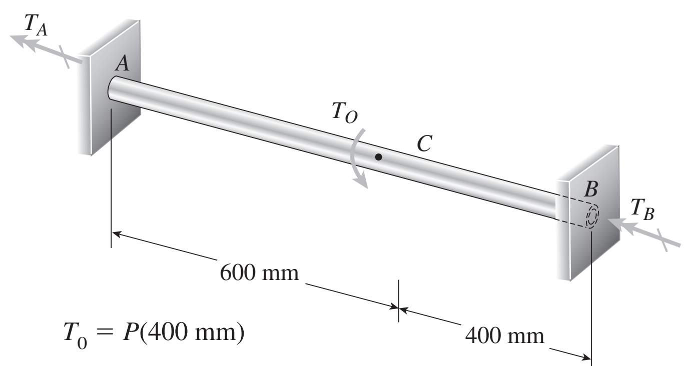


From Eqs. (3-46a and b):

$$T_A = \frac{T_0 L_B}{L}$$

$$T_B = \frac{T_0 L_A}{L}$$

APPLY THE ABOVE FORMULAS TO THE GIVEN SHAFT



$T_0 = P(400\text{ mm})$

$L_B = 400$ mm

$L_A = 600$ mm

$L = L_A + L_B = 1000$ mm

$d_2 = 50$ mm $\quad d_1 = 40$ mm

$\tau_{allow} = 45$ MPa

$$T_A = \frac{T_0 L_B}{L} = \frac{P(0.4\text{ m})(400\text{ mm})}{1000\text{ mm}} = 0.16\,P$$

$$T_B = \frac{T_0 L_A}{L} = \frac{P(0.4\text{ m})(600\text{ mm})}{1000\text{ mm}} = 0.24\,P$$

UNITS: P = Newtons $\quad T$ = Newton meters

The larger torque, and hence the larger shear stress, occurs in part *CB* of the shaft.

$\therefore T_{max} = T_B = 0.24\,P$

SHEAR STRESS IN PART *CB*

$$\tau_{max} = \frac{T_{max}(d/2)}{I_P} \qquad T_{max} = \frac{2\tau_{max} I_P}{d} \qquad \text{(Eq. 1)}$$

UNITS: Newtons and meters

$\pi_{max} = 45 \times 10^6\text{N/m}^2$

$$I_P = \frac{\pi}{32}(d_2^4 - d_1^4) = 362.26 \times 10^{-9}\text{m}^4$$

$d = d_2 = 0.05$ mm

Substitute numerical values into (Eq. 1):

$$0.24P = \frac{2(45 \times 10^6\,\text{N/m}^2)(362.26 \times 10^{-9}\text{m}^4)}{0.05\text{ m}}$$

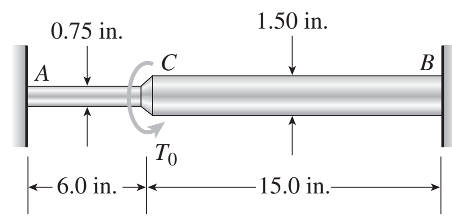
$$= 652.07\text{ N} \cdot \text{m}$$

$$P = \frac{652.07\text{ N} \cdot \text{m}}{0.24\text{ m}} = 2717\text{ N}$$

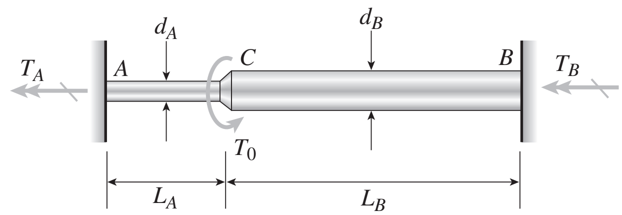
$P_{allow} = 2710$ N $\longleftarrow$

Problem 3.8-5 A stepped shaft *ACB* having solid circular cross sections with two different diameters is held against rotation at the ends (see figure).

If the allowable shear stress in the shaft is 6000 psi, what is the maximum torque $(T_0)_{max}$ that may be applied at section *C*? (*Hint:* Use Eqs. 3-45a and b of Example 3-9 to obtain the reactive torques.)



Solution 3.8-5 **Stepped shaft** ***ACB***



$d_A = 0.75$ in.

$d_B = 1.50$ in.

$L_A = 6.0$ in.

$L_B = 15.0$ in.

$\tau_{allow} = 6000$ psi

Find $(T_0)_{max}$

REACTIVE TORQUES (from Eqs. 3-45a and b)

$$T_A = T_0\left(\frac{L_B I_{PA}}{L_B I_{PA} + L_A I_{PB}}\right) \qquad (1)$$

$$T_B = T_0\left(\frac{L_A I_{PB}}{L_B I_{PA} + L_A I_{PB}}\right) \qquad (2)$$

ALLOWABLE TORQUE BASED UPON SHEAR STRESS IN SEGMENT *AC*

$$\tau_{AC} = \frac{16T_A}{\pi d}$$

$$T_A = \frac{1}{16}\pi d_A^3 \tau_{AC} = \frac{1}{16}\pi d_A^3 \tau_{allow} \qquad (3)$$

Combine Eqs. (1) and (3) and solve for T_0:

$$(T_0)_{AC} = \frac{1}{16}\pi d_A^3 \tau_{allow}\left(1 + \frac{L_A I_{PB}}{L_B I_{PA}}\right)$$

$$= \frac{1}{16}\pi d_A^3 \tau_{allow}\left(1 + \frac{L_A d_B^4}{L_B d_A^4}\right) \qquad (4)$$

Substitute numerical values:

$(T_0)_{AC} = 3678$ lb-in.

ALLOWABLE TORQUE BASED UPON SHEAR STRESS IN SEGMENT *CB*

$$\tau_{CB} = \frac{16T_B}{\pi d_B^3}$$

$$T_B = \frac{1}{16}\pi d_B^3 \tau_{CB} = \frac{1}{16}\pi d_B^3 \tau_{allow} \qquad (5)$$

Combine Eqs. (2) and (5) and solve for T_0:

$$(T_0)_{CB} = \frac{1}{16}\pi d_B^3 \tau_{allow}\left(1 + \frac{L_B I_{PA}}{L_A I_{PB}}\right)$$

$$= \frac{1}{16}\pi d_B^3 \tau_{allow}\left(1 + \frac{L_B d_A^4}{L_A d_B^4}\right) \qquad (6)$$

Substitute numerical values:

$(T_0)_{CB} = 4597$ lb-in.

SEGMENT *AC* GOVERNS

$(T_0)_{max} = 3680$ lb-in. ⟵

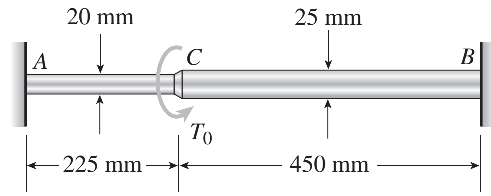
NOTE: From Eqs. (4) and (6) we find that

$$\frac{(T_0)_{AC}}{(T_0)_{CB}} = \left(\frac{L_A}{L_B}\right)\left(\frac{d_B}{d_A}\right)$$

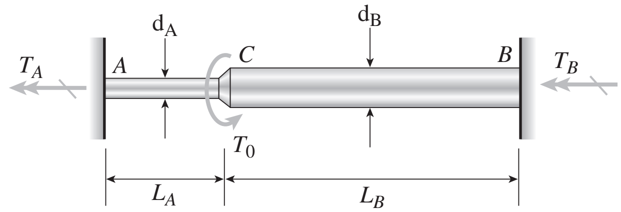
which can be used as a partial check on the results.

Problem 3.8-6 A stepped shaft ACB having solid circular cross sections with two different diameters is held against rotation at the ends (see figure).

If the allowable shear stress in the shaft is 43 MPa, what is the maximum torque $(T_0)_{max}$ that may be applied at section C? (*Hint:* Use Eqs. 3-45a and b of Example 3-9 to obtain the reactive torques.)



Solution 3.8-6 **Stepped shaft** ACB



$d_A = 20$ mm

$d_B = 25$ mm

$L_A = 225$ mm

$L_B = 450$ mm

$\tau_{allow} = 43$ MPa

Find $(T_0)_{max}$

REACTIVE TORQUES (from Eqs. 3-45a and b)

$$T_A = T_0\left(\frac{L_B I_{PA}}{L_B I_{PA} + L_A I_{PB}}\right) \quad (1)$$

$$T_B = T_0\left(\frac{L_A I_{PB}}{L_B I_{PA} + L_A I_{PB}}\right) \quad (2)$$

ALLOWABLE TORQUE BASED UPON SHEAR STRESS IN SEGMENT AC

$$\tau_{AC} = \frac{16 T_A}{\pi d_A^3}$$

$$T_A = \frac{1}{16}\pi d_A^3 \tau_{AC} = \frac{1}{16}\pi d_A^3 \tau_{allow} \quad (3)$$

Combine Eqs. (1) and (3) and solve for T_0:

$$(T_0)_{AC} = \frac{1}{16}\pi d_A^3 \tau_{allow}\left(1 + \frac{L_A I_{PB}}{L_B I_{PA}}\right)$$

$$= \frac{1}{16}\pi d_A^3 \tau_{allow}\left(1 + \frac{L_A d_B^4}{L_B d_A^4}\right) \quad (4)$$

Substitute numerical values:

$(T_0)_{AC} = 150.0 \text{ N} \cdot \text{m}$

ALLOWABLE TORQUE BASED UPON SHEAR STRESS IN SEGMENT CB

$$\tau_{CB} = \frac{16 T_B}{\pi d_B^3}$$

$$T_B = \frac{1}{16}\pi d_B^3 \tau_{CB} = \frac{1}{16}\pi d_B^3 \tau_{allow} \quad (5)$$

Combine Eqs. (2) and (5) and solve for T_0:

$$(T_0)_{CB} = \frac{1}{16}\pi d_B^3 \tau_{allow}\left(1 + \frac{L_B I_{PA}}{L_A I_{PB}}\right)$$

$$= \frac{1}{16}\pi d_B^3 \tau_{allow}\left(1 + \frac{L_B d_A^4}{L_A d_B^4}\right) \quad (6)$$

Substitute numerical values:

$(T_0)_{CB} = 240.0 \text{ N} \cdot \text{m}$

SEGMENT AC GOVERNS

$(T_0)_{max} = 150 \text{ N} \cdot \text{m}$ ←

NOTE: From Eqs. (4) and (6) we find that

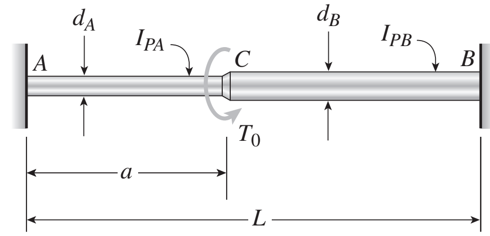
$$\frac{(T_0)_{AC}}{(T_0)_{CB}} = \left(\frac{L_A}{L_B}\right)\left(\frac{d_B}{d_A}\right)$$

which can be used as a partial check on the results.

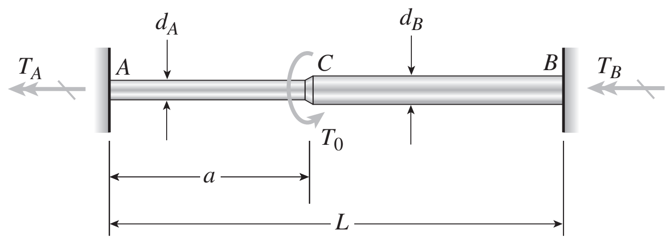
Problem 3.8-7 A stepped shaft ACB is held against rotation at ends A and B and subjected to a torque T_0 acting at section C (see figure). The two segments of the shaft (AC and CB) have diameters d_A and d_B, respectively, and polar moments of inertia I_{PA} and I_{PB}, respectively. The shaft has length L and segment AC has length a.

(a) For what ratio a/L will the maximum shear stresses be the same in both segments of the shaft?

(b) For what ratio a/L will the internal torques be the same in both segments of the shaft? (*Hint:* Use Eqs. 3-45a and b of Example 3-9 to obtain the reactive torques.)



Solution 3.8-7 **Stepped shaft**



SEGMENT AC: d_A, I_{PA} $\quad L_A = a$

SEGMENT CB: d_B, I_{PB} $\quad L_B = L - a$

REACTIVE TORQUES (from Eqs. 3-45a and b)

$$T_A = T_0\left(\frac{L_B I_{PA}}{L_B I_{PA} + L_A I_{PB}}\right); \quad T_B = T_0\left(\frac{L_A I_{PB}}{L_B I_{PA} + L_A I_{PB}}\right)$$

(a) EQUAL SHEAR STRESSES

$$\tau_{AC} = \frac{T_A(d_A/2)}{I_{PA}} \qquad \tau_{CB} = \frac{T_B(d_B/2)}{I_{PB}}$$

$$\tau_{AC} = \tau_{CB} \quad \text{or} \quad \frac{T_A d_A}{I_{PA}} = \frac{T_B d_B}{I_{PB}} \qquad \text{(Eq. 1)}$$

Substitute T_A and T_B into Eq. (1):

$$\frac{L_B I_{PA} d_A}{I_{PA}} = \frac{L_A I_{PB} d_B}{I_{PB}} \quad \text{or} \quad L_B d_A = L_A d_B$$

or $(L-a)d_A = ad_B$

Solve for a/L: $\quad \dfrac{a}{L} = \dfrac{d_A}{d_A + d_B}$ ←

(b) EQUAL TORQUES

$T_A = T_B$ or $L_B I_{PA} = L_A I_{PB}$

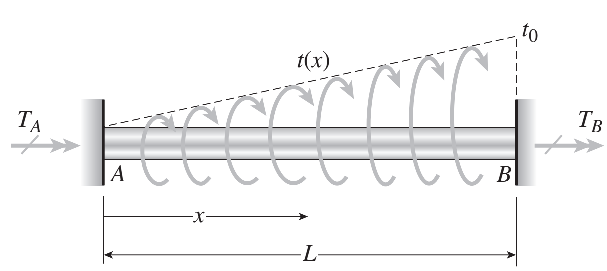
or $(L - a)\, I_{PA} = aI_{PB}$

Solve for a/L: $\quad \dfrac{a}{L} = \dfrac{I_{PA}}{I_{PA} + I_{PB}}$

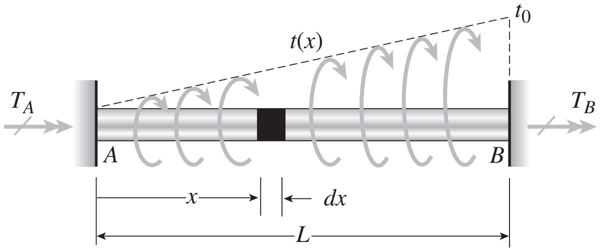
or $\quad \dfrac{a}{L} = \dfrac{d_A^4}{d_A^4 + d_B^4}$ ←

Problem 3.8-8 A circular bar AB of length L is fixed against rotation at the ends and loaded by a distributed torque $t(x)$ that varies linearly in intensity from zero at end A to t_0 at end B (see figure).

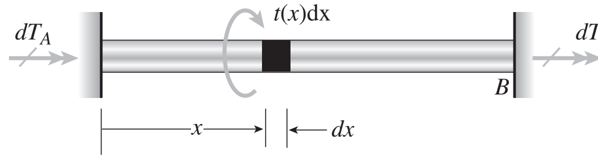
Obtain formulas for the fixed-end torques T_A and T_B.



Solution 3.8-8 Fixed-end bar with triangular load



ELEMENT OF DISTRIBUTED LOAD



$$t(x) = \frac{t_0 x}{L}$$

T_0 = Resultant of distributed torque

$$T_0 = \int_0^L t(x)dx = \int_0^L \frac{t_0 x}{L}\, dx = \frac{t_0 L}{2}$$

EQUILIBRIUM

$$T_A + T_B = T_0 = \frac{t_0 L}{2}$$

dT_A = Elemental reactive torque

dT_B = Elemental reactive torque

From Eqs. (3-46a and b):

$$dT_A = t(x)dx\left(\frac{L-x}{L}\right) \qquad dT_B = t(x)dx\left(\frac{x}{L}\right)$$

REACTIVE TORQUES (FIXED-END TORQUES)

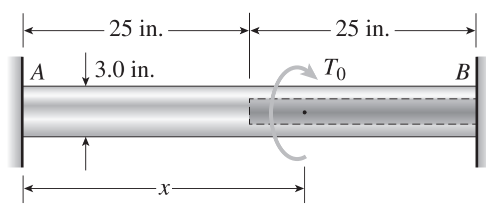
$$T_A = \int dT_A = \int_0^L \left(t_0\frac{x}{L}\right)\left(\frac{L-x}{L}\right)dx = \frac{t_0 L}{6} \quad \longleftarrow$$

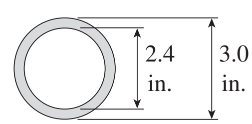
$$T_B = \int dT_B = \int_0^L \left(t_0\frac{x}{L}\right)\left(\frac{x}{L}\right)dx = \frac{t_0 L}{3} \quad \longleftarrow$$

NOTE: $T_A + T_B = \dfrac{t_0 L}{2}$

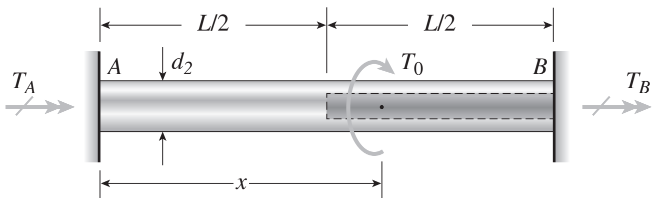
Problem 3.8-9 A circular bar *AB* with ends fixed against rotation has a hole extending for half of its length (see figure). The outer diameter of the bar is $d_2 = 3.0$ in. and the diameter of the hole is $d_1 = 2.4$ in. The total length of the bar is $L = 50$ in.

At what distance x from the left-hand end of the bar should a torque T_0 be applied so that the reactive torques at the supports will be equal?





Solution 3.8-9 Bar with a hole



$L = 50$ in.

$L/2 = 25$ in.

d_2 = outer diameter

= 3.0 in.

d_1 = diameter of hole

= 2.4 in.

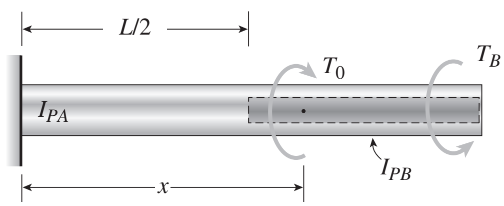
T_0 = Torque applied at distance x

Find x so that $T_A = T_B$

EQUILIBRIUM

$$T_A + T_B = T_0 \qquad \therefore\ T_A = T_B = \frac{T_0}{2} \tag{1}$$

REMOVE THE SUPPORT AT END B



ϕ_B = Angle of twist at B

I_{PA} = Polar moment of inertia at left-hand end

I_{PB} = Polar moment of inertia at right-hand end

$$\phi_B = \frac{T_B(L/2)}{GI_{PB}} + \frac{T_B(L/2)}{GI_{PA}} - \frac{T_0(x - L/2)}{GI_{PB}} - \frac{T_0(L/2)}{GI_{PA}} \tag{2}$$

Substitute Eq. (1) into Eq. (2) and simplify:

$$\phi_B = \frac{T_0}{G}\left[\frac{L}{4I_{PB}} + \frac{L}{4I_{PA}} - \frac{x}{I_{PB}} + \frac{L}{2I_{PB}} - \frac{L}{2I_{PA}}\right]$$

COMPATIBILITY $\phi_B = 0$

$$\therefore\ \frac{x}{I_{PB}} = \frac{3L}{4I_{PB}} - \frac{L}{4I_{PA}}$$

SOLVE FOR x:

$$x = \frac{L}{4}\left(3 - \frac{I_{PB}}{I_{PA}}\right)$$

$$\frac{I_{PB}}{I_{PA}} = \frac{d_2^4 - d_1^4}{d_2^4} = 1 - \left(\frac{d_1}{d_2}\right)^4$$

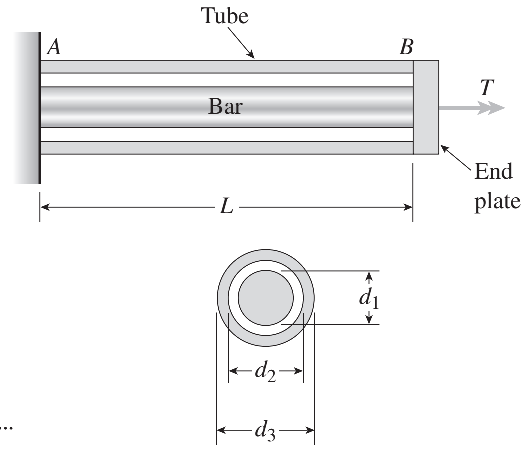
$$x = \frac{L}{4}\left[2 + \left(\frac{d_1}{d_2}\right)^4\right] \longleftarrow$$

SUBSTITUTE NUMERICAL VALUES:

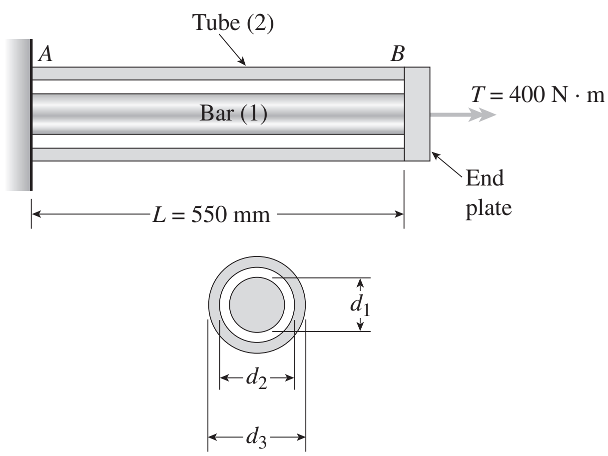
$$x = \frac{50\text{ in.}}{4}\left[2 + \left(\frac{2.4\text{ in.}}{3.0\text{ in.}}\right)^4\right] = 30.12\text{ in.} \longleftarrow$$

Problem 3.8-10 A solid steel bar of diameter $d_1 = 25.0$ mm is enclosed by a steel tube of outer diameter $d_3 = 37.5$ mm and inner diameter $d_2 = 30.0$ mm (see figure). Both bar and tube are held rigidly by a support at end A and joined securely to a rigid plate at end B. The composite bar, which has a length $L = 550$ mm, is twisted by a torque $T = 400$ N · m acting on the end plate.

(a) Determine the maximum shear stresses τ_1 and τ_2 in the bar and tube, respectively.
(b) Determine the angle of rotation ϕ (in degrees) of the end plate, assuming that the shear modulus of the steel is $G = 80$ GPa.
(c) Determine the torsional stiffness k_T of the composite bar. (*Hint:* Use Eqs. 3-44a and b to find the torques in the bar and tube.)



Solution 3.8-10 **Bar enclosed in a tube**



$d_1 = 25.0$ mm $\quad d_2 = 30.0$ mm $\quad d_3 = 37.5$ mm

$G = 80$ GPa

POLAR MOMENTS OF INERTIA

$$\text{Bar: } I_{P1} = \frac{\pi}{32} d_1^4 = 38.3495 \times 10^{-9} \text{ m}^4$$

$$\text{Tube: } I_{P2} = \frac{\pi}{32}\left(d_3^4 - d_2^4\right) = 114.6229 \times 10^{-9} \text{ m}^4$$

TORQUES IN THE BAR (1) AND TUBE (2) FROM EQS. (3-44A AND B)

$$\text{Bar: } T_1 = T\left(\frac{I_{P1}}{I_{P1} + I_{P2}}\right) = 100.2783 \text{ N} \cdot \text{m}$$

$$\text{Tube: } T_2 = T\left(\frac{I_{P2}}{I_{P1} + I_{P2}}\right) = 299.7217 \text{ N} \cdot \text{m}$$

(a) MAXIMUM SHEAR STRESSES

$$\text{Bar: } \tau_1 = \frac{T_1(d_1/2)}{I_{P1}} = 32.7 \text{ MPa} \quad \longleftarrow$$

$$\text{Tube: } \tau_2 = \frac{T_2(d_3/2)}{I_{P2}} = 49.0 \text{ MPa} \quad \longleftarrow$$

(b) ANGLE OF ROTATION OF END PLATE

$$\phi = \frac{T_1 L}{G I_{P1}} = \frac{T_2 L}{G I_{P2}} = 0.017977 \text{ rad}$$

$$\phi = 1.03^\circ \quad \longleftarrow$$

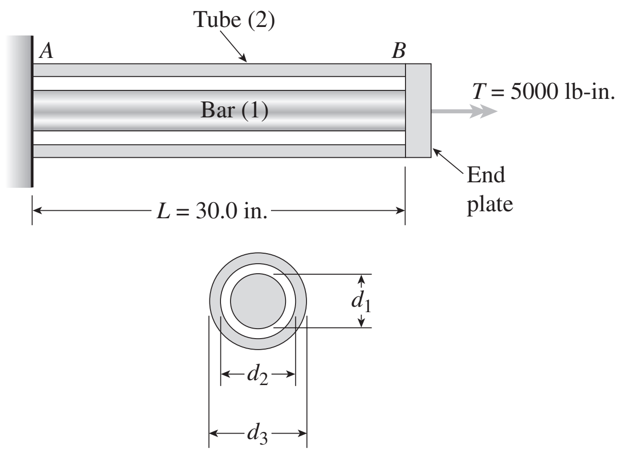
(c) TORSIONAL STIFFNESS

$$k_T = \frac{T}{\phi} = 22.3 \text{ kN} \cdot \text{m} \quad \longleftarrow$$

Problem 3.8-11 A solid steel bar of diameter $d_1 = 1.50$ in. is enclosed by a steel tube of outer diameter $d_3 = 2.25$ in. and inner diameter $d_2 = 1.75$ in. (see figure). Both bar and tube are held rigidly by a support at end A and joined securely to a rigid plate at end B. The composite bar, which has length $L = 30.0$ in., is twisted by a torque $T = 5000$ lb-in. acting on the end plate.

(a) Determine the maximum shear stresses τ_1 and τ_2 in the bar and tube, respectively.
(b) Determine the angle of rotation ϕ (in degrees) of the end plate, assuming that the shear modulus of the steel is $G = 11.6 \times 10^6$ psi.
(c) Determine the torsional stiffness k_T of the composite bar. (*Hint:* Use Eqs. 3-44a and b to find the torques in the bar and tube.)

Solution 3.8-11 **Bar enclosed in a tube**



$d_1 = 1.50$ in. $\quad d_2 = 1.75$ in. $\quad d_3 = 2.25$ in.

$G = 11.6 \times 10^6$ psi

POLAR MOMENTS OF INERTIA

Bar: $I_{P1} = \dfrac{\pi}{32} d_1^4 = 0.497010 \text{ in.}^4$

Tube: $I_{P2} = \dfrac{\pi}{32}\left(d_3^4 - d_2^4\right) = 1.595340 \text{ in.}^4$

TORQUES IN THE BAR (1) AND TUBE (2) FROM EQS. (3-44A AND B)

Bar: $T_1 = T\left(\dfrac{I_{P1}}{I_{P1} + I_{P2}}\right) = 1187.68$ lb-in.

Tube: $T_2 = T\left(\dfrac{I_{P2}}{I_{P1} + I_{P2}}\right) = 3812.32$ lb-in.

(a) MAXIMUM SHEAR STRESSES

Bar: $\tau_1 = \dfrac{T_1(d_1/2)}{I_{P1}} = 1790 \text{ psi} \quad \longleftarrow$

Tube: $\tau_2 = \dfrac{T_2(d_3/2)}{I_{P2}} = 2690 \text{ psi} \quad \longleftarrow$

(b) ANGLE OF ROTATION OF END PLATE

$$\phi = \frac{T_1 L}{G I_{P1}} = \frac{T_2 L}{G I_{P2}} = 0.00618015 \text{ rad}$$

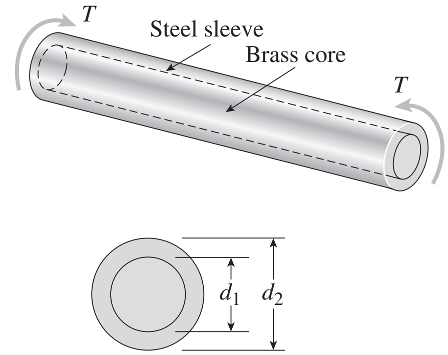
$\phi = 0.354° \quad \longleftarrow$

(c) TORSIONAL STIFFNESS

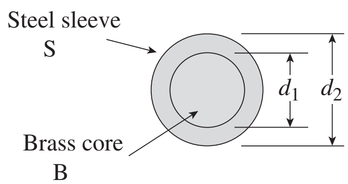
$k_T = \dfrac{T}{\phi} = 809 \text{ k-in.} \quad \longleftarrow$

Problem 3.8-12 The composite shaft shown in the figure is manufactured by shrink-fitting a steel sleeve over a brass core so that the two parts act as a single solid bar in torsion. The outer diameters of the two parts are $d_1 = 40$ mm for the brass core and $d_2 = 50$ mm for the steel sleeve. The shear moduli of elasticity are $G_b = 36$ GPa for the brass and $G_s = 80$ GPa for the steel.

Assuming that the allowable shear stresses in the brass and steel are $\tau_b = 48$ MPa and $\tau_s = 80$ MPa, respectively, determine the maximum permissible torque T_{max} that may be applied to the shaft. (*Hint:* Use Eqs. 3-44a and b to find the torques.)



Solution 3.8-12 **Composite shaft shrink fit**



$d_1 = 40$ mm

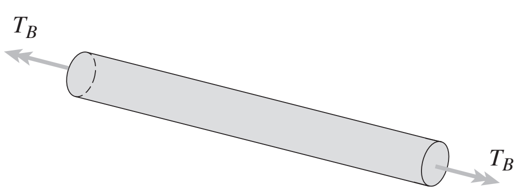
$d_2 = 50$ mm

$G_B = 36$ GPa $G_S = 80$ GPa

Allowable stresses:

$\tau_B = 48$ MPa $\tau_S = 80$ MPa

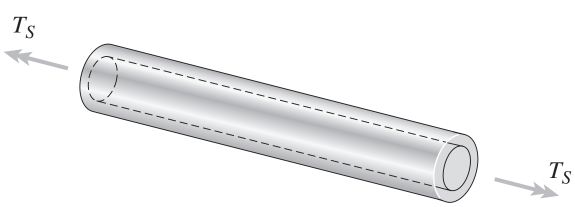
BRASS CORE (ONLY)



$$I_{PB} = \frac{\pi}{32} d_1^4 = 251.327 \times 10^{-9} \text{ m}^4$$

$$G_B I_{PB} = 9047.79 \text{ N} \cdot \text{m}^2$$

STEEL SLEEVE (ONLY)



$$I_{PS} = \frac{\pi}{32}(d_2^4 - d_1^4) = 362.265 \times 10^{-9} \text{ m}^4$$

$$G_S I_{PS} = 28{,}981.2 \text{ N} \cdot \text{m}^2$$

TORQUES

Total torque: $T = T_B + T_S$

$$\text{Eq. (3-44a): } T_B = T\left(\frac{G_B I_{PB}}{G_B I_{PB} + G_S I_{PS}}\right)$$

$$= 0.237918\, T$$

$$\text{Eq. (3-44b): } T_S = T\left(\frac{G_S I_{PS}}{G_S I_{PB} + G_S I_{PS}}\right)$$

$$= 0.762082\, T$$

$T = T_B + T_S$ (CHECK)

ALLOWABLE TORQUE T BASED UPON BRASS CORE

$$\tau_B = \frac{T_B(d_1/2)}{I_{PB}} \qquad T_B = \frac{2\tau_B I_{PB}}{d_1}$$

Substitute numerical values:

$$T_B = 0.237918\, T$$

$$= \frac{2(48 \text{ MPa})(251.327 \times 10^{-9} \text{ m}^4)}{40 \text{ mm}}$$

$$T = 2535 \text{ N} \cdot \text{m}$$

ALLOWABLE TORQUE T BASED UPON STEEL SLEEVE

$$\tau_S = \frac{T_S(d_2/2)}{I_{PS}} \qquad T_S = \frac{2\tau_S I_{PS}}{d_2}$$

SUBSTITUTE NUMERICAL VALUES:

$$T_S = 0.762082\, T$$

$$= \frac{2(80 \text{ MPa})(362.265 \times 10^{-9} \text{ m}^4)}{50 \text{ mm}}$$

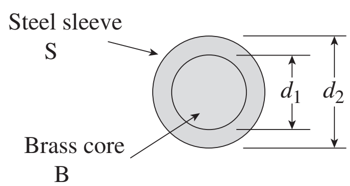
$$T = 1521 \text{ N} \cdot \text{m}$$

STEEL SLEEVE GOVERNS $T_{max} = 1520 \text{ N} \cdot \text{m}$ ←

Problem 3.8-13 The composite shaft shown in the figure is manufactured by shrink-fitting a steel sleeve over a brass core so that the two parts act as a single solid bar in torsion. The outer diameters of the two parts are $d_1 = 1.6$ in. for the brass core and $d_2 = 2.0$ in. for the steel sleeve. The shear moduli of elasticity are $G_b = 5400$ ksi for the brass and $G_s = 12{,}000$ ksi for the steel.

Assuming that the allowable shear stresses in the brass and steel are $\tau_b = 4500$ psi and $\tau_s = 7500$ psi, respectively, determine the maximum permissible torque T_{max} that may be applied to the shaft. (*Hint:* Use Eqs. 3-44a and b to find the torques.)

Solution 3.8-13 Composite shaft shrink fit



$d_1 = 1.6$ in.

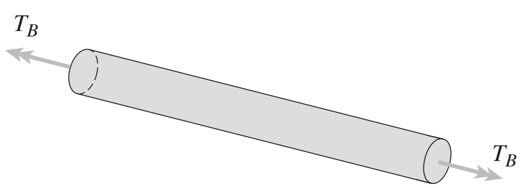
$d_2 = 2.0$ in.

$G_B = 5{,}400$ psi $\quad G_S = 12{,}000$ psi

Allowable stresses:

$\tau_B = 4500$ psi $\quad \tau_S = 7500$ psi

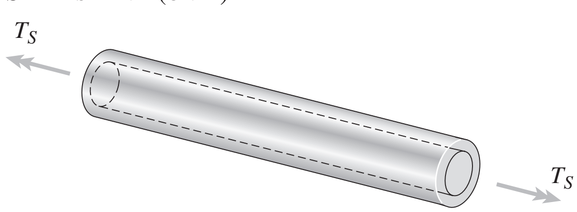
BRASS CORE (ONLY)



$$I_{PB} = \frac{\pi}{32} d_1^4 = 0.643398 \text{ in.}^4$$

$$G_B I_{PB} = 3.47435 \times 10^6 \text{ lb-in.}^2$$

STEEL SLEEVE (ONLY)



$$I_{PS} = \frac{\pi}{32}(d_2^4 - d_1^4) = 0.927398 \text{ in.}^4$$

$$G_S I_{PS} = 11.1288 \times 10^6 \text{ lb-in.}^2$$

TORQUES

Total torque: $T = T_B + T_S$

$$\text{Eq. (3-44a): } T_B = T\left(\frac{G_B I_{PB}}{G_B I_{PB} + G_S I_{PS}}\right)$$

$$= 0.237918\, T$$

$$\text{Eq. (3-44b): } T_S = T\left(\frac{G_S I_{PS}}{G_B I_{PB} + G_S I_{PS}}\right)$$

$$= 0.762082\, T$$

$T = T_B + T_S$ (CHECK)

ALLOWABLE TORQUE T BASED UPON BRASS CORE

$$\tau_B = \frac{T_B(d_1/2)}{I_{PB}} \qquad T_B = \frac{2\tau_B I_{PB}}{d_1}$$

Substitute numerical values:

$$T_B = 0.237918\, T$$

$$= \frac{2(4500 \text{ psi})(0.643398 \text{ in.}^4)}{1.6 \text{ in.}}$$

$T = 15.21$ k-in.

ALLOWABLE TORQUE T BASED UPON STEEL SLEEVE

$$\tau_S = \frac{T_S(d_2/2)}{I_{PS}} \qquad T_S = \frac{2\tau_S I_{PS}}{d_2}$$

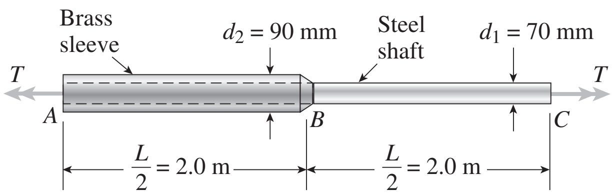
Substitute numerical values:

$$T_S = 0.762082\, T = \frac{2(7500 \text{ psi})(0.927398 \text{ in.}^4)}{2.0 \text{ in.}}$$

$T = 9.13$ k-in.

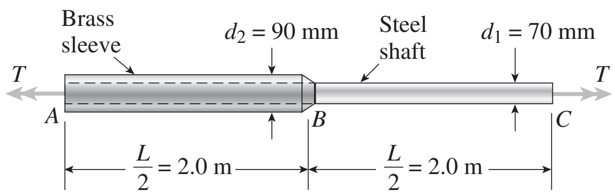
STEEL SLEEVE GOVERNS $\quad T_{max} = 9.13$ k-in. ←

Problem 3.8-14 A steel shaft ($G_s = 80$ GPa) of total length $L = 4.0$ m is encased for one-half of its length by a brass sleeve ($G_b = 40$ GPa) that is securely bonded to the steel (see figure). The outer diameters of the shaft and sleeve are $d_1 = 70$ mm and $d_2 = 90$ mm, respectively.



(a) Determine the allowable torque T_1 that may be applied to the ends of the shaft if the angle of twist ϕ between the ends is limited to 8.0°.
(b) Determine the allowable torque T_2 if the shear stress in the brass is limited to $\tau_b = 70$ MPa.
(c) Determine the allowable torque T_3 if the shear stress in the steel is limited to $\tau_s = 110$ MPa.
(d) What is the maximum allowable torque T_{max} if all three of the preceding conditions must be satisfied?

Solution 3.8-14 **Composite shaft**



PROPERTIES OF THE STEEL SHAFT (s)

$G_S = 80$ GPa $\quad d_1 = 70$ mm

Allowable shear stress: $\tau_S = 110$ MPa

$$I_{PS} = \frac{\pi}{32} d_1^4 = 2.3572 \times 10^{-6} \text{ m}^4$$

$$G_S I_{PS} = 188.574 \times 10^3 \text{ N} \cdot \text{m}^2$$

PROPERTIES OF THE BRASS SLEEVE (b)

$G_b = 40$ GPa $\quad d_2 = 90$ mm $\quad d_1 = 70$ mm

Allowable shear stress: $\tau_b = 70$ MPa

$$I_{PB} = \frac{\pi}{32}(d_2^4 - d_1^4) = 4.0841 \times 10^{-6} \text{ m}^4$$

$$G_b I_{PB} = 163.363 \times 10^3 \text{ N} \cdot \text{m}^2$$

TORQUES IN THE COMPOSITE BAR AB

T_S = Torque in the steel shaft AB

T_b = Torque in the brass sleeve AB

From Eq. (3-44a): $T_S = T\left(\dfrac{G_S I_{PS}}{G_S I_{PS} + G_b I_{Pb}}\right)$

$$T_S = T(0.53582) \qquad \text{(Eq. 1)}$$

$$T_b = T - T_S = T(0.46418) \qquad \text{(Eq. 2)}$$

ANGLE OF TWIST OF THE COMPOSITE BAR AB

$$\phi_{AB} = \frac{T_S(L/2)}{G_S I_{PS}} = \frac{T_b(L/2)}{G_b I_{Pb}}$$

$$= (5.6828 \times 10^{-6})T \qquad \text{(Eq. 3)}$$

UNITS: T = N · m $\quad \phi$ = rad

ANGLE OF TWIST OF PART BC OF THE STEEL SHAFT

$$\phi_{BC} = \frac{T(L/2)}{G_S I_{PS}} = (10.6059 \times 10^{-6})T \qquad \text{(Eq. 4)}$$

ANGLE OF TWIST OF THE ENTIRE SHAFT ABC

$$\phi = \phi_{AB} + \phi_{BC} \qquad \text{(Eqs. 3 and 4)}$$

$$\phi = (16.2887 \times 10^{-6})\, T$$

UNITS: ϕ = rad

T = N · m

(a) ALLOWABLE TORQUE T_1 BASED UPON ANGLE OF TWIST

$$\phi_{allow} = 8.0° = 0.13963 \text{ rad}$$

$$\phi = (16.2887 \times 10^{-6})\, T = 0.13963 \text{ rad}$$

$$T_1 = 8.57 \text{ kN} \cdot \text{m} \quad \longleftarrow$$

(b) ALLOWABLE TORQUE T$_2$ BASED UPON SHEAR STRESS IN THE BRASS SLEEVE

$$\tau_b = \frac{T(d_2/2)}{I_{pb}} \quad \tau_b = 70 \text{ MPa}$$

$$T_b = 0.46418\, T \text{ (From Eq. 2)}$$

$$70 \text{ MPa} = \frac{(0.46418T)(0.045 \text{ m})}{4.0841 \times 10^{-6} \text{ m}^4}$$

Solve for T (Equal to T_2): $T_2 = 13.69$ kN·m $\longleftarrow$

(c) ALLOWABLE TORQUE T_3 BASED UPON SHEAR STRESS IN THE STEEL SHAFT BC

$$\tau_S = \frac{T(d_2/2)}{I_{PS}} \quad \tau_S = 110 \text{ MPa}$$

$$110 \text{ MPa} = \frac{T(0.035 \text{ m})}{2.3572 \times 10^{-6} \text{ m}^4}$$

Solve for T (Equal to T_3):

$T_3 = 7.41$ kN·m $\longleftarrow$

(d) MAXIMUM ALLOWABLE TORQUE

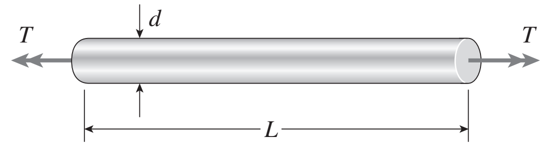
Shear stress in steel governs

$T_{max} = 7.41$ kN · m $\longleftarrow$

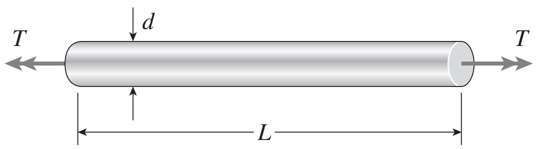
Strain Energy in Torsion

Problem 3.9-1 A solid circular bar of steel ($G = 11.4 \times 10^6$ psi) with length $L = 30$ in. and diameter $d = 1.75$ in. is subjected to pure torsion by torques T acting at the ends (see figure).

(a) Calculate the amount of strain energy U stored in the bar when the maximum shear stress is 4500 psi.
(b) From the strain energy, calculate the angle of twist ϕ (in degrees).



Solution 3.9-1 **Steel bar**



$G = 11.4 \times 10^6$ psi

$L = 30$ in.

$d = 1.75$ in.

$\tau_{max} = 4500$ psi

$$\tau_{max} = \frac{16\,T}{\pi d^3} \qquad T = \frac{\pi d^3 \tau_{max}}{16} \qquad \text{(Eq. 1)}$$

$$I_P = \frac{\pi d^4}{32}$$

(a) STRAIN ENERGY

$$U = \frac{T^2 L}{2GI_P} = \left(\frac{\pi d^3 \tau_{max}}{16}\right)^2 \left(\frac{L}{2G}\right)\left(\frac{32}{\pi d^4}\right)$$

$$= \frac{\pi d^2 L \tau_{max}^2}{16G} \qquad \text{(Eq. 2)}$$

Substitute numerical values:

$U = 32.0$ in.-lb ⟵

(b) ANGLE OF TWIST

$$U = \frac{T\phi}{2} \qquad \phi = \frac{2U}{T}$$

Substitute for T and U from Eqs. (1) and (2):

$$\phi = \frac{2L\tau_{max}}{Gd} \qquad \text{(Eq. 3)}$$

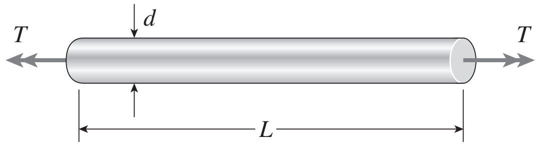
Substitute numerical values:

$\phi = 0.013534 \text{ rad} = 0.775°$ ⟵

Problem 3.9-2 A solid circular bar of copper ($G = 45$ GPa) with length $L = 0.75$ m and diameter $d = 40$ mm is subjected to pure torsion by torques T acting at the ends (see figure).

(a) Calculate the amount of strain energy U stored in the bar when the maximum shear stress is 32 MPa.
(b) From the strain energy, calculate the angle of twist ϕ (in degrees)

Solution 3.9-2 Copper bar



$G = 45$ GPa

$L = 0.75$ m

$d = 40$ mm

$\tau_{max} = 32$ MPa

$$\tau_{max} = \frac{16T}{\pi d^3} \qquad T = \frac{\pi d^3 \tau_{max}}{16} \qquad \text{(Eq. 1)}$$

$$I_P = \frac{\pi d^4}{32}$$

(a) STRAIN ENERGY

$$U = \frac{T^2 L}{2GI_P} = \left(\frac{\pi d^3 \tau_{max}}{16}\right)^2 \left(\frac{L}{2G}\right)\left(\frac{32}{\pi d^4}\right)$$

$$= \frac{\pi d^2 L \tau_{max}^2}{16G} \qquad \text{(Eq. 2)}$$

Substitute numerical values:

$U = 5.36$ J ⟵

(b) ANGLE OF TWIST

$$U = \frac{T\phi}{2} \qquad \phi = \frac{2U}{T}$$

Substitute for T and U from Eqs. (1) and (2):

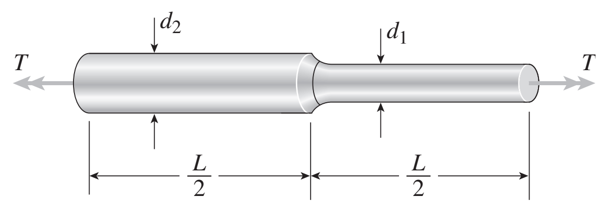
$$\phi = \frac{2L\tau_{max}}{Gd} \qquad \text{(Eq. 3)}$$

Substitute numerical values:

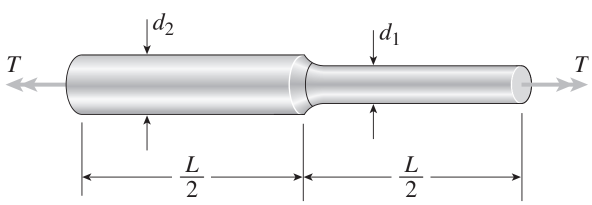
$\phi = 0.026667$ rad $= 1.53°$ ⟵

Problem 3.9-3 A stepped shaft of solid circular cross sections (see figure) has length $L = 45$ in., diameter $d_2 = 1.2$ in., and diameter $d_1 = 1.0$ in. The material is brass with $G = 5.6 \times 10^6$ psi.

Determine the strain energy U of the shaft if the angle of twist is 3.0°.



Solution 3.9-3 Stepped shaft



$d_1 = 1.0$ in.

$d_2 = 1.2$ in.

$L = 45$ in.

$G = 5.6 \times 10^6$ psi (brass)

$\phi = 3.0° = 0.0523599$ rad

STRAIN ENERGY

$$U = \sum \frac{T^2 L}{2GI_P} = \frac{16\, T^2(L/2)}{\pi G d_2^4} + \frac{16\, T^2(L/2)}{\pi G d_1^4}$$

$$= \frac{8T^2 L}{\pi G}\left(\frac{1}{d_2^4} + \frac{1}{d_1^4}\right) \qquad \text{(Eq. 1)}$$

Also, $U = \dfrac{T\phi}{2}$ (Eq. 2)

Equate U from Eqs. (1) and (2) and solve for T:

$$T = \frac{\pi G d_1^4 d_2^4 \phi}{16L(d_1^4 + d_2^4)}$$

$$U = \frac{T\phi}{2} = \frac{\pi G \phi^2}{32L}\left(\frac{d_1^4 d_2^4}{d_1^4 + d_2^4}\right) \qquad \phi = \text{radians}$$

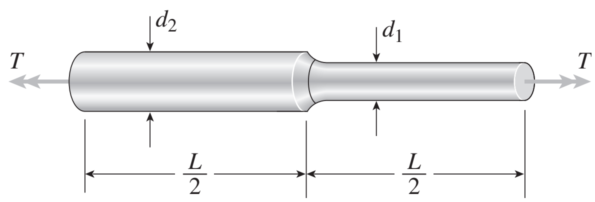
SUBSTITUTE NUMERICAL VALUES:

$U = 22.6$ in.-lb ⟵

Problem 3.9-4 A stepped shaft of solid circular cross sections (see figure) has length $L = 0.80$ m, diameter $d_2 = 40$ mm, and diameter $d_1 = 30$ mm. The material is steel with $G = 80$ GPa.

Determine the strain energy U of the shaft if the angle of twist is 1.0°.

Soluton 3.9-4 Stepped shaft



$d_1 = 30$ mm $\quad d_2 = 40$ mm

$L = 0.80$ m $\quad G = 80$ GPa (steel)

$\phi = 1.0° = 0.0174533$ rad

STRAIN ENERGY

$$U = \sum \frac{T^2L}{2GI_P} = \frac{16T^2(L/2)}{\pi G d_2^4} + \frac{16T^2(L/2)}{\pi G d_1^4}$$

$$= \frac{8T^2L}{\pi G}\left(\frac{1}{d_2^4} + \frac{1}{d_1^4}\right) \qquad \text{(Eq. 1)}$$

Also, $U = \dfrac{T\phi}{2}$ (Eq. 2)

Equate U from Eqs. (1) and (2) and solve for T:

$$T = \frac{\pi G\, d_1^4\, d_2^4\, \phi}{16L(d_1^4 + d_2^4)}$$

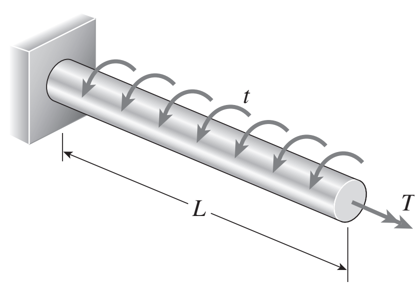
$$U = \frac{T\phi}{2} = \frac{\pi G \phi^2}{32L}\left(\frac{d_1^4\, d_2^4}{d_1^4 + d_2^4}\right) \qquad \phi = \text{radians}$$

SUBSTITUTE NUMERICAL VALUES:

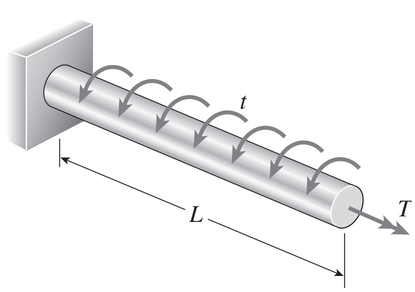
$U = 1.84$ J ←

Problem 3.9-5 A cantilever bar of circular cross section and length L is fixed at one end and free at the other (see figure). The bar is loaded by a torque T at the free end and by a distributed torque of constant intensity t per unit distance along the length of the bar.

(a) What is the strain energy U_1 of the bar when the load T acts alone?
(b) What is the strain energy U_2 when the load t acts alone?
(c) What is the strain energy U_3 when both loads act simultaneously?



Solution 3.9-5 Cantilever bar with distributed torque



G = shear modulus

I_P = polar moment of inertia

T = torque acting at free end

t = torque per unit distance

(a) LOAD T ACTS ALONE (Eq. 3-51a)

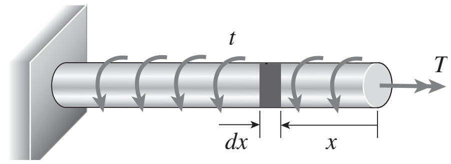
$$U_1 = \frac{T^2L}{2GI_P} \quad \longleftarrow$$

(b) LOAD t ACTS ALONE

From Eq. (3-56) of Example 3-11:

$$U_2 = \frac{t^2L^3}{6GI_P} \quad \longleftarrow$$

(c) BOTH LOADS ACT SIMULTANEOUSLY



At distance x from the free end:

$T(x) = T + tx$

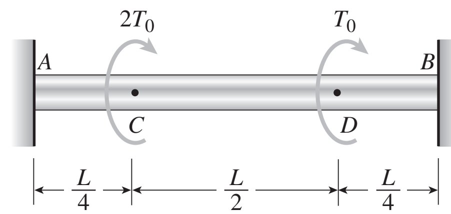
$$U_3 = \int_0^L \frac{[T(x)]^2}{2GI_P}dx = \frac{1}{2GI_P}\int_0^L (T + tx)^2 dx$$

$$= \frac{T^2L}{2GI_P} + \frac{TtL^2}{2GI_P} + \frac{t^2L^3}{6GI_P} \quad \longleftarrow$$

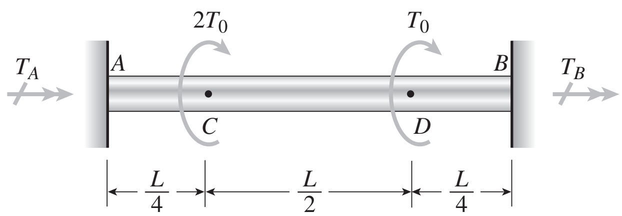
NOTE: U_3 is *not* the sum of U_1 and U_2.

Problem 3.9-6 Obtain a formula for the strain energy U of the statically indeterminate circular bar shown in the figure. The bar has fixed supports at ends A and B and is loaded by torques $2T_0$ and T_0 at points C and D, respectively.

Hint: Use Eqs. 3-46a and b of Example 3-9, Section 3.8, to obtain the reactive torques.



Solution 3.9-6 **Statically indeterminate bar**



REACTIVE TORQUES

From Eq. (3-46a):

$$T_A = \frac{(2T_0)\left(\frac{3L}{4}\right)}{L} + \frac{T_0\left(\frac{L}{4}\right)}{L} = \frac{7T_0}{4}$$

$$T_B = 3T_0 - T_A = \frac{5T_0}{4}$$

INTERNAL TORQUES

$$T_{AC} = -\frac{7T_0}{4} \qquad T_{CD} = \frac{T_0}{4} \qquad T_{DB} = \frac{5T_0}{4}$$

STRAIN ENERGY (from Eq. 3-53)

$$U = \sum_{i=1}^{n} \frac{T_i^2 L_i}{2G_i I_{Pi}}$$

$$= \frac{1}{2GI_p}\left[T_{AC}^2\left(\frac{L}{4}\right) + T_{CD}^2\left(\frac{L}{2}\right) + T_{DB}^2\left(\frac{L}{4}\right)\right]$$

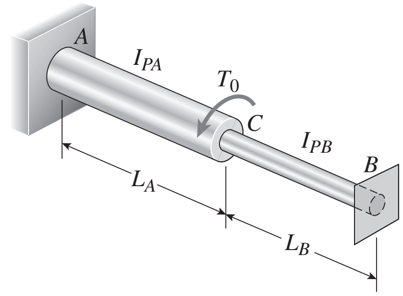
$$= \frac{1}{2GI_P}\left[\left(-\frac{7T_0}{4}\right)^2\left(\frac{L}{4}\right) + \left(\frac{T_0}{4}\right)^2\left(\frac{L}{2}\right) + \left(\frac{5T_0}{4}\right)^2\left(\frac{L}{4}\right)\right]$$

$$U = \frac{19T_0^2L}{32GI_P} \quad \longleftarrow$$

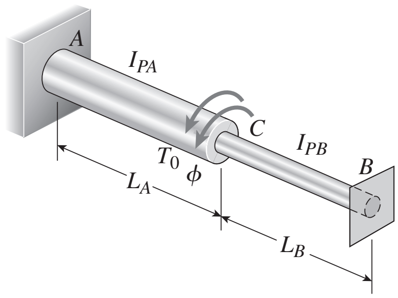
Problem 3.9-7 A statically indeterminate stepped shaft ACB is fixed at ends A and B and loaded by a torque T_0 at point C (see figure). The two segments of the bar are made of the same material, have lengths L_A and L_B, and have polar moments of inertia I_{PA} and I_{PB}.

Determine the angle of rotation ϕ of the cross section at C by using strain energy.

Hint: Use Eq. 3-51b to determine the strain energy U in terms of the angle ϕ. Then equate the strain energy to the work done by the torque T_0. Compare your result with Eq. 3-48 of Example 3-9, Section 3.8.



Solution 3.9-7 Statically indeterminate bar



STRAIN ENERGY (FROM EQ. 3-51B)

$$U = \sum_{i=1}^{n} \frac{GI_{Pi}\phi_i^2}{2L_i} = \frac{GI_{PA}\phi^2}{2L_A} + \frac{GI_{PB}\phi^2}{2L_B}$$

$$= \frac{G\phi^2}{2}\left(\frac{I_{PA}}{L_A} + \frac{I_{PB}}{L_B}\right)$$

WORK DONE BY THE TORQUE T_0

$$W = \frac{T_0\phi}{2}$$

EQUATE U AND W AND SOLVE FOR ϕ

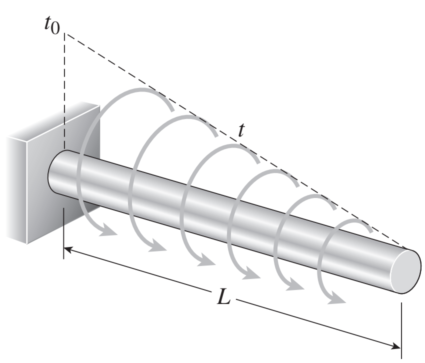
$$\frac{G\phi^2}{2}\left(\frac{I_{PA}}{L_A} + \frac{I_{PB}}{L_B}\right) = \frac{T_0\phi}{2}$$

$$\phi = \frac{T_0 L_A L_B}{G(L_B I_{PA} + L_A I_{PB})} \quad \longleftarrow$$

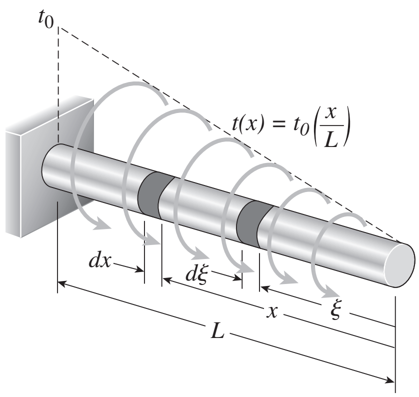
(This result agrees with Eq. (3-48) of Example 3-9, Section 3.8.)

Problem 3.9-8 Derive a formula for the strain energy U of the cantilever bar shown in the figure.

The bar has circular cross sections and length L. It is subjected to a distributed torque of intensity t per unit distance. The intensity varies linearly from $t = 0$ at the free end to a maximum value $t = t_0$ at the support.



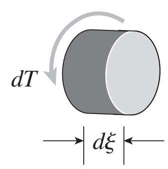
Solution 3.9-8 Cantilever bar with distributed torque



x = distance from right-hand end of the bar

ELEMENT $d\xi$

Consider a differential element $d\xi$ at distance ξ from the right-hand end.

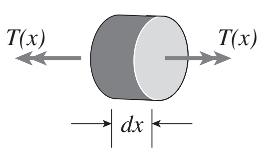


dT = external torque acting on this element

$$dT = t(\xi)d\xi$$

$$= t_0\left(\frac{\xi}{L}\right)d\xi$$

ELEMENT dx AT DISTANCE x



$T(x)$ = internal torque acting on this element

$T(x)$ = total torque from $x = 0$ to $x = x$

$$T(x) = \int_0^x dT = \int_0^x t_0\left(\frac{\xi}{L}\right)d\xi$$

$$= \frac{t_0 x^2}{2L}$$

STRAIN ENERGY OF ELEMENT dx

$$dU = \frac{[T(x)]^2 dx}{2GI_P} = \frac{1}{2GI_P}\left(\frac{t_0}{2L}\right)^2 x^4\,dx$$

$$= \frac{t_0^2}{8L^2 GI_P}x^4\,dx$$

STRAIN ENERGY OF ENTIRE BAR

$$U = \int_0^L dU = \frac{t_0^2}{8L^2GI_P}\int_0^L x^4\,dx$$

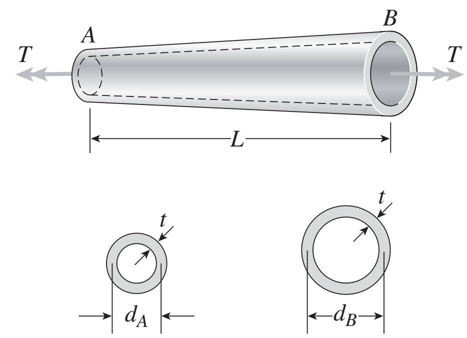
$$= \frac{t_0^2}{8L^2GI_P}\left(\frac{L^5}{5}\right)$$

$$U = \frac{t_0^2 L^3}{40GI_P} \longleftarrow$$

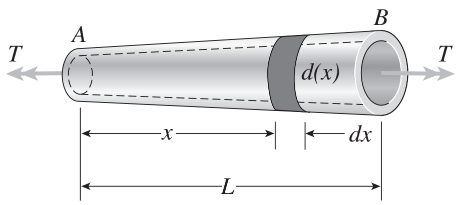
Problem 3.9-9 A thin-walled hollow tube AB of conical shape has constant thickness t and average diameters d_A and d_B at the ends (see figure).

(a) Determine the strain energy U of the tube when it is subjected to pure torsion by torques T.
(b) Determine the angle of twist ϕ of the tube.

Note: Use the approximate formula $I_P \approx \pi d^3 t/4$ for a thin circular ring; see Case 22 of Appendix D.



Solution 3.9-9 Thin-walled, hollow tube



t = thickness

d_A = average diameter at end A

d_B = average diameter at end B

$d(x)$ = average diameter at distance x from end A

$$d(x) = d_A + \left(\frac{d_B - d_A}{L}\right)x$$

POLAR MOMENT OF INERTIA

$$I_P = \frac{\pi d^3 t}{4}$$

$$I_P(x) = \frac{\pi[d(x)]^3 t}{4} = \frac{\pi t}{4}\left[d_A + \left(\frac{d_B - d_A}{L}\right)x\right]^3$$

(a) STRAIN ENERGY (FROM EQ. 3-54)

$$U = \int_0^L \frac{T^2 dx}{2GI_P(x)}$$

$$= \frac{2T^2}{\pi G t}\int_0^L \frac{dx}{\left[d_A + \left(\frac{d_B - d_A}{L}\right)x\right]^3} \qquad \text{(Eq. 1)}$$

From Appendix C:

$$\int \frac{dx}{(a+bx)^3} = -\frac{1}{2b(a+bx)^2}$$

Therefore,

$$\int_0^L \frac{dx}{\left[d_A + \left(\frac{d_B - d_A}{L}\right)x\right]^3}$$

$$= -\frac{1}{\frac{2(d_B - d_A)}{L}\left[d_A + \left(\frac{d_B - d_A}{L}\right)x\right]^2}\Bigg|_0^L$$

$$= -\frac{L}{2(d_B - d_A)(d_B)^2} + \frac{L}{2(d_B - d_A)(d_A)^2}$$

$$= \frac{L(d_A + d_B)}{2d_A^2 d_B^2}$$

Substitute this expression for the integral into the equation for U (Eq. 1):

$$U = \frac{2T^2}{\pi G t}\cdot\frac{L(d_A + d_B)}{2d_A^2 d_B^2} = \frac{T^2 L}{\pi G t}\left(\frac{d_A + d_B}{d_A^2\, d_B^2}\right) \longleftarrow$$

(b) ANGLE OF TWIST

Work of the torque T: $W = \frac{T\phi}{2}$

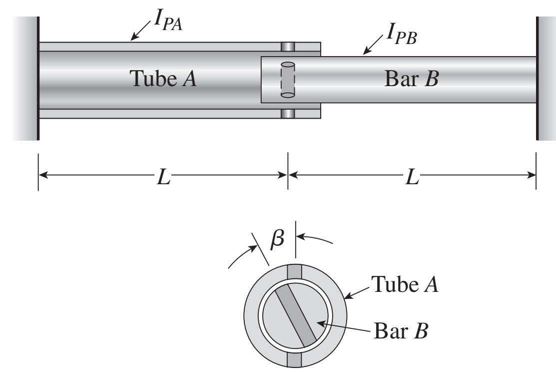
$$W = U \qquad \frac{T\phi}{2} = \frac{T^2 L(d_A + d_B)}{\pi G t\, d_A^2 d_B^2}$$

Solve for ϕ:

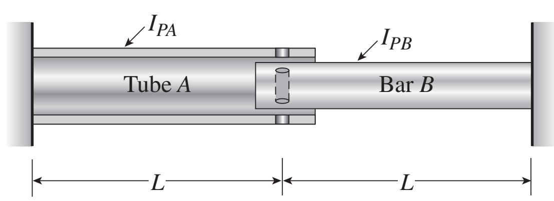
$$\phi = \frac{2TL(d_A + d_B)}{\pi G t\, d_A^2 d_B^2} \longleftarrow$$

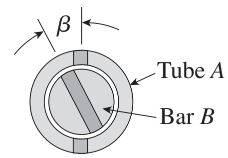
Problem 3.9-10 A hollow circular tube A fits over the end of a solid circular bar B, as shown in the figure. The far ends of both bars are fixed. Initially, a hole through bar B makes an angle β with a line through two holes in tube A. Then bar B is twisted until the holes are aligned, and a pin is placed through the holes.

When bar B is released and the system returns to equilibrium, what is the total strain energy U of the two bars? (Let I_{PA} and I_{PB} represent the polar moments of inertia of bars A and B, respectively. The length L and shear modulus of elasticity G are the same for both bars.)

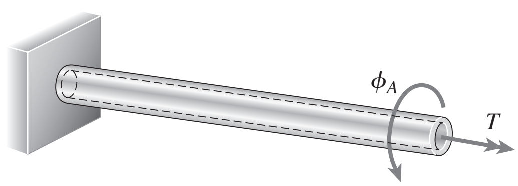


Solution 3.9-10 Circular tube and bar





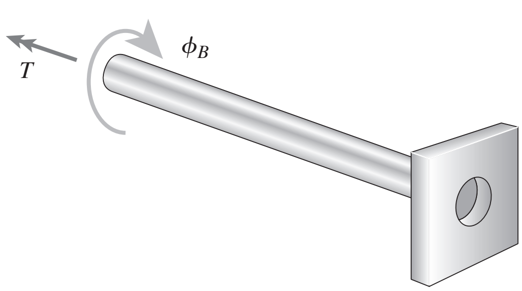
Tube A



T = torque acting on the tube

ϕ_A = angle of twist

Bar B



T = torque acting on the bar

ϕ_B = angle of twist

Compatibility

$$\phi_A + \phi_B = \beta$$

Force-displacement relations

$$\phi_A = \frac{TL}{GI_{PA}} \qquad \phi_B = \frac{TL}{GI_{PB}}$$

Substitute into the equation of compatibility and solve for T:

$$T = \frac{\beta G}{L}\left(\frac{I_{PA}I_{PB}}{I_{PA} + I_{PB}}\right)$$

Strain energy

$$U = \sum \frac{T^2L}{2GI_P} = \frac{T^2L}{2GI_{PA}} + \frac{T^2L}{2GI_{PB}}$$

$$= \frac{T^2L}{2G}\left(\frac{1}{I_{PA}} + \frac{1}{I_{PB}}\right)$$

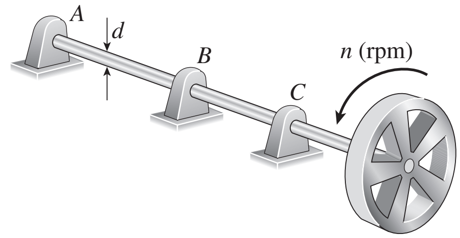
Substitute for T and simplify:

$$U = \frac{\beta^2 G}{2L}\left(\frac{I_{PA}\,I_{PB}}{I_{PA} + I_{PB}}\right) \longleftarrow$$

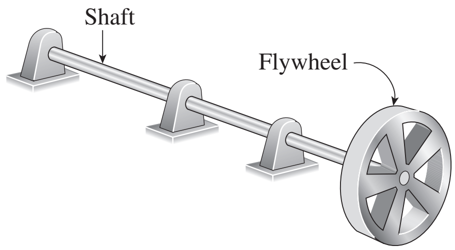
Problem 3.9-11 A heavy flywheel rotating at n revolutions per minute is rigidly attached to the end of a shaft of diameter d (see figure). If the bearing at A suddenly freezes, what will be the maximum angle of twist ϕ of the shaft? What is the corresponding maximum shear stress in the shaft?

(Let L = length of the shaft, G = shear modulus of elasticity, and I_m = mass moment of inertia of the flywheel about the axis of the shaft. Also, disregard friction in the bearings at B and C and disregard the mass of the shaft.)

Hint: Equate the kinetic energy of the rotating flywheel to the strain energy of the shaft.



Solution 3.9-11 **Rotating flywheel**



d = diameter

n = rpm

KINETIC ENERGY OF FLYWHEEL

$$\text{K.E.} = \frac{1}{2} I_m \omega^2$$

$$\omega = \frac{2\pi n}{60}$$

n = rpm

$$\text{K.E.} = \frac{1}{2} I_m \left(\frac{2\pi n}{60}\right)^2 = \frac{\pi^2 n^2 I_m}{1800}$$

UNITS:

I_m = (force)(length)(second)^2

ω = radians per second

K.E. = (length)(force)

STRAIN ENERGY OF SHAFT (FROM EQ. 3-51b)

$$U = \frac{G I_P \phi^2}{2L}$$

$$I_P = \frac{\pi}{32} d^4$$

d = diameter of shaft

$$U = \frac{\pi G d^4 \phi^2}{64L}$$

UNITS:

G = (force)/(length)^2

I_P = (length)^4

ϕ = radians

L = length

U = (length)(force)

EQUATE KINETIC ENERGY AND STRAIN ENERGY

$$\text{K.E.} = U \qquad \frac{\pi^2 n^2 I_m}{1800} = \frac{\pi G d^4 \phi^2}{64L}$$

Solve for ϕ:

$$\phi = \frac{2n}{15d^2}\sqrt{\frac{2\pi I_m L}{G}} \quad \longleftarrow$$

MAXIMUM SHEAR STRESS

$$\tau = \frac{T(d/2)}{I_P} \qquad \phi = \frac{TL}{GI_P}$$

Eliminate T:

$$\tau = \frac{Gd\phi}{2L}$$

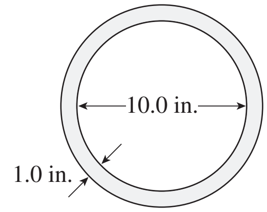
$$\tau_{\max} = \frac{Gd}{2L} \cdot \frac{2n}{15d^2}\sqrt{\frac{2\pi I_m L}{G}}$$

$$\tau_{\max} = \frac{n}{15d}\sqrt{\frac{2\pi G I_m}{L}} \quad \longleftarrow$$

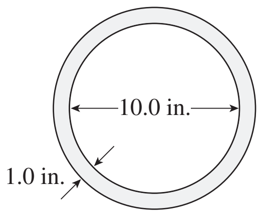
Thin-Walled Tubes

Problem 3.10-1 A hollow circular tube having an inside diameter of 10.0 in. and a wall thickness of 1.0 in. (see figure) is subjected to a torque $T = 1200$ k-in.

Determine the maximum shear stress in the tube using (a) the approximate theory of thin-walled tubes, and (b) the exact torsion theory. Does the approximate theory give conservative or nonconservative results?



Solution 3.10-1 **Hollow circular tube**



$T = 1200$ k-in.

$t = 1.0$ in.

$r =$ radius to median line

$r = 5.5$ in.

$d_2 =$ outside diameter $= 12.0$ in.

$d_1 =$ inside diameter $= 10.0$ in.

APPROXIMATE THEORY (EQ. 3-63)

$$\tau_1 = \frac{T}{2\pi r^2 t} = \frac{1200 \text{ k-in.}}{2\pi(5.5 \text{ in.})^2(1.0 \text{ in.})} = 6314 \text{ psi}$$

$$\tau_{\text{approx}} = 6310 \text{ psi} \longleftarrow$$

EXACT THEORY (EQ. 3-11)

$$\tau_2 = \frac{T(d_2/2)}{I_P} = \frac{Td_2}{2\left(\frac{\pi}{32}\right) d_2^4 - d_1^4}$$

$$= \frac{16(1200 \text{ k-in.})(12.0 \text{ in.})}{\pi[(12.0 \text{ in.})^4 - (10.0 \text{ in.})^4]}$$

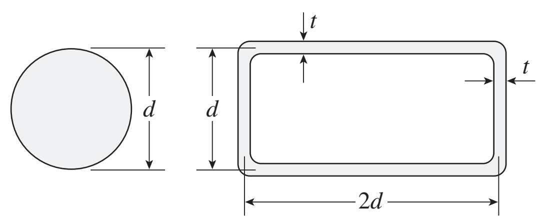
$$= 6831 \text{ psi}$$

$$\tau_{\text{exact}} = 6830 \text{ psi} \longleftarrow$$

Because the approximate theory gives stresses that are too low, it is nonconservative. Therefore, the approximate theory should only be used for very thin tubes.

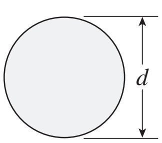
Problem 3.10-2 A solid circular bar having diameter d is to be replaced by a rectangular tube having cross-sectional dimensions $d \times 2d$ to the median line of the cross section (see figure).

Determine the required thickness t_{min} of the tube so that the maximum shear stress in the tube will not exceed the maximum shear stress in the solid bar.



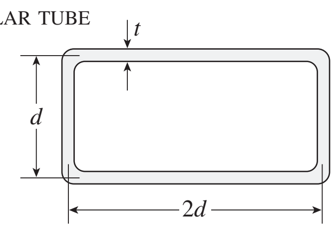
Solution 3.10-2 **Bar and tube**

SOLID BAR



$$\tau_{\max} = \frac{16T}{\pi d^3} \qquad \text{(Eq. 3-12)}$$

RECTANGULAR TUBE



$$A_m = (d)(2d) = 2d^2 \qquad \text{(Eq. 3-64)}$$

$$\tau_{\max} = \frac{T}{2tA_m} = \frac{T}{4td^2} \qquad \text{(Eq. 3-61)}$$

EQUATE THE MAXIMUM SHEAR STRESSES AND SOLVE FOR t

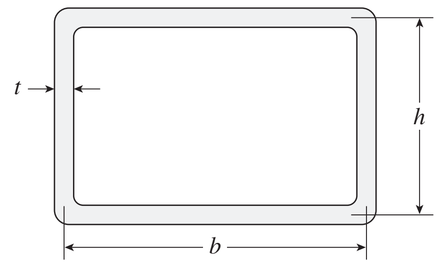
$$\frac{16T}{\pi d^3} = \frac{T}{4td^2}$$

$$t_{\text{min}} = \frac{\pi d}{64} \longleftarrow$$

If $t > t_{\text{min}}$, the shear stress in the tube is less than the shear stress in the bar.

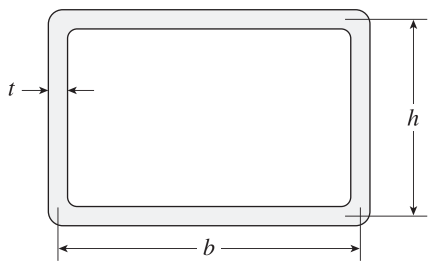
Problem 3.10-3 A thin-walled aluminum tube of rectangular cross section (see figure) has a centerline dimensions $b = 6.0$ in. and $h = 4.0$ in. The wall thickness t is constant and equal to 0.25 in.

(a) Determine the shear stress in the tube due to a torque $T = 15$ k-in.
(b) Determine the angle of twist (in degrees) if the length L of the tube is 50 in. and the shear modulus G is 4.0×10^6 psi.



Use with Prob. 3.10-4

Solution 3.10-3 **Thin-walled tube**



$b = 6.0$ in.

$h = 4.0$ in.

$t = 0.25$ in.

$T = 15$ k-in.

$L = 50$ in.

$G = 4.0 \times 10^6$ psi

Eq. (3-64): $A_m = bh = 24.0 \text{ in.}^2$

Eq. (3-71) with $t_1 = t_2 = t$: $\quad J = \dfrac{2b^2h^2t}{b+h}$

$J = 28.8 \text{ in.}^4$

(a) SHEAR STRESS (EQ. 3-61)

$$\tau = \frac{T}{2tA_m} = 1250 \text{ psi} \quad \longleftarrow$$

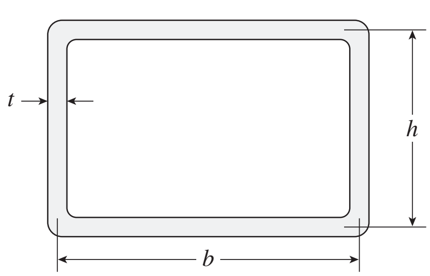
(b) ANGLE OF TWIST (EQ. 3-72)

$$\phi = \frac{TL}{GJ} = 0.0065104 \text{ rad}$$
$$= 0.373° \quad \longleftarrow$$

Problem 3.10-4 A thin-walled steel tube of rectangular cross section (see figure) has centerline dimensions $b = 150$ mm and $h = 100$ mm. The wall thickness t is constant and equal to 6.0 mm.

(a) Determine the shear stress in the tube due to a torque $T = 1650 \text{ N} \cdot \text{m}$.
(b) Determine the angle of twist (in degrees) if the length L of the tube is 1.2 m and the shear modulus G is 75 GPa.

Solution 3.10-4 **Thin-walled tube**



$b = 150$ mm

$h = 100$ mm

$t = 6.0$ mm

$T = 1650 \text{ N} \cdot \text{m}$

$L = 1.2$ m

$G = 75$ GPa

Eq. (3-64): $A_m = bh = 0.015 \text{ m}^2$

Eq. (3-71) with $t_1 = t_2 = t$: $\quad J = \dfrac{2b^2h^2t}{b+h}$

$J = 10.8 \times 10^{-6} \text{ m}^4$

(a) SHEAR STRESS (Eq. 3-61)

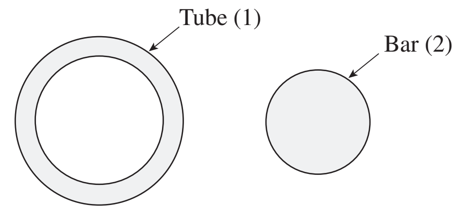
$$\tau = \frac{T}{2tA_m} = 9.17 \text{ MPa} \quad \longleftarrow$$

(b) ANGLE OF TWIST (Eq. 3-72)

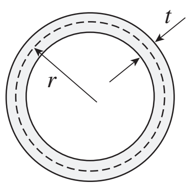
$$\phi = \frac{TL}{GJ} = 0.002444 \text{ rad}$$
$$= 0.140° \quad \longleftarrow$$

Problem 3.10-5 A thin-walled circular tube and a solid circular bar of the same material (see figure) are subjected to torsion. The tube and bar have the same cross-sectional area and the same length.

What is the ratio of the strain energy U_1 in the tube to the strain energy U_2 in the solid bar if the maximum shear stresses are the same in both cases? (For the tube, use the approximate theory for thin-walled bars.)



Solution 3.10-5 THIN-WALLED TUBE (1)



$$A_m = \pi r^2 \quad J = 2\pi r^3 t \quad A = 2\pi r t$$

$$\tau_{max} = \frac{T}{2tA_m} = \frac{T}{2\pi r^2 t}$$

$$T = 2\pi r^2 t \tau_{max}$$

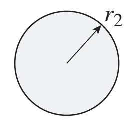
$$U_1 = \frac{T^2 L}{2GJ} = \frac{(2\pi r^2 t \tau_{max})^2 L}{2G(2\pi r^3 t)}$$

$$= \frac{\pi r t \tau_{max}^2 L}{G}$$

But $rt = \dfrac{A}{2\pi}$

$$\therefore\ U_1 = \frac{A\tau_{max}^2 L}{2G}$$

SOLID BAR (2)



$$A = \pi r_2^2 \quad I_P = \frac{\pi}{2} r_2^4$$

$$\tau_{max} = \frac{Tr_2}{I_P} = \frac{2T}{\pi r_2^3} \qquad T = \frac{\pi r_2^3 \tau_{max}}{2}$$

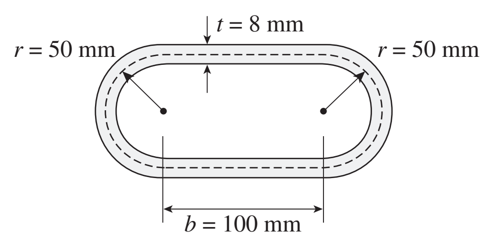
$$U_2 = \frac{T^2 L}{2GI_P} = \frac{(\pi r_2^3 \tau_{max})^2 L}{8G\left(\frac{\pi}{2} r_2^4\right)} = \frac{\pi r_2^2 \tau_{max}^2 L}{4G}$$

But $\pi r_2^2 = A \qquad \therefore\ U_2 = \dfrac{A\tau_{max}^2 L}{4G}$

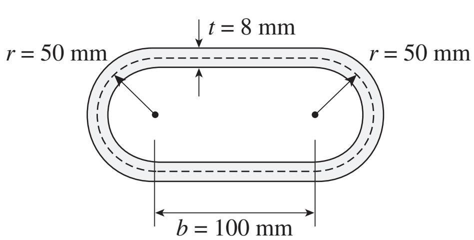
RATIO

$$\frac{U_1}{U_2} = 2 \quad \longleftarrow$$

Problem 3.10-6 Calculate the shear stress τ and the angle of twist ϕ (in degrees) for a steel tube ($G = 76$ GPa) having the cross section shown in the figure. The tube has length $L = 1.5$ m and is subjected to a torque $T = 10$ kN · m.



Solution 3.10-6 **Steel tube**



$G = 76$ GPa

$L = 1.5$ m

$T = 10$ kN · m

$A_m = \pi r^2 + 2br$

$$A_m = \pi(50\ \text{mm})^2 + 2(100\ \text{mm})(50\ \text{mm})$$
$$= 17{,}850\ \text{mm}^2$$

$$L_m = 2b + 2\pi r$$
$$= 2(100\ \text{mm}) + 2\pi(50\ \text{mm})$$
$$= 514.2\ \text{mm}$$

$$J = \frac{4tA_m^2}{L_m} = \frac{4(8\ \text{mm})(17{,}850\ \text{mm}^2)^2}{514.2\ \text{mm}}$$
$$= 19.83 \times 10^6\ \text{mm}^4$$

SHEAR STRESS

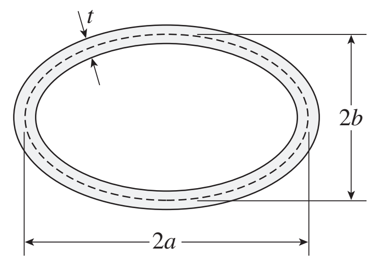
$$\tau = \frac{T}{2tA_m} = \frac{10\ \text{kN}\cdot\text{m}}{2(8\ \text{mm})(17{,}850\ \text{mm}^2)}$$
$$= 35.0\ \text{MPa} \quad \longleftarrow$$

ANGLE OF TWIST

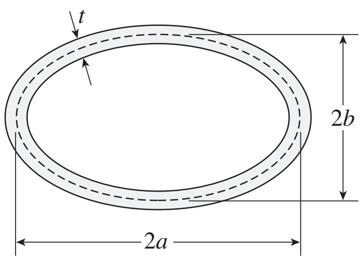
$$\phi = \frac{TL}{GJ} = \frac{(10\ \text{kN}\cdot\text{m})(1.5\ \text{m})}{(76\ \text{GPa})(19.83 \times 10^6\ \text{mm}^4)}$$
$$= 0.00995\ \text{rad}$$
$$= 0.570^\circ \quad \longleftarrow$$

Problem 3.10-7 A thin-walled steel tube having an elliptical cross section with constant thickness t (see figure) is subjected to a torque $T = 18$ k-in.

Determine the shear stress τ and the rate of twist θ (in degrees per inch) if $G = 12 \times 10^6$ psi, $t = 0.2$ in., $a = 3$ in., and $b = 2$ in. (*Note:* See Appendix D, Case 16, for the properties of an ellipse.)



Solution 3.10-7 **Elliptical tube**



$T = 18$ k-in.

$G = 12 \times 10^6$ psi

$t =$ constant

$t = 0.2$ in $\quad a = 3.0$ in. $\quad b = 2.0$ in.

FROM APPENDIX D, CASE 16:

$$A_m = \pi ab = \pi(3.0 \text{ in.})(2.0 \text{ in.}) = 18.850 \text{ in.}^2$$

$$L_m \approx \pi[1.5(a + b) - \sqrt{ab}]$$

$$= \pi[1.5(5.0 \text{ in.}) - \sqrt{6.0 \text{ in.}^2}] = 15.867 \text{ in.}$$

$$J = \frac{4tA_m^2}{L_m} = \frac{4(0.2 \text{ in.})(18.850 \text{ in.}^2)^2}{15.867 \text{ in.}}$$

$$= 17.92 \text{ in.}^4$$

SHEAR STRESS

$$\tau = \frac{T}{2tA_m} = \frac{18 \text{ k-in.}}{2(0.2 \text{ in.})(18.850 \text{ in.}^2)}$$

$$= 2390 \text{ psi} \longleftarrow$$

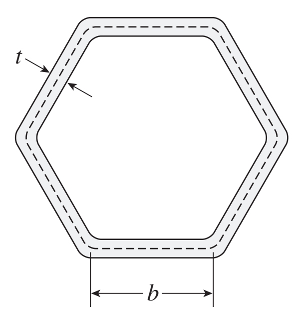
ANGLE OF TWIST PER UNIT LENGTH (RATE OF TWIST)

$$\theta = \frac{\phi}{L} = \frac{T}{GJ} = \frac{18 \text{ k-in.}}{(12 \times 10^6 \text{ psi})(17.92 \text{ in.}^4)}$$

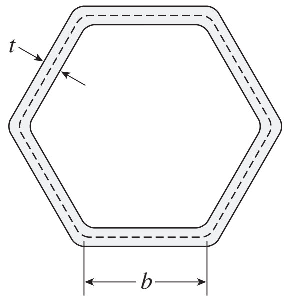
$$\theta = 83.73 \times 10^{-6} \text{ rad/in.} = 0.0048°/\text{in.} \longleftarrow$$

Problem 3.10-8 A torque T is applied to a thin-walled tube having a cross section in the shape of a regular hexagon with constant wall thickness t and side length b (see figure).

Obtain formulas for the shear stress τ and the rate of twist θ.



Solution 3.10-8 **Regular hexagon**



b = Length of side

t = Thickness

$L_m = 6b$

FROM APPENDIX D, CASE 25:

$\beta = 60° \quad n = 6$

$$A_m = \frac{nb^2}{4}\cot\frac{\beta}{2} = \frac{6b^2}{4}\cot 30°$$

$$= \frac{3\sqrt{3}\,b^2}{2}$$

SHEAR STRESS

$$\tau = \frac{T}{2tA_m} = \frac{T\sqrt{3}}{9b^2t} \quad \longleftarrow$$

ANGLE OF TWIST PER UNIT LENGTH (RATE OF TWIST)

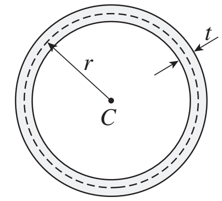
$$J = \frac{4A_m^2 t}{\int_0^{L_m}\frac{d_S}{t}} = \frac{4A_m^2 t}{L_m} = \frac{9b^3t}{2}$$

$$\theta = \frac{T}{GJ} = \frac{2T}{G(9b^3t)} = \frac{2T}{9Gb^3t} \quad \longleftarrow$$

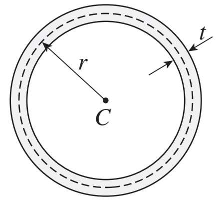
(radians per unit length)

Problem 3.10-9 Compare the angle of twist ϕ_1 for a thin-walled circular tube (see figure) calculated from the approximate theory for thin-walled bars with the angle of twist ϕ_2 calculated from the exact theory of torsion for circular bars.

(a) Express the ratio ϕ_1/ϕ_2 in terms of the nondimensional ratio $\beta = r/t$.
(b) Calculate the ratio of angles of twist for β = 5, 10, and 20. What conclusion about the accuracy of the approximate theory do you draw from these results?



Solution 3.10-9 **Thin-walled tube**



APPROXIMATE THEORY

$$\phi_1 = \frac{TL}{GJ} \qquad J = 2\pi r^3 t \qquad \phi_1 = \frac{TL}{2\pi G r^3 t}$$

EXACT THEORY

$$\phi_2 = \frac{TL}{GI_P} \qquad \text{From Eq. (3-17): } I_p = \frac{\pi r t}{2}(4r^2 + t^2)$$

$$\phi_2 = \frac{TL}{GI_P} = \frac{2TL}{\pi G r t(4r^2 + t^2)}$$

(a) RATIO

$$\frac{\phi_1}{\phi_2} = \frac{4r^2 + t^2}{4r^2} = 1 + \frac{t^2}{4r^2}$$

$$\text{Let } \beta = \frac{r}{t} \qquad \frac{\phi_1}{\phi_2} = 1 + \frac{1}{4\beta^2} \quad \longleftarrow$$

(b)

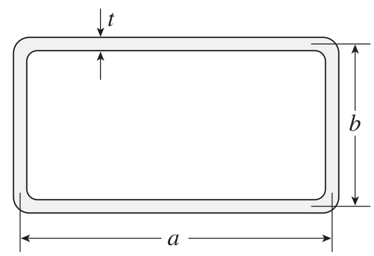
β	ϕ_1/ϕ_2
5	1.0100
10	1.0025
20	1.0006

As the tube becomes thinner and β becomes larger, the ratio ϕ_1/ϕ_2 approaches unity. Thus, the thinner the tube, the more accurate the approximate theory becomes.

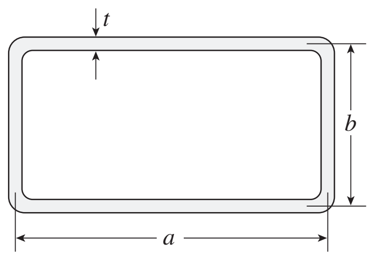
Problem 3.10-10 A thin-walled rectangular tube has uniform thickness t and dimensions $a \times b$ to the median line of the cross section (see figure).

How does the shear stress in the tube vary with the ratio $\beta = a/b$ if the total length L_m of the median line of the cross section and the torque T remain constant?

From your results, show that the shear stress is smallest when the tube is square ($\beta = 1$).



Solution 3.10-10 **Rectangular tube**



t = thickness (constant)

a, b = dimensions of the tube

$$\beta = \frac{a}{b}$$

$$L_m = 2(a + b) = \text{constant}$$

$$T = \text{constant}$$

Shear stress

$$\tau = \frac{T}{2tA_m} \qquad A_m = ab = \beta b^2$$

$$L_m = 2b(1 + \beta) = \text{constant}$$

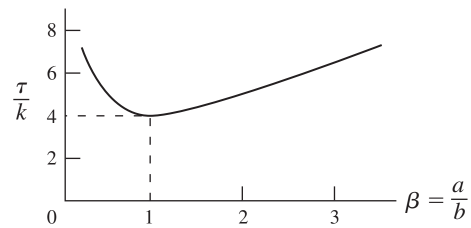
$$b = \frac{L_m}{2(1 + \beta)} \qquad A_m = \beta\left[\frac{L_m}{2(1 + \beta)}\right]^2$$

$$A_m = \frac{\beta L_m^2}{4(1 + \beta)^2}$$

$$\tau = \frac{T}{2tA_m} = \frac{T(4)(1 + \beta)^2}{2t\beta L_m^2} = \frac{2T(1 + \beta)^2}{tL_m^2\beta} \quad \longleftarrow$$

T, t, and L_m are constants.

$$\text{Let } k = \frac{2T}{tL_m^2} = \text{constant} \qquad \tau = k\,\frac{(1 + \beta)^2}{\beta}$$



$$\left(\frac{\tau}{k}\right)_{\min} = 4 \qquad \tau_{\min} = \frac{8T}{tL_m^2}$$

From the graph, we see that τ is minimum when $\beta = 1$ and the tube is square.

Alternate solution

$$\tau = \frac{2T}{tL_m^2}\left[\frac{(1 + \beta)^2}{\beta}\right]$$

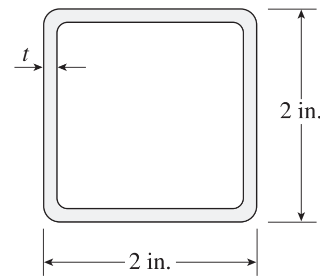
$$\frac{d\tau}{d\beta} = \frac{2T}{tL_m^2}\left[\frac{\beta(2)(1 + \beta) - (1 + \beta)^2(1)}{\beta^2}\right] = 0$$

or $2\beta\,(1+\beta)-(1+\beta)^2 = 0 \quad \therefore \beta = 1$

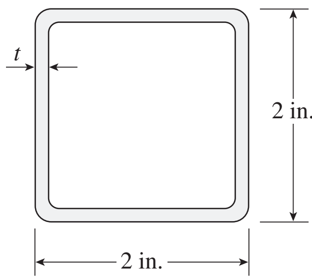
Thus, the tube is square and τ is either a minimum or a maximum. From the graph, we see that τ is a minimum.

Problem 3.10-11 A tubular aluminum bar ($G = 4 \times 10^6$ psi) of square cross section (see figure) with outer dimensions 2 in. × 2 in. must resist a torque $T = 3000$ lb-in.

Calculate the minimum required wall thickness t_{min} if the allowable shear stress is 4500 psi and the allowable rate of twist is 0.01 rad/ft.



Solution 3.10-11 **Square aluminum tube**



Outer dimensions:

2.0 in. × 2.0 in.

$G = 4 \times 10^6$ psi

$T = 3000$ lb-in.

$\tau_{allow} = 4500$ psi

$$\theta_{allow} = 0.01 \text{ rad/ft} = \frac{0.01}{12} \text{ rad/in.}$$

Let b = outer dimension

= 2.0 in.

Centerline dimension = $b - t$

$A_m = (b - t)^2 \qquad L_m = 4(b - t)$

$$J = \frac{4tA_m^2}{L_m} = \frac{4t(b-t)^4}{4(b-t)} = t(b-t)^3$$

THICKNESS t BASED UPON SHEAR STRESS

$$\tau = \frac{T}{2tA_m} \qquad tA_m = \frac{T}{2\tau} \qquad t(b-t)^2 = \frac{T}{2\tau}$$

UNITS: t = in. b = in. T = lb-in. τ = psi

$$t(2.0 \text{ in.} - t)^2 = \frac{3000 \text{ lb-in.}}{2(4500 \text{ psi})} = \frac{1}{3} \text{ in.}^3$$

$$3t(2-t)^2 - 1 = 0$$

Solve for t: $t = 0.0915$ in.

THICKNESS t BASED UPON RATE OF TWIST

$$\theta = \frac{T}{GJ} = \frac{T}{Gt(b-t)^3} \qquad t(b-t)^3 = \frac{T}{G\theta}$$

UNITS: t = in. G = psi θ = rad/in.

$$t(2.0 \text{ in.} - t)^3 = \frac{3000 \text{ lb-in}}{(4 \times 10^6 \text{ psi})(0.01/12 \text{ rad/in.})} = \frac{9}{10}$$

$$10t(2-t)^3 - 9 = 0$$

Solve for t:

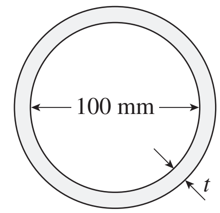
$t = 0.140$ in.

ANGLE OF TWIST GOVERNS

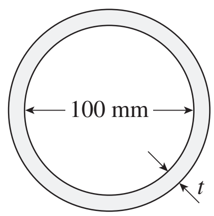
$t_{min} = 0.140$ in. ←

Problem 3.10-12 A thin tubular shaft of circular cross section (see figure) with inside diameter 100 mm is subjected to a torque of 5000 N · m.

If the allowable shear stress is 42 MPa, determine the required wall thickness t by using (a) the approximate theory for a thin-walled tube, and (b) the exact torsion theory for a circular bar.



Solution 3.10-12 **Thin tube**



$T = 5{,}000 \text{ N} \cdot \text{m} \qquad d_1 = \text{inner diameter} = 100 \text{ mm}$

$\tau_{\text{allow}} = 42 \text{ MPa}$

t is in millimeters.

r = Average radius

$= 50 \text{ mm} + \dfrac{t}{2}$

r_1 = Inner radius

$= 50 \text{ mm}$

r_2 = Outer radius

$= 50 \text{ mm} + t \qquad A_m = \pi r^2$

(a) Approximate theory

$$\tau = \frac{T}{2tA_m} = \frac{T}{2t(\pi r^2)} = \frac{T}{2\pi r^2 t}$$

$$42 \text{ MPa} = \frac{5{,}000 \text{ N} \cdot \text{m}}{2\pi\left(50 + \frac{t}{2}\right)^2 t}$$

or

$$t\left(50 + \frac{t}{2}\right)^2 = \frac{5{,}000 \text{ N} \cdot \text{m}}{2\pi(42 \text{ MPa})} = \frac{5 \times 10^6}{84\pi} \text{ mm}^3$$

Solve for t:

$t = 6.66$ mm ←

(b) Exact theory

$$\tau = \frac{Tr_2}{I_p} \qquad I_p = \frac{\pi}{2}(r_2^4 - r_1^4) = \frac{\pi}{2}[(50 + t)^4 - (50)^4]$$

$$42 \text{ MPa} = \frac{(5{,}000 \text{ N} \cdot \text{m})(50 + t)}{\frac{\pi}{2}[(50 + t)^4 - (50)^4]}$$

$$\frac{(50 + t)^4 - (50)^4}{50 + t} = \frac{(5000 \text{ N} \cdot \text{m})(2)}{(\pi)(42 \text{ MPa})} = \frac{5 \times 10^6}{21\pi} \text{ mm}^3$$

Solve for t:

$t = 7.02$ mm ←

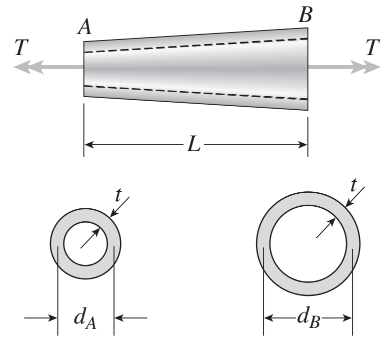
The approximate result is 5% less than the exact result. Thus, the approximate theory is nonconservative and should only be used for thin-walled tubes.

Problem 3.10-13 A long, thin-walled tapered tube AB of circular cross section (see figure) is subjected to a torque T. The tube has length L and constant wall thickness t. The diameter to the median lines of the cross sections at the ends A and B are d_A and d_B, respectively.

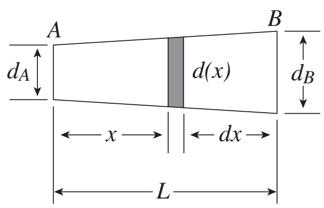
Derive the following formula for the angle of twist of the tube:

$$\phi = \frac{2TL}{\pi G t}\left(\frac{d_A + d_B}{d_A^2 d_B^2}\right)$$

Hint: If the angle of taper is small, we may obtain approximate results by applying the formulas for a thin-walled prismatic tube to a differential element of the tapered tube and then integrating along the axis of the tube.



Solution 3.10-13 **Thin-walled tapered tube**



t = thickness

d_A = average diameter at end A

d_B = average diameter at end B

T = torque

$d(x)$ = average diameter at distance x from end A.

$$d(x) = d_A + \left(\frac{d_B - d_A}{L}\right)x$$

$$J = 2\pi r^3 t = \frac{\pi d^3 t}{4}$$

$$J(x) = \frac{\pi t}{4}[d(x)]^3 = \frac{\pi t}{4}\left[d_A + \left(\frac{d_B - d_A}{L}\right)x\right]^3$$

For element of length dx:

$$d\phi = \frac{T dx}{GJ(x)} = \frac{4T dx}{G\pi t\left[d_A + \left(\frac{d_B - d_A}{L}\right)x\right]^3}$$

For entire tube:

$$\phi = \frac{4T}{\pi GT}\int_0^L \frac{dx}{\left[d_A + \left(\frac{d_B - d_A}{L}\right)x\right]^3}$$

From table of integrals (see Appendix C):

$$\int \frac{dx}{(a + bx)^3} = -\frac{1}{2b(a + bx)^2}$$

$$\phi = \frac{4T}{\pi G t}\left[-\frac{1}{2\left(\frac{d_B - d_A}{L}\right)\left(d_A + \frac{d_B - d_A}{L}\cdot x\right)^2}\right]_0^L$$

$$= \frac{4T}{\pi G t}\left[-\frac{L}{2(d_B - d_A)d_B^2} + \frac{L}{2(d_B - d_A)d_A^2}\right]$$

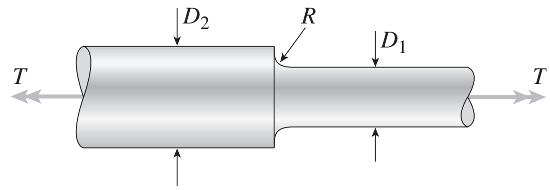
$$\phi = \frac{2TL}{\pi G t}\left(\frac{d_A + d_B}{d_A^2 d_B^2}\right) \longleftarrow$$

Stress Concentrations in Torsion

The problems for Section 3.11 are to be solved by considering the stress-concentration factors.

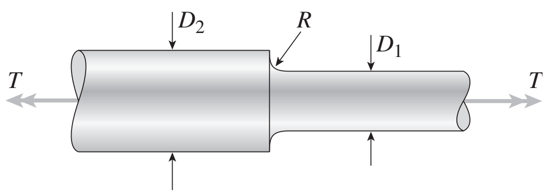
Problem 3.11-1 A stepped shaft consisting of solid circular segments having diameters $D_1 = 2.0$ in. and $D_2 = 2.4$ in. (see figure) is subjected to torques T. The radius of the fillet is $R = 0.1$ in.

If the allowable shear stress at the stress concentration is 6000 psi, what is the maximum permissible torque T_{max}?



Probs. 3.11-1 through 3.11-5

Solution 3.11-1 **Stepped shaft in torsion**



$D_1 = 2.0$ in.

$D_2 = 2.4$ in.

$R = 0.1$ in.

$\tau_{allow} = 6000$ psi

USE FIG. 3-48 FOR THE STRESS-CONCENTRATION FACTOR

$$\frac{R}{D_1} = \frac{0.1 \text{ in.}}{2.0 \text{ in.}} = 0.05 \qquad \frac{D_2}{D_1} = \frac{2.4 \text{ in.}}{2.0 \text{ in.}} = 1.2$$

$$K \approx 1.52 \qquad \tau_{max} = K\tau_{nom} = K\left(\frac{16\,T_{max}}{\pi\,D_1^3}\right)$$

$$T_{max} = \frac{\pi D_1^3 \tau_{max}}{16K}$$

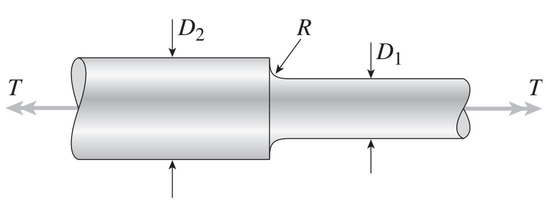
$$= \frac{\pi(2.0 \text{ in.})^3(6000 \text{ psi})}{16(1.52)} = 6200 \text{ lb-in.}$$

$$\therefore\ T_{max} \approx 6200 \text{ lb-in.} \longleftarrow$$

Problem 3.11-2 A stepped shaft with diameters $D_1 = 40$ mm and $D_2 = 60$ mm is loaded by torques $T = 1100$ N · m (see figure).

If the allowable shear stress at the stress concentration is 120 MPa, what is the smallest radius R_{min} that may be used for the fillet?

Solution 3.11-2 **Stepped shaft in torsion**



$D_1 = 40$ mm

$D_2 = 60$ mm

$T = 1100$ N · m

$\tau_{allow} = 120$ MPa

USE FIG. 3-48 FOR THE STRESS-CONCENTRATION FACTOR

$$\tau_{max} = K\tau_{nom} = K\left(\frac{16T}{\pi D_1^3}\right)$$

$$K = \frac{\pi D_1^3 \tau_{max}}{16T} = \frac{\pi(40 \text{ mm})^3(120 \text{ MPa})}{16(1100 \text{ N}\cdot\text{m})} = 1.37$$

$$\frac{D_2}{D_1} = \frac{60 \text{ mm}}{40 \text{ mm}} = 1.5$$

From Fig. (3-48) with $\dfrac{D_2}{D_1} = 1.5$ and $K = 1.37$,

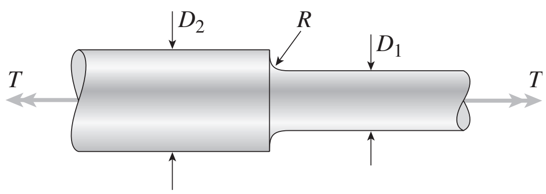
we get $\dfrac{R}{D_1} \approx 0.10$

$$\therefore\ R_{min} \approx 0.10(40 \text{ mm}) = 4.0 \text{ mm} \longleftarrow$$

Problem 3.11-3 A full quarter-circular fillet is used at the shoulder of a stepped shaft having diameter $D_2 = 1.0$ in. (see figure). A torque $T = 500$ lb-in. acts on the shaft.

Determine the shear stress τ_{max} at the stress concentration for values as follows: $D_1 = 0.7$, 0.8, and 0.9 in. Plot a graph showing τ_{max} versus D_1.

Solution 3.11-3 **Stepped shaft in torsion**



D_1 (in.)	D_2/D_1	R (in.)	R/D_1	K	τ_{max} (psi)
0.7	1.43	0.15	0.214	1.20	8900
0.8	1.25	0.10	0.125	1.29	6400
0.9	1.11	0.05	0.056	1.41	4900

$D_2 = 1.0$ in.
$T = 500$ lb-in.
$D_1 = 0.7$, 0.8, and 0.9 in.

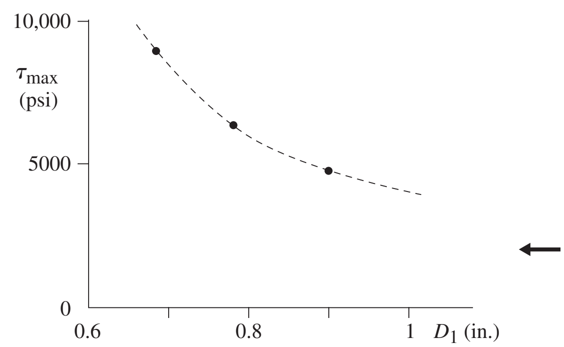
Full quarter-circular fillet ($D_2 = D_1 + 2R$)

$$R = \frac{D_2 - D_1}{2} = 0.5 \text{ in.} - \frac{D_1}{2}$$

USE FIG. 3-48 FOR THE STRESS-CONCENTRATION FACTOR

$$\tau_{max} = K\tau_{nom} = K\left(\frac{16T}{\pi D_1^3}\right)$$

$$= K\frac{16(500 \text{ lb-in.})}{\pi D_1^3} = 2546\frac{K}{D_1^3}$$

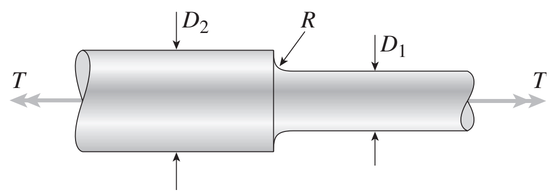


Note that τ_{max} gets smaller as D_1 gets larger, even though K is increasing.

Problem 3.11-4 The stepped shaft shown in the figure is required to transmit 600 kW of power at 400 rpm. The shaft has a full quarter-circular fillet, and the smaller diameter $D_1 = 100$ mm.

If the allowable shear stress at the stress concentration is 100 MPa, at what diameter D_2 will this stress be reached? Is this diameter an upper or a lower limit on the value of D_2?

Solution 3.11-4 Stepped shaft in torsion



$P = 600$ kW $\qquad D_1 = 100$ mm

$n = 400$ rpm $\qquad \tau_{allow} = 100$ MPa

Full quarter-circular fillet

POWER $P = \dfrac{2\pi n T}{60}$ (Eq. 3-42 of Section 3.7)

$P =$ watts $\qquad n =$ rpm $\qquad T =$ Newton meters

$$T = \frac{60P}{2\pi n} = \frac{60(600 \times 10^3 \text{ W})}{2\pi(400 \text{ rpm})} = 14{,}320 \text{ N} \cdot \text{m}$$

USE FIG. 3-48 FOR THE STRESS-CONCENTRATION FACTOR

$$\tau_{max} = K\tau_{nom} = K\left(\frac{16T}{\pi D_1^3}\right)$$

$$K = \frac{\tau_{max}(\pi D_1^3)}{16T}$$

$$= \frac{(100 \text{ MPa})(\pi)(100 \text{ mm})^3}{16(14{,}320 \text{ N} \cdot \text{m})} = 1.37$$

Use the dashed line for a full quarter-circular fillet.

$$\frac{R}{D_1} \approx 0.075 \qquad R \approx 0.075 D_1 = 0.075 \text{ (100 mm)}$$
$$= 7.5 \text{ mm}$$

$$D_2 = D_1 + 2R = 100 \text{ mm} + 2(7.5 \text{ mm}) = 115 \text{ mm}$$

$\therefore D_2 \approx 115$ mm ←

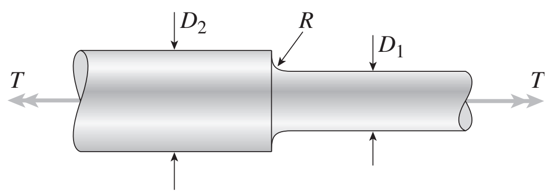
This value of D_2 is a *lower limit* ←

(If D_2 is less than 115 mm, R/D_1 is smaller, K is larger, and τ_{max} is larger, which means that the allowable stress is exceeded.)

Problem 3.11-5 A stepped shaft (see figure) has diameter $D_2 = 1.5$ in. and a full quarter-circular fillet. The allowable shear stress is 15,000 psi and the load $T = 4800$ lb-in.

What is the smallest permissible diameter D_1?

Solution 3.11-5 Stepped shaft in torsion



$D_2 = 1.5$ in.

$\tau_{allow} = 15{,}000$ psi

$T = 4800$ lb-in.

Full quarter-circular fillet $D_2 = D_1 + 2R$

$$R = \frac{D_2 - D_1}{2} = 0.75 \text{ in.} - \frac{D_1}{2}$$

USE FIG. 3-48 FOR THE STRESS-CONCENTRATION FACTOR

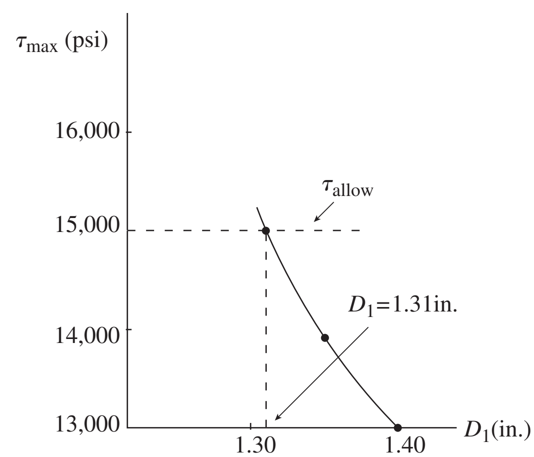
$$\tau_{max} = K\tau_{nom} = K\left(\frac{16T}{\pi D_1^3}\right)$$

$$= \frac{K}{D_1^3}\left[\frac{16(4800 \text{ lb-in.})}{\pi}\right]$$

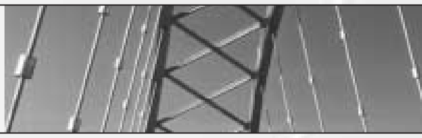
$$= 24{,}450 \frac{K}{D_1^3}$$

Use trial-and-error. Select trial values of D_1

D_1 (in.)	R (in.)	R/D_1	K	τ_{max}(psi)
1.30	0.100	0.077	1.38	15,400
1.35	0.075	0.056	1.41	14,000
1.40	0.050	0.036	1.46	13,000



From the graph, minimum $D_1 \approx 1.31$ in. ←

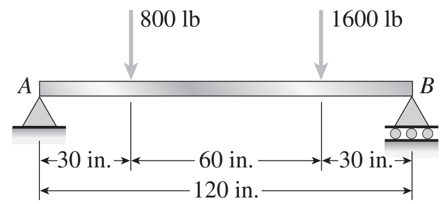


4

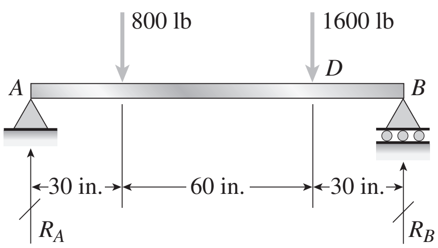
Shear Forces and Bending Moments

Shear Forces and Bending Moments

Problem 4.3-1 Calculate the shear force V and bending moment M at a cross section just to the left of the 1600-lb load acting on the simple beam AB shown in the figure.



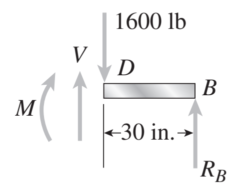
Solution 4.3-1 **Simple beam**



$\Sigma M_A = 0$: $R_B = 1400$ lb

$\Sigma M_B = 0$: $R_A = 1000$ lb

FREE-BODY DIAGRAM OF SEGMENT DB



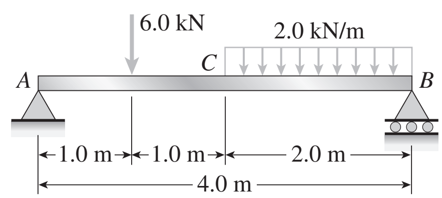
$\Sigma F_{\text{VERT}} = 0$: $V = 1600 \text{ lb} - 1400 \text{ lb}$

$= 200$ lb ⟵

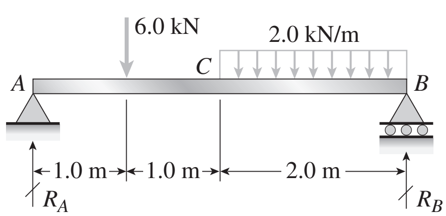
$\Sigma M_D = 0$: $M = (1400 \text{ lb})(30 \text{ in.})$

$= 42{,}000$ lb-in. ⟵

Problem 4.3-2 Determine the shear force V and bending moment M at the midpoint C of the simple beam AB shown in the figure.



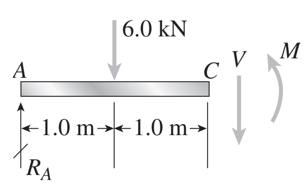
Solution 4.3-2 **Simple beam**



$\Sigma M_A = 0$: $R_B = 4.5$ kN

$\Sigma M_B = 0$: $R_A = 5.5$ kN

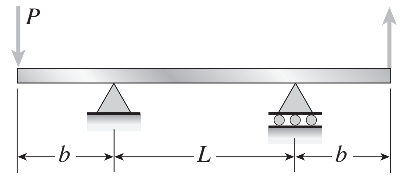
FREE-BODY DIAGRAM OF SEGMENT AC



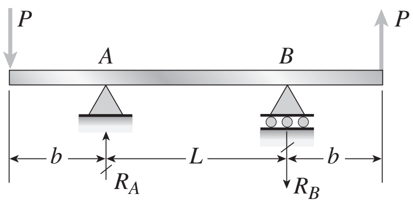
$\Sigma F_{\text{VERT}} = 0$: $V = -0.5$ kN ⟵

$\Sigma M_C = 0$: $M = 5.0 \text{ kN} \cdot \text{m}$ ⟵

Problem 4.3-3 Determine the shear force V and bending moment M at the midpoint of the beam with overhangs (see figure). Note that one load acts downward and the other upward.



Solution 4.3-3 Beam with overhangs

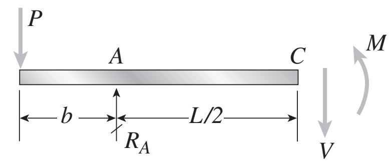


$$\Sigma M_B = 0$$

$$R_A = \frac{1}{L}[P(L + b + b)]$$

$$= P\left(1 + \frac{2b}{L}\right) \quad \text{(upward)}$$

$$\Sigma M_A = 0: \quad R_B = P\left(1 + \frac{2b}{L}\right) \quad \text{(downward)}$$



FREE-BODY DIAGRAM (C IS THE MIDPOINT)

$$\Sigma F_{\text{VERT}} = 0:$$

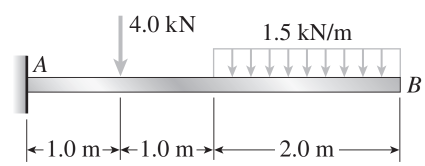
$$V = R_A - P = P\left(1 + \frac{2b}{L}\right) - P = \frac{2bP}{L} \quad \longleftarrow$$

$$\Sigma M_C = 0:$$

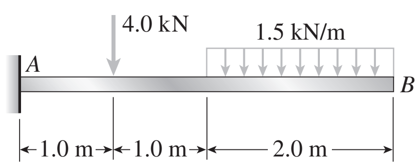
$$M = P\left(1 + \frac{2b}{L}\right)\left(\frac{L}{2}\right) - P\left(b + \frac{L}{2}\right)$$

$$M = \frac{PL}{2} + Pb - Pb - \frac{PL}{2} = 0 \quad \longleftarrow$$

Problem 4.3-4 Calculate the shear force V and bending moment M at a cross section located 0.5 m from the fixed support of the cantilever beam AB shown in the figure.

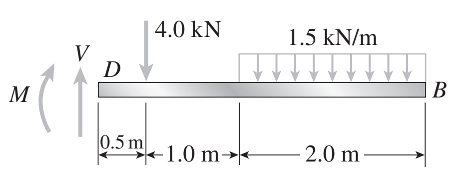


Solution 4.3-4 Cantilever beam



FREE-BODY DIAGRAM OF SEGMENT DB

Point D is 0.5 m from support A.



$$\Sigma F_{\text{VERT}} = 0:$$

$$V = 4.0\text{ kN} + (1.5\text{ kN/m})(2.0\text{ m})$$

$$= 4.0\text{ kN} + 3.0\text{ kN} = 7.0\text{ kN} \quad \longleftarrow$$

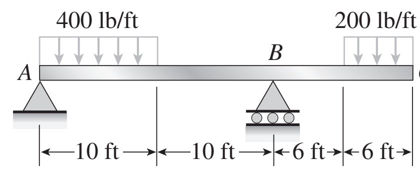
$$\Sigma M_D = 0: \quad M = -(4.0\text{ kN})(0.5\text{ m})$$

$$- (1.5\text{ kN/m})(2.0\text{ m})(2.5\text{ m})$$

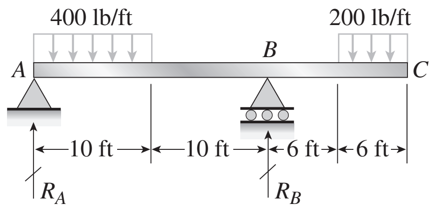
$$= -2.0\text{ kN}\cdot\text{m} - 7.5\text{ kN}\cdot\text{m}$$

$$= -9.5\text{ kN}\cdot\text{m} \quad \longleftarrow$$

Problem 4.3-5 Determine the shear force V and bending moment M at a cross section located 16 ft from the left-hand end A of the beam with an overhang shown in the figure.



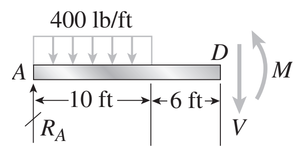
Solution 4.3-5 Beam with an overhang



$\Sigma M_B = 0$: $R_A = 2460\,\text{lb}$

$\Sigma M_A = 0$: $R_B = 2740\,\text{lb}$

FREE-BODY DIAGRAM OF SEGMENT AD



Point D is 16 ft from support A.

$\Sigma F_{\text{VERT}} = 0$:

$V = 2460\text{ lb} - (400\text{ lb/ft})(10\text{ ft})$

$= -1540\text{ lb}$ ←

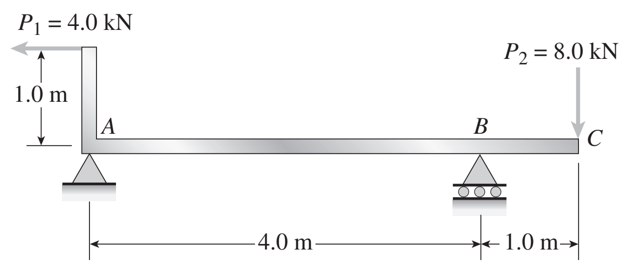
$\Sigma M_D = 0$: $M = (2460\text{ lb})(16\text{ ft})$

$- (400\text{ lb/ft})(10\text{ ft})(11\text{ ft})$

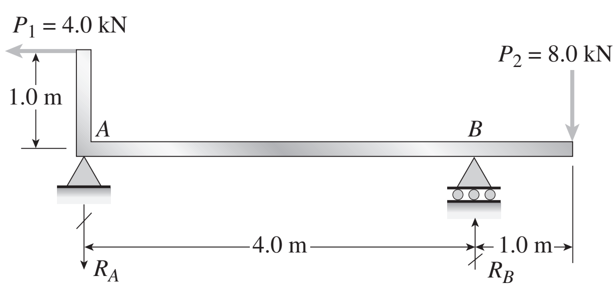
$= -4640\text{ lb-ft}$ ←

Problem 4.3-6 The beam ABC shown in the figure is simply supported at A and B and has an overhang from B to C. The loads consist of a horizontal force $P_1 = 4.0$ kN acting at the end of a vertical arm and a vertical force $P_2 = 8.0$ kN acting at the end of the overhang.

Determine the shear force V and bending moment M at a cross section located 3.0 m from the left-hand support. (*Note:* Disregard the widths of the beam and vertical arm and use centerline dimensions when making calculations.)



Solution 4.3-6 Beam with vertical arm

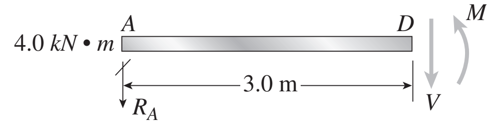


$\Sigma M_B = 0$: $R_A = 1.0\,\text{kN}$ (downward)

$\Sigma M_A = 0$: $R_B = 9.0\,\text{kN}$ (upward)

FREE-BODY DIAGRAM OF SEGMENT AD

Point D is 3.0 m from support A.



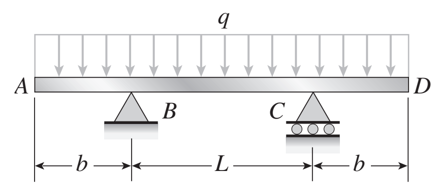
$\Sigma F_{\text{VERT}} = 0$: $V = -R_A = -1.0\text{ kN}$ ←

$\Sigma M_D = 0$: $M = -R_A(3.0\text{ m}) - 4.0\text{ kN}\cdot\text{m}$

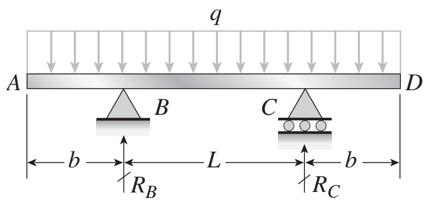
$= -7.0\text{ kN}\cdot\text{m}$ ←

Problem 4.3-7 The beam *ABCD* shown in the figure has overhangs at each end and carries a uniform load of intensity q.

For what ratio b/L will the bending moment at the midpoint of the beam be zero?



Solution 4.3-7 **Beam with overhangs**

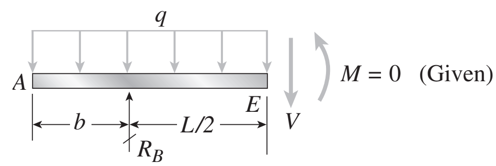


From symmetry and equilibrium of vertical forces:

$$R_B = R_C = q\left(b + \frac{L}{2}\right)$$

FREE-BODY DIAGRAM OF LEFT-HAND HALF OF BEAM:

Point E is at the midpoint of the beam.



$$\Sigma M_E = 0$$

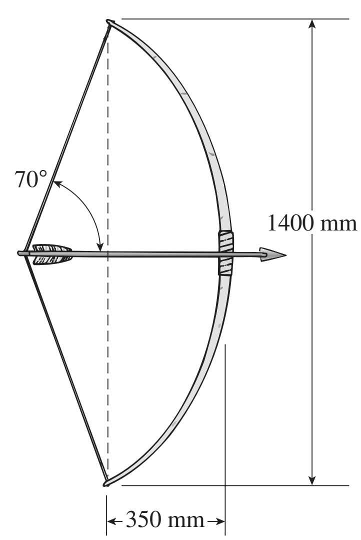
$$-R_B\left(\frac{L}{2}\right) + q\left(\frac{1}{2}\right)\left(b + \frac{L}{2}\right)^2 = 0$$

$$-q\left(b + \frac{L}{2}\right)\left(\frac{L}{2}\right) + q\left(\frac{1}{2}\right)\left(b + \frac{L}{2}\right)^2 = 0$$

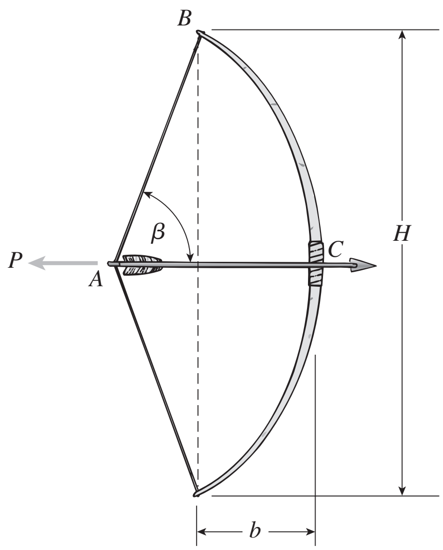
Solve for b/L:

$$\frac{b}{L} = \frac{1}{2} \quad \longleftarrow$$

Problem 4.3-8 At full draw, an archer applies a pull of 130 N to the bowstring of the bow shown in the figure. Determine the bending moment at the midpoint of the bow.



Solution 4.3-8 Archer's bow



$P = 130\,\text{N}$

$\beta = 70°$

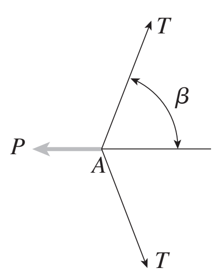
$H = 1400\,\text{mm}$

$= 1.4\,\text{m}$

$b = 350\,\text{mm}$

$= 0.35\,\text{m}$

FREE-BODY DIAGRAM OF POINT A

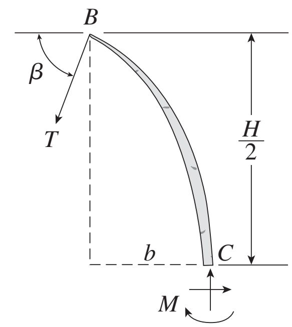


T = tensile force in the bowstring

$\Sigma F_{\text{HORIZ}} = 0\text{:} \quad 2T\cos\beta - P = 0$

$$T = \frac{P}{2\cos\beta}$$

FREE-BODY DIAGRAM OF SEGMENT BC



$\Sigma M_C = 0$

$$T(\cos\beta)\left(\frac{H}{2}\right) + T(\sin\beta)(b) - M = 0$$

$$M = T\left(\frac{H}{2}\cos\beta + b\sin\beta\right)$$

$$= \frac{P}{2}\left(\frac{H}{2} + b\tan\beta\right)$$

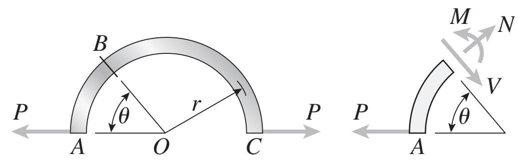
SUBSTITUTE NUMERICAL VALUES:

$$M = \frac{130\text{ N}}{2}\left[\frac{1.4\text{ m}}{2} + (0.35\text{ m})(\tan 70°)\right]$$

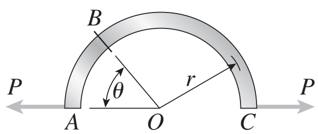
$M = 108\text{ N}\cdot\text{m}$

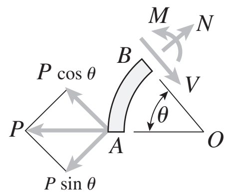
Problem 4.3-9 A curved bar *ABC* is subjected to loads in the form of two equal and opposite forces *P*, as shown in the figure. The axis of the bar forms a semicircle of radius *r*.

Determine the axial force *N*, shear force *V*, and bending moment *M* acting at a cross section defined by the angle θ.



Solution 4.3-9 Curved bar





$\Sigma F_N = 0 \quad \nearrow_{+} \ \swarrow^{-} \qquad N - P\sin\theta = 0$

$N = P\sin\theta$ ←

$\Sigma F_V = 0 \quad \searrow^{+} \ \nwarrow^{-} \qquad V - P\cos\theta = 0$

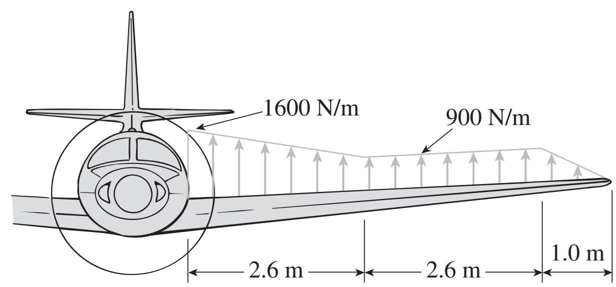
$V = P\cos\theta$ ←

$\Sigma M_O = 0 \qquad M - Nr = 0$

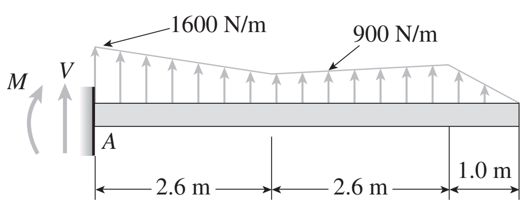
$M = Nr = Pr\sin\theta$ ←

Problem 4.3-10 Under cruising conditions the distributed load acting on the wing of a small airplane has the idealized variation shown in the figure.

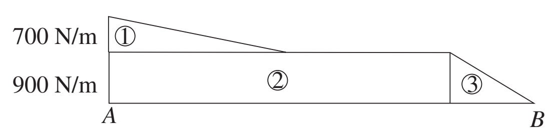
Calculate the shear force *V* and bending moment *M* at the inboard end of the wing.



Solution 4.3-10 Airplane wing



Loading (in three parts)



Shear Force

$\Sigma F_{\text{VERT}} = 0 \quad \uparrow_{+} \ \downarrow^{-}$

$$V + \frac{1}{2}(700\text{ N/m})(2.6\text{ m}) + (900\text{ N/m})(5.2\text{ m}) + \frac{1}{2}(900\text{ N/m})(1.0\text{ m}) = 0$$

$V = -6040\text{ N} = -6.04\text{ kN}$ ←

(Minus means the shear force acts opposite to the direction shown in the figure.)

Bending Moment

$\Sigma M_A = 0$

$$-M + \frac{1}{2}(700\text{ N/m})(2.6\text{ m})\left(\frac{2.6\text{ m}}{3}\right) + (900\text{ N/m})(5.2\text{ m})(2.6\text{ m}) + \frac{1}{2}(900\text{ N/m})(1.0\text{ m})\left(5.2\text{ m} + \frac{1.0\text{ m}}{3}\right) = 0$$

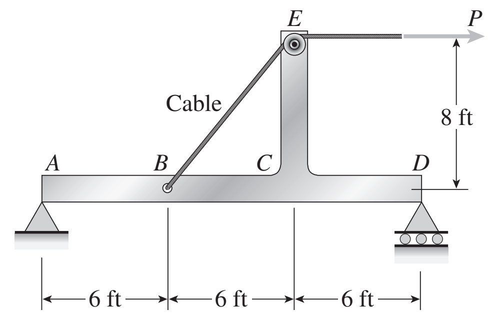
$M = 788.67\text{ N}\cdot\text{m} + 12{,}168\text{ N}\cdot\text{m} + 2490\text{ N}\cdot\text{m}$

$= 15{,}450\text{ N}\cdot\text{m}$

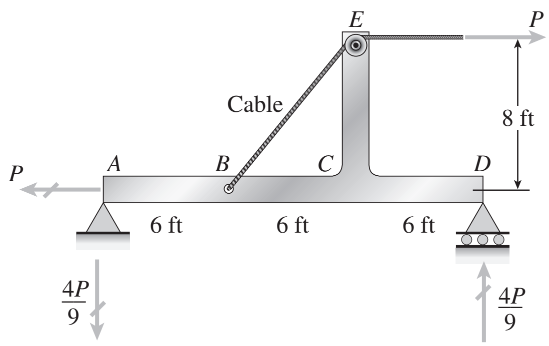
$= 15.45\text{ kN}\cdot\text{m}$ ←

Problem 4.3-11 A beam *ABCD* with a vertical arm *CE* is supported as a simple beam at *A* and *D* (see figure). A cable passes over a small pulley that is attached to the arm at *E*. One end of the cable is attached to the beam at point *B*.

What is the force *P* in the cable if the bending moment in the beam just to the left of point *C* is equal numerically to 640 lb-ft? (*Note:* Disregard the widths of the beam and vertical arm and use centerline dimensions when making calculations.)



Solution 4.3-11 **Beam with a cable**

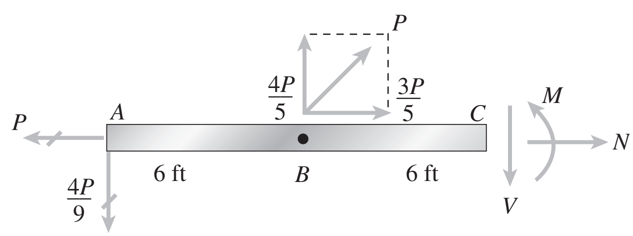


UNITS:

P in lb

M in lb-ft

FREE-BODY DIAGRAM OF SECTION *AC*



$$\Sigma M_C = 0$$

$$M - \frac{4P}{5}(6\text{ ft}) + \frac{4P}{9}(12\text{ ft}) = 0$$

$$M = -\frac{8P}{15}\text{ lb-ft}$$

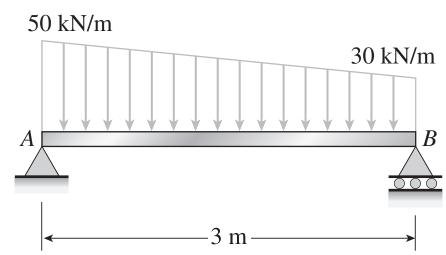
Numerical value of *M* equals 640 lb-ft.

$$\therefore\ 640\text{ lb-ft} = \frac{8P}{15}\text{ lb-ft}$$

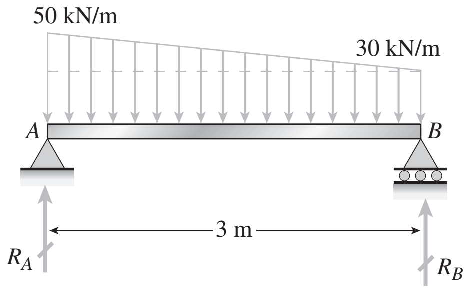
and $P = 1200$ lb ⟵

Problem 4.3-12 A simply supported beam *AB* supports a trapezoidally distributed load (see figure). The intensity of the load varies linearly from 50 kN/m at support *A* to 30 kN/m at support *B*.

Calculate the shear force *V* and bending moment *M* at the midpoint of the beam.



Solution 4.3-12 **Beam with trapezoidal load**



REACTIONS

$\Sigma M_B = 0$ ↺+ $-R_A(3\text{ m}) + (30\text{ kN/m})(3\text{ m})(1.5\text{ m})$

$+ (20\text{ kN/m})(3\text{ m})(1/2)(2\text{ m}) = 0$

$R_A = 65\text{ kN}$

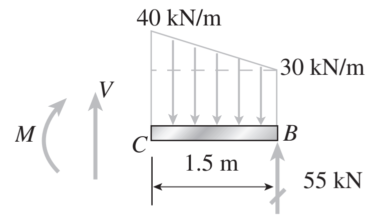
$\Sigma F_{\text{VERT}} = 0$ ↑+

$R_A + R_B - 1/2\,(50\text{ kN/m} + 30\text{ kN/m})(3\text{ m}) = 0$

$R_B = 55\text{ kN}$

FREE-BODY DIAGRAM OF SECTION *CB*

Point *C* is at the midpoint of the beam.



$\Sigma F_{\text{VERT}} = 0$ ↑+ ↓−

$V - (30\text{ kN/m})(1.5\text{ m}) - \frac{1}{2}(10\text{ kN/m})(1.5\text{ m})$

$+ 55\text{ kN} = 0$

$V = -2.5\text{ kN}$ ←

$\Sigma M_C = 0$ ↺+ ↻−

$-M - (30\text{ kN/m})(1.5\text{ m})(0.75\text{ m})$

$-1/2(10\text{ kN/m})(1.5\text{ m})(0.5\text{ m})$

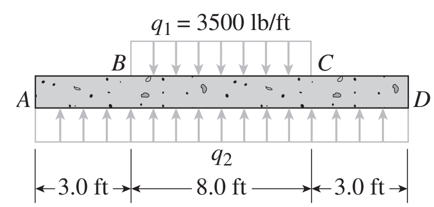
$+ (55\text{ kN})(1.5\text{ m}) = 0$

$M = 45.0\text{ kN}\cdot\text{m}$ ←

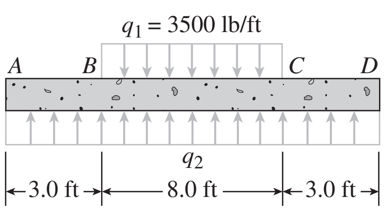
Problem 4.3-13 Beam *ABCD* represents a reinforced-concrete foundation beam that supports a uniform load of intensity $q_1 = 3500$ lb/ft (see figure). Assume that the soil pressure on the underside of the beam is uniformly distributed with intensity q_2.

(a) Find the shear force V_B and bending moment M_B at point *B*.

(b) Find the shear force V_m and bending moment M_m at the midpoint of the beam.



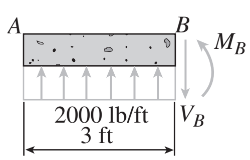
Solution 4.3-13 **Foundation beam**



$\Sigma F_{\text{VERT}} = 0$: $q_2(14\text{ ft}) = q_1(8\text{ ft})$

$\therefore q_2 = \frac{8}{14}\,q_1 = 2000\text{ lb/ft}$

(a) *V* AND *M* AT POINT *B*

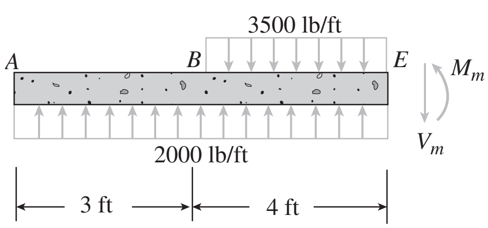


$\Sigma F_{\text{VERT}} = 0$:

$V_B = 6000\text{ lb}$ ←

$\Sigma M_B = 0$: $M_B = 9000\text{ lb-ft}$ ←

(b) *V* AND *M* AT MIDPOINT *E*



$\Sigma F_{\text{VERT}} = 0$: $V_m = (2000\text{ lb/ft})(7\text{ ft}) - (3500\text{ lb/ft})(4\text{ ft})$

$V_m = 0$ ←

$\Sigma M_E = 0$:

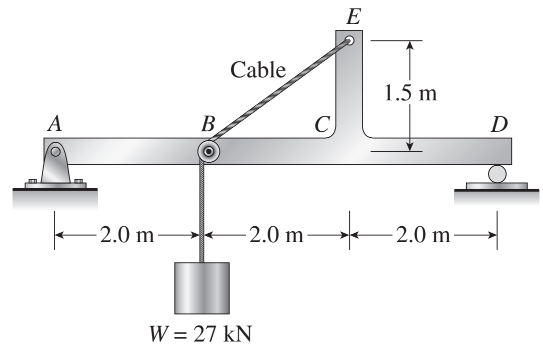
$M_m = (2000\text{ lb/ft})(7\text{ ft})(3.5\text{ ft})$

$- (3500\text{ lb/ft})(4\text{ ft})(2\text{ ft})$

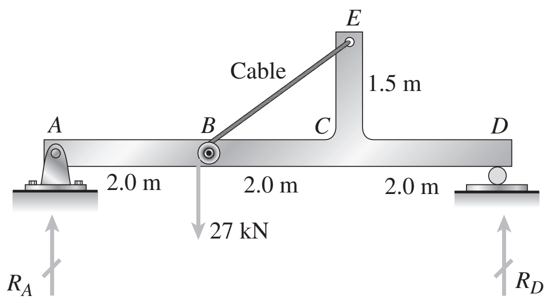
$M_m = 21{,}000\text{ lb-ft}$ ←

Problem 4.3-14 The simply-supported beam *ABCD* is loaded by a weight W = 27 kN through the arrangement shown in the figure. The cable passes over a small frictionless pulley at *B* and is attached at *E* to the end of the vertical arm.

Calculate the axial force *N*, shear force *V*, and bending moment M at section C, which is just to the left of the vertical arm. (*Note:* Disregard the widths of the beam and vertical arm and use centerline dimensions when making calculations.)

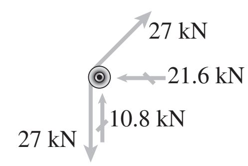


Solution 4.3-14 **Beam with cable and weight**

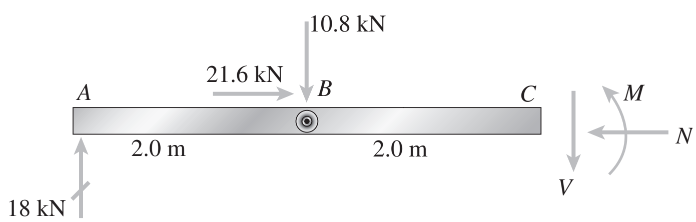


$R_A = 18\,\text{kN}$ $\quad R_D = 9\,\text{kN}$

FREE-BODY DIAGRAM OF PULLEY AT *B*



FREE-BODY DIAGRAM OF SEGMENT *ABC* OF BEAM



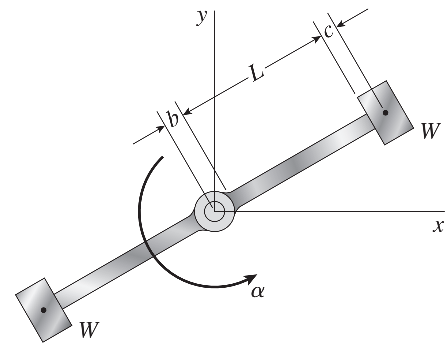
$\Sigma F_{\text{HORIZ}} = 0$: $\quad N = 21.6$ kN (compression) ←

$\Sigma F_{\text{VERT}} = 0$: $\quad V = 7.2$ kN ←

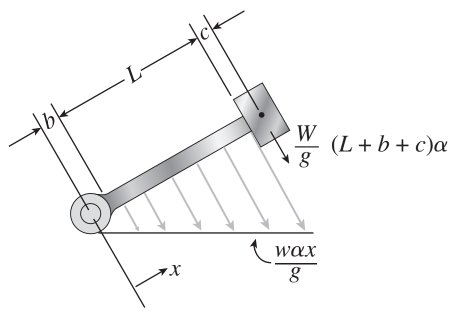
$\Sigma M_C = 0$: $\quad M = 50.4\ \text{kN} \cdot \text{m}$ ←

Problem 4.3-15 The centrifuge shown in the figure rotates in a horizontal plane (the *xy* plane) on a smooth surface about the *z* axis (which is vertical) with an angular acceleration α. Each of the two arms has weight *w* per unit length and supports a weight $W = 2.0\ wL$ at its end.

Derive formulas for the maximum shear force and maximum bending moment in the arms, assuming $b = L/9$ and $c = L/10$.



Solution 4.3-15 **Rotating centrifuge**



Tangential acceleration $= r\alpha$

Inertial force $Mr\alpha = \frac{W}{g} r\alpha$

Maximum *V* and *M* occur at $x = b$.

$$V_{max} = \frac{W}{g}(L + b + c)\alpha + \int_b^{L+b} \frac{w\alpha}{g} x\,dx$$

$$= \frac{W\alpha}{g}(L + b + c) + \frac{wL\alpha}{2g}(L + 2b) \quad \longleftarrow$$

$$M_{max} = \frac{W\alpha}{g}(L + b + c)(L + c) + \int_b^{L+b} \frac{w\alpha}{g} x(x - b)dx$$

$$= \frac{W\alpha}{g}(L + b + c)(L + c) + \frac{w L^2\alpha}{6g}(2L + 3b) \quad \longleftarrow$$

Substitute numerical data:

$$W = 2.0\ wL \quad b = \frac{L}{9} \quad c = \frac{L}{10}$$

$$V_{max} = \frac{91wL^2\alpha}{30g} \quad \longleftarrow$$

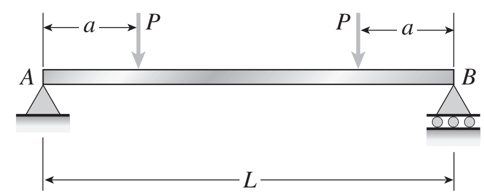
$$M_{max} = \frac{229wL^3\alpha}{75g} \quad \longleftarrow$$

Shear-Force and Bending-Moment Diagrams

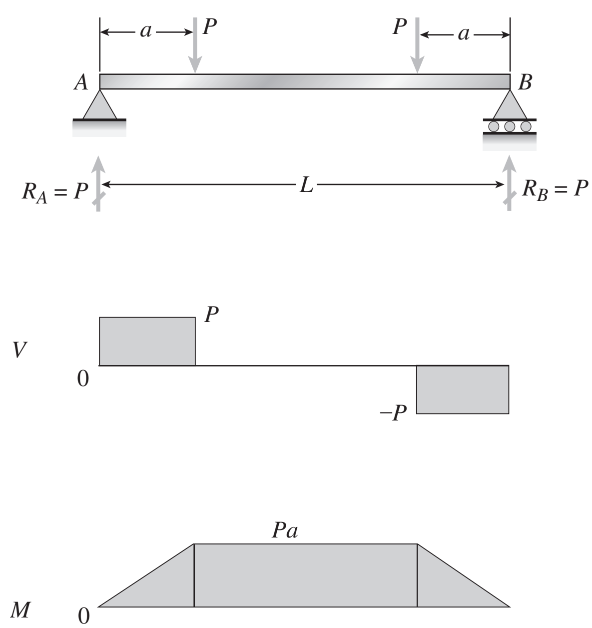
When solving the problems for Section 4.5, draw the shear-force and bending-moment diagrams approximately to scale and label all critical ordinates, including the maximum and minimum values.

Probs. 4.5-1 through 4.5-10 are symbolic problems and Probs. 4.5-11 through 4.5-24 are numerical problems. The remaining problems (4.5-25 through 4.5-30) involve specialized topics, such as optimization, beams with hinges, and moving loads.

Problem 4.5-1 Draw the shear-force and bending-moment diagrams for a simple beam AB supporting two equal concentrated loads P (see figure).

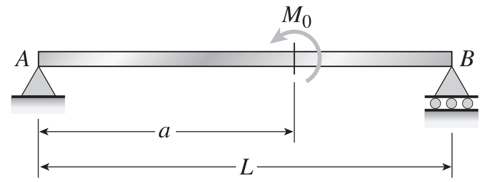


Solution 4.5-1 **Simple beam**

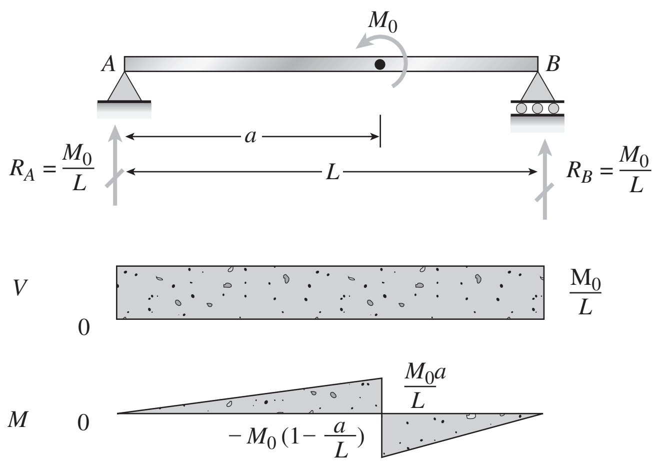


Problem 4.5-2 A simple beam AB is subjected to a counterclockwise couple of moment M_0 acting at distance a from the left-hand support (see figure).

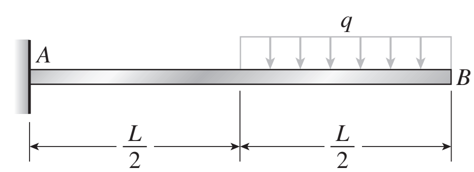
Draw the shear-force and bending-moment diagrams for this beam.



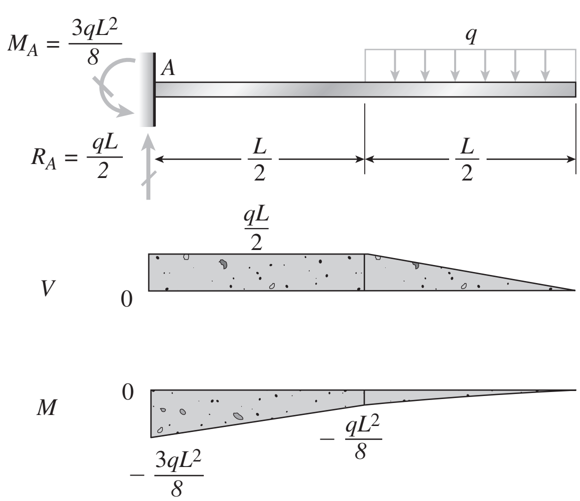
Solution 4.5-2 **Simple beam**



Problem 4.5-3 Draw the shear-force and bending-moment diagrams for a cantilever beam AB carrying a uniform load of intensity q over one-half of its length (see figure).

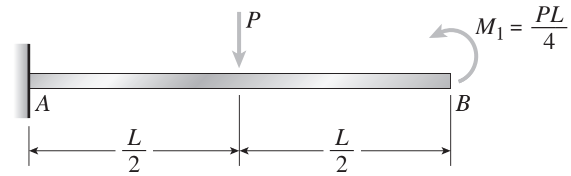


Solution 4.5-3 **Cantilever beam**

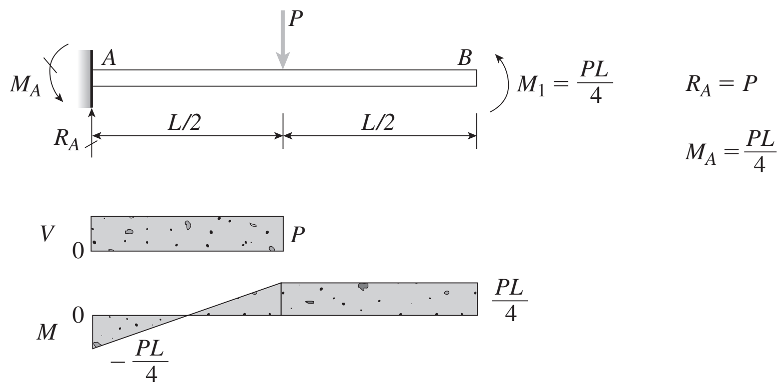


Problem 4.5-4 The cantilever beam AB shown in the figure is subjected to a concentrated load P at the midpoint and a counterclockwise couple of moment $M_1 = PL/4$ at the free end.

Draw the shear-force and bending-moment diagrams for this beam.

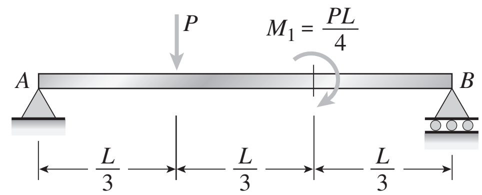


Solution 4.5-4 **Cantilever beam**

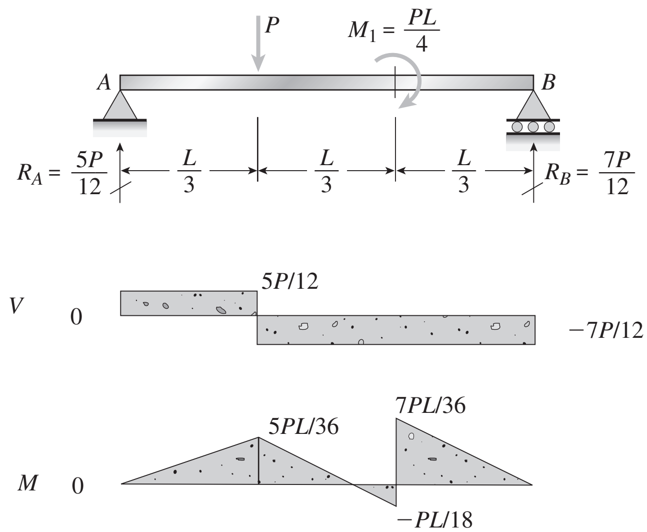


Problem 4.5-5 The simple beam AB shown in the figure is subjected to a concentrated load P and a clockwise couple $M_1 = PL/4$ acting at the third points.

Draw the shear-force and bending-moment diagrams for this beam.

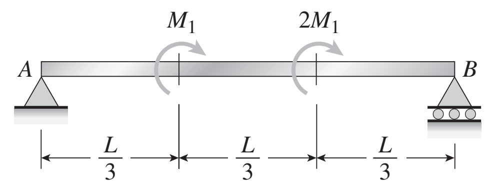


Solution 4.5-5 **Simple beam**

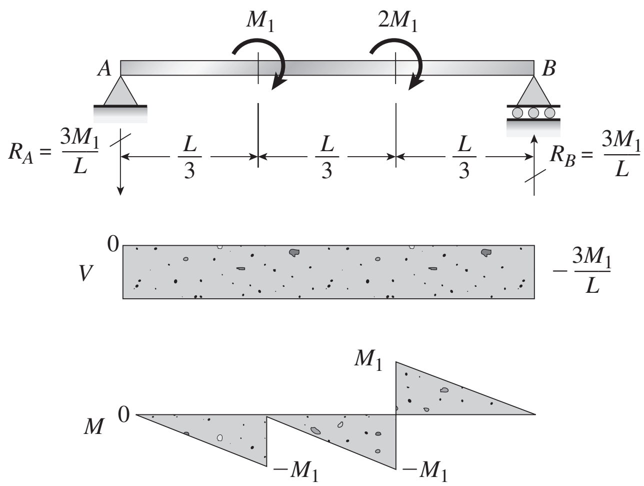


Problem 4.5-6 A simple beam AB subjected to clockwise couples M_1 and $2M_1$ acting at the third points is shown in the figure.

Draw the shear-force and bending-moment diagrams for this beam.

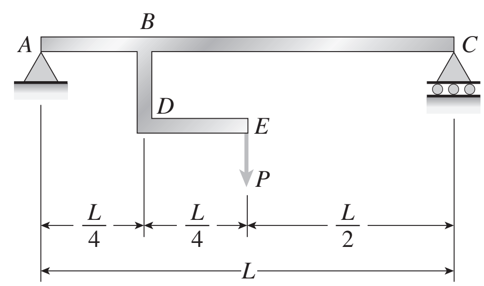


Solution 4.5-6 Simple beam

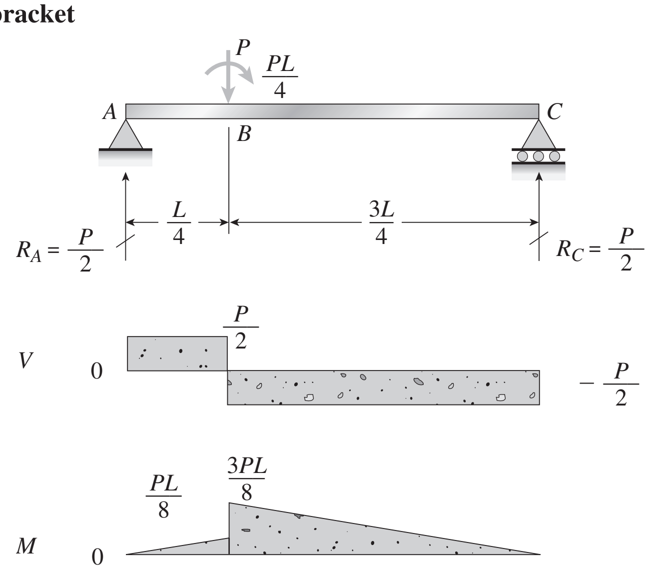


Problem 4.5-7 A simply supported beam ABC is loaded by a vertical load P acting at the end of a bracket BDE (see figure).

Draw the shear-force and bending-moment diagrams for beam ABC.

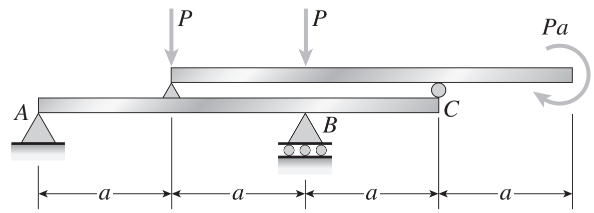


Solution 4.5-7 Beam with bracket

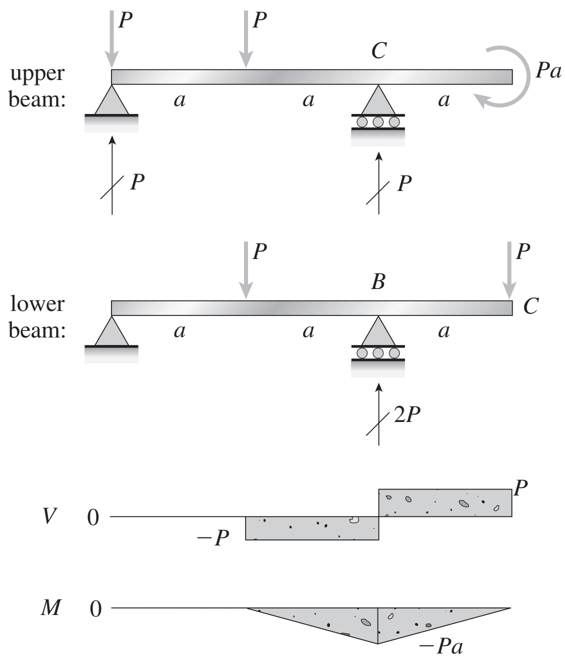


Problem 4.5-8 A beam *ABC* is simply supported at *A* and *B* and has an overhang *BC* (see figure). The beam is loaded by two forces *P* and a clockwise couple of moment *Pa* that act through the arrangement shown.

Draw the shear-force and bending-moment diagrams for beam *ABC*.

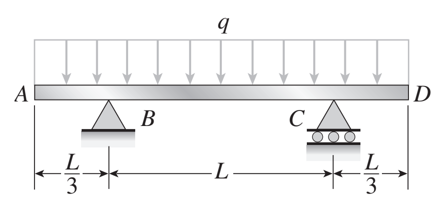


Solution 4.5-8 **Beam with overhang**

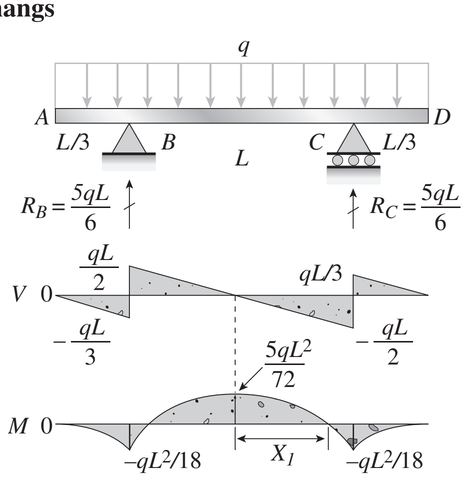


Problem 4.5-9 Beam *ABCD* is simply supported at *B* and *C* and has overhangs at each end (see figure). The span length is *L* and each overhang has length *L*/3. A uniform load of intensity *q* acts along the entire length of the beam.

Draw the shear-force and bending-moment diagrams for this beam.

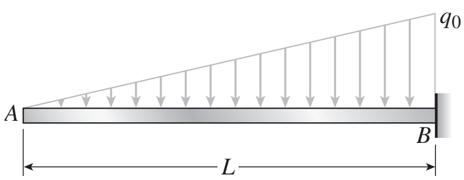


Solution 4.5-9 **Beam with overhangs**

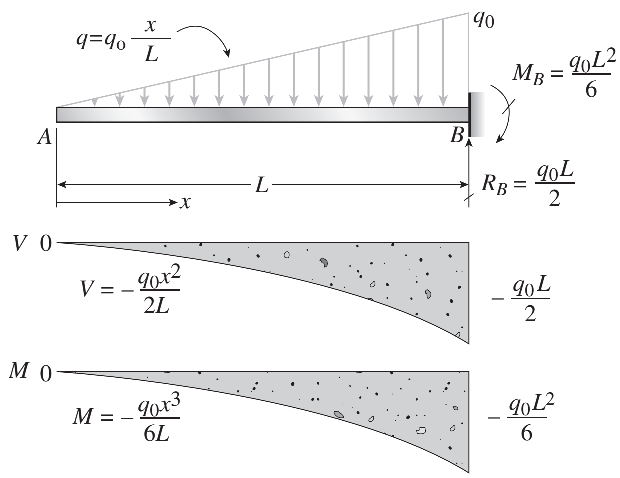


$$x_1 = L\frac{\sqrt{5}}{6} = 0.3727L$$

Problem 4.5-10 Draw the shear-force and bending-moment diagrams for a cantilever beam *AB* supporting a linearly varying load of maximum intensity q_0 (see figure).

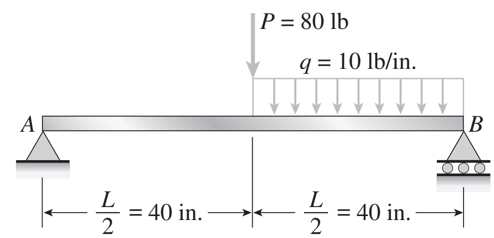


Solution 4.5-10 **Cantilever beam**

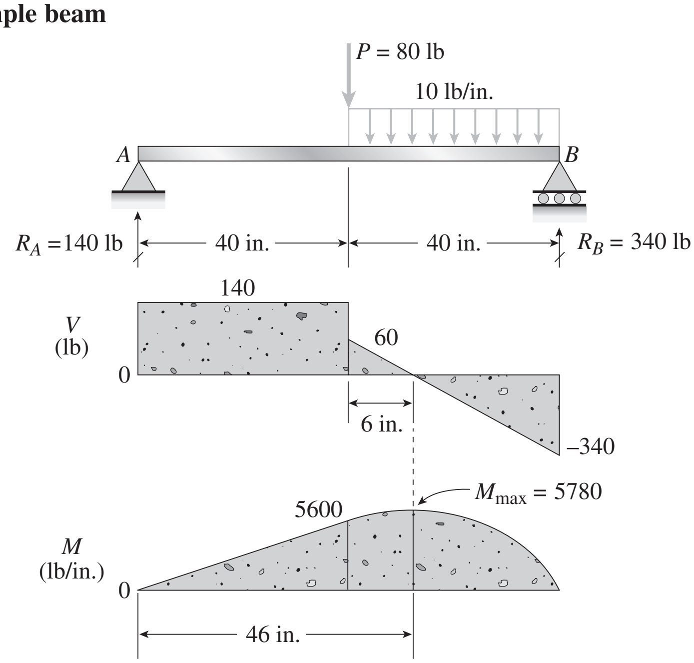


Problem 4.5-11 The simple beam *AB* supports a uniform load of intensity $q = 10$ lb/in. acting over one-half of the span and a concentrated load $P = 80$ lb acting at midspan (see figure).

Draw the shear-force and bending-moment diagrams for this beam.

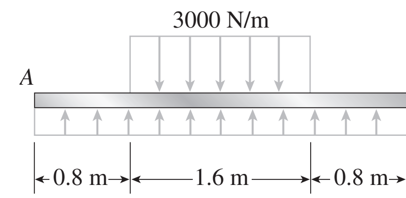


Solution 4.5-11 **Simple beam**

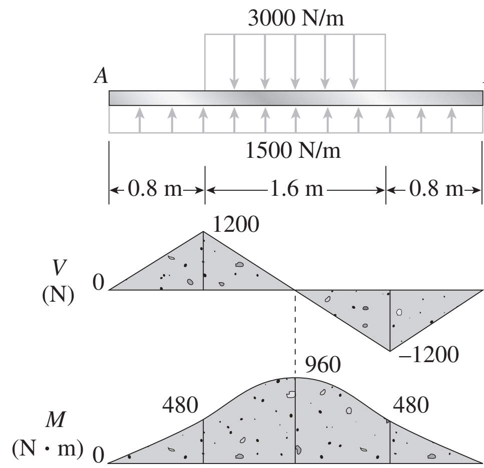


Problem 4.5-12 The beam *AB* shown in the figure supports a uniform load of intensity 3000 N/m acting over half the length of the beam. The beam rests on a foundation that produces a uniformly distributed load over the entire length.

Draw the shear-force and bending-moment diagrams for this beam.

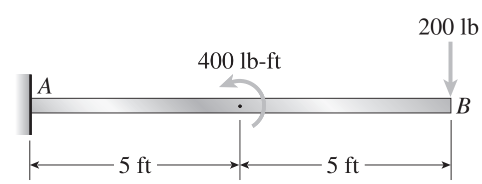


Solution 4.5-12 **Beam with distributed loads**

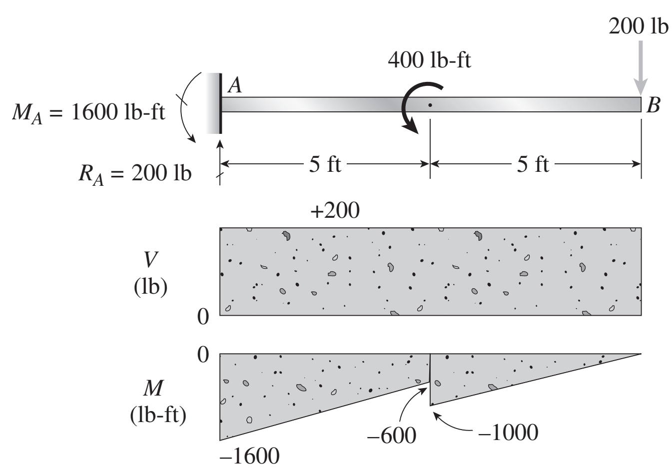


Problem 4.5-13 A cantilever beam *AB* supports a couple and a concentrated load, as shown in the figure.

Draw the shear-force and bending-moment diagrams for this beam.

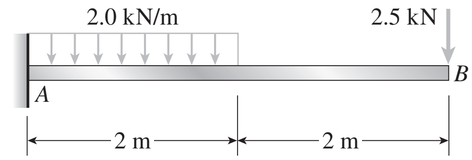


Solution 4.5-13 **Cantilever beam**

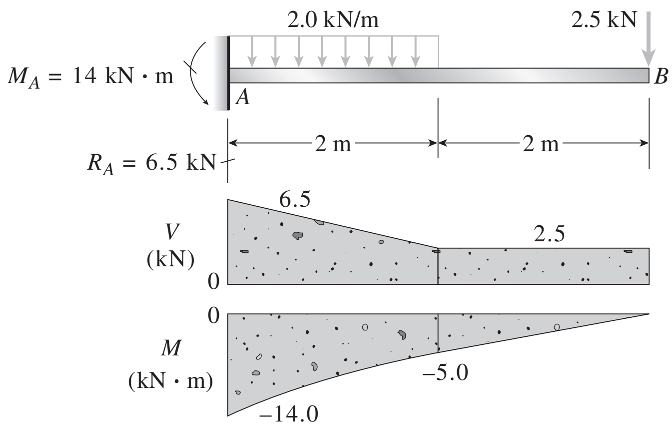


Problem 4.5-14 The cantilever beam *AB* shown in the figure is subjected to a uniform load acting throughout one-half of its length and a concentrated load acting at the free end.

Draw the shear-force and bending-moment diagrams for this beam.

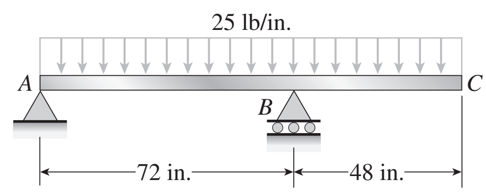


Solution 4.5-14 Cantilever beam

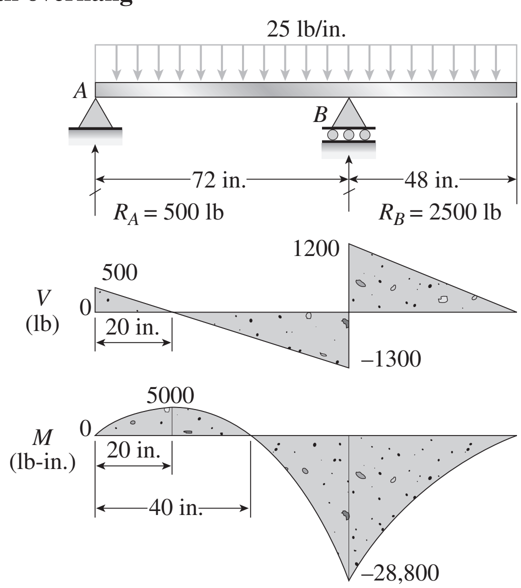


Problem 4.5-15 The uniformly loaded beam *ABC* has simple supports at *A* and *B* and an overhang *BC* (see figure).

Draw the shear-force and bending-moment diagrams for this beam.

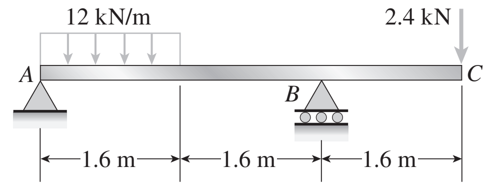


Solution 4.5-15 Beam with an overhang

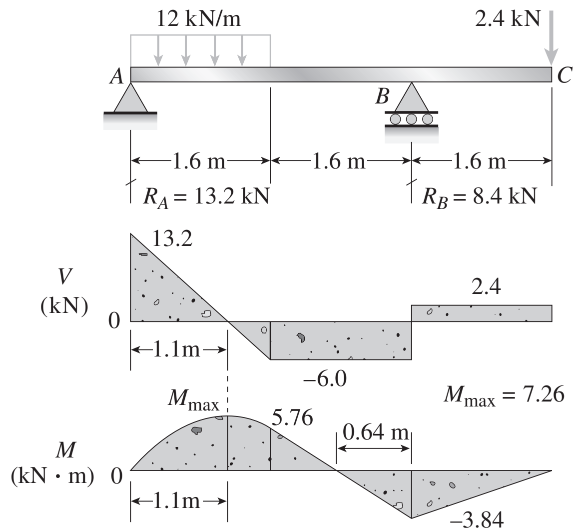


Problem 4.5-16 A beam *ABC* with an overhang at one end supports a uniform load of intensity 12 kN/m and a concentrated load of magnitude 2.4 kN (see figure).

Draw the shear-force and bending-moment diagrams for this beam.

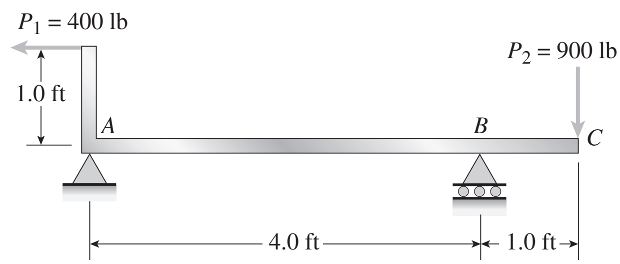


Solution 4.5-16 Beam with an overhang

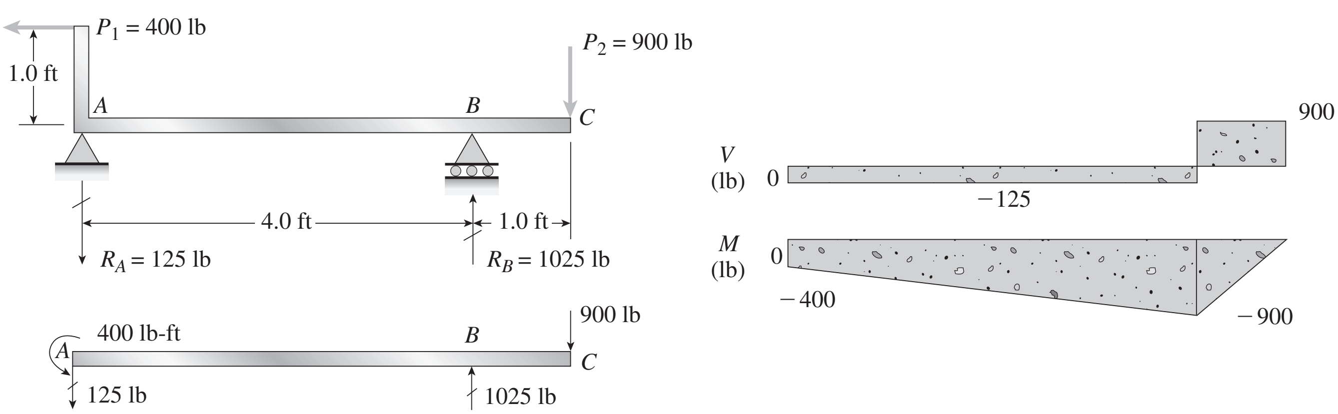


Problem 4.5-17 The beam *ABC* shown in the figure is simply supported at *A* and *B* and has an overhang from *B* to *C*. The loads consist of a horizontal force P_1 = 400 lb acting at the end of the vertical arm and a vertical force P_2 = 900 lb acting at the end of the overhang.

Draw the shear-force and bending-moment diagrams for this beam. (*Note:* Disregard the widths of the beam and vertical arm and use centerline dimensions when making calculations.)

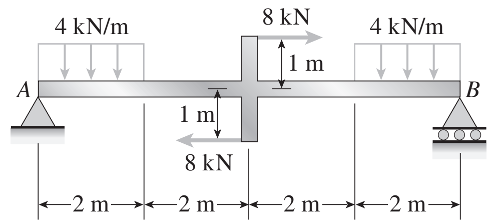


Solution 4.5-17 Beam with vertical arm

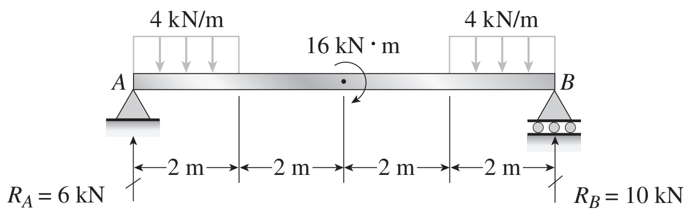


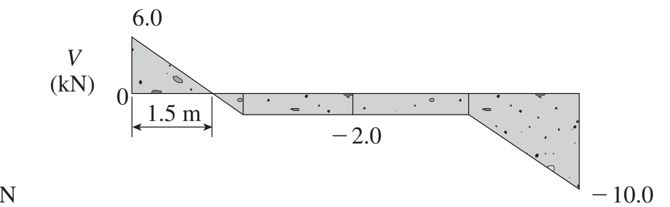
Problem 4.5-18 A simple beam *AB* is loaded by two segments of uniform load and two horizontal forces acting at the ends of a vertical arm (see figure).

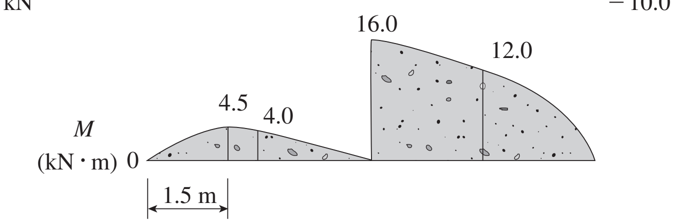
Draw the shear-force and bending-moment diagrams for this beam.



Solution 4.5-18 Simple beam

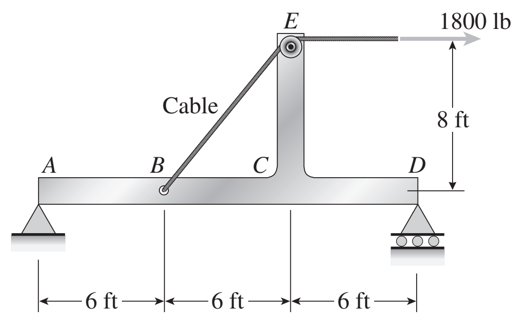




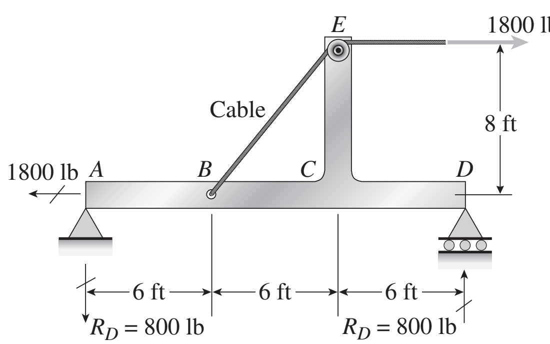


Problem 4.5-19 A beam *ABCD* with a vertical arm *CE* is supported as a simple beam at *A* and *D* (see figure). A cable passes over a small pulley that is attached to the arm at *E*. One end of the cable is attached to the beam at point *B*. The tensile force in the cable is 1800 lb.

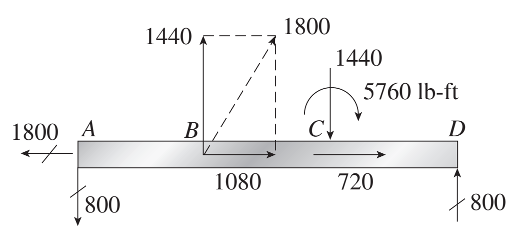
Draw the shear-force and bending-moment diagrams for beam *ABCD*. (*Note:* Disregard the widths of the beam and vertical arm and use center-line dimensions when making calculations.)



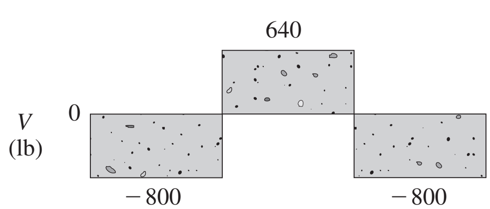
Solution 4.5-19 Beam with a cable

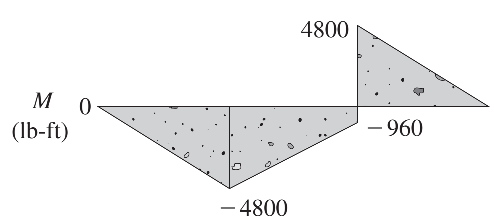


FREE-BODY DIAGRAM OF BEAM *ABCD*



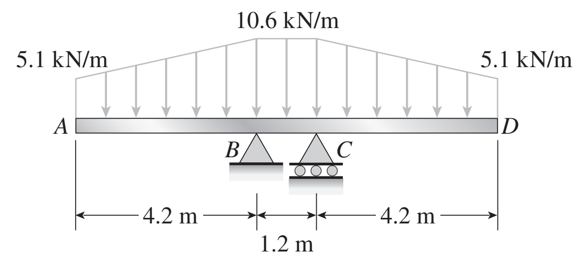
Note: All forces have units of pounds.



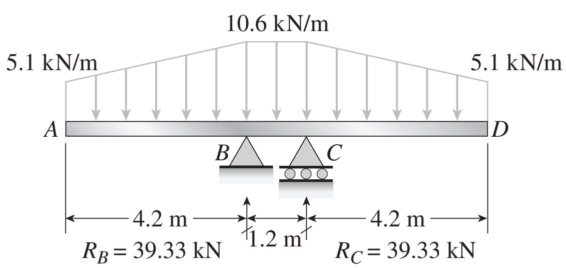


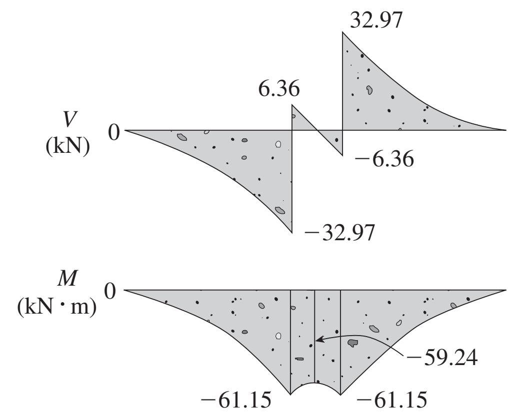
Problem 4.5-20 The beam *ABCD* shown in the figure has overhangs that extend in both directions for a distance of 4.2 m from the supports at *B* and *C*, which are 1.2 m apart.

Draw the shear-force and bending-moment diagrams for this overhanging beam.



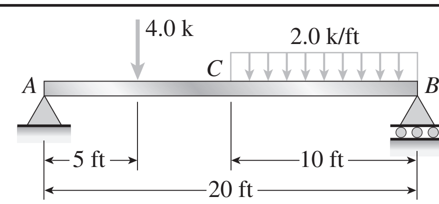
Solution 4.5-20 **Beam with overhangs**



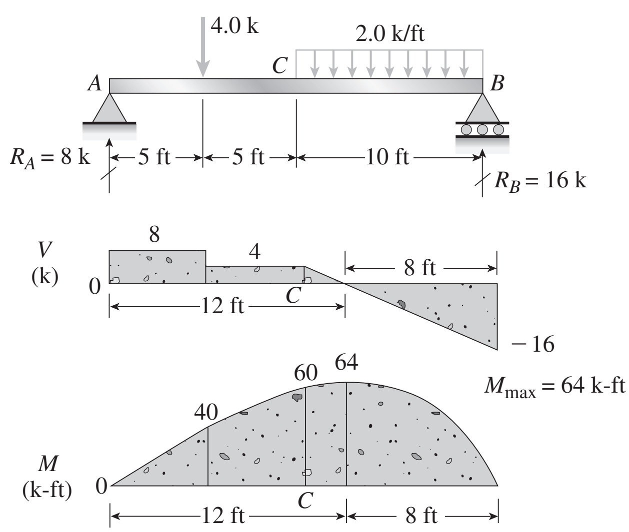


Problem 4.5-21 The simple beam *AB* shown in the figure supports a concentrated load and a segment of uniform load.

Draw the shear-force and bending-moment diagrams for this beam.

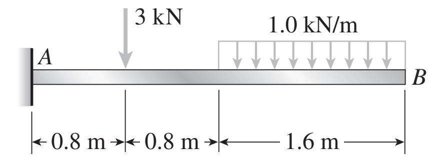


Solution 4.5-21 **Simple beam**

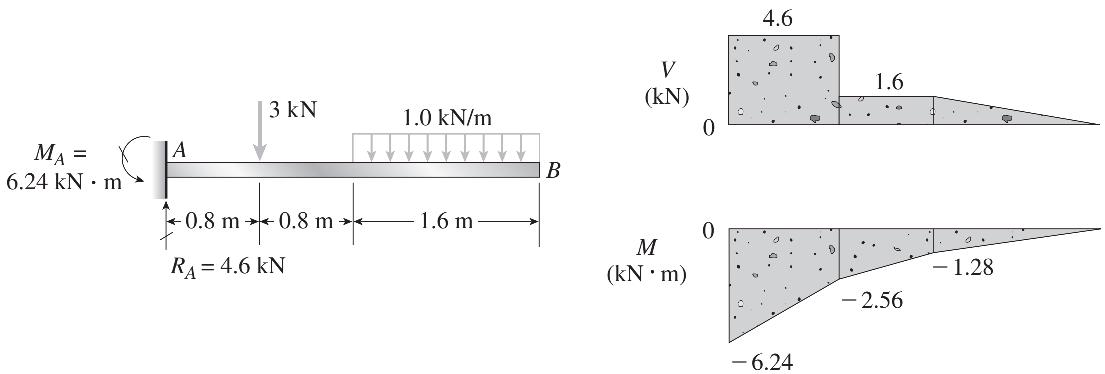


Problem 4.5-22 The cantilever beam shown in the figure supports a concentrated load and a segment of uniform load.

Draw the shear-force and bending-moment diagrams for this cantilever beam.

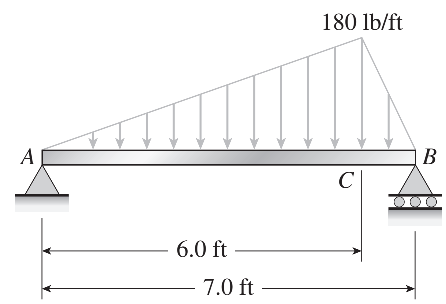


Solution 4.5-22 **Cantilever beam**

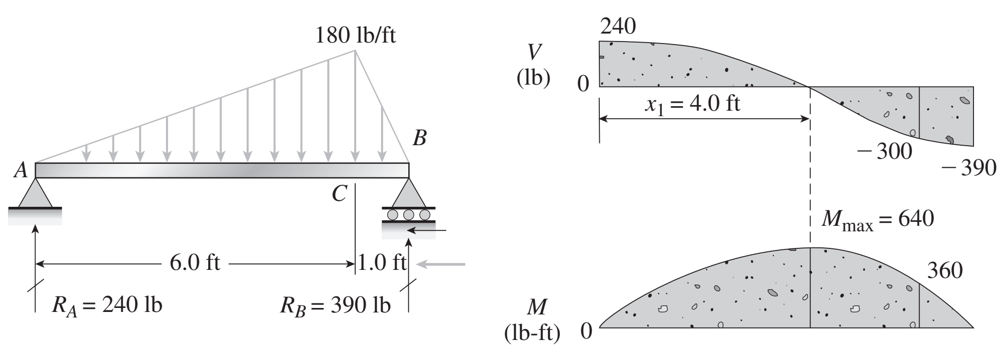


Problem 4.5-23 The simple beam *ACB* shown in the figure is subjected to a triangular load of maximum intensity 180 lb/ft.

Draw the shear-force and bending-moment diagrams for this beam.

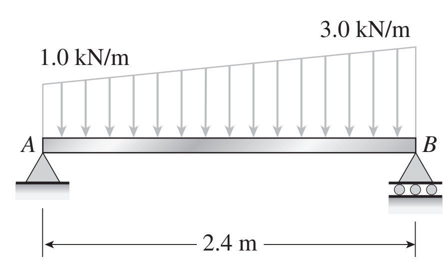


Solution 4.5-23 **Simple beam**

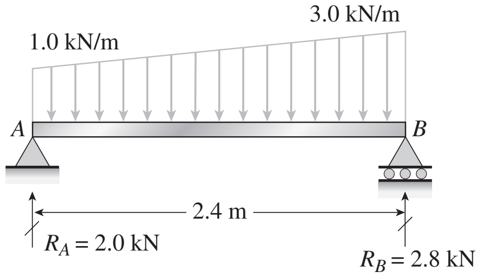


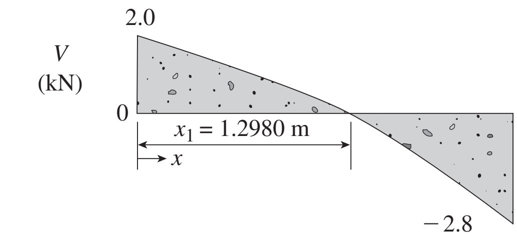
Problem 4.5-24 A beam with simple supports is subjected to a trapezoidally distributed load (see figure). The intensity of the load varies from 1.0 kN/m at support *A* to 3.0 kN/m at support *B*.

Draw the shear-force and bending-moment diagrams for this beam.



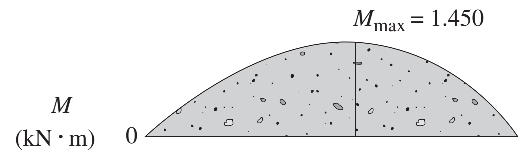
Solution 4.5-24 **Simple beam**





$$V = 2.0 - x - \frac{x^2}{2.4} \quad (x = \text{meters};\ V = \text{kN})$$

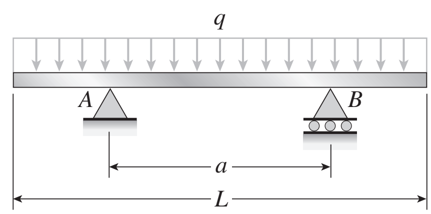
Set $V = 0$: $x_1 = 1.2980\,\text{m}$



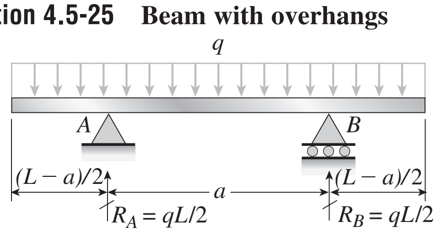
Problem 4.5-25 A beam of length L is being designed to support a uniform load of intensity q (see figure). If the supports of the beam are placed at the ends, creating a simple beam, the maximum bending moment in the beam is $qL^2/8$. However, if the supports of the beam are moved symmetrically toward the middle of the beam (as pictured), the maximum bending moment is reduced.

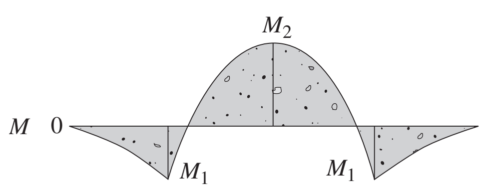
Determine the distance a between the supports so that the maximum bending moment in the beam has the smallest possible numerical value.

Draw the shear-force and bending-moment diagrams for this condition.



Solution 4.5-25 **Beam with overhangs**





The maximum bending moment is smallest when $M_1 = M_2$ (numerically).

$$M_1 = \frac{q(L-a)^2}{8}$$

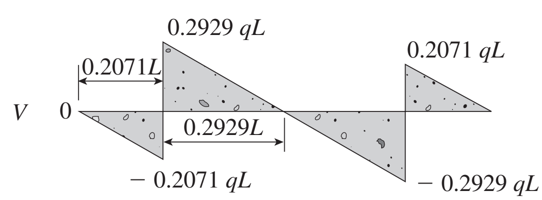
$$M_2 = R_A\left(\frac{a}{2}\right) - \frac{qL^2}{8} = \frac{qL}{8}(2a - L)$$

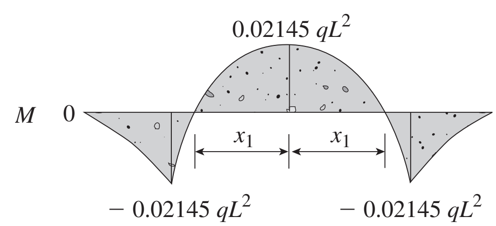
$$M_1 = M_2 \qquad (L-a)^2 = L(2a - L)$$

Solve for a: $a = (2 - \sqrt{2})L = 0.5858L$ ←

$$M_1 = M_2 = \frac{q}{8}(L-a)^2$$

$$= \frac{qL^2}{8}(3 - 2\sqrt{2}) = 0.02145qL^2 \quad \leftarrow$$

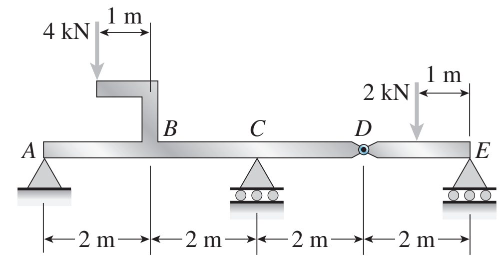




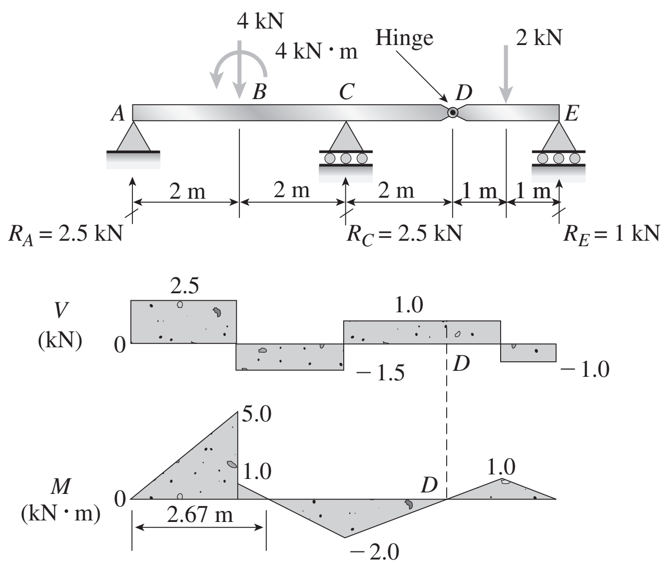
$x_1 = 0.3536\,a$
$= 0.2071\,L$

Problem 4.5-26 The compound beam *ABCDE* shown in the figure consists of two beams (*AD* and *DE*) joined by a hinged connection at *D*. The hinge can transmit a shear force but not a bending moment. The loads on the beam consist of a 4-kN force at the end of a bracket attached at point *B* and a 2-kN force at the midpoint of beam *DE*.

Draw the shear-force and bending-moment diagrams for this compound beam.

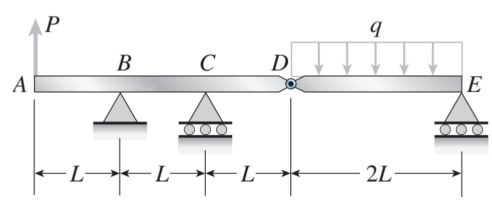


Solution 4.5-26 **Compound beam**

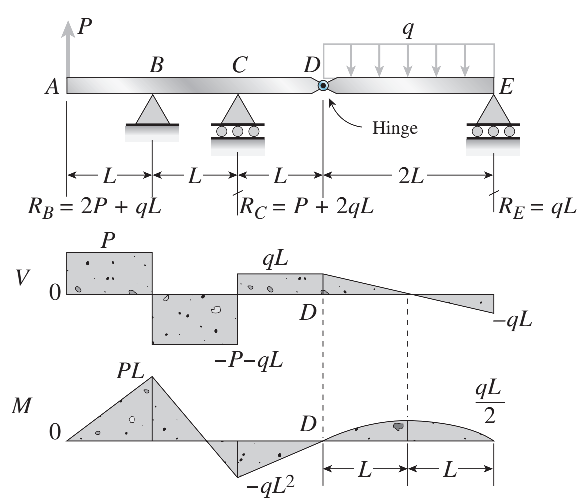


Problem 4.5-27 The compound beam *ABCDE* shown in the figure consists of two beams (*AD* and *DE*) joined by a hinged connection at *D*. The hinge can transmit a shear force but not a bending moment. A force P acts upward at *A* and a uniform load of intensity q acts downward on beam *DE*.

Draw the shear-force and bending-moment diagrams for this compound beam.

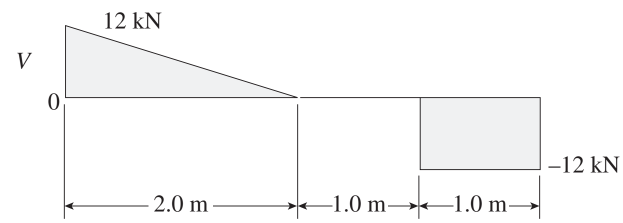


Solution 4.5-27 **Compound beam**

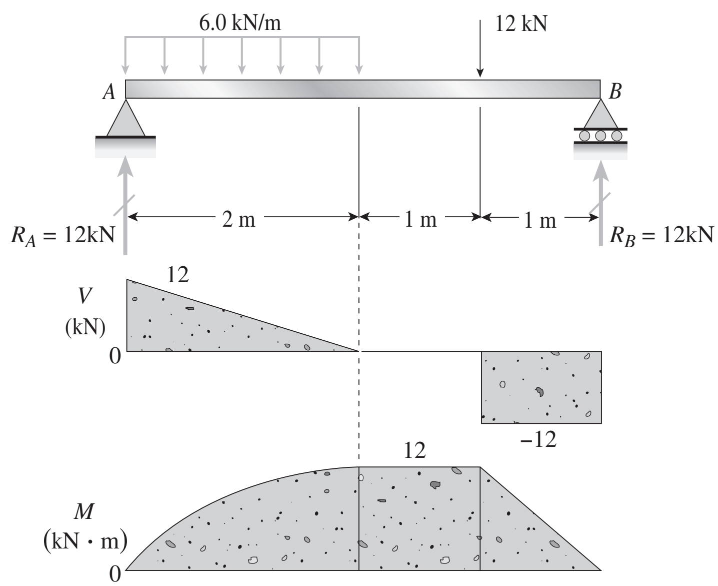


Problem 4.5-28 The shear-force diagram for a simple beam is shown in the figure.

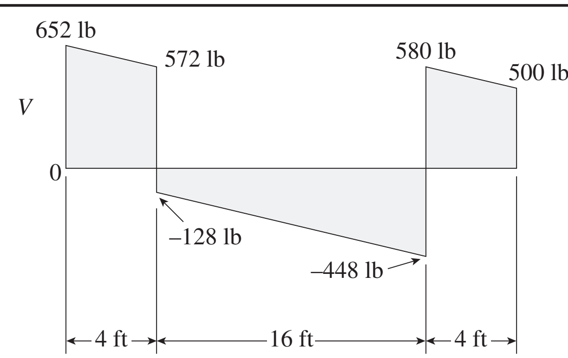
Determine the loading on the beam and draw the bending-moment diagram, assuming that no couples act as loads on the beam.



Solution 4.5-28 **Simple beam (*V* is given)**

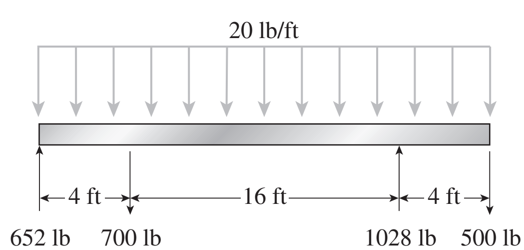


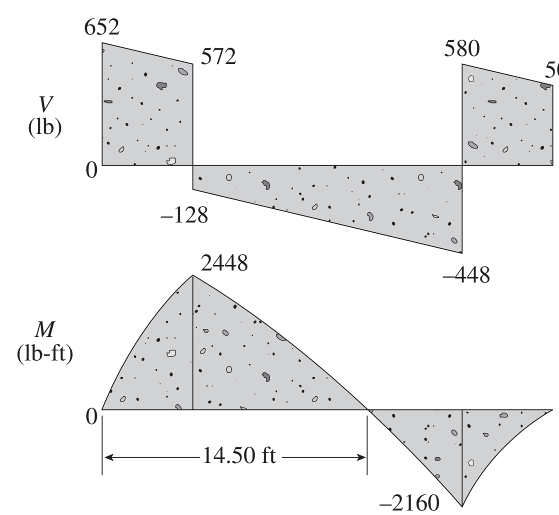
Problem 4.5-29 The shear-force diagram for a beam is shown in the figure. Assuming that no couples act as loads on the beam, determine the forces acting on the beam and draw the bending-moment diagram.



Solution 4.5-29 **Forces on a beam (*V* is given)**

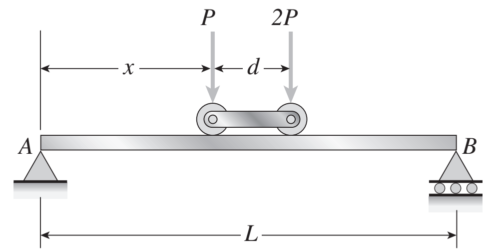
FORCE DIAGRAM



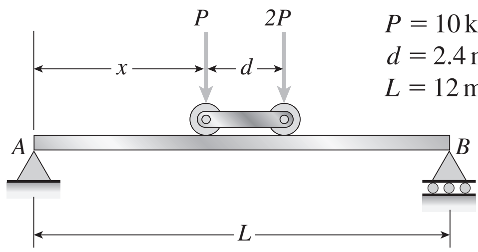


Problem 4.5-30 A simple beam AB supports two connected wheel loads P and $2P$ that are distance d apart (see figure). The wheels may be placed at any distance x from the left-hand support of the beam.

(a) Determine the distance x that will produce the maximum shear force in the beam, and also determine the maximum shear force V_{max}.
(b) Determine the distance x that will produce the maximum bending moment in the beam, and also draw the corresponding bending-moment diagram. (Assume $P = 10$ kN, $d = 2.4$ m, and $L = 12$ m.)



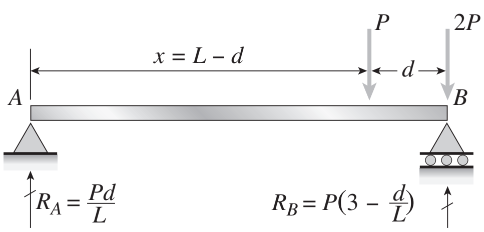
Solution 4.5-30 **Moving loads on a beam**



$P = 10\,\text{kN}$
$d = 2.4\,\text{m}$
$L = 12\,\text{m}$

(a) Maximum shear force

By inspection, the maximum shear force occurs at support B when the larger load is placed close to, but not directly over, that support.

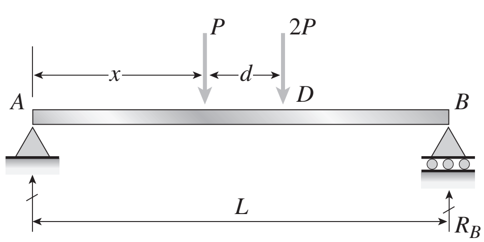


$x = L - d = 9.6$ m ←

$$V_{max} = R_B = P\left(3 - \frac{d}{L}\right) = 28 \text{ kN} \quad \leftarrow$$

(b) Maximum bending moment

By inspection, the maximum bending moment occurs at point D, under the larger load $2P$.



Reaction at support B:

$$R_B = \frac{P}{L}x + \frac{2P}{L}(x + d) = \frac{P}{L}(2d + 3x)$$

Bending moment at D:

$$M_D = R_B(L - x - d)$$

$$= \frac{P}{L}(2d + 3x)(L - x - d)$$

$$= \frac{P}{L}[-3x^2 + (3L - 5d)x + 2d(L - d)] \qquad \text{Eq.(1)}$$

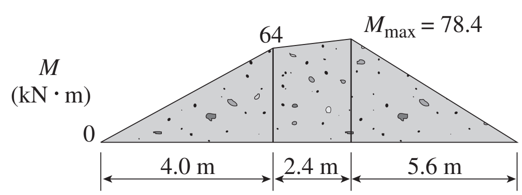
$$\frac{dM_D}{dx} = \frac{P}{L}(-6x + 3L - 5d) = 0$$

Solve for x: $\quad x = \frac{L}{6}\left(3 - \frac{5d}{L}\right) = 4.0 \text{ m} \quad \leftarrow$

Substitute x into Eq (1):

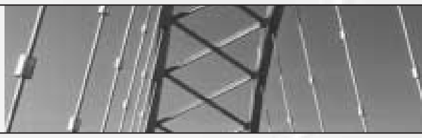
$$M_{max} = \frac{P}{L}\left[-3\left(\frac{L}{6}\right)^2\left(3 - \frac{5d}{L}\right)^2 + (3L - 5d) \times \left(\frac{L}{6}\right)\left(3 - \frac{5d}{L}\right) + 2d(L - d)\right]$$

$$= \frac{PL}{12}\left(3 - \frac{d}{L}\right)^2 = 78.4 \text{ kN} \cdot \text{m} \quad \leftarrow$$



Note: $R_A = \frac{P}{2}\left(3 + \frac{d}{L}\right) = 16 \text{ kN}$

$R_B = \frac{P}{2}\left(3 - \frac{d}{L}\right) = 14 \text{ kN}$

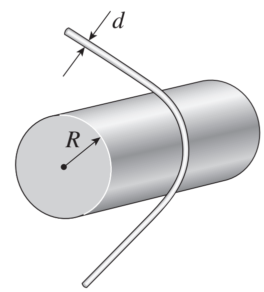


5

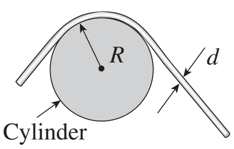
Stresses in Beams (Basic Topics)

Longitudinal Strains in Beams

Problem 5.4-1 Determine the maximum normal strain ϵ_{max} produced in a steel wire of diameter $d = 1/16$ in. when it is bent around a cylindrical drum of radius $R = 24$ in. (see figure).



Solution 5.4-1 **Steel wire**



$$R = 24\text{ in.} \qquad d = \frac{1}{16}\text{ in.}$$

From Eq. (5-4):

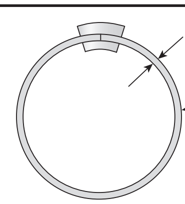
$$\varepsilon_{max} = \frac{y}{\rho}$$

$$= \frac{d/2}{R + d/2} = \frac{d}{2R + d}$$

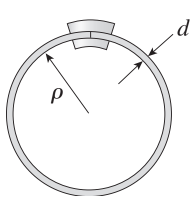
Substitute numerical values:

$$\varepsilon_{max} = \frac{1/16\text{ in.}}{2(24\text{ in.}) + 1/16\text{ in.}} = 1300 \times 10^{-6} \quad \longleftarrow$$

Problem 5.4-2 A copper wire having diameter $d = 3$ mm is bent into a circle and held with the ends just touching (see figure). If the maximum permissible strain in the copper is $\epsilon_{max} = 0.0024$, what is the shortest length L of wire that can be used?



Solution 5.4-2 **Copper wire**



$$d = 3\text{ mm} \qquad \varepsilon_{max} = 0.0024$$

$$L = 2\pi\rho \qquad \rho = \frac{L}{2\pi}$$

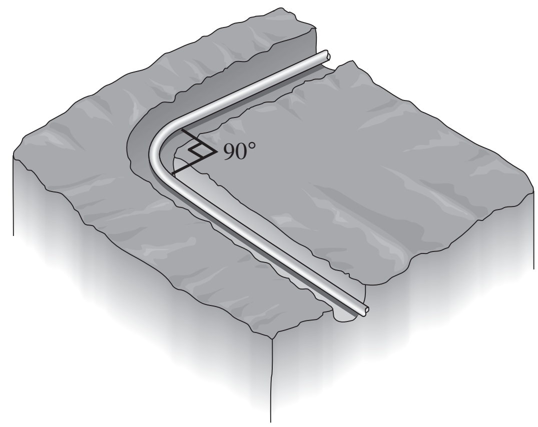
From Eq. (5-4):

$$\varepsilon_{max} = \frac{y}{\rho} = \frac{d/2}{L/2\pi} = \frac{\pi d}{L}$$

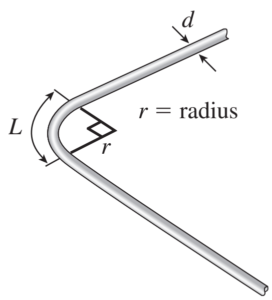
$$L_{min} = \frac{\pi d}{\varepsilon_{max}} = \frac{\pi\,(3\text{ mm})}{0.0024} = 3.93\text{ m} \quad \longleftarrow$$

Problem 5.4-3 A 4.5 in. outside diameter polyethylene pipe designed to carry chemical wastes is placed in a trench and bent around a quarter-circular 90° bend (see figure). The bent section of the pipe is 46 ft long.

Determine the maximum compressive strain ϵ_{max} in the pipe.



Solution 5.4-3 **Polyethylene pipe**



$L =$ length of 90º bend

$L = 46\text{ ft} = 552\text{ in.}$

$d = 4.5\text{ in.}$

$$L = \frac{2\pi r}{4} = \frac{\pi r}{2}$$

Angle equals 90º or $\pi/2$ radians,

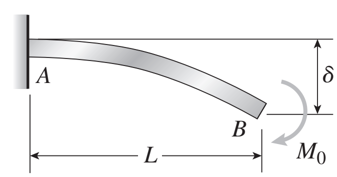
$r = \rho =$ radius of curvature

$$\rho = \frac{L}{\pi/2} = \frac{2L}{\pi} \qquad \varepsilon_{max} = \frac{y}{\rho} = \frac{d/2}{2L/\pi}$$

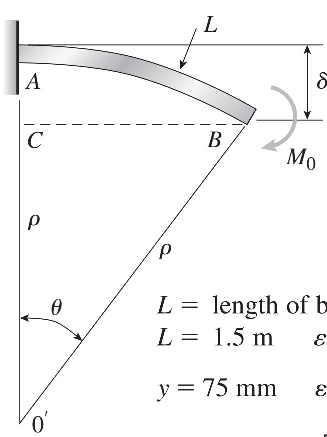
$$\varepsilon_{max} = \frac{\pi d}{4L} = \frac{\pi}{4}\left(\frac{4.5\text{ in.}}{552\text{ in.}}\right) = 6400 \times 10^{-6} \quad \longleftarrow$$

Problem 5.4-4 A cantilever beam AB is loaded by a couple M_0 at its free end (see figure). The length of the beam is $L = 1.5$ m and the longitudinal normal strain at the top surface is 0.001. The distance from the top surface of the beam to the neutral surface is 75 mm.

Calculate the radius of curvature ρ, the curvature κ, and the vertical deflection δ at the end of the beam.



Solution 5.4-4 **Cantilever beam**



$L =$ length of beam

$L = 1.5$ m $\qquad \varepsilon_{max} = 0.001$

$y = 75$ mm $\qquad \varepsilon_{max} = \dfrac{y}{\rho}$

$$\therefore\ \rho = \frac{y}{\varepsilon_{max}} = \frac{75\text{ mm}}{0.001} = 75\text{ m} \quad \longleftarrow$$

$$\kappa = \frac{1}{\rho} = 0.01333\text{ m}^{-1} \quad \longleftarrow$$

Assume that the deflection curve is nearly flat. Then the distance BC is the same as the length L of the beam.

$$\therefore\ \sin\theta = \frac{L}{\rho} = \frac{1.5\text{ m}}{75\text{ m}} = 0.02$$

$\theta =$ arcsin $0.02 = 0.02$ rad

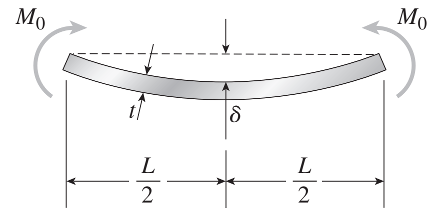
$\delta = \rho(1 - \cos\theta) = (75\text{ m})(1 - \cos(0.02\text{ rad}))$

$= 15.0$ mm $\longleftarrow$

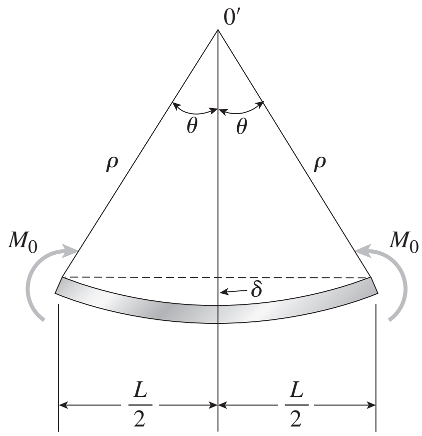
NOTE: $\dfrac{L}{\delta} = 100$, which confirms that the deflection curve is nearly flat.

Problem 5.4-5 A thin strip of steel of length $L = 20$ in. and thickness $t = 0.2$ in. is bent by couples M_0 (see figure). The deflection δ at the midpoint of the strip (measured from a line joining its end points) is found to be 0.25 in.

Determine the longitudinal normal strain ϵ at the top surface of the strip.



Solution 5.4-5 Thin strip of steel



$L = 20$ in. $\quad t = 0.2$ in.
$\delta = 0.25$ in.

The deflection curve is very flat (note that $L/\delta = 80$) and therefore θ is a very small angle.

$$\sin\theta = \frac{L/2}{\rho}$$

For small angles, $\theta = \sin\theta = \dfrac{L/2}{\rho}$ (θ is in radians)

$$\delta = \rho - \rho\cos\theta = \rho(1 - \cos\theta)$$
$$= \rho\left(1 - \cos\frac{L}{2\rho}\right)$$

Substitute numerical values ($\rho =$ inches):

$$0.25 = \rho\left(1 - \cos\frac{10}{\rho}\right)$$

Solve numerically: $\rho = 200.0$ in.

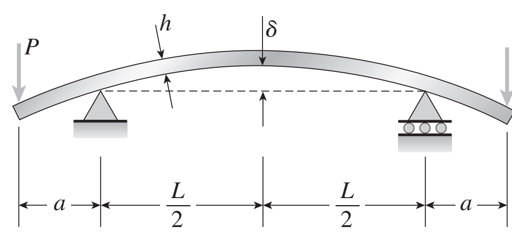
NORMAL STRAIN

$$\varepsilon = \frac{y}{\rho} = \frac{t/2}{\rho} = \frac{0.1 \text{ in.}}{200 \text{ in.}} = 500 \times 10^{-6} \quad \longleftarrow$$

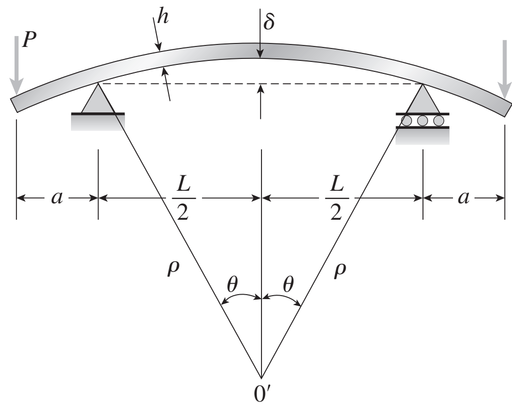
(Shortening at the top surface)

Problem 5.4-6 A bar of rectangular cross section is loaded and supported as shown in the figure. The distance between supports is $L = 1.2$ m and the height of the bar is $h = 100$ mm. The deflection δ at the midpoint is measured as 3.6 mm.

What is the maximum normal strain ϵ at the top and bottom of the bar?



Solution 5.4-6 **Bar of rectangular cross section**



$L = 1.2$ m $\quad h = 100$ mm $\quad \delta = 3.6$ mm

Note that the deflection curve is nearly flat ($L/\delta = 333$) and θ is a very small angle.

$$\sin\theta = \frac{L/2}{\rho}$$

$$\theta = \frac{L/2}{\rho} \text{ (radians)}$$

$$\delta = \rho\,(1 - \cos\theta) = \rho\left(1 - \cos\frac{L}{2\rho}\right)$$

Substitute numerical values ($\rho =$ meters):

$$0.0036 = \rho\left(1 - \cos\frac{0.6}{\rho}\right)$$

Solve numerically: $\rho = 50.00$ m

Normal strain

$$\varepsilon = \frac{y}{\rho} = \frac{h/2}{\rho} = \frac{50 \text{ mm}}{50{,}000 \text{ mm}} = 1000 \times 10^{-6} \quad \longleftarrow$$

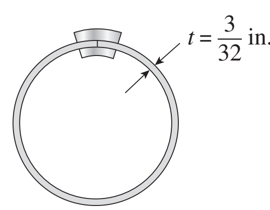
(Elongation on top; shortening on bottom)

Normal Stresses in Beams

Problem 5.5-1 A thin strip of hard copper ($E = 16{,}400$ ksi) having length $L = 80$ in. and thickness $t = 3/32$ in. is bent into a circle and held with the ends just touching (see figure).

(a) Calculate the maximum bending stress σ_{max} in the strip.

(b) Does the stress increase or decrease if the thickness of the strip is increased?



Solution 5.5-1 **Copper strip bent into a circle**

$E = 16{,}400$ ksi $\quad L = 80$ in. $\quad t = 3/32$ in.

(a) Maximum bending stress

$$L = 2\pi r = 2\pi\rho \qquad \rho = \frac{L}{2\pi}$$

From Eq. (5-7): $\sigma = \dfrac{Ey}{\rho} = \dfrac{2\pi E y}{L}$

$$\sigma_{max} = \frac{2\pi E(t/2)}{L} = \frac{\pi E t}{L}$$

Substitute numerical values:

$$\sigma_{max} = \frac{\pi\,(16{,}400 \text{ ksi})(3/32 \text{ in.})}{80 \text{ in.}} = 60.4 \text{ ksi} \quad \longleftarrow$$

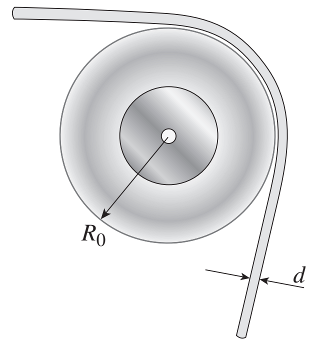
(b) Change in stress

If the thickness t is increased, the stress σ_{max} increases. $\longleftarrow$

Problem 5.5-2 A steel wire (E = 200 GPa) of diameter d = 1.0 mm is bent around a pulley of radius R_0 = 400 mm (see figure).

(a) What is the maximum stress σ_{max} in the wire?

(b) Does the stress increase or decrease if the radius of the pulley is increased?



Solution 5.5-2 **Steel wire bent around a pulley**

E = 200 GPa $\quad d$ = 1.0 mm $\quad R_0$ = 400 mm

(a) MAXIMUM STRESS IN THE WIRE

$$\rho = R_0 + \frac{d}{2} = 400 \text{ mm} + 0.5 \text{ mm} = 400.5 \text{ mm}$$

$$y = \frac{d}{2} = 0.5 \text{ mm}$$

From Eq. (5-7):

$$\sigma_{max} = \frac{Ey}{\rho} = \frac{(200 \text{ GPa})\ (0.5 \text{ mm})}{400.5 \text{ mm}} = 250 \text{ MPa} \quad \longleftarrow$$

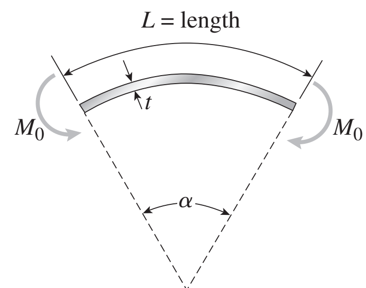
(b) CHANGE IN STRESS

If the radius is increased, the stress σ_{max} decreases. ⟵

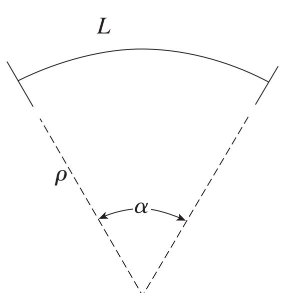
Problem 5.5-3 A thin, high-strength steel rule ($E = 30 \times 10^6$ psi) having thickness t = 0.15 in. and length L = 40 in. is bent by couples M_0 into a circular arc subtending a central angle α = 45° (see figure).

(a) What is the maximum bending stress σ_{max} in the rule?

(b) Does the stress increase or decrease if the central angle is increased?



Solution 5.5-3 **Thin steel rule bent into an arc**



$E = 30 \times 10^6$ psi
t = 0.15 in.
L = 40 in.
α = 45° = 0.78540 rad

(a) MAXIMUM BENDING STRESS

$$L = \rho\alpha \qquad \rho = \frac{L}{\alpha} \qquad \alpha = \text{radians}$$

$$\sigma_{max} = \frac{Ey}{\rho} = \frac{E(t/2)}{L/\alpha} = \frac{Et\alpha}{2L}$$

Substitute numerical values:

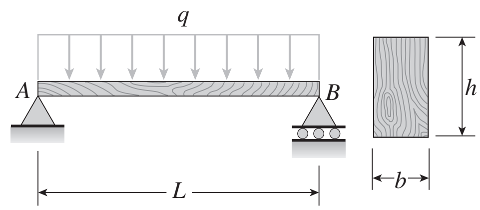
$$\sigma_{max} = \frac{(30 \times 10^6 \text{ psi})\ (0.15 \text{ in.})\ (0.78540 \text{ rad})}{2\ (40 \text{ in.})}$$
$$= 44{,}200 \text{ psi} = 44.2 \text{ ksi} \quad \longleftarrow$$

(b) CHANGE IN STRESS

If the angle α is increased, the stress σ_{max} increases. ⟵

Problem 5.5-4 A simply supported wood beam AB with span length $L = 3.5$ m carries a uniform load of intensity $q = 6.4$ kN/m (see figure).

Calculate the maximum bending stress σ_{max} due to the load q if the beam has a rectangular cross section with width $b = 140$ mm and height $h = 240$ mm.



Solution 5.5-4 **Simple beam with uniform load**

$L = 3.5$ m $\quad q = 6.4$ kN/m
$b = 140$ mm $\quad h = 240$ mm

$$M_{max} = \frac{qL^2}{8} \qquad S = \frac{bh^2}{6}$$

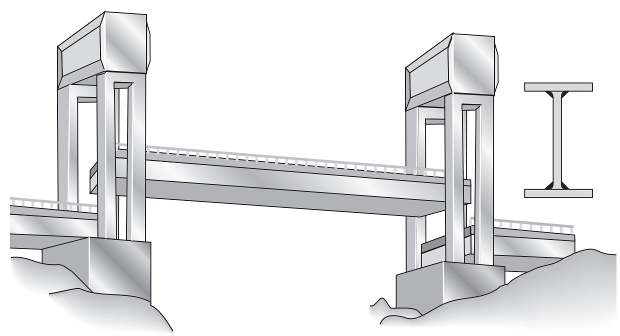
$$\sigma_{max} = \frac{M_{max}}{S} = \frac{3qL^2}{4bh^2}$$

Substitute numerical values:

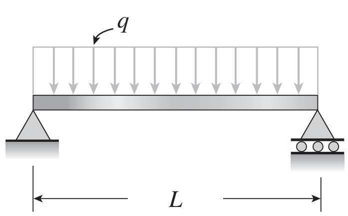
$$\sigma_{max} = \frac{3(6.4 \text{ kN/m})(3.5 \text{ m})^2}{4(140 \text{ mm})(240 \text{ mm})^2} = 7.29 \text{ MPa} \quad \longleftarrow$$

Problem 5.5-5 Each girder of the lift bridge (see figure) is 180 ft long and simply supported at the ends. The design load for each girder is a uniform load of intensity 1.6 k/ft. The girders are fabricated by welding three steel plates so as to form an I-shaped cross section (see figure) having section modulus $S = 3600$ in^3.

What is the maximum bending stress σ_{max} in a girder due to the uniform load?



Solution 5.5-5 **Bridge girder**



$L = 180$ ft $\quad q = 1.6$ k/ft

$S = 3600$ in.^{3}

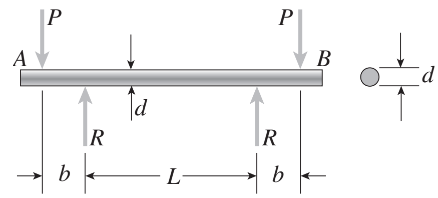
$$M_{max} = \frac{qL^2}{8}$$

$$\sigma_{max} = \frac{M_{max}}{S} = \frac{qL^2}{8S}$$

$$\sigma_{max} = \frac{(1.6 \text{ k/ft})(180 \text{ ft})^2(12 \text{ in./ft})}{8(3600 \text{ in.}^3)} = 21.6 \text{ ksi} \quad \longleftarrow$$

Problem 5.5-6 A freight-car axle AB is loaded approximately as shown in the figure, with the forces P representing the car loads (transmitted to the axle through the axle boxes) and the forces R representing the rail loads (transmitted to the axle through the wheels). The diameter of the axle is $d = 80$ mm, the distance between centers of the rails is L, and the distance between the forces P and R is $b = 200$ mm.

Calculate the maximum bending stress σ_{max} in the axle if $P = 47$ kN.



Solution 5.5-6 Freight-car axle

Diameter $d = 80$ mm
Distance $b = 200$ mm
Load $P = 47$ kN

$$M_{max} = Pb \qquad S = \frac{\pi d^3}{32}$$

MAXIMUM BENDING STRESS

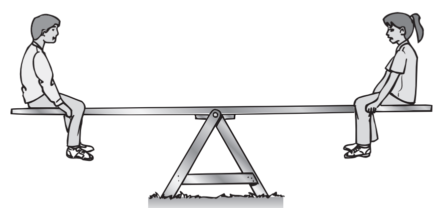
$$\sigma_{max} = \frac{M_{max}}{S} = \frac{32Pb}{\pi d^3}$$

Substitute numerical values:

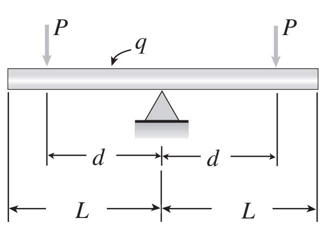
$$\sigma_{max} = \frac{32(47\text{ kN})(200\text{ mm})}{\pi(80\text{ mm})^3} = 187\text{ MPa} \quad \longleftarrow$$

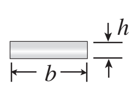
Problem 5.5-7 A seesaw weighing 3 lb/ft of length is occupied by two children, each weighing 90 lb (see figure). The center of gravity of each child is 8 ft from the fulcrum. The board is 19 ft long, 8 in. wide, and 1.5 in. thick.

What is the maximum bending stress in the board?



Solution 5.5-7 Seesaw





$b = 8$ in. $\quad h = 1.5$ in.
$q = 3$ lb/ft $\quad P = 90$ lb $\quad d = 8.0$ ft $\quad L = 9.5$ ft

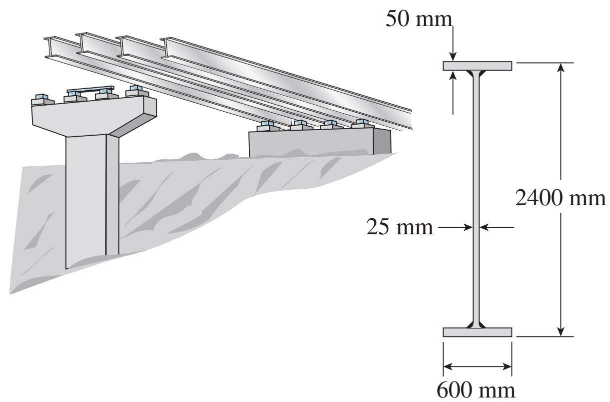
$$M_{max} = Pd + \frac{qL^2}{2} = 720\text{ lb-ft} + 135.4\text{ lb-ft}$$
$$= 855.4\text{ lb-ft} = 10{,}264\text{ lb-in.}$$

$$S = \frac{bh^2}{6} = 3.0\text{ in}^3.$$

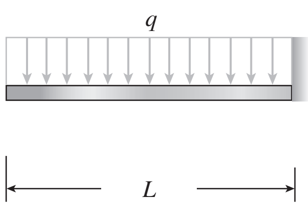
$$\sigma_{max} = \frac{M}{S} = \frac{10{,}264\text{ lb-in.}}{3.0\text{ in.}^3} = 3420\text{ psi} \quad \longleftarrow$$

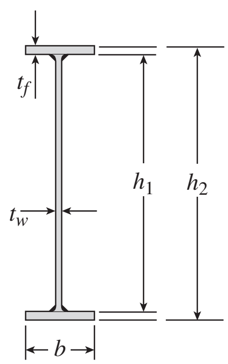
Problem 5.5-8 During construction of a highway bridge, the main girders are cantilevered outward from one pier toward the next (see figure). Each girder has a cantilever length of 46 m and an I-shaped cross section with dimensions as shown in the figure. The load on each girder (during construction) is assumed to be 11.0 kN/m, which includes the weight of the girder.

Determine the maximum bending stress in a girder due to this load.



Solution 5.5-8 Bridge girder





$L = 46$ m
$q = 11.0$ kN/m
$b = 600$ mm $\quad h = 2400$ mm
$t_f = 50$ mm $\quad t_w = 25$ mm
$h_1 = h - 2t_f = 2300$ mm
$b_1 = b - t_w = 575$ mm

$$M_{\max} = \frac{qL^2}{2} = \frac{1}{2}(11.0 \text{ kN/m})(46 \text{ m})^2 = 11{,}638 \text{ kN} \cdot \text{m}$$

$$\sigma_{\max} = \frac{M_{\max}\, c}{I} \qquad c = \frac{h}{2} = 1200 \text{ mm}$$

$$I = \frac{bh^3}{12} - \frac{b_1 h_1^3}{12}$$

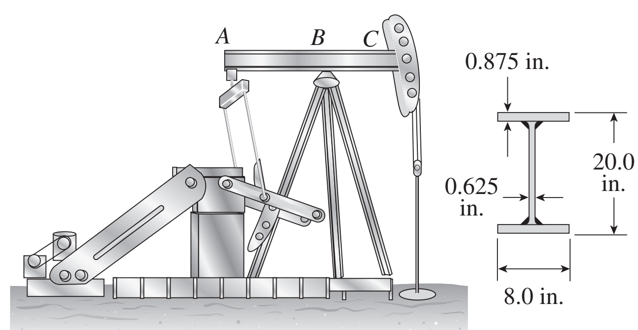
$$= \frac{1}{12}(0.6 \text{ m})(2.4 \text{ m})^3 - \frac{1}{12}(0.575 \text{ m})(2.3 \text{ m})^3$$

$$= 0.6912 \text{ m}^4 - 0.5830 \text{ m}^4 = 0.1082 \text{ m}^4$$

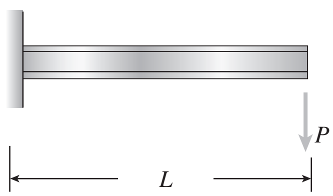
$$\sigma_{\max} = \frac{M_{\max}\, c}{I} = \frac{(11{,}638 \text{ kN} \cdot \text{m})(1.2 \text{ m})}{0.1082 \text{ m}^4}$$

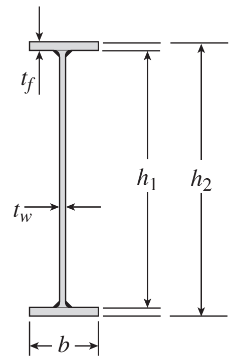
$$= 129 \text{ MPa} \longleftarrow$$

Problem 5.5-9 The horizontal beam *ABC* of an oil-well pump has the cross section shown in the figure. If the vertical pumping force acting at end *C* is 8.8 k, and if the distance from the line of action of that force to point *B* is 14 ft, what is the maximum bending stress in the beam due to the pumping force?



Solution 5.5-9 **Beam in an oil-well pump**





$L = 14$ ft
$P = 8.8$ k
$b = 8.0$ in. $\quad h = 20.0$ in.
$t_f = 0.875$ in. $\quad t_w = 0.625$ in.
$h_1 = h - 2t_f = 18.25$ in.
$b_1 = b - t_w = 7.375$ in.

$$M_{\max} = PL = (8.8 \text{ k})(14 \text{ ft})$$
$$= 123{,}200 \text{ lb-ft} = 1{,}478{,}400 \text{ lb-in.}$$

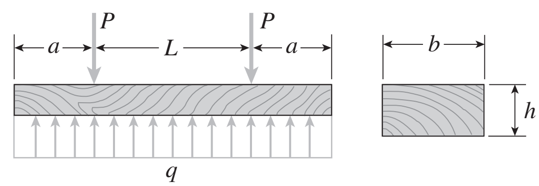
$$\sigma_{\max} = \frac{M_{\max}\, c}{I} \qquad c = \frac{h}{2} = 10.0 \text{ in.}$$

$$I = \frac{bh^3}{12} - \frac{b_1 h_1^3}{12}$$
$$= \frac{1}{12}(8.0 \text{ in.})(20.0 \text{ in.})^3 - \frac{1}{12}(7.375 \text{ in.})(18.25 \text{ in.})^3$$
$$= 5{,}333.3 \text{ in.}^4 - 3{,}735.7 \text{ in.}^4 = 1{,}597.7 \text{ in.}^4$$

$$\sigma_{\max} = \frac{M_{\max}\, c}{I} = \frac{(1.4784 \times 10^6 \text{ lb-in.})(10.0 \text{ in.})}{1{,}597.7 \text{ in.}^4}$$
$$= 9250 \text{ psi} = 9.25 \text{ ksi} \quad \longleftarrow$$

Problem 5.5-10 A railroad tie (or *sleeper*) is subjected to two rail loads, each of magnitude $P = 175$ kN, acting as shown in the figure. The reaction q of the ballast is assumed to be uniformly distributed over the length of the tie, which has cross-sectional dimensions $b = 300$ mm and $h = 250$ mm.

Calculate the maximum bending stress $\sigma_{\max}$ in the tie due to the loads P, assuming the distance $L = 1500$ mm and the overhang length $a = 500$ mm.

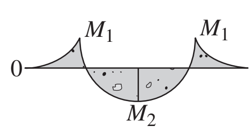


Solution 5.5-10 **Railroad tie (or sleeper)**

DATA $\quad P = 175$ kN $\quad b = 300$ mm $\quad h = 250$ mm
$L = 1500$ mm $\quad a = 500$ mm

$$q = \frac{2P}{L + 2a} \qquad S = \frac{bh^2}{6} = 3.125 \times 10^{-3} \text{ m}^3$$

BENDING-MOMENT DIAGRAM



$$M_1 = \frac{qa^2}{2} = \frac{Pa^2}{L + 2a}$$

$$M_2 = \frac{q}{2}\left(\frac{L}{2} + a\right)^2 - \frac{PL}{2}$$
$$= \frac{P}{L + 2a}\left(\frac{L}{2} + a\right)^2 - \frac{PL}{2}$$
$$= \frac{P}{4}(2a - L)$$

Substitute numerical values:

$M_1 = 17{,}500 \text{ N} \cdot \text{m} \qquad M_2 = -21{,}875 \text{ N} \cdot \text{m}$

$M_{\max} = 21{,}875 \text{ N} \cdot \text{m}$

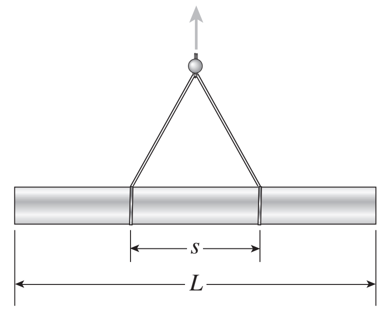
MAXIMUM BENDING STRESS

$$\sigma_{\max} = \frac{M_{\max}}{5} = \frac{21{,}875 \text{ N} \cdot \text{m}}{3.125 \times 10^{-3} \text{ m}^3} = 7.0 \text{ MPa} \quad \longleftarrow$$

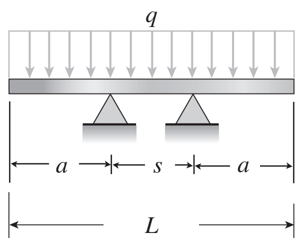
(Tension on top; compression on bottom)

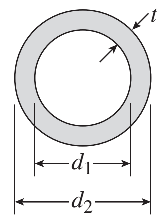
Problem 5.5-11 A fiberglass pipe is lifted by a sling, as shown in the figure. The outer diameter of the pipe is 6.0 in., its thickness is 0.25 in., and its weight density is 0.053 lb/in.^{3} The length of the pipe is $L = 36$ ft and the distance between lifting points is $s = 11$ ft.

Determine the maximum bending stress in the pipe due to its own weight.



Solution 5.5-11 **Pipe lifted by a sling**





$L = 36 \text{ ft} = 432 \text{ in.}$ $d_2 = 6.0 \text{ in.}$ $t = 0.25 \text{ in.}$

$s = 11 \text{ ft} = 132 \text{ in.}$ $d_1 = d_2 - 2t = 5.5 \text{ in.}$

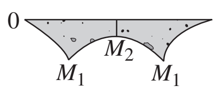
$\gamma = 0.053 \text{ lb/in.}^3$ $A = \dfrac{\pi}{4}(d_2^2 - d_1^2) = 4.5160 \text{ in.}^2$

$a = (L - s)/2 = 150 \text{ in.}$

$$I = \frac{\pi}{64}(d_2^4 - d_1^4) = 18.699 \text{ in.}^4$$

$$q = \gamma A = (0.053 \text{ lb/in.}^3)(4.5160 \text{ in.}^2) = 0.23935 \text{ lb/in.}$$

Bending-moment diagram



$$M_1 = -\frac{qa^2}{2} = -2{,}692.7 \text{ lb-in.}$$

$$M_2 = -\frac{qL}{4}\left(\frac{L}{2} - s\right) = -2{,}171.4 \text{ lb-in.}$$

$M_{max} = 2{,}692.7$ lb-in.

Maximum bending stress

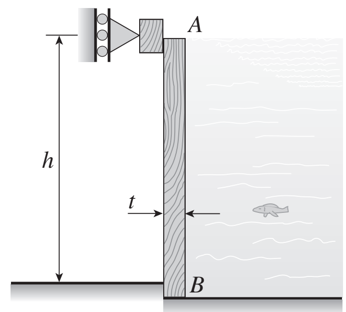
$$\sigma_{max} = \frac{M_{max}\, c}{I} \qquad c = \frac{d_2}{2} = 3.0 \text{ in.}$$

$$\sigma_{max} = \frac{(2{,}692.7 \text{ lb-in.})(3.0 \text{ in.})}{18.699 \text{ in.}^4} = 432 \text{ psi} \quad \longleftarrow$$

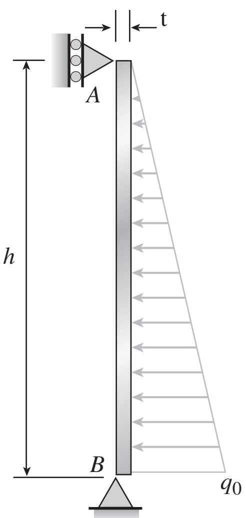
(Tension on top)

Problem 5.5-12 A small dam of height $h = 2.0$ m is constructed of vertical wood beams AB of thickness $t = 120$ mm, as shown in the figure. Consider the beams to be simply supported at the top and bottom.

Determine the maximum bending stress σ_{max} in the beams, assuming that the weight density of water is $\gamma = 9.81\ \text{kN/m}^3$.



Solution 5.5-12 **Vertical wood beam**



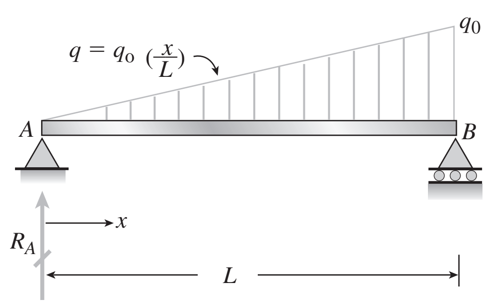
$h = 2.0$ m
$t = 120$ mm
$\gamma = 9.81\ \text{kN/m}^3$ (water)

Let $b =$ width of beam perpendicular to the plane of the figure

Let $q_0 =$ maximum intensity of distributed load

$$q_0 = \gamma b h \qquad S = \frac{bt^2}{6}$$

MAXIMUM BENDING MOMENT



$$R_A = \frac{q_0 L}{6}$$

$$M = R_A x - \frac{q_0 x^3}{6L}$$

$$= \frac{q_0 L x}{6} - \frac{q_0 x^3}{6L}$$

$$\frac{dM}{dx} = \frac{q_0 L}{6} - \frac{q_0 x^2}{2L} = 0 \qquad x = \frac{L}{\sqrt{3}}$$

Substitute $x = L/\sqrt{3}$ into the equation for M:

$$M_{max} = \frac{q_0 L}{6}\left(\frac{L}{\sqrt{3}}\right) - \frac{q_0}{6L}\left(\frac{L^3}{3\sqrt{3}}\right) = \frac{q_0 L^2}{9\sqrt{3}}$$

For the vertical wood beam: $L = h$; $M_{max} = \dfrac{q_0 h^2}{9\sqrt{3}}$

Maximum bending stress

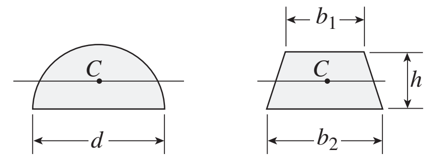
$$\sigma_{max} = \frac{M_{max}}{S} = \frac{2q_0 h^2}{3\sqrt{3}\, bt^2} = \frac{2\gamma h^3}{3\sqrt{3}\, t^2}$$

SUBSTITUTE NUMERICAL VALUES:

$\sigma_{max} = 2.10$ MPa ⟵

NOTE: For $b = 1.0$ m, we obtain $q_0 = 19{,}620$ N/m, $S = 0.0024\ \text{m}^3$, $M_{max} = 5{,}034.5\ \text{N} \cdot \text{m}$, and $\sigma_{max} = M_{max}/S = 2.10$ MPa

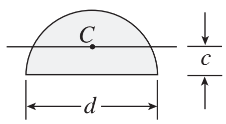
Problem 5.5-13 Determine the maximum tensile stress σ_t (due to pure bending by positive bending moments M) for beams having cross sections as follows (see figure): (a) a semicircle of diameter d, and (b) an isosceles trapezoid with bases $b_1 = b$ and $b_2 = 4b/3$, and altitude h.



(a) (b)

Solution 5.5-13 Maximum tensile stress

(a) SEMICIRCLE



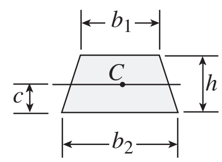
From Appendix D, Case 10:

$$I_C = \frac{(9\pi^2 - 64)r^4}{72\pi} = \frac{(9\pi^2 - 64)d^4}{1152\pi}$$

$$c = \frac{4r}{3\pi} = \frac{2d}{3\pi}$$

$$\sigma_t = \frac{Mc}{I_C} = \frac{768M}{(9\pi^2 - 64)d^3} = 30.93\,\frac{M}{d^3} \quad \longleftarrow$$

(b) TRAPEZOID



$$b_1 = b \qquad b_2 = \frac{4b}{3}$$

From Appendix D, Case 8:

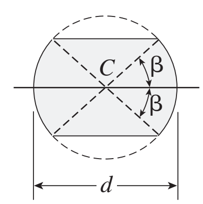
$$I_C = \frac{h^3(b_1^2 + 4b_1b_2 + b_2^2)}{36(b_1 + b_2)}$$

$$= \frac{73bh^3}{756}$$

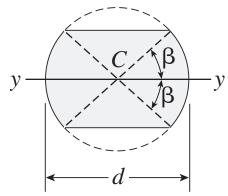
$$c = \frac{h(2b_1 + b_2)}{3(b_1 + b_2)} = \frac{10h}{21}$$

$$\sigma_t = \frac{Mc}{I_C} = \frac{360M}{73bh^2} \quad \longleftarrow$$

Problem 5.5-14 Determine the maximum bending stress σ_{max} (due to pure bending by a moment M) for a beam having a cross section in the form of a circular core (see figure). The circle has diameter d and the angle $\beta = 60°$. (*Hint:* Use the formulas given in Appendix D, Cases 9 and 15.)



Solution 5.5-14 Circular core



From Appendix D, Cases 9 and 15:

$$I_y = \frac{\pi r^4}{4} - \frac{r^4}{2}\left(\alpha - \frac{ab}{r^2} + \frac{2ab^3}{r^4}\right)$$

$$r = \frac{d}{2} \quad \alpha = \frac{\pi}{2} - \beta$$

$\beta =$ radians $\quad \alpha =$ radians $\quad a = r \sin \beta \quad b = r \cos \beta$

$$I_y = \frac{\pi d^4}{64} - \frac{d^4}{32}\left(\frac{\pi}{2} - \beta - \sin\beta\cos\beta + 2\sin\beta\cos^3\beta\right)$$

$$= \frac{\pi d^4}{64} - \frac{d^4}{32}\left(\frac{\pi}{2} - \beta - (\sin\beta\cos\beta)(1 - 2\cos^2\beta)\right)$$

$$= \frac{\pi d^4}{64} - \frac{d^4}{32}\left(\frac{\pi}{2} - \beta - \left(\frac{1}{2}\sin 2\beta\right)(-\cos 2\beta)\right)$$

$$= \frac{\pi d^4}{64} - \frac{d^4}{32}\left(\frac{\pi}{2} - \beta + \frac{1}{4}\sin 4\beta\right)$$

$$= \frac{d^4}{128}(4\beta - \sin 4\beta)$$

MAXIMUM BENDING STRESS

$$\sigma_{max} = \frac{Mc}{I_y} \quad c = r\sin\beta = \frac{d}{2}\sin\beta$$

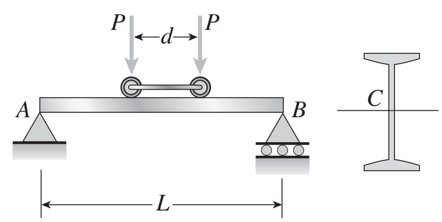
$$\sigma_{max} = \frac{64M\sin\beta}{d^3(4\beta - \sin 4\beta)} \quad \longleftarrow$$

For $\beta = 60° = \pi/3$ *rad:*

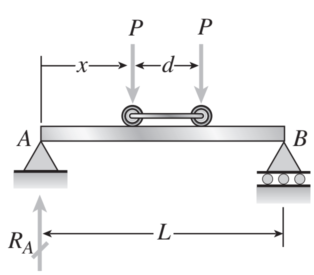
$$\sigma_{max} = \frac{576M}{(8\pi\sqrt{3} + 9)d^3} = 10.96\frac{M}{d^3} \quad \longleftarrow$$

Problem 5.5-15 A simple beam AB of span length $L = 24$ ft is subjected to two wheel loads acting at distance $d = 5$ ft apart (see figure). Each wheel transmits a load $P = 3.0$ k, and the carriage may occupy any position on the beam.

Determine the maximum bending stress σ_{max} due to the wheel loads if the beam is an I-beam having section modulus $S = 16.2$ in.^{3}



Solution 5.5-15 Wheel loads on a beam



$L = 24$ ft $= 288$ in.
$d = 5$ ft $= 60$ in.
$P = 3$ k
$S = 16.2$ in.^{3}

MAXIMUM BENDING MOMENT

$$R_A = \frac{P}{L}(L - x) + \frac{P}{L}(L - x - d) = \frac{P}{L}(2L - d - 2x)$$

$$M = R_A x = \frac{P}{L}(2Lx - dx - 2x^2)$$

$$\frac{dM}{dx} = \frac{P}{L}(2L - d - 4x) = 0 \quad x = \frac{L}{2} - \frac{d}{4}$$

Substitute x into the equation for M:

$$M_{max} = \frac{P}{2L}\left(L - \frac{d}{2}\right)^2$$

MAXIMUM BENDING STRESS

$$\sigma_{max} = \frac{M_{max}}{S} = \frac{P}{2LS}\left(L - \frac{d}{2}\right)^2 \quad \longleftarrow$$

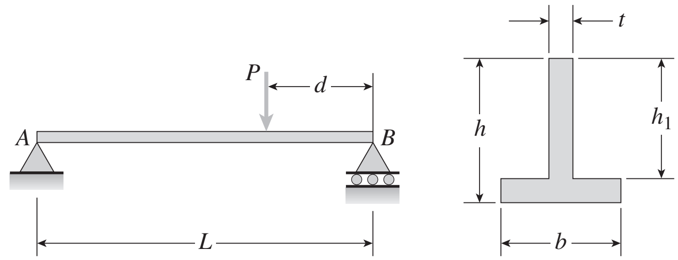
Substitute numerical values:

$$\sigma_{max} = \frac{3\text{k}}{2(288\text{ in.})(16.2\text{ in.}^3)}(288\text{ in.} - 30\text{ in.})^2$$

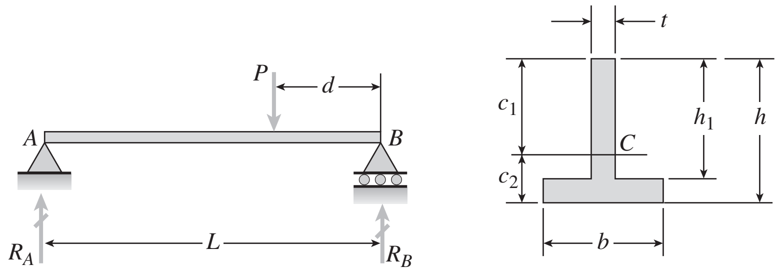
$$= 21.4\text{ ksi} \quad \longleftarrow$$

Problem 5.5-16 Determine the maximum tensile stress σ_t and maximum compressive stress σ_c due to the load P acting on the simple beam AB (see figure).

Data are as follows: $P = 5.4$ kN, $L = 3.0$ m, $d = 1.2$ m, $b = 75$ mm, $t = 25$ mm, $h = 100$ mm, and $h_1 = 75$ mm.



Solution 5.5-16 Simple beam of T-section



$P = 5.4$ kN $\quad L = 3.0$ m

$b = 75$ mm $\quad t = 25$ mm

$d = 1.2$ m $\quad h = 100$ mm $\quad h_1 = 75$ mm

PROPERTIES OF THE CROSS SECTION

$A = 3750\text{ mm}^2$

$c_1 = 62.5$ mm $\quad c_2 = 37.5$ mm

$I_C = 3.3203 \times 10^6\text{ mm}^4$

REACTIONS OF THE BEAM

$R_A = 2.16$ kN $\quad R_B = 3.24$ kN

MAXIMUM BENDING MOMENT

$$M_{\max} = R_A(L - d) = R_B(d) = 3888\text{ N}\cdot\text{m}$$

MAXIMUM TENSILE STRESS

$$\sigma_t = \frac{M_{\max}c_2}{I_C} = \frac{(3888\text{ N}\cdot\text{m})(0.0375\text{ m})}{3.3203 \times 10^6\text{ mm}^4}$$

$$= 43.9\text{ MPa} \longleftarrow$$

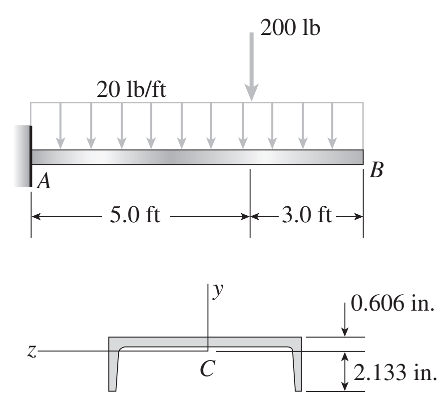
MAXIMUM COMPRESSIVE STRESS

$$\sigma_c = \frac{M_{\max}c_1}{I_C} = \frac{(3888\text{ N}\cdot\text{m})(0.0625\text{ m})}{3.3203 \times 10^6\text{ mm}^4}$$

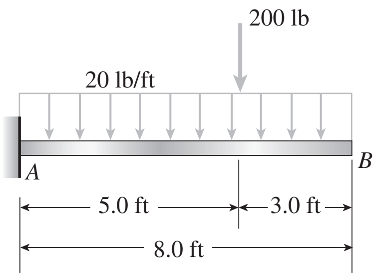
$$= 73.2\text{ MPa} \longleftarrow$$

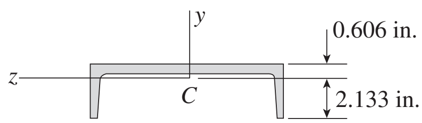
Problem 5.5-17 A cantilever beam AB, loaded by a uniform load and a concentrated load (see figure), is constructed of a channel section.

Find the maximum tensile stress σ_t and maximum compressive stress σ_c if the cross section has the dimensions indicated and the moment of inertia about the z axis (the neutral axis) is $I = 2.81\text{ in.}^4$ (*Note:* The uniform load represents the weight of the beam.)



Solution 5.5-17 **Cantilever beam (channel section)**





$I = 2.81 \text{ in.}^4 \quad c_1 = 0.606 \text{ in.} \quad c_2 = 2.133 \text{ in.}$

$$M_{\max} = (200 \text{ lb})(5.0 \text{ ft}) + (20 \text{ lb/ft})(8.0 \text{ ft})\left(\frac{8.0 \text{ ft}}{2}\right)$$
$$= 1000 \text{ lb-ft} + 640 \text{ lb-ft} = 1640 \text{ lb-ft}$$
$$= 19{,}680 \text{ lb-in.}$$

MAXIMUM TENSILE STRESS

$$\sigma_t = \frac{Mc_1}{I} = \frac{(19{,}680 \text{ lb-in.})(0.606 \text{ in.})}{2.81 \text{ in.}^4}$$
$$= 4{,}240 \text{ psi} \quad \longleftarrow$$

MAXIMUM COMPRESSIVE STRESS

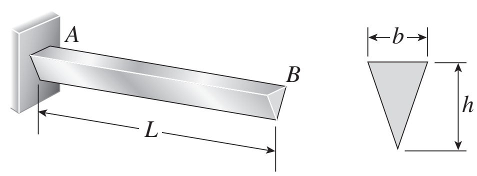
$$\sigma_c = \frac{Mc_2}{I} = \frac{(19{,}680 \text{ lb-in.})(2.133 \text{ in.})}{2.81 \text{ in.}^4}$$
$$= 14{,}940 \text{ psi} \quad \longleftarrow$$

Problem 5.5-18 A cantilever beam *AB* of triangular cross section has length $L = 0.8$ m, width $b = 80$ mm, and height $h = 120$ mm (see figure). The beam is made of brass weighing 85 kN/m^3.

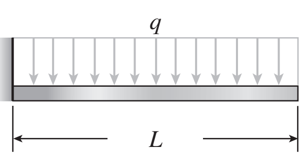
(a) Determine the maximum tensile stress σ_t and maximum compressive stress σ_c due to the beam's own weight.

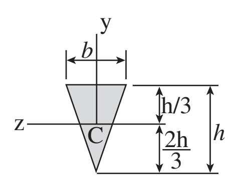
(b) If the width b is doubled, what happens to the stresses?

(c) If the height h is doubled, what happens to the stresses?



Solution 5.5-18 **Triangular beam**





$L = 0.8 \text{ m} \quad b = 80 \text{ mm} \quad h = 120 \text{ mm}$
$\gamma = 85 \text{ kN/m}^3$

(a) MAXIMUM STRESSES

$$q = \gamma A = \gamma\left(\frac{bh}{2}\right) \quad M_{\max} = \frac{qL^2}{2} = \frac{\gamma bhL^2}{4}$$

$$I_z = I_C = \frac{bh^3}{36} \quad c_1 = \frac{h}{3} \quad c_2 = \frac{2h}{3}$$

Tensile stress: $\sigma_t = \dfrac{Mc_1}{I_z} = \dfrac{3\gamma L^2}{h}$

Compressive stress: $\sigma_c = 2\sigma_t$

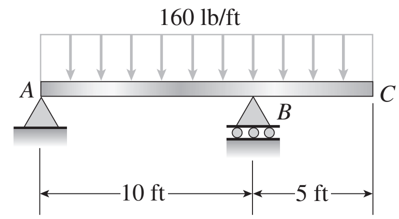
Substitute numerical values: $\sigma_t = 1.36$ MPa $\longleftarrow$
$\sigma_c = 2.72$ MPa $\longleftarrow$

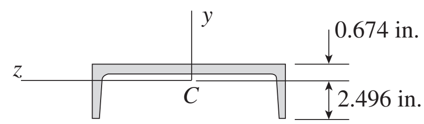
(b) WIDTH b IS DOUBLED
No change in stresses. $\longleftarrow$

(c) HEIGHT h IS DOUBLED
Stresses are reduced by half. $\longleftarrow$

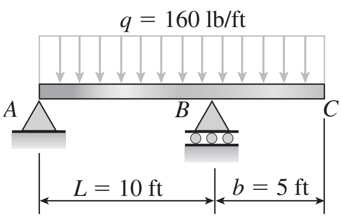
Problem 5.5-19 A beam *ABC* with an overhang from *B* to *C* supports a uniform load of 160 lb/ft throughout its length (see figure). The beam is a channel section with dimensions as shown in the figure. The moment of inertia about the *z* axis (the neutral axis) equals 5.14 in.^{4}

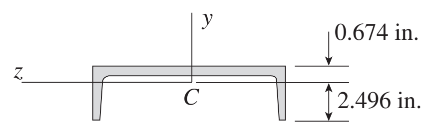
Calculate the maximum tensile stress σ_t and maximum compressive stress σ_c due to the uniform load.

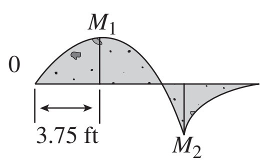




Solution 5.5-19 **Beam with an overhang**







$I_z = 5.14 \text{ in.}^4$

$c_1 = 0.674 \text{ in.} \quad c_2 = 2.496 \text{ in.}$

$R_A = 600 \text{ lb} \quad R_B = 1800 \text{ lb}$

$M_1 = 1125 \text{ lb-ft} = 13{,}500 \text{ lb-in.}$

$M_2 = 2000 \text{ lb-ft} = 24{,}000 \text{ lb-in.}$

At cross section of maximum positive bending moment

$$\sigma_t = \frac{M_1 c_2}{I_z} = \frac{(13{,}500 \text{ lb-in.})(2.496 \text{ in.})}{5.14 \text{ in.}^4} = 6{,}560 \text{ psi}$$

$$\sigma_c = \frac{M_1 c_1}{I_z} = \frac{(13{,}500 \text{ lb-in.})(0.674 \text{ in.})}{5.14 \text{ in.}^4} = 1{,}770 \text{ psi}$$

At cross section of maximum negative bending moment

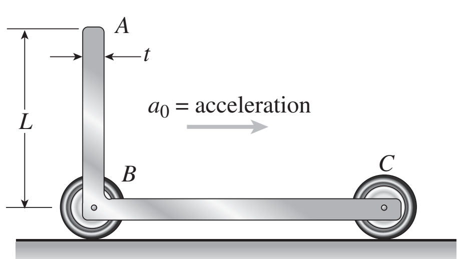
$$\sigma_t = \frac{M_2 c_1}{I_z} = \frac{(24{,}000 \text{ lb-in.})(0.674 \text{ in.})}{5.14 \text{ in.}^4} = 3{,}150 \text{ psi}$$

$$\sigma_c = \frac{M_2 c_2}{I_z} = \frac{(24{,}000 \text{ lb-in.})(2.496 \text{ in.})}{5.14 \text{ in.}^4} = 11{,}650 \text{ psi}$$

Maximum stresses

$\sigma_t = 6{,}560 \text{ psi} \quad \sigma_c = 11{,}650 \text{ psi}$ ←

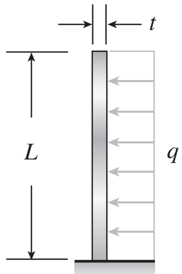
Problem 5.5-20 A frame *ABC* travels horizontally with an acceleration a_0 (see figure). Obtain a formula for the maximum stress σ_{max} in the vertical arm *AB*, which has length *L*, thickness *t*, and mass density *ρ*.



Solution 5.5-20 **Accelerating frame**

L = length of vertical arm
t = thickness of vertical arm
ρ = mass density
a_0 = acceleration
Let b = width of arm perpendicular to the plane of the figure
Let q = inertia force per unit distance along vertical arm

VERTICAL ARM



$$q = \rho b t a_0 \qquad M_{max} = \frac{qL^2}{2} = \frac{\rho b t a_0 L^2}{2}$$

$$S = \frac{bt^2}{6} \qquad \sigma_{max} = \frac{M_{max}}{S} = \frac{3\rho L^2 a_0}{t} \quad \longleftarrow$$

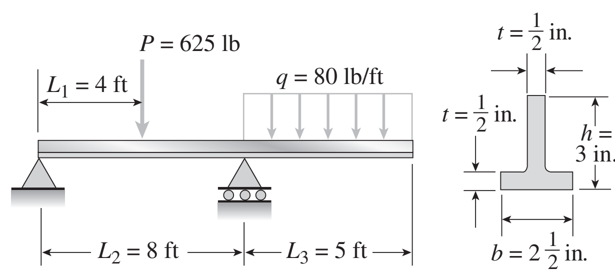
TYPICAL UNITS FOR USE IN THE PRECEDING EQUATION

SI UNITS: $\rho = \text{kg/m}^3 = \text{N} \cdot \text{s}^2/\text{m}^4$
L = meters (m)
$a_0 = \text{m/s}^2$
t = meters (m)
$\sigma_{max} = \text{N/m}^2$ (pascals)

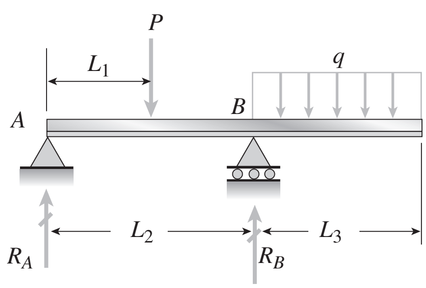
USCS UNITS: $\rho = \text{slug/ft}^3 = \text{lb-s}^2/\text{ft}^4$
L = ft $\quad a_0 = \text{ft/s}^2 \quad t$ = ft
$\sigma_{max} = \text{lb/ft}^2$ (Divide by 144 to obtain psi)

Problem 5.5-21 A beam of T-section is supported and loaded as shown in the figure. The cross section has width b =2 1/2 in., height h = 3 in., and thickness t = 1/2 in.

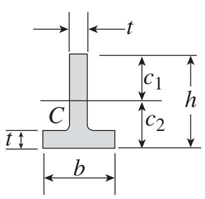
Determine the maximum tensile and compressive stresses in the beam.



Solution 5.5-21 **Beam of T-section**



L_1 = 4 ft = 48 in. $\quad L_2$ = 8 ft = 96 in. $\quad L_3$ = 5 ft = 60 in.
P = 625 lb $\quad q$ = 80 lb/ft = 6.6667 lb/in.



PROPERTIES OF THE CROSS SECTION

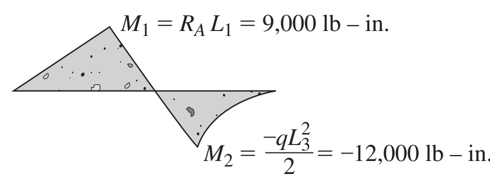
b = 2.5 in. $\quad h$ = 3.0 in. $\quad t$ = 0.5 in.
$A = bt + (h - t)t = 2.50 \text{ in.}^2$
c_1 = 2.0 in. $\quad c_2$ = 1.0 in.

$$I_C = \frac{25}{12} \text{ in.}^4 = 2.0833 \text{ in.}^4$$

REACTIONS

R_A = 187.5 lb (upward)
R_B = 837.5 lb (upward)

BENDING-MOMENT DIAGRAM



AT CROSS SECTION OF MAXIMUM POSITIVE BENDING MOMENT

$$\sigma_t = \frac{M_1 c_2}{I_C} = 4{,}320 \text{ psi} \qquad \sigma_c = \frac{M_1 c_1}{I_C} = 8{,}640 \text{ psi}$$

AT CROSS SECTION OF MAXIMUM NEGATIVE BENDING MOMENT

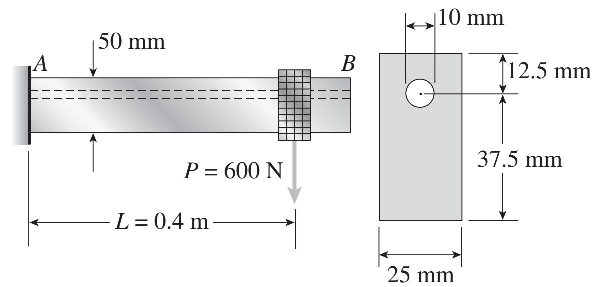
$$\sigma_t = \frac{M_2 c_1}{I_C} = 11{,}520 \text{ psi} \qquad \sigma_c = \frac{M_2 c_2}{I_C} = 5{,}760 \text{ psi}$$

MAXIMUM STRESSES

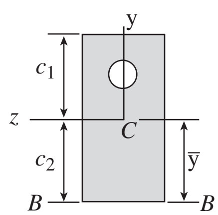
σ_t = 11,520 psi $\quad \sigma_c$ = 8,640 psi $\longleftarrow$

Problem 5.5-22 A cantilever beam *AB* with a rectangular cross section has a longitudinal hole drilled throughout its length (see figure). The beam supports a load $P = 600$ N. The cross section is 25 mm wide and 50 mm high, and the hole has a diameter of 10 mm.

Find the bending stresses at the top of the beam, at the top of the hole, and at the bottom of the beam.



Solution 5.5-22 **Rectangular beam with a hole**



MAXIMUM BENDING MOMENT

$M = PL = (600 \text{ N})(0.4 \text{ m}) = 240 \text{ N} \cdot \text{m}$

PROPERTIES OF THE CROSS SECTION

A_1 = area of rectangle

$= (25 \text{ mm})(50 \text{ mm}) = 1250 \text{ mm}^2$

A_2 = area of hole

$= \dfrac{\pi}{4}(10 \text{ mm})^2 = 78.54 \text{ mm}^2$

A = area of cross section

$= A_1 - A_2 = 1171.5 \text{ mm}^2$

Using line *B-B* as reference axis:

$\sum A_i y_i = A_1(25 \text{ mm}) - A_2(37.5 \text{ mm}) = 28{,}305 \text{ mm}^3$

$$\bar{y} = \frac{\sum A_i y_i}{A} = \frac{28{,}305 \text{ mm}^3}{1171.5 \text{ mm}^2} = 24.162 \text{ mm}$$

Distances to the centroid *C*:

$c_2 = \bar{y} = 24.162 \text{ mm}$

$c_1 = 50 \text{ mm} - c_2 = 25.838 \text{ mm}$

MOMENT OF INERTIA ABOUT THE NEUTRAL AXIS (THE z AXIS)

All dimensions in millimeters.

Rectangle:

$$I_z = I_c + Ad^2$$
$$= \frac{1}{12}(25)(50)^3 + (25)(50)(25 - 24.162)^2$$
$$= 260{,}420 + 878 = 261{,}300 \text{ mm}^4$$

Hole:

$$I_z = I_c + Ad^2 = \frac{\pi}{64}(10)^4 + (78.54)(37.5 - 24.162)^2$$
$$= 490.87 + 13{,}972 = 14{,}460 \text{ mm}^4$$

Cross-section:

$I = 261{,}300 - 14{,}460 = 246{,}800 \text{ mm}^4$

STRESS AT THE TOP OF THE BEAM

$$\sigma_1 = \frac{Mc_1}{I} = \frac{(240 \text{ N} \cdot \text{m})(25.838 \text{ mm})}{246{,}800 \text{ mm}^4}$$
$$= 25.1 \text{ MPa} \longleftarrow$$

(tension)

STRESS AT THE TOP OF THE HOLE

$$\sigma_2 = \frac{My}{I} \qquad y = c_1 - 7.5 \text{ mm} = 18.338 \text{ mm}$$

$$\sigma_2 = \frac{(240 \text{ N} \cdot \text{m})(18.338 \text{ mm})}{246{,}800 \text{ mm}^4} = 17.8 \text{ MPa} \longleftarrow$$

(tension)

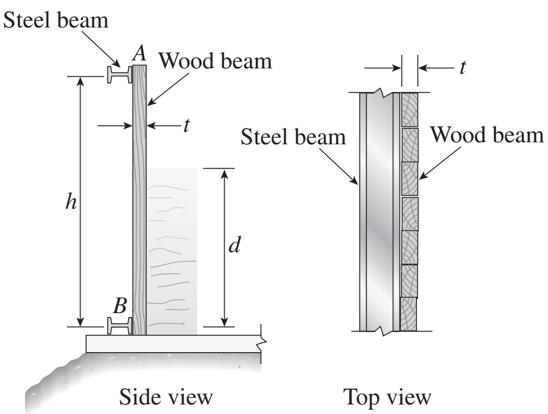
STRESS AT THE BOTTOM OF THE BEAM

$$\sigma_3 = -\frac{Mc_2}{I} = -\frac{(240 \text{ N} \cdot \text{m})(24.162 \text{ mm})}{246{,}800 \text{ mm}^4}$$
$$= -23.5 \text{ MPa} \longleftarrow$$

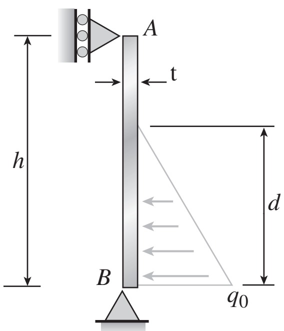
(compression)

Problem 5.5-23 A small dam of height $h = 6$ ft is constructed of vertical wood beams AB, as shown in the figure. The wood beams, which have thickness $t = 2.5$ in., are simply supported by horizontal steel beams at A and B.

Construct a graph showing the maximum bending stress σ_{max} in the wood beams versus the depth d of the water above the lower support at B. Plot the stress σ_{max} (psi) as the ordinate and the depth d (ft) as the abscissa. (*Note:* The weight density γ of water equals 62.4 lb/ft^3.)

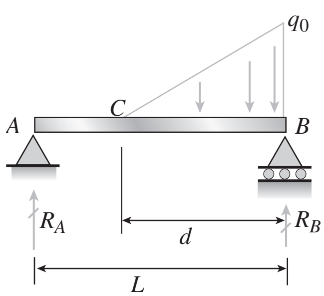


Solution 5.5-23 **Vertical wood beam in a dam**



$h = 6$ ft
$t = 2.5$ in.
$\gamma = 62.4$ lb/ft^3
Let b = width of beam (perpendicular to the figure)
Let q_0 = intensity of load at depth d
$q_0 = \gamma\, bd$

ANALYSIS OF BEAM



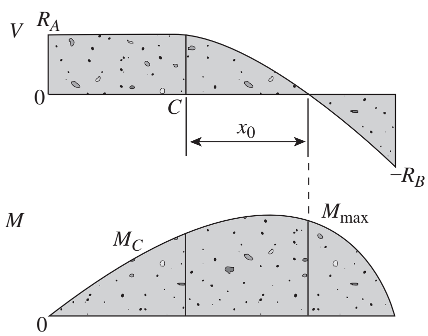
$L = h = 6$ ft

$$R_A = \frac{q_0 d^2}{6L}$$

$$R_B = \frac{q_0 d}{6}\left(3 - \frac{d}{L}\right)$$

$$x_0 = d\sqrt{\frac{d}{3L}}$$

$$M_C = R_A(L - d) = \frac{q_0 d^2}{6}\left(1 - \frac{d}{L}\right)$$



$$M_{max} = \frac{q_0 d^2}{6}\left(1 - \frac{d}{L} + \frac{2d}{3L}\sqrt{\frac{d}{3L}}\right)$$

MAXIMUM BENDING STRESS

Section modulus: $S = \dfrac{1}{6}bt^2$

$$\sigma_{max} = \frac{M_{max}}{S} = \frac{6}{bt^2}\left[\frac{q_0 d^2}{6}\left(1 - \frac{d}{L} + \frac{2d}{3L}\sqrt{\frac{d}{3L}}\right)\right]$$

$$q_0 = \gamma\, bd$$

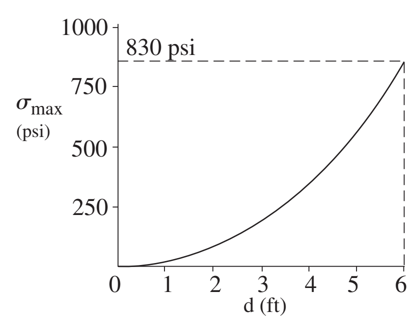
$$\sigma_{max} = \frac{\gamma d^3}{t^2}\left(1 - \frac{d}{L} + \frac{2d}{3L}\sqrt{\frac{d}{3L}}\right) \longleftarrow$$

SUBSTITUTE NUMERICAL VALUES:
d = depth of water (ft) (Max. $d = h = 6$ ft)
$L = h = 6$ ft $\quad \gamma = 62.4$ lb/ft^3 $\quad t = 2.5$ in.
σ_{max} = psi

$$\sigma_{max} = \frac{(62.4)d^3}{(2.5)^2}\left(1 - \frac{d}{6} + \frac{d}{9}\sqrt{\frac{d}{18}}\right)$$

$$= 0.1849d^3(54 - 9d + d\sqrt{2d}) \longleftarrow$$

d (ft)	σ_{max} (psi)
0	0
1	9
2	59
3	171
4	347
5	573
6	830

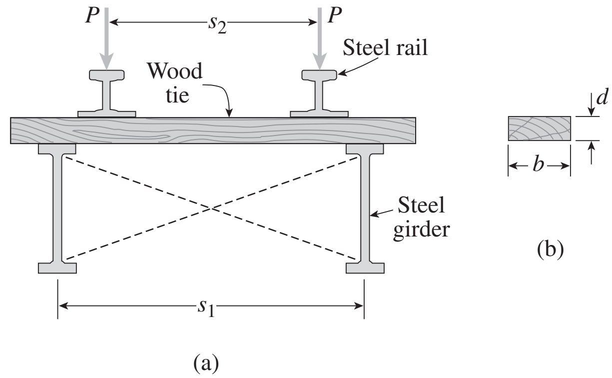


Design of Beams

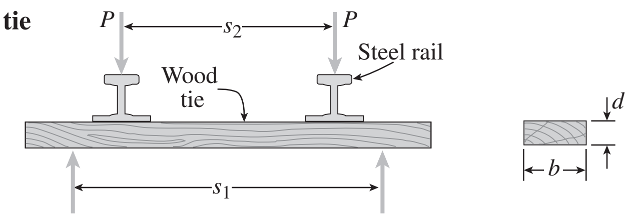
Problem 5.6-1 The cross section of a narrow-gage railway bridge is shown in part (a) of the figure. The bridge is constructed with longitudinal steel girders that support the wood cross ties. The girders are restrained against lateral buckling by diagonal bracing, as indicated by the dashed lines.

The spacing of the girders is $s_1 = 50$ in. and the spacing of the rails is $s_2 = 30$ in. The load transmitted by each rail to a single tie is $P = 1500$ lb. The cross section of a tie, shown in part (b) of the figure, has width $b = 5.0$ in. and depth d.

Determine the minimum value of d based upon an allowable bending stress of 1125 psi in the wood tie. (Disregard the weight of the tie itself.)



Solution 5.6-1 **Railway cross tie**



$s_1 = 50$ in. $\quad b = 5.0$ in. $\quad s_2 = 30$ in.

d = depth of tie $\quad P = 1500$ lb $\quad \sigma_{allow} = 1125$ psi

$$M_{max} = \frac{P(s_1 - s_2)}{2} = 15{,}000 \text{ lb-in.}$$

$$S = \frac{bd^2}{6} = \frac{1}{6}(50 \text{ in.})(d^2) = \frac{5d^2}{6} \qquad d = \text{inches}$$

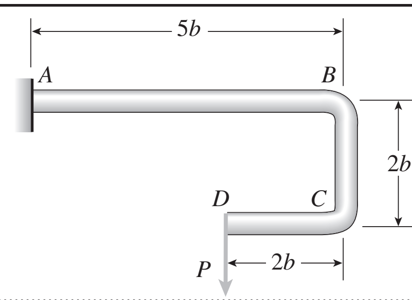
$$M_{max} = \sigma_{allow} S \qquad 15{,}000 = (1125)\left(\frac{5d^2}{6}\right)$$

Solving, $d^2 = 16.0$ in. $\quad d_{min} = 4.0$ in. ←

Note: Symbolic solution: $d^2 = \dfrac{3P(s_1 - s_2)}{b\sigma_{allow}}$

Problem 5.6-2 A fiberglass bracket $ABCD$ of solid circular cross section has the shape and dimensions shown in the figure. A vertical load $P = 36$ N acts at the free end D.

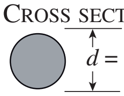
Determine the minimum permissible diameter d_{min} of the bracket if the allowable bending stress in the material is 30 MPa and $b = 35$ mm. (Disregard the weight of the bracket itself.)



Solution 5.6-2 **Fiberglass bracket**

DATA $\quad P = 36$ N $\quad \sigma_{allow} = 30$ MPa $\quad b = 35$ mm

CROSS SECTION



d = diameter $\qquad I = \dfrac{\pi d^4}{64}$

MAXIMUM BENDING MOMENT $\quad M_{max} = P(3b)$

MAXIMUM BENDING STRESS

$$\sigma_{max} = \frac{M_{max}\, c}{I} \qquad c = \frac{d}{2} \qquad \sigma_{allow} = \frac{3Pbd}{2I} = \frac{96\, Pb}{\pi d^3}$$

MINIMUM DIAMETER

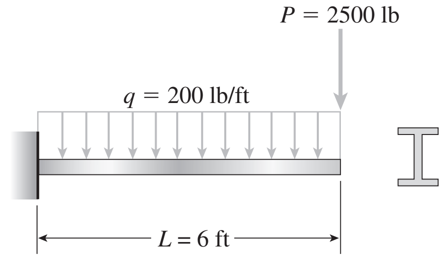
$$d^3 = \frac{96Pb}{\pi\sigma_{allow}} = \frac{(96)(36 \text{ N})(35 \text{ mm})}{\pi(30 \text{ MPa})}$$

$$= 1{,}283.4 \text{ mm}^3$$

$d_{min} = 10.9$ mm ←

Problem 5.6-3 A cantilever beam of length $L = 6$ ft supports a uniform load of intensity $q = 200$ lb/ft and a concentrated load $P = 2500$ lb (see figure).

Calculate the required section modulus S if $\sigma_{allow} = 15{,}000$ psi. Then select a suitable wide-flange beam (W shape) from Table E-1, Appendix E, and recalculate S taking into account the weight of the beam. Select a new beam size if necessary.



Solution 5.6-3 **Cantilever beam**

$P = 2500$ lb $\quad q = 200$ lb/ft $\quad L = 6$ ft

$\sigma_{allow} = 15{,}000$ psi

Required section modulus

$$M_{max} = PL + \frac{qL^2}{2} = 15{,}000 \text{ lb-ft} + 3{,}600 \text{ lb-ft}$$

$$= 18{,}600 \text{ lb-ft} = 223{,}200 \text{ lb-in.}$$

$$S = \frac{M_{max}}{\sigma_{allow}} = \frac{223{,}200 \text{ lb-in.}}{15{,}000 \text{ psi}} = 14.88 \text{ in.}^3$$

Trial section W 8 × 21

$S = 18.2$ in.^{3} $\quad q_0 = 21$ lb/ft

$$M_0 = \frac{q_0 L^2}{2} = 378 \text{ lb-ft} = 4536 \text{ lb-in.}$$

$$M_{max} = 223{,}200 + 4{,}536 = 227{,}700 \text{ lb-in.}$$

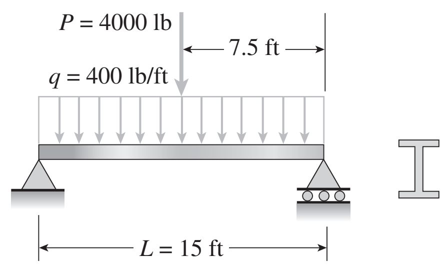
$$\text{Required } S = \frac{M_{max}}{\sigma_{allow}} = \frac{227{,}700 \text{ lb-in.}}{15{,}000 \text{ psi}} = 15.2 \text{ in.}^3$$

15.2 in.$^3 < 18.2$ in.^{3} $\quad \therefore$ Beam is satisfactory.

Use W 8 × 21 ←

Problem 5.6-4 A simple beam of length $L = 15$ ft carries a uniform load of intensity $q = 400$ lb/ft and a concentrated load $P = 4000$ lb (see figure).

Assuming $\sigma_{allow} = 16{,}000$ psi, calculate the required section modulus S. Then select an 8-inch wide-flange beam (W shape) from Table E-1, Appendix E, and recalculate S taking into account the weight of the beam. Select a new 8-inch beam if necessary.



Solution 5.6-4 **Simple beam**

$P = 4000$ lb $\quad q = 400$ lb/ft $\quad L = 15$ ft

$\sigma_{allow} = 16{,}000$ psi $\quad$ use an 8-inch W shape

Required section modulus

$$M_{max} = \frac{PL}{4} + \frac{qL^2}{8} = 15{,}000 \text{ lb-ft} + 11{,}250 \text{ lb-ft}$$

$$= 26{,}250 \text{ lb-ft} = 315{,}000 \text{ lb-in.}$$

$$S = \frac{M_{max}}{\sigma_{allow}} = \frac{315{,}000 \text{ lb-in.}}{16{,}000 \text{ psi}} = 19.69 \text{ in.}^3$$

Trial section W 8 × 28

$S = 24.3$ in.^{3} $\quad q_0 = 28$ lb/ft

$$M_0 = \frac{q_0 L^2}{8} = 787.5 \text{ lb-ft} = 9450 \text{ lb-in.}$$

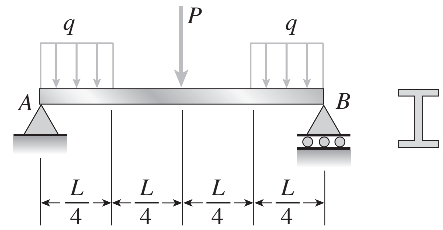
$$M_{max} = 315{,}000 + 9{,}450 = 324{,}450 \text{ lb-in.}$$

$$\text{Required } S = \frac{M_{max}}{\sigma_{allow}} = \frac{324{,}450 \text{ lb-in.}}{16{,}000 \text{ psi}} = 20.3 \text{ in.}^3$$

20.3 in.$^3 < 24.3$ in.^{3} $\quad \therefore$ Beam is satisfactory.

Use W 8 × 28 ←

Problem 5.6-5 A simple beam AB is loaded as shown in the figure on the next page. Calculate the required section modulus S if $\sigma_{allow} = 15{,}000$ psi, $L = 24$ ft, $P = 2000$ lb, and $q = 400$ lb/ft. Then select a suitable I-beam (S shape) from Table E-2, Appendix E, and recalculate S taking into account the weight of the beam. Select a new beam size if necessary.



Solution 5.6-5 Simple beam

$P = 2000$ lb $\quad q = 400$ lb/ft $\quad L = 24$ ft

$\sigma_{allow} = 15{,}000$ psi

REQUIRED SECTION MODULUS

$$M_{max} = \frac{PL}{4} + \frac{qL^2}{32} = 12{,}000 \text{ lb-ft} + 7{,}200 \text{ lb-ft}$$

$$= 19{,}200 \text{ lb-ft} = 230{,}400 \text{ lb-in.}$$

$$S = \frac{M_{max}}{\sigma_{allow}} = \frac{230{,}400 \text{ lb-in.}}{15{,}000 \text{ psi}} = 15.36 \text{ in.}^3$$

TRIAL SECTION S 10×25.4

$S = 24.7$ in.^{3} $\quad q_0 = 25.4$ lb/ft

$$M_0 = \frac{q_0 L^2}{8} = 1829 \text{ lb-ft} = 21{,}950 \text{ lb-in.}$$

$$M_{max} = 230{,}400 + 21{,}950 = 252{,}300 \text{ lb-in.}$$

$$\text{Required } S = \frac{M_{max}}{\sigma_{allow}} = \frac{252{,}300 \text{ lb-in.}}{15{,}000 \text{ psi}} = 16.8 \text{ in.}^3$$

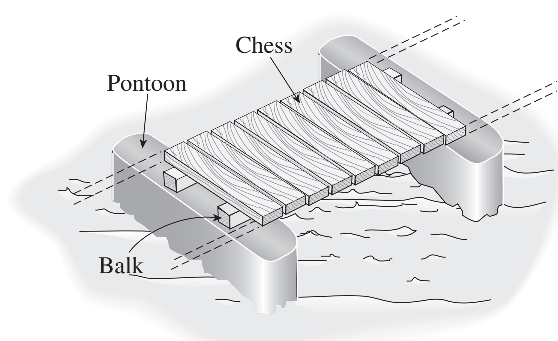
16.8 in.^{3} < 24.7 in.^{3} $\quad \therefore$ Beam is satisfactory.

Use S 10×25.4 ←

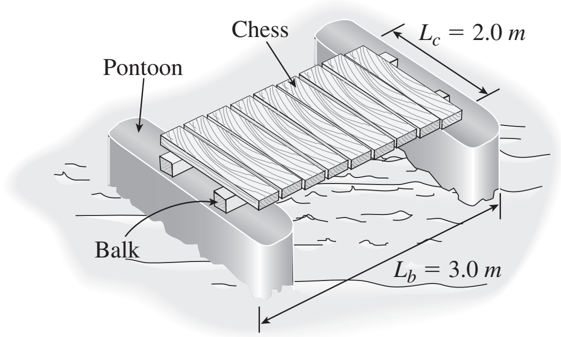
Problem 5.6-6 A pontoon bridge (see figure) is constructed of two longitudinal wood beams, known as *balks*, that span between adjacent pontoons and support the transverse floor beams, which are called *chesses*.

For purposes of design, assume that a uniform floor load of 8.0 kPa acts over the chesses. (This load includes an allowance for the weights of the chesses and balks.) Also, assume that the chesses are 2.0 m long and that the balks are simply supported with a span of 3.0 m. The allowable bending stress in the wood is 16 MPa.

If the balks have a square cross section, what is their minimum required width b_{min}?



Solution 5.6-6 Pontoon bridge



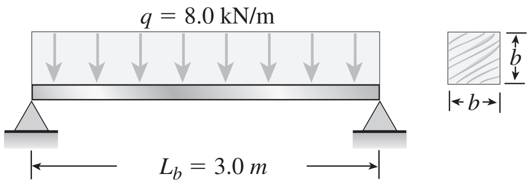
FLOOR LOAD: $w = 8.0$ kPa

ALLOWABLE STRESS: $\sigma_{allow} = 16$ MPa

L_c = length of chesses $\quad L_b$ = length of balks

$= 2.0$ m $\quad = 3.0$ m

LOADING DIAGRAM FOR ONE BALK



$W =$ total load

$= wL_bL_c$

$$q = \frac{W}{2L_b} = \frac{wL_c}{2}$$

$$= \frac{(8.0\text{ kPa})(2.0\text{ m})}{2}$$

$= 8.0$ kN/m

Section modulus $S = \dfrac{b^3}{6}$

$$M_{\max} = \frac{qL_b^2}{8} = \frac{(8.0\text{ kN/m})(3.0\text{ m})^2}{8} = 9{,}000\text{ N}\cdot\text{m}$$

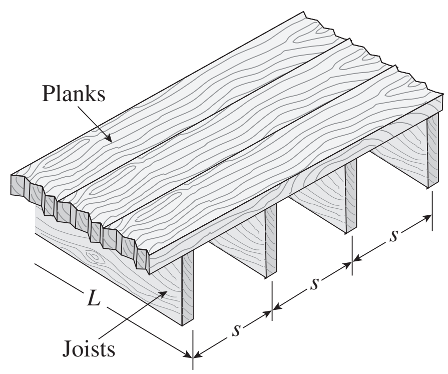
$$S = \frac{M_{\max}}{\sigma_{\text{allow}}} = \frac{9{,}000\text{ N}\cdot\text{m}}{16\text{ MPa}} = 562.5\times10^{-6}\text{ m}^3$$

$$\therefore\ \frac{b^3}{6} = 562.5\times10^{-6}\text{ m}^3 \quad\text{and}\quad b^3 = 3375\times10^{-6}\text{ m}^3$$

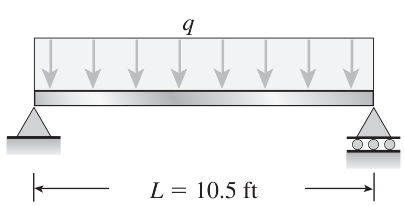
Solving, $b_{\min} = 0.150\text{ m} = 150\text{ mm}$ ←

Problem 5.6-7 A floor system in a small building consists of wood planks supported by 2 in. (nominal width) joists spaced at distance s, measured from center to center (see figure). The span length L of each joist is 10.5 ft, the spacing s of the joists is 16 in., and the allowable bending stress in the wood is 1350 psi. The uniform floor load is 120 lb/ft^2, which includes an allowance for the weight of the floor system itself.

Calculate the required section modulus S for the joists, and then select a suitable joist size (surfaced lumber) from Appendix F, assuming that each joist may be represented as a simple beam carrying a uniform load.



Solution 5.6-7 Floor joists



$\sigma_{\text{allow}} = 1350$ psi
$L = 10.5\text{ ft} = 126$ in.
$w =$ floor load $= 120\text{ lb/ft}^2 = 0.8333\text{ lb/in.}^2$
$s =$ spacing of joists $= 16$ in.
$q = ws = 13.333$ lb/in.

$$M_{\max} = \frac{qL^2}{8} = \frac{1}{8}(13.333\text{ lb/in.})(126\text{ in.})^2 = 26{,}460\text{ lb-in.}$$

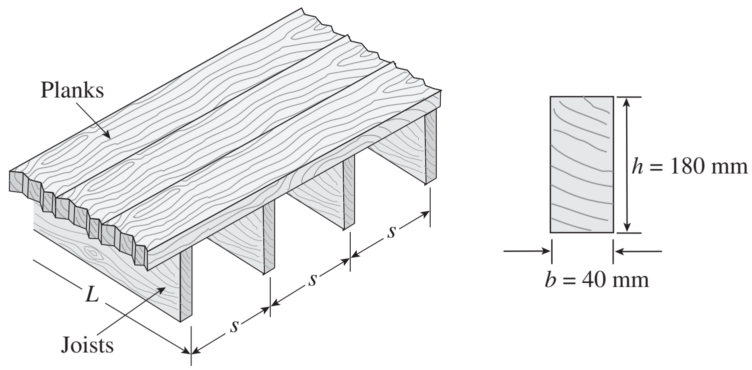
$$\text{Required } S = \frac{M_{\max}}{\sigma_{\text{allow}}} = \frac{26{,}460\text{ lb/in.}}{1350\text{ psi}} = 19.6\text{ in.}^3 \quad\leftarrow$$

From Appendix F: Select 2 × 10 in. joists ←

Problem 5.6-8 The wood joists supporting a plank floor (see figure) are 40 mm × 180 mm in cross section (actual dimensions) and have a span length $L = 4.0$ m. The floor load is 3.6 kPa, which includes the weight of the joists and the floor.

Calculate the maximum permissible spacing s of the joists if the allowable bending stress is 15 MPa. (Assume that each joist may be represented as a simple beam carrying a uniform load.)

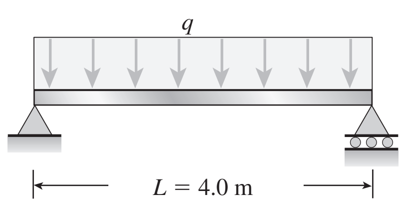
Solution 5.6-8 **Spacing of floor joists**



$L = 4.0$ m

w = floor load = 3.6 kPa $\quad \sigma_{allow} = 15$ MPa

s = spacing of joists



$q = ws$

$$S = \frac{bh^2}{6}$$

$$M_{max} = \frac{qL^2}{8} = \frac{wsL^2}{8}$$

$$S = \frac{M_{max}}{\sigma_{allow}} = \frac{wsL^2}{8\sigma_{allow}} = \frac{bh^2}{6}$$

SPACING OF JOISTS $\quad s_{max} = \dfrac{4\,bh^2\sigma_{allow}}{3wL^2}$ ←

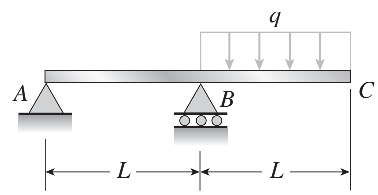
Substitute numerical values:

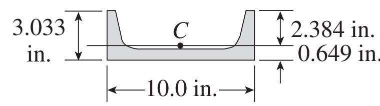
$$s_{max} = \frac{4(40\text{ mm})(180\text{ mm})^2(15\text{ MPa})}{3(3.6\text{ kPa})\,(4.0\text{ m})^2}$$

$$= 0.450\text{ m} = 450\text{ mm} \quad ←$$

Problem 5.6-9 A beam *ABC* with an overhang from *B* to *C* is constructed of a C 10 × 30 channel section (see figure). The beam supports its own weight (30 lb/ft) plus a uniform load of intensity *q* acting on the overhang. The allowable stresses in tension and compression are 18 ksi and 12 ksi, respectively.

Determine the allowable uniform load q_{allow} if the distance *L* equals 3.0 ft.





Solution 5.6-9 **Beam with an overhang**

DATA

C 10 × 30 channel section

$c_1 = 2.384$ in. $\quad c_2 = 0.649$ in.

$I = 3.94$ in.^{4} (from Table E-3)

q_0 = weight of beam *ABC*
= 30 lb/ft = 2.5 lb/in.

q = load on overhang

L = length of overhang
= 3.0 ft = 36 in.

ALLOWABLE STRESSES

$\sigma_t = 18$ ksi $\quad \sigma_c = 12$ ksi

MAXIMUM BENDING MOMENT

M_{max} occurs at support *B*. $M_{max} = \dfrac{(q + q_0)L^2}{2}$

Tension on top; compression on bottom.

ALLOWABLE BENDING MOMENT BASED UPON TENSION

$$M_t = \frac{\sigma_t I}{c_1} = \frac{(18\text{ ksi})(3.94\text{ in.}^4)}{2.384\text{ in.}} = 29{,}750\text{ lb-in.}$$

ALLOWABLE BENDING MOMENT BASED UPON COMPRESSION

$$M_c = \frac{\sigma_c I}{c_2} = \frac{(12\text{ ksi})(3.94\text{ in.}^4)}{0.649\text{ in.}} = 72{,}850\text{ lb-in.}$$

ALLOWABLE BENDING MOMENT

Tension governs. $\quad M_{allow} = 29{,}750$ lb-in.

ALLOWABLE UNIFORM LOAD q

$$M_{max} = \frac{(q + q_0)L^2}{2} \qquad q_{allow} + q_0 = \frac{2M_{allow}}{L^2}$$

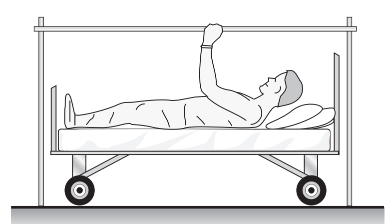
$$q_{allow} = \frac{2M_{allow}}{L^2} - q_0 = \frac{2(29{,}750\text{ lb-in.})}{(36\text{ in.})^2} - 2.5\text{ lb/in.}$$

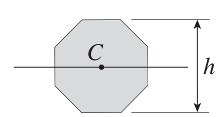
$$= 45.91 - 2.5 = 43.41\text{ lb/in.}$$

$$q_{allow} = (43.41)(12) = 521\text{ lb/ft} \quad \longleftarrow$$

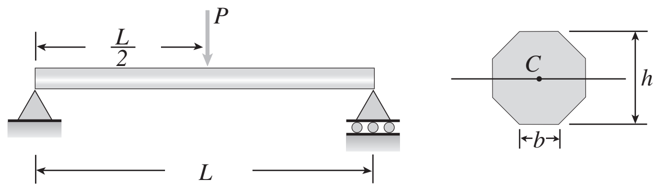
Problem 5.6-10 A so-called "trapeze bar" in a hospital room provides a means for patients to exercise while in bed (see figure). The bar is 2.1 m long and has a cross section in the shape of a regular octagon. The design load is 1.2 kN applied at the midpoint of the bar, and the allowable bending stress is 200 MPa.

Determine the minimum height *h* of the bar. (Assume that the ends of the bar are simply supported and that the weight of the bar is negligible.)





Solution 5.6-10 **Trapeze bar (regular octagon)**



$P = 1.2$ kN $\quad L = 2.1$ m $\quad \sigma_{allow} = 200$ MPa

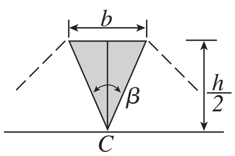
Determine minimum height h.

MAXIMUM BENDING MOMENT

$$M_{max} = \frac{PL}{4} = \frac{(1.2\text{ kN})(2.1\text{ m})}{4} = 630\text{ N}\cdot\text{m}$$

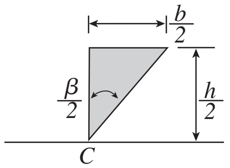
PROPERTIES OF THE CROSS SECTION

Use Appendix D, Case 25, with $n = 8$



b = length of one side

$$\beta = \frac{360°}{n} = \frac{360°}{8} = 45°$$



$$\tan\frac{\beta}{2} = \frac{b}{h}\text{ (from triangle)}$$

$$\cot\frac{\beta}{2} = \frac{h}{b}$$

For $\beta = 45°$: $\dfrac{b}{h} = \tan\dfrac{45°}{2} = 0.41421$

$$\frac{h}{b} = \cot\frac{45°}{2} = 2.41421$$

MOMENT OF INERTIA

$$I_C = \frac{nb^4}{192}\left(\cot\frac{\beta}{2}\right)\left(3\cot^2\frac{\beta}{2} + 1\right)$$

$$I_C = \frac{8b^4}{192}(2.41421)[3(2.41421)^2 + 1] = 1.85948b^4$$

$b = 0.41421h \quad \therefore I_C = 1.85948(0.41421h)^4 = 0.054738h^4$

SECTION MODULUS

$$S = \frac{I_C}{h/2} = \frac{0.054738h^4}{h/2} = 0.109476h^3$$

MINIMUM HEIGHT h

$$\sigma = \frac{M}{S} \qquad S = \frac{M}{\sigma}$$

$$0.109476h^3 = \frac{630\text{N}\cdot\text{m}}{200\text{ MPa}} = 3.15 \times 10^{-6}\text{ m}^3$$

$h^3 = 28.7735 \times 10^{-6}$ m^3 $\quad h = 0.030643$ m

$\therefore h_{min} = 30.6$ mm ←

ALTERNATIVE SOLUTION ($n = 8$)

$$M = \frac{PL}{4} \quad \beta = 45° \quad \tan\frac{\beta}{2} = \sqrt{2} - 1 \quad \cot\frac{\beta}{2} = \sqrt{2} + 1$$

$$b = (\sqrt{2} - 1)h \quad h = (\sqrt{2} + 1)b$$

$$I_C = \left(\frac{11 + 8\sqrt{2}}{12}\right)b^4 = \left(\frac{4\sqrt{2} - 5}{12}\right)h^4$$

$$S = \left(\frac{4\sqrt{2} - 5}{6}\right)h^3 \quad h^3 = \frac{3PL}{2(4\sqrt{2} - 5)\sigma_{allow}} \leftarrow$$

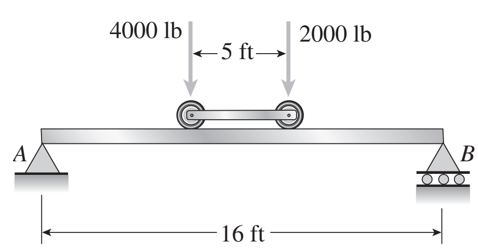
Substitute numerical values:

$h^3 = 28.7735 \times 10^{-6}$ m^3 $\quad h_{min} = 30.643$ mm ←

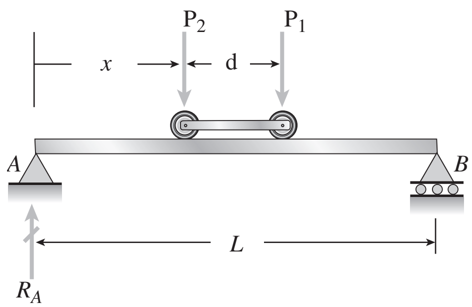
Problem 5.6-11 A two-axle carriage that is part of an overhead traveling crane in a testing laboratory moves slowly across a simple beam AB (see figure). The load transmitted to the beam from the front axle is 2000 lb and from the rear axle is 4000 lb. The weight of the beam itself may be disregarded.

(a) Determine the minimum required section modulus S for the beam if the allowable bending stress is 15.0 ksi, the length of the beam is 16 ft, and the wheelbase of the carriage is 5 ft.

(b) Select a suitable I-beam (S shape) from Table E-2, Appendix E.



Solution 5.6-11 Moving carriage



P_1 = load on front axle
= 2000 lb
P_2 = load on rear axle
= 4000 lb
$L = 16$ ft $d = 5$ ft $\sigma_{allow} = 15$ ksi
x = distance from support A to the larger load P_2 (feet)

$$R_A = P_2\left(\frac{L-x}{L}\right) + P_1\left(\frac{L-x-d}{L}\right)$$

$$= (4000 \text{ lb})\left(1 - \frac{x}{16}\right) + (2000 \text{ lb})\left(1 - \frac{x}{16} - \frac{5}{16}\right)$$

$$= 125(43 - 3x) \quad (x = \text{ft}; R_A = \text{lb})$$

Bending moment under larger load P_2

$$M = R_A\, x = 125(43x - 3x^2) \quad (x = \text{ft}; M = \text{lb-ft})$$

Maximum bending moment

Set $\dfrac{dM}{dx}$ equal to zero and solve for $x = x_m$.

$$\frac{dM}{dx} = 125(43 - 6x) = 0 \quad x = x_m = \frac{43}{6} = 7.1667 \text{ ft}$$

$$M_{max} = (M)_{x=x_m} = 125\left[(43)\left(\frac{43}{6}\right) - 3\left(\frac{43}{6}\right)^2\right]$$

$$= 19,260 \text{ lb-ft} = 231,130 \text{ lb-in.}$$

(a) Minimum section modulus

$$S_{min} = \frac{M_{max}}{\sigma_{allow}} = \frac{231,130 \text{ lb-in.}}{15,000 \text{ psi}} = 15.41 \text{ in.}^3 \longleftarrow$$

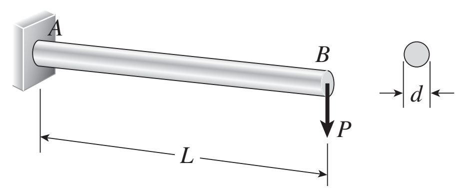
(b) Select on I-beam (S shape)

Table E-2. Select S 8×23 $\longleftarrow$

($S = 16.2$ in.^{3})

Problem 5.6-12 A cantilever beam AB of circular cross section and length $L = 450$ mm supports a load $P = 400$ N acting at the free end (see figure). The beam is made of steel with an allowable bending stress of 60 MPa.

Determine the required diameter d_{min} of the beam, considering the effect of the beam's own weight.



Solution 5.6-12 Cantilever beam

Data $L = 450$ mm $P = 400$ N
$\sigma_{allow} = 60$ MPa
γ = weight density of steel
$= 77.0$ kN/m^3

Weight of beam per unit length

$$q = \gamma\left(\frac{\pi d^2}{4}\right)$$

Maximum bending moment

$$M_{max} = PL + \frac{qL^2}{2} = PL + \frac{\pi \gamma d^2 L^2}{8}$$

Section modulus $S = \dfrac{\pi d^3}{32}$

Minimum diameter

$$M_{max} = \sigma_{allow}\, S$$

$$PL + \frac{\pi \gamma d^2 L^2}{8} = \sigma_{allow}\left(\frac{\pi d^3}{32}\right)$$

Rearrange the equation:

$$\sigma_{allow}\, d^3 - 4\gamma L^2 d^2 - \frac{32\, PL}{\pi} = 0$$

(Cubic equation with diameter d as unknown.)

Substitute numerical values (d = meters):

$$(60 \times 10^6 \text{ N/m}^2)d^3 - 4(77,000 \text{ N/m}^3)(0.45\text{m})^2 d^2$$

$$- \frac{32}{\pi}(400 \text{ N})(0.45 \text{ m}) = 0$$

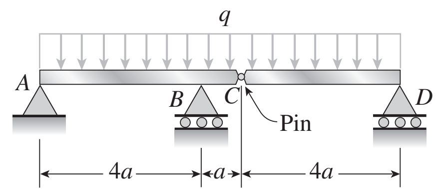
$$60,000d^3 - 62.37d^2 - 1.833465 = 0$$

Solve the equation numerically:

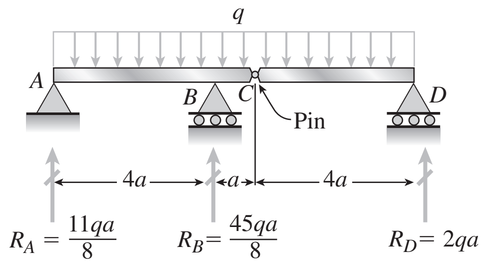
$d = 0.031614$ m $d_{min} = 31.61$ mm $\longleftarrow$

Problem 5.6-13 A compound beam $ABCD$ (see figure) is supported at points A, B, and D and has a splice (represented by the pin connection) at point C. The distance $a = 6.0$ ft and the beam is a W 16 $\times$ 57 wide-flange shape with an allowable bending stress of 10,800 psi.

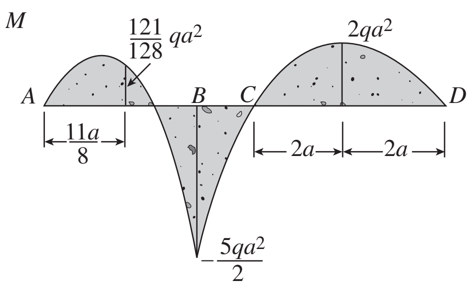
Find the allowable uniform load q_{allow} that may be placed on top of the beam, taking into account the weight of the beam itself.



Solution 5.6-13 Compound beam



Pin connection at point C.



$$M_{max} = \frac{5qa^2}{2} = \sigma_{allow} S$$

$$q_{max} = \frac{2\sigma_{allow} S}{5a^2} \qquad q_{allow} = q_{max} - \text{(weight of beam)}$$

DATA: $a = 6$ ft $= 72$ in. $\quad \sigma_{allow} = 10{,}800$ psi

W 16 $\times$ 57 $\quad S = 92.2$ in.^{3}

ALLOWABLE UNIFORM LOAD

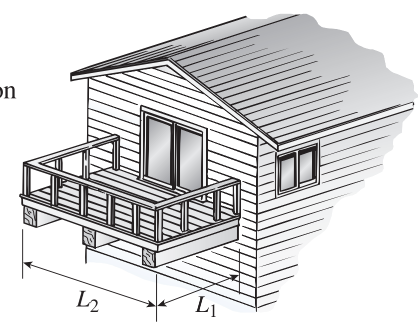
$$q_{max} = \frac{2(10{,}800\text{ psi})(92.2\text{ in.}^3)}{5(72\text{ in.})^2} = 76.833\text{ lb/in.}$$

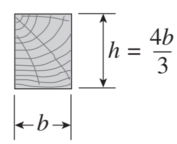
$$= 922\text{ lb/ft}$$

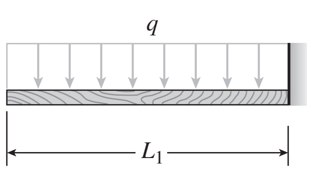
$$q_{allow} = 922\text{ lb/ft} - 57\text{ lb/ft} = 865\text{ lb/ft} \quad \leftarrow$$

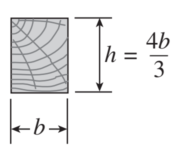
Problem 5.6-14 A small balcony constructed of wood is supported by three identical cantilever beams (see figure). Each beam has length $L_1 = 2.1$ m, width b, and height $h = 4b/3$. The dimensions of the balcon floor are $L_1 \times L_2$, with $L_2 = 2.5$ m. The design load is 5.5 kPa acting over the entire floor area. (This load accounts for all loads except the weights of the cantilever beams, which have a weight density $\gamma = 5.5$ kN/m^3.) The allowable bending stress in the cantilevers is 15 MPa.

Assuming that the middle cantilever supports 50% of the load and each outer cantilever supports 25% of the load, determine the required dimensions b and h.









$L_1 = 2.1$ m $\quad L_2 = 2.5$ m $\quad$ Floor dimensions: $L_1 \times L_2$

Design load $= w = 5.5$ kPa

$\gamma = 5.5$ kN/m^3 (weight density of wood beam)

$\sigma_{allow} = 15$ MPa

MIDDLE BEAM SUPPORTS 50% OF THE LOAD.

$$\therefore\ q = w\left(\frac{L_2}{2}\right) = (5.5\text{ kPa})\left(\frac{2.5\text{ m}}{2}\right) = 6875\text{ N/m}$$

WEIGHT OF BEAM

$$q_0 = \gamma bh = \frac{4\gamma b^2}{3} = \frac{4}{3}(5.5\text{ kN/m}^2)b^2$$

$$= 7333b^2\text{ (N/m)} \qquad (b = \text{meters})$$

Maximum bending moment

$$M_{\max} = \frac{(q + q_0)L_1^2}{2} = \frac{1}{2}(6875\ \text{N/m} + 7333b^2)(2.1\ \text{m})^2$$
$$= 15{,}159 + 16{,}170b^2\ \ (\text{N}\cdot\text{m})$$

$$S = \frac{bh^2}{6} = \frac{8b^3}{27}$$

$$M_{\max} = \sigma_{\text{allow}}\,S$$

$$15{,}159 + 16{,}170b^2 = (15 \times 10^6\ \text{N/m}^2)\left(\frac{8b^3}{27}\right)$$

Rearrange the equation:

$$(120 \times 10^6)b^3 - 436{,}590b^2 - 409{,}300 = 0$$

Solve numerically for dimension b

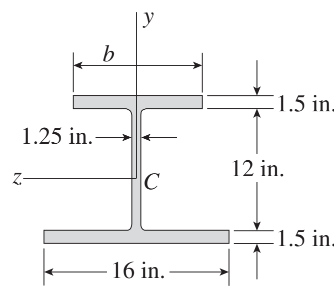
$$b = 0.1517\ \text{m} \qquad h = \frac{4b}{3} = 0.2023\ \text{m}$$

Required dimensions

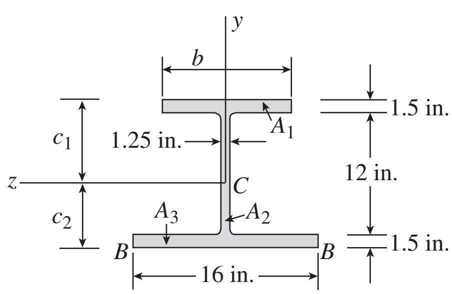
$b = 152$ mm $\quad h = 202$ mm ←

Problem 5.6-15 A beam having a cross section in the form of an unsymmetric wide-flange shape (see figure) is subjected to a negative bending moment acting about the z axis.

Determine the width b of the top flange in order that the stresses at the top and bottom of the beam will be in the ratio 4:3, respectively.



Solution 5.6-15 **Unsymmetric wide-flange beam**



Stresses at top and bottom are in the ratio 4:3.
Find b (inches)

h = height of beam = 15 in.

Locate centroid

$$\frac{\sigma_{\text{top}}}{\sigma_{\text{bottom}}} = \frac{c_1}{c_2} = \frac{4}{3}$$

$$c_1 = \frac{4}{7}h = \frac{60}{7} = 8.57143\ \text{in.}$$

$$c_2 = \frac{3}{7}h = \frac{45}{7} = 6.42857\ \text{in.}$$

Areas of the cross section (in.^{2})

$A_1 = 1.5b \qquad A_2 = (12)(1.25) = 15$ in.^{2}

$A_3 = (16)(1.5) = 24$ in.^{2}

$A = A_1 + A_2 + A_3 = 39 + 1.5b$ (in.^{2})

First moment of the cross-sectional area about the lower edge B-B

$$Q_{BB} = \sum \bar{y}_i A_i = (14.25)(1.5b) + (7.5)(15) + (0.75)(24)$$
$$= 130.5 + 21.375b\ \ (\text{in.}^3)$$

Distance c_2 from line B-B to the centroid C

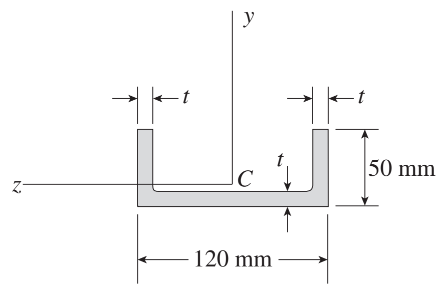
$$c_2 = \frac{Q_{BB}}{A} = \frac{130.5 + 21.375b}{39 + 1.5b} = \frac{45}{7}\ \text{in.}$$

Solve for b

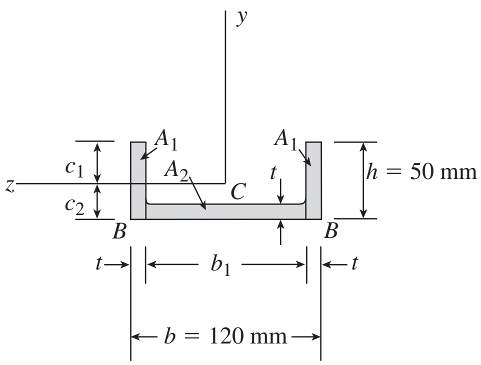
$(39 + 1.5b)(45) = (130.5 + 21.375b)(7)$
$82.125b = 841.5 \qquad b = 10.25$ in. ←

Problem 5.6-16 A beam having a cross section in the form of a channel (see figure) is subjected to a bending moment acting about the z axis.

Calculate the thickness t of the channel in order that the bending stresses at the top and bottom of the beam will be in the ratio 7:3, respectively.



Solution 5.6-16 **Channel beam**



t = thickness (constant) (t is in millimeters)

$b_1 = b - 2t = 120 \text{ mm} - 2t$

Stresses at the top and bottom are in the ratio 7:3.

Determine the thickness t.

LOCATE CENTROID

$$\frac{\sigma_{\text{top}}}{\sigma_{\text{bottom}}} = \frac{c_1}{c_2} = \frac{7}{3}$$

$$c_1 = \frac{7}{10}h = 35 \text{ mm}$$

$$c_2 = \frac{3}{10}h = 15 \text{ mm}$$

AREAS OF THE CROSS SECTION (mm^2)

$A_1 = ht = 50t \qquad A_2 = b_1 t = 120t - 2t^2$

$A = 2A_1 + A_2 = 220t - 2t^2 = 2t(110 - t)$

FIRST MOMENT OF THE CROSS-SECTIONAL AREA ABOUT THE LOWER EDGE B-B

$$Q_{BB} = \sum y_i A_i = (2)\left(\frac{h}{2}\right)(50t) + \left(\frac{t}{2}\right)(b_1)(t)$$

$$= 2(25)(50t) + \left(\frac{t}{2}\right)(120 - 2t)(t)$$

$$= t(2500 + 60t - t^2) \qquad (t = \text{mm};\ Q = \text{mm}^3)$$

DISTANCE c_2 FROM LINE B-B TO THE CENTROID C

$$c_2 = \frac{Q_{BB}}{A} = \frac{t(2500 + 60t - t^2)}{2t(110 - t)}$$

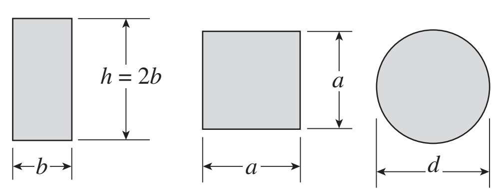
$$= \frac{2500 + 60t - t^2}{2(110 - t)} = 15 \text{ mm}$$

SOLVE FOR t

$2(110 - t)(15) = 2500 + 60t - t^2$

$t^2 - 90t + 800 = 0 \qquad t = 10 \text{ mm} \quad \longleftarrow$

Problem 5.6-17 Determine the ratios of the weights of three beams that have the same length, are made of the same material, are subjected to the same maximum bending moment, and have the same maximum bending stress if their cross sections are (1) a rectangle with height equal to twice the width, (2) a square, and (3) a circle (see figures).



Solution 5.6-17 **Ratio of weights of three beams**

Beam 1: Rectangle ($h = 2b$)
Beam 2: Square (a = side dimension)
Beam 3: Circle (d = diameter)
L, γ, M_{max}, and σ_{max} are the same in all three beams.

S = section modulus $\quad S = \dfrac{M}{\sigma}$

Since M and σ are the same, the section moduli must be the same.

(1) RECTANGLE: $S = \dfrac{bh^2}{6} = \dfrac{2b^3}{3} \quad b = \left(\dfrac{3S}{2}\right)^{1/3}$

$$A_1 = 2b^2 = 2\left(\frac{3S}{2}\right)^{2/3} = 2.6207\,S^{2/3}$$

(2) SQUARE: $S = \dfrac{a^3}{6} \quad a = (6S)^{1/3}$

$$A_2 = a^2 = (6S)^{2/3} = 3.3019\,S^{2/3}$$

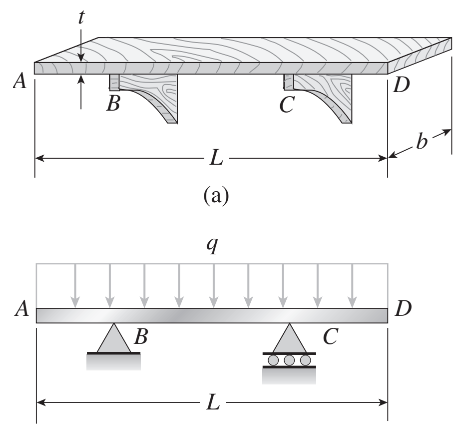
(3) CIRCLE: $S = \dfrac{\pi d^3}{32} \quad d = \left(\dfrac{32S}{\pi}\right)^{1/3}$

$$A_3 = \frac{\pi d^2}{4} = \frac{\pi}{4}\left(\frac{32S}{\pi}\right)^{2/3} = 3.6905\,S^{2/3}$$

Weights are proportional to the cross-sectional areas (since L and γ are the same in all 3 cases).
$W_1 : W_2 : W_3 = A_1 : A_2 : A_3$
$A_1 : A_2 : A_3 = 2.6207 : 3.3019 : 3.6905$
$W_1 : W_2 : W_3 = 1 : 1.260 : 1.408 \quad \longleftarrow$

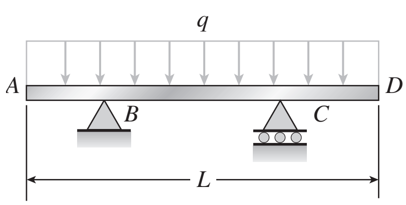
Problem 5.6-18 A horizontal shelf AD of length $L = 900$ mm, width $b = 300$ mm, and thickness $t = 20$ mm is supported by brackets at B and C [see part (a) of the figure]. The brackets are adjustable and may be placed in any desired positions between the ends of the shelf. A uniform load of intensity q, which includes the weight of the shelf itself, acts on the shelf [see part (b) of the figure].

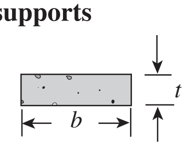
Determine the maximum permissible value of the load q if the allowable bending stress in the shelf is $\sigma_{allow} = 5.0$ MPa and the position of the supports is adjusted for maximum load-carrying capacity.

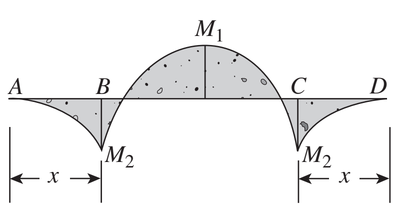


(a)
(b)

Solution 5.6-18 **Shelf with adjustable supports**







$L = 900$ mm
$b = 300$ mm
$t = 20$ mm
$\sigma_{allow} = 5.0$ MPa

For maximum load-carrying capacity, place the supports so that $M_1 = |M_2|$.

Let x = length of overhang

$$M_1 = \frac{qL}{8}(L - 4x) \qquad |M_2| = \frac{qx^2}{2}$$

$$\therefore \frac{qL}{8}(L - 4x) = \frac{qx^2}{2}$$

Solve for x: $x = \dfrac{L}{2}(\sqrt{2} - 1)$

Substitute x into the equation for either M_1 or $|M_2|$:

$$M_{max} = \frac{qL^2}{8}(3 - 2\sqrt{2}) \qquad \text{Eq. (1)}$$

$$M_{max} = \sigma_{allow} S = \sigma_{allow}\left(\frac{bt^2}{6}\right) \qquad \text{Eq. (2)}$$

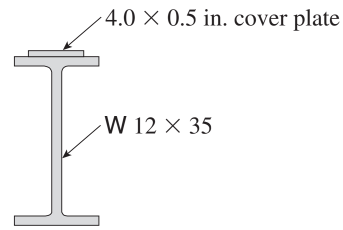
Equate M_{max} from Eqs. (1) and (2) and solve for q:

$$q_{max} = \frac{4bt^2\sigma_{allow}}{3L^2(3 - 2\sqrt{2})}$$

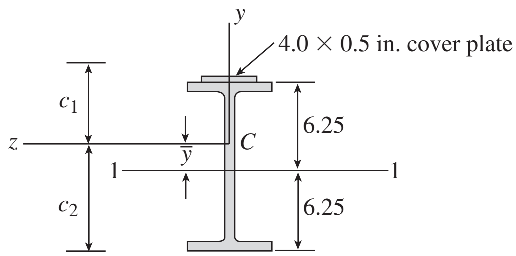
Substitute numerical values:
$q_{max} = 5.76$ kN/m $\quad \longleftarrow$

Problem 5.6-19 A steel plate (called a *cover plate*) having cross-sectional dimensions 4.0 in. $\times$ 0.5 in. is welded along the full length of the top flange of a W 12 $\times$ 35 wide-flange beam (see figure, which shows the beam cross section).

What is the percent increase in section modulus (as compared to the wide-flange beam alone)?



Solution 5.6-19 **Beam with cover plate**



All dimensions in inches.

WIDE-FLANGE BEAM ALONE
(AXIS 1-1 IS CENTROIDAL AXIS)

W 12 $\times$ 35 $\quad d = 12.50$ in.
$A_0 = 10.3 \text{ in.}^2 \quad I_0 = 2.85 \text{ in.}^4 \quad S_0 = 45.6 \text{ in.}^3$

BEAM WITH COVER PLATE
(z AXIS IS CENTROIDAL AXIS)
$A_1 = A_0 + (4.0 \text{ in.})(0.5 \text{ in.}) = 12.3 \text{ in.}^2$
First moment with respect to axis 1-1:

$$Q_1 = \sum \bar{y}_i A_i = (6.25 \text{ in.} + 0.25 \text{ in.})(4.0 \text{ in.})(0.5 \text{ in.})$$
$$= 13.00 \text{ in.}^3$$
$$\bar{y} = \frac{Q_1}{A_1} = \frac{13.00 \text{ in.}^3}{12.3 \text{ in.}^2} = 1.057 \text{ in.}$$
$$c_1 = 6.25 + 0.5 - \bar{y} = 5.693 \text{ in.}$$
$$c_2 = 6.25 + \bar{y} = 7.307 \text{ in.}$$

Moment of inertia about axis 1-1:

$$I_{1-1} = I_0 + \frac{1}{12}(4.0)(0.5)^3 + (4.0)(0.5)(6.25 + 0.25)^2$$
$$= 369.5 \text{ in.}^4$$

Moment of inertia about z axis:

$$I_{1-1} = I_z + A_1\bar{y}^2 \quad I_z = I_{1-1} - A_1\bar{y}^2$$
$$I_z = 369.5 \text{ in.}^4 - (12.3 \text{ in.}^2)(1.057 \text{ in.})^2 = 355.8 \text{ in.}^4$$

SECTION MODULUS (Use the smaller of the two section moduli)

$$S_1 = \frac{I_z}{c_2} = \frac{355.8 \text{ in.}^4}{7.307 \text{ in.}} = 48.69 \text{ in.}^3$$

INCREASE IN SECTION MODULUS

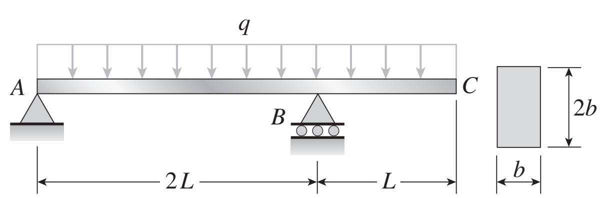
$$\frac{S_1}{S_0} = \frac{48.69}{45.6} = 1.068$$

Percent increase = 6.8% ←

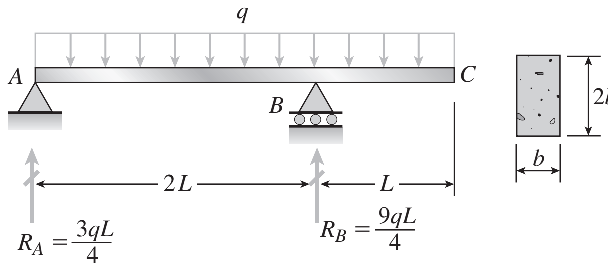
Problem 5.6-20 A steel beam *ABC* is simply supported at *A* and *B* and has an overhang *BC* of length $L = 150$ mm (see figure on the next page). The beam supports a uniform load of intensity $q = 3.5$ kN/m over its entire length of 450 mm. The cross section of the beam is rectangular with width b and height $2b$. The allowable bending stress in the steel is $\sigma_{\text{allow}} = 60$ MPa and its weight density is $\gamma = 77.0 \text{ kN/m}^3$.

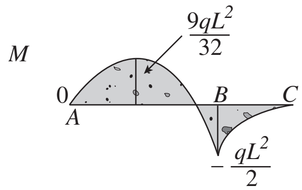
(a) Disregarding the weight of the beam, calculate t he required width b of the rectangular cross section.

(b) Taking into account the weight of the beam, calculate the required width b.



Solution 5.6-20 **Beam with an overhang**





$L = 150$ mm
$q = 3.5$ kN/m
$\sigma_{allow} = 60$ MPa
$\gamma = 77.0$ kN/m^3

$$M_{max} = \frac{qL^2}{2} \qquad S = \frac{bh^2}{6} = \frac{2b^3}{3}$$

(a) DISREGARD THE WEIGHT OF THE BEAM

$$M_{max} = \sigma_{allow} S \qquad \frac{qL^2}{2} = \sigma_{allow}\left(\frac{2b^3}{3}\right)$$

$$b^3 = \frac{3qL^2}{4\sigma_{allow}}$$

Substitute numerical values:

$$b^3 = \frac{3(3.5\text{ kN/m})(150\text{ mm})^2}{4(60\text{ MPa})} = 0.98438 \times 10^{-6}\text{ m}^3$$

$b = 0.00995$ m $= 9.95$ mm ←

(b) INCLUDE THE WEIGHT OF THE BEAM

$q_0 =$ weight of beam per unit length

$q_0 = \gamma(b)(2b) = 2\gamma b^2$

$$M_{max} = \frac{(q - q_0)L^2}{2} = \frac{1}{2}(q + 2\gamma b^2)L^2$$

$$S = \frac{2b^3}{3} \qquad M_{max} = \sigma_{allow} S$$

$$\frac{1}{2}(q + 2\gamma b^2)L^2 = \sigma_{allow}\left(\frac{2b^3}{3}\right)$$

Rearrange the equation:

$4\sigma_{allow}b^3 - 6\gamma L^2 b^2 - 3qL^2 = 0$

Substitute numerical values:

$(240 \times 10^6)b^3 - 10{,}395b^2 - 236.25 = 0$

($b =$ meters)

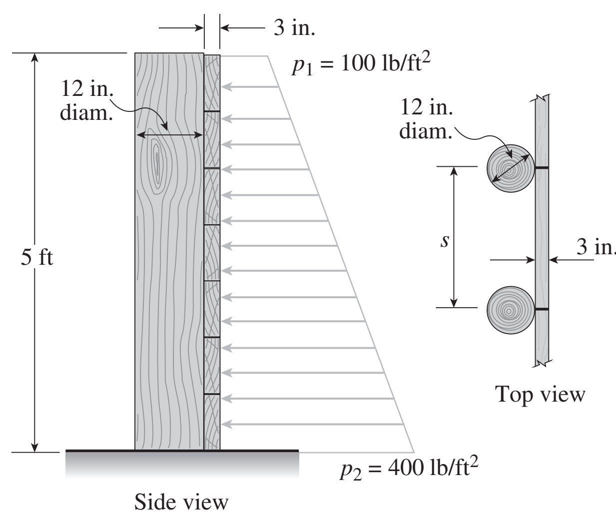
Solve the equation:

$b = 0.00996$ m $= 9.96$ mm ←

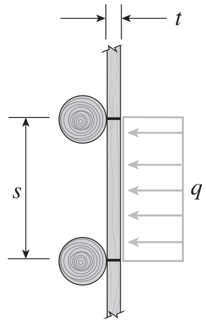
Problem 5.6-21 A retaining wall 5 ft high is constructed of horizontal wood planks 3 in. thick (actual dimension) that are supported by vertical wood piles of 12 in. diameter (actual dimension), as shown in the figure. The lateral earth pressure is $p_1 = 100$ lb/ft^2 at the top of the wall and $p_2 = 400$ lb/ft^2 at the bottom.

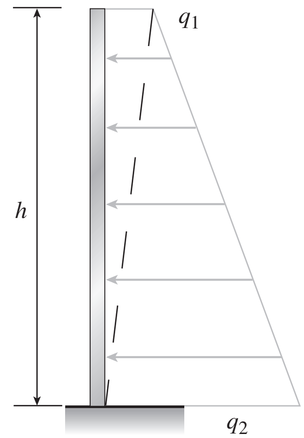
Assuming that the allowable stress in the wood is 1200 psi, calculate the maximum permissible spacing s of the piles.

(*Hint:* Observe that the spacing of the piles may be governed by the load-carrying capacity of either the planks or the piles. Consider the piles to act as cantilever beams subjected to a trapezoidal distribution of load, and consider the planks to act as simple beams between the piles. To be on the safe side, assume that the pressure on the bottom plank is uniform and equal to the maximum pressure.)



Solution 5.6-21 **Retaining wall**





(1) PLANK AT THE BOTTOM OF THE DAM

t = thickness of plank = 3 in.

b = width of plank (perpendicular to the plane of the figure)

p_2 = maximum soil pressure
= 400 lb/ft^2 = 2.778 lb/in.^{2}

s = spacing of piles

$q = p_2 b$ $\quad \sigma_{\text{allow}} = 1200$ psi $\quad S$ = section modulus

$$M_{\max} = \frac{qs^2}{8} = \frac{p_2 bs^2}{8} \qquad S = \frac{bt^2}{6}$$

$$M_{\max} = \sigma_{\text{allow}} S \quad \text{or} \quad \frac{p_2 bs^2}{8} = \sigma_{\text{allow}}\left(\frac{bt^2}{6}\right)$$

Solve for s:

$$s = \sqrt{\frac{4\sigma_{\text{allow}} t^2}{3p_2}} = 72.0 \text{ in.}$$

(2) VERTICAL PILE

h = 5 ft = 60 in.

p_1 = soil pressure at the top
= 100 lb/ft^2 = 0.6944 lb/in.^{2}

$q_1 = p_1 s$

$q_2 = p_2 s$

d = diameter of pile = 12 in.

Divide the trapezoidal load into two triangles (see dashed line).

$$M_{\max} = \frac{1}{2}(q_1)(h)\left(\frac{2h}{3}\right) + \frac{1}{2}(q_2)(h)\left(\frac{h}{3}\right) = \frac{sh^2}{6}(2p_1 + p_2)$$

$$S = \frac{\pi d^3}{32} \qquad M_{\max} = \sigma_{\text{allow}} S \quad \text{or}$$

$$\frac{sh^2}{6}(2p_1 + p_2) = \sigma_{\text{allow}}\left(\frac{\pi d^3}{32}\right)$$

Solve for s:

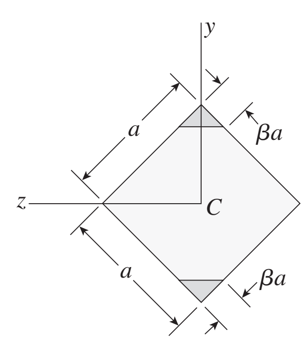
$$s = \frac{3\pi\sigma_{\text{allow}} d^3}{16h^2(2p_1 + p_2)} = 81.4 \text{ in.}$$

PLANK GOVERNS $\quad s_{\max} = 72.0$ in. ⟵

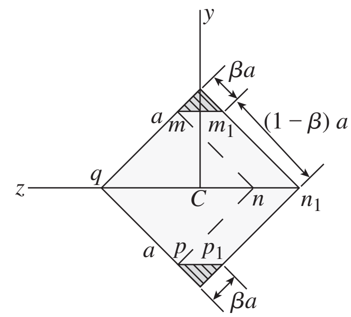
Problem 5.6-22 A beam of square cross section (a = length of each side) is bent in the plane of a diagonal (see figure). By removing a small amount of material at the top and bottom corners, as shown by the shaded triangles in the figure, we can increase the section modulus and obtain a stronger beam, even though the area of the cross section is reduced.

(a) Determine the ratio β defining the areas that should be removed in order to obtain the strongest cross section in bending.

(b) By what percent is the section modulus increased when the areas are removed?



Solution 5.6-22 **Beam of square cross section with corners removed**



a = length of each side
βa = amount removed
Beam is bent about the z axis.

ENTIRE CROSS SECTION (AREA 0)

$$I_0 = \frac{a^4}{12} \qquad c_0 = \frac{a}{\sqrt{2}} \qquad S_0 = \frac{I_0}{c_0} = \frac{a^3\sqrt{2}}{12}$$

SQUARE *mnpq* (AREA 1)

$$I_1 = \frac{(1-\beta)^4 a^4}{12}$$

PARALLELOGRAM *mm,n,n* (AREA 2)

$$I_2 = \frac{1}{3}(\text{base})(\text{height})^3$$

$$I_2 = \frac{1}{3}(\beta a\sqrt{2})\left[\frac{(1-\beta)a}{\sqrt{2}}\right]^3 = \frac{\beta a^4}{6}(1-\beta)^3$$

REDUCED CROSS SECTION (AREA *qmm,n,p,pq*)

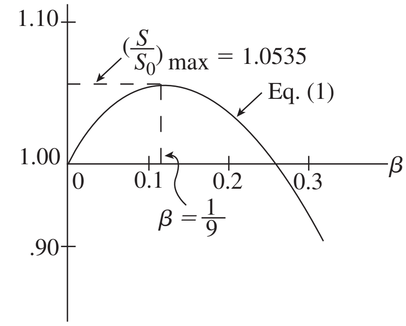
$$I = I_1 + 2I_2 = \frac{a^4}{12}(1+3\beta)(1-\beta)^3$$

$$c = \frac{(1-\beta)a}{\sqrt{2}} \qquad S = \frac{I}{c} = \frac{\sqrt{2}\,a^3}{12}(1+3\beta)(1-\beta)^2$$

RATIO OF SECTION MODULI

$$\frac{S}{S_0} = (1+3\beta)(1-\beta)^2 \qquad \text{Eq. (1)}$$

GRAPH OF EQ. (1)



(a) VALUE OF β FOR A MAXIMUM VALUE OF S/S_0

$$\frac{d}{d\beta}\left(\frac{S}{S_0}\right) = 0$$

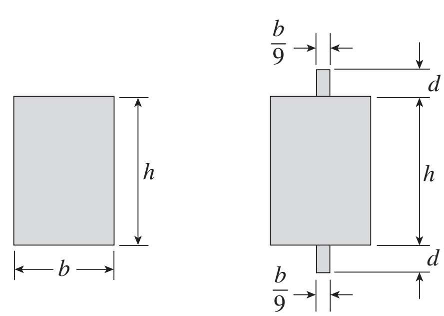
Take the derivative and solve this equation for β.

$$\beta = \frac{1}{9} \quad \longleftarrow$$

(b) MAXIMUM VALUE OF S/S_0
Substitute β = 1/9 into Eq. (1). $(S/S_0)_{max}$ = 1.0535
The section modulus is increased by 5.35% when the triangular areas are removed. ⟵

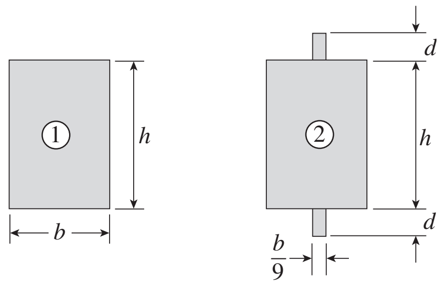
Problem 5.6-23 The cross section of a rectangular beam having width b and height h is shown in part (a) of the figure. For reasons unknown to the beam designer, it is planned to add structural projections of width $b/9$ and height d to the top and bottom of the beam [see part (b) of the figure].

For what values of d is the bending-moment capacity of the beam increased? For what values is it decreased?



(a) (b)

Solution 5.6-23 Beam with projections



(1) ORIGINAL BEAM

$$I_1 = \frac{bh^3}{12} \qquad c_1 = \frac{h}{2} \qquad S_1 = \frac{I_1}{c_1} = \frac{bh^2}{6}$$

(2) BEAM WITH PROJECTIONS

$$I_2 = \frac{1}{12}\left(\frac{8b}{9}\right)h^3 + \frac{1}{12}\left(\frac{b}{9}\right)(h + 2d)^3$$

$$= \frac{b}{108}[8h^3 + (h + 2d)^3]$$

$$c_2 = \frac{h}{2} + d = \frac{1}{2}(h + 2d)$$

$$S_2 = \frac{I_2}{c_2} = \frac{b[8h^3 + (h + 2d)^3]}{54(h + 2d)}$$

RATIO OF SECTION MODULI

$$\frac{S_2}{S_1} = \frac{b[8h^3 + (h + 2d)^3]}{9(h + 2d)(bh^2)} = \frac{8 + \left(1 + \frac{2d}{h}\right)^3}{9\left(1 + \frac{2d}{h}\right)}$$

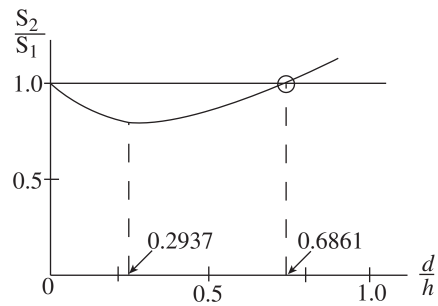
EQUAL SECTION MODULI

Set $\frac{S_2}{S_1} = 1$ and solve numerically for $\frac{d}{h}$.

$$\frac{d}{h} = 0.6861 \quad \text{and} \quad \frac{d}{h} = 0$$

Graph of $\frac{S_2}{S_1}$ versus $\frac{d}{h}$

$\frac{d}{h}$	$\frac{S_2}{S_1}$
0	1.000
0.25	0.8426
0.50	0.8889
0.75	1.0500
1.00	1.2963



Moment capacity is increased when

$\frac{d}{h} > 0.6861$ ←

Moment capacity is decreased when

$\frac{d}{h} < 0.6861$ ←

NOTES:

$\frac{S_2}{S_1} = 1$ when $\left(1 + \frac{2d}{h}\right)^3 - 9\left(1 + \frac{2d}{h}\right) + 8 = 0$

or $\quad \frac{d}{h} = 0.6861$ and 0

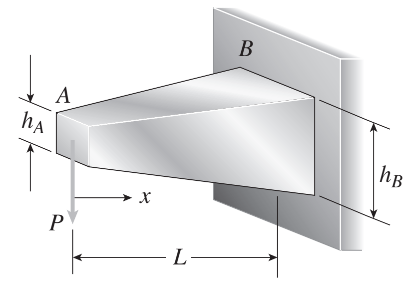
$\frac{S_2}{S_1}$ is minimum when $\frac{d}{h} = \frac{\sqrt[3]{4} - 1}{2} = 0.2937$

$$\left(\frac{S_2}{S_1}\right)_{\min} = 0.8399$$

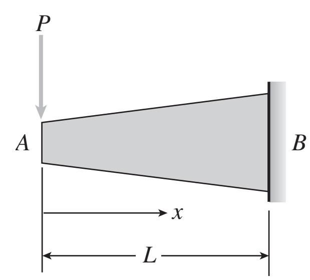
Nonprismatic Beams

Problem 5.7-1 A tapered cantilever beam AB of length L has square cross sections and supports a concentrated load P at the free end (see figure on the next page). The width and height of the beam vary linearly from h_A at the free end to h_B at the fixed end.

Determine the distance x from the free end A to the cross section of maximum bending stress if $h_B = 3h_A$. What is the magnitude σ_{max} of the maximum bending stress? What is the ratio of the maximum stress to the largest stress σ_B at the support?



Solution 5.7-1 Tapered cantilever beam



SQUARE CROSS SECTIONS

h_A = height and width at smaller end
h_B = height and width at larger end
h_x = height and width at distance x

$$\frac{h_B}{h_A} = 3$$

$$h_x = h_A + (h_B - h_A)\left(\frac{x}{L}\right) = h_A\left(1 + \frac{2x}{L}\right)$$

$$S_x = \frac{1}{6}(h_x)^3 = \frac{h_A^3}{6}\left(1 + \frac{2x}{L}\right)^3$$

STRESS AT DISTANCE x

$$\sigma_1 = \frac{M_x}{S_x} = \frac{6Px}{(h_A)^3\left(1 + \frac{2x}{L}\right)^3}$$

AT END A: $x = 0$ $\quad \sigma_A = 0$

AT SUPPORT B: $x = L$

$$\sigma_B = \frac{2PL}{9(h_A)^3}$$

CROSS SECTION OF MAXIMUM STRESS

Set $\frac{d\sigma_1}{dx} = 0$ Evaluate the derivative, set it equal to zero, and solve for x.

$$x = \frac{L}{4} \quad \longleftarrow$$

MAXIMUM BENDING STRESS

$$\sigma_{max} = (\sigma_1)_{x=L/4} = \frac{4PL}{9(h_A)^3} \quad \longleftarrow$$

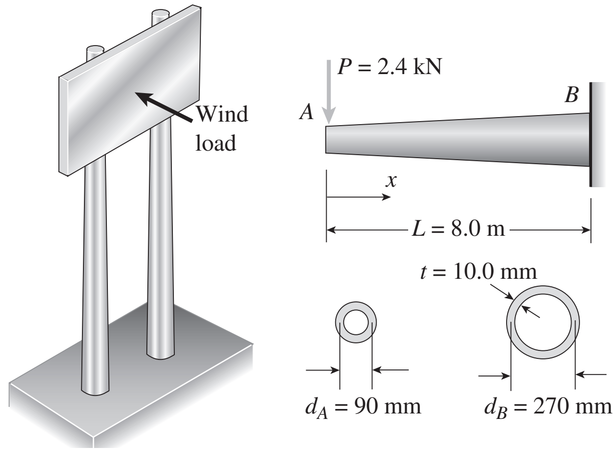
Ratio of σ_{max} *to* σ_B

$$\frac{\sigma_{max}}{\sigma_B} = 2 \quad \longleftarrow$$

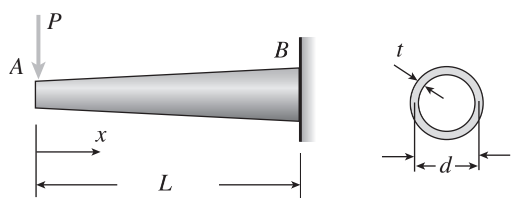
Problem 5.7-2 A tall signboard is supported by two vertical beams consisting of thin-walled, tapered circular tubes (see figure). For purposes of this analysis, each beam may be represented as a cantilever AB of length $L = 8.0$ m subjected to a lateral load $P = 2.4$ kN at the free end. The tubes have constant thickness $t = 10.0$ mm and *average* diameters $d_A = 90$ mm and $d_B = 270$ mm at ends A and B, respectively.

Because the thickness is small compared to the diameters, the moment of inertia at any cross section may be obtained from the formula $I = \pi d^3 t/8$ (see Case 22, Appendix D), and therefore the section modulus may be obtained from the formula $S = \pi d^2 t/4$.

At what distance x from the free end does the maximum bending stress occur? What is the magnitude σ_{max} of the maximum bending stress? What is the ratio of the maximum stress to the largest stress σ_B at the support?



Solution 5.7-2 **Tapered circular tube**



$P = 2.4$ kN
$L = 8.0$ m
$t = 10$ mm
d = average diameter

At end A: $d_A = 90$ mm
At support B: $d_B = 270$ mm

AT DISTANCE x:

$$d_x = d_A + (d_B - d_A)\left(\frac{x}{L}\right) = 90 + 180\frac{x}{L} = 90\left(1 + \frac{2x}{L}\right)$$

$$S_x = \frac{\pi}{4}(d_x)^2(t) = \frac{\pi}{4}(90)^2\left(1 + \frac{2x}{L}\right)^2(10)$$

$$= 20{,}250\pi\left(1 + \frac{2x}{L}\right)^2 \qquad S_x = \text{mm}^3$$

$M_x = Px = 2400x \qquad x = \text{meters}, M_x = \text{N} \cdot \text{m}$

$$\sigma_1 = \frac{M_x}{S_x} = \frac{2400x}{20.25\pi\left(1 + \frac{2x}{L}\right)^2} \qquad L = \text{meters}, \sigma_1 = \text{MPa}$$

AT END A: $x = 0 \qquad \sigma_1 = \sigma_A = 0$

AT SUPPORT B: $\quad x = L = 8.0$ m
$\sigma_1 = \sigma_B = 33.53$ MPa

CROSS SECTION OF MAXIMUM STRESS

Set $\frac{d\sigma_1}{dx} = 0$ Evaluate the derivative, set it equal to zero, and solve for x.

$$x = \frac{L}{2} = 4.0 \text{ m} \quad \longleftarrow$$

MAXIMUM BENDING STRESS

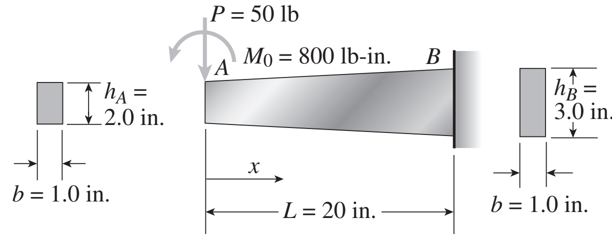
$$\sigma_{max} = (\sigma_1)_{x=L/2} = \frac{2400(4.0)}{(20.25\,\pi)(1+1)^2}$$

$$= 37.73 \text{ MPa} \quad \longleftarrow$$

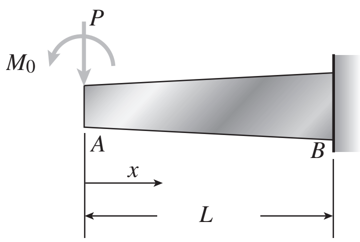
RATIO OF σ_{max} *to* σ_B $\qquad \frac{\sigma_{max}}{\sigma_B} = \frac{9}{8} = 1.125 \quad \longleftarrow$

Problem 5.7-3 A tapered cantilever beam AB having rectangular cross sections is subjected to a concentrated load $P = 50$ lb and a couple $M_0 = 800$ lb-in. acting at the free end (see figure). The width b of the beam is constant and equal to 1.0 in., but the height varies linearly from $h_A = 2.0$ in. at the loaded end to $h_B = 3.0$ in. at the support.

At what distance x from the free end does the maximum bending stress σ_{max} occur? What is the magnitude σ_{max} of the maximum bending stress? What is the ratio of the maximum stress to the largest stress σ_B at the support?



Solution 5.7-3 Tapered cantilever beam



$$P = 50 \text{ lb}$$
$$M_0 = 800 \text{ lb-in.}$$
$$L = 20 \text{ in.}$$
$$h_A = 2.0 \text{ in.}$$
$$h_B = 3.0 \text{ in.}$$
$$b = 1.0 \text{ in.}$$

UNITS: pounds and inches

AT DISTANCE x:

$$h_x = h_A + (h_B - h_A)\frac{x}{L} = 2 + (1)\left(\frac{x}{L}\right) = 2 + \frac{x}{L}$$

$$S_x = \frac{bh_x^2}{6} = \frac{b}{6}\left(2 + \frac{x}{L}\right)^2 = \frac{1}{6}\left(2 + \frac{x}{L}\right)^2$$

$$M_x = Px + M_0 = (50)(x) + 800 = 50(16 + x)$$

$$\sigma_1 = \frac{M_x}{S_x} = \frac{50(16 + x)(6)}{\left(2 + \frac{x}{L}\right)^2} = \frac{120{,}000(16 + x)}{(40 + x)^2}$$

AT END A: $x = 0 \qquad \sigma_1 = \sigma_A = 1200$ psi

AT SUPPORT B: $x = L = 20$ in. $\qquad \sigma_1 = \sigma_B = 1200$ psi

CROSS SECTION OF MAXIMUM STRESS

Set $\dfrac{d\sigma_1}{dx} = 0$ Evaluate the derivative, set it equal to zero, and solve for x.

$x = 8.0$ in. ←

MAXIMUM BENDING STRESS

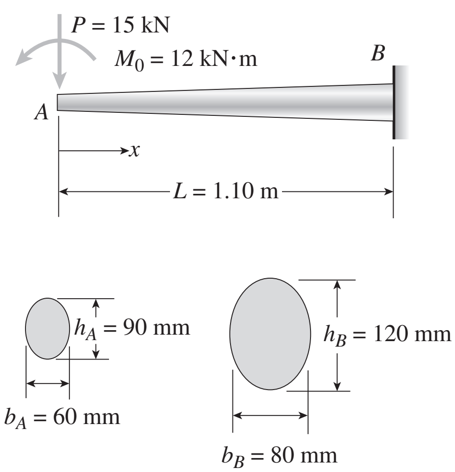
$$\sigma_{max} = (\sigma_1)_{x=8.0} = \frac{(120{,}000)(24)}{(48)^2} = 1250 \text{ psi} \quad \longleftarrow$$

RATIO OF σ_{max} TO σ_B

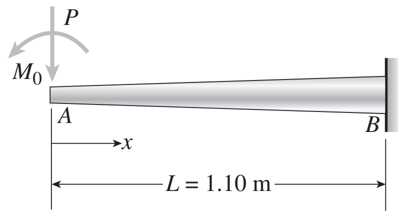
$$\frac{\sigma_{max}}{\sigma_B} = \frac{1250}{1200} = \frac{25}{24} = 1.042 \quad \longleftarrow$$

Problem 5.7-4 The spokes in a large flywheel are modeled as beams fixed at one end and loaded by a force P and a couple M_0 at the other (see figure). The cross sections of the spokes are elliptical with major and minor axes (height and width, respectively) having the lengths shown in the figure. The cross-sectional dimensions vary linearly from end A to end B.

Considering only the effects of bending due to the loads P and M_0, determine the following quantities: (a) the largest bending stress σ_A at end A; (b) the largest bending stress σ_B at end B; (c) the distance x to the cross section of maximum bending stress; and (d) the magnitude σ_{max} of the maximum bending stress.



Solution 5.7-4 Elliptical spokes in a flywheel



$P = 15\text{ kN} = 15{,}000\text{ N}$
$M_0 = 12\text{ kN}\cdot\text{m} = 12{,}000\text{ N}\cdot\text{m}$
$L = 1.1\text{ m}$

UNITS: Newtons, meters

AT END A: $b_A = 0.06$ m, $h_A = 0.09$ m

AT SUPPORT B: $b_B = 0.08$ m, $h_B = 0.12$ m

AT DISTANCE x:

$$b_x = b_A + (b_B - b_A)\frac{x}{L} = 0.06 + 0.02\frac{x}{L} = 0.02\left(3 + \frac{x}{L}\right)$$

$$h_x = h_A + (h_B - h_A)\frac{x}{L} = 0.09 + 0.03\frac{x}{L} = 0.03\left(3 + \frac{x}{L}\right)$$

Case 16, Appendix D: $I = \dfrac{\pi}{64}(bh^3)$

$$I_x = \frac{\pi}{64}(b_x)(h_x)^3 \qquad S_x = \frac{I_x}{h_x/2} = \frac{\pi b_x h_x^2}{32}$$

$$S_x = \frac{\pi}{32}(0.02)\left(3 + \frac{x}{L}\right)(0.03)^2\left(3 + \frac{x}{L}\right)^2$$

$$= \frac{9\pi}{16 \times 10^6}\left(3 + \frac{x}{L}\right)^3$$

$$M_x = M_0 + Px = 12{,}000\text{ N}\cdot\text{m} + (15{,}000\text{ N})x$$
$$= 15{,}000(0.8 + x)$$

$$\sigma_1 = \frac{M_x}{S_x} = \frac{15{,}000(0.8 + x)(16 \times 10^6)}{9\pi\left(3 + \frac{x}{L}\right)^3}$$

$$= \frac{(80 \times 10^9)(0.8 + x)}{3\pi\left(3 + \frac{x}{L}\right)^3}$$

(a) AT END A: $x = 0$

$$\sigma_A = (\sigma_1)_{x=0} = \frac{(80 \times 10^9)(0.8)}{(3\pi)(27)} = 251.5 \times 10^6\text{ N/m}^2$$
$$= 251.5\text{ MPa} \longleftarrow$$

(b) AT END B: $x = L = 1.1$ m

$$\sigma_B = (\sigma_1)_{x=L} = \frac{(80 \times 10^9)(0.8 + 1.1)}{(3\pi)(3 + 1)^3}$$
$$= 252.0 \times 10^6\text{ N/m}^2 = 252.0\text{ MPa} \longleftarrow$$

(c) CROSS SECTION OF MAXIMUM STRESS

Set $\dfrac{d\sigma_1}{dx} = 0$ Evaluate the derivative, set it equal to zero, and solve for x.

$x = 0.45$ m $\longleftarrow$

(d) MAXIMUM BENDING STRESS

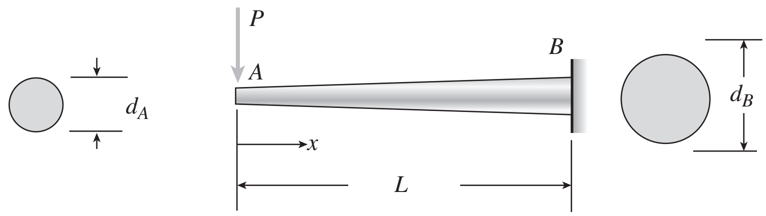
$$\sigma_{max} = (\sigma_1)_{x=0.45} = \frac{(80 \times 10^9)(0.8 + 0.45)}{(3\pi)\left(3 + \frac{0.45}{1.1}\right)^3}$$
$$= 267.8 \times 10^6\text{ N/m}^2 = 267.8\text{ MPa} \longleftarrow$$

Problem 5.7-5 Refer to the tapered cantilever beam of solid circular cross section shown in Fig. 5-24 of Example 5-9.

(a) Considering only the bending stresses due to the load P, determine the range of values of the ratio d_B/d_A for which the maximum normal stress occurs at the support.

(b) What is the maximum stress for this range of values?

Solution 5.7-5 Tapered cantilever beam



FROM EQ. (5-32), EXAMPLE 5-9

$$\sigma_1 = \frac{32Px}{\pi\left[d_A + (d_B - d_A)\left(\frac{x}{L}\right)\right]^3} \qquad \text{Eq. (1)}$$

FIND THE VALUE OF x THAT MAKES σ_1 A MAXIMUM

$$\text{Let } \sigma_1 = \frac{u}{v} \qquad \frac{d\sigma_1}{dx} = \frac{v\left(\frac{du}{dx}\right) - u\left(\frac{dv}{dx}\right)}{v^2} = \frac{N}{D}$$

$$N = \pi\left[d_A + (d_B - d_A)\left(\frac{x}{L}\right)\right]^3[32P]$$

$$-[32Px][\pi][3]\left[d_A + (d_B - d_A)\left(\frac{x}{L}\right)\right]^2\left[\frac{1}{L}(d_B - d_A)\right]$$

After simplification:

$$N = 32\pi P\left[d_A + (d_B - d_A)\left(\frac{x}{L}\right)\right]^2\left[d_A - 2(d_B - d_A)\frac{x}{L}\right]$$

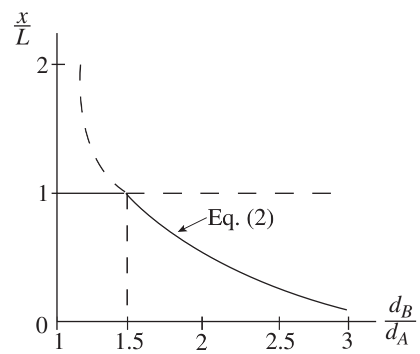
$$D = \pi^2\left[d_A + (d_B - d_A)\frac{x}{L}\right]^6$$

$$\frac{d\sigma_1}{dx} = \frac{N}{D} = \frac{32P\left[d_A - 2(d_B - d_A)\frac{x}{L}\right]}{\pi\left[d_A + (d_B - d_A)\left(\frac{x}{L}\right)\right]^4}$$

$$\frac{d\sigma_1}{dx} = 0 \qquad d_A - 2(d_B - d_A)\left(\frac{x}{L}\right) = 0$$

$$\therefore \frac{x}{L} = \frac{d_A}{2(d_B - d_A)} = \frac{1}{2\left(\frac{d_B}{d_A} - 1\right)} \qquad \text{Eq. (2)}$$

(a) GRAPH OF x/L VERSUS d_B/d_A (EQ. 2)



Maximum bending stress occurs at the support when $1 \le \frac{d_B}{d_A} \le 1.5$ ←

(b) MAXIMUM STRESS (AT SUPPORT B)

Substitute $x/L = 1$ into Eq. (1):

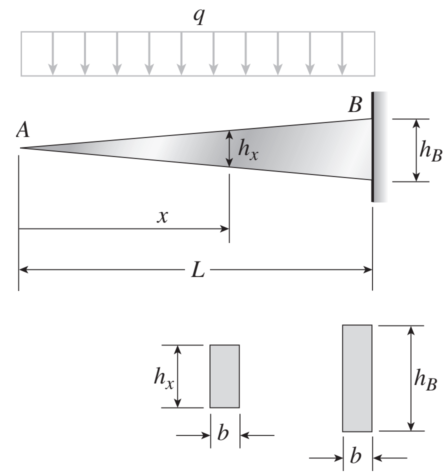
$$\sigma_{max} = \frac{32PL}{\pi d_B^3} \quad \leftarrow$$

Fully Stressed Beams

Problems 5.7-6 to 5.7-8 pertain to fully stressed beams of rectangular cross section. Consider only the bending stresses obtained from the flexure formula and disregard the weights of the beams.

Problem 5.7-6 A cantilever beam AB having rectangular cross sections with constant width b and varying height h_x is subjected to a uniform load of intensity q (see figure).

How should the height h_x vary as a function of x (measured from the free end of the beam) in order to have a fully stressed beam? (Express h_x in terms of the height h_B at the fixed end of the beam.)



Solution 5.7-6 Fully stressed beam with constant width and varying height

h_x = height at distance x
h_B = height at end B
b = width (constant)

AT DISTANCE x: $M = \dfrac{qx^2}{2}$ $\quad S = \dfrac{bh_x^2}{6}$

$$\sigma_{\text{allow}} = \frac{M}{S} = \frac{3qx^2}{bh_x^2}$$

$$h_x = x\sqrt{\frac{3q}{b\sigma_{\text{allow}}}}$$

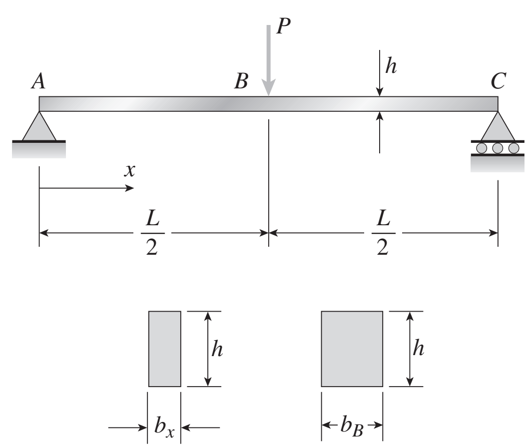
AT THE FIXED END $(x = L)$:

$$h_B = L\sqrt{\frac{3q}{b\sigma_{\text{allow}}}}$$

Therefore, $\dfrac{h_x}{h_B} = \dfrac{x}{L}$ $\quad h_x = \dfrac{h_B x}{L}$ ←

Problem 5.7-7 A simple beam ABC having rectangular cross sections with constant height h and varying width b_x supports a concentrated load P acting at the midpoint (see figure).

How should the width b_x vary as a function of x in order to have a fully stressed beam? (Express b_x in terms of the width b_B at the midpoint of the beam.)



Solution 5.7-7 **Fully stressed beam with constant height and varying width**

$h =$ height of beam (constant)

$b_x =$ width at distance x from end $A\left(0 \le x \le \dfrac{L}{2}\right)$

$b_B =$ width at midpoint $B \quad (x = L/2)$

AT DISTANCE $x \quad M = \dfrac{Px}{2} \quad S = \dfrac{1}{6} b_x h^2$

$$\sigma_{\text{allow}} = \frac{M}{S} = \frac{3Px}{b_x h^2} \quad b_x = \frac{3Px}{\sigma_{\text{allow}}\, h^2}$$

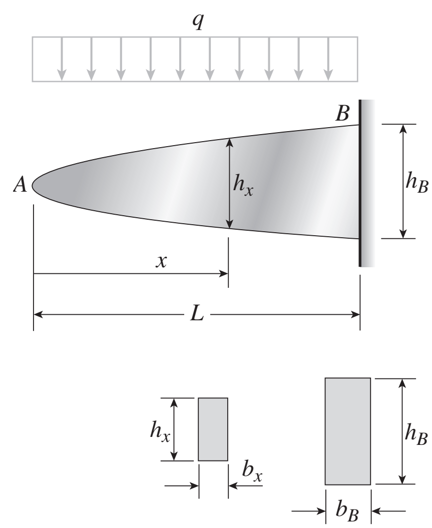
AT MIDPOINT B $(x = L/2)$

$$b_B = \frac{3PL}{2\sigma_{\text{allow}} h^2}$$

Therefore, $\dfrac{b_x}{b_b} = \dfrac{2x}{L}$ and $b_x = \dfrac{2b_B x}{L}$ ←

NOTE: The equation is valid for $0 \le x \le \dfrac{L}{2}$ and the beam is symmetrical about the midpoint.

Problem 5.7-8 A cantilever beam AB having rectangular cross sections with varying width b_x and varying height h_x is subjected to a uniform load of intensity q (see figure). If the width varies linearly with x according to the equation $b_x = b_B x/L$, how should the height h_x vary as a function of x in order to have a fully stressed beam? (Express h_x in terms of the height h_B at the fixed end of the beam.)



Solution 5.7-8 **Fully stressed beam with varying width and varying height**

$h_x =$ height at distance x
$h_B =$ height at end B
$b_x =$ width at distance x
$b_B =$ width at end B

$$b_x = b_B\left(\frac{x}{L}\right)$$

AT DISTANCE x

$$M = \frac{qx^2}{2} \quad S = \frac{b_x h_x^2}{6} = \frac{b_B x}{6L}(h_x)^2$$

$$\sigma_{\text{allow}} = \frac{M}{S} = \frac{3qLx}{b_B h_x^2}$$

$$h_x = \sqrt{\frac{3qLx}{b_B \sigma_{\text{allow}}}}$$

AT THE FIXED END $(x = L)$

$$h_B = \sqrt{\frac{3qL^2}{b_B \sigma_{\text{allow}}}}$$

Therefore, $\dfrac{h_x}{h_B} = \sqrt{\dfrac{x}{L}} \quad h_x = h_B\sqrt{\dfrac{x}{L}}$ ←

Shear Stresses in Rectangular Beams

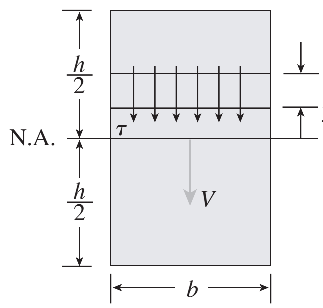
Problem 5.8-1 The shear stresses τ in a rectangular beam are given by Eq. (5-39):

$$\tau = \frac{V}{2I}\left(\frac{h^2}{4} - y_1^2\right)$$

in which V is the shear force, I is the moment of inertia of the cross-sectional area, h is the height of the beam, and y_1 is the distance from the neutral axis to the point where the shear stress is being determined (Fig. 5-30).

By integrating over the cross-sectional area, show that the resultant of the shear stresses is equal to the shear force V.

Solution 5.8-1 **Resultant of the shear stresses**



$$I = \frac{bh^3}{12}$$

$$\tau = \frac{V}{2I}\left(\frac{h^2}{4} - y_1^2\right)$$

$V =$ shear force acting on the cross section

$R =$ resultant of shear stresses τ

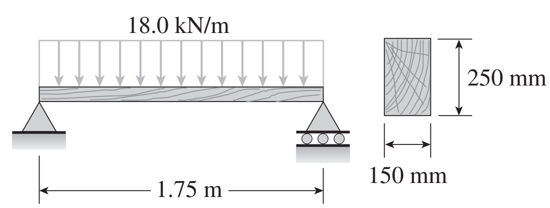
$$R = \int_{-h/2}^{h/2} \tau b dy_1 = 2\int_0^{h/2} \frac{V}{2I}\left(\frac{h^2}{4} - y_1^2\right) b dy_1$$

$$= \frac{12V}{bh^3}(b)\int_0^{h/2}\left(\frac{h^2}{4} - y_1^2\right) dy_1$$

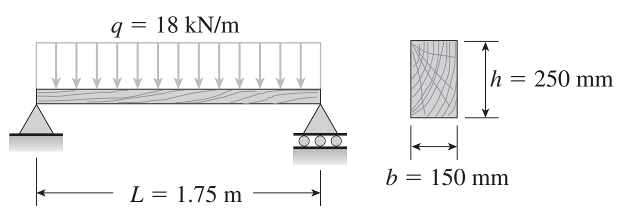
$$= \frac{12V}{h^3}\left(\frac{2h^3}{24}\right) = V$$

$\therefore R = V$ Q.E.D. ←

Problem 5.8-2 Calculate the maximum shear stress τ_{max} and the maximum bending stress σ_{max} in a simply supported wood beam (see figure) carrying a uniform load of 18.0 kN/m (which includes the weight of the beam) if the length is 1.75 m and the cross section is rectangular with width 150 mm and height 250 mm.



Solution 5.8-2 **Wood beam with a uniform load**



MAXIMUM SHEAR STRESS

$$V = \frac{qL}{2} \qquad A = bh$$

$$\tau_{max} = \frac{3V}{2A} = \frac{3qL}{4bh} = \frac{3(18\text{ kN/m})(1.75\text{ m})}{4(150\text{ mm})(250\text{ mm})}$$

$$= 630\text{ kPa} \leftarrow$$

MAXIMUM BENDING STRESS

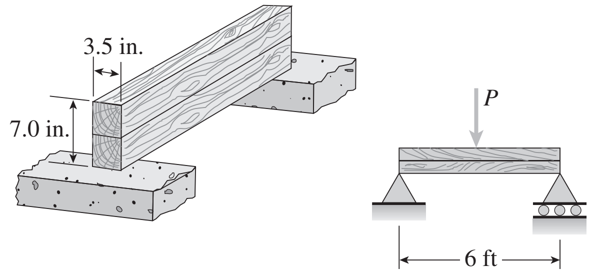
$$M = \frac{qL^2}{8} \qquad S = \frac{bh^2}{6}$$

$$\sigma_{max} = \frac{M}{S} = \frac{3qL^2}{4bh^2} = \frac{3(18\text{ kN/m})(1.75\text{ m})^2}{4(150\text{ mm})(250\text{ mm})^2}$$

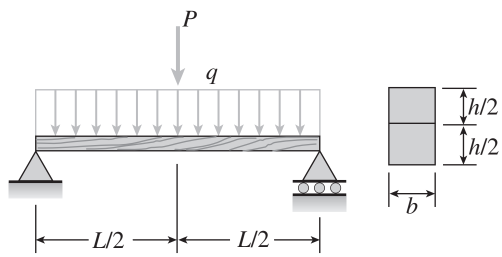
$$= 4.41\text{ MPa} \leftarrow$$

Problem 5.8-3 Two wood beams, each of square cross section (3.5 in. × 3.5 in., actual dimensions) are glued together to form a solid beam of dimensions 3.5 in. × 7.0 in. (see figure). The beam is simply supported with a span of 6 ft.

What is the maximum load P_{max} that may act at the midpoint if the allowable shear stress in the glued joint is 200 psi? (Include the effects of the beam's own weight, assuming that the wood weighs 35 lb/ft^3.)



Solution 5.8-3 **Simple beam with a glued joint**



$L = 6$ ft $= 72$ in. $\quad b = 3.5$ in. $\quad h = 7.0$ in.

$\tau_{allow} = 200$ psi

$$\gamma = (35\text{ lb/ft}^3)\left(\frac{1}{1728}\,\frac{\text{ft}^3}{\text{in.}^3}\right) = \frac{35}{1728}\text{ lb/in.}^3$$

$q =$ weight of beam per unit distance
$\quad = \gamma bh$

MAXIMUM LOAD P_{max}

$$V = \frac{P}{2} + \frac{qL}{2} \qquad A = bh$$

$$\tau_{max} = \frac{3V}{2A} = \frac{3\left(\dfrac{P}{2} + \dfrac{qL}{2}\right)}{2bh} = \frac{3}{4bh}(P + qL)$$

$$P_{max} = \frac{4}{3}bh\tau - qL = \frac{4}{3}bh\tau - \gamma bhL$$

$$= bh\left(\frac{4}{3}\tau - \gamma L\right)$$

SUBSTITUTE NUMERICAL VALUES:

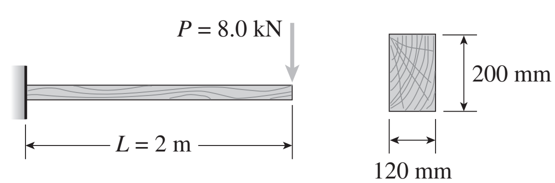
$$P_{max} = (3.5\text{ in.})(7.0\text{ in.}) \times \left[\frac{4}{3}(200\text{ psi}) - \left(\frac{35}{1728}\text{ lb/in.}^3\right)(72\text{ in.})\right]$$

$$= 6500\text{ lb} \quad \longleftarrow$$

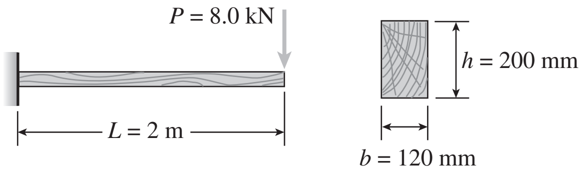
(This result is based solely on the shear stress.)

Problem 5.8-4 A cantilever beam of length $L = 2$ m supports a load $P = 8.0$ kN (see figure). The beam is made of wood with cross-sectional dimensions 120 mm × 200 mm.

Calculate the shear stresses due to the load P at points located 25 mm, 50 mm, 75 mm, and 100 mm from the top surface of the beam. From these results, plot a graph showing the distribution of shear stresses from top to bottom of the beam.



Solution 5.8-4 **Shear stresses in a cantilever beam**



Distance from the top surface (mm)	y_1 (mm)	τ (MPa)	τ (kPa)
0	100	0	0
25	75	0.219	219
50	50	0.375	375
75	25	0.469	469
100 (N.A.)	0	0.500	500

Eq. (5-39): $\tau = \dfrac{V}{2I}\left(\dfrac{h^2}{4} - y_1^2\right)$

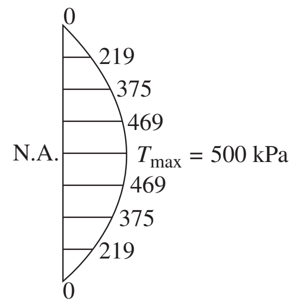
$V = P = 8.0\text{ kN} = 8{,}000\text{ N} \quad I = \dfrac{bh^3}{12} = 80 \times 10^6\text{ mm}^4$

$h = 200\text{ mm} \quad (y_1 = \text{mm})$

$$\tau = \frac{8{,}000}{2(80 \times 10^6)}\left[\frac{(200)^2}{4} - y_1^2\right] \quad (\tau = \text{N/mm}^2 = \text{MPa})$$

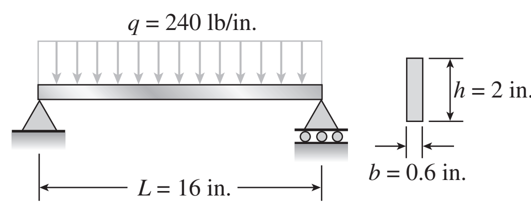
$\tau = 50 \times 10^{-6}(10{,}000 - y_1^2) \quad (y_1 = \text{mm}; \tau = \text{MPa})$

Graph of shear stress τ

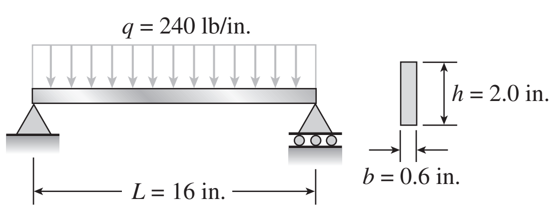


Problem 5.8-5 A steel beam of length $L = 16$ in. and cross-sectional dimensions $b = 0.6$ in. and $h = 2$ in. (see figure) supports a uniform load of intensity $q = 240$ lb/in., which includes the weight of the beam.

Calculate the shear stresses in the beam (at the cross section of maximum shear force) at points located 1/4 in., 1/2 in., 3/4 in., and 1 in. from the top surface of the beam. From these calculations, plot a graph showing the distribution of shear stresses from top to bottom of the beam.



Solution 5.8-5 **Shear stresses in a simple beam**



Distance from the top surface (in.)	y_1 (in.)	τ (psi)
0	1.00	0
0.25	0.75	1050
0.50	0.50	1800
0.75	0.25	2250
1.00 (N.A.)	0	2400

Eq. (5-39): $\tau = \dfrac{V}{2I}\left(\dfrac{h^2}{4} - y_1^2\right)$

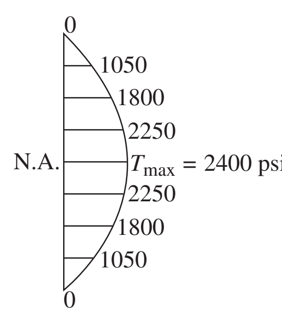
$V = \dfrac{qL}{2} = 1920\text{ lb} \quad I = \dfrac{bh^3}{12} = 0.4\text{ in.}^4$

Units: pounds and inches

$$\tau = \frac{1920}{2(0.4)}\left[\frac{(2)^2}{4} - y_1^2\right] = (2400)(1 - y_1^2)$$

$(\tau = \text{psi}; y_1 = \text{in.})$

Graph of shear stress τ



Problem 5.8-6 A beam of rectangular cross section (width b and height h) supports a uniformly distributed load along its entire length L. The allowable stresses in bending and shear are σ_{allow} and τ_{allow}, respectively.

(a) If the beam is simply supported, what is the span length L_0 below which the shear stress governs the allowable load and above which the bending stress governs?

(b) If the beam is supported as a cantilever, what is the length L_0 below which the shear stress governs the allowable load and above which the bending stress governs?

Solution 5.8-6 Beam of rectangular cross section

$b =$ width $\quad h =$ height $\quad L =$ length

UNIFORM LOAD $\quad q =$ intensity of load

ALLOWABLE STRESSES $\quad \sigma_{allow}$ and τ_{allow}

(a) SIMPLE BEAM

BENDING

$$M_{max} = \frac{qL^2}{8} \qquad S = \frac{bh^2}{6}$$

$$\sigma_{max} = \frac{M_{max}}{S} = \frac{3qL^2}{4bh^2}$$

$$q_{allow} = \frac{4\sigma_{allow}\, bh^2}{3L^2} \qquad (1)$$

SHEAR

$$V_{max} = \frac{qL}{2} \qquad A = bh$$

$$\tau_{max} = \frac{3V}{2A} = \frac{3qL}{4bh}$$

$$q_{allow} = \frac{4\tau_{allow}\, bh}{3L} \qquad (2)$$

Equate (1) and (2) and solve for L_0:

$$L_0 = h\left(\frac{\sigma_{allow}}{\tau_{allow}}\right) \longleftarrow$$

(b) CANTILEVER BEAM

BENDING

$$M_{max} = \frac{qL^2}{2} \qquad S = \frac{bh^2}{6}$$

$$\sigma_{max} = \frac{M_{max}}{S} = \frac{3qL^2}{bh^2} \qquad q_{allow} = \frac{\sigma_{allow}\, bh^2}{3L^2} \qquad (3)$$

SHEAR

$$V_{max} = qL \qquad A = bh$$

$$\tau_{max} = \frac{3V}{2A} = \frac{3qL}{2bh}$$

$$q_{allow} = \frac{2\tau_{allow}\, bh}{3L} \qquad (4)$$

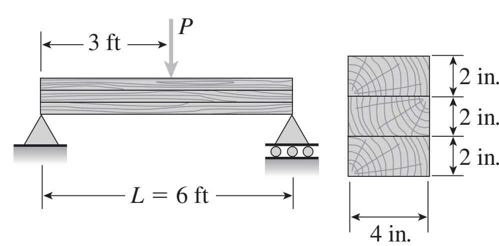
Equate (3) and (4) and solve for L_0:

$$L_0 = \frac{h}{2}\left(\frac{\sigma_{allow}}{\tau_{allow}}\right) \longleftarrow$$

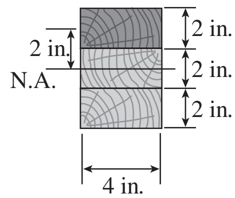
NOTE: If the actual length is less than L_0, the shear stress governs the design. If the length is greater than L_0, the bending stress governs.

Problem 5.8-7 A laminated wood beam on simple supports is built up by gluing together three 2 in. × 4 in. boards (actual dimensions) to form a solid beam 4 in. × 6 in. in cross section, as shown in the figure. The allowable shear stress in the glued joints is 65 psi and the allowable bending stress in the wood is 1800 psi.

If the beam is 6 ft long, what is the allowable load P acting at the midpoint of the beam? (Disregard the weight of the beam.)



Solution 5.8-7 Laminated wood beam on simple supports



$L = 6 \text{ ft} = 72 \text{ in.}$
$\tau_{allow} = 65 \text{ psi}$
$\sigma_{allow} = 1800 \text{ psi}$

ALLOWABLE LOAD BASED UPON SHEAR STRESS IN THE GLUED JOINTS

$$\tau = \frac{VQ}{Ib} \qquad Q = (4 \text{ in.})(2 \text{ in.})(2 \text{ in.}) = 16 \text{ in.}^3$$

$$V = \frac{P}{2} \qquad I = \frac{bh^3}{12} = \frac{1}{12}(4 \text{ in.})(6 \text{ in.})^3 = 72 \text{ in.}^4$$

$$\tau = \frac{(P/2)(16 \text{ in.}^3)}{(72 \text{ in.}^4)(4 \text{ in.})} = \frac{P}{36} \qquad (P = \text{lb}; \tau = \text{psi})$$

$$P_1 = 36\tau_{allow} = 36\ (65 \text{ psi}) = 2340 \text{ lb}$$

ALLOWABLE LOAD BASED UPON BENDING STRESS

$$\sigma = \frac{M}{S} \qquad M = \frac{PL}{4} = P\left(\frac{72 \text{ in.}}{4}\right) = 18P \text{ (lb-in.)}$$

$$S = \frac{bh^2}{6} = \frac{1}{6}(4 \text{ in.})(6 \text{ in.})^2 = 24 \text{ in.}^3$$

$$\sigma = \frac{(18P \text{ lb-in.})}{24 \text{ in.}^3} = \frac{3P}{4} \qquad (P = \text{lb}; \sigma = \text{psi})$$

$$P_2 = \frac{4}{3}\sigma_{allow} = \frac{4}{3}(1800 \text{ psi}) = 2400 \text{ lb}$$

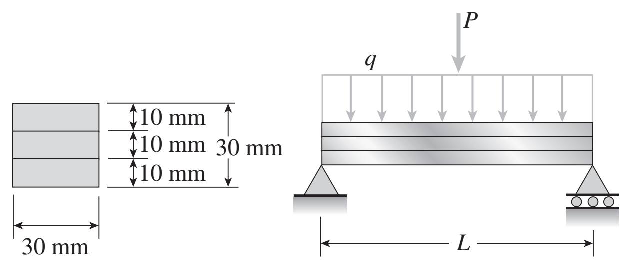
ALLOWABLE LOAD

Shear stress in the glued joints governs.

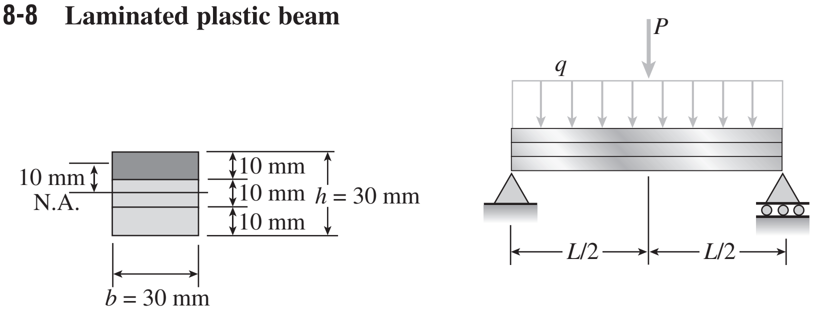
$$P_{allow} = 2340 \text{ lb} \quad \longleftarrow$$

Problem 5.8-8 A laminated plastic beam of square cross section is built up by gluing together three strips, each 10 mm × 30 mm in cross section (see figure). The beam has a total weight of 3.2 N and is simply supported with span length $L = 320$ mm.

Considering the weight of the beam, calculate the maximum permissible load P that may be placed at the midpoint if (a) the allowable shear stress in the glued joints is 0.3 MPa, and (b) the allowable bending stress in the plastic is 8 MPa.



Solution 5.8-8 Laminated plastic beam



$L = 320$ mm
$W = 3.2$ N

$$q = \frac{W}{L} = \frac{3.2 \text{ N}}{320 \text{ mm}} = 10 \text{ N/m}$$

$$I = \frac{bh^3}{12} = \frac{1}{12}(30 \text{ mm})(30 \text{ mm})^3 = 67{,}500 \text{ mm}^4$$

$$S = \frac{bh^2}{6} = \frac{1}{6}(30 \text{ mm})(30 \text{ mm})^2 = 4500 \text{ mm}^3$$

(a) Allowable load based upon shear in glued joints

$\tau_{allow} = 0.3$ MPa

$$\tau = \frac{VQ}{Ib} \qquad V = \frac{P}{2} + \frac{qL}{2} = \frac{P}{2} + 1.6\,\text{N}$$

(V = newtons; P = newtons)

$$Q = (30\text{ mm})(10\text{ mm})(10\text{ mm}) = 3000\text{ mm}^3$$

$$\frac{Q}{Ib} = \frac{3000\text{ mm}^3}{(67{,}500\text{ mm}^4)(30\text{ mm})} = \frac{1}{675\text{ mm}^2}$$

$$\tau = \frac{VQ}{Ib} = \frac{P/2 + 1.6\text{ N}}{675\text{ mm}^2} \qquad (\tau = \text{N/mm}^2 = \text{MPa})$$

Solve for P:

$P = 1350\tau_{allow} - 3.2 = 405\text{ N} - 3.2\text{ N} = 402\text{ N}$ ⟵

(b) Allowable load based upon bending stresses

$\sigma_{allow} = 8$ MPa

$$\sigma = \frac{M_{max}}{S}$$

$$M_{max} = \frac{PL}{4} + \frac{qL^2}{8} = 0.08P + 0.128\ (\text{N}\cdot\text{m})$$

(P = newtons; M = N · m)

$$\sigma = \frac{(0.08P + 0.128)(\text{N}\cdot\text{m})}{4.5 \times 10^{-6}\,\text{m}^3}$$

(σ = N/m² = Pa)

Solve for P:

$P = (56.25 \times 10^{-6})\,\sigma_{allow} - 1.6$

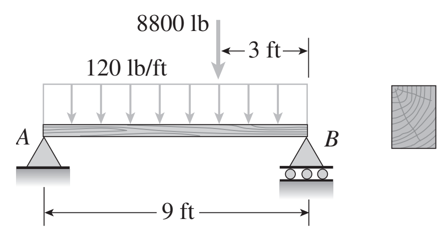
$= (56.25 \times 10^{-6})(8 \times 10^6\text{ Pa}) - 1.6$

$= 450 - 1.6 = 448\text{ N}$ ⟵

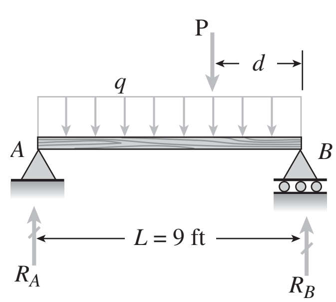
Problem 5.8-9 A wood beam AB on simple supports with span length equal to 9 ft is subjected to a uniform load of intensity 120 lb/ft acting along the entire length of the beam and a concentrated load of magnitude 8800 lb acting at a point 3 ft from the right-hand support (see figure). The allowable stresses in bending and shear, respectively, are 2500 psi and 150 psi.

(a) From the table in Appendix F, select the lightest beam that will support the loads (disregard the weight of the beam).

(b) Taking into account the weight of the beam (weight density = 35 lb/ft³), verify that the selected beam is satisfactory, or, if it is not, select a new beam.



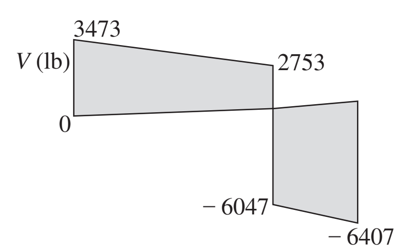
Solution 5.8-9



$q = 120$ lb/ft
$P = 8800$ lb
$d = 3$ ft
$\sigma_{allow} = 2500$ psi
$\tau_{allow} = 150$ psi

$$R_A = \frac{qL}{2} + \frac{P}{3} \qquad R_B = \frac{qL}{2} + \frac{2P}{3}$$

(a) Disregarding the weight of the beam



$$R_A = \frac{(120\text{ lb/ft})(9\text{ ft})}{2} + \frac{8800\text{ lb}}{3} = 3473\text{ lb}$$

$$R_B = 540\text{ lb} + \frac{2}{3}(8800\text{ lb}) = 6407\text{ lb}$$

$$V_{max} = R_B = 6407\text{ lb}$$

(Continued)

Maximum bending moment occurs under the concentrated load.

$$M_{max} = R_B d - \frac{qd^2}{2}$$

$$= (6407 \text{ lb})(3 \text{ ft}) - \frac{1}{2}(120 \text{ lb/ft})(3 \text{ ft})^2$$

$$= 18{,}680 \text{ lb-ft} = 224{,}200 \text{ lb-in.}$$

$$\tau_{max} = \frac{3V}{2A} \quad A_{req} = \frac{3V_{max}}{2\tau_{allow}} = \frac{3(6407 \text{ lb})}{2(150 \text{ psi})} = 64.1 \text{ in.}^2$$

$$\sigma = \frac{M}{S} \quad S_{req} = \frac{M_{max}}{\sigma_{allow}} = \frac{224{,}200 \text{ lb-in.}}{2500 \text{ psi}} = 89.7 \text{ in.}^3$$

From Appendix F: Select 8 × 10 in. beam (nominal dimensions) ←

$A = 71.25 \text{ in.}^2 \quad S = 112.8 \text{ in.}^3$

(b) Considering the weight of the beam

$q_{BEAM} = 17.3$ lb/ft (weight density = 35 lb/ft^3)

$$R_B = 6407 \text{ lb} + \frac{(17.3 \text{ lb/ft})(9 \text{ ft})}{2} = 6407 + 78 = 6485 \text{ lb}$$

$$V_{max} = 6485 \text{ lb} \quad A_{req'd} = \frac{3V_{max}}{2\tau_{allow}} = 64.9 \text{ in.}^2$$

8 × 10 beam is still satisfactory for shear.

$q_{TOTAL} = 120 \text{ lb/ft} + 17.3 \text{ lb/ft} = 137.3 \text{ lb/ft}$

$$M_{max} = R_B d - \frac{qd^2}{2} = (6485 \text{ lb})(3 \text{ ft}) - \frac{1}{2}\left(137.3 \frac{\text{lb}}{\text{ft}}\right)(3 \text{ ft})^2$$

$$= 18{,}837 \text{ lb-ft} = 226{,}050 \text{ lb-in.}$$

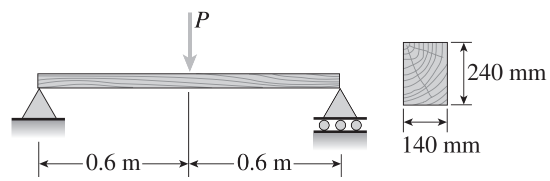
$$S_{req'd} = \frac{M_{max}}{\sigma_{allow}} = \frac{226{,}050 \text{ lb-in.}}{2500 \text{ psi}} = 90.4 \text{ in.}^3$$

8 × 10 beam is still satisfactory for moment.

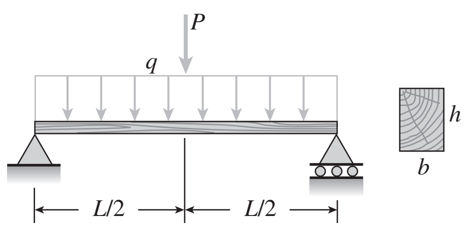
Use 8 × 10 in. beam ←

Problem 5.8-10 A simply supported wood beam of rectangular cross section and span length 1.2 m carries a concentrated load P at midspan in addition to its own weight (see figure). The cross section has width 140 mm and height 240 mm. The weight density of the wood is 5.4 kN/m^3.

Calculate the maximum permissible value of the load P if (a) the allowable bending stress is 8.5 MPa, and (b) the allowable shear stress is 0.8 MPa.



Solution 5.8-10 Simply supported wood beam



$b = 140 \text{ mm} \quad h = 240 \text{ mm} \quad A = bh = 33{,}600 \text{ mm}^2$

$$S = \frac{bh^2}{6} = 1344 \times 10^3 \text{ mm}^3$$

$\gamma = 5.4 \text{ kN/m}^3$

$L = 1.2 \text{ m} \quad q = \gamma bh = 181.44 \text{ N/m}$

(a) Allowable load P based upon bending stress

$$\sigma_{allow} = 8.5 \text{ MPa} \quad \sigma = \frac{M_{max}}{S}$$

$$M_{max} = \frac{PL}{4} + \frac{qL^2}{8} = \frac{P(1.2 \text{ m})}{4} + \frac{(181.44 \text{ N/m})(1.2 \text{ m})^2}{8}$$

$$= 0.3P + 32.66 \text{ N} \cdot \text{m} \quad (P = \text{newtons}; M = \text{N} \cdot \text{m})$$

$$M_{max} = S\sigma_{allow} = (1344 \times 10^3 \text{ mm}^3)(8.5 \text{ MPa}) = 11{,}424 \text{ N} \cdot \text{m}$$

Equate values of M_{max} and solve for P:

$0.3P + 32.66 = 11{,}424 \quad P = 37{,}970$ N

or $P = 38.0$ kN ←

(b) Allowable load P based upon shear stress

$$\tau_{allow} = 0.8 \text{ MPa} \quad \tau = \frac{3V}{2A}$$

$$V = \frac{P}{2} + \frac{qL}{2} = \frac{P}{2} + \frac{(181.44 \text{ N/m})(1.2 \text{ m})}{2}$$

$$= \frac{P}{2} + 108.86 \text{ (N)}$$

$$V = \frac{2A\tau}{3} = \frac{2}{3}(33{,}600 \text{ mm}^2)(0.8 \text{ MPa}) = 17{,}920 \text{ N}$$

Equate values of V and solve for P:

$\frac{P}{2} + 108.86 = 17{,}920 \quad P = 35{,}622$ N

or $P = 35.6$ kN ←

Note: The shear stress governs and $P_{allow} = 35.6$ kN

Problem 5.8-11 A square wood platform, 8 ft $\times$ 8 ft in area, rests on masonry walls (see figure). The deck of the platform is constructed of 2 in. nominal thickness tongue-and-groove planks (actual thickness 1.5 in.; see Appendix F) supported on two 8-ft long beams. The beams have 4 in. $\times$ 6 in. nominal dimensions (actual dimensions 3.5 in. $\times$ 5.5 in.).

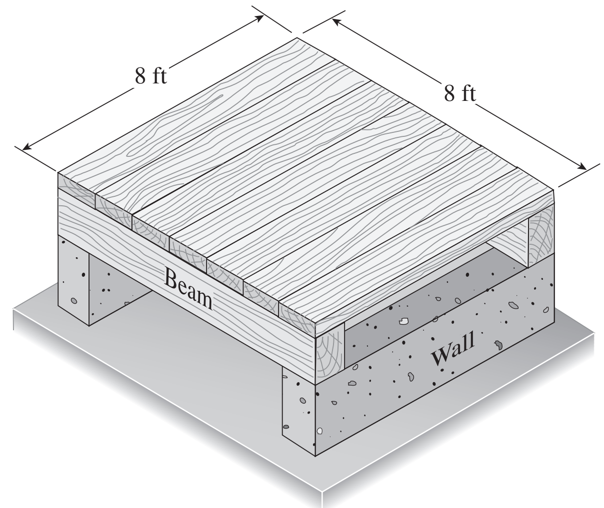
The planks are designed to support a uniformly distributed load w (lb/ft^2) acting over the entire top surface of the platform. The allowable bending stress for the planks is 2400 psi and the allowable shear stress is 100 psi. When analyzing the planks, disregard their weights and assume that their reactions are uniformly distributed over the top surfaces of the supporting beams.

(a) Determine the allowable platform load w_1 (lb/ft^2) based upon the bending stress in the planks.

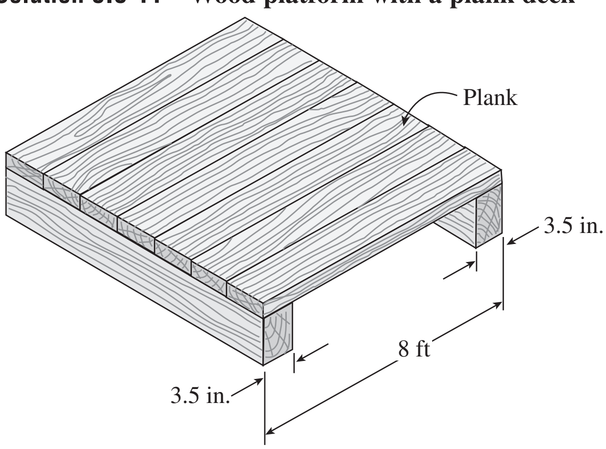
(b) Determine the allowable platform load w_2 (lb/ft^2) based upon the shear stress in the planks.

(c) Which of the preceding values becomes the allowable load w_{allow} on the platform?

(*Hints:* Use care in constructing the loading diagram for the planks, noting especially that the reactions are distributed loads instead of concentrated loads. Also, note that the maximum shear forces occur at the inside faces of the supporting beams.)



Solution 5.8-11 Wood platform with a plank deck



Platform: 8 ft $\times$ 8 ft

t = thickness of planks

= 1.5 in.

w = uniform load on the deck (lb/ft^2)

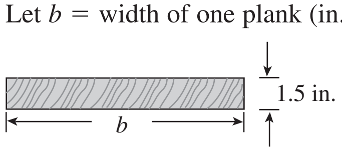
$\sigma_{allow} = 2400$ psi

$\tau_{allow} = 100$ psi

Find w_{allow} (lb/ft^2)

(a) Allowable load based upon bending stress in the planks

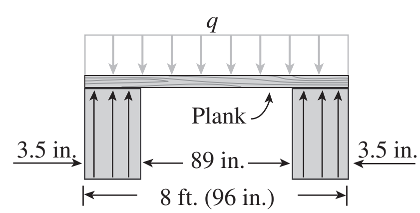
Let b = width of one plank (in.)



$A = 1.5b$ (in.^{2})

$$S = \frac{b}{6}(1.5\text{ in.})^2 = 0.375b\text{ (in.}^3)$$

Free-body diagram of one plank supported on the beams:



Load on one plank:

$$q = \left[\frac{w\text{ (lb/ft}^2)}{144\text{ in.}^2/\text{ft}^2}\right](b\text{ in.}) = \frac{wb}{144}\text{ (lb/in.)}$$

Reaction $\quad R = q\left(\frac{96\text{ in.}}{2}\right) = \left(\frac{wb}{144}\right)(48) = \frac{wb}{3}$

(R = lb; w = lb/ft^2; b = in.)

M_{max} occurs at midspan.

$$M_{max} = R\left(\frac{3.5\text{ in.}}{2} + \frac{89\text{ in.}}{2}\right) - \frac{q(48\text{ in.})^2}{2}$$

$$= \frac{wb}{3}(46.25) - \frac{wb}{144}(1152) = \frac{89}{12}wb$$

(M = lb-in.; w = lb/ft^2; b = in.)

Allowable bending moment:

$M_{allow} = \sigma_{allow}S = (2400\text{ psi})(0.375b) = 900b$ (lb-in.)

Equate M_{max} and M_{allow} and solve for w:

$\frac{89}{12}wb = 900b \quad w_1 = 121$ lb/ft^2 ⟵

(Continued)

(b) ALLOWABLE LOAD BASED UPON SHEAR STRESS IN THE PLANKS

See the free-body diagram in part (a).

V_{max} occurs at the inside face of the support.

$$V_{max} = q\left(\frac{89 \text{ in.}}{2}\right) = 44.5q = (44.5)\left(\frac{wb}{144}\right) = \frac{89\,wb}{288}$$

(V = lb; w = lb/ft^2; b = in.)

Allowable shear force:

$$\tau = \frac{3V}{2A} \quad V_{allow} = \frac{2A\tau_{allow}}{3} = \frac{2(1.5b)(100 \text{ psi})}{3} = 100b \text{ (lb)}$$

EQUATE V_{max} AND V_{allow} AND SOLVE FOR w:

$$\frac{89wb}{288} = 100b \quad w_2 = 324 \text{ lb/ft}^2 \longleftarrow$$

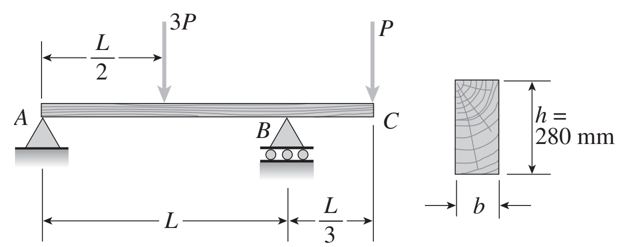
(c) ALLOWABLE LOAD

Bending stress governs. $w_{allow} = 121$ lb/ft^2 $\longleftarrow$

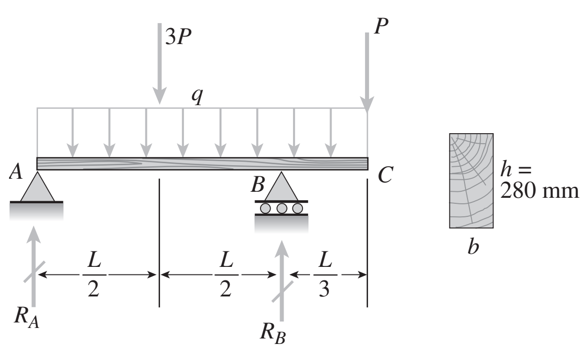
Problem 5.8-12 A wood beam ABC with simple supports at A and B and an overhang BC has height $h = 280$ mm (see figure). The length of the main span of the beam is $L = 3.6$ m and the length of the overhang is $L/3 = 1.2$ m. The beam supports a concentrated load $3P = 15$ kN at the midpoint of the main span and a load $P = 5$ kN at the free end of the overhang. The wood has weight density $\gamma = 5.5$ kN/m^3.

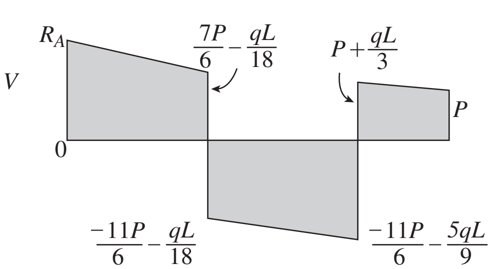
(a) Determine the required width b of the beam based upon an allowable bending stress of 8.2 MPa.

(b) Determine the required width based upon an allowable shear stress of 0.7 MPa.



Solution 5.8-12 **Rectangular beam with an overhang**



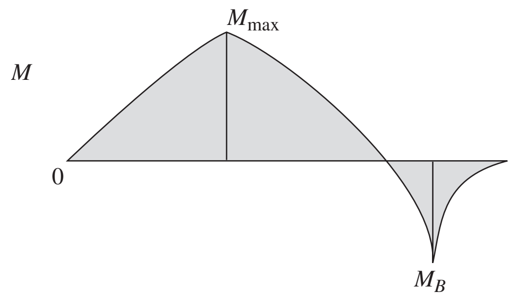


$L = 3.6$ m

$P = 5$ kN

$\gamma = 5.5$ kN/m^3 (for the wood)

$q = \gamma bh$



FIND b

$$R_A = \frac{7P}{6} + \frac{4qL}{9}$$

$$R_B = \frac{17P}{6} + \frac{8qL}{9}$$

$$V_{max} = \frac{11P}{6} + \frac{5qL}{9}$$

$$M_{max} = \frac{7PL}{12} + \frac{7qL^2}{72} \qquad M_B = -\frac{PL}{3} - \frac{qL^2}{18}$$

(a) REQUIRED WIDTH b BASED UPON BENDING STRESS

$$M_{max} = \frac{7PL}{12} + \frac{7qL^2}{72} = \frac{7}{12}(5000\text{ N})(3.6\text{ m}) + \frac{7}{72}(\gamma bh)(3.6\text{ m})^2$$

$$= 10{,}500\text{ N}\cdot\text{m} + \frac{7}{72}(5500\text{ N/m}^3)(b) \times (0.280\text{ m})(3.6\text{ m})^2$$

$$= 10{,}500 + 1940.4b \quad (b = \text{meters}) \quad (M = \text{newton-meters})$$

$$\sigma = \frac{M_{max}}{S} = \frac{6M_{max}}{bh^2} \qquad \sigma_{allow} = 8.2\text{ MPa}$$

$$M_{max} = \frac{bh^2\sigma_{allow}}{6} = \frac{b}{6}(0.280\text{ m})^2(8.2 \times 10^6\text{ Pa})$$

$$= 107{,}150b$$

EQUATE MOMENTS AND SOLVE FOR b:

$10{,}500 + 1940.4b = 107{,}150b$

$b = 0.0998\text{ m} = 99.8\text{ mm}$ ←

(b) REQUIRED WIDTH b BASED UPON SHEAR STRESS

$$V_{max} = \frac{11P}{6} + \frac{5qL}{9}$$

$$= \frac{11}{6}(5000\text{ N}) + \frac{5}{9}(\gamma bh)(3.6\text{ m})$$

$$= 9167\text{ N} + \frac{5}{9}(5500\text{ N/m}^3)(b)(0.280\text{ m})(3.6\text{ m})$$

$$= 9167 + 3080b \quad (b = \text{meters})$$

$$\tau = \frac{3V_{max}}{2A} = \frac{3V_{max}}{2bh} \qquad (V = \text{newtons})$$

$$\tau_{allow} = 0.7\text{ MPa}$$

$$V_{max} = \frac{2bh\tau_{allow}}{3} = \frac{2b}{3}(0.280\text{ m})(0.7 \times 10^6\text{ N/m}^2)$$

$$= 130{,}670b$$

EQUATE SHEAR FORCES AND SOLVE FOR b:

$9167 + 3080b = 130{,}670b$

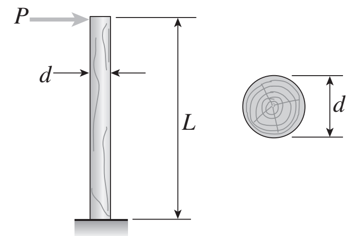
$b = 0.0718\text{ m} = 71.8\text{ mm}$ ←

NOTE: Bending stress governs. $b = 99.8$ mm

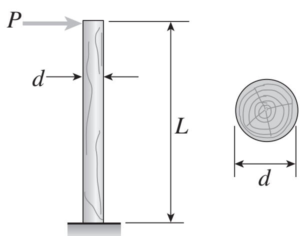
Shear Stresses in Circular Beams

Problem 5.9-1 A wood pole of solid circular cross section (d = diameter) is subjected to a horizontal force $P = 450$ lb (see figure). The length of the pole is $L = 6$ ft, and the allowable stresses in the wood are 1900 psi in bending and 120 psi in shear.

Determine the minimum required diameter of the pole based upon (a) the allowable bending stress, and (b) the allowable shear stress.



Solution 5.9-1 **Wood pole of circular cross section**



$P = 450$ lb $\quad L = 6$ ft $= 72$ in.
$\sigma_{allow} = 1900$ psi
$\tau_{allow} = 120$ psi
Find diameter d

(a) BASED UPON BENDING STRESS

$M_{max} = PL = (450 \text{ lb})(72 \text{ in.}) = 32{,}400$ lb-in.

$$\sigma = \frac{M}{S} = \frac{32M}{\pi d^3} \qquad d^3 = \frac{32M_{max}}{\pi \sigma_{allow}} = 173.7 \text{ in.}^3$$

$d_{min} = 5.58$ in. ⟵

(b) BASED UPON SHEAR STRESS

$V_{max} = 450$ lb

$$\tau = \frac{4V}{3A} = \frac{16V}{3\pi d^2} \qquad d^2 = \frac{16V_{max}}{3\pi \tau_{allow}} = 6.366 \text{ in.}^2$$

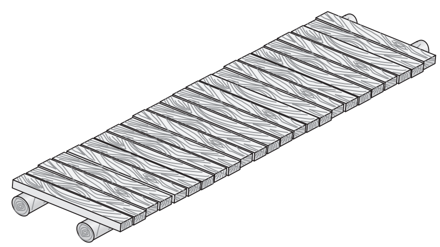
$d_{min} = 2.52$ in. ⟵

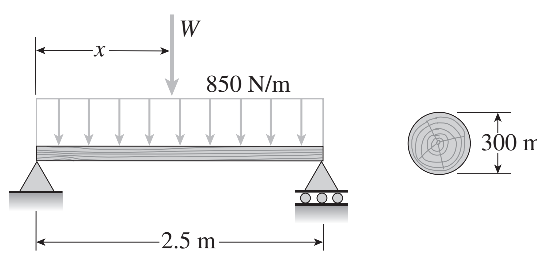
(Bending stress governs.)

Problem 5.9-2 A simple log bridge in a remote area consists of two parallel logs with planks across them (see figure). The logs are Douglas fir with average diameter 300 mm. A truck moves slowly across the bridge, which spans 2.5 m. Assume that the weight of the truck is equally distributed between the two logs.

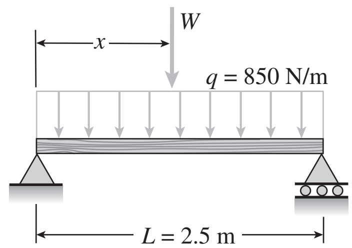
Because the wheelbase of the truck is greater than 2.5 m, only one set of wheels is on the bridge at a time. Thus, the wheel load on one log is equivalent to a concentrated load W acting at any position along the span. In addition, the weight of one log and the planks it supports is equivalent to a uniform load of 850 N/m acting on the log.

Determine the maximum permissible wheel load W based upon (a) an allowable bending stress of 7.0 MPa, and (b) an allowable shear stress of 0.75 MPa.





Solution 5.9-2 **Log bridge**



Diameter $d = 300$ mm
$\sigma_{allow} = 7.0$ MPa
$\tau_{allow} = 0.75$ MPa
Find allowable load W

(a) BASED UPON BENDING STRESS
Maximum moment occurs when wheel is at midspan ($x = L/2$).

$$M_{max} = \frac{WL}{4} + \frac{qL^2}{8} = \frac{W}{4}(2.5\text{ m}) + \frac{1}{8}(850\text{ N/m})(2.5\text{ m})^2$$
$$= 0.625W + 664.1\ (\text{N} \cdot \text{m}) \qquad (W = \text{newtons})$$

$$S = \frac{\pi d^3}{32} = 2.651 \times 10^{-3}\,\text{m}^3$$
$$M_{max} = S\sigma_{allow} = (2.651 \times 10^{-3}\text{ m}^3)(7.0\text{ MPa})$$
$$= 18{,}560\text{ N} \cdot \text{m}$$
$\therefore 0.625W + 664.1 = 18{,}560$
$W = 28{,}600\text{ N} = 28.6\text{ kN}$ ←

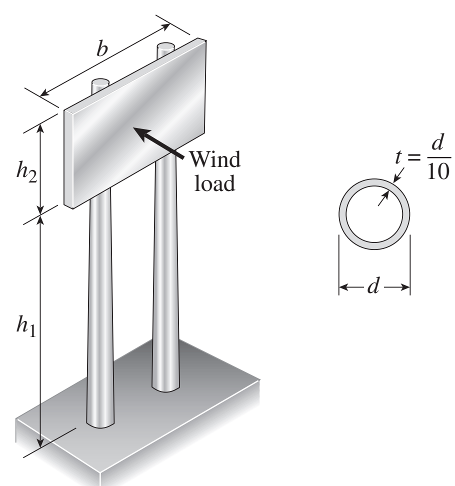
(b) BASED UPON SHEAR STRESS
Maximum shear force occurs when wheel is adjacent to support ($x = 0$).

$$V_{max} = W + \frac{qL}{2} = W + \frac{1}{2}(850\text{ N/m})(2.5\text{ m})$$
$$= W + 1062.5\text{ N} \qquad (W = \text{newtons})$$
$$A = \frac{\pi d^2}{4} = 0.070686\text{ m}^2$$
$$\tau_{max} = \frac{4V_{max}}{3A}$$
$$V_{max} = \frac{3A\,\tau_{allow}}{4} = \frac{3}{4}(0.070686\text{ m}^2)(0.75\text{ MPa})$$
$$= 39{,}760\text{ N}$$
$\therefore W + 1062.5\text{ N} = 39{,}760\text{ N}$
$W = 38{,}700\text{ N} = 38.7\text{ kN}$ ←

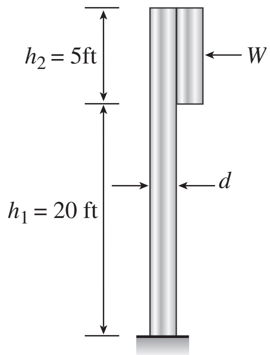
Problem 5.9-3 A sign for an automobile service station is supported by two aluminum poles of hollow circular cross section, as shown in the figure. The poles are being designed to resist a wind pressure of 75 lb/ft^2 against the full area of the sign. The dimensions of the poles and sign are $h_1 = 20$ ft, $h_2 = 5$ ft, and $b = 10$ ft. To prevent buckling of the walls of the poles, the thickness t is specified as one-tenth the outside diameter d.

(a) Determine the minimum required diameter of the poles based upon an allowable bending stress of 7500 psi in the aluminum.

(b) Determine the minimum required diameter based upon an allowable shear stress of 2000 psi.



Solution 5.9-3 **Wind load on a sign**



b = width of sign

$b = 10$ ft

$P = 75 \text{ lb/ft}^2$

$\sigma_{\text{allow}} = 7500$ psi

$\tau_{\text{allow}} = 2000$ psi

d = diameter W = wind force on one pole

$$t = \frac{d}{10} \qquad W = ph_2\left(\frac{b}{2}\right) = 1875 \text{ lb}$$

(a) Required diameter based upon bending stress

$$M_{\max} = W\left(h_1 + \frac{h_2}{2}\right) = 506{,}250 \text{ lb-in.}$$

$$I = \frac{\pi}{64}(d_2^4 - d_2^4) \quad d_2 = d \quad d_1 = d - 2t = \frac{4}{5}d$$

$$I = \frac{\pi}{64}\left[d^4 - \left(\frac{4d}{5}\right)^4\right] = \frac{\pi d^4}{64}\left(\frac{369}{625}\right) = \frac{369\pi d^4}{40{,}000} \text{ (in.}^4)$$

$$c = \frac{d}{2} \qquad (d = \text{inches})$$

$$\sigma = \frac{Mc}{I} = \frac{M(d/2)}{369\pi d^4/40{,}000} = \frac{17.253M}{d^3}$$

$$d^3 = \frac{17.253M_{\max}}{\sigma_{\text{allow}}} = \frac{(17.253)(506{,}250 \text{ lb-in.})}{7500 \text{ psi}}$$

$$= 1164.6 \text{ in.}^3 \qquad d = 10.52 \text{ in.} \longleftarrow$$

(b) Required diameter based upon shear stress

$$V_{\max} = W = 1875 \text{ lb}$$

$$\tau = \frac{4V}{3A}\left(\frac{r_2^2 + r_2r_1 + r_1^2}{r_2^2 + r_1^2}\right) \qquad r_2 = \frac{d}{2}$$

$$r_1 = \frac{d}{2} - t = \frac{d}{2} - \frac{d}{10} = \frac{2d}{5}$$

$$\frac{r_2^2 + r_2r_1 + r_1^2}{r_2^2 + r_1^2} = \frac{\left(\frac{d}{2}\right)^2 + \left(\frac{d}{2}\right)\left(\frac{2d}{5}\right) + \left(\frac{2d}{5}\right)^2}{\left(\frac{d}{2}\right)^2 + \left(\frac{2d}{5}\right)^2} = \frac{61}{41}$$

$$A = \frac{\pi}{4}(d_2^2 - d_1^2) = \frac{\pi}{4}\left[d^2 - \left(\frac{4d}{5}\right)^2\right] = \frac{9\pi d^2}{100}$$

$$\tau = \frac{4V}{3}\left(\frac{61}{41}\right)\left(\frac{100}{9\pi d^2}\right) = 7.0160\frac{V}{d^2}$$

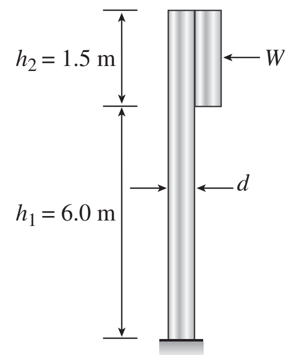
$$d^2 = \frac{7.0160\, V_{\max}}{\tau_{\text{allow}}} = \frac{(7.0160)(1875 \text{ lb})}{2000 \text{ psi}} = 6.5775 \text{ in.}^2$$

$$d = 2.56 \text{ in.} \longleftarrow$$

(Bending stress governs.)

Problem 5.9-4 Solve the preceding problem for a sign and poles having the following dimensions: $h_1 = 6.0$ m, $h_2 = 1.5$ m, $b = 3.0$ m, and $t = d/10$. The design wind pressure is 3.6 kPa, and the allowable stresses in the aluminum are 50 MPa in bending and 14 MPa in shear.

Solution 5.9-4 Wind load on a sign



b = width of sign

$b = 3.0$ m

$p = 3.6$ kPa

$\sigma_{allow} = 50$ MPa

$\tau_{allow} = 16$ MPa

d = diameter $\quad W$ = wind force on one pole

$$t = \frac{d}{10} \qquad W = ph_2\left(\frac{b}{2}\right) = 8.1 \text{ kN}$$

(a) Required diameter based upon bending stress

$$M_{max} = W\left(h_1 + \frac{h_2}{2}\right) = 54.675 \text{ kN} \cdot \text{m}$$

$$\sigma = \frac{Mc}{I} \quad I = \frac{\pi}{64}(d_2^4 - d_1^4) \quad d_2 = d \quad d_1 = d - 2t = \frac{4}{5}d$$

$$I = \frac{\pi}{64}\left[d^4 - \left(\frac{4d}{5}\right)^4\right] = \frac{\pi d^4}{64}\left(\frac{369}{625}\right) = \frac{369\pi d^4}{40{,}000} \ (\text{m}^4)$$

$$c = \frac{d}{2} \qquad (d = \text{meters})$$

$$\sigma = \frac{Mc}{I} = \frac{M(d/2)}{369\pi d^4/40{,}000} = \frac{17.253M}{d^3}$$

$$d^3 = \frac{17.253M_{max}}{\sigma_{allow}} = \frac{(17.253)(54.675 \text{ kN} \cdot \text{m})}{50 \text{ MPa}}$$

$$= 0.018866 \text{ m}^3$$

$d = 0.266$ m $= 266$ mm $\longleftarrow$

(b) Required diameter based upon shear stress

$$V_{max} = W = 8.1 \text{ kN}$$

$$\tau = \frac{4V}{3A}\left(\frac{r_2^2 + r_1 r_2 + r_1^2}{r_2^2 + r_1^2}\right) \qquad r_2 = \frac{d}{2}$$

$$r_1 = \frac{d}{2} - t = \frac{d}{2} - \frac{d}{10} = \frac{2d}{5}$$

$$\frac{r_2^2 + r_1 r_2 + r_1^2}{r_2^2 + r_1^2} = \frac{\left(\frac{d}{2}\right)^2 + \left(\frac{d}{2}\right)\left(\frac{2d}{5}\right) + \left(\frac{2d}{5}\right)^2}{\left(\frac{d}{2}\right)^2 + \left(\frac{2d}{5}\right)^2} = \frac{61}{41}$$

$$A = \frac{\pi}{4}(d_2^2 + d_1^2) = \frac{\pi}{4}\left[d^2 - \left(\frac{4d}{5}\right)^2\right] = \frac{9\pi d^2}{100}$$

$$\tau = \frac{4V}{3}\left(\frac{61}{41}\right)\left(\frac{100}{9\pi d^2}\right) = 7.0160\frac{V}{d^2}$$

$$d^2 = \frac{7.0160 \ V_{max}}{\tau_{allow}} = \frac{(7.0160)(8.1 \text{ kN})}{14 \text{ MPa}}$$

$$= 0.004059 \text{ m}^2$$

$d = 0.06371$ m $= 63.7$ mm $\longleftarrow$

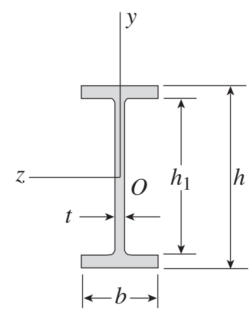
(Bending stress governs)

Shear Stresses in the Webs of Beams with Flanges

Problem 5.10-1 through 5.10-6 A wide-flange beam (see figure) having the cross section described below is subjected to a shear force V. Using the dimensions of the cross section, calculate the moment of inertia and then determine the following quantities:

(a) The maximum shear stress τ_{max} in the web.
(b) The minimum shear stress τ_{min} in the web.
(c) The average shear stress τ_{aver} (obtained by dividing the shear force by the area of the web) and the ratio τ_{max}/τ_{aver}.
(d) The shear force V_{web} carried in the web and the ratio V_{web}/V.

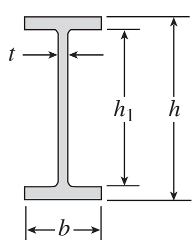
Note: Disregard the fillets at the junctions of the web and flanges and determine all quantities, including the moment of inertia, by considering the cross section to consist of three rectangles.



Probs. 5.10-1 through 5.10-6

Problem 5.10-1 Dimensions of cross section: $b = 6$ in., $t = 0.5$ in., $h = 12$ in., $h_1 = 10.5$ in., and $V = 30$ k.

Solution 5.10-1 Wide-flange beam



$b = 6.0$ in.
$t = 0.5$ in.
$h = 12.0$ in.
$h_1 = 10.5$ in.
$V = 30$ k

MOMENT OF INERTIA (Eq. 5-47)

$$I = \frac{1}{12}(bh^3 - bh_1^3 + th_1^3) = 333.4 \text{ in.}^4$$

(a) MAXIMUM SHEAR STRESS IN THE WEB (Eq. 5-48a)

$$\tau_{max} = \frac{V}{8It}(bh^2 - bh_1^2 + th_1^2) = 5795 \text{ psi} \quad \longleftarrow$$

(b) MINIMUM SHEAR STRESS IN THE WEB (Eq. 5-48b)

$$\tau_{min} = \frac{Vb}{8It}(h^2 - h_1^2) = 4555 \text{ psi} \quad \longleftarrow$$

(c) AVERAGE SHEAR STRESS IN THE WEB (Eq. 5-50)

$$\tau_{aver} = \frac{V}{th_1} = 5714 \text{ psi} \quad \longleftarrow$$

$$\frac{\tau_{max}}{\tau_{aver}} = 1.014 \quad \longleftarrow$$

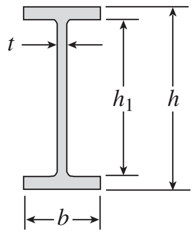
(d) SHEAR FORCE IN THE WEB (Eq. 5-49)

$$V_{web} = \frac{th_1}{3}(2\tau_{max} + \tau_{min}) = 28.25 \text{ k} \quad \longleftarrow$$

$$\frac{V_{web}}{V} = 0.942 \quad \longleftarrow$$

Problem 5.10-2 Dimensions of cross section: $b = 180$ mm, $t = 12$ mm, $h = 420$ mm, $h_1 = 380$ mm, and $V = 125$ kN.

Solution 5.10-2 Wide-flange beam



$b = 180$ mm
$t = 12$ mm
$h = 420$ mm
$h_1 = 380$ mm
$V = 125$ kN

MOMENT OF INERTIA (Eq. 5-47)

$$I = \frac{1}{12}(bh^3 - bh_1^3 + th_1^3) = 343.1 \times 10^6 \text{ mm}^4$$

(a) MAXIMUM SHEAR STRESS IN THE WEB (Eq. 5-48a)

$$\tau_{max} = \frac{V}{8It}(bh^2 - bh_1^2 + th_1^2) = 28.43 \text{ MPa} \longleftarrow$$

(b) MINIMUM SHEAR STRESS IN THE WEB (Eq. 5-48b)

$$\tau_{min} = \frac{Vb}{8It}(h^2 - h_1^2) = 21.86 \text{ MPa} \longleftarrow$$

(c) AVERAGE SHEAR STRESS IN THE WEB (Eq. 5-50)

$$\tau_{aver} = \frac{V}{th_1} = 27.41 \text{ MPa} \longleftarrow$$

$$\frac{\tau_{max}}{\tau_{aver}} = 1.037 \longleftarrow$$

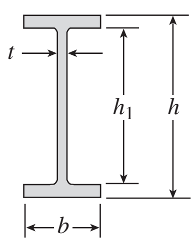
(d) SHEAR FORCE IN THE WEB (Eq. 5-49)

$$V_{web} = \frac{th_1}{3}(2\tau_{max} + \tau_{min}) = 119.7 \text{ kN} \longleftarrow$$

$$\frac{V_{web}}{V} = 0.957 \longleftarrow$$

Problem 5.10-3 Wide-flange shape, W 8 × 28 (see Table E-1, Appendix E); $V = 10$ k.

Solution 5.10-3 Wide-flange beam



W 8 × 28
$b = 6.535$ in.
$t = 0.285$ in.
$h = 8.06$ in.
$h_1 = 7.13$ in.
$V = 10$ k

MOMENT OF INERTIA (Eq. 5-47)

$$I = \frac{1}{12}(bh^3 - bh_1^3 + th_1^3) = 96.36 \text{ in.}^4$$

(a) MAXIMUM SHEAR STRESS IN THE WEB (Eq. 5-48a)

$$\tau_{max} = \frac{V}{8It}(bh^2 - bh_1^2 + th_1^2) = 4861 \text{ psi} \longleftarrow$$

(b) MINIMUM SHEAR STRESS IN THE WEB (Eq. 5-48b)

$$\tau_{min} = \frac{Vb}{8It}(h^2 - h_1^2) = 4202 \text{ psi} \longleftarrow$$

(c) AVERAGE SHEAR STRESS IN THE WEB (Eq. 5-50)

$$\tau_{aver} = \frac{V}{th_1} = 4921 \text{ psi} \longleftarrow$$

$$\frac{\tau_{max}}{\tau_{aver}} = 0.988 \longleftarrow$$

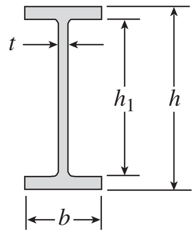
(d) Shear force in the web (Eq. 5-49)

$$V_{web} = \frac{th_1}{3}(2\tau_{max} + \tau_{min}) = 9.432 \text{ k} \longleftarrow$$

$$\frac{V_{web}}{V} = 0.943 \longleftarrow$$

Problem 5.10-4 Dimensions of cross section: $b = 220$ mm, $t = 12$ mm, $h = 600$ mm, $h_1 = 570$ mm, and $V = 200$ kN.

Solution 5.10-4 Wide-flange beam



$b = 220$ mm
$t = 12$ mm
$h = 600$ mm
$h_1 = 570$ mm
$V = 200$ kN

MOMENT OF INERTIA (Eq. 5-47)

$$I = \frac{1}{12}(bh^3 - bh_1^3 + th_1^3) = 750.0 \times 10^6 \text{ mm}^4$$

(a) MAXIMUM SHEAR STRESS IN THE WEB (Eq. 5-48a)

$$\tau_{max} = \frac{V}{8It}(bh^2 - bh_1^2 + th_1^2) = 32.28 \text{ MPa} \quad \longleftarrow$$

(b) MINIMUM SHEAR STRESS IN THE WEB (Eq. 5-48b)

$$\tau_{min} = \frac{Vb}{8It}(h^2 - h_1^2) = 21.45 \text{ MPa} \quad \longleftarrow$$

(c) AVERAGE SHEAR STRESS IN THE WEB (Eq. 5-50)

$$\tau_{aver} = \frac{V}{th_1} = 29.24 \text{ MPa} \quad \longleftarrow$$

$$\frac{\tau_{max}}{\tau_{aver}} = 1.104 \quad \longleftarrow$$

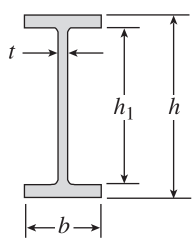
(d) SHEAR FORCE IN THE WEB (Eq. 5-49)

$$V_{web} = \frac{th_1}{3}(2\tau_{max} + \tau_{min}) = 196.1 \text{ kN} \quad \longleftarrow$$

$$\frac{V_{web}}{V} = 0.981 \quad \longleftarrow$$

Problem 5.10-5 Wide-flange shape, W 18 × 71 (see Table E-1, Appendix E); $V = 21$ k.

Solution 5.10-5 Wide-flange beam



W 18 × 71
$b = 7.635$ in.
$t = 0.495$ in.
$h = 18.47$ in.
$h_1 = 16.85$ in.
$V = 21$ k

MOMENT OF INERTIA (Eq. 5-47)

$$I = \frac{1}{12}(bh^3 - bh_1^3 + th_1^3) = 1162 \text{ in.}^4$$

(a) MAXIMUM SHEAR STRESS IN THE WEB (Eq. 5-48a)

$$\tau_{max} = \frac{V}{8It}(bh^2 - bh_1^2 + th_1^2) = 2634 \text{ psi} \quad \longleftarrow$$

(b) MINIMUM SHEAR STRESS IN THE WEB (Eq. 5-48b)

$$\tau_{min} = \frac{Vb}{8It}(h^2 - h_1^2) = 1993 \text{ psi} \quad \longleftarrow$$

(c) AVERAGE SHEAR STRESS IN THE WEB (Eq. 5-50)

$$\tau_{aver} = \frac{V}{th_1} = 2518 \text{ psi} \quad \longleftarrow$$

$$\frac{\tau_{max}}{\tau_{aver}} = 1.046 \quad \longleftarrow$$

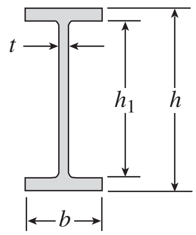
(d) SHEAR FORCE IN THE WEB (Eq. 5-49)

$$V_{web} = \frac{th_1}{3}(2\tau_{max} + \tau_{min}) = 20.19 \text{ k} \quad \longleftarrow$$

$$\frac{V_{web}}{V} = 0.961 \quad \longleftarrow$$

Problem 5.10-6 Dimensions of cross section: $b = 120$ mm, $t = 7$ mm, $h = 350$ mm, $h_1 = 330$ mm, and $V = 60$ kN.

Solution 5.10-6 Wide-flange beam



$b = 120$ mm
$t = 7$ mm
$h = 350$ mm
$h_1 = 330$ mm
$V = 60$ kN

Moment of inertia (Eq. 5-47)

$$I = \frac{1}{12}(bh^3 - bh_1^3 + th_1^3) = 90.34 \times 10^6 \text{ mm}^4$$

(a) Maximum shear stress in the web (Eq. 5-48a)

$$\tau_{max} = \frac{V}{8It}(bh^2 - bh_1^2 + th_1^2) = 28.40 \text{ MPa} \quad \longleftarrow$$

(b) Minimum shear stress in the web (Eq. 5-48b)

$$\tau_{min} = \frac{Vb}{8It}(h^2 - h_1^2) = 19.35 \text{ MPa} \quad \longleftarrow$$

(c) Average shear stress in the web (Eq. 5-50)

$$\tau_{aver} = \frac{V}{th_1} = 25.97 \text{ MPa} \quad \longleftarrow$$

$$\frac{\tau_{max}}{\tau_{aver}} = 1.093 \quad \longleftarrow$$

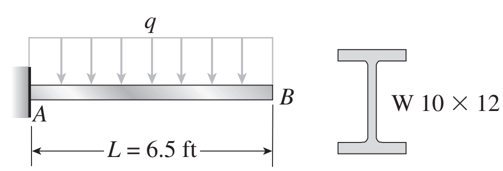
(d) Shear force in the web (Eq. 5-49)

$$V_{web} = \frac{th_1}{3}(2\tau_{max} + \tau_{min}) = 58.63 \text{ kN} \quad \longleftarrow$$

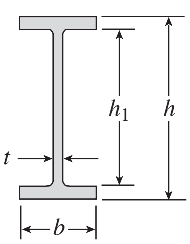
$$\frac{V_{web}}{V} = 0.977 \quad \longleftarrow$$

Problem 5.10-7 A cantilever beam AB of length $L = 6.5$ ft supports a uniform load of intensity q that includes the weight of the beam (see figure). The beam is a steel W 10 × 12 wide-flange shape (see Table E-1, Appendix E).

Calculate the maximum permissible load q based upon (a) an allowable bending stress $\sigma_{allow} = 16$ ksi, and (b) an allowable shear stress $\tau_{allow} = 8.5$ ksi. (*Note:* Obtain the moment of inertia and section modulus of the beam from Table E-1.)



Solution 5.10-7 Cantilever beam



W 10 × 12
From Table E-1:
$b = 3.960$ in.
$t = 0.190$ in.
$h = 9.87$ in.
$h_1 = 9.87 \text{ in.} - 2(0.210 \text{ in.}) = 9.45$ in.
$I = 53.8 \text{ in.}^4$
$S = 10.9 \text{ in.}^3$
$L = 6.5 \text{ ft} = 78$ in.
$\sigma_{allow} = 16{,}000$ psi
$\tau_{allow} = 8{,}500$ psi

(a) Maximum load based upon bending stress

$$M_{max} = \frac{qL^2}{2} \qquad \sigma = \frac{M_{max}}{S} \qquad q = \frac{2S\sigma}{L^2}$$

$$q_{max} = \frac{2S\sigma_{allow}}{L^2} = \frac{2(10.9 \text{ in.}^3)(16{,}000 \text{ psi})}{(78 \text{ in.})^2}$$

$$= 57.33 \text{ lb/in.} = 688 \text{ lb/ft} \quad \longleftarrow$$

(b) Maximum load based upon shear stress

$$V_{max} = qL \qquad \tau_{max} = \frac{V_{max}}{8It}(bh^2 - bh_1^2 + th_1^2) \qquad \text{(Eq. 5-48a)}$$

$$q_{max} = \frac{V_{max}}{L} = \frac{8It(\tau_{allow})}{L(bh^2 - bh_1^2 + th_1^2)}$$

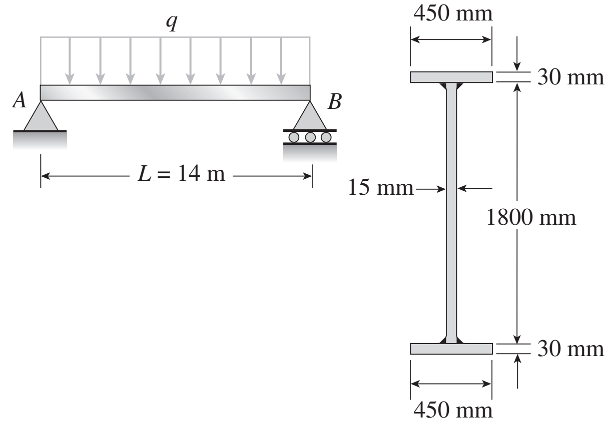
Substitute numerical values:

$$q_{max} = 181.49 \text{ lb/in.} = 2180 \text{ lb/ft} \quad \longleftarrow$$

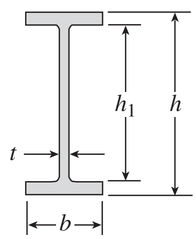
Note: Bending stress governs. $q_{allow} = 688$ lb/ft

Problem 5.10-8 A bridge girder AB on a simple span of length $L = 14$ m supports a uniform load of intensity q that includes the weight of the girder (see figure). The girder is constructed of three plates welded to form the cross section shown.

Determine the maximum permissible load q based upon (a) an allowable bending stress $\sigma_{allow} = 110$ MPa, and (b) an allowable shear stress $\tau_{allow} = 50$ MPa.



Solution 5.10-8 **Bridge girder (simple beam)**



$L = 14$ m
$b = 450$ mm $\quad t = 15$ mm
$h = 1860$ mm $\quad h_1 = 1800$ mm
$\sigma_{allow} = 110$ MPa
$\tau_{allow} = 50$ MPa
$c = h/2 = 930$ mm

Eq. (5-47): $$I = \frac{1}{12}(bh^3 - bh_1^3 + th_1^3)$$
$$= 29.897 \times 10^9 \text{ mm}^4$$
$$S = \frac{I}{c} = \frac{29.897 \times 10^9 \text{ mm}^4}{930 \text{ mm}} = 32.147 \times 10^6 \text{ mm}^3$$

(a) MAXIMUM LOAD BASED UPON BENDING STRESS

$$M_{max} = \frac{qL^2}{8} \qquad \sigma = \frac{M_{max}}{S} \qquad q = \frac{8S\sigma}{L^2}$$
$$q_{max} = \frac{8S\sigma_{allow}}{L^2} = \frac{8(32.147 \times 10^6 \text{ mm}^3)(110 \text{ MPa})}{(14 \text{ m})^2}$$
$$= 144.3 \times 10^3 \text{ N/m} = 144 \text{ kN/m} \longleftarrow$$

(b) MAXIMUM LOAD BASED UPON SHEAR STRESS

$$V_{max} = \frac{qL}{2} \qquad \tau_{max} = \frac{V_{max}}{8It}(bh^2 - bh_1^2 + th_1^2) \qquad \text{(Eq. 5-48a)}$$
$$q_{max} = \frac{2V_{max}}{L} = \frac{16It(\tau_{allow})}{L(bh^2 - bh_1^2 + th_1^2)}$$

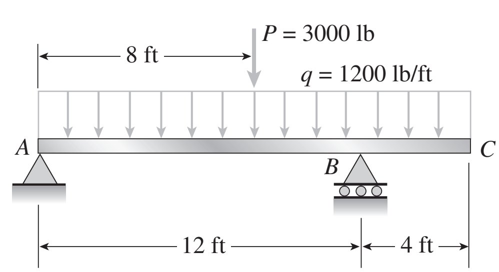
Substitute numerical values:
$q_{max} = 173.8 \times 10^3$ N/m $= 174$ kN/m $\longleftarrow$

NOTE: Bending stress governs. $\quad q_{allow} = 144$ kN/m

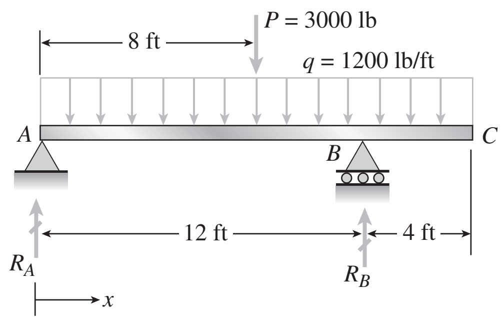
Problem 5.10-9 A simple beam with an overhang supports a uniform load of intensity $q = 1200$ lb/ft and a concentrated load $P = 3000$ lb (see figure). The uniform load includes an allowance for the weight of the beam. The allowable stresses in bending and shear are 18 ksi and 11 ksi, respectively.

Select from Table E-2, Appendix E, the lightest I-beam (S shape) that will support the given loads.

Hint: Select a beam based upon the bending stress and then calculate the maximum shear stress. If the beam is overstressed in shear, select a heavier beam and repeat.



Solution 5.10-9 **Beam with an overhang**



$\sigma_{allow} = 18$ ksi
$\tau_{allow} = 11$ ksi

Select a beam of S shape
$R_A = 7400$ lb
$R_B = 14{,}800$ lb
Maximum shear force: $V_{max} = 10{,}000$ lb at $x = 12$ ft

Maximum bending moment:
$M_{max} = 22{,}820$ lb-ft at $x = 6.167$ ft

REQUIRED SECTION MODULUS

$$S = \frac{M_{max}}{\sigma_{allow}} = \frac{(22{,}820 \text{ lb-ft})(12 \text{ in./ft})}{18{,}000 \text{ psi}} = 15.2 \text{ in.}^3$$

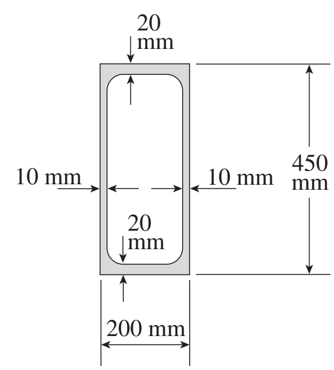
From Table E-2:
Lightest beam is S 8 × 23
$I = 64.9 \text{ in.}^4 \quad S = 16.2 \text{ in.}^3$
$b = 4.171 \text{ in.} \quad t = 0.441 \text{ in.}$
$h = 8.00 \text{ in.} \quad h_1 = 8.00 - 2(0.426) = 7.148 \text{ in.}$

MAXIMUM SHEAR STRESS (Eq. 5-48a)

$$\tau_{max} = \frac{V_{max}}{8It}(bh^2 - bh_1^2 + th_1^2)$$
$$= 3340 \text{ psi} < 11{,}000 \text{ psi} \quad \therefore \text{ ok for shear}$$

Select S 8 × 23 beam ←

Problem 5.10-10 A hollow steel box beam has the rectangular cross section shown in the figure. Determine the maximum allowable shear force V that may act on the beam if the allowable shear stress is 36 MPa.



Solution 5.10-10 **Rectangular box beam**

$\tau_{allow} = 36$ MPa
Find V_{allow}

$$\tau = \frac{VQ}{It}$$

$$V_{allow} = \frac{\tau_{allow} It}{Q}$$

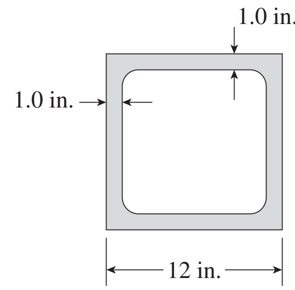
$$I = \frac{1}{12}(200)(450)^3 - \frac{1}{12}(180)(410)^3 = 484.9 \times 10^6 \text{ mm}^4$$

$t = 2(10 \text{ mm}) = 20$ mm

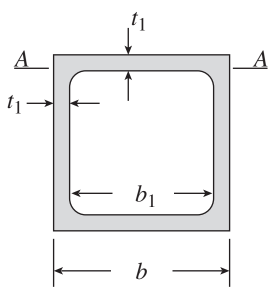
$$Q = (200)\left(\frac{450}{2}\right)\left(\frac{450}{4}\right) - (180)\left(\frac{410}{2}\right)\left(\frac{410}{4}\right)$$
$$= 1.280 \times 10^6 \text{ mm}^3$$

$$V_{allow} = \frac{\tau_{allow} It}{Q}$$
$$= \frac{(36 \text{ MPa})(484.9 \times 10^6 \text{ mm}^4)(20 \text{ mm})}{1.280 \times 10^6 \text{ mm}^3}$$
$$= 273 \text{ kN} \leftarrow$$

Problem 5.10-11 A hollow aluminum box beam has the square cross section shown in the figure. Calculate the maximum and minimum shear stresses τ_{max} and τ_{min} in the webs of the beam due to a shear force $V = 28$ k.



Solution 5.10-11 **Square box beam**



$V = 28 \text{ k} = 28{,}000 \text{ lb}$

$t_1 = 1.0$ in.

$b = 12$ in.

$b_1 = 10$ in.

$$\tau = \frac{VQ}{It} \qquad t = 2t_1 = 2.0 \text{ in.}$$

MOMENT OF INERTIA

$$I = \frac{1}{12}(b^4 - b_1^4) = 894.67 \text{ in.}^4$$

MAXIMUM SHEAR STRESS IN THE WEB (AT NEUTRAL AXIS)

$$Q = A_1\bar{y}_1 - A_2\bar{y}_2 \qquad A_1 = b\left(\frac{b}{2}\right) = \frac{b^2}{2}$$

$$A_2 = b_1\left(\frac{b_1}{2}\right) = \frac{b_1^2}{2}$$

$$\bar{y}_1 = \frac{1}{2}\left(\frac{b}{2}\right) = \frac{b}{4} \qquad \bar{y}_2 = \frac{1}{2}\left(\frac{b_1}{2}\right) = \frac{b_1}{4}$$

$$Q = \left(\frac{b^2}{2}\right)\left(\frac{b}{4}\right) - \left(\frac{b_1^2}{2}\right)\left(\frac{b_1}{4}\right) = \frac{1}{8}(b^3 - b_1^3) = 91.0 \text{ in.}^3$$

$$\tau_{max} = \frac{VQ}{It} = \frac{(28{,}000 \text{ lb})(91.0 \text{ in.}^3)}{(894.67 \text{ in.}^4)(2.0 \text{ in.})} = 1424 \text{ psi}$$

$$= 1.42 \text{ ksi} \longleftarrow$$

MINIMUM SHEAR STRESS IN THE WEB (AT LEVEL A-A)

$$Q = A\bar{y} = (bt_1)\left(\frac{b}{2} - \frac{t_1}{2}\right) = \frac{bt_1}{2}(b - t_1)$$

$$t_1 = \frac{b - b_1}{2} \qquad Q = \frac{b}{8}(b^2 - b_1^2)$$

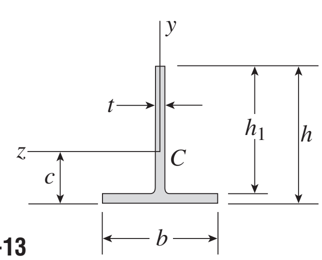
$$Q = \frac{(12 \text{ in.})}{8}[(12 \text{ in.})^2 - (10 \text{ in.})^2] = 66.0 \text{ in.}^3$$

$$\tau_{min} = \frac{VQ}{It} = \frac{(28{,}000 \text{ lb})(66.0 \text{ in.}^3)}{(894.67 \text{ in.}^4)(2.0 \text{ in.})} = 1033 \text{ psi}$$

$$= 1.03 \text{ ksi} \longleftarrow$$

Problem 5.10-12 The T-beam shown in the figure has cross-sectional dimensions as follows: $b = 220$ mm, $t = 15$ mm, $h = 300$ mm, and $h_1 = 275$ mm. The beam is subjected to a shear force $V = 60$ kN.

Determine the maximum shear stress τ_{max} in the web of the beam.



Probs. 5.10-12 and 5.10-13

Solution 5.10-12 **T-beam**

$b = 220$ mm $\quad t = 15$ mm $\quad h = 300$ mm

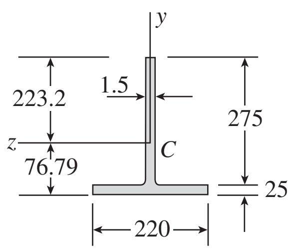
$h_1 = 275$ mm $\quad V = 60$ kN

Find τ_{max}

LOCATE NEUTRAL AXIS
(ALL DIMENSIONS IN MILLIMETERS)

$$c = \frac{\Sigma A\bar{y}}{\Sigma A} = \frac{b(h-h_1)\left(\frac{h-h_1}{2}\right) + th_1\left(h - \frac{h_1}{2}\right)}{b(h-h_1) + th_1}$$

$$= \frac{(220)(25)\left(\frac{25}{2}\right) + (15)(275)\left(300 - \frac{275}{2}\right)}{(220)(25) + (15)(275)} = 76.79 \text{ mm}$$



MOMENT OF INERTIA ABOUT THE z-AXIS

$$I_{web} = \frac{1}{3}(15)(223.2)^3 + \frac{1}{3}(15)(76.79 - 25)^3$$
$$= 56.29 \times 10^6 \text{ mm}^4$$

$$I_{flange} = \frac{1}{12}(220)(25)^3 + (220)(25)\left(76.79 - \frac{25}{2}\right)^2$$
$$= 23.02 \times 10^6 \text{ mm}^4$$

$$I = I_{web} + I_{flange} = 79.31 \times 10^6 \text{ mm}^4$$

FIRST MOMENT OF AREA ABOVE THE z AXIS

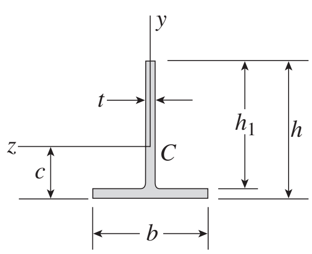
$$Q = (15)(223.2)\left(\frac{223.2}{2}\right)$$
$$= 373.6 \times 10^3 \text{ mm}^3$$

MAXIMUM SHEAR STRESS

$$\tau_{max} = \frac{VQ}{It} = \frac{(60 \text{ kN})(373.6 \times 10^3 \text{ mm}^3)}{(79.31 \times 10^6 \text{ mm}^4)(15 \text{ mm})}$$
$$= 18.8 \text{ MPa} \longleftarrow$$

Problem 5.10-13 Calculate the maximum shear stress τ_{max} in the web of the T-beam shown in the figure if $b = 10$ in., $t = 0.6$ in., $h = 8$ in., $h_1 = 7$ in., and the shear force $V = 5000$ lb.

Solution 5.10-13 **T-beam**



$b = 10$ in.

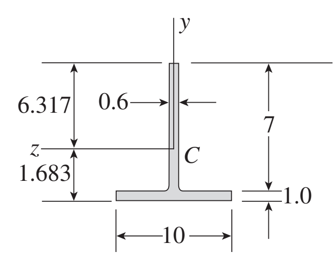
$t = 0.6$ in.

$h = 8$ in.

$h_1 = 7$ in.

$V = 5000$ lb

Find τ_{max}



LOCATE NEUTRAL AXIS
(ALL DIMENSIONS IN INCHES)

$$c = \frac{\Sigma A\bar{y}}{\Sigma A} = \frac{b(h-h_1)\left(\frac{h-h_1}{2}\right) + th_1\left(h - \frac{h_1}{2}\right)}{b(h-h_1) + th_1}$$

$$= \frac{(10)(1)(0.5) + (0.6)(7)(4.5)}{10(1) + (0.6)(7)} = 1.683 \text{ in.}$$

MOMENT OF INERTIA ABOUT THE z-AXIS

$$I_{web} = \frac{1}{3}(0.6)(6.317)^3 + \frac{1}{3}(0.6)(1.683 - 1.0)^3$$
$$= 50.48 \text{ in.}^4$$

$$I_{flange} = \frac{1}{12}(10)(1.0)^3 + (10)(1.0)(1.683 - 0.5)^2$$
$$= 14.83 \text{ in.}^4$$

$$I = I_{web} + I_{flange} = 65.31 \text{ in}^4.$$

FIRST MOMENT OF AREA ABOVE THE z AXIS

$$Q = (0.6)(6.317)\left(\frac{6.317}{2}\right) = 11.97 \text{ in.}^3$$

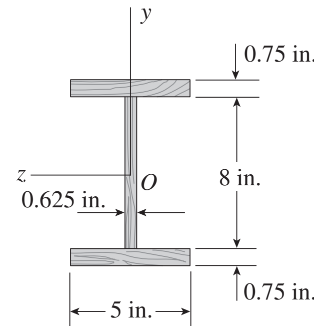
MAXIMUM SHEAR STRESS

$$\tau_{max} = \frac{VQ}{It} = \frac{(5000 \text{ lb})(11.97 \text{ in.}^3)}{(65.31 \text{ in.}^4)(0.6 \text{ in.})} = 1530 \text{ psi} \longleftarrow$$

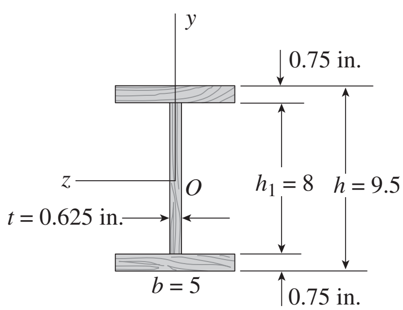
Built-Up Beams

Problem 5.11-1 A prefabricated wood I-beam serving as a floor joist has the cross section shown in the figure. The allowable load in shear for the glued joints between the web and the flanges is 65 lb/in. in the longitudinal direction.

Determine the maximum allowable shear force V_{max} for the beam.



Solution 5.11-1 **Wood I-beam**



All dimensions in inches.

Find V_{max} based upon shear in the glued joints.

Allowable load in shear for the glued joints is 65 lb/in.

$\therefore \quad f_{allow} = 65$ lb/in.

$$f = \frac{VQ}{I} \qquad V_{max} = \frac{f_{allow} I}{Q}$$

$$I = \frac{bh^3}{12} - \frac{(b-t)h_1^3}{12} = \frac{1}{12}(5)(9.5)^3 - \frac{1}{12}(4.375)(8)^3$$

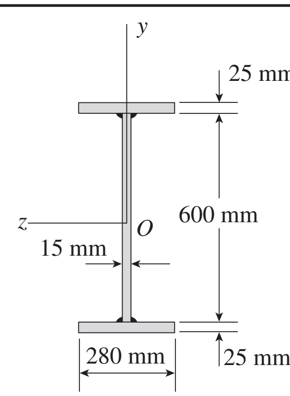
$$= 170.57 \text{ in.}^4$$

$$Q = Q_{flange} = A_f d_f = (5)(0.75)(4.375) = 16.406 \text{ in.}^3$$

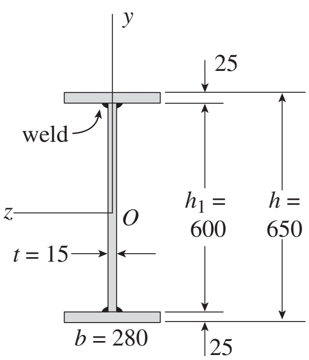
$$V_{max} = \frac{f_{allow} I}{Q} = \frac{(65 \text{ lb/in.})(170.57 \text{ in.}^4)}{16.406 \text{ in.}^3} = 676 \text{ lb} \quad \longleftarrow$$

Problem 5.11-2 A welded steel girder having the cross section shown in the figure is fabricated of two 280 mm × 25 mm flange plates and a 600 mm × 15 mm web plate. The plates are joined by four fillet welds that run continuously for the length of the girder. Each weld has an allowable load in shear of 900 kN/m.

Calculate the maximum allowable shear force V_{max} for the girder.



Solution 5.11-2 **Welded steel girder**



All dimensions in millimeters.

Allowable load in shear for one weld is 900 kN/m.

$\therefore\ f_{\text{allow}} = 2(900) = 1800$ kN/m

$$f = \frac{VQ}{I} \qquad V_{\max} = \frac{f_{\text{allow}} I}{Q}$$

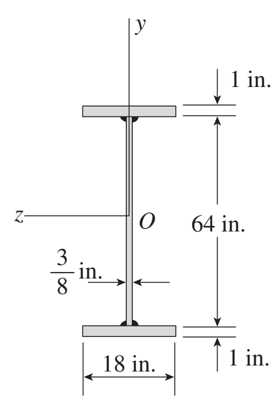
$$I = \frac{bh^3}{12} - \frac{(b-t)h_1^3}{12} = \frac{1}{12}(280)(650)^3 - \frac{1}{12}(265)(600)^3$$
$$= 1638 \times 10^6 \text{ mm}^4$$

$$Q = Q_{\text{flange}} = A_f d_f = (280)(25)(312.5) = 2.1875 \times 10^6 \text{ mm}^3$$

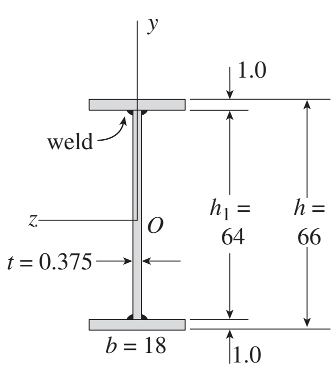
$$V_{\max} = \frac{f_{\text{allow}} I}{Q} = \frac{(1800 \text{ kN/m})(1638 \times 10^6 \text{ mm}^4)}{2.1875 \times 10^6 \text{ mm}^3}$$
$$= 1.35 \text{ MN} \quad \longleftarrow$$

Problem 5.11-3 A welded steel girder having the cross section shown in the figure is fabricated of two 18 in. $\times$ 1 in. flange plates and a 64 in. $\times$ 3/8 in. web plate. The plates are joined by four longitudinal fillet welds that run continuously throughout the length of the girder.

If the girder is subjected to a shear force of 300 kips, what force F (per inch of length of weld) must be resisted by each weld?



Solution 5.11-3 **Welded steel girder**



All dimensions in inches.

$V = 300$ k
$F =$ force per inch of length of one weld

$$f = \text{shear flow} \qquad f = 2F = \frac{VQ}{I} \qquad F = \frac{VQ}{2I}$$

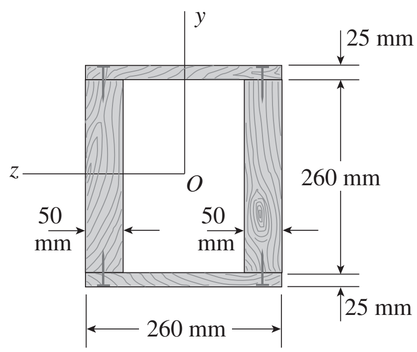
$$I = \frac{bh^3}{12} - \frac{(b-t)h_1^3}{12} = \frac{1}{12}(18)(66)^3 - \frac{1}{12}(17.625)(64)^3$$
$$= 46{,}220 \text{ in.}^4$$

$$Q = Q_{\text{flange}} = A_f d_f = (18)(1.0)(32.5) = 585 \text{ in.}^3$$

$$F = \frac{VQ}{2I} = \frac{(300 \text{ k})(585 \text{ in.}^3)}{2(46{,}220 \text{ in.}^4)} = 1900 \text{ lb/in.} \quad \longleftarrow$$

Problem 5.11-4 A box beam of wood is constructed of two 260 mm × 50 mm boards and two 260 mm × 25 mm boards (see figure). The boards are nailed at a longitudinal spacing $s = 100$ mm.

If each nail has an allowable shear force $F = 1200$ N, what is the maximum allowable shear force V_{max}?



Solution 5.11-4 Wood box beam

All dimensions in millimeters.

$b = 260 \quad b_1 = 260 - 2(50) = 160$
$h = 310 \quad h_1 = 260$

s = nail spacing = 100 mm
F = allowable shear force for one nail = 1200 N
f = shear flow between one flange and both webs

$$f_{\text{allow}} = \frac{2F}{s} = \frac{2(1200\text{ N})}{100\text{ mm}} = 24\text{ kN/m}$$

$$f = \frac{VQ}{I} \qquad V_{\max} = \frac{f_{\text{allow}} I}{Q}$$

$$I = \frac{1}{12}(bh^3 - b_1 h_1^3) = 411.125 \times 10^6\text{ mm}^4$$

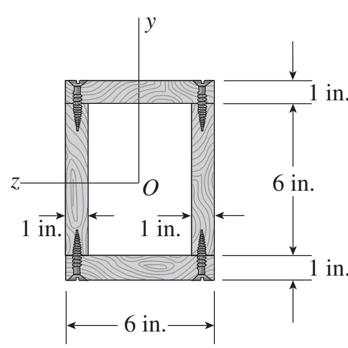
$$Q = Q_{\text{flange}} = A_f d_f = (260)(25)(142.5) = 926.25 \times 10^3\text{ mm}^3$$

$$V_{\max} = \frac{f_{\text{allow}} I}{Q} = \frac{(24\text{ kN/m})(411.125 \times 10^6\text{ mm}^4)}{926.25 \times 10^3\text{ mm}^3}$$

$$= 10.7\text{ kN} \quad \longleftarrow$$

Problem 5.11-5 A box beam constructed of four wood boards of size 6 in. × 1 in. (actual dimensions) is shown in the figure. The boards are joined by screws for which the allowable load in shear is $F = 250$ lb per screw.

Calculate the maximum permissible longitudinal spacing s_{max} of the screws if the shear force V is 1200 lb.



Solution 5.11-5 Wood box beam

All dimensions in inches.

$b = 6.0 \quad b_1 = 6.0 - 2(1.0) = 4.0$
$h = 8.0 \quad h_1 = 6.0$
F = allowable shear force for one screw = 250 lb
V = shear force = 1200 lb
s = longitudinal spacing of the screws
f = shear flow between one flange and both webs

$$f = \frac{VQ}{I} = \frac{2F}{s} \qquad \therefore\ s_{\max} = \frac{2FI}{VQ}$$

$$I = \frac{1}{12}(bh^3 - b_1 h_1^3) = 184\text{ in.}^4$$

$$Q = Q_{\text{flange}} = A_f d_f = (6.0)(1.0)(3.5) = 21\text{ in.}^3$$

$$s_{\max} = \frac{2FI}{VQ} = \frac{2(250\text{ lb})(184\text{ in.}^4)}{(1200\text{ lb})(21\text{ in.}^3)}$$

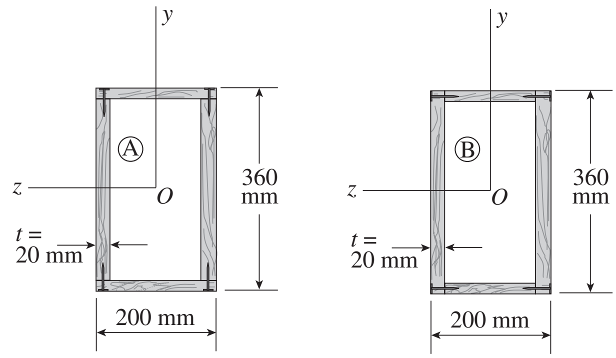
$$= 3.65\text{ in.} \quad \longleftarrow$$

Problem 5.11-6 Two wood box beams (beams A and B) have the same outside dimensions (200 mm × 360 mm) and the same thickness (t = 20 mm) throughout, as shown in the figure on the next page. Both beams are formed by nailing, with each nail having an allowable shear load of 250 N. The beams are designed for a shear force V = 3.2 kN.

(a) What is the maximum longitudinal spacing s_A for the nails in beam A?

(b) What is the maximum longitudinal spacing s_B for the nails in beam B?

(c) Which beam is more efficient in resisting the shear force?



Solution 5.11-6 **Two wood box beams**

Cross-sectional dimensions are the same.

All dimensions in millimeters.

$b = 200 \qquad b_1 = 200 - 2(20) = 160$
$h = 360 \qquad h_1 = 360 - 2(20) = 320$
$t = 20$
F = allowable load per nail = 250 N
V = shear force = 3.2 kN

$$I = \frac{1}{12}(bh^3 - b_1h_1^3) = 340.69 \times 10^6 \text{ mm}^4$$

s = longitudinal spacing of the nails
f = shear flow between one flange and both webs

$$f = \frac{2F}{s} = \frac{VQ}{I} \qquad \therefore\ s_{\max} = \frac{2FI}{VQ}$$

(a) Beam A

$$Q = A_p d_p = (bt)\left(\frac{h-t}{2}\right) = (200)(20)\left(\frac{1}{2}\right)(340)$$
$$= 680 \times 10^3 \text{ mm}^3$$

$$s_A = \frac{2FI}{VQ} = \frac{(2)(250\text{ N})(340.7 \times 10^6 \text{ mm}^4)}{(3.2\text{ kN})(680 \times 10^3 \text{ mm}^3)}$$
$$= 78.3 \text{ mm} \longleftarrow$$

(b) Beam B

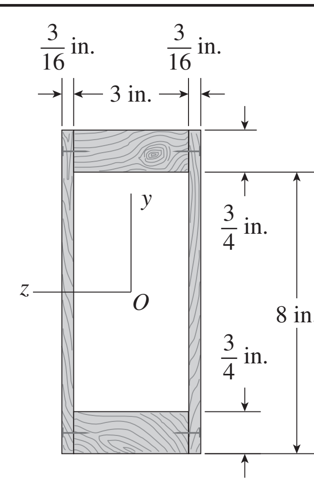
$$Q = A_f d_f = (b - 2t)(t)\left(\frac{h-t}{2}\right) = (160)(20)\frac{1}{2}(340)$$
$$= 544 \times 10^3 \text{ mm}^3$$

$$s_B = \frac{2FI}{VQ} = \frac{(2)(250\text{ N})(340.7 \times 10^6 \text{ mm}^4)}{(3.2\text{ kN})(544 \times 10^3 \text{ mm}^3)}$$
$$= 97.9 \text{ mm} \longleftarrow$$

(c) Beam B is more efficient because the shear flow on the contact surfaces is smaller and therefore fewer nails are needed. ⟵

Problem 5.11-7 A hollow wood beam with plywood webs has the cross-sectional dimensions shown in the figure. The plywood is attached to the flanges by means of small nails. Each nail has an allowable load in shear of 30 lb.

Find the maximum allowable spacing s of the nails at cross sections where the shear force V is equal to (a) 200 lb and (b) 300 lb.



Solution 5.11-7 Wood beam with plywood webs

All dimensions in inches.

$b = 3.375 \quad b_1 = 3.0$
$h = 8.0 \quad h_1 = 6.5$
F = allowable shear force for one nail = 30 lb
s = longitudinal spacing of the nails
f = shear flow between one flange and both webs

$$f = \frac{VQ}{I} = \frac{2F}{s} \qquad \therefore\ s_{\max} = \frac{2FI}{VQ}$$

$$I = \frac{1}{12}(bh^3 - b_1h_1^3) = 75.3438 \text{ in.}^4$$

$$Q = Q_{\text{flange}} = A_f d_f = (3.0)(0.75)(3.625) = 8.1563 \text{ in.}^3$$

(a) $V = 200$ lb

$$s_{\max} = \frac{2FI}{VQ} = \frac{2(30\text{ lb})(75.344\text{ in.}^4)}{(200\text{ lb})(8.1563\text{ in.}^3)} = 2.77 \text{ in.} \longleftarrow$$

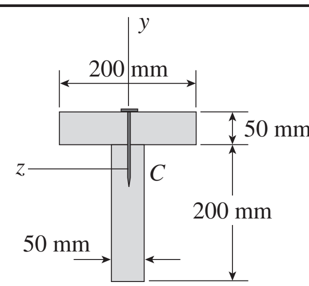
(b) $V = 300$ lb

By proportion,

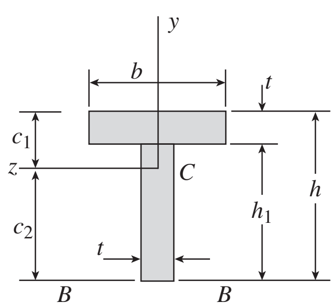
$$s_{\max} = (2.77\text{ in.})\left(\frac{200}{300}\right) = 1.85 \text{ in.} \longleftarrow$$

Problem 5.11-8 A beam of T cross section is formed by nailing together two boards having the dimensions shown in the figure.

If the total shear force V acting on the cross section is 1600 N and each nail may carry 750 N in shear, what is the maximum allowable nail spacing s?



Solution 5.11-8 T-beam (nailed)



All dimensions in millimeters.
$V = 1600$ N
F = allowable load per nail
$F = 750$ N
$b = 200$ mm $\quad t = 50$ mm
$h = 250$ mm $\quad h_1 = 200$ mm
s = nail spacing
Find $s_{\max}$

LOCATION OF NEUTRAL AXIS (z AXIS)

Use the lower edge of the cross section (line B-B) as a reference axis.

$$Q_{BB} = (h_1 t)\left(\frac{h_1}{2}\right) + (bt)\left(h - \frac{t}{2}\right)$$
$$= (200)(50)(100) + (200)(50)(225)$$
$$= 3.25 \times 10^6 \text{ mm}^3$$

$$A = bt + h_1 t = t(b + h_1) = (50)(400)$$
$$= 20 \times 10^3 \text{ mm}^2$$

$$c_2 = \frac{Q_{BB}}{A} = \frac{3.25 \times 10^6\text{ mm}^3}{20 \times 10^3\text{ mm}^2} = 162.5 \text{ mm}$$

$$c_1 = h - c_2 = 250 - 162.5 = 87.5 \text{ mm}$$

MOMENT OF INERTIA ABOUT THE NEUTRAL AXIS

$$I = \frac{1}{3}tc_2^3 + \frac{1}{3}t(h_1 - c_2)^3 + \frac{1}{12}bt^3 + bt\left(c_1 - \frac{t}{2}\right)^2$$
$$= \frac{1}{3}(50)(162.5)^3 + \frac{1}{3}(50)(37.5)^3 + \frac{1}{12}(200)(50)^3$$
$$+ (200)(50)(62.5)^2$$
$$= 113.541 \times 10^6 \text{ mm}^4$$

FIRST MOMENT OF AREA OF FLANGE

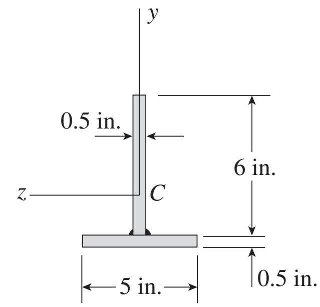
$$Q = bt\left(c_1 - \frac{t}{2}\right) = (200)(50)(62.5) = 625 \times 10^3 \text{ mm}^3$$

MAXIMUM ALLOWABLE SPACING OF NAILS

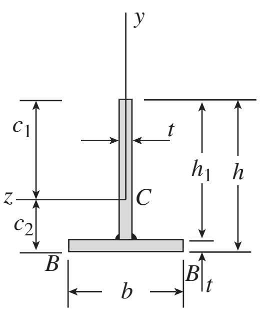
$$f = \frac{VQ}{I} = \frac{F}{s}$$

$$s_{\max} = \frac{F_{\text{allow}} I}{VQ} = \frac{(750\text{ N})(113.541 \times 10^6\text{ mm}^4)}{(1600\text{ N})(625 \times 10^3\text{ mm}^3)}$$
$$= 85.2 \text{ mm} \longleftarrow$$

Problem 5.11-9 The T-beam shown in the figure is fabricated by welding together two steel plates. If the allowable load for each weld is 2.0 k/in. in the longitudinal direction, what is the maximum allowable shear force V?



Solution 5.11-9 **T-beam (welded)**



All dimensions in inches.

F = allowable load per inch of weld
$F = 2.0$ k/in.
$b = 5.0 \qquad t = 0.5$
$h = 6.5 \qquad h_1 = 6.0$
V = shear force
Find $V_{\max}$

LOCATION OF NEUTRAL AXIS (z AXIS)

Use the lower edge of the cross section (line B-B) as a reference axis.

$$Q_{BB} = (bt)\left(\frac{t}{2}\right) + (h_1 t)\left(h - \frac{h_1}{2}\right)$$
$$= (5)(0.5)(0.25) + (6)(0.5)(3.5) = 11.25 \text{ in.}^3$$

$$A = bt + h_1 t = (5)(0.5) + (6)(0.5) = 5.5 \text{ in.}^2$$

$$c_2 = \frac{Q_{BB}}{A} = \frac{11.125 \text{ in.}^3}{5.5 \text{ in.}^2} = 2.0227 \text{ in.}$$
$$c_1 = h - c_2 = 4.4773 \text{ in.}$$

MOMENT OF INERTIA ABOUT THE NEUTRAL AXIS

$$I = \frac{1}{3}tc_1^3 + \frac{1}{3}t(c_2 - t)^3 + \frac{1}{12}bt^3 + (bt)\left(c_2 - \frac{t}{2}\right)^2$$
$$= \frac{1}{3}(0.5)(4.4773)^3 + \frac{1}{3}(0.5)(1.5227)^3 + \frac{1}{12}(5)(0.5)^3$$
$$+ (5)(0.5)(1.7727)^2 = 23.455 \text{ in.}^4$$

FIRST MOMENT OF AREA OF FLANGE

$$Q = bt\left(c_2 - \frac{t}{2}\right) = (5)(0.5)(1.7727) = 4.4318 \text{ in.}^3$$

SHEAR FLOW AT WELDS

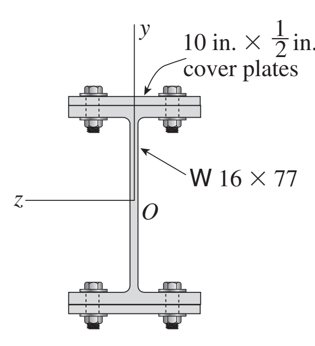
$$f = 2F = \frac{VQ}{I}$$

MAXIMUM ALLOWABLE SHEAR FORCE

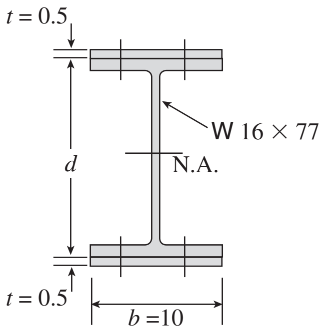
$$V_{\max} = \frac{2FI}{Q} = \frac{2(2.0 \text{ k/in.})(23.455 \text{ in.}^4)}{4.4318 \text{ in.}^3} = 21.2 \text{ k} \longleftarrow$$

Problem 5.11-10 A steel beam is built up from a W 16 × 77 wide-flange beam and two 10 in. × 1/2 in. cover plates (see figure on the next page). The allowable load in shear on each bolt is 2.1 kips.

What is the required bolt spacing s in the longitudinal direction if the shear force V = 30 kips? (*Note:* Obtain the dimensions and moment of inertia of the W shape from Table E-1.)



Solution 5.11-10 **Beam with cover plates**



All dimensions in inches.
Wide-flange beam (W 16 × 77):
$d = 16.52$ in.
$I_{beam} = 1110$ in.^{4}
Cover plates:
$b = 10$ in. $\quad t = 0.5$ in.
F = allowable load per bolt
$= 2.1$ k
V = shear force
$= 30$ k
s = spacing of bolts in the longitudinal direction
Find s_{max}

Moment of inertia about the neutral axis

$$I = I_{beam} + 2\left[\frac{1}{12}bt^3 + (bt)\left(\frac{d}{2} + \frac{t}{2}\right)^2\right]$$

$$= 1110 \text{ in.}^4 + 2\left[\frac{1}{12}(10)(0.5)^3 + (10)(0.5)(8.51)^2\right]$$

$$= 1834 \text{ in.}^4$$

First moment of area of a cover plate

$$Q = bt\left(\frac{d+t}{2}\right) = (10)(0.5)(8.51) = 42.55 \text{ in.}^3$$

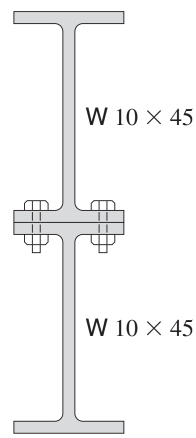
Maximum spacing of bolts

$$f = \frac{VQ}{I} = \frac{2F}{s} \qquad s = \frac{2FI}{VQ}$$

$$s_{max} = \frac{2(2.1 \text{ k})(1834 \text{ in.}^4)}{(30 \text{ k})(42.55 \text{ in.}^3)} = 6.03 \text{ in.} \longleftarrow$$

Problem 5.11-11 Two W 10 × 45 steel wide-flange beams are bolted together to form a built-up beam as shown in the figure.

What is the maximum permissible bolt spacing s if the shear force $V = 20$ kips and the allowable load in shear on each bolt is $F = 3.1$ kips? (*Note:* Obtain the dimensions and properties of the W shapes from Table E-1.)



Solution 5.11-11 **Built-up steel beam**

All dimensions in inches.
W 10 × 45: $\quad I_1 = 248$ in.^{4} $\quad d = 10.10$ in.
$A = 13.3$ in.^{2}
$V = 20$ k $\quad F = 3.1$ k

Find maximum allowable bolt spacing s_{max}.

Moment of inertia of built-up beam

$$I = 2\left[I_1 + A\left(\frac{d}{2}\right)^2\right] = 2[248 + (13.3)(5.05)^2]$$

$$= 1174.4 \text{ in.}^4$$

First moment of area of one beam

$$Q = A\left(\frac{d}{2}\right) = (13.3)(5.05) = 67.165 \text{ in.}^3$$

Maximum spacing of bolts in the longitudinal direction

$$f = \frac{VQ}{I} = \frac{2F}{s} \qquad s = \frac{2FI}{VQ}$$

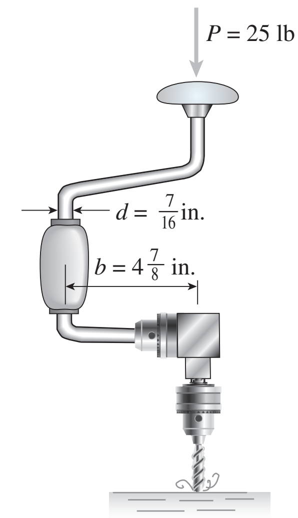
$$s_{max} = \frac{2(3.1 \text{ k})(1174.4 \text{ in.}^4)}{(20 \text{ k})(67.165 \text{ in.}^3)} = 5.42 \text{ in.} \longleftarrow$$

Beams with Axial Loads

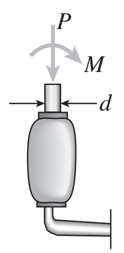
When solving the problems for Section 5.12, assume that the bending moments are not affected by the presence of lateral deflections.

Problem 5.12-1 While drilling a hole with a brace and bit, you exert a downward force $P = 25$ lb on the handle of the brace (see figure). The diameter of the crank arm is $d = 7/16$ in. and its lateral offset is $b = 4\text{-}7/8$ in.

Determine the maximum tensile and compressive stresses σ_t and σ_c, respectively, in the crank.



Solution 5.12-1 **Brace and bit**



$P = 25$ lb (compression)

$M = Pb = (25 \text{ lb})(4\ 7/8 \text{ in.})$

$= 121.9$ lb-in.

$d =$ diameter

$d = 7/16$ in.

$$A = \frac{\pi d^2}{4} = 0.1503 \text{ in.}^2$$

$$S = \frac{\pi d^3}{32} = 0.008221 \text{ in.}^3$$

MAXIMUM STRESSES

$$\sigma_t = -\frac{P}{A} + \frac{M}{S} = -\frac{25 \text{ lb}}{0.1503 \text{ in.}^2} + \frac{121.9 \text{ lb-in.}}{0.008221 \text{ in.}^3}$$

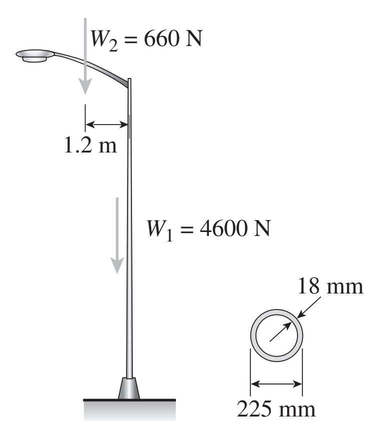
$$= -166 \text{ psi} + 14{,}828 \text{ psi} = 14{,}660 \text{ psi} \quad \longleftarrow$$

$$\sigma_c = -\frac{P}{A} - \frac{M}{S} = -166 \text{ psi} - 14{,}828 \text{ psi}$$

$$= -14{,}990 \text{ psi} \quad \longleftarrow$$

Problem 5.12-2 An aluminum pole for a street light weighs 4600 N and supports an arm that weighs 660 N (see figure). The center of gravity of the arm is 1.2 m from the axis of the pole. The outside diameter of the pole (at its base) is 225 mm and its thickness is 18 mm.

Determine the maximum tensile and compressive stresses σ_t and σ_c, respectively, in the pole (at its base) due to the weights.



Solution 5.12-2 Aluminum pole for a street light

W_1 = weight of pole = 4600 N

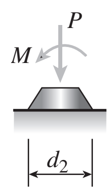
W_2 = weight of arm = 660 N

b = distance between axis of pole and center of gravity of arm = 1.2 m

d_2 = outer diameter of pole = 225 mm

d_1 = inner diameter of pole = 225 mm − 2(18 mm) = 189 mm

AT BASE OF POLE



$P = W_1 + W_2 = 5260$ N

$M = W_2 b = 792$ N · m

PROPERTIES OF THE CROSS SECTION

$$A = \frac{\pi}{4}(d_2^2 - d_1^2) = 11{,}706 \text{ mm}^2$$

$$I = \frac{\pi}{64}(d_2^4 - d_1^4) = 63.17 \times 10^6 \text{ mm}^4$$

$$c = \frac{d_2}{2} = 112.5 \text{ mm}$$

MAXIMUM STRESSES

$$\sigma_t = -\frac{P}{A} + \frac{Mc}{I} = -\frac{5260 \text{ N}}{11{,}706 \text{ mm}^2} + \frac{(792 \text{ N}\cdot\text{m})(112.5 \text{ mm})}{63.17 \times 10^6 \text{ mm}^4}$$

$$= -0.4493 \text{ MPa} + 1.4105 \text{ MPa}$$

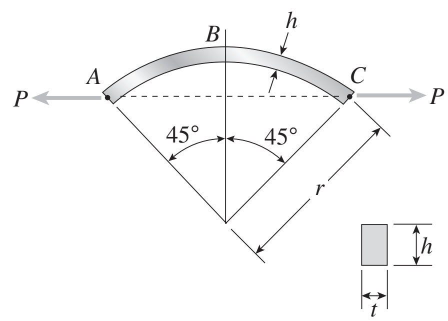
$$= 0.961 \text{ MPa} = 961 \text{ kPa} \longleftarrow$$

$$\sigma_c = -\frac{P}{A} - \frac{Mc}{I} = -0.4493 \text{ MPa} - 1.4105 \text{ MPa}$$

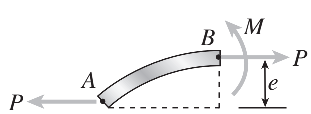
$$= -1.860 \text{ MPa} = -1860 \text{ kPa} \longleftarrow$$

Problem 5.12-3 A curved bar *ABC* having a circular axis (radius $r = 12$ in.) is loaded by forces $P = 400$ lb (see figure). The cross section of the bar is rectangular with height h and thickness t.

If the allowable tensile stress in the bar is 12,000 psi and the height $h = 1.25$ in., what is the minimum required thickness t_{min}?



Solution 5.12-3 Curved bar



r = radius of curved bar

$e = r - r\cos 45°$

$$= r\left(1 - \frac{1}{\sqrt{2}}\right)$$

$$M = Pe = \frac{Pr}{2}(2 - \sqrt{2})$$

CROSS SECTION

h = height t = thickness $A = ht$ $S = \frac{1}{6}th^2$

TENSILE STRESS

$$\sigma_t = \frac{P}{A} + \frac{M}{S} = \frac{P}{ht} + \frac{3Pr(2-\sqrt{2})}{th^2}$$

$$= \frac{P}{ht}\left[1 + 3(2-\sqrt{2})\frac{r}{h}\right]$$

MINIMUM THICKNESS

$$t_{min} = \frac{P}{h\sigma_{allow}}\left[1 + 3(2-\sqrt{2})\frac{r}{h}\right]$$

SUBSTITUTE NUMERICAL VALUES:

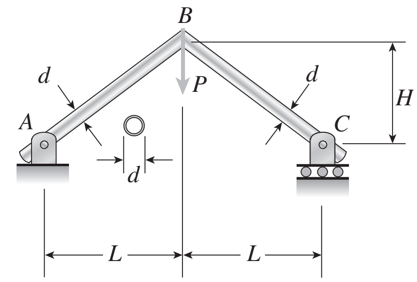
$P = 400$ lb $\sigma_{allow} = 12{,}000$ psi

$r = 12$ in. $h = 1.25$ in.

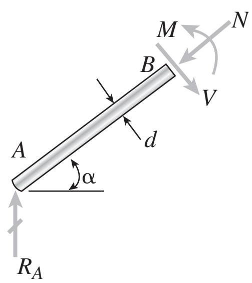
$t_{min} = 0.477$ in. $\longleftarrow$

Problem 5.12-4 A rigid frame *ABC* is formed by welding two steel pipes at *B* (see figure). Each pipe has cross-sectional area $A = 11.31 \times 10^3$ mm², moment of inertia $I = 46.37 \times 10^6$ mm⁴, and outside diameter $d = 200$ mm.

Find the maximum tensile and compressive stresses σ_t and σ_c, respectively, in the frame due to the load $P = 8.0$ kN if $L = H = 1.4$ m.



Solution 5.12-4 Rigid frame



Load P at midpoint B

REACTIONS: $R_A = R_C = \dfrac{P}{2}$

BAR AB:

$$\tan\alpha = \frac{H}{L}$$

$$\sin\alpha = \frac{H}{\sqrt{H^2 + L^2}}$$

d = diameter
$c = d/2$

AXIAL FORCE: $N = R_A \sin\alpha = \dfrac{P}{2}\sin\alpha$

BENDING MOMENT: $M = R_A L = \dfrac{PL}{2}$

TENSILE STRESS

$$\sigma_t = -\frac{N}{A} + \frac{Mc}{I} = -\frac{P\sin\alpha}{2A} + \frac{PLd}{4I}$$

SUBSTITUTE NUMERICAL VALUES:

$P = 8.0$ kN $\quad L = H = 1.4$ m $\quad \alpha = 45°$
$\sin\alpha = 1/\sqrt{2} \quad d = 200$ mm
$A = 11.31 \times 10^3$ mm² $\quad I = 46.37 \times 10^6$ mm⁴

$$\sigma_t = -\frac{(8.0\text{ kN})(1/\sqrt{2})}{2(11.31\times 10^3\text{ mm}^2)} + \frac{(8.0\text{ kN})(1.4\text{ m})(200\text{ mm})}{4(46.37\times 10^6\text{ mm}^4)}$$

$$= -0.250\text{ MPa} + 12.08\text{ MPa}$$

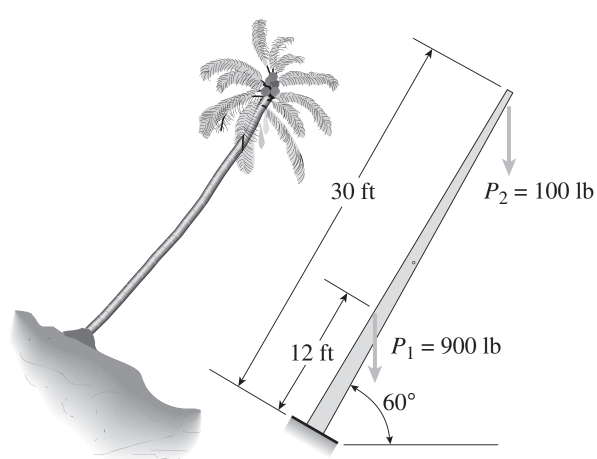
$$= 11.83\text{ MPa (tension)} \quad \longleftarrow$$

$$\sigma_c = -\frac{N}{A} - \frac{Mc}{I} = -0.250\text{ MPa} - 12.08\text{ MPa}$$

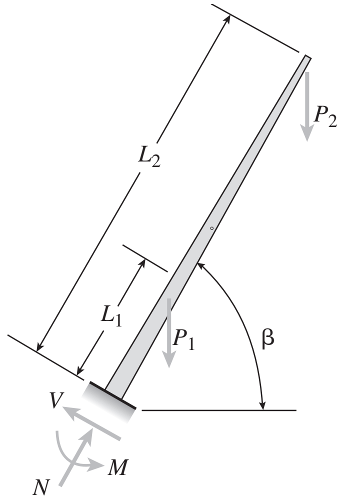
$$= -12.33\text{ MPa (compression)} \quad \longleftarrow$$

Problem 5.12-5 A palm tree weighing 1000 lb is inclined at an angle of 60° (see figure). The weight of the tree may be resolved into two resultant forces, a force $P_1 = 900$ lb acting at a point 12 ft from the base and a force $P_2 = 100$ lb acting at the top of the tree, which is 30 ft long. The diameter at the base of the tree is 14 in.

Calculate the maximum tensile and compressive stresses σ_t and σ_c, respectively, at the base of the tree due to its weight.



Solution 5.12-5 Palm tree



FREE-BODY DIAGRAM

$P_1 = 900$ lb
$P_2 = 100$ lb
$L_1 = 12$ ft $= 144$ in.
$L_2 = 30$ ft $= 360$ in.
$d = 14$ in.

$$A = \frac{\pi d^2}{4} = 153.94 \text{ in.}^2$$

$$S = \frac{\pi d^3}{32} = 269.39 \text{ in.}^3$$

$$M = P_1L_1 \cos 60^\circ + P_2L_2 \cos 60^\circ$$
$$= [(900 \text{ lb})(144 \text{ in.}) + (100 \text{ lb})(360 \text{ in.})] \cos 60^\circ$$
$$= 82{,}800 \text{ lb-in.}$$
$$N = (P_1 + P_2) \sin 60^\circ = (1000 \text{ lb}) \sin 60^\circ = 866 \text{ lb}$$

MAXIMUM TENSILE STRESS

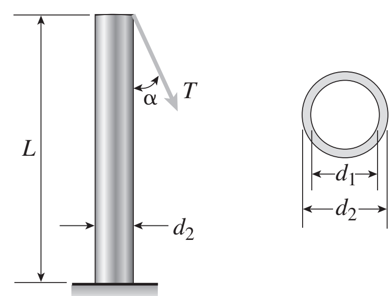
$$\sigma_t = -\frac{N}{A} + \frac{M}{S} = -\frac{866 \text{ lb}}{153.94 \text{ in.}^2} + \frac{82{,}800 \text{ lb-in.}}{269.39 \text{ in.}^3}$$
$$= -5.6 \text{ psi} + 307.4 \text{ psi} = 302 \text{ psi} \quad \longleftarrow$$

MAXIMUM COMPRESSIVE STRESS

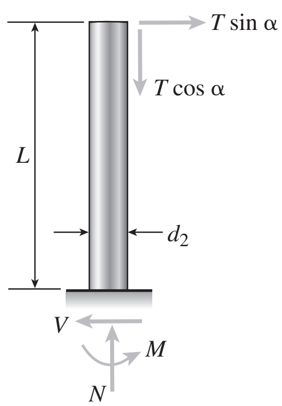
$$\sigma_c = -5.6 \text{ psi} - 307.4 \text{ psi} = -313 \text{ psi} \quad \longleftarrow$$

Problem 5.12-6 A vertical pole of aluminum is fixed at the base and pulled at the top by a cable having a tensile force T (see figure). The cable is attached at the outer surface of the pole and makes an angle $\alpha = 25°$ at the point of attachment. The pole has length $L = 2.0$ m and a hollow circular cross section with outer diameter $d_2 = 260$ mm and inner diameter $d_1 = 200$ mm.

Determine the allowable tensile force T_{allow} in the cable if the allowable compressive stress in the aluminum pole is 90 MPa.



Solution 5.12-6 Aluminum pole



$\alpha = 25^\circ$
$L = 2.0$ m
$d_2 = 260$ mm
$d_1 = 200$ mm
$(\sigma_c)_{allow} = 90$ MPa

CROSS SECTION

$$A = \frac{\pi}{4}(d_2^2 - d_1^2) = 21{,}677 \text{ mm}^2 = 21.677 \times 10^{-3} \text{ mm}^2$$

$$I = \frac{\pi}{64}(d_2^4 - d_1^4) = 145{,}778 \times 10^3 \text{ mm}^4$$
$$= 145.778 \times 10^{-6} \text{ m}^4$$

$$c = \frac{d_2}{2} = 130 \text{ mm} = 0.13 \text{ m}$$

AT THE BASE OF THE POLE

$$N = T \cos \alpha = 0.90631T \qquad (N, T = \text{newtons})$$

$$M = (T \cos \alpha)\left(\frac{d_2}{2}\right) + (T \sin \alpha)(L)$$
$$= 0.11782\, T + 0.84524\, T$$
$$= 0.96306\, T \qquad (M = \text{newton meters})$$

COMPRESSIVE STRESS

$$\sigma_c = \frac{N}{A} + \frac{Mc}{I} = \frac{0.90631T}{21.677 \times 10^{-3}\ \text{m}^2} + \frac{(0.96306T)(0.13\ \text{m})}{145.778 \times 10^{-6}\ \text{m}^4}$$

$$= 41.82\ T + 858.83\ T$$

$$= 900.64\ T \qquad (\sigma_c = \text{pascals})$$

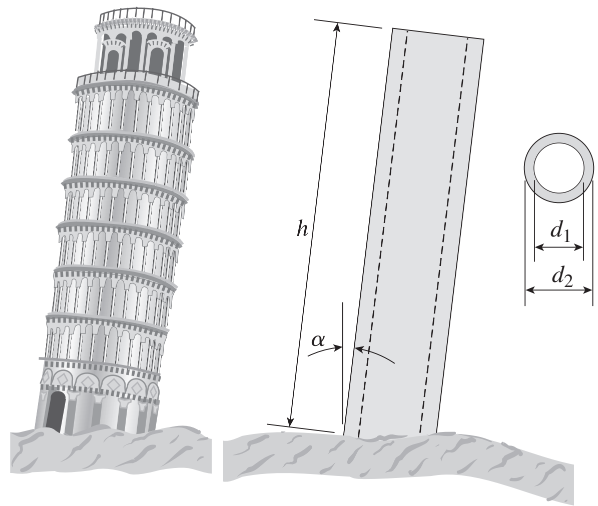
ALLOWABLE TENSILE FORCE

$$T_{\text{allow}} = \frac{(\sigma_c)_{\text{allow}}}{900.64} = \frac{90 \times 10^6\ \text{pascals}}{900.64}$$

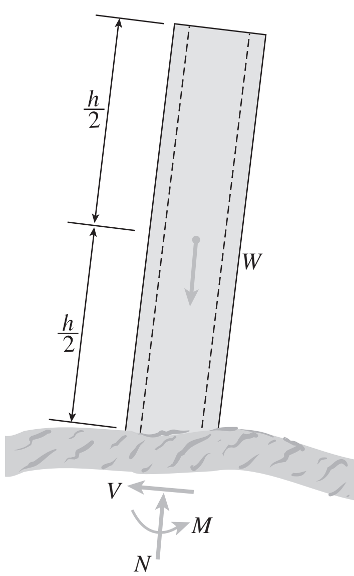
$$= 99{,}900\ \text{N} = 99.9\ \text{kN} \quad \longleftarrow$$

Problem 5.12-7 Because of foundation settlement, a circular tower is leaning at an angle α to the vertical (see figure). The structural core of the tower is a circular cylinder of height h, outer diameter d_2, and inner diameter d_1. For simplicity in the analysis, assume that the weight of the tower is uniformly distributed along the height.

Obtain a formula for the maximum permissible angle α if there is to be no tensile stress in the tower.



Solution 5.12-7 Leaning tower



W = weight of tower
α = angle of tilt

CROSS SECTION

$$A = \frac{\pi}{4}(d_2^2 - d_1^2)$$

$$I = \frac{\pi}{64}(d_2^4 - d_1^4)$$

$$= \frac{\pi}{64}(d_2^2 - d_1^2)(d_2^2 + d_1^2)$$

$$\frac{I}{A} = \frac{d_2^2 + d_1^2}{16}$$

$$c = \frac{d_2}{2}$$

AT THE BASE OF THE TOWER

$$N = W\cos\alpha \qquad M = W\left(\frac{h}{2}\right)\sin\alpha$$

TENSILE STRESS (EQUAL TO ZERO)

$$\sigma_t = -\frac{N}{A} + \frac{Mc}{I} = -\frac{W\cos\alpha}{A} + \frac{W}{I}\left(\frac{h}{2}\sin\alpha\right)\left(\frac{d_2}{2}\right) = 0$$

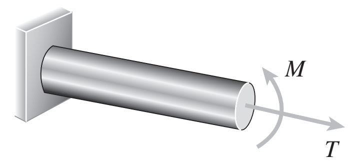
$$\therefore \frac{\cos\alpha}{A} = \frac{hd_2\sin\alpha}{4I} \qquad \tan\alpha = \frac{4I}{hd_2A} = \frac{d_2^2 + d_1^2}{4hd_2}$$

MAXIMUM ANGLE α

$$\alpha = \arctan\frac{d_2^2 + d_1^2}{4hd_2} \quad \longleftarrow$$

Problem 5.12-8 A steel bar of solid circular cross section is subjected to an axial tensile force $T = 26$ kN and a bending moment $M = 3.2$ kN · m (see figure).

Based upon an allowable stress in tension of 120 MPa, determine the required diameter d of the bar. (Disregard the weight of the bar itself.)



Solution 5.12-8 Circular bar

$T = 26$ kN $\quad M = 3.2$ kN · m

$\sigma_{allow} = 120$ MPa $\quad d =$ diameter

$$A = \frac{\pi d^2}{4} \qquad S = \frac{\pi d^3}{32}$$

Tensile stress

$$\sigma_t = \frac{T}{A} + \frac{M}{S} = \frac{4T}{\pi d^2} + \frac{32M}{\pi d^3}$$

or $\pi d^3 \sigma_{allow} - 4Td - 32M = 0$

$(\pi)(120 \text{ MPa})d^3 - 4(26 \text{ kN})d - 32(3.2 \text{ kN} \cdot \text{m}) = 0$

(d = meters)

$(120{,}000{,}000 \text{ N/m}^2)(\pi)d^3 - (104{,}000 \text{ N})d$
$\quad - 102{,}400 \text{ N} \cdot \text{m} = 0$

Simplify the equation:

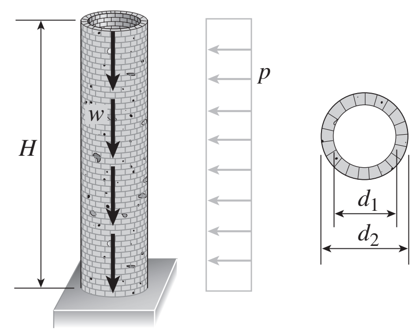
$(15{,}000\ \pi)\ d^3 - 13d - 12.8 = 0$

Solve numerically for the required diameter:

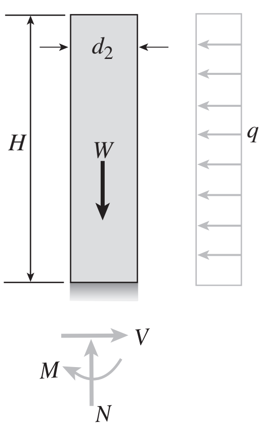
$d = 0.0662 \text{ m} = 66.2 \text{ mm}$ ←

Problem 5.12-9 A cylindrical brick chimney of height H weighs $w = 825$ lb/ft of height (see figure). The inner and outer diameters are $d_1 = 3$ ft and $d_2 = 4$ ft, respectively. The wind pressure against the side of the chimney is $p = 10$ lb/ft^2 of projected area.

Determine the maximum height H if there is to be no tension in the brickwork.



Solution 5.12-9 Brick chimney



p = wind pressure
q = intensity of load = pd_2
d_2 = outer diameter
d_1 = inner diameter
W = total weight of chimney = wH

Cross section

$$A = \frac{\pi}{4}(d_2^2 - d_1^2)$$

$$I = \frac{\pi}{64}(d_2^4 - d_1^4) = \frac{\pi}{64}(d_2^2 - d_1^2)(d_2^2 + d_1^2)$$

$$\frac{I}{A} = \frac{1}{16}(d_2^2 + d_1^2) \qquad c = \frac{d_2}{2}$$

At base of chimney

$$N = W = wH \qquad M = qH\left(\frac{H}{2}\right) = \frac{1}{2}pd_2H^2$$

Tensile stress (equal to zero)

$$\sigma_t = -\frac{N}{A} + \frac{Md_2}{2I} = 0 \qquad \text{or} \qquad \frac{M}{N} = \frac{2I}{Ad_2}$$

$$\frac{pd_2H^2}{2wH} = \frac{d_2^2 + d_1^2}{8d_2}$$

Solve for H $\qquad H = \dfrac{w(d_2^2 + d_1^2)}{4pd_2^2}$ ←

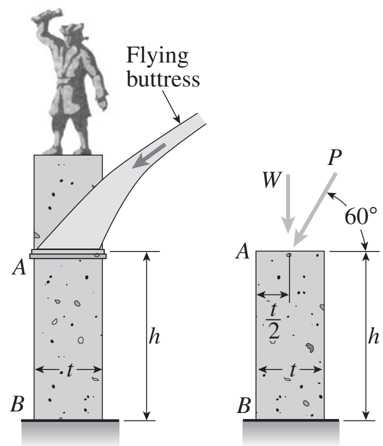
Substitute numerical values

$w = 825$ lb/ft $\quad d_2 = 4$ ft $\quad d_1 = 3$ ft $\quad p = 10$ lb/ft^2

$H_{max} = 32.2$ ft ←

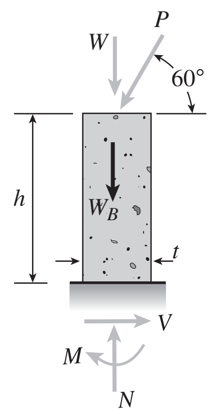
Problem 5.12-10 A flying buttress transmits a load $P = 25$ kN, acting at an angle of 60° to the horizontal, to the top of a vertical buttress AB (see figure). The vertical buttress has height $h = 5.0$ m and rectangular cross section of thickness $t = 1.5$ m and width $b = 1.0$ m (perpendicular to the plane of the figure). The stone used in the construction weighs $\gamma = 26$ kN/m^3.

What is the required weight W of the pedestal and statue above the vertical buttress (that is, above section A) to avoid any tensile stresses in the vertical buttress?



Solution 5.12-10 **Flying buttress**

FREE-BODY DIAGRAM OF VERTICAL BUTTRESS



$P = 25$ kN
$h = 5.0$ m
$t = 1.5$ m
$b =$ width of buttress perpendicular to the figure
$b = 1.0$ m
$\gamma = 26$ kN/m^3
$W_B =$ weight of vertical buttress
$= bth\gamma$
$= 195$ kN

CROSS SECTION

$$A = bt = (1.0\text{ m})(1.5\text{ m}) = 1.5\text{ m}^2$$

$$S = \frac{1}{6}bt^2 = \frac{1}{6}(1.0\text{ m})(1.5\text{ m})^2 = 0.375\text{ m}^3$$

AT THE BASE

$$N = W + W_B + P\sin 60^\circ$$
$$= W + 195\text{ kN} + (25\text{ kN})\sin 60^\circ$$
$$= W + 216.651\text{ kN}$$

$$M = (P\cos 60^\circ)h = (25\text{ kN})(\cos 60^\circ)(5.0\text{ m})$$
$$= 62.5\text{ kN}\cdot\text{m}$$

TENSILE STRESS (EQUAL TO ZERO)

$$\sigma_t = -\frac{N}{A} + \frac{M}{S}$$
$$= -\frac{W + 216.651\text{ kN}}{1.5\text{ m}^2} + \frac{62.5\text{ kN}\cdot\text{m}}{0.375\text{ m}^3} = 0$$

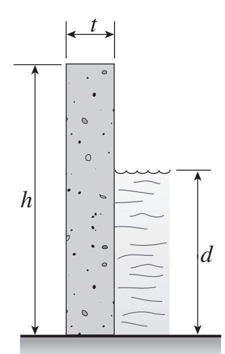
or $-W - 216.651\text{ kN} + 250\text{ kN} = 0$

$W = 33.3$ kN ⟵

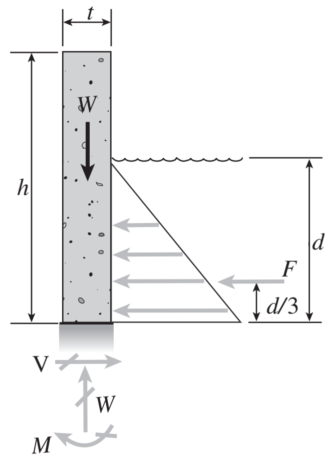
Problem 5.12-11 A plain concrete wall (i.e., a wall with no steel reinforcement) rests on a secure foundation and serves as a small dam on a creek (see figure). The height of the wall is $h = 6.0$ ft and the thickness of the wall is $t = 1.0$ ft.

(a) Determine the maximum tensile and compressive stresses σ_t and σ_c, respectively, at the base of the wall when the water level reaches the top ($d = h$). Assume plain concrete has weight density $\gamma_c = 145$ lb/ft^3.

(b) Determine the maximum permissible depth d_{max} of the water if there is to be no tension in the concrete.



Solution 5.12-11 Concrete wall



h = height of wall
t = thickness of wall
b = width of wall (perpendicular to the figure)
γ_c = weight density of concrete
γ_w = weight density of water
d = depth of water
W = weight of wall
$W = bht\gamma_c$
F = resultant force for the water pressure

MAXIMUM WATER PRESSURE $= \gamma_w d$

$$F = \frac{1}{2}(d)(\gamma_w d)(b) = \frac{1}{2}bd^2\gamma_w$$

$$M = F\left(\frac{d}{3}\right) = \frac{1}{6}bd^3\gamma_w$$

$$A = bt \qquad S = \frac{1}{6}bt^2$$

STRESSES AT THE BASE OF THE WALL
(d = DEPTH OF WATER)

$$\sigma_t = -\frac{W}{A} + \frac{M}{S} = -h\gamma_c + \frac{d^3\gamma_w}{t^2} \qquad \text{Eq. (1)}$$

$$\sigma_c = -\frac{W}{A} - \frac{M}{S} = -h\gamma_c - \frac{d^3\gamma_w}{t^2} \qquad \text{Eq. (2)}$$

(a) STRESSES AT THE BASE WHEN $d = h$

$h = 6.0$ ft $= 72$ in. $\qquad d = 72$ in.
$t = 1.0$ ft $= 12$ in.

$$\gamma_c = 145 \text{ lb/ft}^3 = \frac{145}{1728} \text{ lb/in.}^3$$

$$\gamma_w = 62.4 \text{ lb/ft}^3 = \frac{62.4}{1728} \text{ lb/in.}^3$$

Substitute numerical values into Eqs. (1) and (2):
$\sigma_t = -6.042$ psi $+ 93.600$ psi $= 87.6$ psi ⟵
$\sigma_c = -6.042$ psi $- 93.600$ psi $= -99.6$ psi ⟵

(b) MAXIMUM DEPTH FOR NO TENSION

Set $\sigma_t = 0$ in Eq. (1):

$$-h\gamma_c + \frac{d^3\gamma_w}{t^2} = 0 \qquad d^3 = ht^2\left(\frac{\gamma_c}{\gamma_w}\right)$$

$$d^3 = (72 \text{ in.})(12 \text{ in.})^2\left(\frac{145}{62.4}\right) = 24{,}092 \text{ in.}^3$$

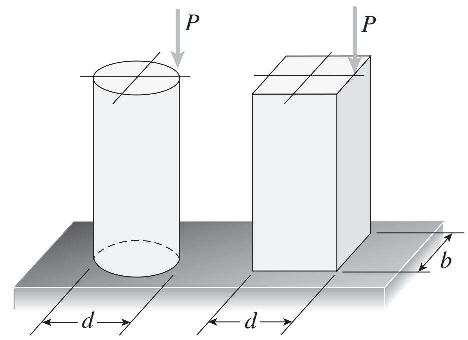
$d_{max} = 28.9$ in. ⟵

Eccentric Axial Loads

Problem 5.12-12 A circular post and a rectangular post are each compressed by loads that produce a resultant force P acting at the edge of the cross section (see figure). The diameter of the circular post and the depth of the rectangular post are the same.

(a) For what width b of the rectangular post will the maximum tensile stresses be the same in both posts?

(b) Under the conditions described in part (a), which post has the larger compressive stress?



Solution 5.12-12 **Two posts in compression**

CIRCULAR POST

$$A = \frac{\pi d^2}{4} \quad S = \frac{\pi d^3}{32} \quad M = \frac{Pd}{2}$$

Tension: $\sigma_t = -\frac{P}{A} + \frac{M}{S} = -\frac{4P}{\pi d^2} + \frac{16P}{\pi d^2} = \frac{12P}{\pi d^2}$

Compression: $\sigma_c = -\frac{P}{A} - \frac{M}{S} = -\frac{4P}{\pi d^2} - \frac{16P}{\pi d^2}$

$$= -\frac{20P}{\pi d^2}$$

RECTANGULAR POST

$$A = bd \quad S = \frac{bd^2}{6} \quad M = \frac{Pd}{2}$$

Tension: $\sigma_t = -\frac{P}{A} + \frac{M}{S} = -\frac{P}{bd} + \frac{3P}{bd} = \frac{2P}{bd}$

Compression: $\sigma_c = -\frac{P}{A} - \frac{M}{S} = -\frac{P}{bd} - \frac{3P}{bd} = -\frac{4P}{bd}$

EQUAL MAXIMUM TENSILE STRESSES

$$\frac{12P}{\pi d^2} = \frac{2P}{bd} \quad \text{or} \quad \frac{6}{\pi d} = \frac{1}{b} \qquad \text{(Eq. 1)}$$

(a) Determine the width b of the rectangular post

From Eq. (1): $b = \frac{\pi d}{6}$ ←

(b) Compressive stresses

Circular post: $\sigma_c = -\frac{20P}{\pi d^2}$

Rectangular post: $\sigma_c = -\frac{4P}{bd} = -\frac{4P}{(\pi d/6)d}$

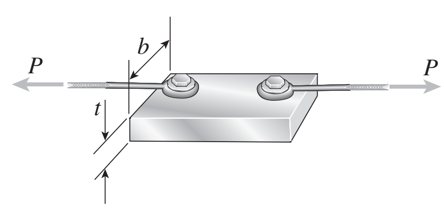
$$= -\frac{24P}{\pi d^2}$$

Rectangular post has the larger compressive stress. ←

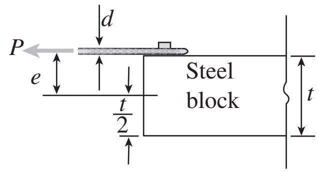
Problem 5.12-13 Two cables, each carrying a tensile force $P = 1200$ lb, are bolted to a block of steel (see figure). The block has thickness $t = 1$ in. and width $b = 3$ in.

(a) If the diameter d of the cable is 0.25 in., what are the maximum tensile and compressive stresses σ_t and σ_c, respectively, in the block?

(b) If the diameter of the cable is increased (without changing the force P), what happens to the maximum tensile and compressive stresses?



Solution 5.12-13 **Steel block loaded by cables**



$P = 1200 \text{ lb} \qquad d = 0.25 \text{ in.}$

$t = 1.0 \text{ in.} \qquad e = \dfrac{t}{2} + \dfrac{d}{2} = 0.625 \text{ in.}$

b = width of block
= 3.0 in.

CROSS SECTION OF BLOCK

$A = bt = 3.0 \text{ in.}^2 \qquad I = \dfrac{1}{12}bt^3 = 0.25 \text{ in.}^4$

(a) MAXIMUM TENSILE STRESS (AT TOP OF BLOCK)

$$y = \frac{t}{2} = 0.5 \text{ in.}$$

$$\sigma_t = \frac{P}{A} + \frac{Pey}{I}$$
$$= \frac{1200 \text{ lb}}{3 \text{ in.}^2} + \frac{(1200 \text{ lb})(0.625 \text{ in.})(0.5 \text{ in.})}{0.25 \text{ in.}^4}$$
$$= 400 \text{ psi} + 1500 \text{ psi} = 1900 \text{ psi} \quad \longleftarrow$$

MAXIMUM COMPRESSIVE STRESS (AT BOTTOM OF BLOCK)

$$y = -\frac{t}{2} = -0.5 \text{ in.}$$

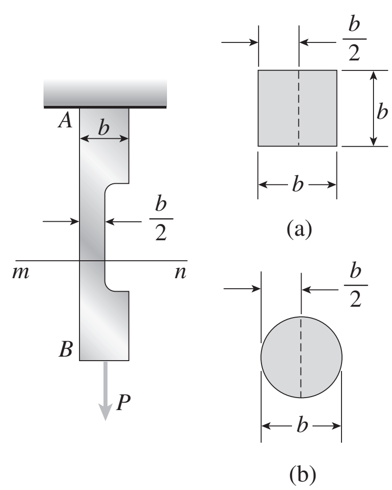
$$\sigma_c = \frac{P}{A} + \frac{Pey}{I}$$
$$= \frac{1200 \text{ lb}}{3 \text{ in.}^2} + \frac{(1200 \text{ lb})(0.625 \text{ in.})(-0.5 \text{ in.})}{0.25 \text{ in.}^4}$$
$$= 400 \text{ psi} - 1500 \text{ psi} = -1100 \text{ psi} \quad \longleftarrow$$

(b) IF d IS INCREASED, the eccentricity e increases and both stresses increase in magnitude.

Problem 5.12-14 A bar AB supports a load P acting at the centroid of the end cross section (see figure). In the middle region of the bar the cross-sectional area is reduced by removing one-half of the bar.

(a) If the end cross sections of the bar are square with sides of length b, what are the maximum tensile and compressive stresses σ_t and σ_c, respectively, at cross section mn within the reduced region?

(b) If the end cross sections are circular with diameter b, what are the maximum stresses σ_t and σ_c?



Solution 5.12-14 Bar with reduced cross section

(a) Square bar

Cross section *mn* is a rectangle.

$$A = (b)\left(\frac{b}{2}\right) = \frac{b^2}{2} \qquad I = \frac{1}{12}(b)\left(\frac{b}{2}\right)^3 = \frac{b^4}{96}$$

$$M = P\left(\frac{b}{4}\right) \qquad c = \frac{b}{4}$$

Stresses

$$\sigma_t = \frac{P}{A} + \frac{Mc}{I} = \frac{2P}{b^2} + \frac{6P}{b^2} = \frac{8P}{b^2} \longleftarrow$$

$$\sigma_c = \frac{P}{A} - \frac{Mc}{I} = \frac{2P}{b^2} - \frac{6P}{b^2} = -\frac{4P}{b^2} \longleftarrow$$

(b) Circular bar

Cross section *mn* is a semicircle

$$A = \frac{1}{2}\left(\frac{\pi b^2}{4}\right) = \frac{\pi b^2}{8} = 0.3927\ b^2$$

From Appendix D, Case 10:

$$I = 0.1098\left(\frac{b}{2}\right)^4 = 0.006860\ b^4$$

$$M = P\left(\frac{2b}{3\pi}\right) = 0.2122\ Pb$$

For tension:

$$c_t = \frac{4r}{3\pi} = \frac{2b}{3\pi} = 0.2122\ b$$

For compression:

$$c_c = r - c_t = \frac{b}{2} - \frac{2b}{3\pi} = 0.2878\ b$$

Stresses

$$\sigma_t = \frac{P}{A} + \frac{Mc_t}{I} = \frac{P}{0.3927\ b^2} + \frac{(0.2122\ Pb)(0.2122\ b)}{0.006860\ b^4}$$

$$= 2.546\,\frac{P}{b^2} + 6.564\,\frac{P}{b^2} = 9.11\,\frac{P}{b^2} \longleftarrow$$

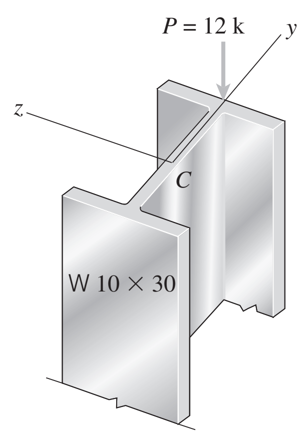
$$\sigma_c = \frac{P}{A} - \frac{Mc_c}{I} = \frac{P}{0.3927 b^2} - \frac{(0.2122\,Pb)(0.2878\,b)}{0.006860\,b^4}$$

$$= 2.546\,\frac{P}{b^2} - 8.903\,\frac{P}{b^2} = -6.36\,\frac{P}{b^2} \longleftarrow$$

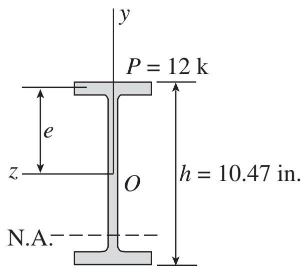
Problem 5.12-15 A short column constructed of a W 10 × 30 wide-flange shape is subjected to a resultant compressive load $P = 12$ k having its line of action at the midpoint of one flange (see figure).

(a) Determine the maximum tensile and compressive stresses σ_t and σ_c, respectively, in the column.

(b) Locate the neutral axis under this loading condition.



Solution 5.12-15 Column of wide-flange shape



W 10 × 30 $\quad A = 8.84 \text{ in.}^2$

$I = 170 \text{ in.}^4 \quad t_f = 0.510 \text{ in.}$

$$e = \frac{h}{2} - \frac{t_f}{2} = 4.98 \text{ in.}$$

(a) Maximum stresses

$$\sigma_t = -\frac{P}{A} + \frac{Pe(h/2)}{I} = -1357 \text{ psi} + 1840 \text{ psi}$$
$$= 480 \text{ psi} \quad \longleftarrow$$

$$\sigma_c = -\frac{P}{A} - \frac{Pe(h/2)}{I} = -1357 \text{ psi} - 1840 \text{ psi}$$
$$= -3200 \text{ psi} \quad \longleftarrow$$

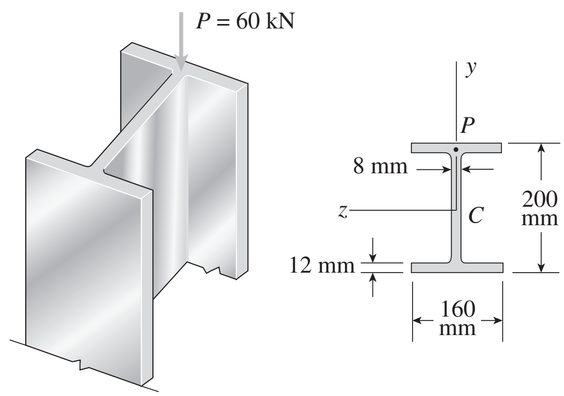
(b) Neutral axis (see figure)

$$y_0 = -\frac{I}{Ae} = -3.86 \text{ in.} \quad \longleftarrow$$

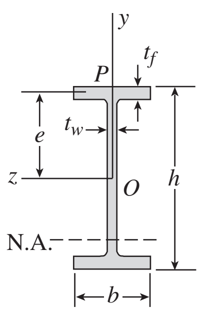
Problem 5.12-16 A short column of wide-flange shape is subjected to a compressive load that produces a resultant force $P = 60$ kN acting at the midpoint of one flange (see figure).

(a) Determine the maximum tensile and compressive stresses σ_t and σ_c, respectively, in the column.

(b) Locate the neutral axis under this loading condition.



Solution 5.12-16 Column of wide-flange shape



$b = 160 \text{ mm} \quad t_w = 8 \text{ mm}$

$h = 200 \text{ mm} \quad t_f = 12 \text{ mm}$

$$P = 60 \text{ kN} \qquad e = \frac{h}{2} - \frac{t_f}{2} = 94 \text{ mm}$$

$$A = 2bt_f + (h - 2t_f)\, t_w = 5248 \text{ mm}^2$$

$$I = \frac{1}{12}bh^3 - \frac{1}{12}(b - t_w)(h - 2t_f)^3$$
$$= 37.611 \times 10^6 \text{ mm}^4$$

(a) Maximum stresses

$$\sigma_t = -\frac{P}{A} + \frac{Pe(h/2)}{I}$$
$$= -\frac{60 \text{ kN}}{5248 \text{ mm}^2} + \frac{(60 \text{ kN})(94 \text{ mm})(100 \text{ mm})}{37.611 \times 10^6 \text{ mm}^4}$$
$$= -11.43 \text{ MPa} + 15.00 \text{ MPa}$$
$$= 3.57 \text{ MPa} \quad \longleftarrow$$

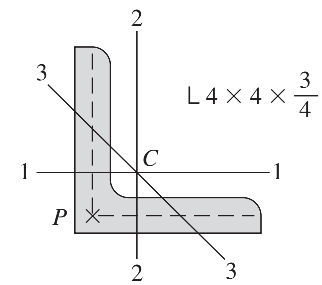
$$\sigma_c = -11.43 \text{ MPa} - 15.00 \text{ MPa}$$
$$= -26.4 \text{ MPa} \quad \longleftarrow$$

(b) Neutral axis (see figure)

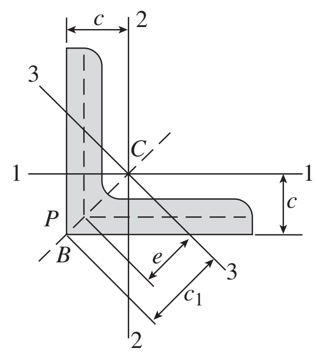
$$y_0 = -\frac{I}{Ae} = -\frac{37.611 \times 10^6 \text{ mm}^4}{(5248 \text{ mm}^2)(94 \text{ mm})}$$
$$= -76.2 \text{ mm} \quad \longleftarrow$$

Problem 5.12-17 A tension member constructed of an L 4 × 4 × ¾ inch angle section (see Appendix E) is subjected to a tensile load $P = 15$ kips that acts through the point where the midlines of the legs intersect (see figure).

Determine the maximum tensile stress σ_t in the angle section.



Solution 5.12-17 **Angle section in tension**



Bending occurs about axis 3-3.

$\text{L}\, 4 \times 4 \times \dfrac{3}{4}$

$A = 5.44 \text{ in.}^2 \qquad t = \text{thickness of legs}$

$c = 1.27 \text{ in.} \qquad\quad = 0.75 \text{ in.}$

$e = \text{eccentricity of load } P$

$$= \left(c - \frac{t}{2}\right)\sqrt{2}$$

$$= (1.27 - 0.375)\sqrt{2}$$

$$= 1.266 \text{ in.}$$

$P = 15 \text{ k (tensile load)}$

$c_1 = \text{distance from centroid } C \text{ to corner } B \text{ of angle}$

$$= c\sqrt{2} = (1.27 \text{ in.})\sqrt{2} = 1.796 \text{ in.}$$

$I_3 = Ar_{\text{min}}^2 \qquad \text{(see Table E-4)}$

$r_{\text{min}} = 0.778 \text{ in.}$

$I_3 = (5.44 \text{ in.}^2)(0.778 \text{ in.})^2 = 3.293 \text{ in.}^4$

$M = Pe = (15 \text{ k})(1.266 \text{ in.}) = 18.94 \text{ k-in.}$

Maximum tensile stress

Maximum tensile stress occurs at corner B.

$$\sigma_t = \frac{P}{A} + \frac{Mc_1}{I_3}$$

$$= \frac{15 \text{ k}}{5.44 \text{ in}^2} + \frac{(18.99 \text{ k-in.})(1.796 \text{ in.})}{3.293 \text{ in.}^4}$$

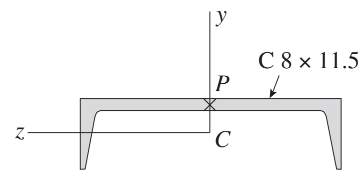
$$= 2.76 \text{ ksi} + 10.36 \text{ ksi}$$

$$= 13.1 \text{ ksi} \longleftarrow$$

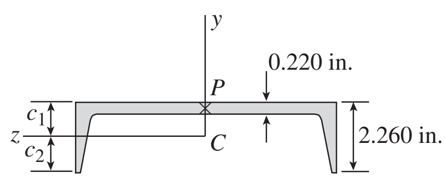
Problem 5.12-18 A short length of a C 8×11.5 channel is subjected to an axial compressive force P that has its line of action through the midpoint of the web of the channel (see figure).

(a) Determine the equation of the neutral axis under this loading condition.

(b) If the allowable stresses in tension and compression are 10,000 psi and 8,000 psi, respectively, find the maximum permissible load P_{max}.



Solution 5.12-18 Channel in compression



C 8 × 11.5

$A = 3.38$ in.^{2} $h = 2.260$ in. $t_w = 0.220$ in.

$I_z = 1.32$ in.^{4} $c_1 = 0.571$ in. $c_2 = 1.689$ in.

Eccentricity of the load

$$e = c_1 - \frac{t_w}{2} = 0.571 - 0.110 = 0.461 \text{ in.}$$

(a) Location of the neutral axis

$$y_0 = -\frac{I}{Ae} = -\frac{1.32 \text{ in.}^4}{(3.38 \text{ in.}^2)(0.461 \text{ in.})}$$

$$= -0.847 \text{ in.} \longleftarrow$$

(b) Maximum load based upon tensile stress

$\sigma_{allow} = 10{,}000$ psi (P = pounds)

$$\sigma_t = -\frac{P}{A} + \frac{Pec_2}{I}$$

$$= -\frac{P}{3.38 \text{ in.}^2} + \frac{P(0.461 \text{ in.})(1.689 \text{ in.})}{1.32 \text{ in.}^4}$$

$$10{,}000 = -\frac{P}{3.38} + \frac{P}{1.695} = 0.2941P$$

$$P = 34{,}000 \text{ lb} = 34 \text{ k}$$

Maximum load based upon compressive stress

$\sigma_{allow} = 8000$ psi (P = pounds)

$$\sigma_c = -\frac{P}{A} - \frac{Pec_1}{I}$$

$$= -\frac{P}{3.38 \text{ in.}^2} - \frac{P(0.461 \text{ in.})(0.571 \text{ in.})}{1.32 \text{ in.}^4}$$

$$8000 = \frac{P}{3.38} - \frac{P}{5.015} = 0.4953P$$

$$P = 16{,}200 \text{ lb} = 16.2 \text{ k}$$

Compression governs. $P_{max} = 16.2$ k $\longleftarrow$

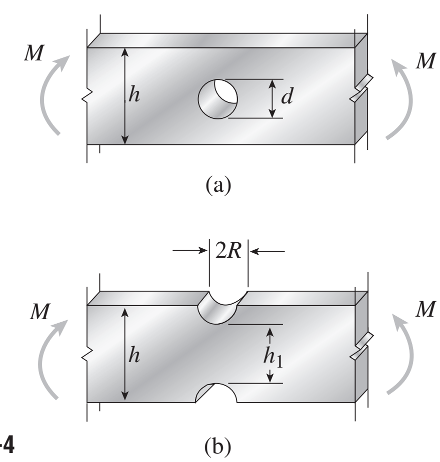
Stress Concentrations

The problems for Section 5.13 are to be solved considering the stress-concentration factors.

Problem 5.13-1 The beams shown in the figure are subjected to bending moments $M = 2100$ lb-in. Each beam has a rectangular cross section with height $h = 1.5$ in. and width $b = 0.375$ in. (perpendicular to the plane of the figure).

(a) For the beam with a hole at midheight, determine the maximum stresses for hole diameters $d = 0.25$, 0.50, 0.75, and 1.00 in.

(b) For the beam with two identical notches (inside height $h_1 = 1.25$ in.), determine the maximum stresses for notch radii $R = 0.05$, 0.10, 0.15, and 0.20 in.



Probs. 5.13-1 through 5.13-4

Solution 5.13-1

$M = 2100$ lb-in. $\quad h = 1.5$ in. $\quad b = 0.375$ in.

(a) Beam with a hole

$$\frac{d}{h} \le \frac{1}{2} \quad \text{Eq. (5-57):} \quad \sigma_C = \frac{6Mh}{b(h^3 - d^3)} = \frac{50{,}400}{3.375 - d^3} \qquad (1)$$

$$\frac{d}{h} \ge \frac{1}{2} \quad \text{Eq. (5-56):} \quad \sigma_B = \frac{12Md}{b(h^3 - d^3)} = \frac{67{,}200\,d}{3.375 - d^3} \qquad (2)$$

d (in.)	$\frac{d}{h}$	σ_C Eq.(1) (psi)	σ_B Eq.(2) (psi)	σ_{max} (psi)
0.25	0.1667	15,000	—	15,000
0.50	0.3333	15,500	—	15,500
0.75	0.5000	17,100	17,100	17,100
1.00	0.6667	—	28,300	28,300

Note: The larger the hole, the larger the stress.

(b) Beam with notches

$$h_1 = 1.25 \text{ in.} \qquad \frac{h}{h_1} = \frac{1.5 \text{ in.}}{1.25 \text{ in.}} = 1.2$$

Eq. (5-58):

$$\sigma_{nom} = \frac{6M}{bh_1^2} = 21{,}500 \text{ psi}$$

$$\sigma_{max} = K\sigma_{nom}$$

R (in.)	$\frac{R}{h_1}$	K (Fig. 5-50)	σ_{max} (psi)
0.05	0.04	3.0	65,000
0.10	0.08	2.3	49,000
0.15	0.12	2.1	45,000
0.20	0.16	1.9	41,000

Note: The larger the notch radius, the smaller the stress.

Problem 5.13-2 The beams shown in the figure are subjected to bending moments $M = 250$ N · m. Each beam has a rectangular cross section with height $h = 44$ mm and width $b = 10$ mm (perpendicular to the plane of the figure).

(a) For the beam with a hole at midheight, determine the maximum stresses for hole diameters $d = 10$, 16, 22, and 28 mm.

(b) For the beam with two identical notches (inside height $h_1 = 40$ mm), determine the maximum stresses for notch radii $R = 2$, 4, 6, and 8 mm.

Solution 5.13-2

$M = 250$ N · m $\quad h = 44$ mm $\quad b = 10$ mm

(a) Beam with a hole

$$\frac{d}{h} \le \frac{1}{2} \quad \text{Eq. (5-57):} \quad \sigma_C = \frac{6Mh}{b(h^3 - d^3)} = \frac{6.6 \times 10^6}{85{,}180 - d^3} \text{ MPa} \quad (1)$$

$$\frac{d}{h} \ge \frac{1}{2} \quad \text{Eq. (5-56):} \quad \sigma_B = \frac{12Md}{b(h^3 - d^3)} = \frac{300 \times 10^3 d}{85{,}180 - d^3} \text{ MPa} \quad (2)$$

d (mm)	$\frac{d}{h}$	σ_C Eq.(1) (MPa)	σ_B Eq.(2) (MPa)	σ_{max} (MPa)
10	0.227	78	—	78
16	0.364	81	—	81
22	0.500	89	89	89
28	0.636	—	133	133

Note: The larger the hole, the larger the stress.

(b) Beam with notches

$$h_1 = 40 \text{ mm} \qquad \frac{h}{h_1} = \frac{44 \text{ mm}}{40 \text{ mm}} = 1.1$$

$$\text{Eq. (5-58):} \quad \sigma_{nom} = \frac{6M}{bh_1^2} = 93.8 \text{ MPa}$$

$$\sigma_{max} = K\sigma_{nom}$$

R (mm)	$\frac{R}{h_1}$	K (Fig. 5-50)	σ_{max} (MPa)
2	0.05	2.6	240
4	0.10	2.1	200
6	0.15	1.8	170
8	0.20	1.7	160

Note: The larger the notch radius, the smaller the stress.

Problem 5.13-3 A rectangular beam with semicircular notches, as shown in part (b) of the figure, has dimensions $h = 0.88$ in. and $h_1 = 0.80$ in. The maximum allowable bending stress in the metal beam is $\sigma_{max} = 60$ ksi, and the bending moment is $M = 600$ lb-in.

Determine the minimum permissible width b_{min} of the beam.

Solution 5.13-3 Beam with semicircular notches

$h = 0.88$ in. $\qquad h_1 = 0.80$ in.

$\sigma_{max} = 60$ ksi $\qquad M = 600$ lb-in.

$$h = h_1 + 2R \qquad R = \frac{1}{2}(h - h_1) = 0.04 \text{ in.}$$

$$\frac{R}{h_1} = \frac{0.04 \text{ in.}}{0.80 \text{ in.}} = 0.05$$

From Fig. 5-50: $K \approx 2.57$

$$\sigma_{max} = K\sigma_{nom} = K\left(\frac{6M}{bh_1^2}\right)$$

$$60 \text{ ksi} = 2.57\left[\frac{6(600 \text{ lb-in.})}{b(0.80 \text{ in.})^2}\right]$$

Solve for b:

$$b_{min} \approx 0.24 \text{ in.} \quad \longleftarrow$$

Problem 5.13-4 A rectangular beam with semicircular notches, as shown in part (b) of the figure, has dimensions $h = 120$ mm and $h_1 = 100$ mm. The maximum allowable bending stress in the plastic beam is $\sigma_{max} = 6$ MPa, and the bending moment is $M = 150$ N · m.

Determine the minimum permissible width b_{min} of the beam.

Solution 5.13-4 **Beam with semicircular notches**

$h = 120$ mm $\quad h_1 = 100$ mm

$\sigma_{max} = 6$ MPa $\quad M = 150$ N · m

$$h = h_1 + 2R \quad R = \frac{1}{2}(h - h_1) = 10 \text{ mm}$$

$$\frac{R}{h_1} = \frac{10 \text{ mm}}{100 \text{ mm}} = 0.10$$

From Fig. 5-50: $K \approx 2.20$

$$\sigma_{max} = K\sigma_{nom} = K\left(\frac{6M}{bh_1^2}\right)$$

$$6 \text{ MPa} = (2.20)\left[\frac{6(150 \text{ N} \cdot \text{m})}{b(100 \text{ mm})^2}\right]$$

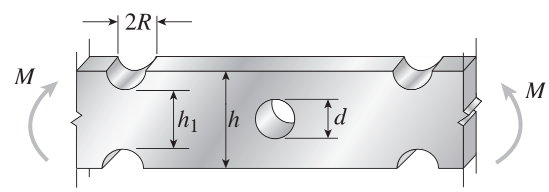
Solve for b:

$b_{min} \approx 33$ mm ←

Problem 5.13-5 A rectangular beam with notches and a hole (see figure) has dimensions $h = 5.5$ in., $h_1 = 5$ in., and width $b = 1.6$ in. The beam is subjected to a bending moment $M = 130$ k-in., and the maximum allowable bending stress in the material (steel) is $\sigma_{max} = 42{,}000$ psi.

(a) What is the smallest radius R_{min} that should be used in the notches?

(b) What is the diameter d_{max} of the largest hole that should be drilled at the midheight of the beam?



Solution 5.13-5 **Beam with notches and a hole**

$h = 5.5$ in. $\quad h_1 = 5$ in. $\quad b = 1.6$ in.

$M = 130$ k-in. $\quad \sigma_{max} = 42{,}000$ psi

(a) Minimum notch radius

$$\frac{h}{h_1} = \frac{5.5 \text{ in.}}{5 \text{ in.}} = 1.1$$

$$\sigma_{nom} = \frac{6M}{bh_1^2} = 19{,}500 \text{ psi}$$

$$K = \frac{\sigma_{max}}{\sigma_{nom}} = \frac{42{,}000 \text{ psi}}{19{,}500 \text{ psi}} = 2.15$$

From Fig. 5-50, with $K = 2.15$ and $\frac{h}{h_1} = 1.1$, we get

$$\frac{R}{h_1} \approx 0.090$$

$\therefore R_{min} \approx 0.090h_1 = 0.45$ in. ←

(b) Largest hole diameter

Assume $\frac{d}{h} > \frac{1}{2}$ and use Eq. (5-56).

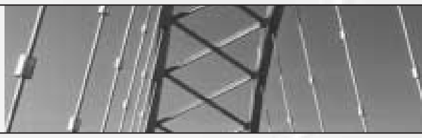
$$\sigma_B = \frac{12Md}{b(h^3 - d^3)}$$

$$42{,}000 \text{ psi} = \frac{12(130 \text{ k-in.})d}{(1.6 \text{ in.})[(5.5 \text{ in.})^3 - d^3]} \quad \text{or}$$

$$d^3 + 23.21d - 166.4 = 0$$

Solve numerically:

$d_{max} = 4.13$ in. ←



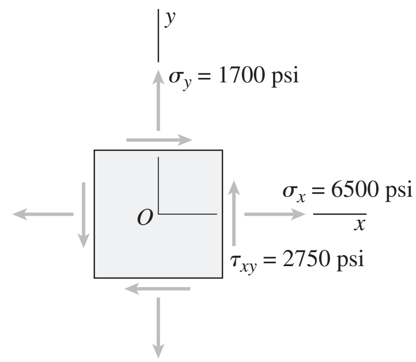
7

Analysis of Stress and Strain

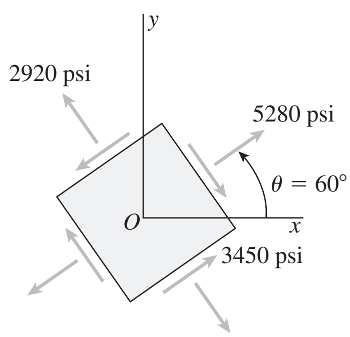
Plane Stress

Problem 7.2-1 An element in *plane stress* is subjected to stresses $\sigma_x = 6500$ psi, $\sigma_y = 1700$ psi, and $\tau_{xy} = 2750$ psi, as shown in the figure.

Determine the stresses acting on an element oriented at an angle $\theta = 60°$ from the x axis, where the angle θ is positive when counterclockwise. Show these stresses on a sketch of an element oriented at the angle θ.



Solution 7.2-1 **Plane stress (angle θ)**



$\sigma_x = 6500$ psi $\quad \sigma_y = 1700$ psi $\quad \tau_{xy} = 2750$ psi

$\theta = 60°$

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

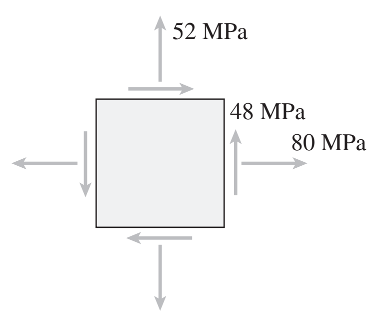
$$= 5280 \text{ psi} \longleftarrow$$

$$\tau_{x_1y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

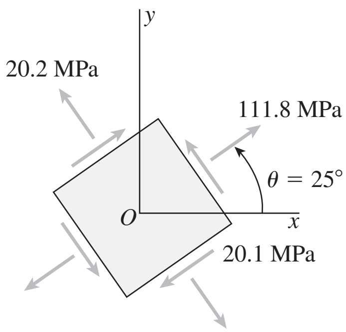
$$= -3450 \text{ psi} \longleftarrow$$

$$\sigma_{y_1} = \sigma_x + \sigma_y - \sigma_{x_1} = 2920 \text{ psi} \longleftarrow$$

Problem 7.2-2 Solve the preceding problem for $\sigma_x = 80$ MPa, $\sigma_y = 52$ MPa, $\tau_{xy} = 48$ MPa, and $\theta = 25°$ (see figure).



Solution 7.2-2 **Plane stress (angle θ)**



$\sigma_x = 80$ MPa $\quad \sigma_y = 52$ MPa $\quad \tau_{xy} = 48$ MPa

$\theta = 25°$

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

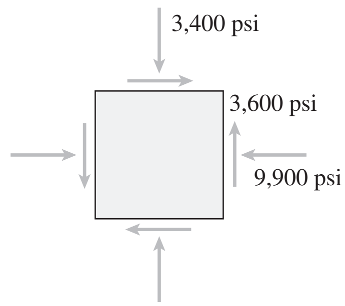
$$= 111.8 \text{ MPa} \longleftarrow$$

$$\tau_{x_1y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

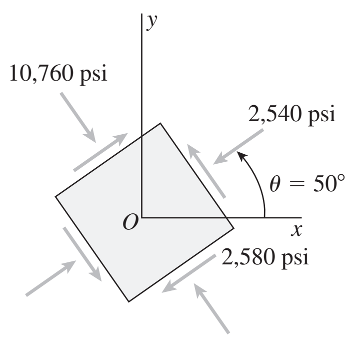
$$= 20.1 \text{ MPa} \longleftarrow$$

$$\sigma_{y_1} = \sigma_x + \sigma_y - \sigma_{x_1} = 20.2 \text{ MPa} \longleftarrow$$

Problem 7.2-3 Solve Problem 7.2-1 for $\sigma_x = -9{,}900$ psi, $\sigma_y = -3{,}400$ psi, $\tau_{xy} = 3{,}600$ psi, and $\theta = 50°$ (see figure).



Solution 7.2-3 **Plane stress (angle θ)**



$\sigma_x = -9900$ psi $\quad \sigma_y = -3400$ psi $\quad \tau_{xy} = 3600$ psi

$\theta = 50°$

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$= -2540 \text{ psi} \longleftarrow$$

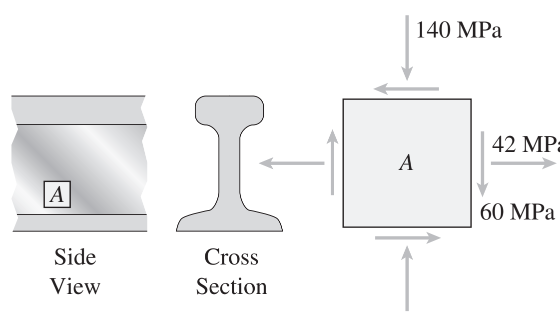
$$\tau_{x_1y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$= 2580 \text{ psi} \longleftarrow$$

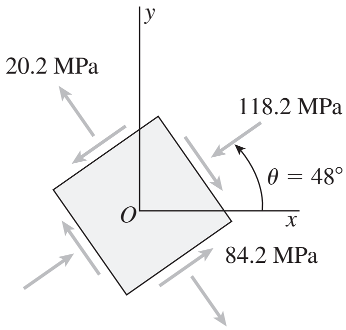
$$\sigma_{y_1} = \sigma_x + \sigma_y - \sigma_{x_1} = -10{,}760 \text{ psi} \longleftarrow$$

Problem 7.2-4 The stresses acting on element *A* in the web of a train rail are found to be 42 MPa tension in the horizontal direction and 140 MPa compression in the vertical direction (see figure). Also, shear stresses of magnitude 60 MPa act in the directions shown.

Determine the stresses acting on an element oriented at a counterclockwise angle of 48° from the horizontal. Show these stresses on a sketch of an element oriented at this angle.



Solution 7.2-4 **Plane stress (angle θ)**



$\sigma_x = 42$ MPa $\quad \sigma_y = -140$ MPa $\quad \tau_{xy} = -60$ MPa

$\theta = 48°$

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

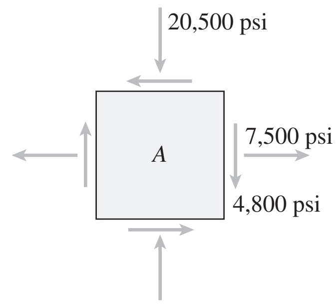
$$= -118.2 \text{ MPa} \longleftarrow$$

$$\tau_{x_1y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

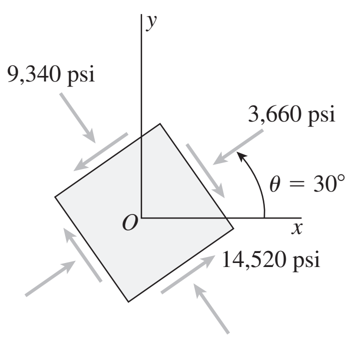
$$= -84.2 \text{ MPa} \longleftarrow$$

$$\sigma_{y_1} = \sigma_x + \sigma_y - \sigma_{x_1} = 20.2 \text{ MPa} \longleftarrow$$

Problem 7.2-5 Solve the preceding problem if the normal and shear stresses acting on element *A* are 7,500 psi, 20,500 psi, and 4,800 psi (in the directions shown in the figure) and the angle is 30° (counterclockwise).



Solution 7.2-5 **Plane stress (angle θ)**



$\sigma_x = 7{,}500$ psi $\quad \sigma_y = -20{,}500$ psi

$\tau_{xy} = -4{,}800$ psi

$\theta = 30°$

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$= -3{,}660 \text{ psi} \longleftarrow$$

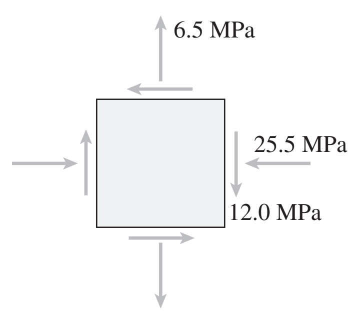
$$\tau_{x_1y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$= -14,520 \text{ psi} \longleftarrow$$

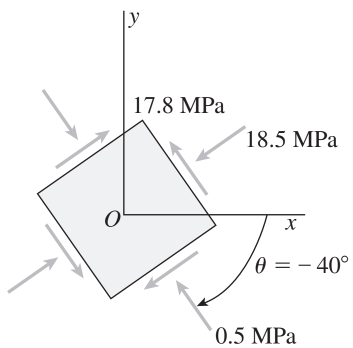
$$\sigma_{y_1} = \sigma_x + \sigma_y - \sigma_{x_1} = -9{,}340 \text{ psi} \longleftarrow$$

Problem 7.2-6 An element in *plane stress* from the fuselage of an airplane is subjected to compressive stresses of magnitude 25.5 MPa in the horizontal direction and tensile stresses of magnitude 6.5 MPa in the vertical direction (see figure). Also, shear stresses of magnitude 12.0 MPa act in the directions shown.

Determine the stresses acting on an element oriented at a clockwise angle of 40° from the horizontal. Show these stresses on a sketch of an element oriented at this angle.



Solution 7.2-6 Plane stress (angle θ)



$\sigma_x = -25.5$ MPa $\quad \sigma_y = 6.5$ MPa

$\tau_{xy} = -12.0$ MPa

$\theta = -40°$

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$= -0.5 \text{ MPa} \longleftarrow$$

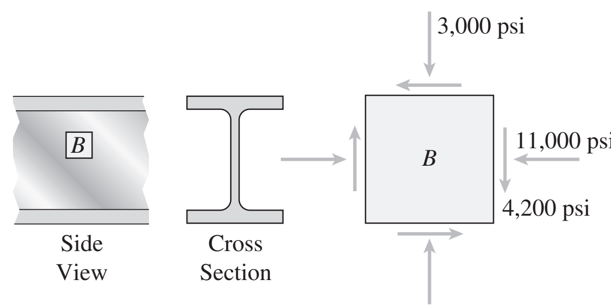
$$\tau_{x_1y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$= -17.8 \text{ MPa} \longleftarrow$$

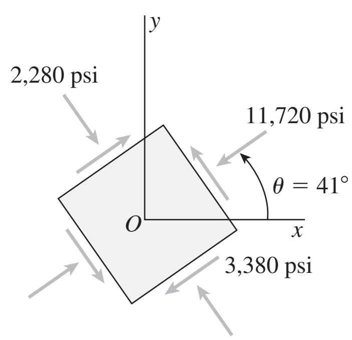
$$\sigma_{y_1} = \sigma_x + \sigma_y - \sigma_{x_1} = -18.5 \text{ MPa} \longleftarrow$$

Problem 7.2-7 The stresses acting on element *B* in the web of a wide-flange beam are found to be 11,000 psi compression in the horizontal direction and 3,000 psi compression in the vertical direction (see figure). Also, shear stresses of magnitude 4,200 psi act in the directions shown.

Determine the stresses acting on an element oriented at a counterclockwise angle of 41° from the horizontal. Show these stresses on a sketch of an element oriented at this angle.



Solution 7.2-7 Plane stress (angle θ)



$\sigma_x = -11{,}000$ psi $\quad \sigma_y = -3{,}000$ psi

$\tau_{xy} = -4{,}200$ psi

$\theta = 41°$

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

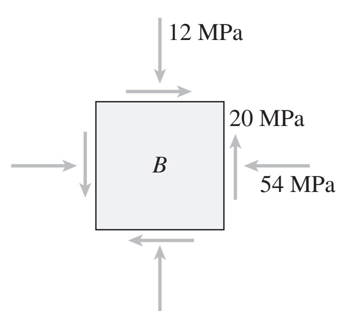
$$= -11{,}720 \text{ psi} \longleftarrow$$

$$\tau_{x_1y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

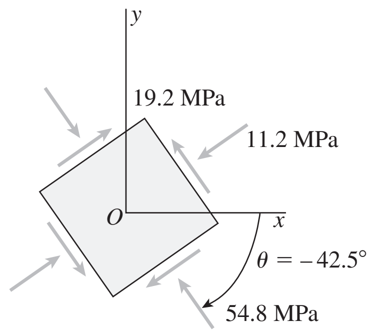
$$= 3{,}380 \text{ psi} \longleftarrow$$

$$\sigma_{y_1} = \sigma_x + \sigma_y - \sigma_{x_1} = -2{,}280 \text{ psi} \longleftarrow$$

Problem 7.2-8 Solve the preceding problem if the normal and shear stresses acting on element *B* are 54 MPa, 12 MPa, and 20 MPa (in the directions shown in the figure) and the angle is 42.5° (clockwise).



Solution 7.2-8 **Plane stress (angle θ)**



$\sigma_x = -54 \text{ MPa} \quad \sigma_y = -12 \text{ MPa} \quad \tau_{xy} = 20 \text{ MPa}$
$\theta = -42.5°$

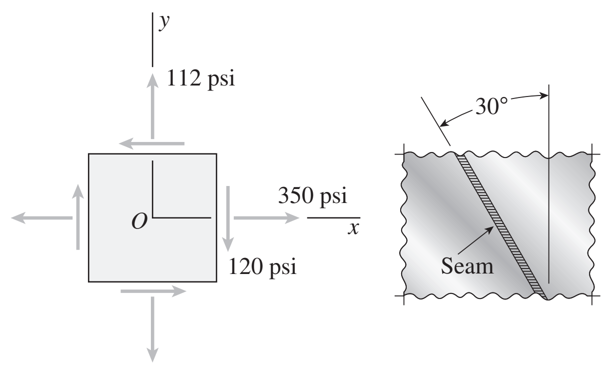
$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$
$$= -54.8 \text{ MPa} \longleftarrow$$

$$\tau_{x_1y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$
$$= -19.2 \text{ MPa} \longleftarrow$$

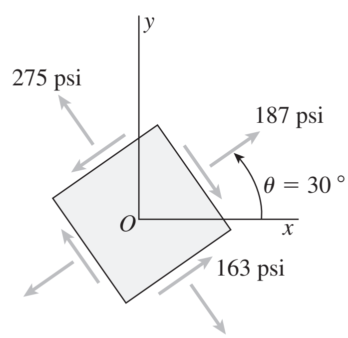
$$\sigma_{y_1} = \sigma_x + \sigma_y - \sigma_{x_1} = -11.2 \text{ MPa} \longleftarrow$$

Problem 7.2-9 The polyethylene liner of a settling pond is subjected to stresses $\sigma_x = 350$ psi, $\sigma_y = 112$ psi, and $\tau_{xy} = -120$ psi, as shown by the plane-stress element in the first part of the figure.

Determine the normal and shear stresses acting on a seam oriented at an angle of 30° to the element, as shown in the second part of the figure. Show these stresses on a sketch of an element having its sides parallel and perpendicular to the seam.



Solution 7.2-9 **Plane stress (angle θ)**



$\sigma_x = 350 \text{ psi} \quad \sigma_y = 112 \text{ psi} \quad \tau_{xy} = -120 \text{ psi}$
$\theta = 30°$

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$
$$= 187 \text{ psi} \longleftarrow$$

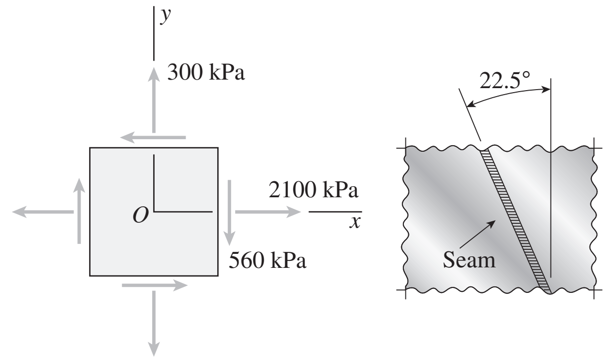
$$\tau_{x_1y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$
$$= -163 \text{ psi} \longleftarrow$$

$$\sigma_{y_1} = \sigma_x + \sigma_y - \sigma_{x_1} = 275 \text{ psi} \longleftarrow$$

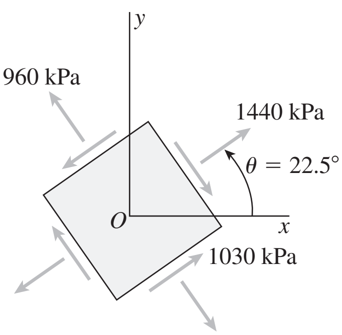
The normal stress on the seam equals 187 psi tension. $\longleftarrow$

The shear stress on the seam equals 163 psi, acting clockwise against the seam. $\longleftarrow$

Problem 7.2-10 Solve the preceding problem if the normal and shear stresses acting on the element are $\sigma_x = 2100$ kPa, $\sigma_y = 300$ kPa, and $\tau_{xy} = -560$ kPa, and the seam is oriented at an angle of 22.5° to the element (see figure).



Solution 7.2-10 **Plane stress (angle θ)**



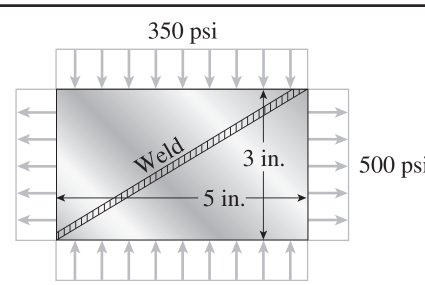
$\sigma_x = 2100$ kPa $\quad \sigma_y = 300$ kPa $\quad \tau_{xy} = -560$ kPa
$\theta = 22.5°$

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$
$$= 1440 \text{ kPa} \longleftarrow$$
$$\tau_{x_1y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$
$$= -1030 \text{ kPa} \longleftarrow$$
$$\sigma_{y_1} = \sigma_x + \sigma_y - \sigma_{x_1} = 960 \text{ kPa} \longleftarrow$$

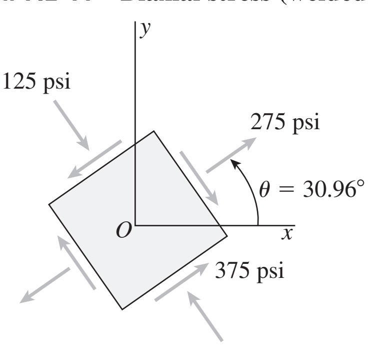
The normal stress on the seam equals 1440 kPa tension. ⟵
The shear stress on the seam equals 1030 kPa, acting clockwise against the seam. ⟵

Problem 7.2-11 A rectangular plate of dimensions 3.0 in. × 5.0 in. is formed by welding two triangular plates (see figure). The plate is subjected to a tensile stress of 500 psi in the long direction and a compressive stress of 350 psi in the short direction.

Determine the normal stress σ_w acting perpendicular to the line of the weld and the shear stress τ_w acting parallel to the weld. (Assume that the normal stress σ_w is positive when it acts in tension against the weld and the shear stress τ_w is positive when it acts counterclockwise against the weld.)



Solution 7.2-11 **Biaxial stress (welded joint)**

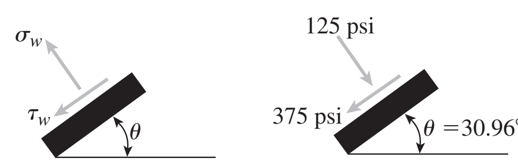


$\sigma_x = 500$ psi $\quad \sigma_y = -350$ psi $\quad \tau_{xy} = 0$

$$\theta = \arctan\frac{3 \text{ in.}}{5 \text{ in.}} = \arctan 0.6 = 30.96°$$

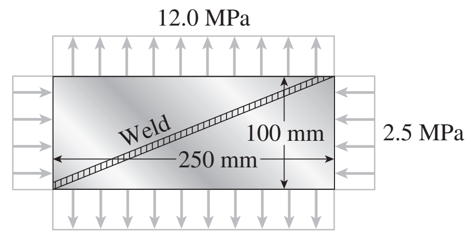
$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$
$$= 275 \text{ psi}$$
$$\tau_{x_1y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta = -375 \text{ psi}$$
$$\sigma_{y_1} = \sigma_x + \sigma_y - \sigma_{x_1} = -125 \text{ psi}$$

STRESSES ACTING ON THE WELD

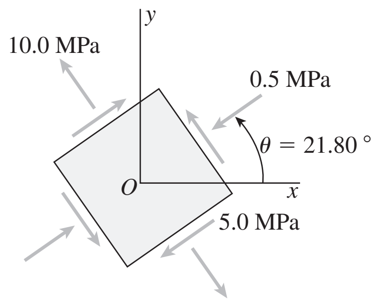


$$\sigma_w = -125 \text{ psi} \longleftarrow$$
$$\tau_w = 375 \text{ psi} \longleftarrow$$

Problem 7.2-12 Solve the preceding problem for a plate of dimensions 100 mm × 250 mm subjected to a compressive stress of 2.5 MPa in the long direction and a tensile stress of 12.0 MPa in the short direction (see figure).



Solution 7.2-12 **Biaxial stress (welded joint)**



$\sigma_x = -2.5\text{ MPa} \qquad \sigma_y = 12.0\text{ MPa} \qquad \tau_{xy} = 0$

$$\theta = \arctan\frac{100\text{ mm}}{250\text{ mm}} = \arctan 0.4 = 21.80°$$

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$
$$= -0.5\text{ MPa}$$

$$\tau_{x_1y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta = 5.0\text{ MPa}$$

$$\sigma_{y_1} = \sigma_x + \sigma_y - \sigma_{x_1} = 10.0\text{ MPa}$$

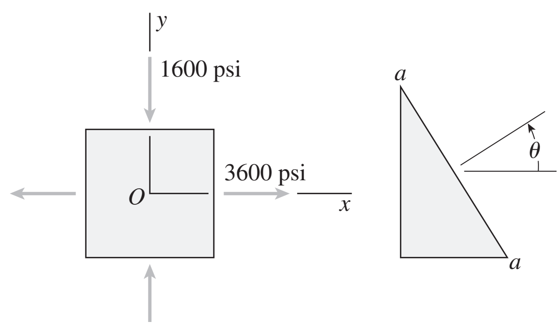
STRESSES ACTING ON THE WELD

$\sigma_w = 10.0\text{ MPa}$ ←

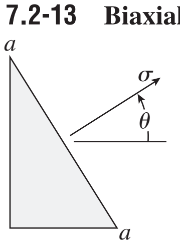
$\tau_w = -5.0\text{ MPa}$ ←

Problem 7.2-13 At a point on the surface of a machine the material is in *biaxial stress* with $\sigma_x = 3600$ psi and $\sigma_y = -1600$ psi, as shown in the first part of the figure. The second part of the figure shows an inclined plane *aa* cut through the same point in the material but oriented at an angle θ.

Determine the value of the angle θ between zero and 90° such that no normal stress acts on plane *aa*. Sketch a stress element having plane *aa* as one of its sides and show all stresses acting on the element.



Solution 7.2-13 **Biaxial stress**



$\sigma_x = 3600$ psi
$\sigma_y = -1600$ psi
$\tau_{xy} = 0$

Find angle θ for $\sigma = 0$.
σ = normal stress on plane *a-a*

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$
$$= 1000 + 2600\cos 2\theta\text{ (psi)}$$

For $\sigma_{x_1} = 0$, we obtain $\cos 2\theta = -\dfrac{1000}{2600}$

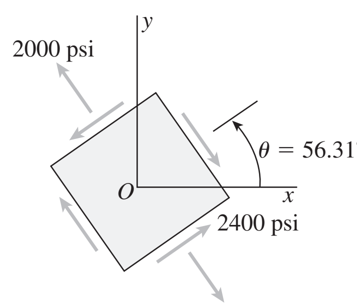
$\therefore \quad 2\theta = 112.62° \quad$ and $\quad \theta = 56.31°$ ←

STRESS ELEMENT

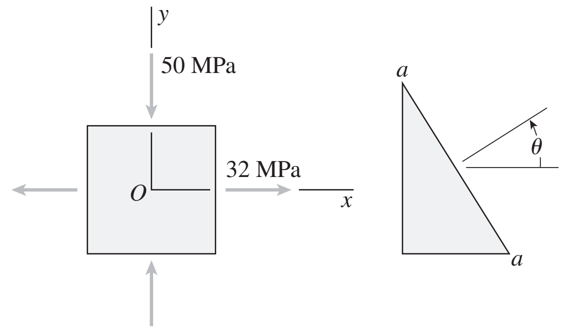
$\sigma_{x_1} = 0 \qquad \theta = 56.31°$

$\sigma_{y_1} = \sigma_x + \sigma_y - \sigma_{x_1} = 2000$ psi ←

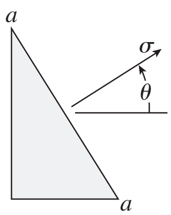
$$\tau_{x_1y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$
$$= -2400\text{ psi}$$ ←



Problem 7.2-14 Solve the preceding problem for $\sigma_x = 32$ MPa and $\sigma_y = -50$ MPa (see figure).



Solution 7.2-14 **Biaxial stress**



$\sigma_x = 32$ MPa
$\sigma_y = -50$ MPa
$\tau_{xy} = 0$

Find angles θ for $\sigma = 0$.
σ = normal stress on plane *a-a*

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$= -9 + 41\cos 2\theta \text{ (MPa)}$$

For $\sigma_{x_1} = 0$, we obtain $\cos 2\theta = \dfrac{9}{41}$

$\therefore \quad 2\theta = 77.32°$ and $\theta = 38.66°$ ⟵

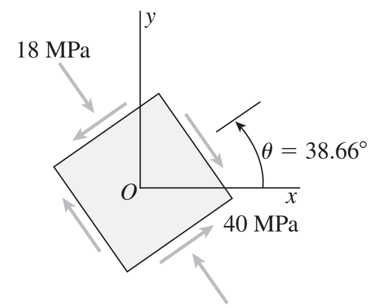
STRESS ELEMENT

$\sigma_{x_1} = 0 \qquad \theta = 38.66°$

$\sigma_{y_1} = \sigma_x + \sigma_y - \sigma_{x_1} = -18$ MPa ⟵

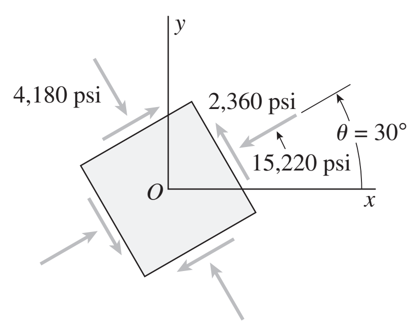
$$\tau_{x_1y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$= -40 \text{ MPa}$$ ⟵



Problem 7.2-15 An element in *plane stress* from the frame of a racing car is oriented at a known angle θ (see figure). On this inclined element, the normal and shear stresses have the magnitudes and directions shown in the figure.

Determine the normal and shear stresses acting on an element whose sides are parallel to the *xy* axes; that is, determine σ_x, σ_y, and τ_{xy}. Show the results on a sketch of an element oriented at $\theta = 0°$.



Solution 7.2-15 **Plane stress**

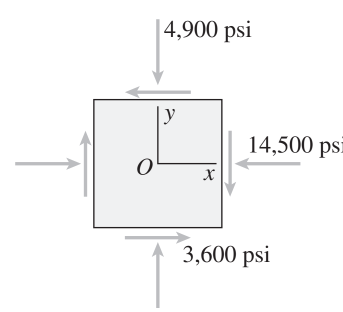
Transform from $\theta = 30°$ to $\theta = 0°$.
Let: $\sigma_x = -15{,}220$ psi, $\sigma_y = -4{,}180$ psi,
$\tau_{xy} = 2{,}360$ psi, and $\theta = -30°$.

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$= -14{,}500 \text{ psi}$$

$$\tau_{x_1y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta = -3{,}600 \text{ psi}$$

$$\sigma_{y_1} = \sigma_x + \sigma_y - \sigma_{x_1} = -4{,}900 \text{ psi}$$



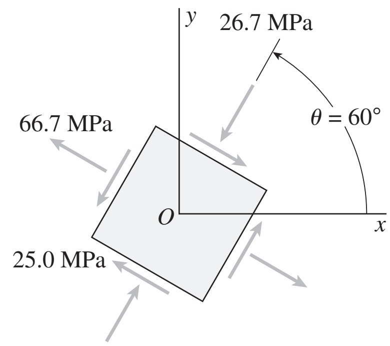
FOR $\theta = 0$:

$\sigma_x = \sigma_{x_1} = -14{,}500$ psi ⟵

$\sigma_y = \sigma_{y_1} = -4{,}900$ psi ⟵

$\tau_{xy} = \tau_{x_1y_1} = -3{,}600$ psi ⟵

Problem 7.2-16 Solve the preceding problem for the element shown in the figure.

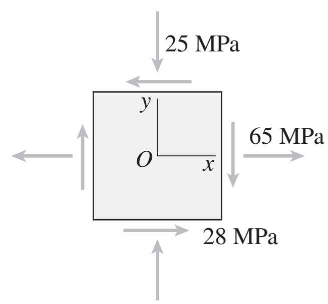


Solution 7.2-16 **Plane stress**

Transform from $\theta = 60°$ to $\theta = 0°$.
Let: $\sigma_x = -26.7$ MPa, $\sigma_y = 66.7$ MPa,
$\tau_{xy} = -25.0$ MPa, and $\theta = -60°$.

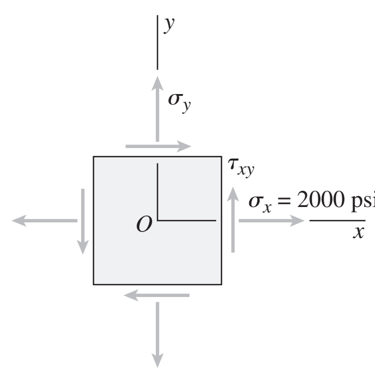
$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$
$$= 65 \text{ MPa}$$
$$\tau_{x_1y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta = -28 \text{ MPa}$$
$$\sigma_{y_1} = \sigma_x + \sigma_y - \sigma_{x_1} = -25 \text{ MPa}$$

FOR $\theta = 0$:
$\sigma_x = \sigma_{x_1} = 65$ MPa ⟵
$\sigma_y = \sigma_{y_1} = -25$ MPa ⟵
$\tau_{xy} = \tau_{x_1y_1} = -28$ MPa ⟵



Problem 7.2-17 A plate in *plane stress* is subjected to normal stresses σ_x and σ_y and shear stress τ_{xy}, as shown in the figure. At counterclockwise angles $\theta = 40°$ and $\theta = 80°$ from the x axis the normal stress is 5000 psi tension.

If the stress σ_x equals 2000 psi tension, what are the stresses σ_y and τ_{xy}?



Solution 7.2-17 **Plane stress**

$\sigma_x = 2000$ psi $\quad \sigma_y = ? \quad \tau_{xy} = ?$
At $\theta = 40°$ and $\theta = 80°$; $\sigma_{x_1} = 5000$ psi (tension)
Find σ_y and τ_{xy}.

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

FOR $\theta = 40°$:

$\sigma_{x_1} = 5000$

$$= \frac{2000 + \sigma_y}{2} + \frac{2000 - \sigma_y}{2}\cos 80° + \tau_{xy}\sin 80°$$

or $\quad 0.41318\sigma_y + 0.98481\tau_{xy} = 3826.4$ psi $\qquad$ (1)

FOR $\theta = 80°$:

$\sigma_{x_1} = 5000$

$$= \frac{2000 + \sigma_y}{2} + \frac{2000 - \sigma_y}{2}\cos 160° + \tau_{xy}\sin 160°$$

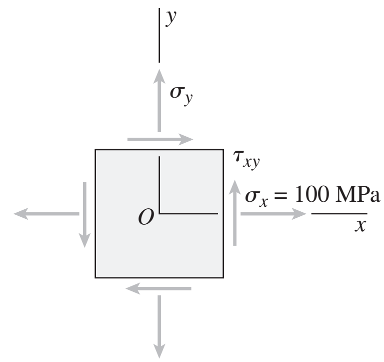
or $\quad 0.96985\sigma_y + 0.34202\tau_{xy} = 4939.7$ psi $\qquad$ (2)

SOLVE EQS. (1) AND (2):

$\sigma_y = 4370$ psi $\qquad \tau_{xy} = 2050$ psi ⟵

Problem 7.2-18 The surface of an airplane wing is subjected to *plane stress* with normal stresses σ_x and σ_y and shear stress τ_{xy}, as shown in the figure. At a counterclockwise angle $\theta = 30°$ from the x axis the normal stress is 35 MPa tension, and at an angle $\theta = 50°$ it is 10 MPa compression.

If the stress σ_x equals 100 MPa tension, what are the stresses σ_y and τ_{xy}?



Solution 7.2-18 **Plane stress**

$\sigma_x = 100$ MPa $\quad \sigma_y = ?\quad \tau_{xy} = ?$

At $\theta = 30°$, $\sigma_{x_1} = 35$ MPa (tension)

At $\theta = 50°$, $\sigma_{x_1} = -10$ MPa (compression)

Find σ_y and τ_{xy}

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

FOR $\theta = 30°$:

$$\sigma_{x_1} = 35$$

$$= \frac{100 + \sigma_y}{2} + \frac{100 - \sigma_y}{2}\cos 60° + \tau_{xy}\sin 60°$$

or $\quad 0.25\sigma_y + 0.86603\tau_{xy} = -40 \text{ MPa} \quad (1)$

FOR $\theta = 50°$:

$$\sigma_{x_1} = -10$$

$$= \frac{100 + \sigma_y}{2} + \frac{100 - \sigma_y}{2}\cos 100° + \tau_{xy}\sin 100°$$

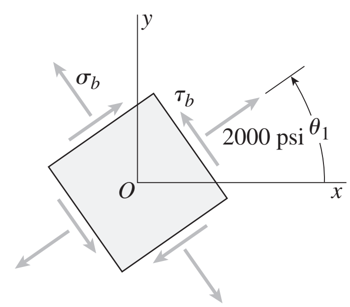
or $\quad 0.58682\sigma_y + 0.98481\tau_{xy} = -51.318 \text{ MPa} \quad (2)$

SOLVE EQS. (1) AND (2):

$\sigma_y = -19.3$ MPa $\quad \tau_{xy} = -40.6$ MPa ⟵

Problem 7.2-19 At a point in a structure subjected to *plane stress*, the stresses are $\sigma_x = -4000$ psi, $\sigma_y = 2500$ psi, and $\tau_{xy} = 2800$ psi (the sign convention for these stresses is shown in Fig. 7-1). A stress element located at the same point in the structure, but oriented at a counterclockwise angle θ_1 with respect to the x axis, is subjected to the stresses shown in the figure (σ_b, τ_b, and 2000 psi).

Assuming that the angle θ_1 is between zero and 90°, calculate the normal stress σ_b, the shear stress τ_b, and the angle θ_1.



Solution 7.2-19 **Plane stress**

$\sigma_x = -4000$ psi $\quad \sigma_y = 2500$ psi $\quad \tau_{xy} = 2800$ psi

FOR $\theta = \theta_1$:

$\sigma_{x_1} = 2000$ psi $\quad \sigma_{y_1} = \sigma_b \quad \tau_{xy} = \tau_b$

Find σ_b, τ_b, and θ_1

STRESS σ_b

$\sigma_b = \sigma_x + \sigma_y - 2000 \text{ psi} = -3500$ psi ⟵

ANGLE θ_1

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$2000 \text{ psi} = -750 - 3250\cos 2\theta_1 + 2800\sin 2\theta_1$

or $-65\cos 2\theta_1 + 56\sin 2\theta_1 - 55 = 0$

Solve numerically:

$2\theta_1 = 89.12° \quad \theta_1 = 44.56°$ ⟵

SHEAR STRESS τ_b

$$\tau_b = \tau_{x_1y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta_1 + \tau_{xy}\cos 2\theta_1$$

$= 3290$ psi ⟵

Principal Stresses and Maximum Shear Stresses

When solving the problems for Section 7.3, consider only the in-plane stresses (the stresses in the xy plane).

Problem 7.3-1 An element in plane stress is subjected to stresses $\sigma_x = 6500$ psi, $\sigma_y = 1700$ psi, and $\tau_{xy} = 2750$ psi (see the figure for Problem 7.2-1).

Determine the principal stresses and show them on a sketch of a properly oriented element.

Solution 7.3-1 **Principal stresses**

$\sigma_x = 6500$ psi $\quad \sigma_y = 1700$ psi $\quad \tau_{xy} = 2750$ psi

PRINCIPAL STRESSES

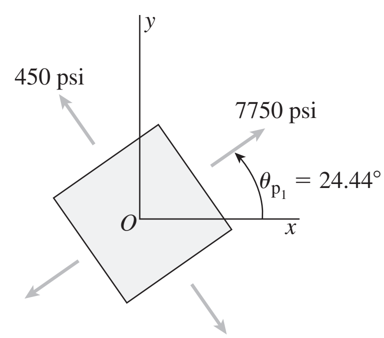
$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = 1.1458$$

$2\theta_p = 48.89°$ and $\theta_p = 24.44°$
$2\theta_p = 228.89°$ and $\theta_p = 114.44°$

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

For $2\theta_p = 48.89°$: $\sigma_{x_1} = 7750$ psi
For $2\theta_p = 228.89°$: $\sigma_{x_1} = 450$ psi

$$\left.\begin{array}{r}\text{Therefore, } \sigma_1 = 7750 \text{ psi and } \theta_{p_1} = 24.44° \\ \sigma_2 = 450 \text{ psi and } \theta_{p_2} = 114.44°\end{array}\right\} \longleftarrow$$



Problem 7.3-2 An element in plane stress is subjected to stresses $\sigma_x = 80$ MPa, $\sigma_y = 52$ MPa, and $\tau_{xy} = 48$ MPa (see the figure for Problem 7.2-2).

Determine the principal stresses and show them on a sketch of a properly oriented element.

Solution 7.3-2 **Principal stresses**

$\sigma_x = 80$ MPa $\quad \sigma_y = 52$ MPa $\quad \tau_{xy} = 48$ MPa

PRINCIPAL STRESSES

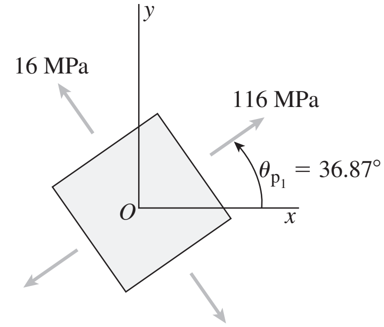
$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = 3.429$$

$2\theta_p = 73.74°$ and $\theta_p = 36.87°$
$2\theta_p = 253.74°$ and $\theta_p = 126.87°$

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

For $2\theta_p = 73.74°$: $\sigma_{x_1} = 116$ MPa
For $2\theta_p = 253.74°$: $\sigma_{x_1} = 16$ MPa

$$\left.\begin{array}{r}\text{Therefore, } \sigma_1 = 116 \text{ MPa and } \theta_{p_1} = 36.87° \\ \sigma_2 = 16 \text{ MPa and } \theta_{p_2} = 126.87°\end{array}\right\} \longleftarrow$$



Problem 7.3-3 An element in plane stress is subjected to stresses $\sigma_x = -9{,}900$ psi, $\sigma_y = -3{,}400$ psi, and $\tau_{xy} = 3{,}600$ psi (see the figure for Problem 7.2-3).

Determine the principal stresses and show them on a sketch of a properly oriented element.

Solution 7.3-3 Principal stresses

$\sigma_x = -9900$ psi $\quad \sigma_y = -3400$ psi $\quad \tau_{xy} = 3600$ psi

PRINCIPAL STRESSES

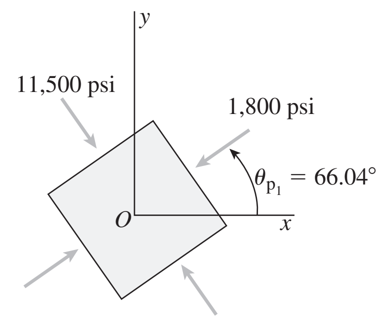
$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = -1.1077$$

$2\theta_p = -47.92°$ and $\theta_p = -23.96°$
$2\theta_p = 132.08°$ and $\theta_p = 66.04°$

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

For $2\theta_p = -47.92°$: $\sigma_{x_1} = -11{,}500$ psi
For $2\theta_p = 132.08°$: $\sigma_{x_1} = -1{,}800$ psi

$$\left.\begin{aligned}\text{Therefore, } \sigma_1 &= -1{,}800 \text{ psi and } \theta_{p_1} = 66.04° \\ \sigma_2 &= -11{,}500 \text{ psi and } \theta_{p_2} = -23.96°\end{aligned}\right\} \longleftarrow$$



Problem 7.3-4 An element in plane stress is subjected to stresses $\sigma_x = 42$ MPa, $\sigma_y = -140$ MPa, and $\tau_{xy} = -60$ MPa (see the figure for Problem 7.2-4).

Determine the principal stresses and show them on a sketch of a properly oriented element.

Solution 7.3-4 Principal stresses

$\sigma_x = 42$ MPa $\quad \sigma_y = -140$ MPa
$\tau_{xy} = -60$ MPa

PRINCIPAL STRESSES

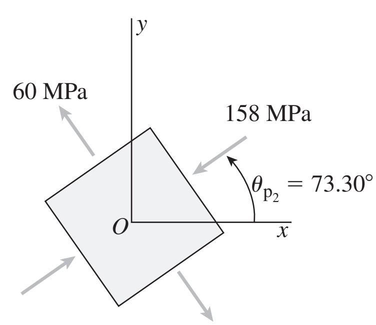
$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = -0.6593$$

$2\theta_p = -33.40°$ and $\theta_p = -16.70°$
$2\theta_p = 146.60°$ and $\theta_p = 73.30°$

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

For $2\theta_p = -33.40°$: $\sigma_{x_1} = 60$ MPa
For $2\theta_p = 146.60°$: $\sigma_{x_1} = -158$ MPa

$$\left.\begin{aligned}\text{Therefore, } \sigma_1 &= 60 \text{ MPa and } \theta_{p_1} = -16.70° \\ \sigma_2 &= -158 \text{ MPa and } \theta_{p_2} = 73.30°\end{aligned}\right\} \longleftarrow$$



Problem 7.3-5 An element in plane stress is subjected to stresses $\sigma_x = 7{,}500$ psi, $\sigma_y = -20{,}500$ psi, and $\tau_{xy} = -4{,}800$ psi (see the figure for Problem 7.2-5).

Determine the maximum shear stresses and associated normal stresses and show them on a sketch of a properly oriented element.

Solution 7.3-5 Maximum shear stresses

$\sigma_x = 7{,}500$ psi $\quad \sigma_y = -20{,}500$ psi

$\tau_{xy} = -4{,}800$ psi

PRINCIPAL ANGLES

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = -0.3429$$

$2\theta_p = -18.92°$ and $\theta_p = -9.46°$

$2\theta_p = 161.08°$ and $\theta_p = 80.54°$

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

For $2\theta_p = -18.92°$: $\quad \sigma_{x_1} = 8{,}300$ psi

For $2\theta_p = 161.08°$: $\quad \sigma_{x_1} = -21{,}300$ psi

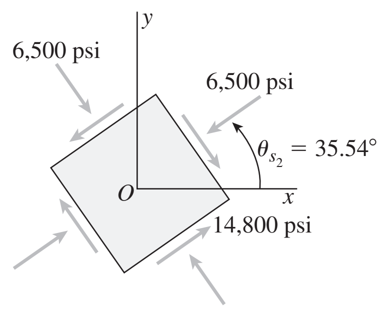
Therefore, $\theta_{p_1} = -9.46°$

MAXIMUM SHEAR STRESSES

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 14{,}800 \text{ psi}$$

$$\left.\begin{aligned}\theta_{s_1} &= \theta_{p_1} - 45° = -54.46° \text{ and } \tau = 14{,}800 \text{ psi}\\ \theta_{s_2} &= \theta_{p_1} + 45° = 35.54° \text{ and } \tau = -14{,}800 \text{ psi}\end{aligned}\right\} \longleftarrow$$

$$\sigma_{\text{aver}} = \frac{\sigma_x + \sigma_y}{2} = -6{,}500 \text{ psi} \quad \longleftarrow$$



Problem 7.3-6 An element in plane stress is subjected to stresses $\sigma_x = -25.5$ MPa, $\sigma_y = 6.5$ MPa, and $\tau_{xy} = -12.0$ MPa (see the figure for Problem 7.2-6).

Determine the maximum shear stresses and associated normal stresses and show them on a sketch of a properly oriented element.

Solution 7.3-6 Maximum shear stresses

$\sigma_x = -25.5$ MPa $\quad \sigma_y = 6.5$ MPa

$\tau_{xy} = -12.0$ MPa

PRINCIPAL ANGLES

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = 0.7500$$

$2\theta_p = 36.87°$ and $\theta_p = 18.43°$

$2\theta_p = 216.87°$ and $\theta_p = 108.43°$

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

For $2\theta_p = 36.87°$: $\quad \sigma_{x_1} = -29.5$ MPa

For $2\theta_p = 216.87°$: $\quad \sigma_{x_1} = 10.5$ MPa

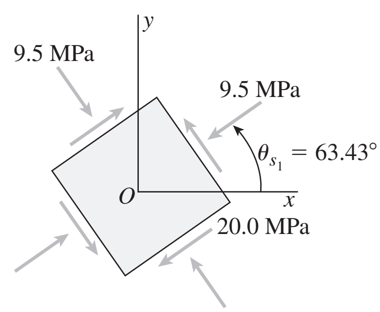
Therefore, $\theta_{p_1} = 108.4°$

MAXIMUM SHEAR STRESSES

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 20.0 \text{ MPa}$$

$$\left.\begin{aligned}\theta_{s_1} &= \theta_{p_1} - 45° = 63.48° \text{ and } \tau = 20.0 \text{ MPa}\\ \theta_{s_2} &= \theta_{p_1} + 45° = 153.43° \text{ and } \tau = -20.0 \text{ MPa}\end{aligned}\right\} \longleftarrow$$

$$\sigma_{\text{aver}} = \frac{\sigma_x + \sigma_y}{2} = -9.5 \text{ MPa} \quad \longleftarrow$$



Problem 7.3-7 An element in plane stress is subjected to stresses $\sigma_x = -11{,}000$ psi, $\sigma_y = -3{,}000$ psi, and $\tau_{xy} = -4200$ psi (see the figure for Problem 7.2-7).

Determine the maximum shear stresses and associated normal stresses and show them on a sketch of a properly oriented element.

Solution 7.3-7 Maximum shear stresses

$\sigma_x = -11{,}000$ psi $\quad \sigma_y = -3{,}000$ psi

$\tau_{xy} = -4{,}200$ psi

PRINCIPAL ANGLES

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = 1.0500$$

$2\theta_p = 46.40°$ and $\theta_p = 23.20°$

$2\theta_p = 226.40°$ and $\theta_p = 113.20°$

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

For $2\theta_p = 46.40°$: $\quad \sigma_{x_1} = -12{,}800$ psi

For $2\theta_p = 226.40°$: $\quad \sigma_{x_1} = -1{,}200$ psi

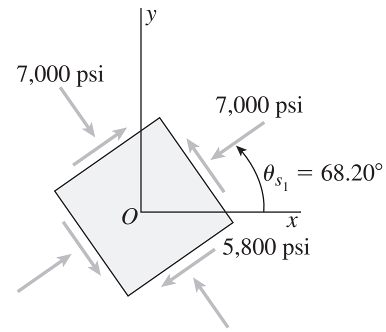
Therefore, $\theta_{p_1} = 113.20°$

MAXIMUM SHEAR STRESSES

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 5{,}800 \text{ psi}$$

$$\left.\begin{aligned}\theta_{s_1} &= \theta_{p_1} - 45° = 68.20° \text{ and } \tau = 5{,}800 \text{ psi}\\ \theta_{s_2} &= \theta_{p_1} + 45° = 158.20° \text{ and } \tau = -5{,}800 \text{ psi}\end{aligned}\right\} \longleftarrow$$

$$\sigma_{aver} = \frac{\sigma_x + \sigma_y}{2} = -7{,}000 \text{ psi} \quad \longleftarrow$$



Problem 7.3-8 An element in plane stress is subjected to stresses $\sigma_x = -54$ MPa, $\sigma_y = -12$ MPa, and $\tau_{xy} = 20$ MPa (see the figure for Problem 7.2-8).

Determine the maximum shear stresses and associated normal stresses and show them on a sketch of a properly oriented element.

Solution 7.3-8 Maximum shear stresses

$\sigma_x = -54$ MPa $\quad \sigma_y = -12$ MPa

$\tau_{xy} = 20$ MPa

PRINCIPAL ANGLES

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = -0.9524$$

$2\theta_p = -43.60°$ and $\theta_p = -21.80°$

$2\theta_p = 136.40°$ and $\theta_p = 68.20°$

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

For $2\theta_p = -43.60°$: $\quad \sigma_{x_1} = -62$ MPa

For $2\theta_p = 136.40°$: $\quad \sigma_{x_1} = -4.0$ MPa

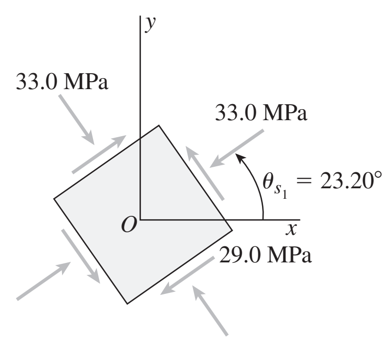
Therefore, $\theta_{p_1} = 68.20°$

MAXIMUM SHEAR STRESSES

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 29.0 \text{ MPa}$$

$$\left.\begin{aligned}\theta_{s_1} &= \theta_{p_1} - 45° = 23.20° \text{ and } \tau = 29.0 \text{ MPa}\\ \theta_{s_2} &= \theta_{p_1} + 45° = 113.20° \text{ and } \tau = -29.0 \text{ MPa}\end{aligned}\right\} \longleftarrow$$

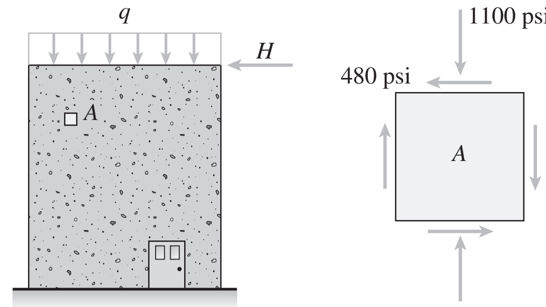
$$\sigma_{aver} = \frac{\sigma_x + \sigma_y}{2} = -33.0 \text{ MPa} \quad \longleftarrow$$



Problem 7.3-9 A shear wall in a reinforced concrete building is subjected to a vertical uniform load of intensity q and a horizontal force H, as shown in the first part of the figure. (The force H represents the effects of wind and earthquake loads.) As a consequence of these loads, the stresses at point A on the surface of the wall have the values shown in the second part of the figure (compressive stress equal to 1100 psi and shear stress equal to 480 psi).

(a) Determine the principal stresses and show them on a sketch of a properly oriented element.

(b) Determine the maximum shear stresses and associated normal stresses and show them on a sketch of a properly oriented element.



Solution 7.3-9 **Shear wall**

$\sigma_x = 0 \quad \sigma_y = -1100 \text{ psi} \quad \tau_{xy} = -480 \text{ psi}$

(a) PRINCIPAL STRESSES

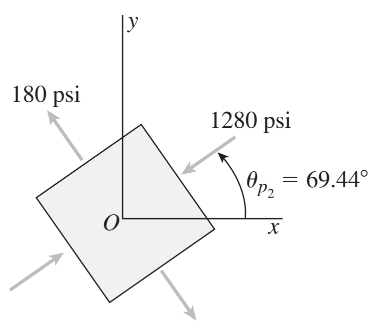
$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = -0.87273$$

$2\theta_p = -41.11°$ and $\theta_p = -20.56°$
$2\theta_p = 138.89°$ and $\theta_p = 69.44°$

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

For $2\theta_p = -41.11°$: $\sigma_{x_1} = 180$ psi
For $2\theta_p = 138.89°$: $\sigma_{x_1} = -1280$ psi

$$\left.\begin{aligned}\text{Therefore, } \sigma_1 &= 180 \text{ psi and } \theta_{p_1} = -20.56° \\ \sigma_2 &= -1280 \text{ psi and } \theta_{p_2} = 69.44°\end{aligned}\right\} \longleftarrow$$

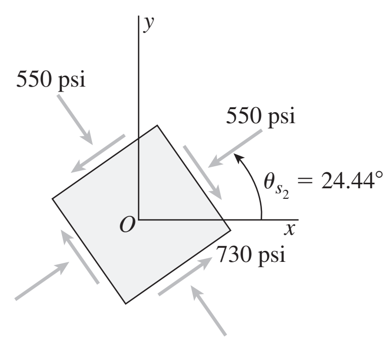


(b) MAXIMUM SHEAR STRESSES

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 730 \text{ psi}$$

$$\left.\begin{aligned}\theta_{s_1} &= \theta_{p_1} - 45° = -65.56° \text{ and } \tau = 730 \text{ psi} \\ \theta_{s_2} &= \theta_{p_1} + 45° = 24.44° \quad \text{and } \tau = -730 \text{ psi}\end{aligned}\right\} \longleftarrow$$

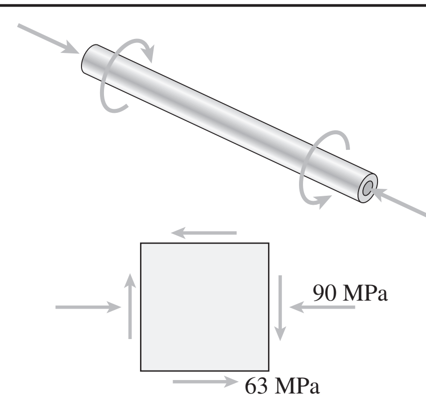
$$\sigma_{aver} = \frac{\sigma_x + \sigma_y}{2} = -550 \text{ psi} \quad \longleftarrow$$



Problem 7.3-10 A propeller shaft subjected to combined torsion and axial thrust is designed to resist a shear stress of 63 MPa and a compressive stress of 90 MPa (see figure).

(a) Determine the principal stresses and show them on a sketch of a properly oriented element.

(b) Determine the maximum shear stresses and associated normal stresses and show them on a sketch of a properly oriented element.



Solution 7.3-10 Propeller shaft

$\sigma_x = -90$ MPa $\quad \sigma_y = 0 \quad \tau_{xy} = -63$ MPa

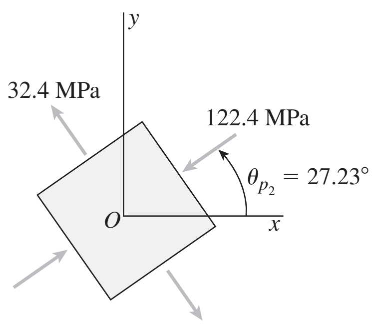
(a) PRINCIPAL STRESSES

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = 1.4000$$

$2\theta_p = 54.46°$ and $\theta_p = 27.23°$
$2\theta_p = 234.46°$ and $\theta_p = 117.23°$

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

For $2\theta_p = 54.46°$: $\sigma_{x_1} = -122.4$ MPa
For $2\theta_p = 234.46°$: $\sigma_{x_1} = 32.4$ MPa



Therefore,

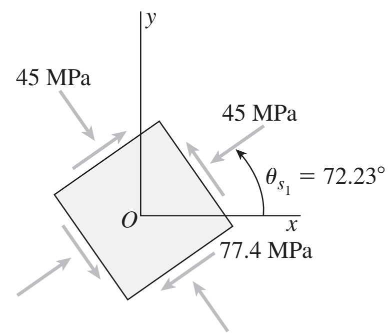
$$\left.\begin{aligned}\sigma_1 &= 32.4 \text{ MPa and } \theta_{p_1} = 117.23° \\ \sigma_2 &= -122.4 \text{ MPa and } \theta_{p_2} = 27.23°\end{aligned}\right\} \longleftarrow$$

(b) MAXIMUM SHEAR STRESSES

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 77.4 \text{ MPa}$$

$$\left.\begin{aligned}\theta_{s_1} &= \theta_{p_1} - 45° = 72.23° \text{ and } \tau = 77.4 \text{ MPa} \\ \theta_{s_2} &= \theta_{p_1} + 45° = 162.23° \text{ and } \tau = -77.4 \text{ MPa}\end{aligned}\right\} \longleftarrow$$

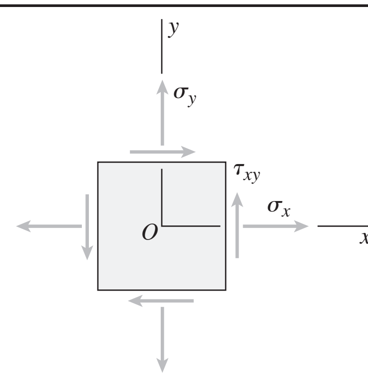
$$\sigma_{\text{aver}} = \frac{\sigma_x + \sigma_y}{2} = -45 \text{ MPa} \longleftarrow$$



Problems 7.3-11 through 7.3-16 An element in *plane stress* (see figure) is subjected to stresses σ_x, σ_y, and τ_{xy}.

(a) Determine the principal stresses and show them on a sketch of a properly oriented element.

(b) Determine the maximum shear stresses and associated normal stresses and show them on a sketch of a properly oriented element.



Data for 7.3-11 $\sigma_x = 3500$ psi, $\sigma_y = 1120$ psi, $\tau_{xy} = -1200$ psi

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Solution 7.3-11 Plane stress

$\sigma_x = 3500$ psi $\quad \sigma_y = 1120$ psi $\quad \tau_{xy} = -1200$ psi

(a) PRINCIPAL STRESSES

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = -1.0084$$

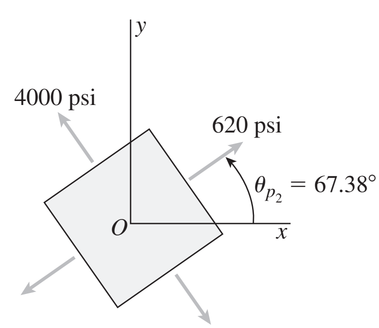
$2\theta_p = -45.24°$ and $\theta_p = -22.62°$
$2\theta_p = 134.76°$ and $\theta_p = 67.38°$

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

For $2\theta_p = -45.24°$: $\sigma_{x_1} = 4000$ psi
For $2\theta_p = 134.76°$: $\sigma_{x_1} = 620$ psi

Therefore,

$$\left.\begin{aligned}\sigma_1 &= 4000 \text{ psi and } \theta_{p_1} = -22.62° \\ \sigma_2 &= 620 \text{ psi and } \theta_{p_2} = 67.38°\end{aligned}\right\} \longleftarrow$$

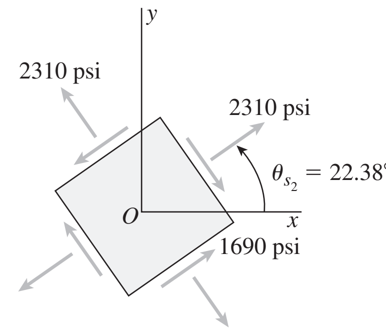


(b) Maximum shear stresses

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 1690 \text{ psi}$$

$$\left.\begin{aligned} \theta_{s_1} &= \theta_{p_1} - 45° = -67.62° \text{ and } \tau = 1690 \text{ psi} \\ \theta_{s_2} &= \theta_{p_1} + 45° = 22.38° \text{ and } \tau = -1690 \text{ psi} \end{aligned}\right\} \longleftarrow$$

$$\sigma_{aver} = \frac{\sigma_x + \sigma_y}{2} = 2310 \text{ psi} \quad \longleftarrow$$



Data for 7.3-12 $\sigma_x = 2100$ kPa, $\sigma_y = 300$ kPa, $\tau_{xy} = -560$ kPa

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Solution 7.3-12 **Plane stress**

$\sigma_x = 2100$ kPa $\quad \sigma_y = 300$ kPa $\quad \tau_{xy} = -560$ kPa

(a) Principal stresses

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = -0.6222$$

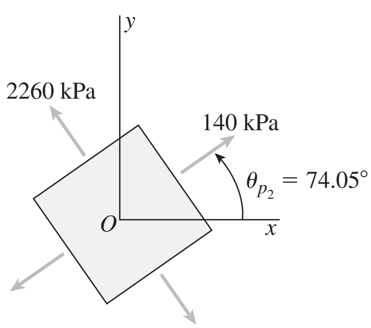
$$2\theta_p = -31.89° \quad \text{and} \quad \theta_p = -15.95°$$
$$2\theta_p = 148.11° \quad \text{and} \quad \theta_p = 74.05°$$

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

For $2\theta_p = -31.89°$: $\quad \sigma_{x_1} = 2260$ kPa

For $2\theta_p = 148.11°$: $\quad \sigma_{x_1} = 140$ kPa

$$\left.\begin{aligned} \text{Therefore, } \sigma_1 &= 2260 \text{ kPa and } \theta_{p_1} = -15.95° \\ \sigma_2 &= 140 \text{ kPa and } \theta_{p_2} = 74.05° \end{aligned}\right\} \longleftarrow$$

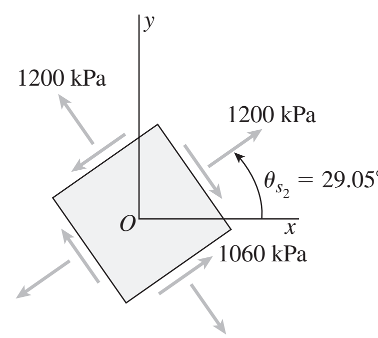


(b) Maximum shear stresses

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 1060 \text{ kPa}$$

$$\left.\begin{aligned} \theta_{s_1} &= \theta_{p_1} - 45° = -60.95° \text{ and } \tau = 1060 \text{ kPa} \\ \theta_{s_2} &= \theta_{p_1} + 45° = 29.05° \text{ and } \tau = -1060 \text{ kPa} \end{aligned}\right\} \longleftarrow$$

$$\sigma_{aver} = \frac{\sigma_x + \sigma_y}{2} = 1200 \text{ kPa} \quad \longleftarrow$$



Data for 7.3-13 $\sigma_x = 15{,}000$ psi, $\sigma_y = 1{,}000$ psi, $\tau_{xy} = 2{,}400$ psi

Solution 7.3-13 Plane stress

$\sigma_x = 15{,}000$ psi $\quad \sigma_y = 1{,}000$ psi $\quad \tau_{xy} = 2{,}400$ psi

(a) PRINCIPAL STRESSES

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = 0.34286$$

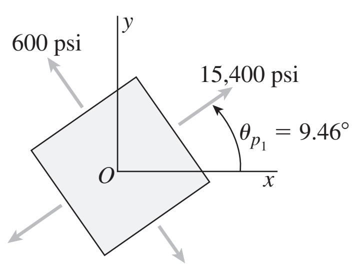
$2\theta_p = 18.92°$ and $\theta_p = 9.46°$
$2\theta_p = 198.92°$ and $\theta_p = 99.46°$

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

For $2\theta_p = 18.92°$: $\quad \sigma_{x_1} = 15{,}400$ psi

For $2\theta_p = 198.92°$: $\quad \sigma_{x_1} = 600$ psi

Therefore, $\sigma_1 = 15{,}400$ psi and $\theta_{p_1} = 9.46°$
$\sigma_2 = 600$ psi and $\theta_{p_2} = 99.96°$ ←

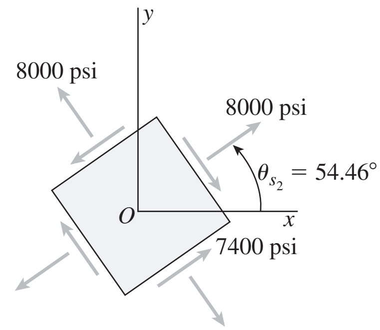


(b) MAXIMUM SHEAR STRESSES

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 7{,}400 \text{ psi}$$

$\theta_{s_1} = \theta_{p_1} - 45° = -35.54°$ and $\tau = 7{,}400$ psi
$\theta_{s_2} = \theta_{p_1} + 45° = 54.46°$ and $\tau = -7{,}400$ psi ←

$$\sigma_{aver} = \frac{\sigma_x + \sigma_y}{2} = 8{,}000 \text{ psi} \leftarrow$$



Data for 7.3-14 $\sigma_x = 16$ MPa, $\sigma_y = -96$ MPa, $\tau_{xy} = -42$ MPa

Solution 7.3-14 Plane stress

$\sigma_x = 16$ MPa $\quad \sigma_y = -96$ MPa $\quad \tau_{xy} = -42$ MPa

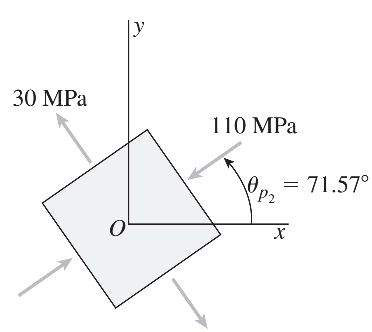
(a) PRINCIPAL STRESSES

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = -0.7500$$

$2\theta_p = -36.87°$ and $\theta_p = -18.43°$
$2\theta_p = 143.13°$ and $\theta_p = 71.57°$

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

For $2\theta_p = -36.87°$: $\quad \sigma_{x_1} = 30$ MPa
For $2\theta_p = 143.13°$: $\quad \sigma_{x_1} = -110$ MPa
Therefore, $\sigma_1 = 30$ MPa and $\theta_{p_1} = -18.43°$
$\sigma_2 = -110$ MPa and $\theta_{p_2} = 71.57°$ ←

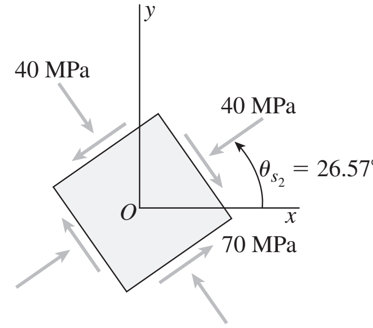


(b) MAXIMUM SHEAR STRESSES

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 70 \text{ MPa}$$

$\theta_{s_1} = \theta_{p_1} - 45° = -63.43°$ and $\tau = 70$ MPa
$\theta_{s_2} = \theta_{p_1} + 45° = 26.57°$ and $\tau = -70$ MPa ←

$$\sigma_{aver} = \frac{\sigma_x + \sigma_y}{2} = -40 \text{ MPa} \leftarrow$$



Data for 7.3-15 $\sigma_x = -3000$ psi, $\sigma_y = -12{,}000$ psi, $\tau_{xy} = 6000$ psi

Solution 7.3-15 Plane stress

$\sigma_x = -3000$ psi $\quad \sigma_y = -12{,}000$ psi
$\tau_{xy} = 6000$ psi

(a) PRINCIPAL STRESSES

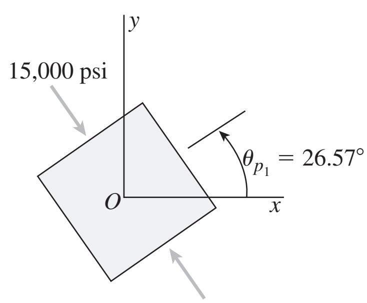
$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = 1.3333$$

$2\theta_p = 53.13°$ and $\theta_p = 26.57°$
$2\theta_p = 233.13°$ and $\theta_p = 116.57°$

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

For $2\theta_p = 53.13°$: $\quad \sigma_{x_1} = 0$
For $2\theta_p = 233.13°$: $\quad \sigma_{x_1} = -15{,}000$ psi
Therefore,

$$\left.\begin{array}{l}\sigma_1 = 0 \text{ and } \theta_{p_1} = 26.57° \\ \sigma_2 = -15{,}000 \text{ psi and } \theta_{p_2} = 116.57°\end{array}\right\} \longleftarrow$$

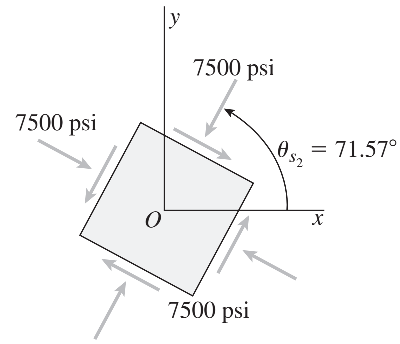


(b) MAXIMUM SHEAR STRESSES

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 7500 \text{ psi}$$

$$\left.\begin{array}{l}\theta_{s_1} = \theta_{p_1} - 45° = -18.43° \text{ and } \tau = 7500 \text{ psi} \\ \theta_{s_2} = \theta_{p_1} + 45° = 71.57° \text{ and } \tau = -7500 \text{ psi}\end{array}\right\} \longleftarrow$$

$$\sigma_{\text{aver}} = \frac{\sigma_x + \sigma_y}{2} = -7500 \text{ psi} \longleftarrow$$



Data for 7.3-16 $\sigma_x = -100$ MPa, $\sigma_y = 50$ MPa, $\tau_{xy} = -50$ MPa

Solution 7.3-16 Plane stress

$\sigma_x = -100$ MPa $\quad \sigma_y = 50$ MPa $\quad \tau_{xy} = -50$ MPa

(a) PRINCIPAL STRESSES

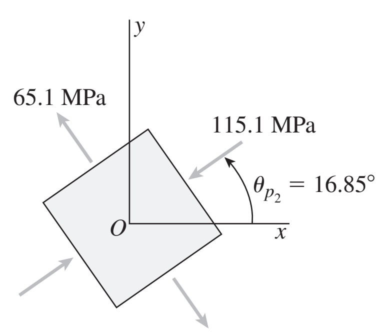
$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = 0.66667$$

$2\theta_p = 33.69°$ and $\theta_p = 16.85°$
$2\theta_p = 213.69°$ and $\theta_p = 106.85°$

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

For $2\theta_p = 33.69°$: $\quad \sigma_{x_1} = -115.1$ MPa
For $2\theta_p = 213.69°$: $\sigma_{x_1} = 65.1$ MPa
Therefore,

$$\left.\begin{array}{l}\sigma_1 = 65.1 \text{ MPa and } \theta_{p_1} = 106.85° \\ \sigma_2 = -115.1 \text{ MPa and } \theta_{p_2} = 16.85°\end{array}\right\} \longleftarrow$$

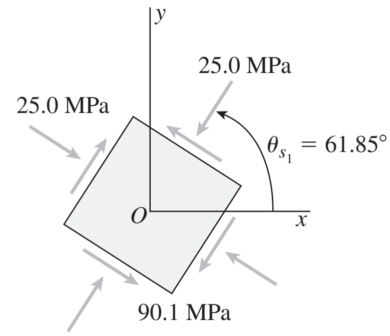


(b) MAXIMUM SHEAR STRESSES

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 90.1 \text{ MPa}$$

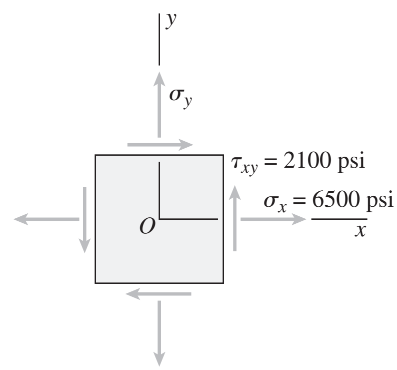
$$\left.\begin{array}{l}\theta_{s_1} = \theta_{p_1} - 45° = 61.85° \text{ and } \tau = 90.1 \text{ MPa} \\ \theta_{s_2} = \theta_{p_1} + 45° = 151.85° \text{ and } \tau = -90.1 \text{ MPa}\end{array}\right\} \longleftarrow$$

$$\sigma_{\text{aver}} = \frac{\sigma_x + \sigma_y}{2} = -25.0 \text{ MPa} \longleftarrow$$



Problem 7.3-17 At a point on the surface of a machine component the stresses acting on the x face of a stress element are $\sigma_x = 6500$ psi and $\tau_{xy} = 2100$ psi (see figure).

What is the allowable range of values for the stress σ_y if the maximum shear stress is limited to $\tau_0 = 2900$ psi?



Solution 7.3-17 Allowable range of values

$\sigma_x = 6500$ psi $\quad \tau_{xy} = 2100$ psi $\quad \sigma_y = ?$

Find the allowable range of values for σ_y if the maximum allowable shear stresses is $\tau_0 = 2900$ psi.

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \qquad \text{Eq. (1)}$$

or

$$\tau_{\max}^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2 \qquad \text{Eq. (2)}$$

SOLVE FOR σ_y

$$\sigma_y = \sigma_x \pm 2\sqrt{\tau_{\max}^2 - \tau_{xy}^2}$$

Substitute numerical values:

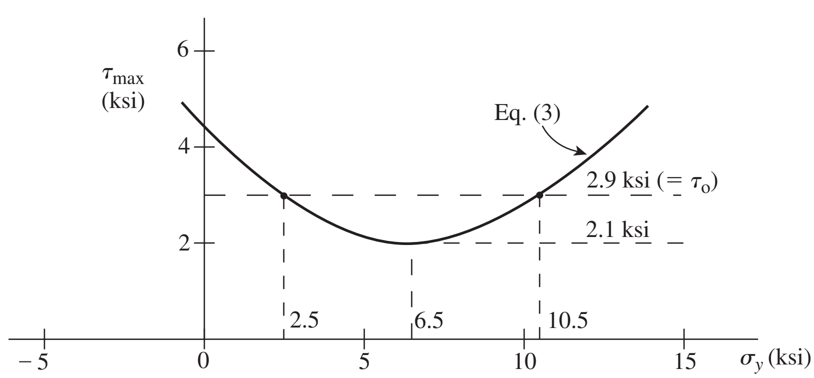
$$\sigma_y = 6500 \text{ psi} \pm 2\sqrt{(2900 \text{ psi})^2 - (2100 \text{ psi})^2}$$
$$= 6500 \text{ psi} \pm 4000 \text{ psi}$$

Therefore, $2500 \text{ psi} \le \sigma_y \le 10{,}500 \text{ psi}$ $\longleftarrow$

GRAPH OF $\tau_{\max}$

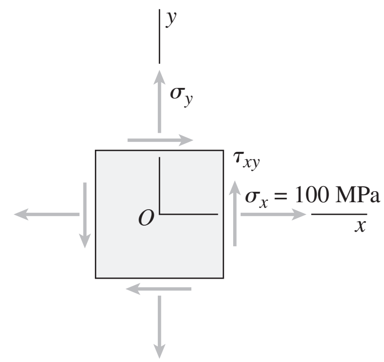
From Eq. (1):

$$\tau_{\max} = \sqrt{\left(\frac{6500 - \sigma_y}{2}\right)^2 + (2100)^2} \qquad \text{Eq. (3)}$$



Problem 7.3-18 At a point on the surface of a machine component the stresses acting on the x face of a stress element are $\sigma_x = 45$ MPa and $\tau_{xy} = 30$ MPa (see figure).

What is the allowable range of values for the stress σ_y if the maximum shear stress is limited to $\tau_0 = 34$ MPa?



Solution 7.3-18 **Allowable range of values**

$\sigma_x = 45$ MPa $\quad \tau_{xy} = 30$ MPa $\quad \sigma_y = ?$

Find the allowable range of values for σ_y if the maximum allowable shear stresses is $\tau_0 = 34$ MPa.

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \qquad \text{Eq. (1)}$$

or

$$\tau_{\max}^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2 \qquad \text{Eq. (2)}$$

SOLVE FOR σ_y

$$\sigma_y = \sigma_x \pm 2\sqrt{\tau_{\max}^2 - \tau_{xy}^2}$$

Substitute numerical values:

$$\sigma_y = 45 \text{ MPa} \pm 2\sqrt{(34 \text{ MPa})^2 - (30 \text{ MPa})^2}$$

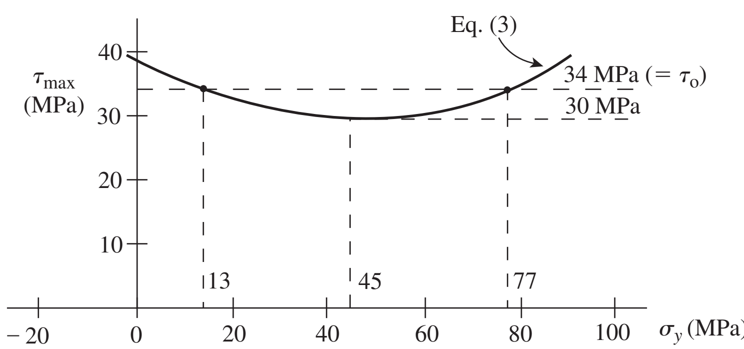
$$= 45 \text{ MPa} \pm 32 \text{ MPa}$$

Therefore, $13 \text{ MPa} \le \sigma_y \le 77 \text{ MPa}$ ⟵

GRAPH OF $\tau_{\max}$

From Eq. (1):

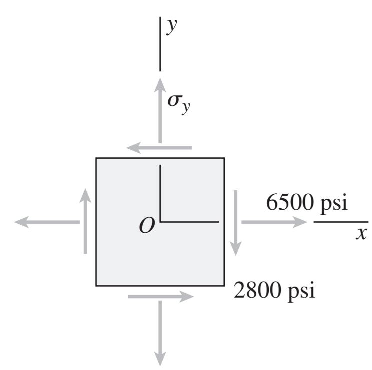
$$\tau_{\max} = \sqrt{\left(\frac{45 - \sigma_y}{2}\right)^2 + (30)^2} \qquad \text{Eq. (3)}$$



Problem 7.3-19 An element in *plane stress* is subjected to stresses $\sigma_x = 6500$ psi and $\tau_{xy} = -2800$ psi (see figure). It is known that one of the principal stresses equals 7300 psi in tension.

(a) Determine the stress σ_y.

(b) Determine the other principal stress and the orientation of the principal planes; then show the principal stresses on a sketch of a properly oriented element.



Solution 7.3-19 Plane stress

$\sigma_x = 6500$ psi $\quad \tau_{xy} = -2800$ psi $\quad \sigma_y = ?$
One principal stress $= 7300$ psi (tension)

(a) STRESS σ_y

Because σ_x is smaller than the given principal stress, we know that the given stress is the larger principal stress.

$\sigma_1 = 7300$ psi

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Substitute numerical values and solve for σ_y:

$\sigma_y = -2500$ psi ⟵

(b) PRINCIPAL STRESSES

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = -0.62222$$

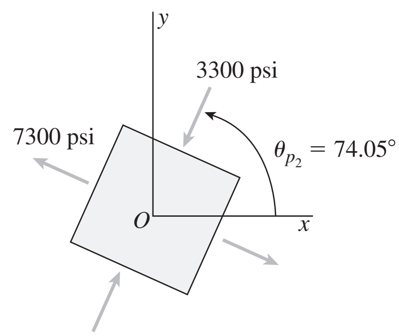
$2\theta_p = -31.891°$ and $\theta_p = -15.945°$
$2\theta_p = 148.109°$ and $\theta_p = 74.053°$

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

For $2\theta_p = -31.891°$: $\quad \sigma_{x_1} = 7300$ psi
For $2\theta_p = 148.109°$: $\quad \sigma_{x_1} = -3300$ psi

Therefore,

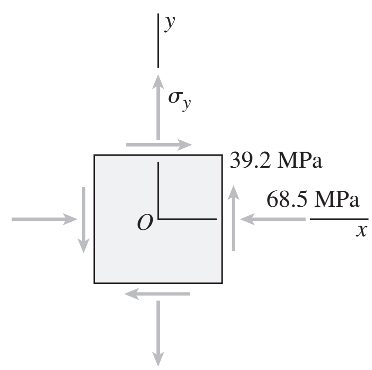
$$\left.\begin{array}{l}\sigma_1 = 7300 \text{ psi and } \theta_{p_1} = -15.95° \\ \sigma_2 = -3300 \text{ psi and } \theta_{p_2} = 74.05°\end{array}\right\} \longleftarrow$$



Problem 7.3-20 An element in *plane stress* is subjected to stresses $\sigma_x = -68.5$ MPa and $\tau_{xy} = 39.2$ MPa (see figure). It is known that one of the principal stresses equals 26.3 MPa in tension.

(a) Determine the stress σ_y.

(b) Determine the other principal stress and the orientation of the principal planes; then show the principal stresses on a sketch of a properly oriented element.



Solution 7.3-20 Plane stress

$\sigma_x = -68.5$ MPa $\quad \tau_{xy} = 39.2$ MPa $\quad \sigma_y = ?$
One principal stress $= 26.3$ MPa (tension)

(a) STRESS σ_y

Because σ_x is smaller than the given principal stress, we know that the given stress is the larger principal stress.

$\sigma_1 = 26.3$ MPa

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Substitute numerical values and solve for σ_y:

$\sigma_y = 10.1$ MPa ⟵

(b) PRINCIPAL STRESSES

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = -0.99746$$

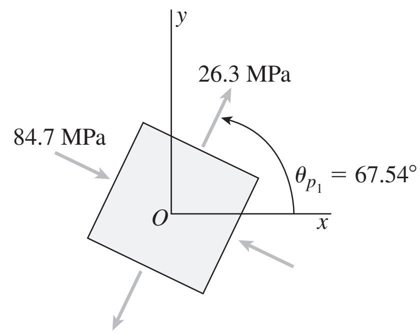
$2\theta_p = -44.93°$ and $\theta_p = -22.46°$
$2\theta_p = 135.07°$ and $\theta_p = 67.54°$

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

For $2\theta_p = -44.93°$: $\sigma_{x_1} = -84.7$ MPa
For $2\theta_p = 135.07°$: $\sigma_{x_1} = 26.3$ MPa

Therefore,

$$\left.\begin{array}{l}\sigma_1 = 26.3 \text{ MPa and } \theta_{p_1} = 67.54° \\ \sigma_2 = -84.7 \text{ MPa and } \theta_{p_2} = -22.46°\end{array}\right\} \longleftarrow$$

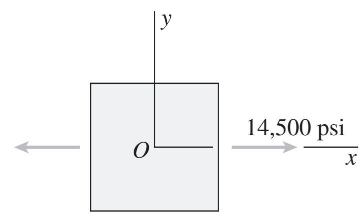


Mohr's Circle for Plane Stress

The problems for Section 7.4 are to be solved using Mohr's circle. Consider only the in-plane stresses (the stresses in the xy *plane).*

Problem 7.4-1 An element in *uniaxial stress* is subjected to tensile stresses $\sigma_x = 14{,}500$ psi, as shown in the figure.

Using Mohr's circle, determine (a) the stresses acting on an element oriented at a counterclockwise angle $\theta = 24°$ from the x axis and (b) the maximum shear stresses and associated normal stresses. Show all results on sketches of properly oriented elements.



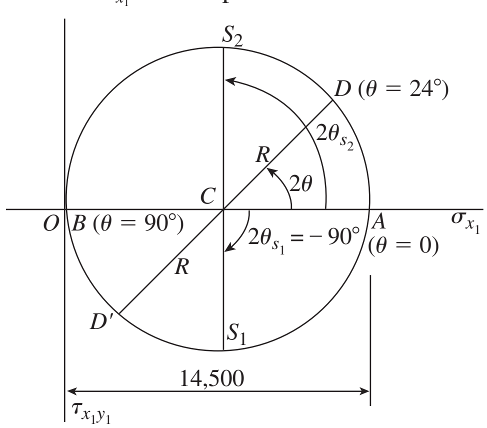
Solution 7.4-1 **Uniaxial stress**

$\sigma_x = 14{,}500$ psi $\quad \sigma_y = 0 \quad \tau_{xy} = 0$

(a) ELEMENT AT $\theta = 24°$ (All stresses in psi)

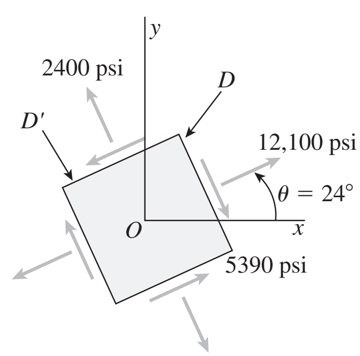
$2\theta = 48° \quad \theta = 24° \quad R = 7250$ psi

Point C: $\sigma_{x_1} = 7250$ psi



Point D: $\sigma_{x_1} = R + R\cos 2\theta = 12{,}100$ psi
$\tau_{x_1y_1} = R\sin 2\theta = -5390$ psi

Point D': $\sigma_{x_1} = R - R\cos 2\theta = 2400$ psi
$\tau_{x_1y_1} = 5390$ psi



(b) MAXIMUM SHEAR STRESSES

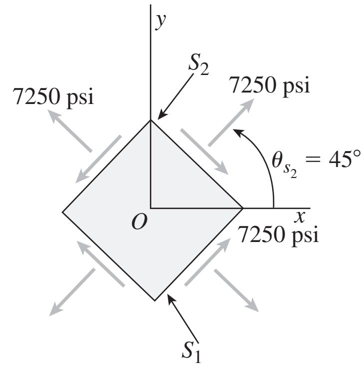
Point S_1: $2\theta_{s_1} = -90°$ $\theta_{s_1} = -45°$

$\tau_{max} = R = 7250$ psi

Point S_2: $2\theta_{s_2} = 90°$ $\theta_{s_2} = 45°$

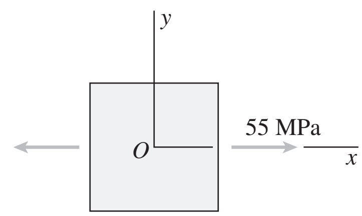
$\tau_{min} = -R = -7250$ psi

$\sigma_{aver} = R = 7250$ psi



Problem 7.4-2 An element in *uniaxial stress* is subjected to tensile stresses $\sigma_x = 55$ MPa, as shown in the figure.

Using Mohr's circle, determine (a) the stresses acting on an element oriented at an angle $\theta = -30°$ from the x axis (minus means clockwise) and (b) the maximum shear stresses and associated normal stresses. Show all results on sketches of properly oriented elements.



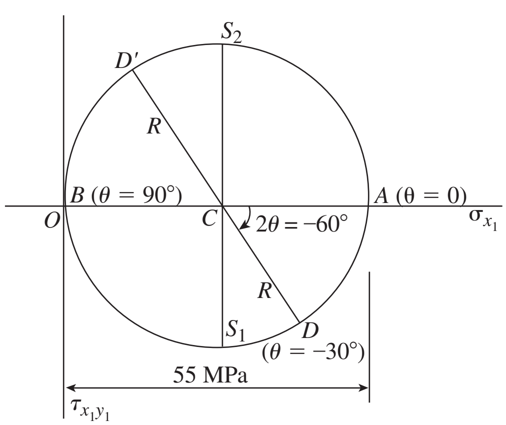
Solution 7.4-2 **Uniaxial stress**

$\sigma_x = 55$ MPa $\sigma_y = 0$ $\tau_{xy} = 0$

(a) ELEMENT AT $\theta = -30°$ (All stresses in MPa)

$2\theta = -60°$ $\theta = -30°$ $R = 27.5$ MPa

Point C: $\sigma_{x_1} = 27.5$ MPa



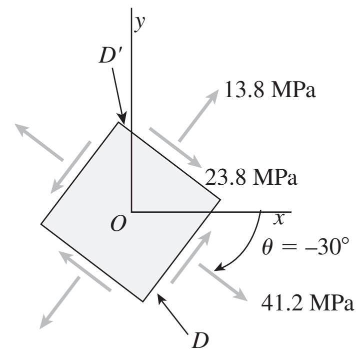
Point D: $\sigma_{x_1} = R + R\cos|2\theta|$

$= R(1 + \cos 60°) = 41.2$ MPa

$\tau_{x_1y_1} = R\sin|2\theta| = R\sin 60° = 23.8$ MPa

Point D': $\sigma_{x_1} = R - R\cos|2\theta| = 13.8$ MPa

$\tau_{x_1y_1} = -R\sin|2\theta| = -23.8$ MPa



(b) MAXIMUM SHEAR STRESSES

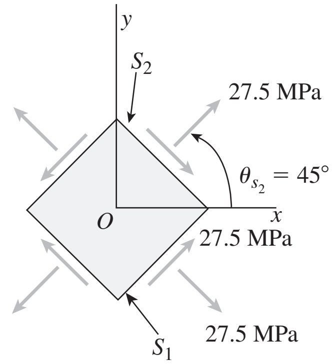
Point S_1: $2\theta_{s_1} = -90°$ $\theta_{s_1} = -45°$

$\tau_{max} = R = 27.5$ MPa

Point S_2: $2\theta_{s_2} = 90°$ $\theta_{s_2} = 45°$

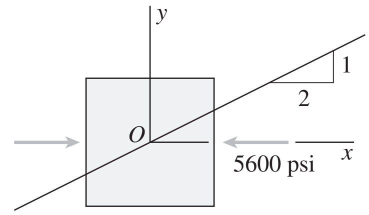
$\tau_{min} = -R = -27.5$ MPa

$\sigma_{aver} = R = 27.5$ MPa



Problem 7.4-3 An element in *uniaxial stress* is subjected to compressive stresses of magnitude 5600 psi, as shown in the figure.

Using Mohr's circle, determine (a) the stresses acting on an element oriented at a slope of 1 on 2 (see figure) and (b) the maximum shear stresses and associated normal stresses. Show all results on sketches of properly oriented elements.

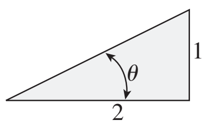


Solution 7.4-3 **Uniaxial stress**

$\sigma_x = -5600$ psi $\quad \sigma_y = 0 \quad \tau_{xy} = 0$

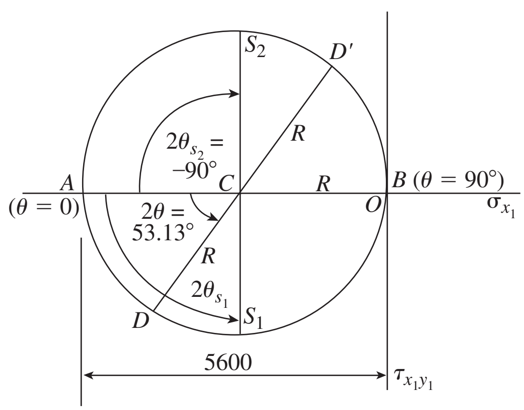
(a) ELEMENT AT A SLOPE OF 1 ON 2

(All stresses in psi) $\quad \theta = \arctan \dfrac{1}{2} = 26.565°$



$$2\theta = 53.130°$$
$$\theta = 26.57°$$
$$R = 2800 \text{ psi}$$

Point C: $\sigma_{x_1} = -2800$ psi

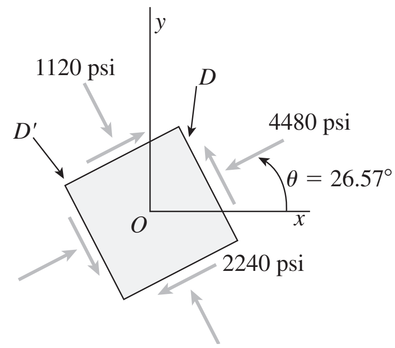


Point D: $\sigma_{x_1} = -R - R\cos 2\theta = -4480$ psi

$\tau_{x_1y_1} = R \sin 2\theta = 2240$ psi

Point D': $\sigma_{x_1} = -R + R\cos 2\theta = -1120$ psi

$\tau_{x_1y_1} = -R \sin 2\theta = -2240$ psi



(b) MAXIMUM SHEAR STRESSES

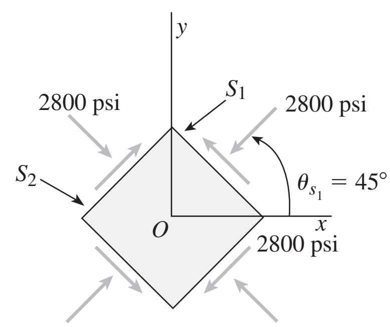
Point S_1: $2\theta_{s_1} = 90° \qquad \theta_{s_1} = 45°$

$\tau_{max} = R = 2800$ psi

Point S_2: $2\theta_{s_2} = -90° \qquad \theta_{s_2} = -45°$

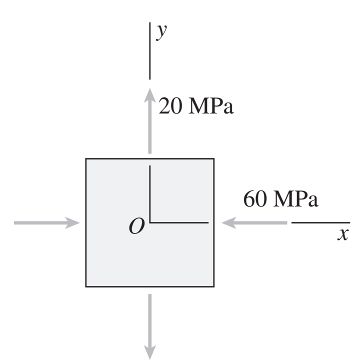
$\tau_{min} = -R = -2800$ psi

$\sigma_{aver} = -R = -2800$ psi



Problem 7.4-4 An element in *biaxial stress* is subjected to stresses $\sigma_x = -60$ MPa and $\sigma_y = 20$ MPa, as shown in the figure.

Using Mohr's circle, determine (a) the stresses acting on an element oriented at a counterclockwise angle $\theta = 22.5°$ from the x axis and (b) the maximum shear stresses and associated normal stresses. Show all results on sketches of properly oriented elements.



Solution 7.4-4 **Biaxial stress**

$\sigma_x = -60$ MPa $\quad \sigma_y = 20$ MPa $\quad \tau_{xy} = 0$

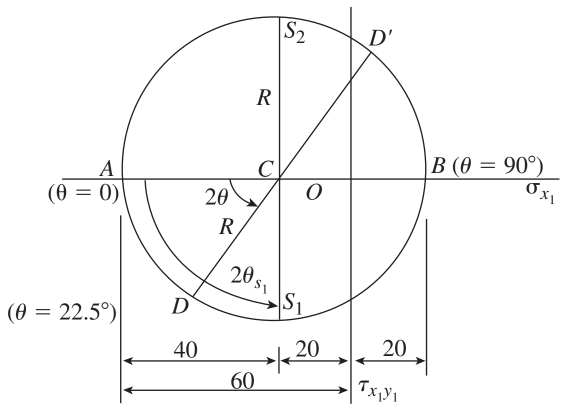
(a) ELEMENT AT $\theta = 22.5°$

(All stresses in MPa)

$2\theta = 45° \qquad \theta = 22.5°$

$2R = 60 + 20 = 80$ MPa $\qquad R = 40$ MPa

Point C: $\sigma_{x_1} = -20$ MPa

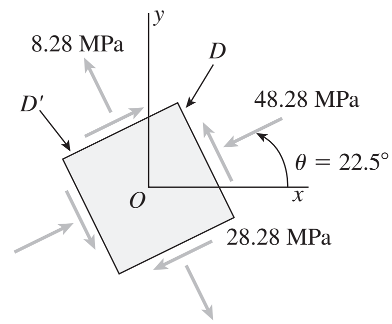


Point D: $\sigma_{x_1} = -20 - R\cos 2\theta = -48.28$ MPa

$\tau_{x_1y_1} = R\sin 2\theta = 28.28$ MPa

Point D': $\sigma_{x_1} = R\cos 2\theta - 20 = 8.28$ MPa

$\tau_{x_1y_1} = -R\sin 2\theta = -28.28$ MPa



(b) MAXIMUM SHEAR STRESSES

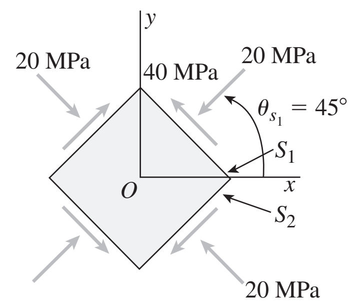
Point S_1: $2\theta_{s_1} = 90° \qquad \theta_{s_1} = 45°$

$\tau_{max} = R = 40$ MPa

Point S_2: $2\theta_{s_2} = -90° \qquad \theta_{s_2} = -45°$

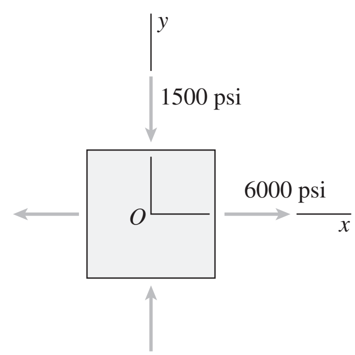
$\tau_{min} = -R = -40$ MPa

$\sigma_{aver} = -20$ MPa



Problem 7.4-5 An element in *biaxial stress* is subjected to stresses $\sigma_x = 6000$ psi and $\sigma_y = -1500$ psi, as shown in the figure.

Using Mohr's circle, determine (a) the stresses acting on an element oriented at a counterclockwise angle $\theta = 60°$ from the x axis and (b) the maximum shear stresses and associated normal stresses. Show all results on sketches of properly oriented elements.



Solution 7.4-5 **Biaxial stress**

$\sigma_x = 6000$ psi $\quad \sigma_y = -1500$ psi $\quad \tau_{xy} = 0$

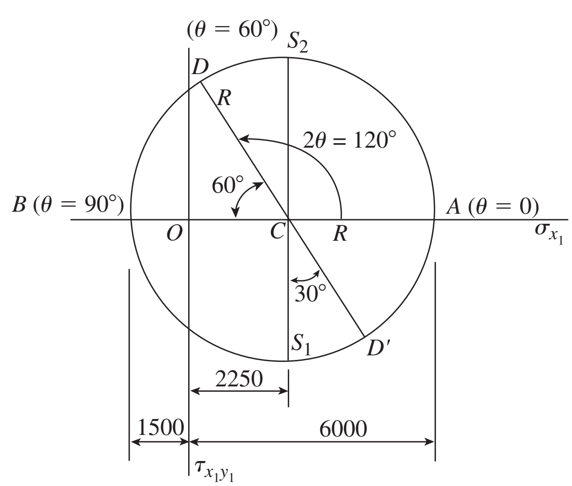
(a) ELEMENT AT $\theta = 60°$

(All stresses in psi)

$2\theta = 120° \qquad \theta = 60°$

$2R = 7500$ psi $\qquad R = 3750$ psi

Point C: $\sigma_{x_1} = 2250$ psi

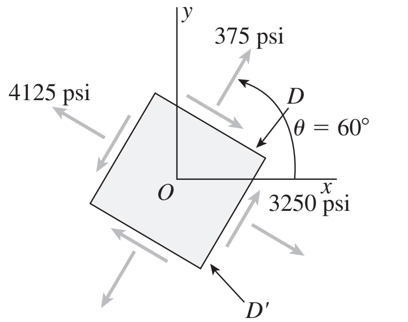


Point D: $\sigma_{x_1} = 2250 - R\cos 60° = 375$ psi

$\tau_{x_1y_1} = -R\sin 60° = -3248$ psi

Point D': $\sigma_{x_1} = 2250 + R\cos 60° = 4125$ psi

$\tau_{x_1y_1} = R\sin 60° = 3248$ psi



(b) Maximum shear stresses

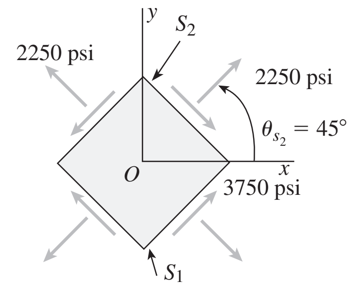
Point S_1: $2\theta_{s_1} = -90°$ $\quad \theta_{s_1} = -45°$

$\tau_{max} = R = 3750$ psi

Point S_2: $2\theta_{s_2} = 90°$ $\quad \theta_{s_2} = 45°$

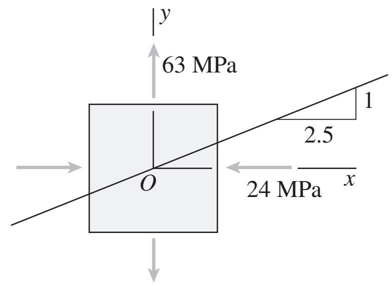
$\tau_{min} = -R = -3750$ psi

$\sigma_{aver} = 2250$ psi



Problem 7.4-6 An element in *biaxial stress* is subjected to stresses $\sigma_x = -24$ MPa and $\sigma_y = 63$ MPa, as shown in the figure.

Using Mohr's circle, determine (a) the stresses acting on an element oriented at a slope of 1 on 2.5 (see figure) and (b) the maximum shear stresses and associated normal stresses. Show all results on sketches of properly oriented elements.

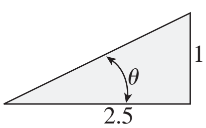


Solution 7.4-6 Biaxial stress

$\sigma_x = -24$ MPa $\quad \sigma_y = 63$ MPa $\quad \tau_{xy} = 0$

(a) Element at a slope of 1 on 2.5

(All stresses in MPa) $\quad \theta = \arctan \dfrac{1}{2.5} = 21.801°$

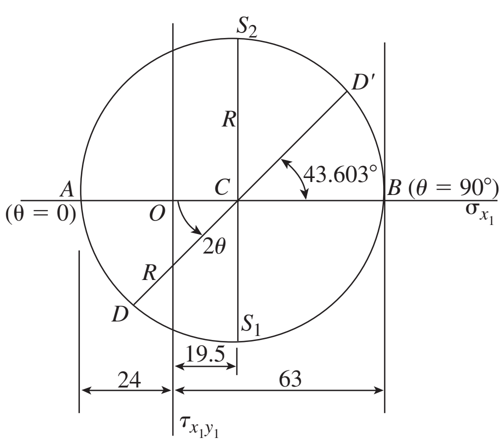


$$2\theta = 43.603°$$
$$\theta = 21.801°$$
$$2R = 87 \text{ MPa}$$
$$R = 43.5 \text{ MPa}$$

Point C: $\sigma_{x_1} = 19.5$ MPa

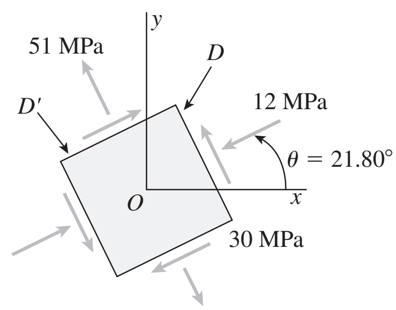
Point D: $\sigma_{x_1} = -R\cos 2\theta + 19.5 = -12$ MPa

$\tau_{x_1y_1} = R\sin 2\theta = 30$ MPa



Point D': $\sigma_{x_1} = 19.5 + R\cos 2\theta = 51$ MPa

$\tau_{x_1y_1} = -R\sin 2\theta = -30$ MPa



(b) Maximum shear stresses

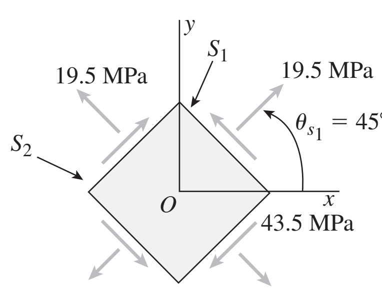
Point S_1: $2\theta_{s_1} = 90°$ $\quad \theta_{s_1} = 45°$

$\tau_{max} = R = 43.5$ MPa

Point S_2: $2\theta_{s_2} = -90°$ $\quad \theta_{s_2} = -45°$

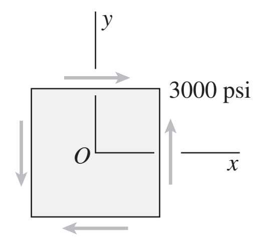
$\tau_{min} = -R = -43.5$ MPa

$\sigma_{aver} = 19.5$ MPa



Problem 7.4-7 An element in *pure shear* is subjected to stresses $\tau_{xy} = 3000$ psi, as shown in the figure.

Using Mohr's circle, determine (a) the stresses acting on an element oriented at a counterclockwise angle $\theta = 70°$ from the x axis and (b) the principal stresses. Show all results on sketches of properly oriented elements.



Solution 7.4-7 Pure shear

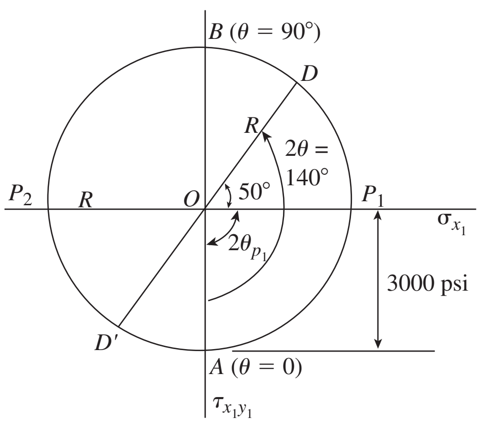
$\sigma_x = 0 \quad \sigma_y = 0 \quad \tau_{xy} = 3000$ psi

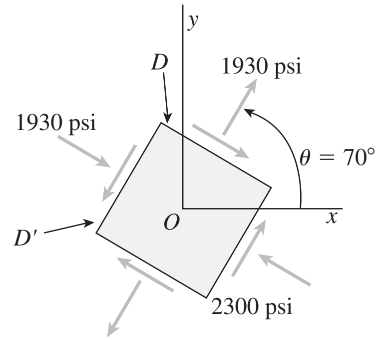
(a) ELEMENT AT $\theta = 70°$

(All stresses in psi)

$2\theta = 140° \qquad \theta = 70° \qquad R = 3000$ psi

Origin O is at center of circle.





Point D: $\sigma_{x_1} = R\cos 50° = 1928$ psi

$\tau_{x_1y_1} = -R\sin 50° = -2298$ psi

Point D': $\sigma_{x_1} = -R\cos 50° = -1928$ psi

$\tau_{x_1y_1} = R\sin 50° = 2298$ psi

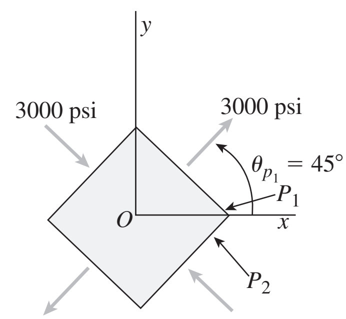
(b) PRINCIPAL STRESSES

Point P_1: $2\theta_{p_1} = 90° \qquad \theta_{p_1} = 45°$

$\sigma_1 = R = 3000$ psi

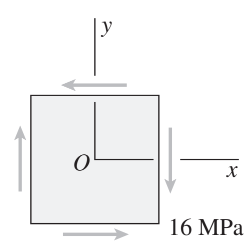
Point P_2: $2\theta_{p_2} = -90° \qquad \theta_{p_2} = -45°$

$\sigma_2 = -R = -3000$ psi



Problem 7.4-8 An element in *pure shear* is subjected to stresses $\tau_{xy} = -16$ MPa, as shown in the figure.

Using Mohr's circle, determine (a) the stresses acting on an element oriented at a counterclockwise angle $\theta = 20°$ from the x axis and (b) the principal stresses. Show all results on sketches of properly oriented elements.

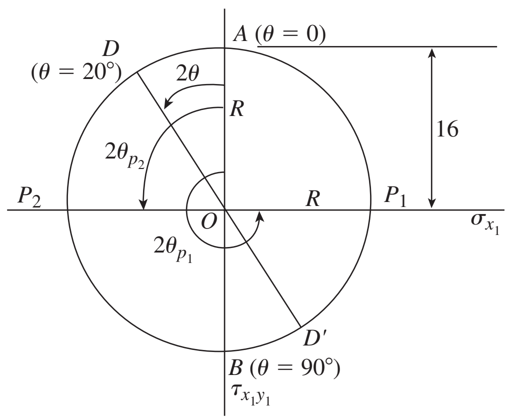


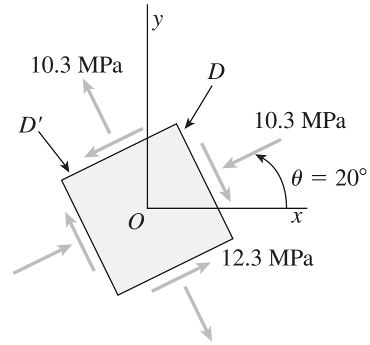
Solution 7.4-8 Pure shear

$\sigma_x = 0 \quad \sigma_y = 0 \quad \tau_{xy} = -16$ MPa

(a) ELEMENT AT $\theta = 20°$

(All stresses in MPa)
$2\theta = 40° \quad \theta = 20° \quad R = 16$ MPa
Origin O is at center of circle.





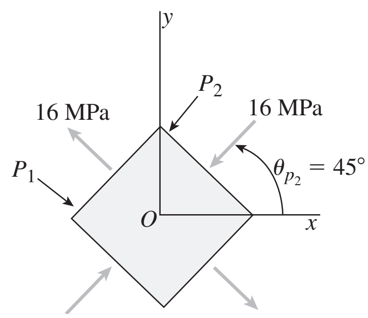
Point D: $\sigma_{x_1} = -R \sin 2\theta = -10.28$ MPa
$\tau_{x_1y_1} = -R \cos 2\theta = -12.26$ MPa

Point D': $\sigma_{x_1} = R \sin 2\theta = 10.28$ MPa
$\tau_{x_1y_1} = R \cos 2\theta = 12.26$ MPa

(b) PRINCIPAL STRESSES

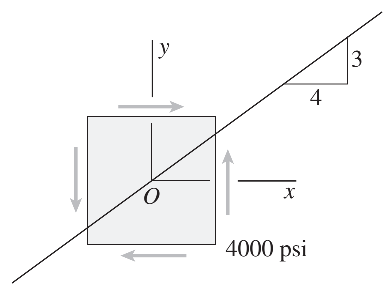
Point P_1: $2\theta_{p_1} = 270° \quad \theta_{p_1} = 135°$
$\sigma_1 = R = 16$ MPa

Point P_2: $2\theta_{p_2} = 90° \quad \theta_{p_2} = 45°$
$\sigma_2 = -R = -16$ MPa



Problem 7.4-9 An element in *pure shear* is subjected to stresses $\tau_{xy} = 4000$ psi, as shown in the figure.

Using Mohr's circle, determine (a) the stresses acting on an element oriented at a slope of 3 on 4 (see figure) and (b) the principal stresses. Show all results on sketches of properly oriented elements.

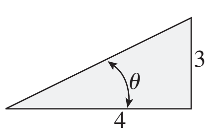


Solution 7.4-9 Pure shear

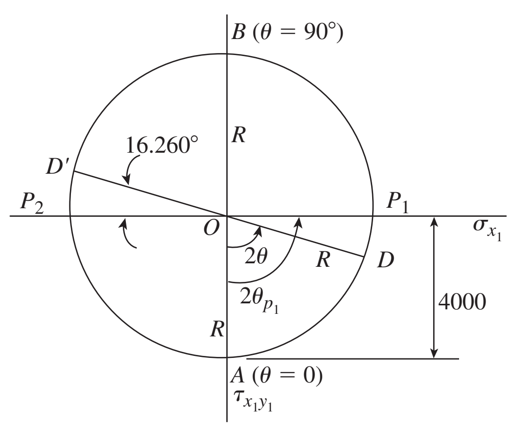
$\sigma_x = 0 \quad \sigma_y = 0 \quad \tau_{xy} = 4000$ psi

(a) ELEMENT AT A SLOPE OF 3 ON 4

(All stresses in psi) $\quad \theta = \arctan \dfrac{3}{4} = 36.870°$



$2\theta = 73.740° \quad \theta = 36.870°$
$R = 4000$ psi
Origin O is at center of circle.

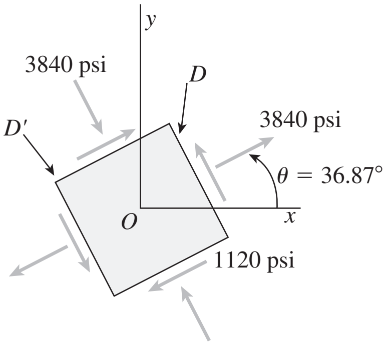


Point D: $\sigma_{x_1} = R\cos 16.260° = 3840$ psi

$\tau_{x_1y_1} = R\sin 16.260° = 1120$ psi

Point D': $\sigma_{x_1} = -R\cos 16.260° = -3840$ psi

$\tau_{x_1y_1} = -R\sin 16.260° = -1120$ psi



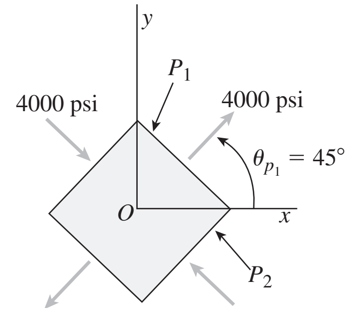
(b) Principal stresses

Point P_1: $2\theta_{p_1} = 90°$ $\theta_{p_1} = 45°$

$\sigma_1 = R = 4000$ psi

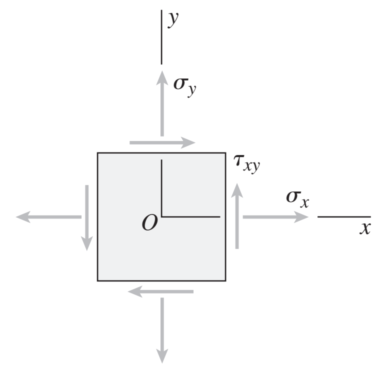
Point P_2: $2\theta_{p_2} = -90°$ $\theta_{p_2} = -45°$

$\sigma_2 = -R = -4000$ psi



Problems 7.4-10 through 7.4-15 An element in *plane stress* is subjected to stresses σ_x, σ_y, and τ_{xy} (see figure).

Using Mohr's circle, determine the stresses acting on an element oriented at an angle θ from the x axis. Show these stresses on a sketch of an element oriented at the angle θ. (*Note:* The angle θ is positive when counterclockwise and negative when clockwise.)



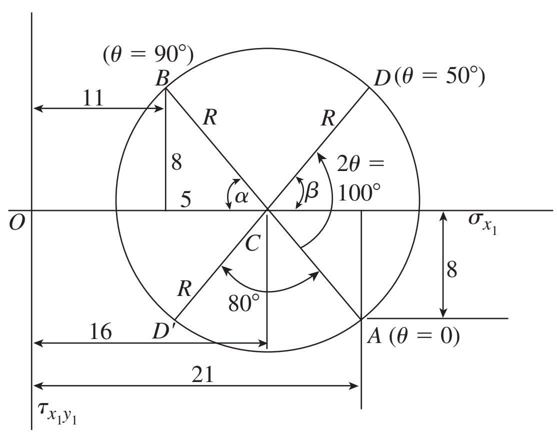
Data for 7.4-10 $\sigma_x = 21$ MPa, $\sigma_y = 11$ MPa, $\tau_{xy} = 8$ MPa, $\theta = 50°$

Solution 7.4-10 Plane stress (angle θ)

$\sigma_x = 21$ MPa $\sigma_y = 11$ MPa

$\tau_{xy} = 8$ MPa $\theta = 50°$

(All stresses in MPa)



$$R = \sqrt{(5)^2 + (8)^2} = 9.4340 \text{ MPa}$$

$$\alpha = \arctan\frac{8}{5} = 57.99°$$

$\beta = 2\theta - \alpha = 100° - \alpha = 42.01°$

Point D ($\theta = 50°$):

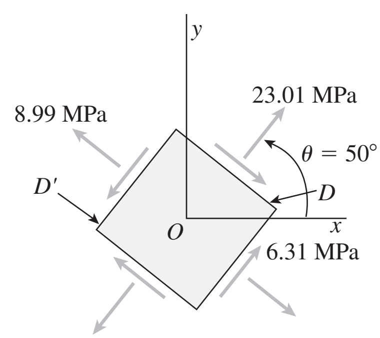
$\sigma_{x_1} = 16 + R\cos\beta = 23.01$ MPa

$\tau_{x_1y_1} = -R\sin\beta = -6.31$ MPa

Point D' ($\theta = -40°$):

$\sigma_{x_1} = 16 - R\cos\beta = 8.99$ MPa

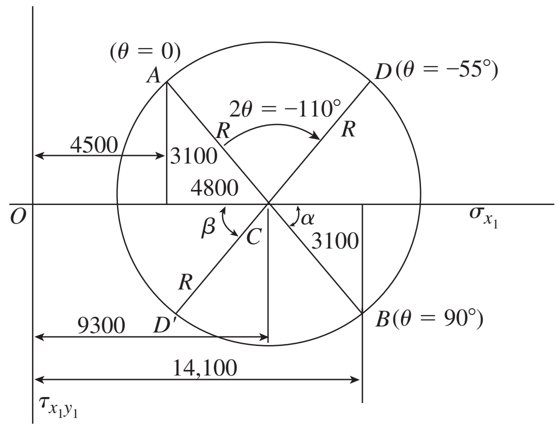
$\tau_{x_1y_1} = R\sin\beta = 6.31$ MPa



Data for 7.4-11 $\sigma_x = 4500$ psi, $\sigma_y = 14{,}100$ psi, $\tau_{xy} = -3100$ psi, $\theta = -55°$

Solution 7.4-11 Plane stress (angle θ)

$\sigma_x = 4500$ psi $\quad \sigma_y = 14{,}100$ psi
$\tau_{xy} = -3100$ psi $\quad \theta = -55°$
(All stresses in psi)



$$R = \sqrt{(4800)^2 + (3100)^2} = 5714 \text{ psi}$$

$$\alpha = \arctan \frac{3100}{4800} = 32.86°$$

$\beta = 180° - 110° - \alpha = 37.14°$

Point D ($\theta = -55°$):

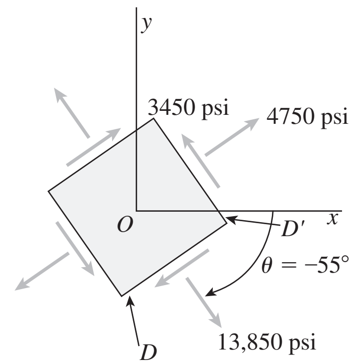
$$\sigma_{x_1} = 9300 + R\cos\beta = 13{,}850 \text{ psi}$$

$$\tau_{x_1y_1} = -R\sin\beta = -3450 \text{ psi}$$

Point D' ($\theta = 35°$):

$$\sigma_{x_1} = 9300 - R\cos\beta = 4750 \text{ psi}$$

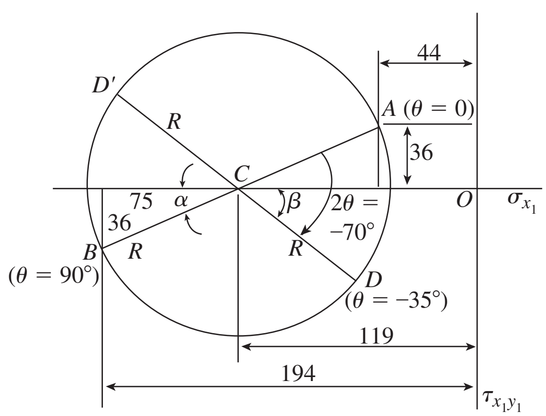
$$\tau_{x_1y_1} = R\sin\beta = 3450 \text{ psi}$$



Data for 7.4-12 $\sigma_x = -44$ MPa, $\sigma_y = -194$ MPa, $\tau_{xy} = -36$ MPa, $\theta = -35°$

Solution 7.4-12 Plane stress (angle θ)

$\sigma_x = -44$ MPa $\quad \sigma_y = -194$ MPa
$\tau_{xy} = -36$ MPa $\quad \theta = -35°$
(All stresses in MPa)



$$R = \sqrt{(75)^2 + (36)^2} = 83.19 \text{ MPa}$$

$$\alpha = \arctan \frac{36}{75} = 25.64°$$

$\beta = 70° - \alpha = 44.36°$

Point D ($\theta = -35°$):

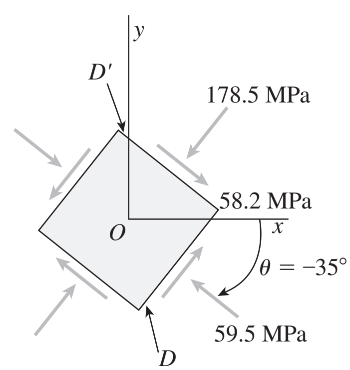
$$\sigma_{x_1} = -119 + R\cos\beta = -59.5 \text{ MPa}$$

$$\tau_{x_1y_1} = R\sin\beta = 58.2 \text{ MPa}$$

Point D' ($\theta = 55°$):

$$\sigma_{x_1} = -119 - R\cos\beta = -178.5 \text{ MPa}$$

$$\tau_{x_1y_1} = -R\sin\beta = -58.2 \text{ MPa}$$



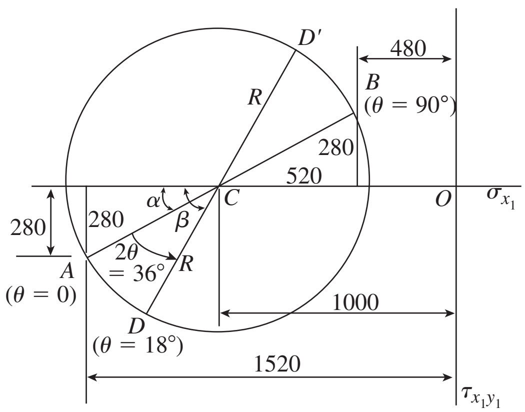
Data for 7.4-13 $\sigma_x = -1520$ psi, $\sigma_y = -480$ psi, $\tau_{xy} = 280$ psi, $\theta = 18°$

Solution 7.4-13 Plane stress (angle θ)

$\sigma_x = -1520$ psi $\quad \sigma_y = -480$ psi

$\tau_{xy} = 280$ psi $\quad \theta = 18°$

(All stresses in psi)



$$R = \sqrt{(520)^2 + (280)^2} = 590.6 \text{ psi}$$

$$\alpha = \arctan \frac{280}{520} = 28.30°$$

$\beta = \alpha + 36° = 64.30°$

Point D ($\theta = 18°$):

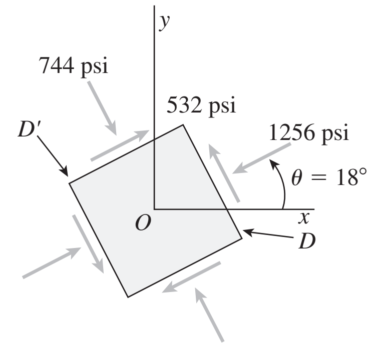
$$\sigma_{x_1} = -1000 - R\cos\beta = -1256 \text{ psi}$$

$$\tau_{x_1y_1} = R\sin\beta = 532 \text{ psi}$$

Point D' ($\theta = 108°$):

$$\sigma_{x_1} = -1000 + R\cos\beta = -744 \text{ psi}$$

$$\tau_{x_1y_1} = -R\sin\beta = -532 \text{ psi}$$



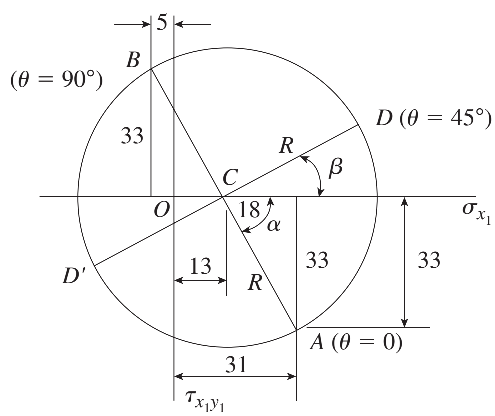
Data for 7.4-14 $\sigma_x = 31$ MPa, $\sigma_y = -5$ MPa, $\tau_{xy} = 33$ MPa, $\theta = 45°$

Solution 7.4-14 Plane stress (angle θ)

$\sigma_x = 31$ MPa $\quad \sigma_y = -5$ MPa

$\tau_{xy} = 33$ MPa $\quad \theta = 45°$

(All stresses in MPa)



$$R = \sqrt{(18)^2 + (33)^2} = 37.590 \text{ MPa}$$

$$\alpha = \arctan \frac{33}{18} = 61.390°$$

$$\beta = 90° - \alpha = 28.610°$$

Point D ($\theta = 45°$):

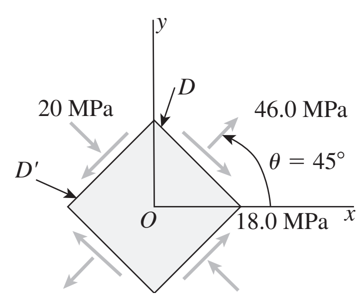
$$\sigma_{x_1} = 13 + R\cos\beta = 46.0 \text{ MPa}$$

$$\tau_{x_1y_1} = -R\sin\beta = -18.0 \text{ MPa}$$

Point D' ($\theta = 135°$):

$$\sigma_{x_1} = 13 - R\cos\beta = -20.0 \text{ MPa}$$

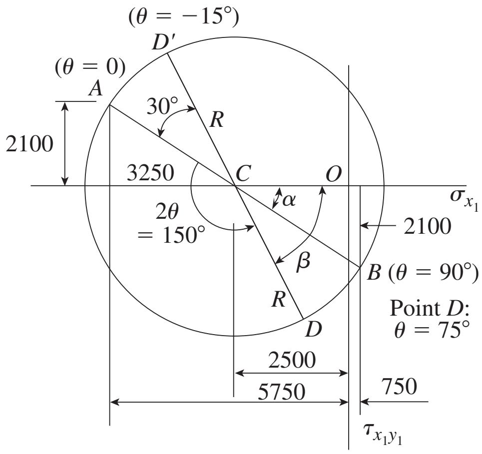
$$\tau_{x_1y_1} = R\sin\beta = 18.0 \text{ MPa}$$



Data for 7.4-15 $\sigma_x = -5750$ psi, $\sigma_y = 750$ psi, $\tau_{xy} = -2100$ psi, $\theta = 75°$

Solution 7.4-15 Plane stress (angle θ))

$\sigma_x = -5750$ psi $\quad \sigma_y = 750$ psi
$\tau_{xy} = -2100$ psi $\quad \theta = 75°$
(All stresses in psi)



$$R = \sqrt{(3250)^2 + (2100)^2} = 3869 \text{ psi}$$

$$\alpha = \arctan \frac{2100}{3250} = 32.87°$$

$$\beta = \alpha + 30° = 62.87°$$

Point D ($\theta = 75°$):

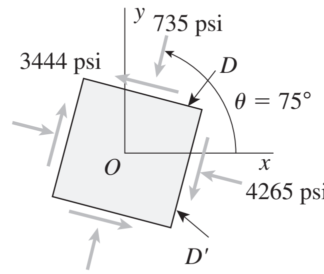
$$\sigma_{x_1} = -2500 + R\cos\beta = -735 \text{ psi}$$

$$\tau_{x_1y_1} = R\sin\beta = 3444 \text{ psi}$$

Point D'($\theta = -15°$):

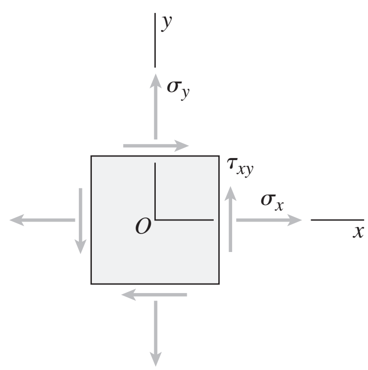
$$\sigma_{x_1} = -2500 - R\cos\beta = -4265 \text{ psi}$$

$$\tau_{x_1y_1} = -R\sin\beta = -3444 \text{ psi}$$



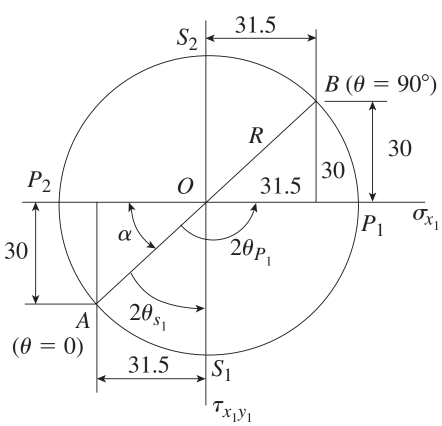
Problems 7.4-16 through 7.4-23 An element in *plane stress* is subjected to stresses σ_x, σ_y, and τ_{xy} (see figure).

Using Mohr's circle, determine (a) the principal stresses and (b) the maximum shear stresses and associated normal stresses. Show all results on sketches of properly oriented elements.



Data for 7.4-16 $\sigma_x = -31.5$ MPa, $\sigma_y = 31.5$ MPa, $\tau_{xy} = 30$ MPa

Solution 7.4-16 Principal stresses



$\sigma_x = -31.5$ MPa $\quad \sigma_y = 31.5$ MPa
$\tau_{xy} = 30$ MPa
(All stresses in MPa)

$$R = \sqrt{(31.5)^2 + (30.0)^2} = 43.5 \text{ MPa}$$

$$\alpha = \arctan \frac{30}{31.5} = 43.60°$$

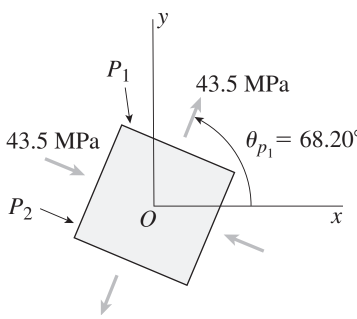
(a) PRINCIPAL STRESSES

$2\theta_{p_1} = 180° - \alpha = 136.40° \quad \theta_{p_1} = 68.20°$

$2\theta_{p_2} = -\alpha = -43.60° \quad \theta_{p_2} = -21.80°$

Point P_1: $\sigma_1 = R = 43.5$ MPa

Point P_2: $\sigma_2 = -R = -43.5$ MPa



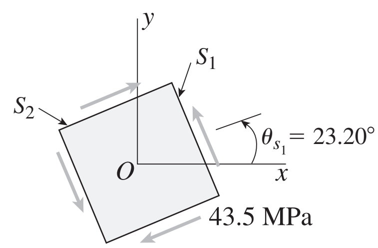
(b) MAXIMUM SHEAR STRESSES

$2\theta_{s_1} = 90° - \alpha = 46.40° \quad \theta_{s_1} = 23.20°$

$2\theta_{s_2} = 2\theta_{s_1} + 180° = 226.40° \quad \theta_{s_2} = 113.20°$

Point S_1: $\sigma_{aver} = 0 \quad \tau_{max} = R = 43.5$ MPa

Point S_2: $\sigma_{aver} = 0 \quad \tau_{min} = -43.5$ MPa

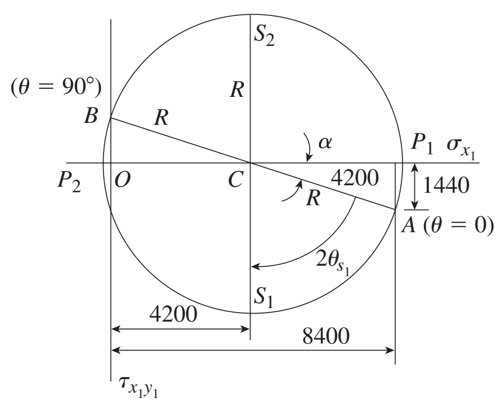


Data for 7.4-17 $\sigma_x = 8400$ psi, $\sigma_y = 0$, $\tau_{xy} = 1440$ psi

Solution 7.4-17 Principal stresses

$\sigma_x = 8400$ psi $\quad \sigma_y = 0 \quad \tau_{xy} = 1440$ psi

(All stresses in psi)



$R = \sqrt{(4200)^2 + (1440)^2} = 4440$ psi

$\alpha = \arctan \dfrac{1440}{4200} = 18.92°$

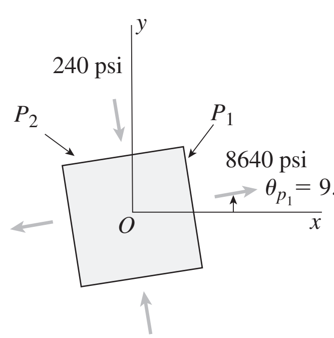
(a) PRINCIPAL STRESSES

$2\theta_{p_1} = \alpha = 18.92° \quad \theta_{p_1} = 9.46°$

$2\theta_{p_2} = 180° + \alpha = 198.92° \quad \theta_{p_2} = 99.46°$

Point P_1: $\sigma_1 = 4200 + R = 8640$ psi

Point P_2: $\sigma_2 = 4200 - R = -240$ psi



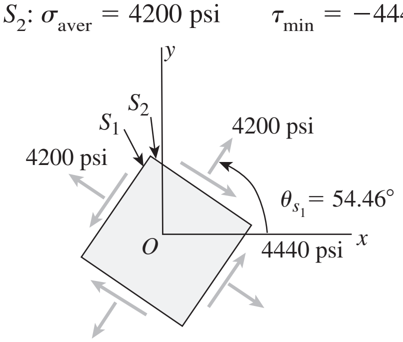
(b) MAXIMUM SHEAR STRESSES

$2\theta_{s_1} = -(90° - \alpha) = -71.08° \quad \theta_{s_1} = -35.54°$

$2\theta_{s_2} = 90° + \alpha = 108.92° \quad \theta_{s_2} = 54.46°$

Point S_1: $\sigma_{aver} = 4200$ psi $\quad \tau_{max} = R = 4440$ psi

Point S_2: $\sigma_{aver} = 4200$ psi $\quad \tau_{min} = -4440$ psi



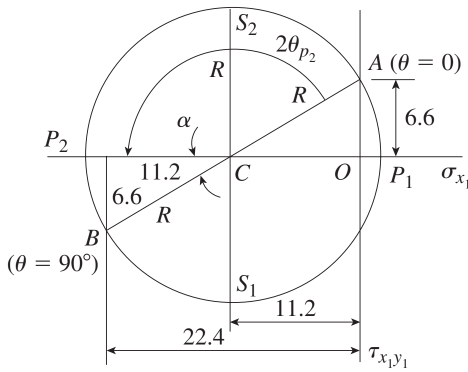
Data for 7.4-18 $\sigma_x = 0$, $\sigma_y = -22.4$ MPa, $\tau_{xy} = -6.6$ MPa

Solution 7.4-18 Principal stresses

$\sigma_x = 0 \quad \sigma_y = -22.4$ MPa

$\tau_{xy} = -6.6$ MPa

(All stresses in MPa)



$R = \sqrt{(11.2)^2 + (6.6)^2} = 13.0$ MPa

$\alpha = \arctan \dfrac{6.6}{11.2} = 30.51°$

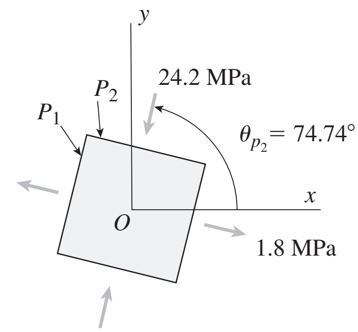
(a) Principal stresses

$2\theta_{p_1} = -\alpha = -30.51° \quad \theta_{p_1} = -15.26°$

$2\theta_{p_2} = 180° - \alpha = 149.49° \quad \theta_{p_2} = 74.74°$

Point P_1: $\sigma_1 = R - 11.2 = 1.8$ MPa

Point P_2: $\sigma_2 = -11.2 - R = -24.2$ MPa



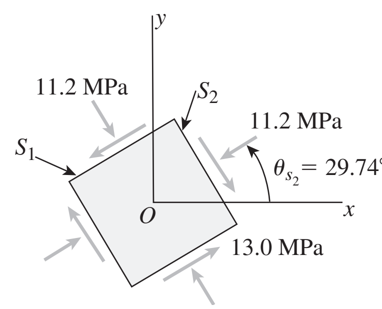
(b) Maximum shear stresses

$2\theta_{s_1} = -\alpha - 90° = -120.51° \quad \theta_{s_1} = -60.26°$

$2\theta_{s_2} = 90° - \alpha = 59.49° \quad \theta_{s_2} = 29.74°$

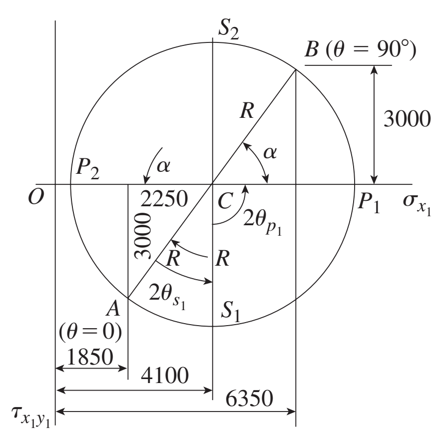
Point S_1: $\sigma_{aver} = -11.2$ MPa $\quad \tau_{max} = R = 13.0$ MPa

Point S_2: $\sigma_{aver} = -11.2$ MPa $\quad \tau_{min} = -13.0$ MPa



Data for 7.4-19 $\sigma_x = 1850$ psi, $\sigma_y = 6350$ psi, $\tau_{xy} = 3000$ psi

Solution 7.4-19 Principal stresses



$\sigma_x = 1850$ psi $\quad \sigma_y = 6350$ psi $\quad \tau_{xy} = 3000$ psi

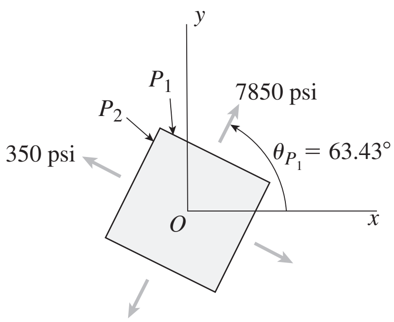
(All stresses in psi)

$R = \sqrt{(2250)^2 + (3000)^2} = 3750$ psi

$\alpha = \arctan \dfrac{3000}{2260} = 53.13°$

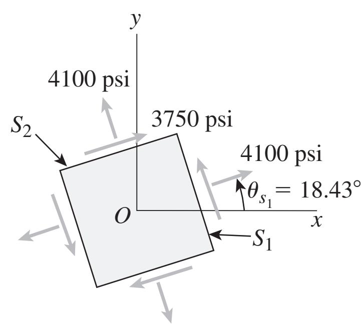
(a) PRINCIPAL STRESSES

$2\theta_{p_1} = 180° - \alpha = 126.87°$ $\theta_{p_1} = 63.43°$
$2\theta_{p_2} = -\alpha = -53.13°$ $\theta_{p_2} = -26.57°$
Point P_1: $\sigma_1 = 4100 + R = 7850$ psi
Point P_2: $\sigma_2 = 4100 - R = 350$ psi



(b) MAXIMUM SHEAR STRESSES

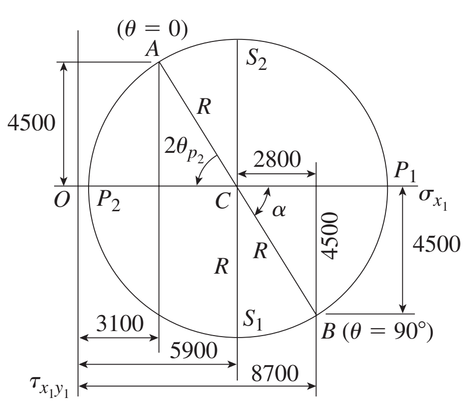
$2\theta_{s_1} = 90° - \alpha = 36.87°$ $\theta_{s_1} = 18.43°$
$2\theta_{s_2} = 270° - \alpha = 216.87°$ $\theta_{s_2} = 108.43°$
Point S_1: $\sigma_{aver} = 4100$ psi $\tau_{max} = R = 3750$ psi
Point S_2: $\sigma_{aver} = 4100$ psi $\tau_{min} = -3750$ psi



Data for 7.4-20 $\sigma_x = 3100$ kPa, $\sigma_y = 8700$ kPa, $\tau_{xy} = -4500$ kPa

Solution 7.4-20 Principal stresses

$\sigma_x = 3100$ kPa $\sigma_y = 8700$ kPa
$\tau_{xy} = -4500$ kPa
(All stresses in kPa)

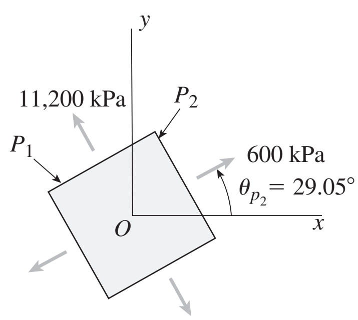


$R = \sqrt{(2800)^2 + (4500)^2} = 5300$ kPa

$\alpha = \arctan \dfrac{4500}{2800} = 58.11°$

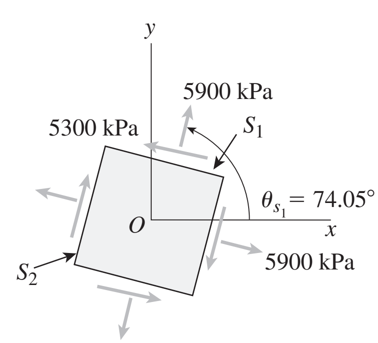
(a) PRINCIPAL STRESSES

$2\theta_{p_1} = \alpha + 180° = 238.11°$ $\theta_{p_1} = 119.05°$
$2\theta_{p_2} = \alpha = 58.11°$ $\theta_{p_2} = 29.05°$
Point P_1: $\sigma_1 = 5900 + R = 11{,}200$ kPa
Point P_2: $\sigma_2 = 5900 - R = 600$ kPa



(b) MAXIMUM SHEAR STRESSES

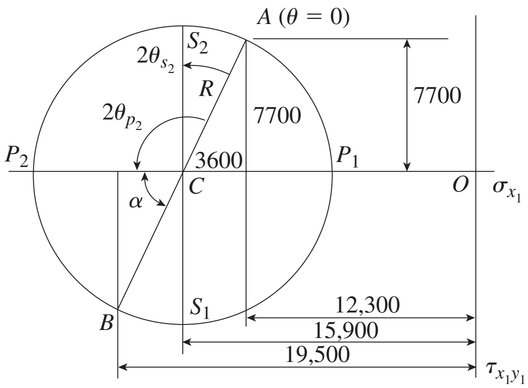
$2\theta_{s_1} = 90° + \alpha = 148.11°$ $\theta_{s_1} = 74.05°$
$2\theta_{s_2} = 270° + \alpha = 328.11°$ $\theta_{s_2} = 164.05°$
Point S_1: $\sigma_{aver} = 5900$ kPa $\tau_{max} = R = 5300$ kPa
Point S_2: $\sigma_{aver} = 5900$ kPa $\tau_{min} = -5300$ kPa



Data for 7.4-21 $\sigma_x = -12{,}300$ psi, $\sigma_y = -19{,}500$ psi, $\tau_{xy} = -7700$ psi

Solution 7.4-21 Principal stresses

$\sigma_x = -12{,}300$ psi $\quad \sigma_y = -19{,}500$ psi
$\tau_{xy} = -7700$ psi
(All stresses in psi)

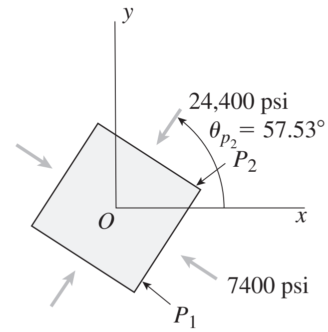


$$R = \sqrt{(3600)^2 + (7700)^2} = 8500 \text{ psi}$$

$$\alpha = \arctan\frac{7700}{3600} = 64.94°$$

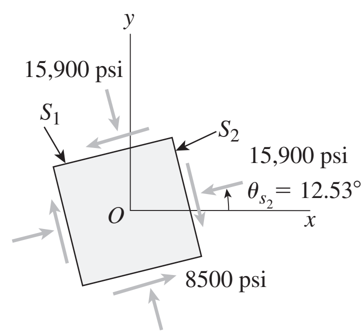
(a) PRINCIPAL STRESSES

$2\theta_{p_1} = -\alpha = -64.94° \quad \theta_{p_1} = -32.47°$
$2\theta_{p_2} = 180° - \alpha = 115.06° \quad \theta_{p_2} = 57.53°$
Point P_1: $\sigma_1 = -15{,}900 + R = -7400$ psi
Point P_2: $\sigma_2 = -15{,}900 - R = -24{,}400$ psi



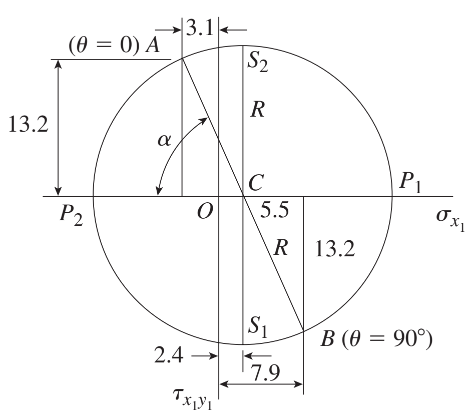
(b) MAXIMUM SHEAR STRESSES

$2\theta_{s_1} = 270° - \alpha = 205.06° \qquad \theta_{s_1} = 102.53°$
$2\theta_{s_2} = 90° - \alpha = 25.06° \qquad \theta_{s_2} = 12.53°$
Point S_1: $\sigma_{aver} = -15{,}900$ psi $\quad \tau_{max} = R = 8500$ psi
Point S_2: $\sigma_{aver} = -15{,}900$ psi $\quad \tau_{min} = -8500$ psi



Data for 7.4-22 $\sigma_x = -3.1$ MPa, $\sigma_y = 7.9$ MPa, $\tau_{xy} = -13.2$ MPa

Solution 7.4-22 Principal stresses



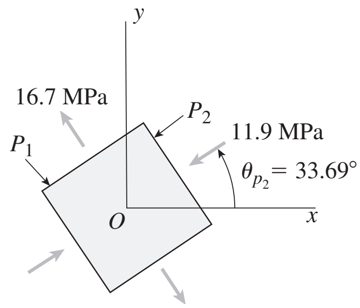
$\sigma_x = -3.1$ MPa $\quad \sigma_y = 7.9$ MPa
$\tau_{xy} = -13.2$ MPa
(All stresses in MPa)

$$R = \sqrt{(5.5)^2 + (13.2)^2} = 14.3 \text{ MPa}$$

$$\alpha = \arctan\frac{13.2}{5.5} = 67.38°$$

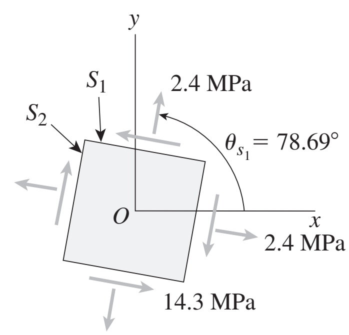
(a) PRINCIPAL STRESSES

$2\theta_{p_1} = 180° + \alpha = 247.38° \quad \theta_{p_1} = 123.69°$
$2\theta_{p_2} = \alpha = 67.38° \quad \theta_{p_2} = 33.69°$
Point P_1: $\sigma_1 = 2.4 + R = 16.7$ MPa
Point P_2: $\sigma_2 = -R + 2.4 = -11.9$ MPa



(b) MAXIMUM SHEAR STRESSES

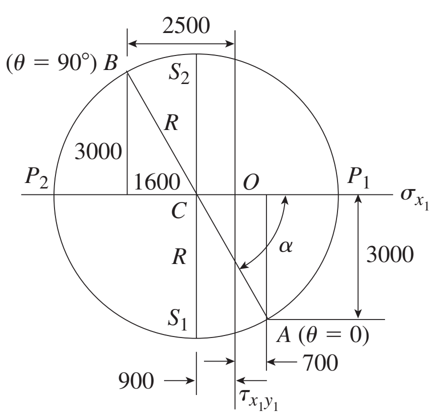
$2\theta_{s_1} = \alpha + 90° = 157.38° \quad \theta_{s_1} = 78.69°$
$2\theta_{s_2} = -90° + \alpha = -22.62° \quad \theta_{s_2} = -11.31°$
Point S_1: $\sigma_{aver} = 2.4$ MPa $\quad \tau_{max} = R = 14.3$ MPa
Point S_2: $\sigma_{aver} = 2.4$ MPa $\quad \tau_{min} = -14.3$ MPa



Data for 7.4-23 $\sigma_x = 700$ psi, $\sigma_y = -2500$ psi, $\tau_{xy} = 3000$ psi

Solution 7.4-23 Principal stresses

$\sigma_x = 700$ psi $\quad \sigma_y = -2500$ psi
$\tau_{xy} = 3000$ psi
(All stresses in psi)

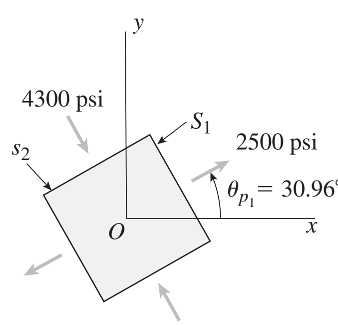


$R = \sqrt{(1600)^2 + (3000)^2} = 3400$ psi

$\alpha = \arctan \dfrac{3000}{1600} = 61.93°$

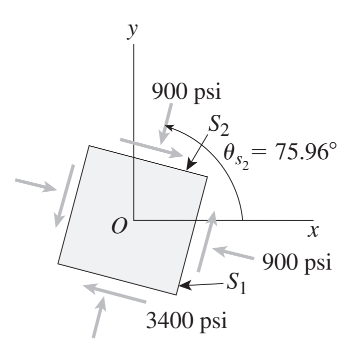
(a) PRINCIPAL STRESSES

$2\theta_{p_1} = \alpha = 61.93° \quad \theta_{p_1} = 30.96°$
$2\theta_{p_2} = 180° + \alpha = 241.93° \quad \theta_{p_2} = 120.96°$
Point P_1: $\sigma_1 = -900 + R = 2500$ psi
Point P_2: $\sigma_2 = -900 - R = -4300$ psi



(b) MAXIMUM SHEAR STRESSES

$2\theta_{s_1} = -90° + \alpha = -28.07° \quad \theta_{s_1} = -14.04°$
$2\theta_{s_2} = 90° + \alpha = 151.93° \quad \theta_{s_2} = 75.96°$
Point S_1: $\sigma_{aver} = -900$ psi $\quad \tau_{max} = R = 3400$ psi
Point S_2: $\sigma_{aver} = -900$ psi $\quad \tau_{min} = -3400$ psi

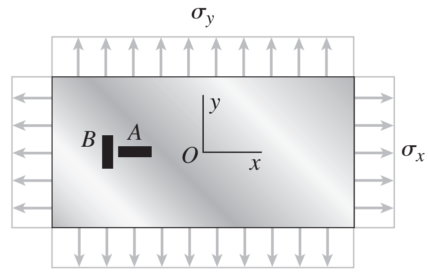


Hooke's Law for Plane Stress

When solving the problems for Section 7.5, assume that the material is linearly elastic with modulus of elasticity E and Poisson's ratio ν.

Problem 7.5-1 A rectangular steel plate with thickness $t = 0.25$ in. is subjected to uniform normal stresses σ_x and σ_y, as shown in the figure. Strain gages A and B, oriented in the x and y directions, respectively, are attached to the plate. The gage readings give normal strains $\epsilon_x = 0.0010$ (elongation) and $\epsilon_y = -0.0007$ (shortening).

Knowing that $E = 30 \times 10^6$ psi and $\nu = 0.3$, determine the stresses σ_x and σ_y and the change Δt in the thickness of the plate.



Probs. 7.5-1 and 7.5-2

Solution 7.5-1 Rectangular plate in biaxial stress

$t = 0.25$ in. $\quad \varepsilon_x = 0.0010 \quad \varepsilon_y = -0.0007$
$E = 30 \times 10^6$ psi $\quad \nu = 0.3$

Substitute numerical values:

Eq. (7-40a):

$$\sigma_x = \frac{E}{(1-\nu^2)}(\varepsilon_x + \nu\varepsilon_y) = 26{,}040 \text{ psi} \quad \longleftarrow$$

Eq. (7-40b):

$$\sigma_y = \frac{E}{(1-\nu^2)}(\varepsilon_y + \nu\varepsilon_x) = -13{,}190 \text{ psi} \quad \longleftarrow$$

Eq. (7-39c):

$$\varepsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y) = -128.5 \times 10^{-6}$$

$$\Delta t = \varepsilon_z t = -32.1 \times 10^{-6} \text{ in.} \quad \longleftarrow$$

(Decrease in thickness)

Problem 7.5-2 Solve the preceding problem if the thickness of the steel plate is $t = 10$ mm, the gage readings are $\epsilon_x = 480 \times 10^{-6}$ (elongation) and $\epsilon_y = 130 \times 10^{-6}$ (elongation), the modulus is $E = 200$ GPa, and Poisson's ratio is $\nu = 0.30$.

Solution 7.5-2 Rectangular plate in biaxial stress

$t = 10$ mm $\quad \varepsilon_x = 480 \times 10^{-6}$
$\varepsilon_y = 130 \times 10^{-6}$
$E = 200$ GPa $\quad \nu = 0.3$

Substitute numerical values:

Eq. (7-40a):

$$\sigma_x = \frac{E}{(1-\nu^2)}(\varepsilon_x + \nu\varepsilon_y) = 114.1 \text{ MPa} \quad \longleftarrow$$

Eq. (7-40b):

$$\sigma_y = \frac{E}{(1-\nu^2)}(\varepsilon_y + \nu\varepsilon_x) = 60.2 \text{ MPa} \quad \longleftarrow$$

Eq. (7-39c):

$$\varepsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y) = -261.4 \times 10^{-6}$$

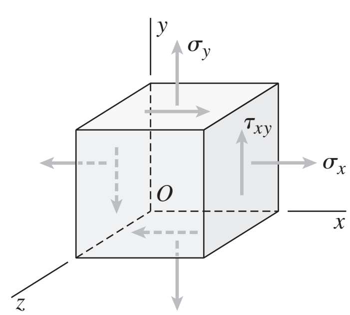
$$\Delta t = \varepsilon_z t = -2610 \times 10^{-6} \text{ mm} \quad \longleftarrow$$

(Decrease in thickness)

Problem 7.5-3 Assume that the normal strains ϵ_x and ϵ_y for an element in *plane stress* (see figure) are measured with strain gages.

(a) Obtain a formula for the normal strain ϵ_z in the z direction in terms of ϵ_x, ϵ_y, and Poisson's ratio ν.

(b) Obtain a formula for the dilatation e in terms of ϵ_x, ϵ_y, and Poisson's ratio ν.



Solution 7.5-3 **Plane stress**

Given: ε_x, ε_y, ν

(a) NORMAL STRAIN ε_z

Eq. (7-34c): $\varepsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y)$

Eq. (7-36a): $\sigma_x = \frac{E}{(1-\nu^2)}(\varepsilon_x + \nu\varepsilon_y)$

Eq. (7-36b): $\sigma_y = \frac{E}{(1-\nu^2)}(\varepsilon_y + \nu\varepsilon_x)$

Substitute σ_x and σ_y into the first equation and simplify:

$$\varepsilon_z = -\frac{\nu}{1-\nu}(\varepsilon_x + \varepsilon_y) \longleftarrow$$

(b) DILATATION

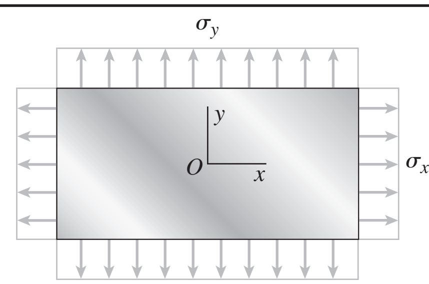
Eq. (7-47): $e = \frac{1-2\nu}{E}(\sigma_x + \sigma_y)$

Substitute σ_x and σ_y from above and simplify:

$$e = \frac{1-2\nu}{1-\nu}(\varepsilon_x + \varepsilon_y) \longleftarrow$$

Problem 7.5-4 A magnesium plate in *biaxial stress* is subjected to tensile stresses $\sigma_x = 24$ MPa and $\sigma_y = 12$ MPa (see figure). The corresponding strains in the plate are $\epsilon_x = 440 \times 10^{-6}$ and $\epsilon_y = 80 \times 10^{-6}$.

Determine Poisson's ratio ν and the modulus of elasticity E for the material.



Probs. 7.5-4 through 7.5-7

Solution 7.5-4 **Biaxial stress**

$\sigma_x = 24$ MPa $\quad \sigma_y = 12$ MPa
$\varepsilon_x = 440 \times 10^{-6} \quad \varepsilon_y = 80 \times 10^{-6}$

POISSON'S RATIO AND MODULUS OF ELASTICITY

Eq. (7-39a): $\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y)$

Eq. (7-39b): $\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x)$

Substitute numerical values:
$E(440 \times 10^{-6}) = 24 \text{ MPa} - \nu(12 \text{ MPa})$
$E(80 \times 10^{-6}) = 12 \text{ MPa} - \nu(24 \text{ MPa})$
Solve simultaneously:
$\nu = 0.35 \quad E = 45 \text{ GPa} \longleftarrow$

Problem 7.5-5 Solve the preceding problem for a steel plate with $\sigma_x = 10{,}800$ psi (tension), $\sigma_y = -5400$ psi (compression), $\epsilon_x = 420 \times 10^{-6}$ (elongation), and $\epsilon_y = -300 \times 10^{-6}$ (shortening).

Solution 7.5-5 **Biaxial stress**

$\sigma_x = 10{,}800$ psi $\quad \sigma_y = -5400$ psi
$\varepsilon_x = 420 \times 10^{-6} \quad \varepsilon_y = -300 \times 10^{-6}$

POISSON'S RATIO AND MODULUS OF ELASTICITY

Eq. (7-39a): $\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y)$

Eq. (7-39b): $\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x)$

Substitute numerical values:
$E(420 \times 10^{-6}) = 10{,}800 \text{ psi} - \nu(-5400 \text{ psi})$
$E(-300 \times 10^{-6}) = -5400 \text{ psi} - \nu(10{,}800 \text{ psi})$
Solve simultaneously:
$\nu = 1/3 \quad E = 30 \times 10^6 \text{ psi} \longleftarrow$

Problem 7.5-6 A rectangular plate in *biaxial stress* (see figure) is subjected to normal stresses $\sigma_x = 90$ MPa (tension) and $\sigma_y = -20$ MPa (compression). The plate has dimensions $400 \times 800 \times 20$ mm and is made of steel with $E = 200$ GPa and $\nu = 0.30$.

(a) Determine the maximum in-plane shear strain γ_{max} in the plate.
(b) Determine the change Δt in the thickness of the plate.
(c) Determine the change ΔV in the volume of the plate.

Solution 7.5-6 **Biaxial stress**

$\sigma_x = 90$ MPa $\quad \sigma_y = -20$ MPa
$E = 200$ GPa $\quad \nu = 0.30$
Dimensions of Plate: 400 mm × 800 mm × 20 mm
Shear Modulus (Eq. 7-38):

$$G = \frac{E}{2(1+\nu)} = 76.923 \text{ GPa}$$

(a) MAXIMUM IN-PLANE SHEAR STRAIN

Principal stresses: $\sigma_1 = 90$ MPa $\quad \sigma_2 = -20$ MPa

Eq. (7-26): $\tau_{max} = \dfrac{\sigma_1 - \sigma_2}{2} = 55.0$ MPa

Eq. (7-35): $\gamma_{max} = \dfrac{\tau_{max}}{G} = 715 \times 10^{-6}$ ←

(b) CHANGE IN THICKNESS

Eq. (7-39c): $\varepsilon_z = -\dfrac{\nu}{E}(\sigma_x + \sigma_y) = -105 \times 10^{-6}$

$\Delta t = \varepsilon_z t = -2100 \times 10^{-6}$ mm ←
(Decrease in thickness)

(c) CHANGE IN VOLUME

From Eq. (7-47): $\Delta V = V_0\left(\dfrac{1-2\nu}{E}\right)(\sigma_x + \sigma_y)$

$V_0 = (400)(800)(20) = 6.4 \times 10^6 \text{ mm}^3$

Also, $\left(\dfrac{1-2\nu}{E}\right)(\sigma_x + \sigma_y) = 140 \times 10^{-6}$

$\therefore\ \Delta V = (6.4 \times 10^6 \text{ mm}^3)(140 \times 10^{-6})$
$= 896 \text{ mm}^3$ ←
(Increase in volume)

Problem 7.5-7 Solve the preceding problem for an aluminum plate with $\sigma_x = 12{,}000$ psi (tension), $\sigma_y = -3{,}000$ psi (compression), dimensions $20 \times 30 \times 0.5$ in., $E = 10.5 \times 10^6$ psi, and $\nu = 0.33$.

Solution 7.5-7 **Biaxial stress**

$\sigma_x = 12{,}000$ psi $\quad \sigma_y = -3{,}000$ psi
$E = 10.5 \times 10^6$ psi $\quad \nu = 0.33$
Dimensions of Plate: 20 in. × 30 in. × 0.5 in.
Shear Modulus (Eq. 7-38):

$$G = \frac{E}{2(1+\nu)} = 3.9474 \times 10^6 \text{ psi}$$

(a) MAXIMUM IN-PLANE SHEAR STRAIN

Principal stresses: $\sigma_1 = 12{,}000$ psi
$\sigma_2 = -3{,}000$ psi

Eq. (7-26): $\tau_{max} = \dfrac{\sigma_1 - \sigma_2}{2} = 7{,}500$ psi

Eq. (7-35): $\gamma_{max} = \dfrac{\tau_{max}}{G} = 1{,}900 \times 10^{-6}$ ←

(b) CHANGE IN THICKNESS

Eq. (7-39c): $\varepsilon_z = -\dfrac{\nu}{E}(\sigma_x + \sigma_y) = -282.9 \times 10^{-6}$

$\Delta t = \varepsilon_z t = -141 \times 10^{-6}$ in. ←
(Decrease in thickness)

(c) CHANGE IN VOLUME

From Eq. (7-47): $\Delta V = V_0\left(\dfrac{1-2\nu}{E}\right)(\sigma_x + \sigma_y)$

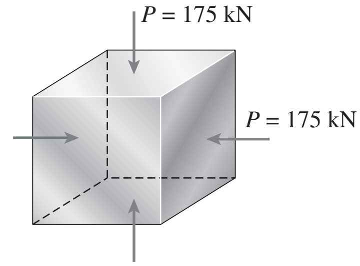
$V_0 = (20)(30)(0.5) = 300 \text{ in.}^3$

Also, $\left(\dfrac{1-2\nu}{E}\right)(\sigma_x + \sigma_y) = 291.4 \times 10^{-6}$

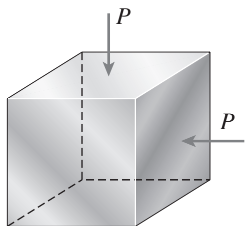
$\therefore\ \Delta V = (300 \text{ in.}^3)(291.4 \times 10^{-6})$
$= 0.0874 \text{ in.}^3$ ←
(Increase in volume)

Problem 7.5-8 A brass cube 50 mm on each edge is compressed in two perpendicular directions by forces $P = 175$ kN (see figure).

Calculate the change ΔV in the volume of the cube and the strain energy U stored in the cube, assuming $E = 100$ GPa and $\nu = 0.34$.



Solution 7.5-8 **Biaxial stress-cube**



Side $b = 50$ mm $P = 175$ kN

$E = 100$ GPa $\nu = 0.34$ (Brass)

$$\sigma_x = \sigma_y = -\frac{P}{b^2} = -\frac{(175\text{ kN})}{(50\text{ mm})^2} = -70.0\text{ MPa}$$

CHANGE IN VOLUME

$$\text{Eq. (7-47): } e = \frac{1-2\nu}{E}(\sigma_x + \sigma_y) = -448 \times 10^{-6}$$

$$V_0 = b^3 = (50\text{ mm})^3 = 125 \times 10^3\text{ mm}^3$$

$$\Delta V = eV_0 = -56\text{ mm}^3 \longleftarrow$$

(Decrease in volume)

STRAIN ENERGY

$$\text{Eq. (7-50): } u = \frac{1}{2E}(\sigma_x^2 + \sigma_y^2 - 2\nu\sigma_x\sigma_y)$$

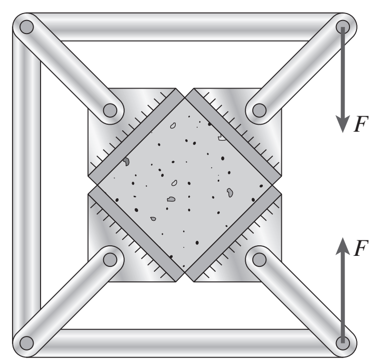
$$= 0.03234\text{ MPa}$$

$$U = uV_0 = (0.03234\text{ MPa})(125 \times 10^3\text{ mm}^3)$$

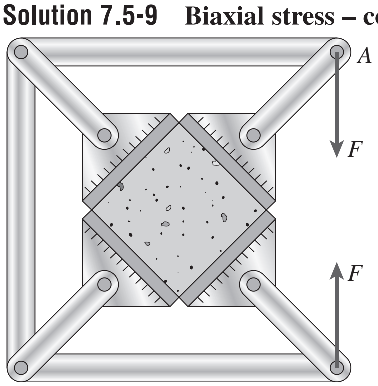
$$= 4.04\text{ J} \longleftarrow$$

Problem 7.5-9 A 4.0-inch cube of concrete ($E = 3.0 \times 10^6$ psi, $\nu = 0.1$) is compressed in *biaxial stress* by means of a framework that is loaded as shown in the figure.

Assuming that each load F equals 20 k, determine the change ΔV in the volume of the cube and the strain energy U stored in the cube.



Solution 7.5-9 **Biaxial stress – concrete cube**



$b = 4$ in.

$E = 3.0 \times 10^6$ psi

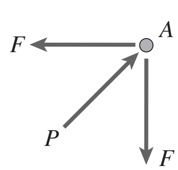
$\nu = 0.1$

$F = 20$ kips

Joint A:

$P = F\sqrt{2}$

$= 28.28$ kips



$$\sigma_x = \sigma_y = -\frac{P}{b^2} = -1768\text{ psi}$$

CHANGE IN VOLUME

$$\text{Eq. (7-47): } e = \frac{1-2\nu}{E}(\sigma_x + \sigma_y) = -0.0009429$$

$$V_0 = b^3 = (4\text{ in.})^3 = 64\text{ in.}^3$$

$$\Delta V = eV_0 = -0.0603\text{ in.}^3 \longleftarrow$$

(Decrease in volume)

STRAIN ENERGY

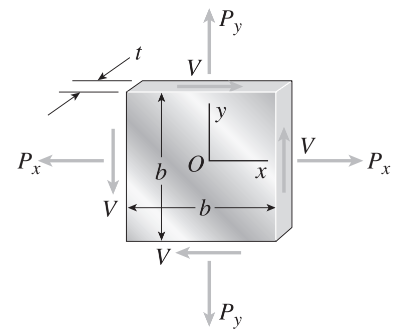
$$\text{Eq. (7-50): } u = \frac{1}{2E}(\sigma_x^2 + \sigma_y^2 - 2\nu\sigma_x\sigma_y)$$

$$= 0.9377\text{ psi}$$

$$U = uV_0 = 60.0\text{ in.-lb} \longleftarrow$$

Problem 7.5-10 A square plate of width b and thickness t is loaded by normal forces P_x and P_y, and by shear forces V, as shown in the figure. These forces produce uniformly distributed stresses acting on the side faces of the plate.

Calculate the change ΔV in the volume of the plate and the strain energy U stored in the plate if the dimensions are $b = 600$ mm and $t = 40$ mm, the plate is made of magnesium with $E = 45$ GPa and $\nu = 0.35$, and the forces are $P_x = 480$ kN, $P_y = 180$ kN, and $V = 120$ kN.



Probs. 7.5-10 and 7.5-11

Solution 7.5-10 Square plate in plane stress

$b = 600$ mm $\quad t = 40$ mm

$E = 45$ GPa $\quad \nu = 0.35$ (magnesium)

$P_x = 480$ kN $\quad \sigma_x = \dfrac{P_x}{bt} = 20.0$ MPa

$P_y = 180$ kN $\quad \sigma_y = \dfrac{P_y}{bt} = 7.5$ MPa

$V = 120$ kN $\quad \tau_{xy} = \dfrac{V}{bt} = 5.0$ MPa

CHANGE IN VOLUME

Eq. (7-47): $e = \dfrac{1 - 2\nu}{E}(\sigma_x + \sigma_y) = 183.33 \times 10^{-6}$

$V_0 = b^2 t = 14.4 \times 10^6\ \text{mm}^3$

$\Delta V = eV_0 = 2640\ \text{mm}^3$ ←

(Increase in volume)

STRAIN ENERGY

Eq. (7-50): $u = \dfrac{1}{2E}(\sigma_x^2 + \sigma_y^2 - 2\nu\sigma_x\sigma_y) + \dfrac{\tau_{xy}^2}{2G}$

$$G = \frac{E}{2(1 + \nu)} = 16.667 \text{ GPa}$$

Substitute numerical values:

$u = 4653$ Pa

$U = uV_0 = 67.0\ \text{N} \cdot \text{m} = 67.0$ J ←

Problem 7.5-11 Solve the preceding problem for an aluminum plate with $b = 12$ in., $t = 1.0$ in., $E = 10{,}600$ ksi, $\nu = 0.33$, $P_x = 90$ k, $P_y = 20$ k, and $V = 15$ k.

Solution 7.5-11 Square plate in plane stress

$b = 12.0$ in. $\quad t = 1.0$ in.

$E = 10{,}600$ ksi $\quad \nu = 0.33$ (aluminum)

$P_x = 90$ k $\quad \sigma_x = \dfrac{P_x}{bt} = 7500$ psi

$P_y = 20$ k $\quad \sigma_y = \dfrac{P_y}{bt} = 1667$ psi

$V = 15$ k $\quad \tau_{xy} = \dfrac{V}{bt} = 1250$ psi

CHANGE IN VOLUME

Eq. (7-47): $e = \dfrac{1 - 2\nu}{E}(\sigma_x + \sigma_y) = 294 \times 10^{-6}$

$V_0 = b^2 t = 144\ \text{in.}^3$

$\Delta V = eV_0 = 0.0423\ \text{in.}^3$ ←

(Increase in volume)

STRAIN ENERGY

Eq. (7-50): $u = \dfrac{1}{2E}(\sigma_x^2 + \sigma_y^2 - 2\nu\sigma_x\sigma_y) + \dfrac{\tau_{xy}^2}{2G}$

$$G = \frac{E}{2(1 + \nu)} = 3985 \text{ ksi}$$

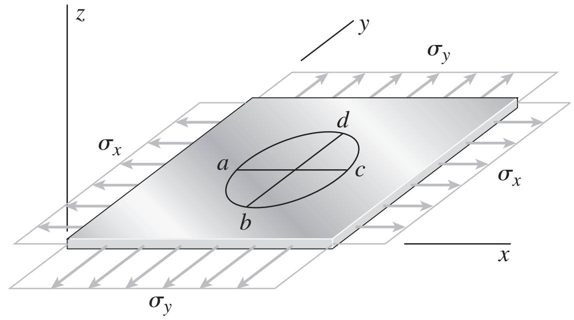
Substitute numerical values:

$u = 2.591$ psi

$U = uV_0 = 373$ in.-lb ←

Problem 7.5-12 A circle of diameter $d = 200$ mm is etched on a brass plate (see figure). The plate has dimensions $400 \times 400 \times 20$ mm. Forces are applied to the plate, producing uniformly distributed normal stresses $\sigma_x = 42$ MPa and $\sigma_y = 14$ MPa.

Calculate the following quantities: (a) the change in length Δac of diameter ac; (b) the change in length Δbd of diameter bd; (c) the change Δt in the thickness of the plate; (d) the change ΔV in the volume of the plate, and (e) the strain energy U stored in the plate. (Assume $E = 100$ GPa and $\nu = 0.34$.)



Solution 7.5-12 Plate in biaxial stress

$\sigma_x = 42$ MPa $\quad \sigma_y = 14$ MPa
Dimensions: $400 \times 400 \times 20$ (mm)
Diameter of circle: $d = 200$ mm
$E = 100$ GPa $\quad \nu = 0.34$ (Brass)

(a) Change in length of diameter in x direction

Eq. (7-39a): $\varepsilon_x = \dfrac{1}{E}(\sigma_x - \nu\sigma_y) = 372.4 \times 10^{-6}$

$\Delta ac = \varepsilon_x d = 0.0745$ mm ⟵
(increase)

(b) Change in length of diameter in y direction

Eq. (7-39b): $\varepsilon_y = \dfrac{1}{E}(\sigma_y - \nu\sigma_x) = -2.80 \times 10^{-6}$

$\Delta bd = \varepsilon_y d = -560 \times 10^{-6}$ mm ⟵
(decrease)

(c) Change in thickness

Eq. (7-39c): $\varepsilon_z = -\dfrac{\nu}{E}(\sigma_x + \sigma_y) = -190.4 \times 10^{-6}$

$\Delta t = \varepsilon_z t = -0.00381$ mm ⟵
(decrease)

(d) Change in volume

Eq. (7-47): $e = \dfrac{1 - 2\nu}{E}(\sigma_x + \sigma_y) = 179.2 \times 10^{-6}$

$V_0 = (400)(400)(20) = 3.2 \times 10^6 \text{ mm}^3$
$\Delta V = eV_0 = 573 \text{ mm}^3$ ⟵
(increase)

(e) Strain energy

Eq. (7-50): $u = \dfrac{1}{2E}(\sigma_x^2 + \sigma_y^2 - 2\nu\sigma_x\sigma_y)$

$= 7.801 \times 10^{-3}$ MPa

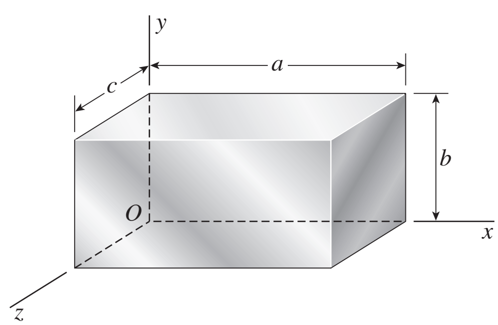
$U = uV_0 = 25.0 \text{ N} \cdot \text{m} = 25.0 \text{ J}$ ⟵

Triaxial Stress

When solving the problems for Section 7.6, assume that the material is linearly elastic with modulus of elasticity E and Poisson's ratio ν.

Problem 7.6-1 An element of aluminum in the form of a rectangular parallelepiped (see figure) of dimensions $a = 6.0$ in., $b = 4.0$ in, and $c = 3.0$ in. is subjected to *triaxial stresses* $\sigma_x = 12{,}000$ psi, $\sigma_y = -4{,}000$ psi, and $\sigma_z = -1{,}000$ psi acting on the x, y, and z faces, respectively.

Determine the following quantities: (a) the maximum shear stress τ_{max} in the material; (b) the changes Δa, Δb, and Δc in the dimensions of the element; (c) the change ΔV in the volume; and (d) the strain energy U stored in the element. (Assume $E = 10{,}400$ ksi and $\nu = 0.33$.)



Probs. 7.6-1 and 7.6-2

Solution 7.6-1 **Triaxial stress**

$\sigma_x = 12{,}000$ psi $\quad \sigma_y = -4{,}000$ psi
$\sigma_z = -1{,}000$ psi
$a = 6.0$ in. $\quad b = 4.0$ in. $\quad c = 3.0$ in.
$E = 10{,}400$ ksi $\quad \nu = 0.33$ (aluminum)

(a) Maximum shear stress

$\sigma_1 = 12{,}000$ psi $\quad \sigma_2 = -1{,}000$ psi
$\sigma_3 = -4{,}000$ psi

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} = 8{,}000 \text{ psi} \quad \longleftarrow$$

(b) Changes in dimensions

Eq. (7-53a): $\varepsilon_x = \dfrac{\sigma_x}{E} - \dfrac{\nu}{E}(\sigma_y + \sigma_z) = 1312.5 \times 10^{-6}$

Eq. (7-53b): $\varepsilon_y = \dfrac{\sigma_y}{E} - \dfrac{\nu}{E}(\sigma_z + \sigma_x) = -733.7 \times 10^{-6}$

Eq. (7-53c): $\varepsilon_z = \dfrac{\sigma_z}{E} - \dfrac{\nu}{E}(\sigma_x + \sigma_y) = -350.0 \times 10^{-6}$

$$\left.\begin{array}{lll} \Delta a = a\varepsilon_x = 0.0079 \text{ in.} & \text{(increase)} \\ \Delta b = b\varepsilon_y = -0.0029 \text{ in.} & \text{(decrease)} \\ \Delta c = c\varepsilon_z = -0.0011 \text{ in.} & \text{(decrease)} \end{array}\right\} \longleftarrow$$

(c) Change in volume

Eq. (7-56):

$$e = \frac{1 - 2\nu}{E}(\sigma_x + \sigma_y + \sigma_z) = 228.8 \times 10^{-6}$$

$V = abc$
$\Delta V = e\,(abc) = 0.0165 \text{ in.}^3$ (increase) $\longleftarrow$

(d) Strain energy

Eq. (7-57a): $u = \dfrac{1}{2}(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z)$
$= 9.517$ psi

$U = u\,(abc) = 685$ in.-lb $\longleftarrow$

Problem 7.6-2 Solve the preceding problem if the element is steel ($E = 200$ GPa, $\nu = 0.30$) with dimensions $a = 300$ mm, $b = 150$ mm, and $c = 150$ mm and the stresses are $\sigma_x = -60$ MPa, $\sigma_y = -40$ MPa, and $\sigma_z = -40$ MPa.

Solution 7.6-2 **Triaxial stress**

$\sigma_x = -60$ MPa $\quad \sigma_y = -40$ MPa
$\sigma_z = -40$ MPa
$a = 300$ mm $\quad b = 150$ mm $\quad c = 150$ mm
$E = 200$ GPa $\quad \nu = 0.30$ (steel)

(a) Maximum shear stress

$\sigma_1 = -40$ MPa $\quad \sigma_2 = -40$ MPa
$\sigma_3 = -60$ MPa

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} = 10.0 \text{ MPa} \quad \longleftarrow$$

(b) Changes in dimensions

Eq. (7-53a): $\varepsilon_x = \dfrac{\sigma_x}{E} - \dfrac{\nu}{E}(\sigma_y + \sigma_z) = -180.0 \times 10^{-6}$

Eq. (7-53b): $\varepsilon_y = \dfrac{\sigma_y}{E} - \dfrac{\nu}{E}(\sigma_z + \sigma_x) = -50.0 \times 10^{-6}$

Eq. (7-53c): $\varepsilon_z = \dfrac{\sigma_z}{E} - \dfrac{\nu}{E}(\sigma_x + \sigma_y) = -50.0 \times 10^{-6}$

$\Delta a = a\varepsilon_x = -0.0540$ mm (decrease)
$\Delta b = b\varepsilon_y = -0.0075$ mm (decrease) $\longleftarrow$
$\Delta c = c\varepsilon_z = -0.0075$ mm (decrease)

(c) Change in volume

Eq. (7-56):

$$e = \frac{1 - 2\nu}{E}(\sigma_x + \sigma_y + \sigma_z) = -280.0 \times 10^{-6}$$

$V = abc$
$\Delta V = e\,(abc) = -1890 \text{ mm}^3$ (decrease) $\longleftarrow$

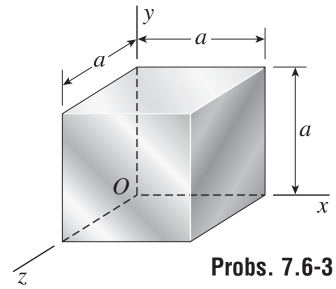
(d) Strain energy

Eq. (7-57a): $u = \dfrac{1}{2}(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z)$
$= 0.00740$ MPa

$U = u\,(abc) = 50.0 \text{ N} \cdot \text{m} = 50.0 \text{ J}$ $\longleftarrow$

Problem 7.6-3 A cube of cast iron with sides of length $a = 4.0$ in. (see figure) is tested in a laboratory under *triaxial stress*. Gages mounted on the testing machine show that the compressive strains in the material are $\epsilon_x = -225 \times 10^{-6}$ and $\epsilon_y = \epsilon_z = -37.5 \times 10^{-6}$.

Determine the following quantities: (a) the normal stresses σ_x, σ_y, and σ_z acting on the x, y, and z faces of the cube; (b) the maximum shear stress τ_{max} in the material; (c) the change ΔV in the volume of the cube; and (d) the strain energy U stored in the cube. (Assume $E = 14{,}000$ ksi and $\nu = 0.25$.)



Probs. 7.6-3 and 7.6-4

Solution 7.6-3 Triaxial stress (cube)

$\varepsilon_x = -225 \times 10^{-6}$ $\quad \varepsilon_y = -37.5 \times 10^{-6}$
$\varepsilon_z = -37.5 \times 10^{-6}$ $\quad a = 4.0$ in.
$E = 14{,}000$ ksi $\quad \nu = 0.25$ (cast iron)

(a) NORMAL STRESSES

Eq. (7-54a):

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_x + \nu(\varepsilon_y + \varepsilon_z)]$$

$$= -4200 \text{ psi} \quad \longleftarrow$$

In a similar manner, Eqs. (7-54 b and c) give

$\sigma_y = -2100$ psi $\quad \sigma_z = -2100$ psi $\longleftarrow$

(b) MAXIMUM SHEAR STRESS

$\sigma_1 = -2100$ psi $\quad \sigma_2 = -2100$ psi
$\sigma_3 = -4200$ psi

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} = 1050 \text{ psi} \quad \longleftarrow$$

(c) CHANGE IN VOLUME

Eq. (7-55): $e = \varepsilon_x + \varepsilon_y + \varepsilon_z = -0.000300$
$V = a^3$
$\Delta V = ea^3 = -0.0192$ in.^{3} (decrease) $\longleftarrow$

(d) STRAIN ENERGY

$$\text{Eq. (7-57a): } u = \frac{1}{2}(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z)$$

$$= 0.55125 \text{ psi}$$

$$U = ua^3 = 35.3 \text{ in.-lb} \quad \longleftarrow$$

Problem 7.6-4 Solve the preceding problem if the cube is granite ($E = 60$ GPa, $\nu = 0.25$) with dimensions $a = 75$ mm and compressive strains $\epsilon_x = -720 \times 10^{-6}$ and $\epsilon_y = \epsilon_z = -270 \times 10^{-6}$.

Solution 7.6-4 Triaxial stress (cube)

$\varepsilon_x = -720 \times 10^{-6}$ $\quad \varepsilon_y = -270 \times 10^{-6}$
$\varepsilon_z = -270 \times 10^{-6}$ $\quad a = 75$ mm $\quad E = 60$ GPa
$\nu = 0.25$ (Granite)

(a) NORMAL STRESSES

Eq. (7-54a):

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_x + \nu(\varepsilon_x + \varepsilon_z)]$$

$$= -64.8 \text{ MPa} \quad \longleftarrow$$

In a similar manner, Eqs. (7-54 b and c) give

$\sigma_y = -43.2$ MPa $\quad \sigma_z = -43.2$ MPa $\longleftarrow$

(b) MAXIMUM SHEAR STRESS

$\sigma_1 = -43.2$ MPa $\quad \sigma_2 = -43.2$ MPa
$\sigma_3 = -64.8$ MPa

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} = 10.8 \text{ MPa} \quad \longleftarrow$$

(c) CHANGE IN VOLUME

Eq. (7-55): $e = \varepsilon_x + \varepsilon_y + \varepsilon_z = -1260 \times 10^{-6}$
$V = a^3$
$\Delta V = ea^3 = -532$ mm^3 (decrease) $\longleftarrow$

(d) STRAIN ENERGY

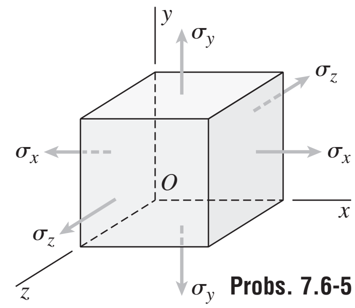
$$\text{Eq. (7-57a): } u = \frac{1}{2}(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z)$$

$$= 0.03499 \text{ MPa} = 34.99 \text{ kPa}$$

$$U = ua^3 = 14.8 \text{ N} \cdot \text{m} = 14.8 \text{ J} \quad \longleftarrow$$

Problem 7.6-5 An element of aluminum in *triaxial stress* (see figure) is subjected to stresses $\sigma_x = 5200$ psi (tension), $\sigma_y = -4750$ psi (compression), and $\sigma_z = -3090$ psi (compression). It is also known that the normal strains in the x and y directions are $\epsilon_x = 713.8 \times 10^{-6}$ (elongation) and $\epsilon_y = -502.3 \times 10^{-6}$ (shortening).

What is the bulk modulus K for the aluminum?



Probs. 7.6-5 and 7.6-6

Solution 7.6-5 Triaxial stress (bulk modulus)

$\sigma_x = 5200$ psi $\quad \sigma_y = -4750$ psi
$\sigma_z = -3090$ psi $\quad \varepsilon_x = 713.8 \times 10^{-6}$
$\varepsilon_y = -502.3 \times 10^{-6}$

Find K.

Eq. (7-53a): $\varepsilon_x = \dfrac{\sigma_x}{E} - \dfrac{\nu}{E}(\sigma_y + \sigma_z)$

Eq. (7-53b): $\varepsilon_y = \dfrac{\sigma_y}{E} - \dfrac{\nu}{E}(\sigma_z + \sigma_x)$

Substitute numerical values and rearrange:

$(713.8 \times 10^{-6})\, E = 5200 + 7840\, \nu \qquad (1)$

$(-502.3 \times 10^{-6})\, E = -4750 - 2110\, \nu \qquad (2)$

Units: E = psi

Solve simultaneously Eqs. (1) and (2):

$E = 10.801 \times 10^6$ psi $\qquad \nu = 0.3202$

Eq. (7-61): $K = \dfrac{E}{3(1 - 2\nu)} = 10.0 \times 10^6$ psi $\longleftarrow$

Problem 7.6-6 Solve the preceding problem if the material is nylon subjected to compressive stresses $\sigma_x = -4.5$ MPa, $\sigma_y = -3.6$ MPa, and $\sigma_z = -2.1$ MPa, and the normal strains are $\epsilon_x = -740 \times 10^{-6}$ and $\epsilon_y = -320 \times 10^{-6}$ (shortenings).

Solution 7.6-6 Triaxial stress (bulk modulus)

$\sigma_x = -4.5$ MPa $\quad \sigma_y = -3.6$ MPa
$\sigma_z = -2.1$ MPa $\quad \varepsilon_x = -740 \times 10^{-6}$
$\varepsilon_y = -320 \times 10^{-6}$

Find K.

Eq. (7-53a): $\varepsilon_x = \dfrac{\sigma_x}{E} - \dfrac{\nu}{E}(\sigma_y + \sigma_z)$

Eq. (7-53b): $\varepsilon_y = \dfrac{\sigma_y}{E} - \dfrac{\nu}{E}(\sigma_z + \sigma_x)$

Substitute numerical values and rearrange:

$(-740 \times 10^{-6})\, E = -4.5 + 5.7\, \nu \qquad (1)$

$(-320 \times 10^{-6})\, E = -3.6 + 6.6\, \nu \qquad (2)$

Units: E = MPa

Solve simultaneously Eqs. (1) and (2):

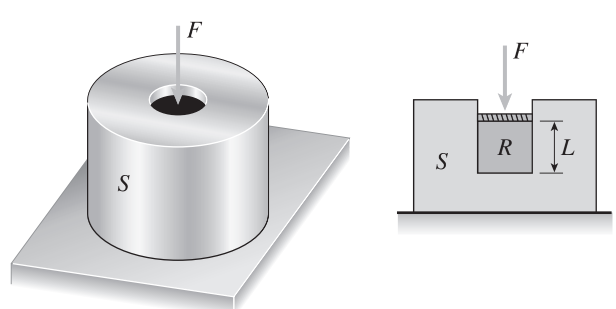
$E = 3{,}000$ MPa $= 3.0$ GPa $\qquad \nu = 0.40$

Eq. (7-61): $K = \dfrac{E}{3(1 - 2\nu)} = 5.0$ GPa $\longleftarrow$

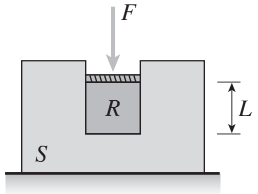
Problem 7.6-7 A rubber cylinder R of length L and cross-sectional area A is compressed inside a steel cylinder S by a force F that applies a uniformly distributed pressure to the rubber (see figure).

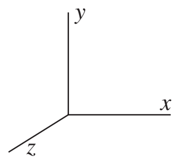
(a) Derive a formula for the lateral pressure p between the rubber and the steel. (Disregard friction between the rubber and the steel, and assume that the steel cylinder is rigid when compared to the rubber.)

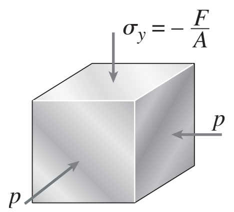
(b) Derive a formula for the shortening δ of the rubber cylinder.



Solution 7.6-7 **Rubber cylinder**







$\sigma_x = -p \qquad \sigma_y = -\dfrac{F}{A}$

$\sigma_z = -p$

$\varepsilon_x = \varepsilon_z = 0$

(a) LATERAL PRESSURE

Eq. (7-53a): $\varepsilon_x = \dfrac{\sigma_x}{E} - \dfrac{\nu}{E}(\sigma_y + \sigma_z)$

OR $\quad 0 = -p - \nu\left(-\dfrac{F}{A} - p\right)$

Solve for p: $\quad p = \dfrac{\nu}{1-\nu}\left(\dfrac{F}{A}\right) \quad \longleftarrow$

(b) SHORTENING

Eq. (7-53b): $\varepsilon_y = \dfrac{\sigma_y}{E} - \dfrac{\nu}{E}(\sigma_z + \sigma_x)$

$= -\dfrac{F}{EA} - \dfrac{\nu}{E}(-2p)$

Substitute for p and simplify:

$$\varepsilon_y = \frac{F}{EA}\,\frac{(1+\nu)(-1+2\nu)}{1-\nu}$$

(Positive ε_y represents an increase in strain, that is, elongation.)

$\delta = -\varepsilon_y L$

$\delta = \dfrac{(1+\nu)(1-2\nu)}{(1-\nu)}\left(\dfrac{FL}{EA}\right) \quad \longleftarrow$

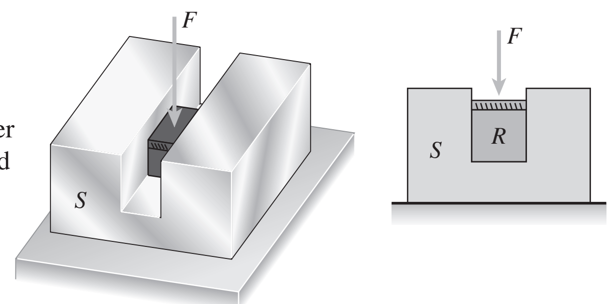
(Positive δ represents a shortening of the rubber cylinder.)

Problem 7.6-8 A block R of rubber is confined between plane parallel walls of a steel block S (see figure). A uniformly distributed pressure p_0 is applied to the top of the rubber block by a force F.

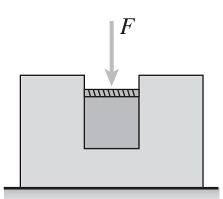
(a) Derive a formula for the lateral pressure p between the rubber and the steel. (Disregard friction between the rubber and the steel, and assume that the steel block is rigid when compared to the rubber.)

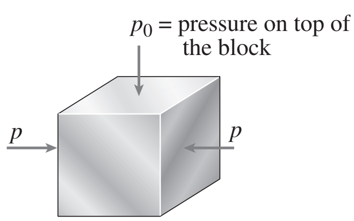
(b) Derive a formula for the dilatation e of the rubber.

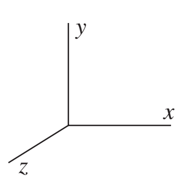
(c) Derive a formula for the strain-energy density u of the rubber.



Solution 7.6-8 **Block of rubber**







$\sigma_x = -p$

$\sigma_y = -p_0 \qquad \sigma_z = 0$

$\varepsilon_x = 0 \qquad \varepsilon_y \neq 0 \qquad \varepsilon_z \neq 0$

(a) LATERAL PRESSURE

Eq. (7-53a): $\varepsilon_x = \dfrac{\sigma_x}{E} - \dfrac{\nu}{E}(\sigma_y + \sigma_z)$

OR $\quad 0 = -p - \nu(-p_0) \qquad \therefore p = \nu p_0 \quad \longleftarrow$

(b) DILATATION

Eq. (7-56): $e = \dfrac{1-2\nu}{E}(\sigma_x + \sigma_y + \sigma_z)$

$= \dfrac{1-2\nu}{E}(-p - p_0)$

Substitute for p:

$$e = -\frac{(1+\nu)(1-2\nu)p_0}{E} \quad \longleftarrow$$

(c) STRAIN ENERGY DENSITY

Eq. (7-57b):

$u = \dfrac{1}{2E}(\sigma_x^2 + \sigma_y^2 + \sigma_z^2) - \dfrac{\nu}{E}(\sigma_x\sigma_y + \sigma_x\sigma_z + \sigma_y\sigma_z)$

Substitute for σ_x, σ_y, σ_z, and p:

$u = \dfrac{(1-\nu^2)p_0^2}{2E} \quad \longleftarrow$

Problem 7.6-9 A solid spherical ball of brass ($E = 15 \times 10^6$ psi, $\nu = 0.34$) is lowered into the ocean to a depth of 10,000 ft. The diameter of the ball is 11.0 in.

Determine the decrease Δd in diameter, the decrease ΔV in volume, and the strain energy U of the ball.

Solution 7.6-9 Brass sphere

$E = 15 \times 10^6$ psi $\quad \nu = 0.34$

Lowered in the ocean to depth $h = 10{,}000$ ft

Diameter $d = 11.0$ in.

Sea water: $\gamma = 63.8$ lb/ft^3

Pressure: $\sigma_0 = \gamma h = 638{,}000$ lb/ft^2 $= 4431$ psi

DECREASE IN DIAMETER

Eq. (7-59): $\varepsilon_0 = \dfrac{\sigma_0}{E}(1 - 2\nu) = 94.53 \times 10^{-6}$

$$\Delta d = \varepsilon_0 d = 1.04 \times 10^{-3} \text{ in.} \quad \longleftarrow$$

(decrease)

DECREASE IN VOLUME

Eq. (7-60): $e = 3\varepsilon_0 = 283.6 \times 10^{-6}$

$$V_0 = \frac{4}{3}\pi r^3 = \frac{4}{3}(\pi)\left(\frac{11.0 \text{ in.}}{2}\right)^3 = 696.9 \text{ in.}^3$$

$\Delta V = eV_0 = 0.198$ in.^{3} $\longleftarrow$

(decrease)

STRAIN ENERGY

Use Eq. (7-57b) with $\sigma_x = \sigma_y = \sigma_z = \sigma_0$:

$$u = \frac{3(1 - 2\nu)\sigma_0^2}{2E} = 0.6283 \text{ psi}$$

$U = uV_0 = 438$ in.-lb $\longleftarrow$

Problem 7.6-10 A solid steel sphere ($E = 210$ GPa, $\nu = 0.3$) is subjected to hydrostatic pressure p such that its volume is reduced by 0.4%.

(a) Calculate the pressure p.

(b) Calculate the volume modulus of elasticity K for the steel.

(c) Calculate the strain energy U stored in the sphere if its diameter is $d = 150$ mm.

Solution 7.6-10 Steel sphere

$E = 210$ GPa $\quad \nu = 0.3$

Hydrostatic Pressure. V_0 = Initial volume

$\Delta V = 0.004V_0$

Dilatation: $e = \dfrac{\Delta V}{V_0} = 0.004$

(a) PRESSURE

Eq. (7-60): $e = \dfrac{3\sigma_0(1 - 2\nu)}{E}$

or $\quad \sigma_0 = \dfrac{Ee}{3(1 - 2\nu)} = 700$ MPa

Pressure $p = \sigma_0 = 700$ MPa $\longleftarrow$

(b) VOLUME MODULUS OF ELASTICITY

Eq. (7-63): $K = \dfrac{\sigma_0}{E} = \dfrac{700 \text{ MPa}}{0.004} = 175$ GPa $\longleftarrow$

(c) STRAIN ENERGY (d = diameter)

$d = 150$ mm $\quad r = 75$ mm

From Eq. (7-57b) with $\sigma_x = \sigma_y = \sigma_z = \sigma_0$:

$$u = \frac{3(1 - 2\nu)\sigma_0^2}{2E} = 1.40 \text{ MPa}$$

$$V_0 = \frac{4\pi r^3}{3} = 1767 \times 10^{-6} \text{ m}^3$$

$U = uV_0 = 2470$ N · m $= 2470$ J $\longleftarrow$

Problem 7.6-11 A solid bronze sphere (volume modulus of elasticity $K = 14.5 \times 10^6$ psi) is suddenly heated around its outer surface. The tendency of the heated part of the sphere to expand produces uniform tension in all directions at the center of the sphere.

If the stress at the center is 12,000 psi, what is the strain? Also, calculate the unit volume change e and the strain-energy density u at the center.

Solution 7.6-11 **Bronze sphere (heated)**

$K = 14.5 \times 10^6$ psi

$\sigma_0 = 12{,}000$ psi (tension at the center)

STRAIN AT THE CENTER OF THE SPHERE

Eq. (7-59): $\varepsilon_0 = \dfrac{\sigma_0}{E}(1 - 2\nu)$

Eq. (7-61): $K = \dfrac{E}{3(1 - 2\nu)}$

Combine the two equations:

$\varepsilon_0 = \dfrac{\sigma_0}{3K} = 276 \times 10^{-6}$ ←

UNIT VOLUME CHANGE AT THE CENTER

Eq. (7-62): $e = \dfrac{\sigma_0}{K} = 828 \times 10^{-6}$ ←

STRAIN ENERGY DENSITY AT THE CENTER

Eq. (7-57b) with $\sigma_x = \sigma_y = \sigma_z = \sigma_0$:

$u = \dfrac{3(1 - 2\nu)\sigma_0^2}{2E} = \dfrac{\sigma_0^2}{2K}$

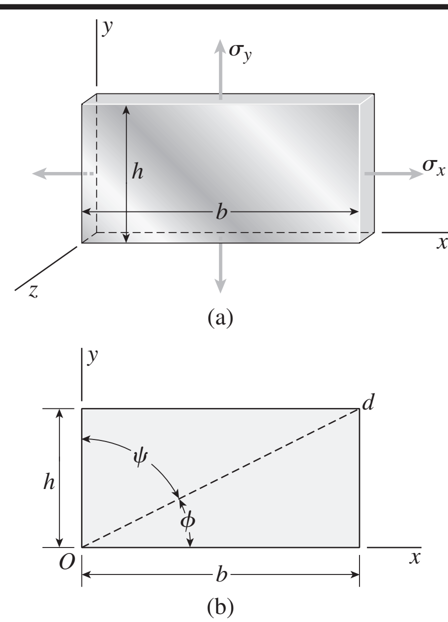
$u = 4.97$ psi ←

Plane Strain

When solving the problems for Section 7.7, consider only the in-plane strains (the strains in the xy plane) unless stated otherwise. Use the transformation equations of plane strain except when Mohr's circle is specified (Problems 7.7-23 through 7.7-28).

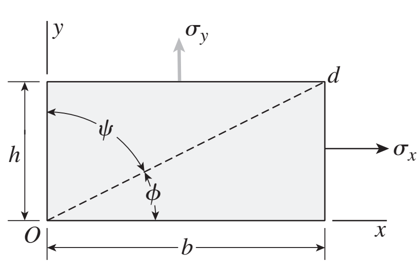
Problem 7.7-1 A thin rectangular plate in *biaxial stress* is subjected to stresses σ_x and σ_y, as shown in part (a) of the figure on the next page. The width and height of the plate are $b = 8.0$ in. and $h = 4.0$ in., respectively. Measurements show that the normal strains in the x and y directions are $\epsilon_x = 195 \times 10^{-6}$ and $\epsilon_y = -125 \times 10^{-6}$, respectively.

With reference to part (b) of the figure, which shows a two-dimensional view of the plate, determine the following quantities: (a) the increase Δd in the length of diagonal Od; (b) the change $\Delta\phi$ in the angle ϕ between diagonal Od and the x axis; and (c) the change $\Delta\psi$ in the angle ψ between diagonal Od and the y axis.



Probs. 7.7-1 and 7.7-2

Solution 7.7-1 **Plate in biaxial stress**



$b = 8.0$ in. $\quad h = 4.0$ in. $\quad \varepsilon_x = 195 \times 10^{-6}$

$\varepsilon_y = -125 \times 10^{-6} \quad \gamma_{xy} = 0$

$\phi = \arctan \dfrac{h}{b} = 26.57°$

$L_d = \sqrt{b^2 + h^2} = 8.944$ in.

(a) INCREASE IN LENGTH OF DIAGONAL

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

For $\theta = \phi = 26.57°$, $\varepsilon_{x_1} = 130.98 \times 10^{-6}$

$\Delta d = \varepsilon_{x_1} L_d = 0.00117$ in. ←

(b) CHANGE IN ANGLE ϕ

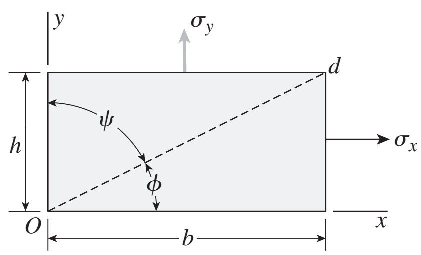
Eq. (7-68): $\alpha = -(\varepsilon_x - \varepsilon_y)\sin\theta\cos\theta - \gamma_{xy}\sin^2\theta$
For $\theta = \phi = 26.57°$: $\alpha = -128.0 \times 10^{-6}$ rad
Minus sign means line Od rotates clockwise (angle ϕ decreases).
$\Delta\phi = 128 \times 10^{-6}$ rad (decrease) ←

(c) CHANGE IN ANGLE ψ

Angle ψ increases the same amount that ϕ decreases.
$\Delta\psi = 128 \times 10^{-6}$ rad (increase) ←

Problem 7.7-2 Solve the preceding problem if $b = 160$ mm, $h = 60$ mm, $\epsilon_x = 410 \times 10^{-6}$, and $\epsilon_y = -320 \times 10^{-6}$.

Solution 7.7-2 Plate in biaxial stress



$b = 160$ mm $h = 60$ mm $\varepsilon_x = 410 \times 10^{-6}$
$\varepsilon_y = -320 \times 10^{-6}$ $\gamma_{xy} = 0$

$$\phi = \arctan\frac{h}{b} = 20.56°$$

$$L_d = \sqrt{b^2 + h^2} = 170.88 \text{ mm}$$

(a) INCREASE IN LENGTH OF DIAGONAL

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

For $\theta = \phi = 20.56°$: $\varepsilon_{x_1} = 319.97 \times 10^{-6}$
$\Delta d = \varepsilon_{x_1} L_d = 0.0547$ mm ←

(b) CHANGE IN ANGLE ϕ

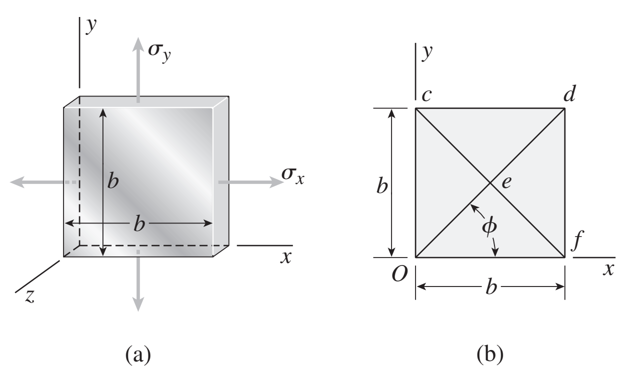
Eq. (7-68): $\alpha = -(\varepsilon_x - \varepsilon_y)\sin\theta\cos\theta - \gamma_{xy}\sin^2\theta$
For $\theta = \phi = 20.56°$: $\alpha = -240.0 \times 10^{-6}$ rad
Minus sign means line Od rotates clockwise (angle ϕ decreases).
$\Delta\phi = 240 \times 10^{-6}$ rad (decrease) ←

(c) CHANGE IN ANGLE ψ

Angle ψ increases the same amount that ϕ decreases.
$\Delta\psi = 240 \times 10^{-6}$ rad (increase) ←

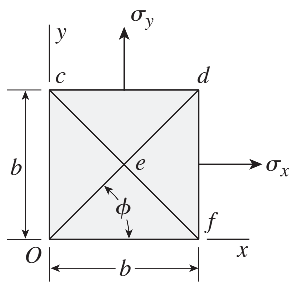
Problem 7.7-3 A thin square plate in *biaxial stress* is subjected to stresses σ_x and σ_y, as shown in part (a) of the figure. The width of the plate is $b = 12.0$ in. Measurements show that the normal strains in the x and y directions are $\epsilon_x = 427 \times 10^{-6}$ and $\epsilon_y = 113 \times 10^{-6}$, respectively.

With reference to part (b) of the figure, which shows a two-dimensional view of the plate, determine the following quantities: (a) the increase Δd in the length of diagonal Od; (b) the change $\Delta\phi$ in the angle ϕ between diagonal Od and the x axis; and (c) the shear strain γ associated with diagonals Od and cf (that is, find the decrease in angle ced).



Probs. 7.7-3 and 7.7-4

Solution 7.7-3 **Square plate in biaxial stress**



$b = 12.0$ in. $\quad \varepsilon_x = 427 \times 10^{-6}$

$\varepsilon_y = 113 \times 10^{-6}$

$\phi = 45°\quad \gamma_{xy} = 0$

$L_d = b\sqrt{2} = 16.97$ in.

(a) Increase in length of diagonal

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

For $\theta = \phi = 45°$: $\varepsilon_{x_1} = 270 \times 10^{-6}$

$\Delta d = \varepsilon_{x_1} L_d = 0.00458$ in. ←

(b) Change in angle ϕ

Eq. (7-68): $\alpha = -(\varepsilon_x - \varepsilon_y)\sin\theta\cos\theta - \gamma_{xy}\sin^2\theta$

For $\theta = \phi = 45°$: $\alpha = -157 \times 10^{-6}$ rad

Minus sign means line *Od* rotates clockwise (angle ϕ decreases).

$\Delta\phi = 157 \times 10^{-6}$ rad (decrease) ←

(c) Shear strain between diagonals

Eq. (7-71b): $$\frac{\gamma_{x_1y_1}}{2} = -\frac{\varepsilon_x - \varepsilon_y}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

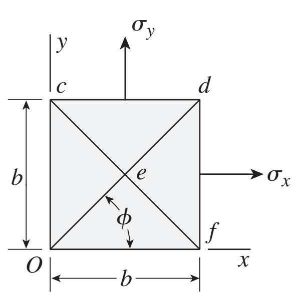
For $\theta = \phi = 45°$: $\gamma_{x_1y_1} = -314 \times 10^{-6}$ rad

(Negative strain means angle *ced* increases)

$\gamma = -314 \times 10^{-6}$ rad ←

Problem 7.7-4 Solve the preceding problem if $b = 225$ mm, $\epsilon_x = 845 \times 10^{-6}$, and $\epsilon_y = 211 \times 10^{-6}$.

Solution 7.7-4 **Square plate in biaxial stress**



$b = 225$ mm $\quad \varepsilon_x = 845 \times 10^{-6}$

$\varepsilon_y = 211 \times 10^{-6}\quad \phi = 45°\quad \gamma_{xy} = 0$

$L_d = b\sqrt{2} = 318.2$ mm

(a) Increase in length of diagonal

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

For $\theta = \phi = 45°$: $\varepsilon_{x_1} = 528 \times 10^{-6}$

$\Delta d = \varepsilon_{x_1} L_d = 0.168$ mm ←

(b) Change in angle ϕ

Eq. (7-68): $\alpha = -(\varepsilon_x - \varepsilon_y)\sin\theta\cos\theta - \gamma_{xy}\sin^2\theta$

For $\theta = \phi = 45°$: $\alpha = -317 \times 10^{-6}$ rad

Minus sign means line *Od* rotates clockwise (angle ϕ decreases).

$\Delta\phi = 317 \times 10^{-6}$ rad (decrease) ←

(c) Shear strain between diagonals

Eq. (7-71b): $$\frac{\gamma_{x_1y_1}}{2} = -\frac{\varepsilon_x - \varepsilon_y}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

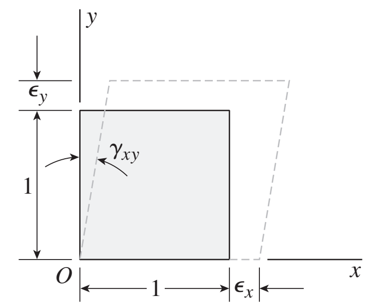
For $\theta = \phi = 45°$: $\gamma_{x_1y_1} = -634 \times 10^{-6}$ rad

(Negative strain means angle *ced* increases)

$\gamma = -634 \times 10^{-6}$ rad ←

Problem 7.7-5 An element of material subjected to *plane strain* (see figure) has strains as follows: $\epsilon_x = 220 \times 10^{-6}$, $\epsilon_y = 480 \times 10^{-6}$, and $\gamma_{xy} = 180 \times 10^{-6}$.

Calculate the strains for an element oriented at an angle $\theta = 50°$ and show these strains on a sketch of a properly oriented element.



Probs. 7.7-5 through 7.7-10

Solution 7.7-5 Element in plane strain

$\varepsilon_x = 220 \times 10^{-6}$ $\quad \varepsilon_y = 480 \times 10^{-6}$

$\gamma_{xy} = 180 \times 10^{-6}$

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

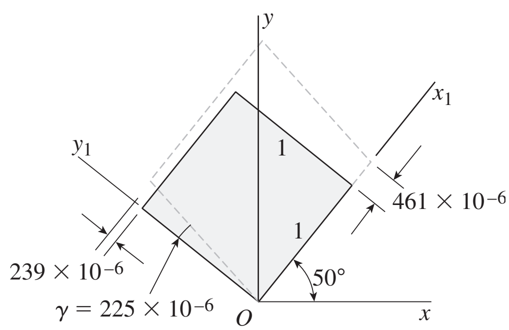
$$\frac{\gamma_{x_1y_1}}{2} = -\frac{\varepsilon_x - \varepsilon_y}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

$$\varepsilon_{y_1} = \varepsilon_x + \varepsilon_y - \varepsilon_{x_1}$$

For $\theta = 50°$:

$\varepsilon_{x_1} = 461 \times 10^{-6}$ $\quad \gamma_{x_1y_1} = 225 \times 10^{-6}$

$\varepsilon_{y_1} = 239 \times 10^{-6}$



Problem 7.7-6 Solve the preceding problem for the following data: $\epsilon_x = 420 \times 10^{-6}$, $\epsilon_y = -170 \times 10^{-6}$, $\gamma_{xy} = 310 \times 10^{-6}$, and $\theta = 37.5°$.

Solution 7.7-6 Element in plane strain

$\varepsilon_x = 420 \times 10^{-6}$ $\quad \varepsilon_y = -170 \times 10^{-6}$

$\gamma_{xy} = 310 \times 10^{-6}$

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

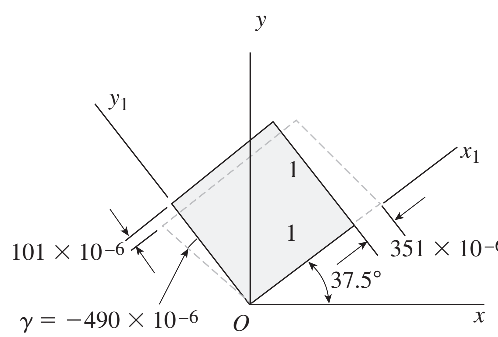
$$\frac{\gamma_{x_1y_1}}{2} = -\frac{\varepsilon_x - \varepsilon_y}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

$$\varepsilon_{y_1} = \varepsilon_x + \varepsilon_y - \varepsilon_{x_1}$$

For $\theta = 37.5°$:

$\varepsilon_{x_1} = 351 \times 10^{-6}$ $\quad \gamma_{x_1y_1} = -490 \times 10^{-6}$

$\varepsilon_{y_1} = -101 \times 10^{-6}$



Problem 7.7-7 The strains for an element of material in *plane strain* (see figure) are as follows: $\epsilon_x = 480 \times 10^{-6}$, $\epsilon_y = 140 \times 10^{-6}$, and $\gamma_{xy} = -350 \times 10^{-6}$.

Determine the principal strains and maximum shear strains, and show these strains on sketches of properly oriented elements.

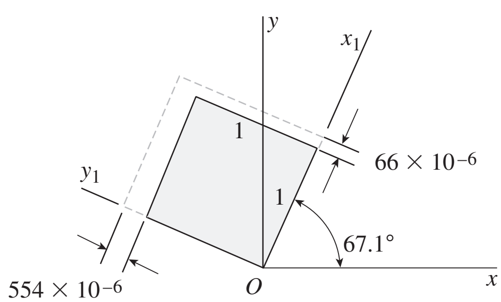
Solution 7.7-7 **Element in plane strain**

$\varepsilon_x = 480 \times 10^{-6} \quad \varepsilon_y = 140 \times 10^{-6}$
$\gamma_{xy} = -350 \times 10^{-6}$

PRINCIPAL STRAINS

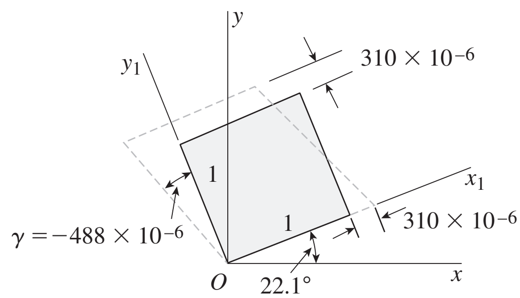
$$\varepsilon_{1,2} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$
$$= 310 \times 10^{-6} \pm 244 \times 10^{-6}$$
$$\varepsilon_1 = 554 \times 10^{-6} \qquad \varepsilon_2 = 66 \times 10^{-6}$$
$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} = -1.0294$$
$$2\theta_p = -45.8^\circ \quad \text{and} \quad 134.2^\circ$$
$$\theta_p = -22.9^\circ \quad \text{and} \quad 67.1^\circ$$

For $\theta_p = -22.9^\circ$:
$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$
$$= 554 \times 10^{-6}$$
$$\therefore \theta_{p_1} = -22.9^\circ \qquad \varepsilon_1 = 554 \times 10^{-6} \quad \longleftarrow$$
$$\theta_{p_2} = 67.1^\circ \qquad \varepsilon_2 = 66 \times 10^{-6} \quad \longleftarrow$$



MAXIMUM SHEAR STRAINS

$$\frac{\gamma_{\max}}{2} = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$
$$= 244 \times 10^{-6}$$
$$\gamma_{\max} = 488 \times 10^{-6}$$
$$\theta_{s_1} = \theta_{p_1} - 45^\circ = -67.9^\circ \text{ or } 112.1^\circ$$
$$\gamma_{\max} = 488 \times 10^{-6} \quad \longleftarrow$$
$$\theta_{s_2} = \theta_{s_1} + 90^\circ = 22.1^\circ$$
$$\gamma_{\min} = -488 \times 10^{-6} \quad \longleftarrow$$
$$\varepsilon_{\text{aver}} = \frac{\varepsilon_x + \varepsilon_y}{2} = 310 \times 10^{-6}$$



Problem 7.7-8 Solve the preceding problem for the following strains: $\epsilon_x = 120 \times 10^{-6}$, $\epsilon_y = -450 \times 10^{-6}$, and $\gamma_{xy} = -360 \times 10^{-6}$.

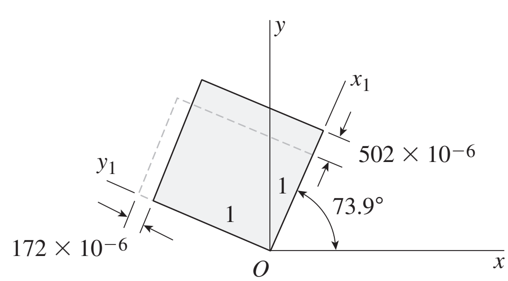
Solution 7.7-8 **Element in plane strain**

$\varepsilon_x = 120 \times 10^{-6} \quad \varepsilon_y = -450 \times 10^{-6}$
$\gamma_{xy} = -360 \times 10^{-6}$

PRINCIPAL STRAINS

$$\varepsilon_{1,2} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$
$$= -165 \times 10^{-6} \pm 377 \times 10^{-6}$$
$$\varepsilon_1 = 172 \times 10^{-6} \qquad \varepsilon_2 = -502 \times 10^{-6}$$
$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} = -0.6316$$
$$2\theta_p = 327.7^\circ \quad \text{and} \quad 147.7^\circ$$
$$\theta_p = 163.9^\circ \quad \text{and} \quad 73.9^\circ$$

For $\theta_p = 163.9^\circ$:
$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$
$$= 172 \times 10^{-6}$$
$$\therefore \theta_{p_1} = 163.9^\circ \qquad \varepsilon_1 = 172 \times 10^{-6} \quad \longleftarrow$$
$$\theta_{p_2} = 73.9^\circ \qquad \varepsilon_2 = -502 \times 10^{-6} \quad \longleftarrow$$



MAXIMUM SHEAR STRAINS

$$\frac{\gamma_{max}}{2} = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$= 337 \times 10^{-6}$$

$$\gamma_{max} = 674 \times 10^{-6}$$

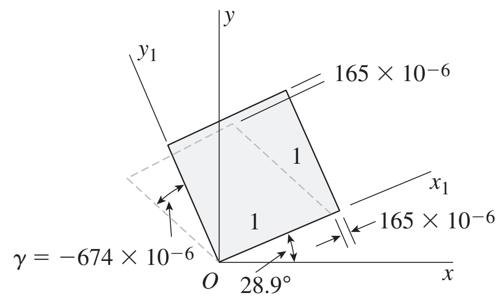
$$\theta_{s_1} = \theta_{p_1} - 45° = 118.9°$$

$$\gamma_{max} = 674 \times 10^{-6} \quad \longleftarrow$$

$$\theta_{s_2} = \theta_{s_1} - 90° = 28.9°$$

$$\gamma_{min} = -674 \times 10^{-6} \quad \longleftarrow$$

$$\varepsilon_{aver} = \frac{\varepsilon_x + \varepsilon_y}{2} = -165 \times 10^{-6}$$



Problem 7.7-9 An element of material in *plane strain* (see figure) is subjected to strains $\epsilon_x = 480 \times 10^{-6}$, $\epsilon_y = 70 \times 10^{-6}$, and $\gamma_{xy} = 420 \times 10^{-6}$.

Determine the following quantities: (a) the strains for an element oriented at an angle $\theta = 75°$, (b) the principal strains, and (c) the maximum shear strains. Show the results on sketches of properly oriented elements.

Solution 7.7-9 Element in plane strain

$\varepsilon_x = 480 \times 10^{-6} \qquad \varepsilon_y = 70 \times 10^{-6}$

$\gamma_{xy} = 420 \times 10^{-6}$

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

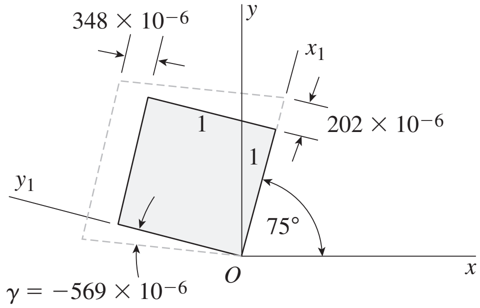
$$\frac{\gamma_{x_1y_1}}{2} = -\frac{\varepsilon_x - \varepsilon_y}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

$$\varepsilon_{y_1} = \varepsilon_x + \varepsilon_y - \varepsilon_{x_1}$$

FOR $\theta = 75°$:

$\varepsilon_{x_1} = 202 \times 10^{-6} \qquad \gamma_{x_1y_1} = -569 \times 10^{-6}$

$\varepsilon_{y_1} = 348 \times 10^{-6}$



PRINCIPAL STRAINS

$$\varepsilon_{1,2} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$= 275 \times 10^{-6} \pm 293 \times 10^{-6}$$

$$\varepsilon_1 = 568 \times 10^{-6} \qquad \varepsilon_2 = -18 \times 10^{-6}$$

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} = 1.0244$$

$2\theta_p = 45.69°$ and $225.69°$

$\theta_p = 22.85°$ and $112.85°$

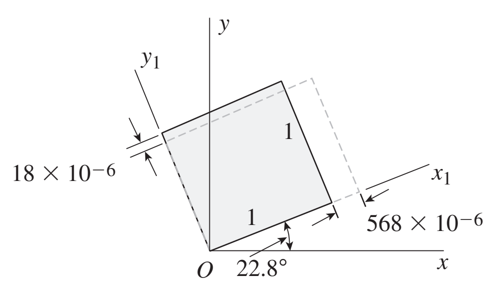
For $\theta_p = 22.85°$:

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$= 568 \times 10^{-6}$$

$\therefore \quad \theta_{p_1} = 22.8° \qquad \varepsilon_1 = 568 \times 10^{-6} \quad \longleftarrow$

$\theta_{p_2} = 112.8° \qquad \varepsilon_2 = -18 \times 10^{-6} \quad \longleftarrow$



MAXIMUM SHEAR STRAINS

$$\frac{\gamma_{max}}{2} = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} = 293 \times 10^{-6}$$

$\gamma_{max} = 587 \times 10^{-6}$

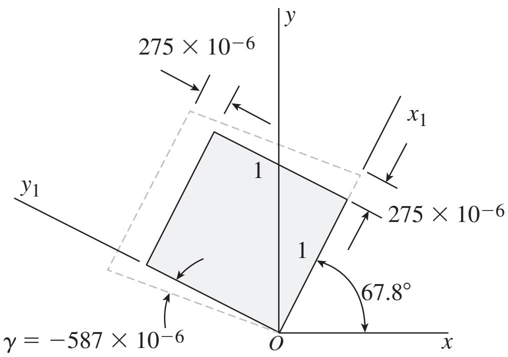
$\theta_{s_1} = \theta_{p_1} - 45° = -22.2°$ or $157.8°$

$\gamma_{max} = 587 \times 10^{-6}$ ←

$\theta_{s_2} = \theta_{s_1} + 90° = 67.8°$

$\gamma_{min} = -587 \times 10^{-6}$ ←

$$\varepsilon_{aver} = \frac{\varepsilon_x + \varepsilon_y}{2} = 275 \times 10^{-6}$$



Problem 7.7-10 Solve the preceding problem for the following data: $\epsilon_x = -1120 \times 10^{-6}$, $\epsilon_y = -430 \times 10^{-6}$, $\gamma_{xy} = 780 \times 10^{-6}$, and $\theta = 45°$.

Solution 7.7-10 Element in plane strain

$\varepsilon_x = -1120 \times 10^{-6}$ $\quad \varepsilon_y = -430 \times 10^{-6}$

$\gamma_{xy} = 780 \times 10^{-6}$

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

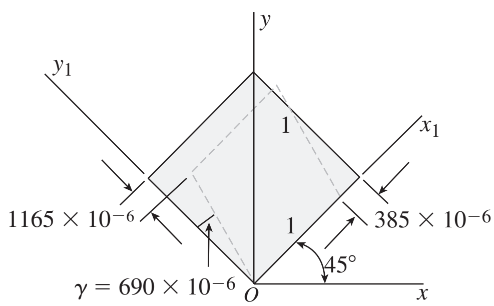
$$\frac{\gamma_{x_1y_1}}{2} = -\frac{\varepsilon_x - \varepsilon_y}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

$\varepsilon_{y_1} = \varepsilon_x + \varepsilon_y - \varepsilon_{x_1}$

FOR $\theta = 45°$:

$\varepsilon_{x_1} = -385 \times 10^{-6}$ $\quad \gamma_{x_1y_1} = 690 \times 10^{-6}$

$\varepsilon_{y_1} = -1165 \times 10^{-6}$



PRINCIPAL STRAINS

$$\varepsilon_{1,2} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$= -775 \times 10^{-6} \pm 521 \times 10^{-6}$

$\varepsilon_1 = -254 \times 10^{-6}$ $\quad \varepsilon_2 = -1296 \times 10^{-6}$

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} = -1.1304$$

$2\theta_p = 131.5°$ and $311.5°$

$\theta_p = 65.7°$ and $155.7°$

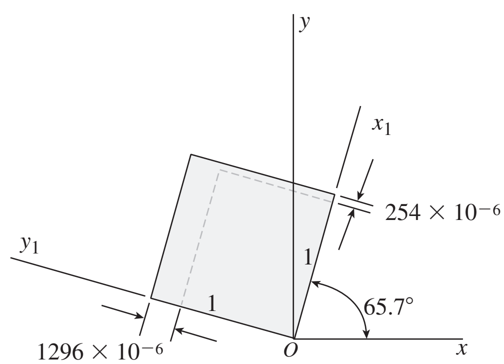
For $\theta_p = 65.7°$:

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$= -254 \times 10^{-6}$

$\therefore \theta_{p_1} = 65.7°$ $\quad \varepsilon_1 = -254 \times 10^{-6}$ ←

$\theta_{p_2} = 155.7°$ $\quad \varepsilon_2 = -1296 \times 10^{-6}$ ←



MAXIMUM SHEAR STRAINS

$$\frac{\gamma_{max}}{2} = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$= 521 \times 10^{-6}$

$\gamma_{max} = 1041 \times 10^{-6}$

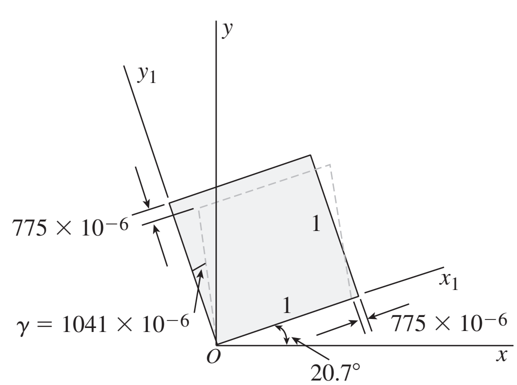
$\theta_{s_1} = \theta_{p_1} - 45° = 20.7°$

$\gamma_{max} = 1041 \times 10^{-6}$ ←

$\theta_{s_2} = \theta_{s_1} + 90° = 110.7°$

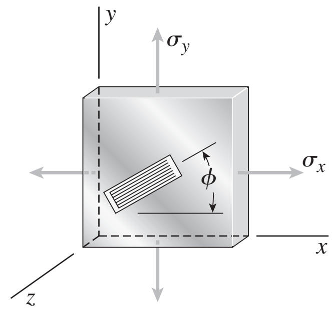
$\gamma_{min} = -1041 \times 10^{-6}$ ←

$$\varepsilon_{aver} = \frac{\varepsilon_x + \varepsilon_y}{2} = -775 \times 10^{-6}$$



Problem 7.7-11 A steel plate with modulus of elasticity $E = 30 \times 10^6$ psi and Poisson's ratio $\nu = 0.30$ is loaded in *biaxial stress* by normal stresses σ_x and σ_y (see figure). A strain gage is bonded to the plate at an angle $\phi = 30°$.

If the stress σ_x is 18,000 psi and the strain measured by the gage is $\epsilon = 407 \times 10^{-6}$, what is the maximum in-plane shear stress $(\tau_{max})_{xy}$ and shear strain $(\gamma_{max})_{xy}$? What is the maximum shear strain $(\gamma_{max})_{xz}$ in the xz plane? What is the maximum shear strain $(\gamma_{max})_{yz}$ in the yz plane?



Probs. 7.7-11 and 7.7-12

Solution 7.7-11 Steel plate in biaxial stress

$\sigma_x = 18{,}000$ psi $\quad \gamma_{xy} = 0 \quad \sigma_y = ?$
$E = 30 \times 10^6$ psi $\quad \nu = 0.30$
Strain gage: $\phi = 30° \quad \varepsilon = 407 \times 10^{-6}$

UNITS: All stresses in psi.

STRAIN IN BIAXIAL STRESS (EQS. 7-39)

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) = \frac{1}{30 \times 10^6}(18{,}000 - 0.3\sigma_y) \qquad (1)$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) = \frac{1}{30 \times 10^6}(\sigma_y - 5400) \qquad (2)$$

$$\varepsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y) = -\frac{0.3}{30 \times 10^6}(18{,}000 + \sigma_y) \qquad (3)$$

STRAINS AT ANGLE $\phi = 30°$ (Eq. 7-71a)

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$407 \times 10^{-6} = \left(\frac{1}{2}\right)\left(\frac{1}{30 \times 10^6}\right)(12{,}600 + 0.7\sigma_y) + \left(\frac{1}{2}\right)\left(\frac{1}{30 \times 10^6}\right)(23{,}400 - 1.3\sigma_y)\cos 60°$$

Solve for σ_y: $\quad \sigma_y = 2400$ psi $\qquad (4)$

MAXIMUM IN-PLANE SHEAR STRESS

$$(\tau_{max})_{xy} = \frac{\sigma_x - \sigma_y}{2} = 7800 \text{ psi} \quad \longleftarrow$$

STRAINS FROM EQS. (1), (2), AND (3)

$\varepsilon_x = 576 \times 10^{-6} \quad \varepsilon_y = -100 \times 10^{-6}$
$\varepsilon_z = -204 \times 10^{-6}$

MAXIMUM SHEAR STRAINS (EQ. 7-75)

xy plane: $$\frac{(\gamma_{max})_{xy}}{2} = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$\gamma_{xy} = 0 \quad (\gamma_{max})_{xy} = 676 \times 10^{-6} \quad \longleftarrow$

xz plane: $$\frac{(\gamma_{max})_{xz}}{2} = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_z}{2}\right)^2 + \left(\frac{\gamma_{xz}}{2}\right)^2}$$

$\gamma_{xz} = 0 \quad (\gamma_{max})_{xz} = 780 \times 10^{-6} \quad \longleftarrow$

yz plane: $$\frac{(\gamma_{max})_{yz}}{2} = \sqrt{\left(\frac{\varepsilon_y - \varepsilon_z}{2}\right)^2 + \left(\frac{\gamma_{yz}}{2}\right)^2}$$

$\gamma_{yz} = 0 \quad (\gamma_{max})_{yz} = 104 \times 10^{-6} \quad \longleftarrow$

Problem 7.7-12 Solve the preceding problem if the plate is made of aluminum with $E = 72$ GPa and $\nu = 1/3$, the stress σ_x is 86.4 MPa, the angle ϕ is 21°, and the strain ϵ is 946×10^{-6}.

Solution 7.7-12 Aluminum plate in biaxial stress

$\sigma_x = 86.4$ MPa $\quad \gamma_{xy} = 0 \quad \sigma_y = ?$
$E = 72$ GPa $\quad \nu = 1/3$
Strain gage: $\phi = 21° \quad \varepsilon = 946 \times 10^{-6}$

UNITS: All stresses in MPa.

STRAINS IN BIAXIAL STRESS (EQS. 7-39)

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) = \frac{1}{72{,}000}\left(86.4 - \frac{1}{3}\sigma_y\right) \qquad (1)$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) = \frac{1}{72{,}000}(\sigma_y - 28.8) \qquad (2)$$

$$\varepsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y) = -\frac{1/3}{72{,}000}(86.4 + \sigma_y) \qquad (3)$$

STRAINS AT ANGLE $\phi = 21°$ (EQ. 7-71a)

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$946 \times 10^{-6} = \left(\frac{1}{2}\right)\left(\frac{1}{72{,}000}\right)\left(57.6 + \frac{2}{3}\sigma_y\right) + \left(\frac{1}{2}\right)\left(\frac{1}{72{,}000}\right)\left(115.2 - \frac{4}{3}\sigma_y\right)\cos 42°$$

Solve for σ_y: $\quad \sigma_y = 21.55$ MPa $\qquad$ (4)

MAXIMUM IN-PLANE SHEAR STRESS

$$(\tau_{max})_{xy} = \frac{\sigma_x - \sigma_y}{2} = 32.4 \text{ MPa} \longleftarrow$$

STRAINS FROM EQS. (1), (2), AND (3)

$\varepsilon_x = 1100 \times 10^{-6} \qquad \varepsilon_y = -101 \times 10^{-6}$
$\varepsilon_z = -500 \times 10^{-6}$

MAXIMUM SHEAR STRAINS (EQ. 7-75)

$$xy \text{ plane: } \frac{(\gamma_{max})_{xy}}{2} = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$\gamma_{xy} = 0 \qquad (\gamma_{max})_{xy} = 1200 \times 10^{-6} \longleftarrow$$

$$xz \text{ plane: } \frac{(\gamma_{max})_{xz}}{2} = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_z}{2}\right)^2 + \left(\frac{\gamma_{xz}}{2}\right)^2}$$

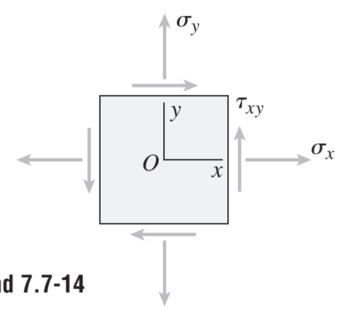
$$\gamma_{xz} = 0 \qquad (\gamma_{max})_{xz} = 1600 \times 10^{-6} \longleftarrow$$

$$yz \text{ plane: } \frac{(\gamma_{max})_{yz}}{2} = \sqrt{\left(\frac{\varepsilon_y - \varepsilon_z}{2}\right)^2 + \left(\frac{\gamma_{yz}}{2}\right)^2}$$

$$\gamma_{yz} = 0 \qquad (\gamma_{max})_{yz} = 399 \times 10^{-6} \longleftarrow$$

Problem 7.7-13 An element in *plane stress* is subjected to stresses $\sigma_x = -8400$ psi, $\sigma_y = 1100$ psi, and $\tau_{xy} = -1700$ psi (see figure). The material is aluminum with modulus of elasticity $E = 10{,}000$ ksi and Poisson's ratio $\nu = 0.33$.

Determine the following quantities: (a) the strains for an element oriented at an angle $\theta = 30°$, (b) the principal strains, and (c) the maximum shear strains. Show the results on sketches of properly oriented elements.



Probs. 7.7-13 and 7.7-14

Solution 7.7-13 Element in plane stress

$\sigma_x = -8400$ psi $\quad \sigma_y = 1100$ psi
$\tau_{xy} = -1700$ psi $\quad E = 10{,}000$ ksi $\quad \nu = 0.33$

HOOKE'S LAW (EQS. 7-34 AND 7-35)

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) = -876.3 \times 10^{-6}$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) = 387.2 \times 10^{-6}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{2\tau_{xy}(1+\nu)}{E} = -452.2 \times 10^{-6}$$

FOR $\theta = 30°$:

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

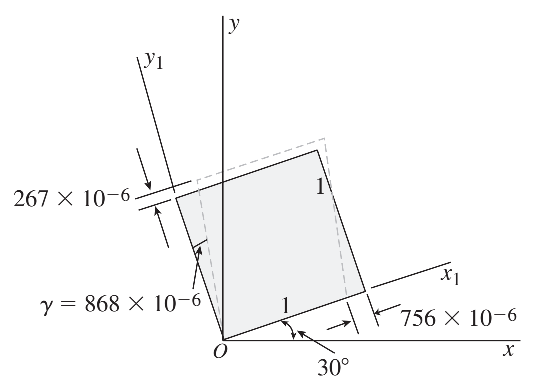
$$= -756 \times 10^{-6}$$

$$\frac{\gamma_{x_1y_1}}{2} = -\frac{\varepsilon_x - \varepsilon_y}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

$$= 434 \times 10^{-6}$$

$$\gamma_{x_1y_1} = 868 \times 10^{-6}$$

$$\varepsilon_{y_1} = \varepsilon_x + \varepsilon_y - \varepsilon_{x_1} = 267 \times 10^{-6}$$

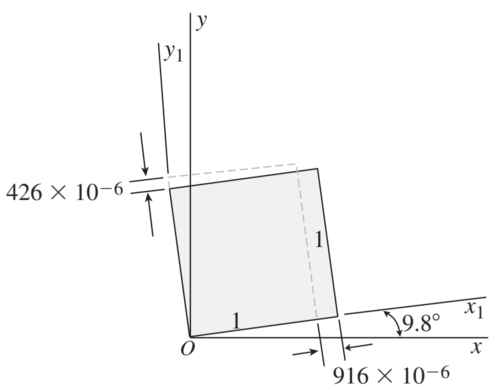


PRINCIPAL STRAINS

$$\varepsilon_{1,2} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$
$$= -245 \times 10^{-6} \pm 671 \times 10^{-6}$$
$$\varepsilon_1 = 426 \times 10^{-6} \qquad \varepsilon_2 = -916 \times 10^{-6}$$
$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} = 0.3579$$
$$2\theta_p = 19.7^\circ \quad \text{and} \quad 199.7^\circ$$
$$\theta_p = 9.8^\circ \quad \text{and} \quad 99.8^\circ$$

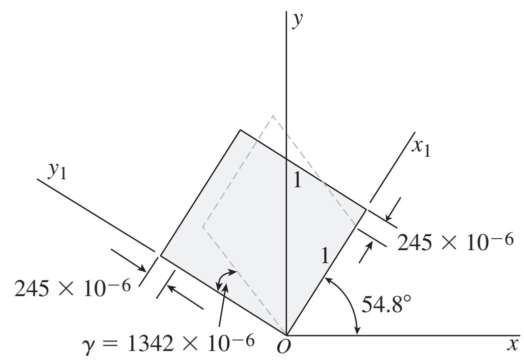
For $\theta_p = 9.8^\circ$:

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$
$$= -916 \times 10^{-6}$$
$$\therefore\ \theta_{p_1} = 99.8^\circ \qquad \varepsilon_1 = 426 \times 10^{-6} \quad \longleftarrow$$
$$\theta_{p_2} = 9.8^\circ \qquad \varepsilon_2 = -916 \times 10^{-6} \quad \longleftarrow$$



MAXIMUM SHEAR STRAINS

$$\frac{\gamma_{max}}{2} = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$
$$= 671 \times 10^{-6}$$
$$\gamma_{max} = 1342 \times 10^{-6}$$
$$\theta_{s_1} = \theta_{p_1} - 45^\circ = 54.8^\circ$$
$$\gamma_{max} = 1342 \times 10^{-6} \quad \longleftarrow$$
$$\theta_{s_2} = \theta_{s_1} + 90^\circ = 144.8^\circ$$
$$\gamma_{min} = -1342 \times 10^{-6} \quad \longleftarrow$$
$$\varepsilon_{aver} = \frac{\varepsilon_x + \varepsilon_y}{2} = -245 \times 10^{-6}$$



Problem 7.7-14 Solve the preceding problem for the following data: $\sigma_x = -150$ MPa, $\sigma_y = -210$ MPa, $\tau_{xy} = -16$ MPa, and $\theta = 50^\circ$. The material is brass with $E = 100$ GPa and $\nu = 0.34$.

Solution 7.7-14 Element in plane stress

$\sigma_x = -150$ MPa $\quad \sigma_y = -210$ MPa
$\tau_{xy} = -16$ MPa $\quad E = 100$ GPa $\quad \nu = 0.34$

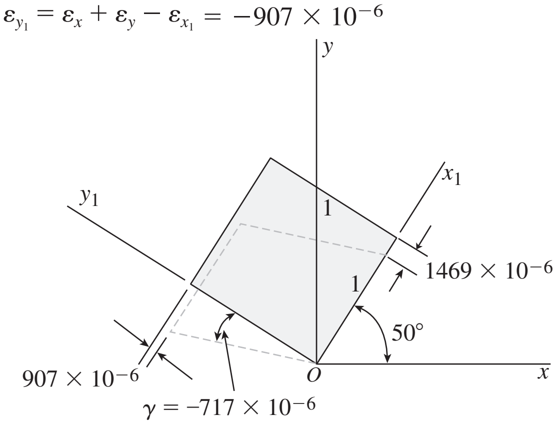
HOOKE'S LAW (EQS. 7-34 AND 7-35)

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) = -786 \times 10^{-6}$$
$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) = -1590 \times 10^{-6}$$
$$\gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{2\tau_{xy}(1+\nu)}{E} = -429 \times 10^{-6}$$

FOR $\theta = 50^\circ$:

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$
$$= -1469 \times 10^{-6}$$

$$\frac{\gamma_{x_1y_1}}{2} = -\frac{\varepsilon_x - \varepsilon_y}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$
$$= -358.5 \times 10^{-6}$$
$$\gamma_{x_1y_1} = -717 \times 10^{-6}$$
$$\varepsilon_{y_1} = \varepsilon_x + \varepsilon_y - \varepsilon_{x_1} = -907 \times 10^{-6}$$



PRINCIPAL STRAINS

$$\varepsilon_{1,2} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$= -1188 \times 10^{-6} \pm 456 \times 10^{-6}$

$\varepsilon_1 = -732 \times 10^{-6} \quad \varepsilon_2 = -1644 \times 10^{-6}$

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} = -0.5333$$

$2\theta_p = 151.9°$ and $331.9°$

$\theta_p = 76.0°$ and $166.0°$

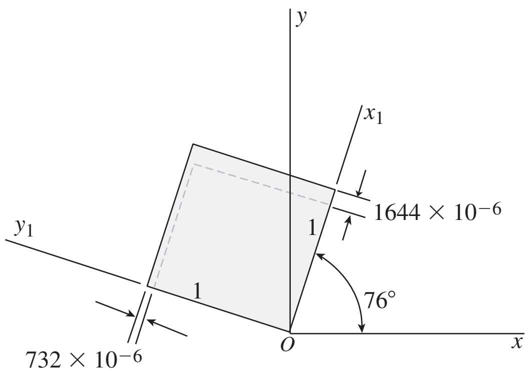
For $\theta_p = 76.0°$:

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$= -1644 \times 10^{-6}$

$\therefore\ \theta_{p_1} = 166.0° \quad \varepsilon_1 = -732 \times 10^{-6}$ ←

$\theta_{p_2} = 76.0° \quad \varepsilon_2 = -1644 \times 10^{-6}$ ←



MAXIMUM SHEAR STRAINS

$$\frac{\gamma_{max}}{2} = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$= 456 \times 10^{-6}$

$\gamma_{max} = 911 \times 10^{-6}$

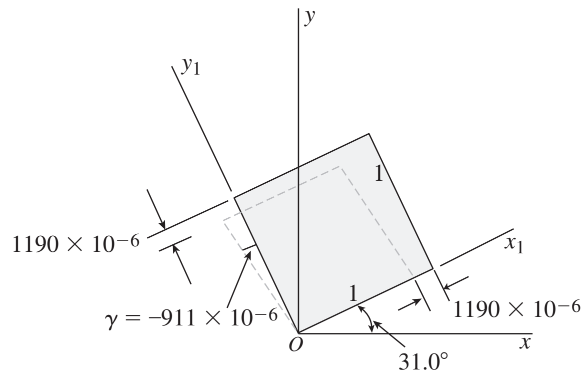
$\theta_{s_1} = \theta_{p_1} - 45° = 121.0°$

$\gamma_{max} = 911 \times 10^{-6}$ ←

$\theta_{s_2} = \theta_{s_1} - 90° = 31.0°$

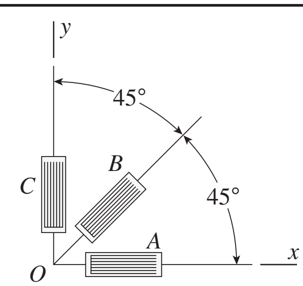
$\gamma_{min} = -911 \times 10^{-6}$ ←

$$\varepsilon_{aver} = \frac{\varepsilon_x + \varepsilon_y}{2} = -1190 \times 10^{-6}$$



Problem 7.7-15 During a test of an airplane wing, the strain gage readings from a 45° rosette (see figure) are as follows: gage A, 520×10^{-6}; gage B, 360×10^{-6}; and gage C, -80×10^{-6}.

Determine the principal strains and maximum shear strains, and show them on sketches of properly oriented elements.



Probs. 7.7-15 and 7.7-16

Solution 7.7-15 45° strain rosette

$\varepsilon_A = 520 \times 10^{-6} \quad \varepsilon_B = 360 \times 10^{-6}$

$\varepsilon_C = -80 \times 10^{-6}$

FROM EQS. (7-77) AND (7-78) OF EXAMPLE 7-8:

$\varepsilon_x = \varepsilon_A = 520 \times 10^{-6} \quad \varepsilon_y = \varepsilon_C = -80 \times 10^{-6}$

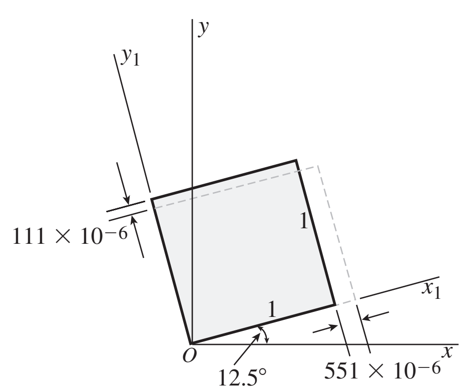
$\gamma_{xy} = 2\varepsilon_B - \varepsilon_A - \varepsilon_C = 280 \times 10^{-6}$

PRINCIPAL STRAINS

$$\varepsilon_{1,2} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$= 220 \times 10^{-6} \pm 331 \times 10^{-6}$

$\varepsilon_1 = 551 \times 10^{-6} \quad \varepsilon_2 = -111 \times 10^{-6}$



$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} = 0.4667$$
$2\theta_p = 25.0°$ and $205.0°$
$\theta_p = 12.5°$ and $102.5°$

For $\theta_p = 12.5°$:
$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$
$= 551 \times 10^{-6}$

$\therefore\ \theta_{p_1} = 12.5°$ $\quad \varepsilon_1 = 551 \times 10^{-6}$ ←

$\theta_{p_2} = 102.5°$ $\quad \varepsilon_2 = -111 \times 10^{-6}$ ←

MAXIMUM SHEAR STRAINS

$$\frac{\gamma_{max}}{2} = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$
$= 331 \times 10^{-6}$

$\gamma_{max} = 662 \times 10^{-6}$

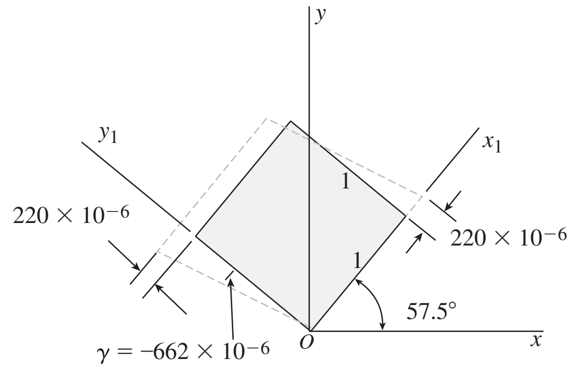
$\theta_{s_1} = \theta_{p_1} - 45° = -32.5°$ or $147.5°$

$\gamma_{max} = 662 \times 10^{-6}$ ←

$\theta_{s_2} = \theta_{s_1} + 90° = 57.5°$

$\gamma_{min} = -662 \times 10^{-6}$ ←

$$\varepsilon_{aver} = \frac{\varepsilon_x + \varepsilon_y}{2} = 220 \times 10^{-6}$$



Problem 7.7-16 A 45° strain rosette (see figure) mounted on the surface of an automobile frame gives the following readings: gage *A*, 310×10^{-6}; gage *B*, 180×10^{-6}; and gage *C*, -160×10^{-6}.

Determine the principal strains and maximum shear strains, and show them on sketches of properly oriented elements.

Solution 7.7-16 **45° strain rosette**

$\varepsilon_A = 310 \times 10^{-6}$ $\quad \varepsilon_B = 180 \times 10^{-6}$

$\varepsilon_C = -160 \times 10^{-6}$

FROM EQS. (7-77) AND (7-78) OF EXAMPLE 7-8:

$\varepsilon_x = \varepsilon_A = 310 \times 10^{-6}$ $\quad \varepsilon_y = \varepsilon_C = -160 \times 10^{-6}$

$\gamma_{xy} = 2\varepsilon_B - \varepsilon_A - \varepsilon_C = 210 \times 10^{-6}$

PRINCIPAL STRAINS

$$\varepsilon_{1,2} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$
$= 75 \times 10^{-6} \pm 257 \times 10^{-6}$

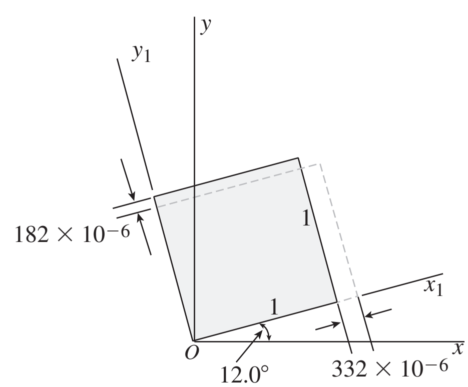
$\varepsilon_1 = 332 \times 10^{-6}$ $\quad \varepsilon_2 = -182 \times 10^{-6}$

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} = 0.4468$$
$2\theta_p = 24.1°$ and $204.1°$
$\theta_p = 12.0°$ and $102.0°$

For $\theta_p = 12.0°$:
$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$
$= 332 \times 10^{-6}$

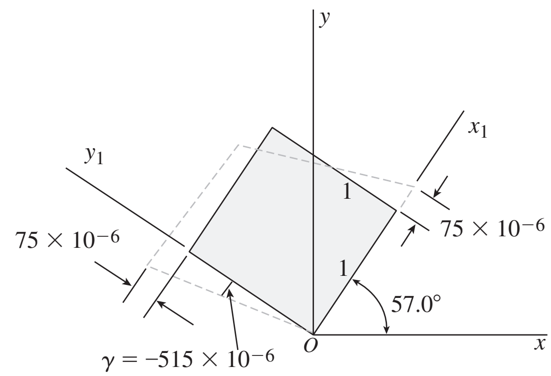
$\therefore\ \theta_{p_1} = 12.0°$ $\quad \varepsilon_1 = 332 \times 10^{-6}$ ←

$\theta_{p_2} = 102.0°$ $\quad \varepsilon_2 = -182 \times 10^{-6}$ ←



Maximum shear strains

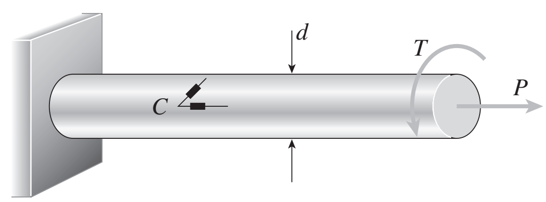
$$\frac{\gamma_{max}}{2} = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$
$$= 257 \times 10^{-6}$$
$$\gamma_{max} = 515 \times 10^{-6}$$
$$\theta_{s_1} = \theta_{p_1} - 45° = -33.0° \text{ or } 147.0°$$
$$\gamma_{max} = 515 \times 10^{-6} \longleftarrow$$
$$\theta_{s_2} = \theta_{s_1} + 90° = 57.0°$$
$$\gamma_{min} = -515 \times 10^{-6} \longleftarrow$$
$$\varepsilon_{aver} = \frac{\varepsilon_x + \varepsilon_y}{2} = 75 \times 10^{-6}$$

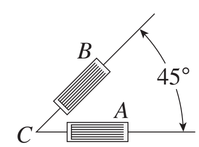


Problem 7.7-17 A solid circular bar of diameter $d = 1.5$ in. is subjected to an axial force P and a torque T (see figure). Strain gages A and B mounted on the surface of the bar give readings $\epsilon_a = 100 \times 10^{-6}$ and $\epsilon_b = -55 \times 10^{-6}$. The bar is made of steel having $E = 30 \times 10^6$ psi and $\nu = 0.29$.

(a) Determine the axial force P and the torque T.

(b) Determine the maximum shear strain γ_{max} and the maximum shear stress τ_{max} in the bar.





Solution 7.7-17 Circular bar (plane stress)

Bar is subjected to a torque T and an axial force P.

$E = 30 \times 10^6$ psi $\quad \nu = 0.29$

Diameter $d = 1.5$ in.

Strain gages

At $\theta = 0°$: $\quad \varepsilon_A = \varepsilon_x = 100 \times 10^{-6}$

At $\theta = 45°$: $\quad \varepsilon_B = -55 \times 10^{-6}$

Element in plane stress

$$\sigma_x = \frac{P}{A} = \frac{4P}{\pi d^2} \qquad \sigma_y = 0 \qquad \tau_{xy} = -\frac{16T}{\pi d^3}$$
$$\varepsilon_x = 100 \times 10^{-6} \qquad \varepsilon_y = -\nu\varepsilon_x = -29 \times 10^{-6}$$

Axial force P

$$\varepsilon_x = \frac{\sigma_x}{E} = \frac{4P}{\pi d^2 E} \qquad P = \frac{\pi d^2 E \varepsilon_x}{4} = 5300 \text{ lb} \longleftarrow$$

Shear strain

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{2\tau_{xy}(1+\nu)}{E} = -\frac{32T(1+\nu)}{\pi d^3 E}$$
$$= -(0.1298 \times 10^{-6})T \qquad (T = \text{lb-in.})$$

Strain at $\theta = 45°$

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta \qquad (1)$$
$$\varepsilon_{x_1} = \varepsilon_B = -55 \times 10^{-6} \qquad 2\theta = 90°$$

Substitute numerical values into Eq. (1):

$$-55 \times 10^{-6} = 35.5 \times 10^{-6} - (0.0649 \times 10^{-6})T$$

Solve for T: $\quad T = 1390$ lb-in. $\longleftarrow$

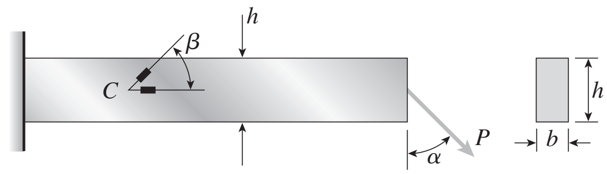
Maximum shear strain and maximum shear stress

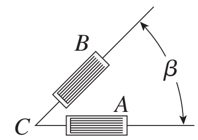
$$\gamma_{xy} = -(0.1298 \times 10^{-6})T = -180.4 \times 10^{-6} \text{ rad}$$

Eq. (7-75): $$\frac{\gamma_{max}}{2} = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$
$$= 111 \times 10^{-6} \text{ rad}$$
$$\gamma_{max} = 222 \times 10^{-6} \text{ rad} \longleftarrow$$
$$\tau_{max} = G\gamma_{max} = 2580 \text{ psi} \longleftarrow$$

Problem 7.7-18 A cantilever beam of rectangular cross section (width $b = 25$ mm, height $h = 100$ mm) is loaded by a force P that acts at the midheight of the beam and is inclined at an angle α to the vertical (see figure). Two strain gages are placed at point C, which also is at the midheight of the beam. Gage A measures the strain in the horizontal direction and gage B measures the strain at an angle $\beta = 60°$ to the horizontal. The measured strains are $\epsilon_a = 125 \times 10^{-6}$ and $\epsilon_b = -375 \times 10^{-6}$.

Determine the force P and the angle α, assuming the material is steel with $E = 200$ GPa and $\nu = 1/3$.





Solution 7.7-18 **Cantilever beam (plane stress)**

Beam loaded by a force P acting at an angle α.

$E = 200$ GPa $\quad \nu = 1/3 \quad b = 25$ mm

$h = 100$ mm

Axial force $F = P \sin \alpha$

Shear force $V = P \cos \alpha$

(At the neutral axis, the bending moment produces no stresses.)

STRAIN GAGES

At $\theta = 0°$: $\quad \varepsilon_A = \varepsilon_x = 125 \times 10^{-6}$

At $\theta = 60°$: $\quad \varepsilon_B = -375 \times 10^{-6}$

ELEMENT IN PLANE STRESS

$$\sigma_x = \frac{F}{A} = \frac{P \sin \alpha}{bh} \qquad \sigma_y = 0$$

$$\tau_{xy} = -\frac{3V}{2A} = -\frac{3P \cos \alpha}{2bh}$$

$$\varepsilon_x = 125 \times 10^{-6} \qquad \varepsilon_y = -\nu\varepsilon_x = -41.67 \times 10^{-6}$$

HOOKE'S LAW

$$\varepsilon_x = \frac{\sigma_x}{E} = \frac{P \sin \alpha}{bhE}$$

$$P \sin \alpha = bhE\varepsilon_x = 62{,}500 \text{ N} \tag{1}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = -\frac{3P \cos \alpha}{2bhG} = -\frac{3(1 + \nu)P \cos \alpha}{bhE}$$

$$= -(8.0 \times 10^{-9})P \cos \alpha \tag{2}$$

FOR $\theta = 60°$:

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2} \cos 2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta \tag{3}$$

$$\varepsilon_{x_1} = \varepsilon_B = -375 \times 10^{-6} \qquad 2\theta = 120°$$

Substitute into Eq. (3):

$$-375 \times 10^{-6} = 41.67 \times 10^{-6} - 41.67 \times 10^{-6} - (3.464 \times 10^{-9})P \cos \alpha$$

$$\text{or } P \cos \alpha = 108{,}260 \text{ N} \tag{4}$$

SOLVE EQS. (1) AND (4):

$\tan \alpha = 0.5773 \qquad \alpha = 30°$ ←

$P = 125$ kN ←

Problem 7.7-19 Solve the preceding problem if the cross-sectional dimensions are $b = 1.0$ in. and $h = 3.0$ in., the gage angle is $\beta = 75°$, the measured strains are $\epsilon_a = 171 \times 10^{-6}$ and $\epsilon_b = -266 \times 10^{-6}$, and the material is a magnesium alloy with modulus $E = 6.0 \times 10^6$ psi and Poisson's ratio $\nu = 0.35$.

Solution 7.7-19 Cantilever beam (plane stress)

Beam loaded by a force P acting at an angle α.
$E = 6.0 \times 10^6$ psi $\quad \nu = 0.35 \quad b = 1.0$ in.
$h = 3.0$ in.
Axial force $F = P \sin \alpha \quad$ Shear force $V = P \cos \alpha$
(At the neutral axis, the bending moment produces no stresses.)

STRAIN GAGES

At $\theta = 0°$: $\quad \varepsilon_A = \varepsilon_x = 171 \times 10^{-6}$
At $\theta = 75°$: $\quad \varepsilon_B = -266 \times 10^{-6}$

ELEMENT IN PLANE STRESS

$$\sigma_x = \frac{F}{A} = \frac{P \sin \alpha}{bh} \qquad \sigma_y = 0$$

$$\tau_{xy} = -\frac{3V}{2A} = -\frac{3P \cos \alpha}{2bh}$$

$$\varepsilon_x = 171 \times 10^{-6} \qquad \varepsilon_y = -\nu\varepsilon_x = -59.85 \times 10^{-6}$$

HOOKE'S LAW

$$\varepsilon_x = \frac{\sigma_x}{E} = \frac{P \sin \alpha}{bhE}$$

$$P \sin \alpha = bhE\varepsilon_x = 3078 \text{ lb} \qquad (1)$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = -\frac{3P \cos \alpha}{2bhG} = -\frac{3(1+\nu)P \cos \alpha}{bhE}$$

$$= -(225.0 \times 10^{-9})P \cos \alpha \qquad (2)$$

FOR $\theta = 75°$:

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2} \cos 2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta \qquad (3)$$

$$\varepsilon_{x_1} = \varepsilon_B = -266 \times 10^{-6} \qquad 2\theta = 150°$$

Substitute into Eq. (3):

$$-266 \times 10^{-6} = 55.575 \times 10^{-6} - 99.961 \times 10^{-6} - (56.25 \times 10^{-9})P \cos \alpha$$

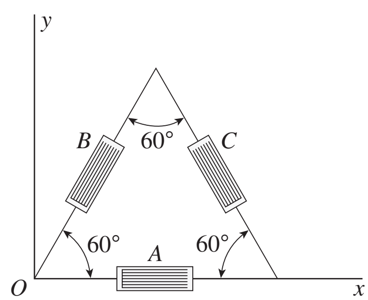
or $P \cos \alpha = 3939.8$ lb $\qquad (4)$

SOLVE EQS. (1) AND (4):

$\tan \alpha = 0.7813 \qquad \alpha = 38°$ ⟵
$P = 5000$ lb ⟵

Problem 7.7-20 A 60° strain rosette, or *delta rosette*, consists of three electrical-resistance strain gages arranged as shown in the figure. Gage A measures the normal strain ϵ_a in the direction of the x axis. Gages B and C measure the strains ϵ_b and ϵ_c in the inclined directions shown.

Obtain the equations for the strains ϵ_x, ϵ_y, and γ_{xy} associated with the xy axes.



Solution 7.7-20 Delta rosette (60° strain rosette)

STRAIN GAGES

Gage A at $\theta = 0°$ $\quad$ Strain $= \varepsilon_A$
Gage B at $\theta = 60°$ $\quad$ Strain $= \varepsilon_B$
Gage C at $\theta = 120°$ $\quad$ Strain $= \varepsilon_C$

FOR $\theta = 0°$: $\quad \varepsilon_x = \varepsilon_A$ ⟵

FOR $\theta = 60°$:

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2} \cos 2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta$$

$$\varepsilon_B = \frac{\varepsilon_A + \varepsilon_y}{2} + \frac{\varepsilon_A - \varepsilon_y}{2} (\cos 120°) + \frac{\gamma_{xy}}{2} (\sin 120°)$$

$$\varepsilon_B = \frac{\varepsilon_A}{4} + \frac{3\varepsilon_y}{4} + \frac{\gamma_{xy}\sqrt{3}}{4} \qquad (1)$$

FOR $\theta = 120°$:

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2} \cos 2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta$$

$$\varepsilon_C = \frac{\varepsilon_A + \varepsilon_y}{2} + \frac{\varepsilon_A - \varepsilon_y}{2} (\cos 240°) + \frac{\gamma_{xy}}{2} (\sin 240°)$$

$$\varepsilon_C = \frac{\varepsilon_A}{4} + \frac{3\varepsilon_y}{4} - \frac{\gamma_{xy}\sqrt{3}}{4} \qquad (2)$$

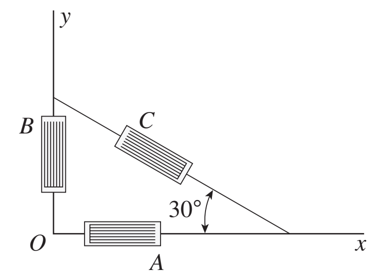
SOLVE EQS. (1) AND (2):

$$\varepsilon_y = \frac{1}{3}(2\varepsilon_B + 2\varepsilon_C - \varepsilon_A) \quad ⟵$$

$$\gamma_{xy} = \frac{2}{\sqrt{3}}(\varepsilon_B - \varepsilon_C) \quad ⟵$$

Problem 7.7-21 On the surface of a structural component in a space vehicle, the strains are monitored by means of three strain gages arranged as shown in the figure. During a certain maneuver, the following strains were recorded: $\epsilon_a = 1100 \times 10^{-6}$, $\epsilon_b = 200 \times 10^{-6}$, and $\epsilon_c = 200 \times 10^{-6}$.

Determine the principal strains and principal stresses in the material, which is a magnesium alloy for which $E = 6000$ ksi and $\nu = 0.35$. (Show the principal strains and principal stresses on sketches of properly oriented elements.)



Solution 7.7-21 **30-60-90° strain rosette**

Magnesium alloy: $E = 6000$ ksi $\quad \nu = 0.35$

STRAIN GAGES

Gage A at $\theta = 0°$ $\quad \varepsilon_A = 1100 \times 10^{-6}$
Gage B at $\theta = 90°$ $\quad \varepsilon_B = 200 \times 10^{-6}$
Gage C at $\theta = 150°$ $\quad \varepsilon_C = 200 \times 10^{-6}$

FOR $\theta = 0°$: $\quad \varepsilon_x = \varepsilon_A = 1100 \times 10^{-6}$

FOR $\theta = 90°$: $\quad \varepsilon_y = \varepsilon_B = 200 \times 10^{-6}$

FOR $\theta = 150°$:

$$\varepsilon_{x_1} = \varepsilon_C = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$200 \times 10^{-6} = 650 \times 10^{-6} + 225 \times 10^{-6} - 0.43301\gamma_{xy}$$

Solve for γ_{xy}: $\quad \gamma_{xy} = 1558.9 \times 10^{-6}$

PRINCIPAL STRAINS

$$\varepsilon_{1,2} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$= 650 \times 10^{-6} \pm 900 \times 10^{-6}$$

$\varepsilon_1 = 1550 \times 10^{-6} \quad \varepsilon_2 = -250 \times 10^{-6}$

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} = \sqrt{3} = 1.7321$$

$2\theta_p = 60° \quad \theta_p = 30°$

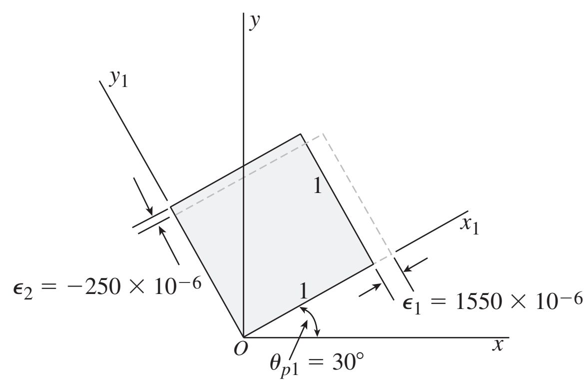
For $\theta_p = 30°$:

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$= 1550 \times 10^{-6}$$

$\therefore\ \theta_{p_1} = 30° \quad \varepsilon_1 = 1550 \times 10^{-6}$ ←

$\theta_{p_2} = 120° \quad \varepsilon_2 = -250 \times 10^{-6}$ ←

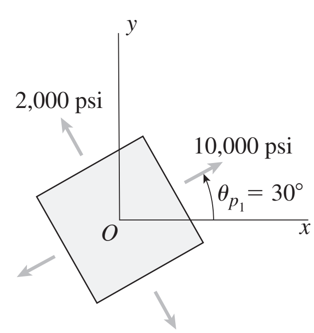


PRINCIPAL STRESSES (see Eqs. 7-36)

$$\sigma_1 = \frac{E}{1 - \nu^2}(\varepsilon_1 + \nu\varepsilon_2) \qquad \sigma_2 = \frac{E}{1 - \nu^2}(\varepsilon_2 + \nu\varepsilon_1)$$

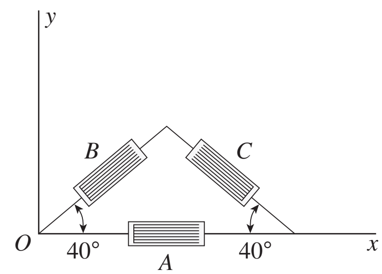
Substitute numerical values:

$\sigma_1 = 10{,}000$ psi $\quad \sigma_2 = 2{,}000$ psi ←



Problem 7.7-22 The strains on the surface of an experimental device made of pure aluminum (E = 70 GPa, ν = 0.33) and tested in a space shuttle were measured by means of strain gages. The gages were oriented as shown in the figure, and the measured strains were $\epsilon_a = 1100 \times 10^{-6}$, $\epsilon_b = 1496 \times 10^{-6}$, and $\epsilon_c = -39.44 \times 10^{-6}$.

What is the stress σ_x in the x direction?



Solution 7.7-22 **40-40-100° strain rosette**

Pure aluminum: E = 70 GPa $\nu = 0.33$

STRAIN GAGES

Gage A at $\theta = 0°$ $\varepsilon_A = 1100 \times 10^{-6}$
Gage B at $\theta = 40°$ $\varepsilon_B = 1496 \times 10^{-6}$
Gage C at $\theta = 140°$ $\varepsilon_C = -39.44 \times 10^{-6}$

FOR $\theta = 0°$: $\varepsilon_x = \varepsilon_A = 1100 \times 10^{-6}$

FOR $\theta = 40°$:

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

Substitute $\varepsilon_{x_1} = \varepsilon_B = 1496 \times 10^{-6}$ and $\varepsilon_x = 1100 \times 10^{-6}$; then simplify and rearrange:

$0.41318\varepsilon_y + 0.49240\gamma_{xy} = 850.49 \times 10^{-6}$ (1)

FOR $\theta = 140°$:

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

Substitute $\varepsilon_{x_1} = \varepsilon_C = -39.44 \times 10^{-6}$ and $\varepsilon_x = 1100 \times 10^{-6}$; then simplify and rearrange:

$0.41318\varepsilon_y - 0.49240\gamma_{xy} = -684.95 \times 10^{-6}$ (2)

SOLVE EQS. (1) AND (2):

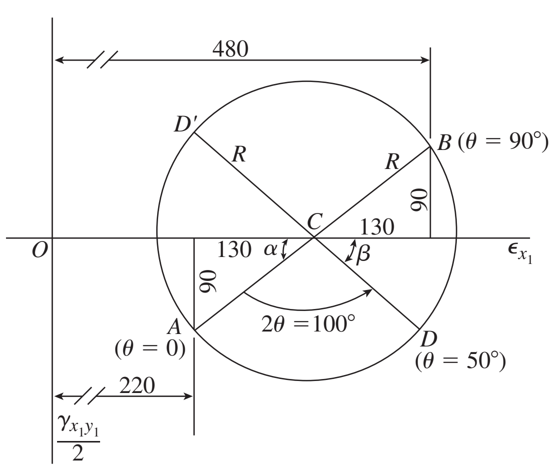
$\varepsilon_y = 200.3 \times 10^{-6}$ $\gamma_{xy} = 1559.2 \times 10^{-6}$

HOOKE'S LAW

$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y) = 91.6 \text{ MPa} \quad \longleftarrow$$

Problem 7.7-23 Solve Problem 7.7-5 by using Mohr's circle for plane strain.

Solution 7.7-23 **Element in plane strain**



$\varepsilon_x = 220 \times 10^{-6}$ $\varepsilon_y = 480 \times 10^{-6}$

$\gamma_{xy} = 180 \times 10^{-6}$ $\dfrac{\gamma_{xy}}{2} = 90 \times 10^{-6}$ $\theta = 50°$

$R = \sqrt{(130 \times 10^{-6})^2 + (90 \times 10^{-6})^2}$
$= 158.11 \times 10^{-6}$

$\alpha = \arctan \dfrac{90}{130} = 34.70°$

$\beta = 180° - \alpha - 2\theta = 45.30°$

POINT C: $\varepsilon_{x_1} = 350 \times 10^{-6}$

POINT D $(\theta = 50°)$:
$\varepsilon_{x_1} = 350 \times 10^{-6} + R \cos \beta = 461 \times 10^{-6}$

$\dfrac{\gamma_{x_1y_1}}{2} = R \sin \beta = 112.4 \times 10^{-6}$

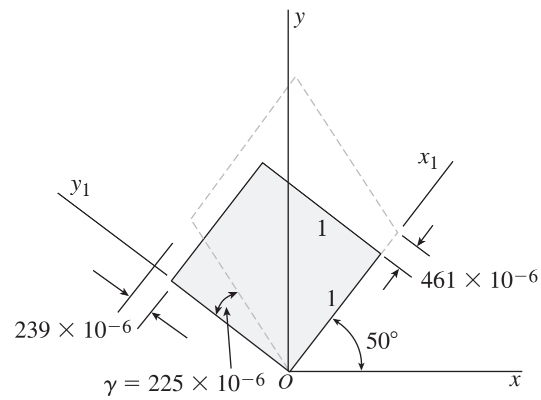
$\gamma_{x_1y_1} = 225 \times 10^{-6}$

POINT D' $(\theta = 140°)$:

$\varepsilon_{x_1} = 350 \times 10^{-6} - R \cos \beta = 239 \times 10^{-6}$

$\dfrac{\gamma_{x_1y_1}}{2} = -R \sin \beta = -112.4 \times 10^{-6}$

$\gamma_{x_1y_1} = -225 \times 10^{-6}$

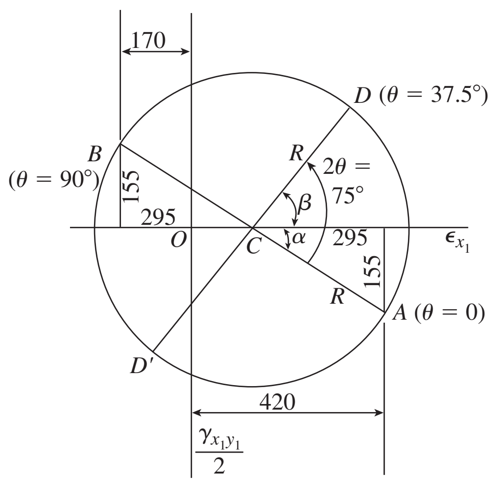


Problem 7.7-24 Solve Problem 7.7-6 by using Mohr's circle for plane strain.

Solution 7.7-24 Element in plane strain

$\varepsilon_x = 420 \times 10^{-6}$ $\quad \varepsilon_y = -170 \times 10^{-6}$

$\gamma_{xy} = 310 \times 10^{-6}$ $\quad \dfrac{\gamma_{xy}}{2} = 155 \times 10^{-6}$ $\quad \theta = 37.5°$



$R = \sqrt{(295 \times 10^{-6})^2 + (155 \times 10^{-6})^2}$
$= 333.24 \times 10^{-6}$

$\alpha = \arctan \dfrac{155}{295} = 27.72°$

$\beta = 2\theta - \alpha = 47.28°$

POINT C: $\varepsilon_{x_1} = 125 \times 10^{-6}$

POINT D $(\theta = 37.5°)$:

$\varepsilon_{x_1} = 125 \times 10^{-6} + R \cos \beta = 351 \times 10^{-6}$

$\dfrac{\gamma_{x_1y_1}}{2} = -R \sin \beta = -244.8 \times 10^{-6}$

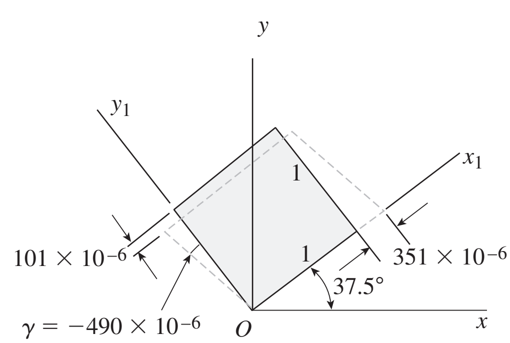
$\gamma_{x_1y_1} = -490 \times 10^{-6}$

POINT D' $(\theta = 127.5°)$:

$\varepsilon_{x_1} = 125 \times 10^{-6} - R \cos \beta = -101 \times 10^{-6}$

$\dfrac{\gamma_{x_1y_1}}{2} = R \sin \beta = 244.8 \times 10^{-6}$

$\gamma_{x_1y_1} = 490 \times 10^{-6}$

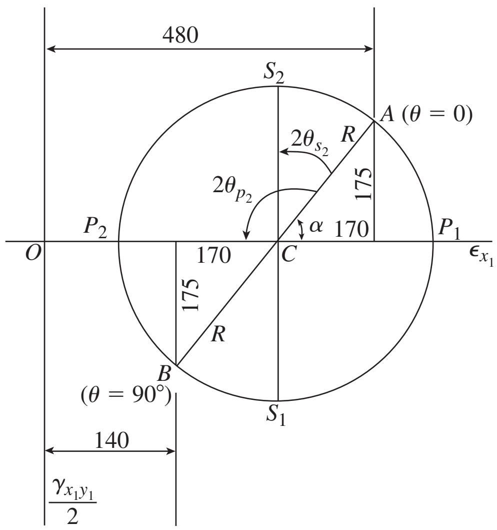


Problem 7.7-25 Solve Problem 7.7-7 by using Mohr's circle for plane strain.

Solution 7.7-25 Element in plane strain

$\varepsilon_x = 480 \times 10^{-6}$ $\quad \varepsilon_y = 140 \times 10^{-6}$

$\gamma_{xy} = -350 \times 10^{-6}$ $\quad \dfrac{\gamma_{xy}}{2} = -175 \times 10^{-6}$



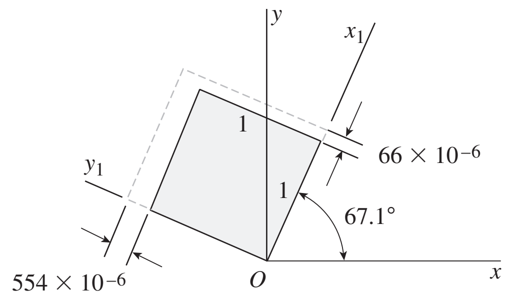
$R = \sqrt{(175 \times 10^{-6})^2 + (170 \times 10^{-6})^2}$
$\quad = 243.98 \times 10^{-6}$

$\alpha = \arctan \dfrac{175}{170} = 45.83°$

POINT C: $\quad \varepsilon_{x_1} = 310 \times 10^{-6}$

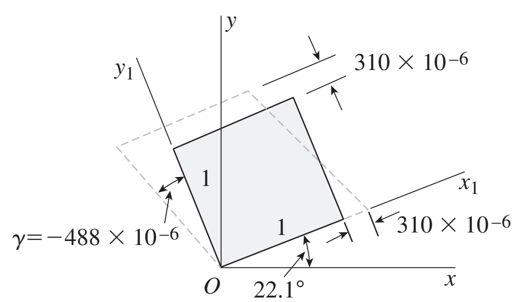
PRINCIPAL STRAINS

$2\theta_{p_2} = 180° - \alpha = 134.2° \quad \theta_{p_2} = 67.1°$
$2\theta_{p_1} = 2\theta_{p_2} + 180° = 314.2° \quad \theta_{p_1} = 157.1°$
Point P_1: $\varepsilon_1 = 310 \times 10^{-6} + R = 554 \times 10^{-6}$
Point P_2: $\varepsilon_2 = 310 \times 10^{-6} - R = 66 \times 10^{-6}$



MAXIMUM SHEAR STRAINS

$2\theta_{s_2} = 90° - \alpha = 44.17° \quad \theta_{s_2} = 22.1°$
$2\theta_{s_1} = 2\theta_{s_2} + 180° = 224.17° \quad \theta_{s_1} = 112.1°$
Point S_1: $\varepsilon_{\text{aver}} = 310 \times 10^{-6}$
$\gamma_{\max} = 2R = 488 \times 10^{-6}$
Point S_2: $\varepsilon_{\text{aver}} = 310 \times 10^{-6}$
$\gamma_{\min} = -488 \times 10^{-6}$

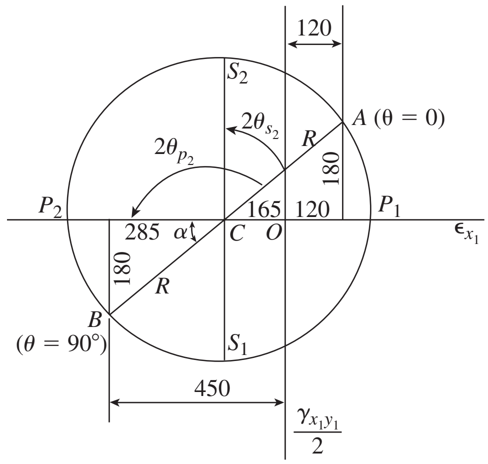


Problem 7.7-26 Solve Problem 7.7-8 by using Mohr's circle for plane strain.

Solution 7.7-26 Element in plane strain

$\varepsilon_x = 120 \times 10^{-6}$ $\quad \varepsilon_y = -450 \times 10^{-6}$

$\gamma_{xy} = -360 \times 10^{-6}$ $\quad \dfrac{\gamma_{xy}}{2} = -180 \times 10^{-6}$



$$R = \sqrt{(285 \times 10^{-6})^2 + (180 \times 10^{-6})^2}$$
$$= 337.08 \times 10^{-6}$$

$$\alpha = \arctan \frac{180}{285} = 32.28°$$

POINT C: $\quad \varepsilon_{x_1} = -165 \times 10^{-6}$

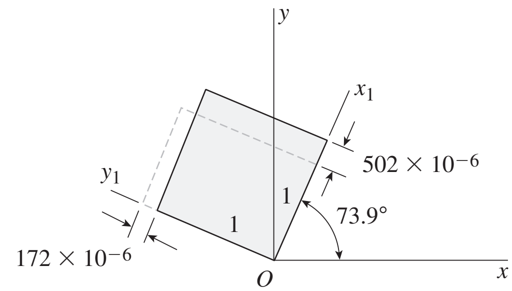
PRINCIPAL STRAINS

$2\theta_{p_2} = 180° - \alpha = 147.72°$ $\quad \theta_{p_2} = 73.9°$

$2\theta_{p_1} = 2\theta_{p_2} + 180° = 327.72°$ $\quad \theta_{p_1} = 163.9°$

Point P_1: $\varepsilon_1 = R - 165 \times 10^{-6} = 172 \times 10^{-6}$

Point P_2: $\varepsilon_2 = -165 \times 10^{-6} - R = -502 \times 10^{-6}$



MAXIMUM SHEAR STRAINS

$2\theta_{s_2} = 90° - \alpha = 57.72°$ $\quad \theta_{s_2} = 28.9°$

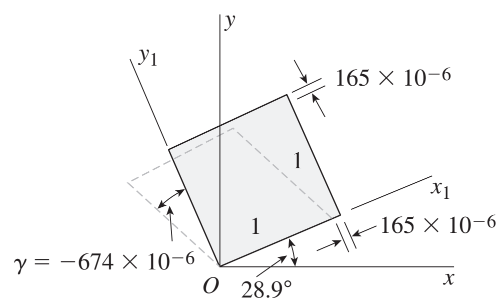
$2\theta_{s_1} = 2\theta_{s_2} + 180° = 237.72°$ $\quad \theta_{s_1} = 118.9°$

Point S_1: $\varepsilon_{aver} = -165 \times 10^{-6}$

$\gamma_{max} = 2R = 674 \times 10^{-6}$

Point S_2: $\varepsilon_{aver} = -165 \times 10^{-6}$

$\gamma_{min} = -674 \times 10^{-6}$

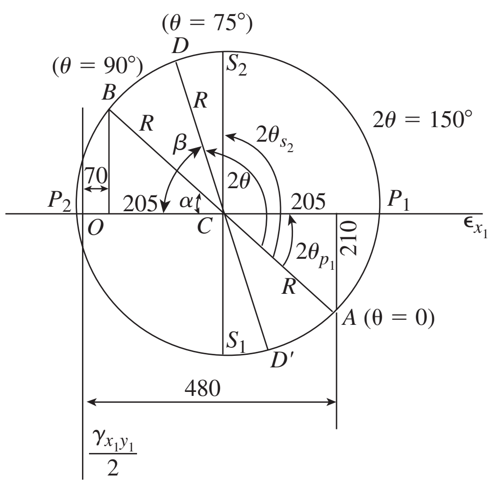


Problem 7.7-27 Solve Problem 7.7-9 by using Mohr's circle for plane strain.

Solution 7.7-27 Element in plane strain

$\varepsilon_x = 480 \times 10^{-6} \quad \varepsilon_y = 70 \times 10^{-6}$

$\gamma_{xy} = 420 \times 10^{-6} \quad \dfrac{\gamma_{xy}}{2} = 210 \times 10^{-6} \quad \theta = 75°$



$$R = \sqrt{(205 \times 10^{-6})^2 + (210 \times 10^{-6})^2}$$
$$= 293.47 \times 10^{-6}$$

$$\alpha = \arctan \frac{210}{205} = 45.69°$$

$$\beta = \alpha + 180° - 2\theta = 75.69°$$

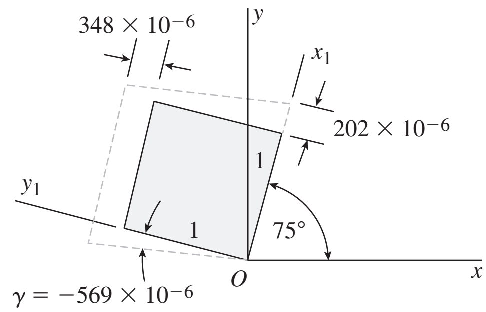
Point C: $\varepsilon_{x_1} = 275 \times 10^{-6}$

Point D $(\theta = 75°)$:

$$\varepsilon_{x_1} = 275 \times 10^{-6} - R\cos\beta = 202 \times 10^{-6}$$
$$\frac{\gamma_{x_1y_1}}{2} = -R\sin\beta = -284.36 \times 10^{-6}$$
$$\gamma_{x_1y_1} = -569 \times 10^{-6}$$

Point D' $(\theta = 165°)$:

$$\varepsilon_{x_1} = 275 \times 10^{-6} + R\cos\beta = 348 \times 10^{-6}$$
$$\frac{\gamma_{x_1y_1}}{2} = R\sin\beta = 284.36 \times 10^{-6}$$
$$\gamma_{x_1y_1} = 569 \times 10^{-6}$$



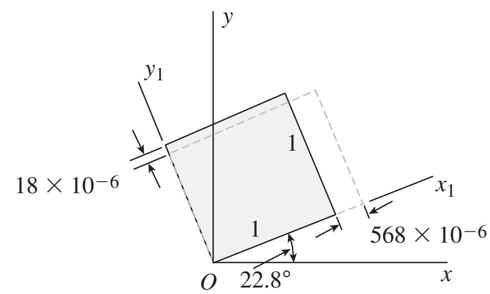
Principal strains

$2\theta_{p_1} = \alpha = 45.69° \quad \theta_{p_1} = 22.8°$

$2\theta_{p_2} = 2\theta_{p_1} + 180° = 225.69° \quad \theta_{p_2} = 112.8°$

Point P_1: $\varepsilon_1 = 275 \times 10^{-6} + R = 568 \times 10^{-6}$

Point P_2: $\varepsilon_2 = 275 \times 10^{-6} - R = -18 \times 10^{-6}$



Maximum shear strains

$2\theta_{s_2} = 90° + \alpha = 135.69° \quad \theta_{s_2} = 67.8°$

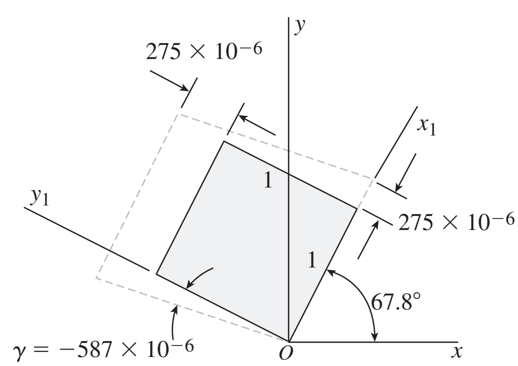
$2\theta_{s_1} = 2\theta_{s_2} + 180° = 315.69° \quad \theta_{s_1} = 157.8°$

Point S_1: $\varepsilon_{aver} = 275 \times 10^{-6}$

$\gamma_{max} = 2R = 587 \times 10^{-6}$

Point S_2: $\varepsilon_{aver} = 275 \times 10^{-6}$

$\gamma_{min} = -587 \times 10^{-6}$

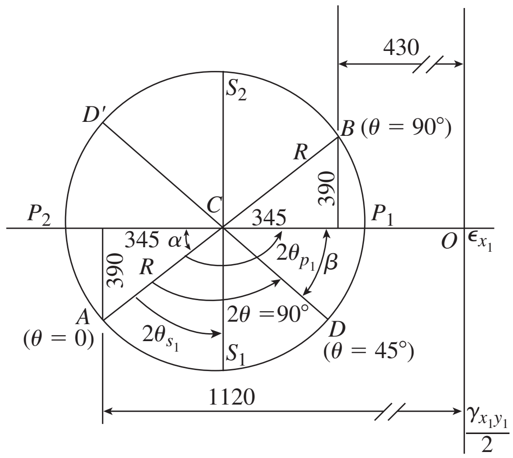


Problem 7.7-28 Solve Problem 7.7-10 by using Mohr's circle for plane strain.

Solution 7.7-28 Element in plane strain

$\varepsilon_x = -1120 \times 10^{-6}$ $\quad \varepsilon_y = -430 \times 10^{-6}$

$\gamma_{xy} = 780 \times 10^{-6}$ $\quad \dfrac{\gamma_{xy}}{2} = 390 \times 10^{-6}$ $\quad \theta = 45°$



$R = \sqrt{(345 \times 10^{-6})^2 + (390 \times 10^{-6})^2}$
$= 520.70 \times 10^{-6}$

$\alpha = \arctan \dfrac{390}{345} = 48.50°$

$\beta = 180° - \alpha - 2\theta = 41.50°$

POINT C: $\quad \varepsilon_{x_1} = -775 \times 10^{-6}$

POINT D (θ = 45°):

$\varepsilon_{x_1} = -775 \times 10^{-6} + R\cos\beta = -385 \times 10^{-6}$

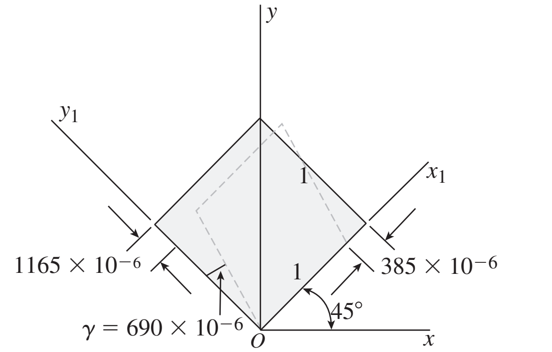
$\dfrac{\gamma_{x_1y_1}}{2} = R\sin\beta = 345 \times 10^{-6}$ $\quad \gamma_{x_1y_1} = 690 \times 10^{-6}$

POINT D'(θ = 135°):

$\varepsilon_{x_1} = -775 \times 10^{-6} - R\cos\beta = -1165 \times 10^{-6}$

$\dfrac{\gamma_{x_1y_1}}{2} = -R\sin\beta = -345 \times 10^{-6}$

$\gamma_{x_1y_1} = -690 \times 10^{-6}$



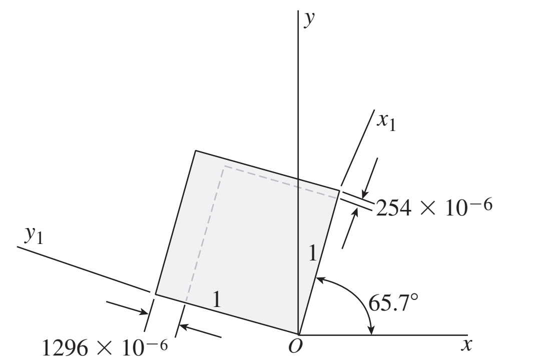
PRINCIPAL STRAINS

$2\theta_{p_1} = 180° - \alpha = 131.50°$ $\quad \theta_{p_1} = 65.7°$

$2\theta_{p_2} = 2\theta_{p_1} + 180° = 311.50°$ $\quad \theta_{p_2} = 155.7°$

Point P_1: $\varepsilon_1 = -775 \times 10^{-6} + R = -254 \times 10^{-6}$

Point P_2: $\varepsilon_2 = -775 \times 10^{-6} - R = -1296 \times 10^{-6}$



MAXIMUM SHEAR STRAINS

$2\theta_{s_1} = 90° - \alpha = 41.50°$ $\quad \theta_{s_1} = 20.7°$

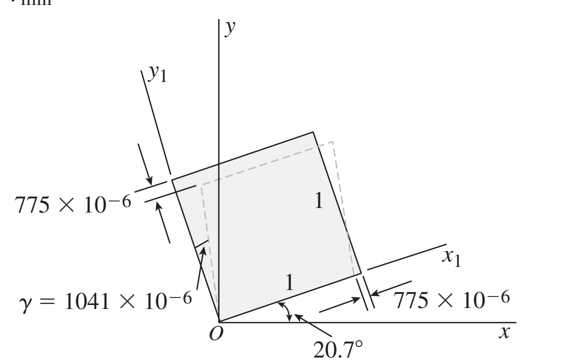
$2\theta_{s_2} = 2\theta_{s_1} + 180° = 221.50°$ $\quad \theta_{s_2} = 110.7°$

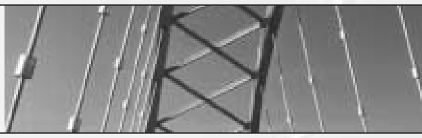
Point S_1: $\varepsilon_{aver} = -775 \times 10^{-6}$

$\gamma_{max} = 2R = 1041 \times 10^{-6}$

Point S_2: $\varepsilon_{aver} = -775 \times 10^{-6}$

$\gamma_{min} = -1041 \times 10^{-6}$





9

Deflections of Beams

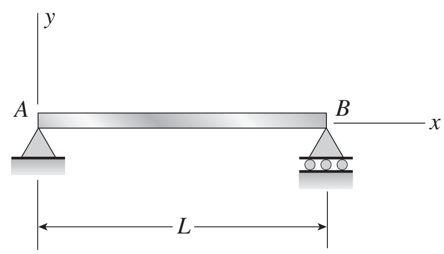
Differential Equations of the Deflection Curve

The beams described in the problems for Section 9.2 have constant flexural rigidity EI.

Problem 9.2-1 The deflection curve for a simple beam AB (see figure) is given by the following equation:

$$v = -\frac{q_0 x}{360LEI}(7L^4 - 10L^2x^2 + 3x^4)$$

Describe the load acting on the beam.



Probs. 9.2-1 and 9.2-2

Solution 9.2-1 **Simple beam**

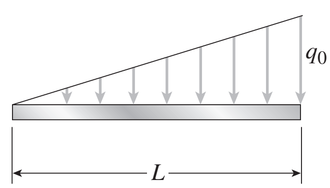
$$v = -\frac{q_0 x}{360\,LEI}(7L^4 - 10\,L^2x^2 + 3x^4)$$

Take four consecutive derivatives and obtain:

$$v'''' = -\frac{q_0 x}{LEI}$$

From Eq. (9-12c): $q = -EIv'''' = \dfrac{q_0 x}{L}$ ←

The load is a downward triangular load of maximum intensity q_0. ←



Problem 9.2-2 The deflection curve for a simple beam AB (see figure) is given by the following equation:

$$v = -\frac{q_0 L^4}{\pi^4 EI}\sin\frac{\pi x}{L}$$

(a) Describe the load acting on the beam.
(b) Determine the reactions R_A and R_B at the supports.
(c) Determine the maximum bending moment M_{max}.

Solution 9.2-2 **Simple beam**

$$v = -\frac{q_0 L^4}{\pi^4 EI}\sin\frac{\pi x}{L}$$

$$v' = -\frac{q_0 L^3}{\pi^3 EI}\cos\frac{\pi x}{L}$$

$$v'' = \frac{q_0 L^2}{\pi^2 EI}\sin\frac{\pi x}{L}$$

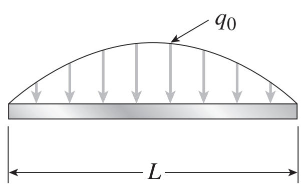
$$v''' = \frac{q_0 L}{\pi EI}\cos\frac{\pi x}{L}$$

$$v'''' = -\frac{q_0}{EI}\sin\frac{\pi x}{L}$$

(a) LOAD (EQ. 9-12c)

$$q = -EIv'''' = q_0 \sin\frac{\pi x}{L} \quad \longleftarrow$$

The load has the shape of a sine curve, acts downward, and has maximum intensity q_0. ⟵



(b) REACTIONS (EQ. 9-12b)

$$V = EIv''' = -\frac{q_0 L}{\pi}\cos\frac{\pi x}{L}$$

$$\text{At } x = 0: \quad V = R_A = -\frac{q_0 L}{\pi} \quad \longleftarrow$$

$$\text{At } x = L: \quad V = -R_B = -\frac{q_0 L}{\pi};\ R_B = \frac{q_0 L}{\pi} \quad \longleftarrow$$

(c) MAXIMUM BENDING MOMENT (EQ. 9-12a)

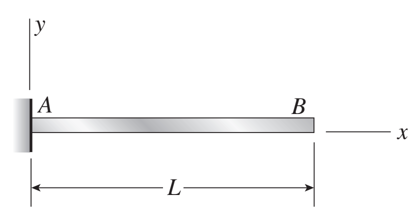
$$M = EIv'' = \frac{q_0 L^2}{\pi^2}\sin\frac{\pi x}{L}$$

For maximum moment, $x = \frac{L}{2};\ M_{\max} = \frac{q_0 L^2}{\pi^2}$ ⟵

Problem 9.2-3 The deflection curve for a cantilever beam *AB* (see figure) is given by the following equation:

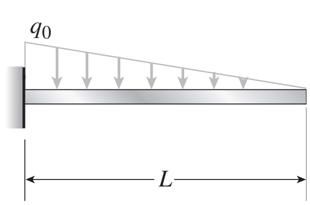
$$v = -\frac{q_0 x^2}{120LEI}(10L^3 - 10L^2x + 5Lx^2 - x^3)$$

Describe the load acting on the beam.



Probs. 9.2-3 and 9.2-4

Solution 9.2-3 **Cantilever beam**



$$v = -\frac{q_0 x^2}{120\,LEI}(10\,L^3 - 10\,L^2x + 5\,Lx^2 - x^3)$$

Take four consecutive derivatives and obtain:

$$v'''' = -\frac{q_0}{LEI}(L - x)$$

From Eq. (9-12c):

$$q = -EIv'''' = q_0\left(1 - \frac{x}{L}\right) \quad \longleftarrow$$

The load is a downward triangular load of maximum intensity q_0. ⟵

Problem 9.2-4 The deflection curve for a cantilever beam AB (see figure) is given by the following equation:

$$v = -\frac{q_0 x^2}{360L^2EI}(45L^4 - 40L^3x + 15L^2x^2 - x^4)$$

(a) Describe the load acting on the beam.
(b) Determine the reactions R_A and M_A at the support.

Solution 9.2-4 Cantilever beam

$$v = -\frac{q_0 x^2}{360\,L^2EI}(45L^4 - 40\,L^3x + 15\,L^2x^2 - x^4)$$

$$v' = -\frac{q_0}{60\,L^2EI}(15\,L^4x - 20\,L^3x^2 + 10\,L^2x^3 - x^5)$$

$$v'' = -\frac{q_0}{12\,L^2EI}(3\,L^4 - 8\,L^3x + 6\,L^2x^2 - x^4)$$

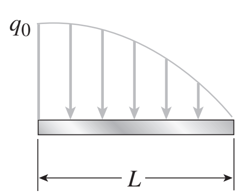
$$v''' = -\frac{q_0}{3\,L^2EI}(-2\,L^3 + 3\,L^2x - x^3)$$

$$v'''' = -\frac{q_0}{L^2EI}(L^2 - x^2)$$

(a) Load (Eq. 9-12c)

$$q = -EIv'''' = q_0\left(1 - \frac{x^2}{L^2}\right) \longleftarrow$$

The load is a downward parabolic load of maximum intensity q_0. ←



(b) Reactions R_A and M_A (Eq. 9-12b and Eq. 9-12a)

$$V = EIv''' = -\frac{q_0}{3L^2}(-2\,L^3 + 3\,L^2x - x^3)$$

At $x = 0$: $\quad V = R_A = \dfrac{2q_0L}{3} \longleftarrow$

$$M = EIv'' = -\frac{q_0}{12L^2}(3\,L^4 - 8\,L^3x + 6\,L^2x^2 - x^4)$$

At $x = 0$: $\quad M = M_A = -\dfrac{q_0L^2}{4} \longleftarrow$

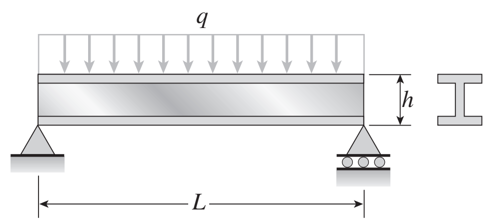
Note: Reaction R_A is positive upward. Reaction M_A is positive clockwise (minus means M_A is counterclockwise).

Deflection Formulas

Problems 9.3-1 through 9.3-7 require the calculation of deflections using the formulas derived in Examples 9-1, 9-2, and 9-3. All beams have constant flexural rigidity EI.

Problem 9.3-1 A wide-flange beam (W 12 × 35) supports a uniform load on a simple span of length $L = 14$ ft (see figure).

Calculate the maximum deflection δ_{max} at the midpoint and the angles of rotation θ at the supports if $q = 1.8$ k/ft and $E = 30 \times 10^6$ psi. Use the formulas of Example 9-1.



Probs. 9.3-1, 9.3-2 and 9.3-3

Solution 9.3-1 **Simple beam (uniform load)**

W 12 × 35 $\quad L = 14\text{ ft} = 168\text{ in.}$

$q = 1.8\text{ k/ft} = 150\text{ lb/in.}\quad E = 30 \times 10^6\text{ psi}$

$I = 285\text{ in.}^4$

MAXIMUM DEFLECTION (EQ. 9-18)

$$\delta_{max} = \frac{5qL^4}{384EI} = \frac{5(150\text{ lb/in.})(168\text{ in.})^4}{384(30 \times 10^6\text{ psi})(285\text{ in.}^4)}$$

$$= 0.182\text{ in.} \longleftarrow$$

ANGLE OF ROTATION AT THE SUPPORTS (EQS. 9-19 AND 9-20)

$$\theta = \theta_A = \theta_B = \frac{qL^3}{24EI} = \frac{(150\text{ lb/in.})(168\text{ in.})^3}{24(30 \times 10^6\text{ psi})(285\text{ in.}^4)}$$

$$= 0.003466\text{ rad} = 0.199^\circ \longleftarrow$$

Problem 9.3-2 A uniformly loaded steel wide-flange beam with simple supports (see figure) has a downward deflection of 10 mm at the midpoint and angles of rotation equal to 0.01 radians at the ends.

Calculate the height h of the beam if the maximum bending stress is 90 MPa and the modulus of elasticity is 200 GPa. (*Hint:* Use the formulas of Example 9-1.)

Solution 9.3-2 **Simple beam (uniform load)**

$\delta = \delta_{max} = 10\text{ mm} \qquad \theta = \theta_A = \theta_B = 0.01\text{ rad}$

$\sigma = \sigma_{max} = 90\text{ MPa} \qquad E = 200\text{ GPa}$

Calculate the height h of the beam.

$$\text{Eq. (9-18): } \delta = \delta_{max} = \frac{5qL^4}{384EI} \quad \text{or} \quad q = \frac{384EI\delta}{5L^4} \qquad (1)$$

$$\text{Eq. (9-19): } \theta = \theta_A = \frac{qL^3}{24EI} \quad \text{or} \quad q = \frac{24EI\theta}{L^3} \qquad (2)$$

$$\text{Equate (1) and (2) and solve for } L\text{: } L = \frac{16\delta}{5\theta} \qquad (3)$$

$$\text{Flexure formula: } \sigma = \frac{Mc}{I} = \frac{Mh}{2I}$$

Maximum bending moment:

$$M = \frac{qL^2}{8} \qquad \therefore \sigma = \frac{qL^2h}{16I} \qquad (4)$$

$$\text{Solve Eq. (4) for } h\text{: } \quad h = \frac{16I\sigma}{qL^2} \qquad (5)$$

Substitute for q from (2) and for L from (3):

$$h = \frac{32\sigma\delta}{15E\theta^2} \longleftarrow$$

Substitute numerical values:

$$h = \frac{32(90\text{ MPa})(10\text{ mm})}{15(200\text{ GPa})(0.01\text{ rad})^2} = 96\text{ mm} \longleftarrow$$

Problem 9.3-3 What is the span length L of a uniformly loaded simple beam of wide-flange cross section (see figure) if the maximum bending stress is 12,000 psi, the maximum deflection is 0.1 in., the height of the beam is 12 in., and the modulus of elasticity is 30×10^6 psi? (Use the formulas of Example 9-1.)

Solution 9.3-3 Simple beam (uniform load)

$\sigma = \sigma_{max} = 12{,}000$ psi $\quad \delta = \delta_{max} = 0.1$ in.
$h = 12$ in. $\quad E = 30 \times 10^6$ psi

Calculate the span length L.

Eq. (9-18): $\delta = \delta_{max} = \dfrac{5qL^4}{384\,EI}$ or $q = \dfrac{384\,EI\delta}{5L^4}$ (1)

Flexure formula: $\sigma = \dfrac{Mc}{I} = \dfrac{Mh}{2I}$

Maximum bending moment:

$M = \dfrac{qL^2}{8} \quad \therefore \sigma = \dfrac{qL^2h}{16I}$ (2)

Solve Eq. (2) for q: $q = \dfrac{16\,I\sigma}{L^2h}$ (3)

Equate (1) and (2) and solve for L:

$L^2 = \dfrac{24\,Eh\delta}{5\sigma} \quad L = \sqrt{\dfrac{24\,Eh\delta}{5\sigma}}$ ←

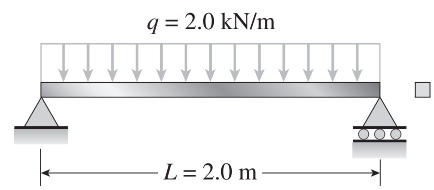
Substitute numerical values:

$$L^2 = \frac{24(30 \times 10^6 \text{ psi})(12 \text{ in.})(0.1 \text{ in.})}{5(12{,}000 \text{ psi})} = 14{,}400 \text{ in.}^2$$

$L = 120$ in. $= 10$ ft ←

Problem 9.3-4 Calculate the maximum deflection δ_{max} of a uniformly loaded simple beam (see figure) if the span length $L = 2.0$ m, the intensity of the uniform load $q = 2.0$ kN/m, and the maximum bending stress $\sigma = 60$ MPa.

The cross section of the beam is square, and the material is aluminum having modulus of elasticity $E = 70$ GPa. (Use the formulas of Example 9-1.)



Solution 9.3-4 Simple beam (uniform load)

$L = 2.0$ m $\quad q = 2.0$ kN/m
$\sigma = \sigma_{max} = 60$ MPa $\quad E = 70$ GPa

CROSS SECTION (square; b = width)

$I = \dfrac{b^4}{12} \quad S = \dfrac{b^3}{6}$

Maximum deflection (Eq. 9-18): $\delta = \dfrac{5qL^4}{384\,EI}$ (1)

Substitute for I: $\delta = \dfrac{5qL^4}{32\,Eb^4}$ (2)

Flexure formula with $M = \dfrac{qL^2}{8}$: $\quad \sigma = \dfrac{M}{S} = \dfrac{qL^2}{8S}$

Substitute for S: $\sigma = \dfrac{3qL^2}{4b^3}$ (3)

Solve for b^3: $b^3 = \dfrac{3qL^2}{4\sigma}$ (4)

Substitute b into Eq. (2): $\delta_{max} = \dfrac{5L\sigma}{24E}\left(\dfrac{4L\sigma}{3q}\right)^{1/3}$ ←

(The term in parentheses is nondimensional.)

Substitute numerical values:

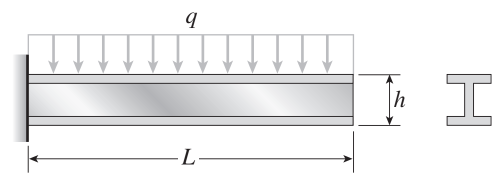
$$\frac{5L\sigma}{24E} = \frac{5(2.0 \text{ m})(60 \text{ MPa})}{24(70 \text{ GPa})} = \frac{1}{2800} m = \frac{1}{2.8} \text{ mm}$$

$$\left(\frac{4\,L\sigma}{3q}\right)^{1/3} = \left[\frac{4(2.0 \text{ m})(60 \text{ MPa})}{3(2000 \text{ N/m})}\right]^{1/3} = 10(80)^{1/3}$$

$\delta_{max} = \dfrac{10(80)^{1/3}}{2.8}$ mm $= 15.4$ mm ←

Problem 9.3-5 A cantilever beam with a uniform load (see figure) has a height h equal to 1/8 of the length L. The beam is a steel wide-flange section with $E = 28 \times 10^6$ psi and an allowable bending stress of 17,500 psi in both tension and compression.

Calculate the ratio δ/L of the deflection at the free end to the length, assuming that the beam carries the maximum allowable load. (Use the formulas of Example 9-2.)



Solution 9.3-5 Cantilever beam (uniform load)

$$\frac{h}{L} = \frac{1}{8} \qquad E = 28 \times 10^6 \text{ psi} \qquad \sigma = 17{,}500 \text{ psi}$$

Calculate the ratio δ/L.

Maximum deflection (Eq. 9-26): $\delta_{\max} = \dfrac{qL^4}{8\,EI}$ (1)

$$\therefore \quad \frac{\delta}{L} = \frac{qL^3}{8EI} \tag{2}$$

Flexure formula with $M = \dfrac{qL^2}{2}$:

$$\sigma = \frac{Mc}{I} = \left(\frac{qL^2}{2}\right)\left(\frac{h}{2I}\right) = \frac{qL^2h}{4I}$$

Solve for q:

$$q = \frac{4I\sigma}{L^2h} \tag{3}$$

Substitute q from (3) into (2):

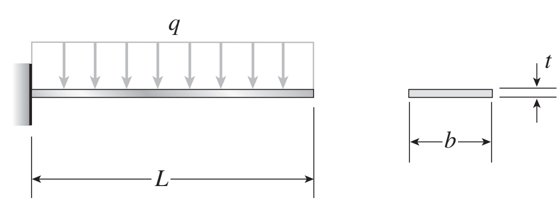
$$\frac{\delta}{L} = \frac{\sigma}{2E}\left(\frac{L}{h}\right) \longleftarrow$$

Substitute numerical values:

$$\frac{\delta}{L} = \frac{17{,}500 \text{ psi}}{2(28 \times 10^6 \text{ psi})}(8) = \frac{1}{400} \longleftarrow$$

Problem 9.3-6 A gold-alloy microbeam attached to a silicon wafer behaves like a cantilever beam subjected to a uniform load (see figure). The beam has length $L = 27.5$ μm and rectangular cross section of width $b = 4.0$ μm and thickness $t = 0.88$ μm. The total load on the beam is 17.2 μN.

If the deflection at the end of the beam is 2.46 μm, what is the modulus of elasticity E_g of the gold alloy? (Use the formulas of Example 9-2.)



Solution 9.3-6 Gold-alloy microbeam

Cantilever beam with a uniform load.

$L = 27.5\ \mu\text{m} \qquad b = 4.0\ \mu\text{m} \qquad t = 0.88\ \mu\text{m}$

$qL = 17.2\ \mu\text{N} \qquad \delta_{\max} = 2.46\ \mu\text{m}$

Determine Eq.

Eq. (9-26): $\delta = \dfrac{qL^4}{8\,E_qI}$ or $E_q = \dfrac{qL^4}{8\,I\delta_{\max}}$

$$I = \frac{bt^3}{12} \qquad E_q = \frac{3\,qL^4}{2\,bt^3\delta_{\max}} \longleftarrow$$

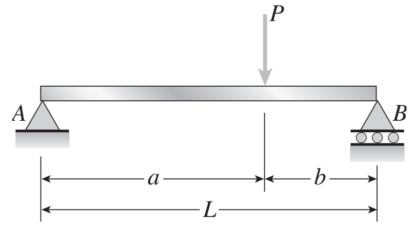
Substitute numerical values:

$$E_q = \frac{3(17.2\ \mu\text{N})(27.5\ \mu\text{m})^3}{2(4.0\ \mu\text{m})(0.88\ \mu\text{m})^3(2.46\ \mu\text{m})}$$

$$= 80.02 \times 10^9\ \text{N/m}^2 \quad \text{or} \quad E_q = 80.0\ \text{GPa} \longleftarrow$$

Problem 9.3-7 Obtain a formula for the ratio δ_C/δ_{max} of the deflection at the midpoint to the maximum deflection for a simple beam supporting a concentrated load P (see figure).

From the formula, plot a graph of δ_C/δ_{max} versus the ratio a/L that defines the position of the load ($0.5 < a/L < 1$). What conclusion do you draw from the graph? (Use the formulas of Example 9-3.)



Solution 9.3-7 **Simple beam (concentrated load)**

Eq. (9-35): $\delta_C = \dfrac{Pb(3L^2 - 4b^2)}{48EI} \quad (a \geq b)$

Eq. (9-34): $\delta_{max} = \dfrac{Pb(L^2 - b^2)^{3/2}}{9\sqrt{3}\,LEI} \quad (a \geq b)$

$$\frac{\delta_c}{\delta_{max}} = \frac{(3\sqrt{3}L)(3L^2 - 4b^2)}{16(L^2 - b^2)^{3/2}} \quad (a \geq b)$$

Replace the distance b by the distance a by substituting $L - a$ for b:

$$\frac{\delta_c}{\delta_{max}} = \frac{(3\sqrt{3}L)(-L^2 + 8ab - 4a^2)}{16(2aL - a^2)^{3/2}}$$

Divide numerator and denominator by L^2:

$$\frac{\delta_c}{\delta_{max}} = \frac{(3\sqrt{3}L)\left(-1 + 8\frac{a}{L} - 4\frac{a^2}{L^2}\right)}{16L\left(2\frac{a}{L} - \frac{a^2}{L^2}\right)^{3/2}}$$

$$\frac{\delta_c}{\delta_{max}} = \frac{(3\sqrt{3})\left(-1 + 8\frac{a}{L} - 4\frac{a^2}{L^2}\right)}{16\left(2\frac{a}{L} - \frac{a^2}{L^2}\right)^{3/2}} \longleftarrow$$

Alternative form of the ratio

Let $\beta = \dfrac{a}{L}$

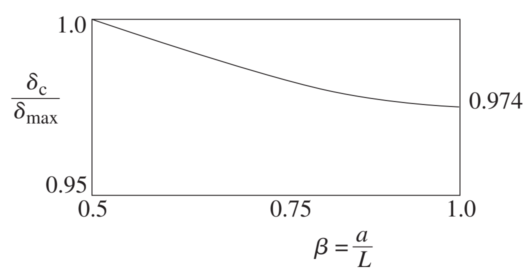
$$\frac{\delta_c}{\delta_{max}} = \frac{(3\sqrt{3})(-1 + 8\beta - 4\beta^2)}{16(2\beta - \beta^2)^{3/2}} \longleftarrow$$

Graph of δ_c/δ_{max} versus $\beta = a/L$

Because $a \geq b$, the ratio β versus from 0.5 to 1.0.

β	$\dfrac{\delta_c}{\delta_{max}}$
0.5	1.0
0.6	0.996
0.7	0.988
0.8	0.981
0.9	0.976
1.0	0.974

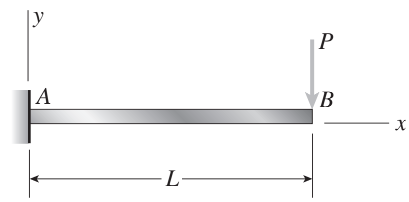
Note: The deflection δ_c at the midpoint of the beam is almost as large as the maximum deflection δ_{max}. The greatest difference is only 2.6% and occurs when the load reaches the end of the beam ($\beta = 1$).



Deflections by Integration of the Bending-Moment Equation

Problems 9.3-8 through 9.3-16 are to be solved by integrating the second-order differential equation of the deflection curve (the bending-moment equation). The origin of coordinates is at the left-hand end of each beam, and all beams have constant flexural rigidity EI.

Problem 9.3-8 Derive the equation of the deflection curve for a cantilever beam AB supporting a load P at the free end (see figure). Also, determine the deflection δ_B and angle of rotation θ_B at the free end. (*Note:* Use the second-order differential equation of the deflection curve.)



Solution 9.3-8 Cantilever beam (concentrated load)

BENDING-MOMENT EQUATION (EQ. 9-12a)

$$EIv'' = M = -P(L - x)$$

$$EIv' = -PLx + \frac{Px^2}{2} + C_1$$

B.C. $v'(0) = 0 \quad \therefore C_2 = 0$

$$EIv = -\frac{PLx^2}{2} + \frac{Px^3}{6} + C_2$$

B.C. $v(0) = 0 \quad \therefore C_1 = 0$

$$v = -\frac{Px^2}{6EI}(3L - x) \quad \longleftarrow$$

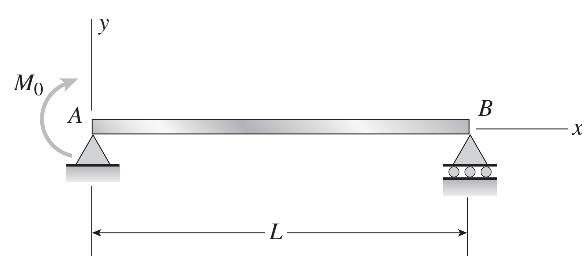
$$v' = -\frac{Px}{2EI}(2L - x)$$

$$\delta_B = -v(L) = \frac{PL^3}{3EI} \quad \longleftarrow$$

$$\theta_B = -v'(L) = \frac{PL^2}{2EI} \quad \longleftarrow$$

(These results agree with Case 4, Table G-1.)

Problem 9.3-9 Derive the equation of the deflection curve for a simple beam AB loaded by a couple M_0 at the left-hand support (see figure). Also, determine the maximum deflection δ_{max}. (*Note:* Use the second-order differential equation of the deflection curve.)



Solution 9.3-9 Simple beam (couple M_0)

BENDING-MOMENT EQUATION (EQ. 9-12a)

$$EIv'' = M = M_0\left(1 - \frac{x}{L}\right)$$

$$EIv' = M_0\left(x - \frac{x^2}{2L}\right) + C_1$$

$$EIv = M_0\left(\frac{x^2}{2} - \frac{x^3}{6L}\right) + C_1x + C_2$$

B.C. $v(0) = 0 \quad \therefore C_2 = 0$

B.C. $v(L) = 0 \quad \therefore C_1 = -\frac{M_0L}{3}$

$$v = -\frac{M_0x}{6\,LEI}(2L^2 - 3Lx + x^2) \quad \longleftarrow$$

MAXIMUM DEFLECTION

$$v' = -\frac{M_0}{6\,LEI}(2\,L^2 - 6\,Lx + 3\,x^2)$$

Set $v' = 0$ and solve for x:

$$x_1 = L\left(1 - \frac{\sqrt{3}}{3}\right) \quad \longleftarrow$$

Substitute x_1 into the equation for v:

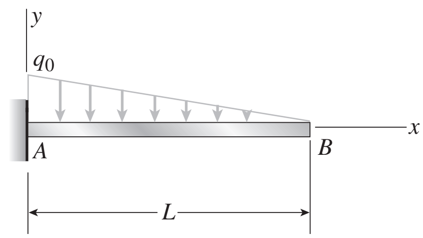
$$\delta_{max} = -(v)_{x=x_1}$$

$$= \frac{M_0L^2}{9\sqrt{3}EI} \quad \longleftarrow$$

(These results agree with Case 7, Table G-2.)

Problem 9.3-10 A cantilever beam AB supporting a triangularly distributed load of maximum intensity q_0 is shown in the figure.

Derive the equation of the deflection curve and then obtain formulas for the deflection δ_B and angle of rotation θ_B at the free end. (*Note:* Use the second-order differential equation of the deflection curve.)



Solution 9.3-10 **Cantilever beam (triangular load)**

BENDING-MOMENT EQUATION (EQ. 9-12a)

$$EIv'' = M = -\frac{q_0}{6L}(L - x)^3$$

$$EIv' = \frac{q_0}{24L}(L - x)^4 + C_1$$

B.C. $v'(0) = 0 \quad \therefore c_2 = -\frac{q_0 L^3}{24}$

$$EIv = -\frac{q_0}{120L}(L - x)^5 - \frac{q_0 L^3 x}{24} + C^2$$

B.C. $v(0) = 0 \quad \therefore c_2 = \frac{q_0 L^4}{120}$

$$v = -\frac{q_0 x^2}{120\,LEI}(10L^3 - 10L^2 x + 5Lx^2 - x^3) \quad \longleftarrow$$

$$v' = -\frac{q_0 x}{24\,LEI}(4L^3 - 6L^2 x + 4Lx^2 - x^3)$$

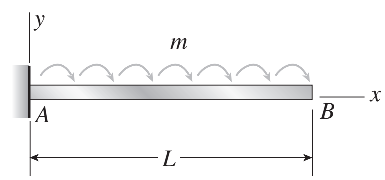
$$\delta_B = -v(L) = \frac{q_0 L^4}{30\,EI} \quad \longleftarrow$$

$$\theta_B = -v'(L) = \frac{q_0 L^3}{24\,EI} \quad \longleftarrow$$

(These results agree with Case 8, Table G-1.)

Problem 9.3-11 A cantilever beam AB is acted upon by a uniformly distributed moment (bending moment, not torque) of intensity m per unit distance along the axis of the beam (see figure).

Derive the equation of the deflection curve and then obtain formulas for the deflection δ_B and angle of rotation θ_B at the free end. (*Note:* Use the second-order differential equation of the deflection curve.)



Solution 9.3-11 **Cantilever beam (distributed moment)**

BENDING-MOMENT EQUATION (EQ. 9-12a)

$$EIv'' = M = -m(L - x)$$

$$EIv' = -m\left(Lx - \frac{x^2}{2}\right) + C_1$$

B.C. $v'(0) = 0 \quad \therefore C_1 = 0$

$$EIv = -m\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right) + C_2$$

B.C. $v(0) = 0 \quad \therefore C_2 = 0$

$$v = -\frac{mx^2}{6\,EI}(3L - x) \quad \longleftarrow$$

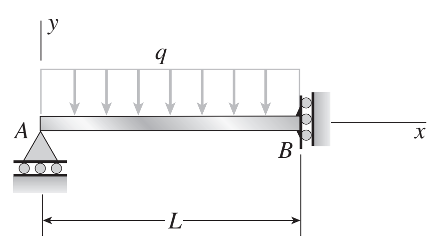
$$v' = -\frac{mx}{2EI}(2L - x)$$

$$\delta_B = -v(L) = \frac{mL^3}{3\,EI} \quad \longleftarrow$$

$$\theta_B = -v'(L) = \frac{mL^2}{2\,EI} \quad \longleftarrow$$

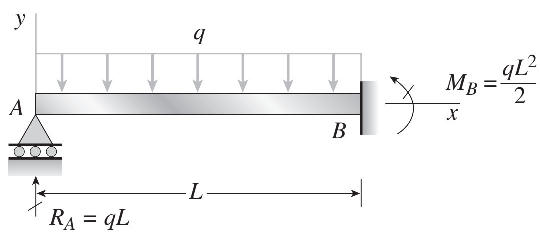
Problem 9.3-12 The beam shown in the figure has a roller support at A and a guided support at B. The guided support permits vertical movement but no rotation.

Derive the equation of the deflection curve and determine the deflection δ_B at end B due to the uniform load of intensity q. (*Note:* Use the second-order differential equation of the deflection curve.)



Solution 9.3-12 **Beam with a guided support**

REACTIONS AND DEFLECTION CURVE



BENDING-MOMENT EQUATION (EQ. 9-12a)

$$EIv'' = M = qLx - \frac{qx^2}{2}$$

$$EIv' = \frac{qLx^2}{2} - \frac{qx^3}{6} + C_1$$

B.C. $v(L) = 0 \quad \therefore \; C_1 = -\frac{qL^3}{3}$

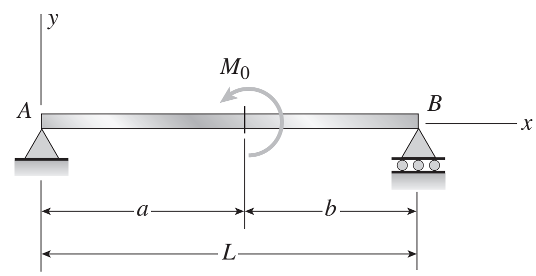
$$EIv = \frac{qLx^3}{6} - \frac{qx^4}{24} - \frac{qL^3x}{3} + C_2$$

B.C. $v(0) = 0 \quad \therefore C_2 = 0$

$$v = -\frac{qx}{24\,EI}(8L^3 - 4Lx^2 + x^3) \quad \longleftarrow$$

$$\delta_B = -v(L) = \frac{5\,qL^4}{24\,EI} \quad \longleftarrow$$

Problem 9.3-13 Derive the equations of the deflection curve for a simple beam AB loaded by a couple M_0 acting at distance a from the left-hand support (see figure). Also, determine the deflection δ_0 at the point where the load is applied. (*Note:* Use the second-order differential equation of the deflection curve.)



Solution 9.3-13 **Simple beam (couple M_0)**

BENDING-MOMENT EQUATION (EQ. 9-12a)

$$EIv'' = M = \frac{M_0 x}{L} \quad (0 \le x \le a)$$

$$EIv' = \frac{M_0 x^2}{2L} + C_1 \quad (0 \le x \le a)$$

$$EIv'' = M = -\frac{M_0}{L}(L - x) \quad (a \le x \le L)$$

$$EIv' = -\frac{M_0}{L}\left(Lx - \frac{x^2}{2}\right) + C_2 \quad (a \le x \le L)$$

B.C. 1 $(v')_{\text{Left}} = (v')_{\text{Right}}$ at $x = a$

$\therefore C_2 = C_1 + M_0 a$

$$EIv = \frac{M_0 x^3}{6L} + C_1 x + C_3 \quad (0 \le x \le a)$$

B.C. 2 $v(0) = 0 \quad \therefore C_3 = 0$

$$EIv = -\frac{M_0 x^2}{2} + \frac{M_0 x^3}{6L} + C_1 x + M_0 ax + C_4$$

$(a \le x \le L)$

B.C. 3 $v(L) = 0 \quad \therefore C_4 = -M_0 L\left(a - \frac{L}{3}\right) - C_1 L$

B.C. 4 $(v)_{\text{Left}} = (v)_{\text{Right}}$ at $x = a$

$$\therefore C_4 = -\frac{M_0 a^2}{2}$$

$$C_1 = \frac{M_0}{6L}(2L^2 - 6aL + 3a^2)$$

$$v = -\frac{M_0 x}{6\,LEI}(6aL - 3a^2 - 2L^2 - x^2) \quad (0 \le x \le a) \quad \longleftarrow$$

$$v = -\frac{M_0}{6\,LEI}(3a^2L - 3a^2x - 2L^2x + 3Lx^2 - x^3) \quad \longleftarrow$$

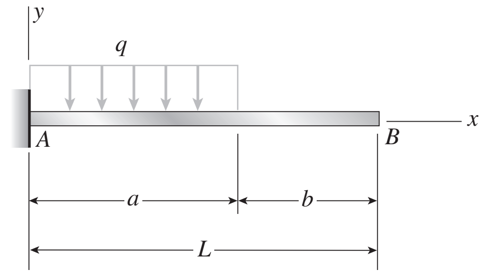
$(a \le x \le L)$

$$\delta_0 = -v(a) = \frac{M_0\,a(L - a)(2a - L)}{3\,LEI}$$

$$= \frac{M_0\,ab(2a - L)}{3LEI} \quad \longleftarrow$$

NOTE: δ_0 is positive downward. The pending results agree with Case 9, Table G-2.

Problem 9.3-14 Derive the equations of the deflection curve for a cantilever beam AB carrying a uniform load of intensity q over part of the span (see figure). Also, determine the deflection δ_B at the end of the beam. (*Note:* Use the second-order differential equation of the deflection curve.)



Solution 9.3-14 Cantilever beam (partial uniform load)

BENDING-MOMENT EQUATION (EQ. 9-12a)

$$EIv'' = M = -\frac{q}{2}(a-x)^2 = -\frac{q}{2}(a^2 - 2ax + x^2) \quad (0 \le x \le a)$$

$$EIv' = -\frac{q}{2}\left(a^2x - ax^2 + \frac{x^3}{3}\right) + C_1 \quad (0 \le x \le a)$$

B.C. 1 $v'(0) = 0 \quad \therefore C_1 = 0$

$EIv'' = M = 0 \qquad (a \le x \le L)$

$EIv' = C_2 \qquad (a \le x \le L)$

B.C. 2 $(v')_{\text{Left}} = (v')_{\text{Right}}$ at $x = a$

$$\therefore C_2 = -\frac{qa^3}{6}$$

$$EIv = -\frac{q}{2}\left(\frac{a^2x^2}{2} - \frac{ax^3}{3} + \frac{x^4}{12}\right) + C_3 \quad (0 \le x \le a)$$

B.C. 3 $v(0) = 0 \quad \therefore C_3 = 0$

$$EIv = C_2x + C_4 = -\frac{qa^3x}{6} + C_4 \quad (a \le x \le L)$$

B.C. 4 $(v)_{\text{Left}} = (v)_{\text{Right}}$ at $x = a$

$$\therefore C_4 = \frac{qa^4}{24}$$

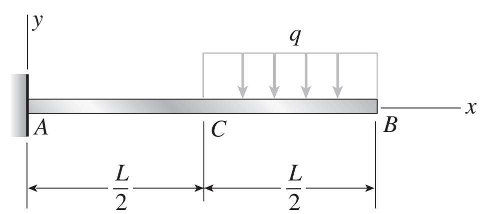
$$v = -\frac{qx^2}{24EI}(6a^2 - 4ax + x^2) \quad (0 \le x \le a) \longleftarrow$$

$$v = -\frac{qa^3}{24EI}(4x - a) \; (a \le x \le L) \longleftarrow$$

$$\delta_B = -v(L) = \frac{qa^3}{24EI}(4L - a) \longleftarrow$$

(These results agree with Case 2, Table G-1.)

Problem 9.3-15 Derive the equations of the deflection curve for a cantilever beam AB supporting a uniform load of intensity q acting over one-half of the length (see figure). Also, obtain formulas for the deflections δ_B and δ_C at points B and C, respectively. (*Note:* Use the second-order differential equation of the deflection curve.)



Solution 9.3-15 Cantilever beam (partial uniform load)

BENDING-MOMENT EQUATION (EQ. 9-12a)

$$EIv'' = M = -\frac{qL}{8}(3L - 4x) \quad \left(0 \le x \le \frac{L}{2}\right)$$

$$EIv' = -\frac{qL}{8}(3Lx - 2x^2) + C_1 \quad \left(0 \le x \le \frac{L}{2}\right)$$

B.C. 1 $v'(0) = 0 \quad \therefore C_1 = 0$

$$EIv'' = M = -\frac{q}{2}(L^2 - 2Lx + x^2) \quad \left(\frac{L}{2} \le x \le L\right)$$

$$EIv' = -\frac{q}{2}\left(L^2x - Lx^2 + \frac{x^3}{3}\right) + C_2 \quad \left(\frac{L}{2} \le x \le L\right)$$

B.C. 2 $(v')_{\text{Left}} = (v')_{\text{Right}}$ at $x = \dfrac{L}{2}$

$$\therefore\ C_2 = \frac{qL^3}{48}$$

$$EIv = -\frac{qL}{8}\left(\frac{3Lx^2}{2} - \frac{2x^3}{3}\right) + C_3 \quad \left(0 \le x \le \frac{L}{2}\right)$$

B.C. 3 $v(0) = 0 \quad \therefore C_3 = 0$

$$EIv = -\frac{q}{2}\left(\frac{L^2x^2}{2} - \frac{Lx^3}{3} + \frac{x^4}{12}\right) + \frac{qL^3}{48}x + C_4 \quad \left(\frac{L}{2} \le x \le L\right)$$

B.C. 4 $(v)_{\text{Left}} = (v)_{\text{Right}}$ at $x = \dfrac{L}{2}$

$$\therefore\ C_4 = -\frac{qL^4}{384}$$

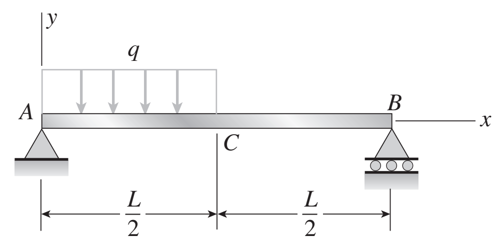
$$v = -\frac{qLx^2}{48\,EI}(9L - 4x) \quad \left(0 \le x \le \frac{L}{2}\right) \quad \longleftarrow$$

$$\delta_C = -v\left(\frac{L}{2}\right) = \frac{7qL^4}{192EI} \quad \longleftarrow$$

$$v = -\frac{q}{384\,EI}(16x^4 - 64\,Lx^3 + 96\,L^2x^2 - 8\,L^3x + L^4) \quad \left(\frac{L}{2} \le x \le L\right) \quad \longleftarrow$$

$$\delta_B = -v(L) = \frac{41qL^4}{384EI} \quad \longleftarrow$$

Problem 9.3-16 Derive the equations of the deflection curve for a simple beam *AB* with a uniform load of intensity *q* acting over the left-hand half of the span (see figure). Also, determine the deflection δ_C at the midpoint of the beam. (*Note:* Use the second-order differential equation of the deflection curve.)



Solution 9.3-16 Simple beam (partial uniform load)

BENDING-MOMENT EQUATION (EQ. 9-12a)

$$EIv'' = M = \frac{3qLx}{8} - \frac{qx^2}{2} \quad \left(0 \le x \le \frac{L}{2}\right)$$

$$EIv' = \frac{3qLx^2}{16} - \frac{qx^3}{6} + C_1 \quad \left(0 \le x \le \frac{L}{2}\right)$$

$$EIv'' = M = \frac{qL^2}{8} - \frac{qLx}{8} \quad \left(\frac{L}{2} \le x \le L\right)$$

$$EIv' = \frac{qL^2x}{8} - \frac{qLx^2}{16} + C_2 \quad \left(\frac{L}{2} \le x \le L\right)$$

B.C. 1 $(v')_{\text{Left}} = (v')_{\text{Right}}$ at $x = \dfrac{L}{2}$

$$\therefore\ C_2 = C_1 - \frac{qL^2}{48}$$

$$EIv = \frac{qLx^3}{16} - \frac{qx^4}{24} + C_1x + C_3 \quad \left(0 \le x \le \frac{L}{2}\right)$$

B.C. 2 $v(0) = 0 \quad \therefore C_3 = 0$

$$EIv = \frac{qL^2x^2}{16} - \frac{qLx^3}{48} + C_1x - \frac{qL^3x}{48} + C_4 \quad \left(\frac{L}{2} \le x \le L\right)$$

B.C. 3 $v(L) = 0 \quad \therefore C_4 = -C_1L - \dfrac{qL^4}{48}$

B.C. 4 $(v)_{\text{Left}} = (v)_{\text{Right}}$ at $x = \dfrac{L}{2}$

$$\therefore\ C_1 = -\frac{3qL^3}{128}$$

$$v = -\frac{qx}{384EI}(9L^3 - 24Lx^2 + 16x^3) \left(0 \le x \le \frac{L}{2}\right) \quad \longleftarrow$$

$$v = -\frac{qL}{384EI}(8x^3 - 24Lx^2 + 17L^2x - L^3) \quad \longleftarrow \quad \left(\frac{L}{2} \le x \le L\right)$$

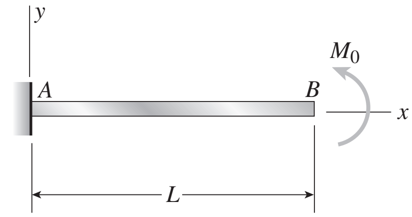
$$\delta_C = -v\left(\frac{L}{2}\right) = \frac{5qL^4}{768EI} \quad \longleftarrow$$

(These results agree with Case 2, Table G-2.)

Differential Equations of the Deflection Curve

The beams described in the problems for Section 9.4 have constant flexural rigidity EI. Also, the origin of coordinates is at the left-hand end of each beam.

Problem 9.4-1 Derive the equation of the deflection curve for a cantilever beam AB when a couple M_0 acts counterclockwise at the free end (see figure). Also, determine the deflection δ_B and slope θ_B at the free end. Use the third-order differential equation of the deflection curve (the shear-force equation).



Solution 9.4-1 Cantilever beam (couple M_0)

SHEAR-FORCE EQUATION (EQ. 9-12 b).

$EIv''' = V = 0$

$EIv'' = C_1$

B.C. 1 $M = M_0 \quad EIv'' = M = M_0 = C_1$

$EIv' = C_1x + C_2 = M_0x + C_2$

B.C. 2 $v'(0) = 0 \quad \therefore C_2 = 0$

$$EIv = \frac{M_0x^2}{2} + C_3$$

B.C. 3 $v(0) = 0 \quad \therefore C_3 = 0$

$$v = \frac{M_0x^2}{2EI} \quad \leftarrow$$

$$v' = \frac{M_0x}{EI}$$

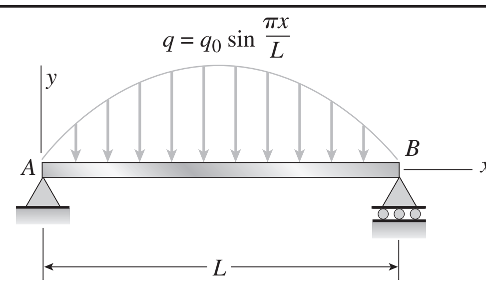
$$\delta_B = v(L) = \frac{M_0L^2}{2EI} \text{ (upward)} \quad \leftarrow$$

$$\theta_B = v'(L) = \frac{M_0L}{EI} \text{ (counterclockwise)} \quad \leftarrow$$

(These results agree with Case 6, Table G-1.)

Problem 9.4-2 A simple beam AB is subjected to a distributed load of intensity $q = q_0 \sin \pi x/L$, where q_0 is the maximum intensity of the load (see figure).

Derive the equation of the deflection curve, and then determine the deflection δ_{max} at the midpoint of the beam. Use the fourth-order differential equation of the deflection curve (the load equation).



Solution 9.4-2 Simple beam (sine load)

LOAD EQUATION (EQ. 9-12 c).

$$EIv'''' = -q = -q_0 \sin \frac{\pi x}{L}$$

$$EIv''' = q_0\left(\frac{L}{\pi}\right)\cos\frac{\pi x}{L} + C_1$$

$$EIv'' = q_0\left(\frac{L}{\pi}\right)^2 \sin\frac{\pi x}{L} + C_1x + C_2$$

B.C. 1 $EIv'' = M \quad EIv''(0) = 0 \quad \therefore C_2 = 0$

B.C. 2 $EIv''(L) = 0 \quad \therefore C_1 = 0$

$$EIv' = -q_0\left(\frac{L}{\pi}\right)^3 \cos\frac{\pi x}{L} + C_3$$

$$EIv = -q_0\left(\frac{L}{\pi}\right)^4 \sin\frac{\pi x}{L} + C_3x + C_4$$

B.C. 3 $v(0) = 0 \quad \therefore C_4 = 0$

B.C. 4 $v(L) = 0 \quad \therefore C_3 = 0$

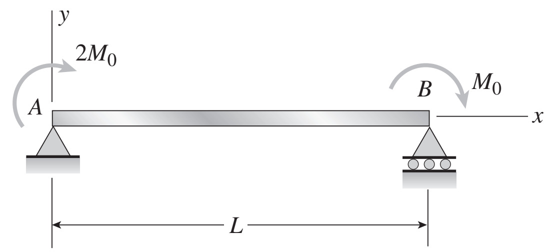
$$v = -\frac{q_0L^4}{\pi^4EI}\sin\frac{\pi x}{L} \quad \leftarrow$$

$$\delta_{max} = -v\left(\frac{L}{2}\right) = \frac{q_0L^4}{\pi^4EI} \quad \leftarrow$$

(These results agree with Case 13, Table G-2.)

Problem 9.4-3 The simple beam AB shown in the figure has moments $2M_0$ and M_0 acting at the ends.

Derive the equation of the deflection curve, and then determine the maximum deflection δ_{max}. Use the third-order differential equation of the deflection curve (the shear-force equation).



Solution 9.4-3 Simple beam with two couples

Reaction at support A: $R_A = \dfrac{3M_0}{L}$ (downward)

Shear force in beam: $V = -R_A = -\dfrac{3M_0}{L}$

SHEAR-FORCE EQUATION (EQ. 9-12 b)

$$EIv''' = V = -\frac{3M_0}{L}$$

$$EIv'' = -\frac{3M_0 x}{L} + C_1$$

B.C. 1 $EIv'' = M \quad EIv''(0) = 2M_0 \quad \therefore C_1 = 2M_0$

$$EIv' = -\frac{3M_0 x^2}{2L} + 2M_0 x + C_2$$

$$EIv = -\frac{M_0 x^3}{2L} + M_0 x^2 + C_2 x + C_3$$

B.C. 2 $v(0) = 0 \quad \therefore C_3 = 0$

B.C. 3 $v(L) = 0 \quad \therefore C_2 = -\dfrac{M_0 L}{2}$

$$v = -\frac{M_0 x}{2LEI}(L^2 - 2Lx + x^2) = -\frac{M_0 x}{2LEI}(L - x)^2 \longleftarrow$$

$$v' = -\frac{M_0}{2LEI}(L - x)(L - 3x)$$

MAXIMUM DEFLECTION

Set $v' = 0$ and solve for x:

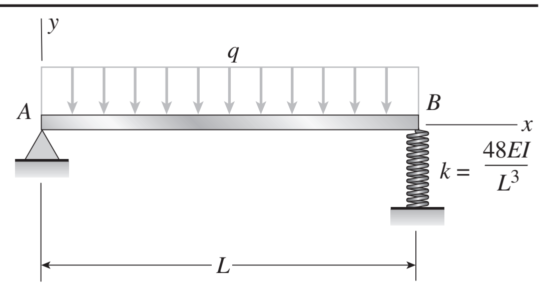
$x_1 = L$ and $x_2 = \dfrac{L}{3}$

Maximum deflection occurs at $x_2 = \dfrac{L}{3}$.

$$\delta_{max} = -v\left(\frac{L}{3}\right) = \frac{2M_0 L^2}{27EI} \quad \text{(downward)} \longleftarrow$$

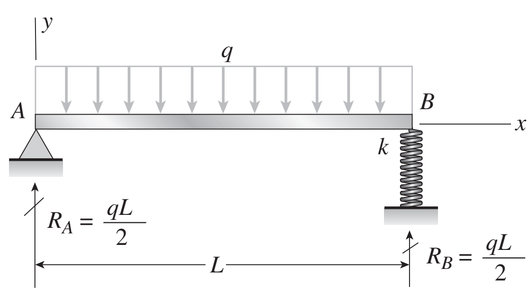
Problem 9.4-4 A simple beam with a uniform load is pin supported at one end and spring supported at the other. The spring has stiffness $k = 48EI/L^3$.

Derive the equation of the deflection curve by starting with the third-order differential equation (the shear-force equation). Also, determine the angle of rotation θ_A at support A.



Solution 9.4-4 Beam with a spring support

REACTIONS



DEFLECTIONS AT END B

$$k = \frac{48EI}{L^3} \qquad \delta_B = \frac{R_B}{k} = \frac{qL}{2k} = \frac{qL^4}{96EI}$$

SHEAR-FORCE EQUATION (EQ. 9-12 b)

$$V = R_A - qx = \frac{q}{2}(L - 2x)$$

$$EIv''' = V = \frac{q}{2}(L - 2x)$$

$$EIv'' = \frac{q}{2}(Lx - x^2) + C_1$$

B.C. 1 $EIv'' = M \quad EIv''(0) = 0 \quad \therefore C_1 = 0$

$$EIv' = \frac{q}{2}\left(\frac{Lx^2}{2} - \frac{x^3}{3}\right) + C_2$$

$$EIv = \frac{q}{2}\left(\frac{Lx^3}{6} - \frac{x^4}{12}\right) + C_2 x + C_3$$

B.C. 2 $v(0) = 0 \quad \therefore C_3 = 0$

B.C. 3 $v(L) = -\delta_B = -\dfrac{qL^4}{96EI}$

$$\therefore C_2 = -\frac{5qL^3}{96}$$

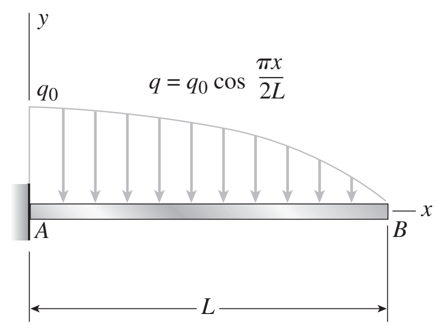
$$v = -\frac{qx}{96EI}(5L^3 - 8Lx^2 + 4x^3) \quad \longleftarrow$$

$$v' = -\frac{q}{96EI}(5L^3 - 24Lx^2 + 16x^3)$$

$$\theta_A = -v'(0) = \frac{5qL^3}{96EI} \quad \text{(clockwise)} \quad \longleftarrow$$

Problem 9.4-5 The distributed load acting on a cantilever beam AB has an intensity q given by the expression $q_0 \cos \pi x/2L$, where q_0 is the maximum intensity of the load (see figure).

Derive the equation of the deflection curve, and then determine the deflection δ_B at the free end. Use the fourth-order differential equation of the deflection curve (the load equation).



Solution 9.4-5 Cantilever beam (cosine load)

LOAD EQUATION (EQ. 9-12 c)

$$EIv'''' = -q = -q_0 \cos\frac{\pi x}{2L}$$

$$EIv''' = -q_0\left(\frac{2L}{\pi}\right)\sin\frac{\pi x}{2L} + C_1$$

B.C. 1 $EIv''' = V \quad EIv'''(L) = 0 \quad \therefore C_1 = \dfrac{2q_0 L}{\pi}$

$$EIv'' = q_0\left(\frac{2L}{\pi}\right)^2 \cos\frac{\pi x}{2L} + \frac{2q_0 Lx}{\pi} + C_2$$

B.C. 2 $EIv'' = M \quad EIv''(L) = 0 \quad \therefore C_2 = -\dfrac{2q_0 L^2}{\pi}$

$$EIv' = q_0\left(\frac{2L}{\pi}\right)^3 \sin\frac{\pi x}{2L} + \frac{q_0 Lx^2}{\pi} - \frac{2q_0 L^2 x}{\pi} + C_3$$

B.C. 3 $v'(0) = 0 \quad \therefore C_3 = 0$

$$EIv = -q_0\left(\frac{2L}{\pi}\right)^4 \cos\frac{\pi x}{2L} + \frac{q_0 Lx^3}{3\pi} - \frac{q_0 L^2 x^2}{\pi} + C_4$$

B.C. 4 $v(0) = 0 \quad \therefore C_4 = \dfrac{16q_0 L^4}{\pi^4}$

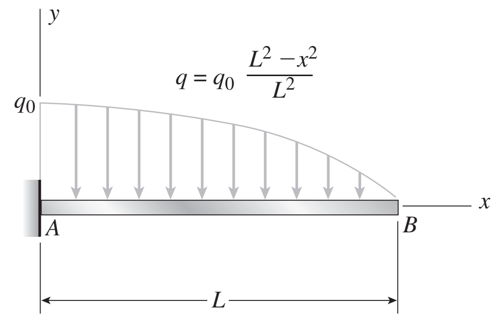
$$v = -\frac{q_0 L}{3\pi^4 EI}\left(48L^3 \cos\frac{\pi x}{2L} - 48L^3 + 3\pi^3 Lx^2 - \pi^3 x^3\right) \quad \longleftarrow$$

$$\delta_B = -v(L) = \frac{2q_0 L^4}{3\pi^4 EI}(\pi^3 - 24) \quad \longleftarrow$$

(These results agree with Case 10, Table G-1.)

Problem 9.4-6 A cantilever beam AB is subjected to a parabolically varying load of intensity $q = q_0(L^2 - x^2)/L^2$, where q_0 is the maximum intensity of the load (see figure).

Derive the equation of the deflection curve, and then determine the deflection δ_B and angle of rotation θ_B at the free end. Use the fourth-order differential equation of the deflection curve (the load equation).



Solution 9.4-6 **Cantilever beam (parabolic load)**

LOAD EQUATION (EQ. 9-12 c)

$$EIv'''' = -q = -\frac{q_0}{L^2}(L^2 - x^2)$$

$$EIv''' = -\frac{q_0}{L^2}\left(L^2x - \frac{x^3}{3}\right) + C_1$$

B.C. 1 $EIv''' = V \quad EIv'''(L) = 0 \quad \therefore C_1 = \dfrac{2q_0L}{3}$

$$EIv'' = -\frac{q_0}{L^2}\left(\frac{L^2x^2}{2} - \frac{x^4}{12}\right) + \frac{2q_0L}{3}x + C_2$$

B.C. 2 $EIv'' = M \quad EIv''(L) = 0 \quad \therefore C_2 = -\dfrac{q_0L^2}{4}$

$$EIv' = -\frac{q_0}{L^2}\left(\frac{L^2x^3}{6} - \frac{x^5}{60}\right) + \frac{q_0Lx^2}{3} - \frac{q_0L^2x}{4} + C_3$$

B.C. 3 $v'(0) = 0 \quad \therefore C_3 = 0$

$$EIv = -\frac{q_0}{L^2}\left(\frac{L^2x^4}{24} - \frac{x^6}{360}\right) + \frac{q_0Lx^3}{9} - \frac{q_0L^2x^2}{8} + C_4$$

B.C. 4 $v(0) = 0 \quad \therefore C_4 = 0$

$$v = -\frac{q_0x^2}{360L^2EI}(45L^4 - 40L^3x + 15L^2x^2 - x^4) \longleftarrow$$

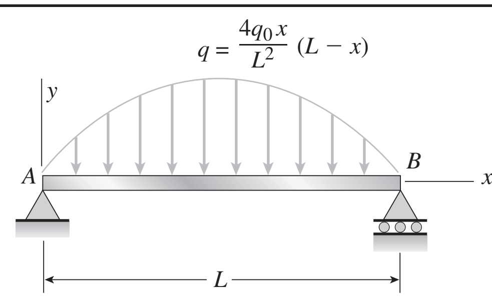
$$\delta_B = -v(L) = \frac{19q_0L^4}{360EI} \longleftarrow$$

$$v' = -\frac{q_0x}{60L^2EI}(15L^4 - 20L^3x + 10L^2x^2 - x^4)$$

$$\theta_B = -v'(L) = \frac{q_0L^3}{15EI} \longleftarrow$$

Problem 9.4-7 A beam on simple supports is subjected to a parabolically distributed load of intensity $q = 4q_0x(L - x)/L^2$, where q_0 is the maximum intensity of the load (see figure).

Derive the equation of the deflection curve, and then determine the maximum deflection δ_{max}. Use the fourth-order differential equation of the deflection curve (the load equation).



Solution 9.4-7 **Single beam (parabolic load)**

LOAD EQUATION (EQ. 9-12 c)

$$EIv'''' = -q = -\frac{4q_0x}{L^2}(L - x) = -\frac{4q_0}{L^2}(Lx - x^2)$$

$$EIv''' = -\frac{2q_0}{3L^2}(3Lx^2 - 2x^3) + C_1$$

$$EIv'' = -\frac{q_0}{3L^2}(2Lx^3 - x^4) + C_1x + C_2$$

B.C. 1 $EIv'' = M \quad EIv''(0) = 0 \quad \therefore C_2 = 0]$

B.C. 2 $EIv''(L) = 0 \quad \therefore C_1 = \dfrac{q_0L}{3}$

$$EIv' = -\frac{q_0}{30L^2}(-5L^3x^2 + 5Lx^4 - 2x^5) + C_3$$

B.C. 3 (Symmetry) $v'\left(\dfrac{L}{2}\right) = 0 \quad \therefore C_3 = -\dfrac{q_0L^3}{30}$

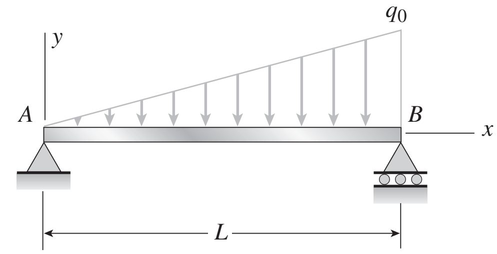
$$EIv = -\frac{q_0}{30L^2}\left(L^5x - \frac{5L^3x^3}{3} + Lx^5 - \frac{x^6}{3}\right) + C_4$$

B.C. 4 $v(0) = 0 \quad \therefore C_4 = 0$

$$v = -\frac{q_0x}{90L^2EI}(3L^5 - 5L^3x^2 + 3Lx^4 - x^5) \longleftarrow$$

$$\delta_{max} = -v\left(\frac{L}{2}\right) = \frac{61q_0L^4}{5760EI} \longleftarrow$$

Problem 9.4-8 Derive the equation of the deflection curve for a simple beam AB carrying a triangularly distributed load of maximum intensity q_0 (see figure). Also, determine the maximum deflection δ_{max} of the beam. Use the fourth-order differential equation of the deflection curve (the load equation).



Solution 9.4-8 Simple beam (triangular load)

LOAD EQUATION (EQ. 9-12 c)

$$EIv'''' = -q = -\frac{q_0 x}{L} \quad EIv''' = -\frac{q_0 x^2}{2L} + C_1$$

$$EIv'' = -\frac{q_0 x^3}{6L} + C_1 x + C_2$$

B.C. 1 $EIv'' = M \quad EIv''(0) = 0 \quad \therefore C_2 = 0$

B.C. 2 $EIv''(L) = 0 \quad \therefore C_1 = \frac{q_0 L}{6}$

$$EIv' = -\frac{q_0 x^4}{24L} + \frac{q_0 L x^2}{12} + C_3$$

$$EIv = -\frac{q_0 x^5}{120L} + \frac{q_0 L x^3}{36} + C_3 x + C_4$$

B.C. 3 $v(0) = 0 \quad \therefore C_4 = 0$

B.C. 4 $v(L) = 0 \quad \therefore C_3 = -\frac{7 q_0 L^3}{360}$

$$v = -\frac{q_0 x}{360 LEI}(7L^4 - 10L^2x^2 + 3x^4) \leftarrow$$

$$v' = -\frac{q_0}{360 LEI}(7L^4 - 30L^2x^2 + 15x^4)$$

MAXIMUM DEFLECTION

Set $v' = 0$ and solve for x:

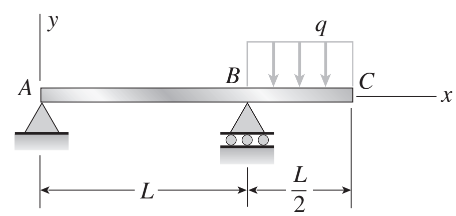
$$x_1^2 = L^2\left(1 - \sqrt{\frac{8}{15}}\right) \quad x_1 = 0.51933L$$

$$\delta_{max} = -v(x_1) = \frac{q_0 L^4}{225EI}\left(\frac{5}{3} + \frac{2}{3} - \sqrt{\frac{8}{15}}\right)^{1/2} \leftarrow$$

$$= 0.006522\frac{q_0 L^4}{EI} \leftarrow$$

(These results agree with Case 11, Table G-2.)

Problem 9.4-9 Derive the equations of the deflection curve for an overhanging beam ABC subjected to a uniform load of intensity q acting on the overhang (see figure). Also, obtain formulas for the deflection δ_C and angle of rotation θ_C at the end of the overhang. Use the fourth-order differential equation of the deflection curve (the load equation).



Solution 9.4-9 Beam with an overhang

LOAD EQUATION (EQ. 9-12 c)

$EIv'''' = -q = 0 \qquad (0 \le x \le L)$
$EIv''' = C_1 \qquad (0 \le x \le L)$
$EIv'' = C_1 x + C_2 \qquad (0 \le x \le L)$

B.C. 1 $EIv'' = M \quad EIv''(0) = 0 \quad \therefore C_2 = 0$

$$EIv'''' = -q \qquad \left(L \le x \le \frac{3L}{2}\right)$$

$$EIv''' = -qx + C_3 \qquad \left(L \le x \le \frac{3L}{2}\right)$$

B.C. 2 $EIv''' = V \quad EIv'''\left(\frac{3L}{2}\right) = 0 \quad \therefore C_3 = \frac{3qL}{2}$

$$EIv'' = -\frac{qx^2}{2} + \frac{3qLx}{2} + C_4 \quad \left(L \le x \le \frac{3L}{2}\right)$$

B.C. 3 $EIv'' = M \quad EIv''\left(\frac{3L}{2}\right) = 0 \quad \therefore C_4 = -\frac{9qL^2}{8}$

B.C. 4 $EI(v'')_{Left} = EI(v'')_{Right}$ at $x = L$

$$C_1 L = -\frac{qL^2}{2} + \frac{3qL^2}{2} - \frac{9qL^2}{8} \quad \therefore C_1 = -\frac{qL}{8}$$

$$EIv' = -\frac{qLx^2}{16} + C_5 \quad (0 \le x \le L)$$

$$EIv' = -\frac{qx^3}{6} + \frac{3qLx^2}{4} - \frac{9qL^2x}{8} + C_6$$

$$\left(L \le x \le \frac{3L}{2}\right)$$

B.C. 5 $(v')_{\text{Left}} = (v')_{\text{Right}}$ at $x = L$

$$\therefore C_6 = C_5 + \frac{23qL^3}{48} \qquad \text{(a)}$$

$$EIv = -\frac{qLx^3}{48} + C_5 x + C_7 \quad (0 \le x \le L)$$

B.C. 6 $v(0) = 0$ $\therefore C_7 = 0$

B.C. 7 $v(L) = 0$ for $0 \le x \le L$ $\therefore C_5 = \frac{qL^3}{48}$

From Eq.(a): $C_6 = \frac{qL^3}{2}$

$$EIv = -\frac{qx^4}{24} + \frac{3qLx^3}{12} - \frac{9qL^2x^2}{16} + \frac{qL^3x}{2} + C_8 \quad \left(L \le x \le \frac{3L}{2}\right)$$

B.C. 8 $v(L) = 0$ for $L \le x \le \frac{3L}{2}$ $\therefore C_8 = -\frac{7qL^4}{48}$

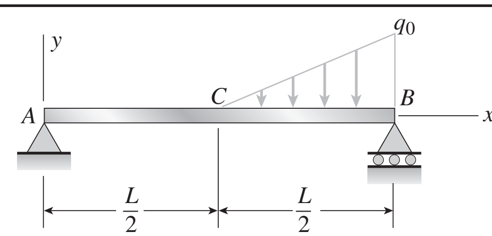
$$v = -\frac{qLx}{48EI}(L^2 - x^2) \quad (0 \le x \le L) \longleftarrow$$

$$v = -\frac{q(L-x)}{48EI}(7L^3 - 17L^2x + 10Lx^2 - 2x^3) \longleftarrow \quad \left(L \le x \le \frac{3L}{2}\right)$$

$$\delta_C = -v\left(\frac{3L}{2}\right) = \frac{11qL^4}{384EI} \longleftarrow$$

$$\theta_C = -v'\left(\frac{3L}{2}\right) = \frac{qL^3}{16EI} \longleftarrow$$

Problem 9.4-10 Derive the equations of the deflection curve for a simple beam AB supporting a triangularly distributed load of maximum intensity q_0 acting on the right-hand half of the beam (see figure). Also, determine the angles of rotation θ_A and θ_B at the ends and the deflection δ_C at the midpoint. Use the fourth-order differential equation of the deflection curve (the load equation).



Solution 9.4-10 **Simple beam (triangular load)**

LOAD EQUATION (EQ. 9-12 c)

Left-hand half (part AC): $0 \le x \le \frac{L}{2}$

Right-hand half (part CB): $\frac{1}{2} \le x \le L$

PART AC $q = 0$

$$EIv'''' = -q = 0 \qquad EIv''' = C_1$$

$$EIv'' = C_1 x + C_2 \quad EIv' = C_1\left(\frac{x^2}{2}\right) + C_2 x + C_3$$

$$EIv = C_1\left(\frac{x^3}{6}\right) + C_2\left(\frac{x^2}{2}\right) + C_3 x + C_4$$

PART CB $q = \frac{q_0}{L}(2x - L)$

$$EIv'''' = -q = \frac{q_0}{L}(L - 2x)$$

$$EIv''' = \frac{q_0}{L}(Lx - x^2) + C_5$$

$$EIv'' = \frac{q_0}{L}\left(\frac{Lx^2}{2} - \frac{x^3}{3}\right) + C_5 x + C_6$$

$$EIv' = \frac{q_0}{L}\left(\frac{Lx^3}{6} - \frac{x^4}{12}\right) + C_5\left(\frac{x^2}{2}\right) + C_6 x + C_7$$

$$EIv = \frac{q_0}{L}\left(\frac{Lx^4}{24} - \frac{x^5}{60}\right) + C_5\left(\frac{x^3}{6}\right) + C_6\left(\frac{x^2}{2}\right) + C_7 x + C_8$$

BOUNDARY CONDITIONS

B.C. 1 $EIv''' = V$ $EI(v''')_{AC} = EI(v''')_{BC}$ at $x = \frac{L}{2}$

$$C_1 - C_5 = \frac{q_0 L}{4} \qquad (1)$$

B.C. 2 $EIv'' = M$ $EIv''(0) = 0$

$$C_2 = 0 \qquad (2)$$

B.C. 3 $EIv'(L) = 0$ $C_5 L + C_6 = -\frac{q_0 L^2}{6}$ (3)

B.C. 4 $(EIv'')_{AC} = (EIv'')_{CB}$ for $x = \frac{L}{2}$

$$C_1 L - C_5 L - 2C_6 = \frac{q_0 L^2}{6} \qquad (4)$$

B.C. 5 $(v')_{AC} = (v')_{CB}$ for $x = \dfrac{L}{2}$

$$C_1 L^2 + 8C_3 - C_5 L^2 - 4C_6 L - 8C_7 = \frac{q_0 L^3}{8} \qquad (5)$$

B.C. 6 $v(0) = 0 \quad C_4 = 0 \qquad (6)$

B.C. 7 $v(L) = 0$

$$C_5 L^3 + 3C_6 L^2 + 6C_7 L + 6C_8 = -\frac{3q_0 L^4}{20} \qquad (7)$$

B.C. 8 $(v)_{AC} = (v)_{CB}$ for $x = \dfrac{L}{2}$

$$C_1 L^3 + 24C_3 L - C_5 L^3 - 6C_6 L^2 - 24C_7 L - 48C_8 = \frac{q_0 L^4}{10} \qquad (8)$$

SOLVE EQS. (1) THROUGH (B):

$$C_1 = \frac{q_0 L}{24} \quad C_2 = 0 \quad C_3 = -\frac{37q_0 L^3}{5760}$$

$$C_4 = 0 \quad C_5 = -\frac{5q_0 L}{24} \quad C_6 = \frac{q_0 L^2}{24}$$

$$C_7 = -\frac{67q_0 L^3}{5760} \quad C_B = \frac{q_0 L^4}{1920}$$

Substitute constants into equations for v and v'.

DEFLECTION CURVE FOR PART $AC \left(0 \le x \le \dfrac{L}{2}\right)$

$$v = -\frac{q_0 L x}{5760EI}(37L^2 - 40x^2) \quad \longleftarrow$$

$$v' = -\frac{q_0 L}{5760EI}(37L^2 - 120x^2)$$

$$\theta_A = -v'(0) = \frac{37q_0 L^3}{5760\,EI} \quad \longleftarrow$$

$$\delta_C = -v\left(\frac{L}{2}\right) = \frac{3q_0 L^4}{1280\,EI} \quad \longleftarrow$$

DEFLECTION CURVE FOR PART $CB \left(\dfrac{L}{2} \le x \le L\right)$

$$v = -\frac{q_0}{5760\,LEI}[L^2 x(37L^2 - 40x^2) + 3(2x - L)^5] \quad \longleftarrow$$

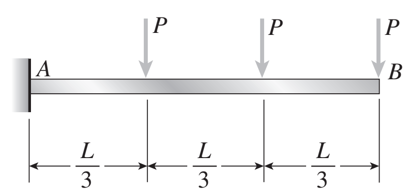
$$v' = -\frac{q_0}{5760\,LEI}[L^2(37L^2 - 120x^2) + 30(2x - L)^4]$$

$$\theta_B = v'(L) = \frac{53q_0 L^3}{5760EI} \quad \longleftarrow$$

Method of Superposition

The problems for Section 9.5 are to be solved by the method of superposition. All beams have constant flexural rigidity EI.

Problem 9.5-1 A cantilever beam AB carries three equally spaced concentrated loads, as shown in the figure. Obtain formulas for the angle of rotation θ_B and deflection δ_B at the free end of the beam.



Solution 9.5-1 Cantilever beam with 3 loads

Table G-1, Cases 4 and 5

$$\theta_B = \frac{P\left(\frac{L}{3}\right)^2}{2EI} + \frac{P\left(\frac{2L}{3}\right)^2}{2\,EI} + \frac{PL^2}{2\,EI} = \frac{7PL^2}{9\,EI} \quad \longleftarrow$$

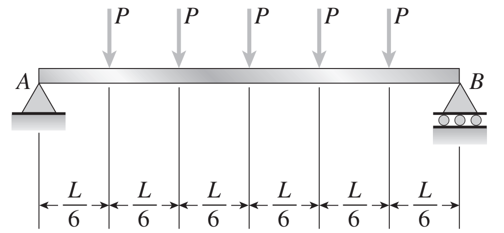
$$\delta_B = \frac{P\left(\frac{L}{3}\right)^2}{6\,EI}\left(3L - \frac{L}{3}\right) + \frac{P\left(\frac{2L}{3}\right)^2}{6\,EI}\left(3L - \frac{2L}{3}\right) + \frac{PL^3}{3\,EI} = \frac{5PL^3}{9\,EI} \quad \longleftarrow$$

Problem 9.5-2 A simple beam AB supports five equally spaced loads P (see figure).

(a) Determine the deflection δ_1 at the midpoint of the beam.

(b) If the same total load ($5P$) is distributed as a uniform load on the beam, what is the deflection δ_2 at the midpoint?

(c) Calculate the ratio of δ_1 to δ_2.



Solution 9.5-2 Simple beam with 5 loads

(a) Table G-2, Cases 4 and 6

$$\delta_1 = \frac{P\left(\frac{L}{6}\right)}{24\,EI}\left[3L^2 - 4\left(\frac{L}{6}\right)^2\right] + \frac{P\left(\frac{L}{3}\right)}{24\,EI}\left[3L^2 - 4\left(\frac{L}{3}\right)^2\right] + \frac{PL^3}{48\,EI}$$

$$= \frac{11PL^3}{144\,EI} \quad \longleftarrow$$

(b) Table G-2, Case 1 $qL = 5P$

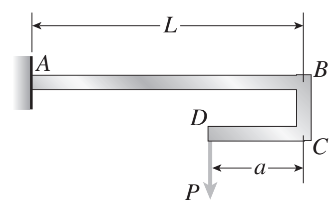
$$\delta_2 = \frac{5qL^4}{384\,EI} = \frac{25\,PL^3}{384\,EI} \quad \longleftarrow$$

(c) $$\frac{\delta_1}{\delta_2} = \frac{11}{144}\left(\frac{384}{25}\right) = \frac{88}{75} = 1.173 \quad \longleftarrow$$

Problem 9.5-3 The cantilever beam AB shown in the figure has an extension BCD attached to its free end. A force P acts at the end of the extension.

(a) Find the ratio a/L so that the vertical deflection of point B will be zero.

(b) Find the ratio a/L so that the angle of rotation at point B will be zero.



Solution 9.5-3 Cantilever beam with extension

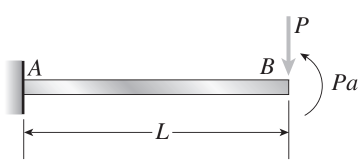


Table G-1, Cases 4 and 6

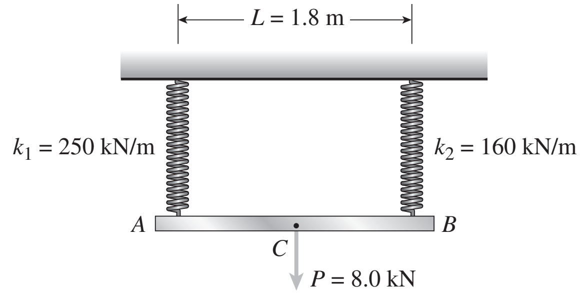
(a) $\delta_B = \dfrac{PL^3}{3EI} - \dfrac{PaL^2}{2EI} = 0 \qquad \dfrac{a}{L} = \dfrac{2}{3} \quad \longleftarrow$

(b) $\theta_B = \dfrac{PL^2}{2EI} - \dfrac{PaL}{EI} = 0 \qquad \dfrac{a}{L} = \dfrac{1}{2} \quad \longleftarrow$

Problem 9.5-4 Beam ACB hangs from two springs, as shown in the figure. The springs have stiffnesses k_1 and k_2 and the beam has flexural rigidity EI.

What is the downward displacement of point C, which is at the midpoint of the beam, when the load P is applied?

Data for the structure are as follows: $P = 8.0$ kN, $L = 1.8$ m, $EI = 216$ kN·m^2, $k_1 = 250$ kN/m, and $k_2 = 160$ kN/m.



Solution 9.5-4 **Beam hanging from springs**

$P = 8.0$ kN $\quad L = 1.8$ m
$EI = 216$ kN · m^2 $\quad k_1 = 250$ kN/m
$k_2 = 160$ kN/m

Stretch of springs:

$$\delta_A = \frac{P/2}{k_1} \qquad \delta_B = \frac{P/2}{k_2}$$

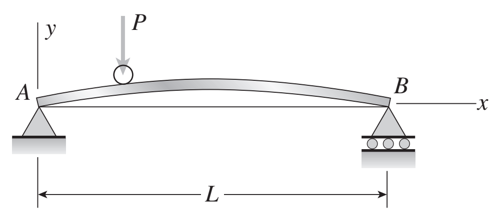
Table G-2, Case 4

$$\delta_C = \frac{PL^3}{48\,EI} + \frac{1}{2}\left(\frac{P/2}{k_1} + \frac{P/2}{k_2}\right)$$
$$= \frac{PL^3}{48EI} + \frac{P}{4}\left(\frac{1}{k_1} + \frac{1}{k_2}\right) \longleftarrow$$

Substitute numerical values:

$$\delta_C = \frac{(8.0\text{ kN})(1.8\text{ m})^3}{48\,(216\text{ kN}\cdot\text{m}^2)} + \frac{8.0\text{ kN}}{4}\left(\frac{1}{250\text{ kN/m}} + \frac{1}{160\text{ kN/m}}\right) \longleftarrow$$
$$= 4.5\text{ mm} + 20.5\text{ mm}$$
$$= 25\text{ mm} \longleftarrow$$

Problem 9.5-5 What must be the equation $y = f(x)$ of the axis of the slightly curved beam AB (see figure) *before* the load is applied in order that the load P, moving along the bar, always stays at the same level?



Solution 9.5-5 **Slightly curved beam**

Let x = distance to load P
δ = downward deflection at load P

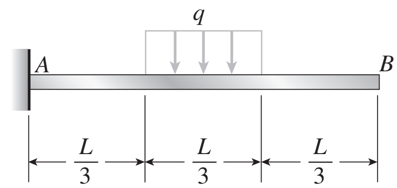
Table G-2, Case 5:

$$\delta = \frac{P(L-x)\,x}{6LEI}\,[L^2 - (L-x)^2 - x^2] = \frac{Px^2\,(L-x)^2}{3LEI}$$

Initial upward displacement of the beam must equal δ.

$$\therefore\ y = \frac{Px^2\,(L-x)^2}{3LEI} \longleftarrow$$

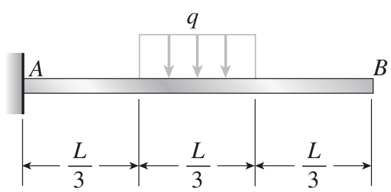
Problem 9.5-6 Determine the angle of rotation θ_B and deflection δ_B at the free end of a cantilever beam AB having a uniform load of intensity q acting over the middle third of its length (see figure).



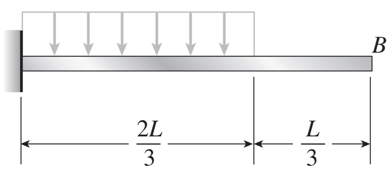
Solution 9.5-6 Cantilever beam (partial uniform load)

q = intensity of uniform load

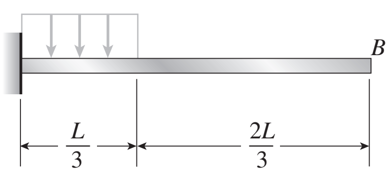
Original load on the beam:



Load No. 1:



Load No. 2:



SUPERPOSITION: Original load = Load No. 1 minus Load No. 2

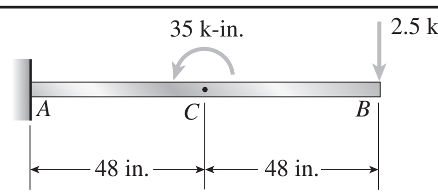
Table G-1, Case 2

$$\theta_B = \frac{q}{6EI}\left(\frac{2L}{3}\right)^3 - \frac{q}{6EI}\left(\frac{L}{3}\right)^3 = \frac{7qL^3}{162EI} \quad \longleftarrow$$

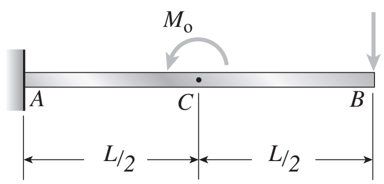
$$\delta_B = \frac{q}{24EI}\left(\frac{2L}{3}\right)^3\left(4L - \frac{2L}{3}\right) - \frac{q}{24EI}\left(\frac{1}{3}\right)^3\left(4L - \frac{L}{3}\right)$$

$$= \frac{23qL^4}{648EI} \quad \longleftarrow$$

Problem 9.5-7 The cantilever beam ACB shown in the figure has flexural rigidity $EI = 2.1 \times 10^6$ k-in.^{2} Calculate the downward deflections δ_C and δ_B at points C and B, respectively, due to the simultaneous action of the moment of 35 k-in. applied at point C and the concentrated load of 2.5 k applied at the free end B.



Solution 9.5-7 **Cantilever beam (two loads)**



$EI = 2.1 \times 10^6$ k-in.^{2}
$M_0 = 35$ k-in.
$P = 2.5$ k
$L = 96$ in.

Table G-1, Cases 4.6, and 7

$$\delta_C = -\frac{M_0(L/2)^2}{2EI} + \frac{P(L/2)^2}{6EI}\left(3L - \frac{L}{2}\right)$$

$$= -\frac{M_0L^2}{8EI} + \frac{5PL^3}{48EI} \quad (+ = \text{downward deflection})$$

$$\delta_B = -\frac{M_0(L/2)}{2EI}\left(2L - \frac{L}{2}\right) + \frac{PL^3}{3EI}$$

$$= -\frac{3M_0L^2}{8EI} + \frac{PL^3}{3EI} \quad (+ = \text{downward deflection})$$

Substitute numerical values:

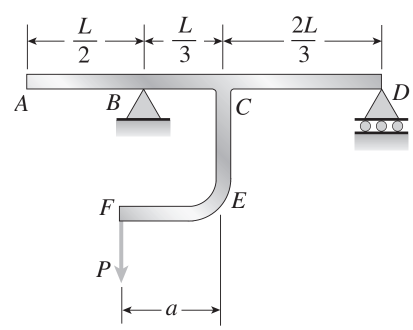
$\delta_C = -0.01920$ in. $+ 0.10971$ in.
$= 0.0905$ in. ←

$\delta_B = -0.05760$ in. $+ 0.35109$ in.
$= 0.293$ in. ←

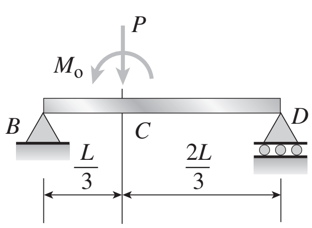
Problem 9.5-8 A beam *ABCD* consisting of a simple span *BD* and an overhang *AB* is loaded by a force *P* acting at the end of the bracket *CEF* (see figure).

(a) Determine the deflection δ_A at the end of the overhang.

(b) Under what conditions is this deflection upward? Under what conditions is it downward?



Solution 9.5-8 **Beam with bracket and overhang**



Consider part BD of the beam.
$M_0 = Pa$

Table G-2, Cases 5 and 9

$$\theta_B = \frac{P(L/3)(2L/3)(5L/3)}{6LEI} + \frac{Pa}{6LEI}\left[6\left(\frac{L^2}{3}\right) - 3\left(\frac{L^2}{9}\right) - 2L^2\right]$$

$$= \frac{PL}{162EI}(10L - 9a) \quad (+ = \text{clockwise angle})$$

(a) Deflection at the end of the overhang

$$\delta_A = \theta_B\left(\frac{L}{2}\right) = \frac{PL^2}{324\,EI}(10L - 9a) \quad \leftarrow$$

$(+ = \text{upward deflection})$

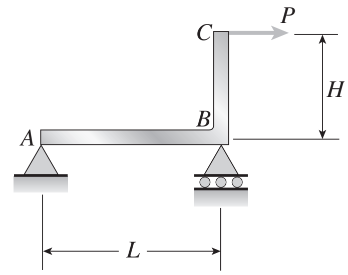
(b) Deflection is upward when $\frac{a}{L} < \frac{10}{9}$ and downward when $\frac{a}{L} > \frac{10}{9}$ ←

Problem 9.5-9 A horizontal load P acts at end C of the bracket ABC shown in the figure.

(a) Determine the deflection δ_C of point C.

(b) Determine the maximum upward deflection δ_{max} of member AB.

Note: Assume that the flexural rigidity EI is constant throughout the frame. Also, disregard the effects of axial deformations and consider only the effects of bending due to the load P.



Solution 9.5-9 **Bracket** ABC

BEAM AB

$$M_0 = PH$$

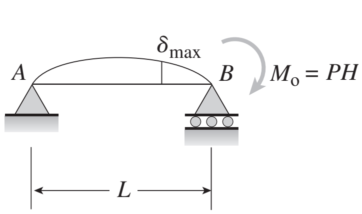


Table G-2, Case 7: $\theta_B = \dfrac{M_0 L}{3EI} = \dfrac{PHL}{3EI}$

(a) ARM BC Table G-1, Case 4

$$\delta_C = \frac{PH^3}{3EI} + \theta_B H = \frac{PH^3}{3EI} + \frac{PH^2L}{3EI}$$

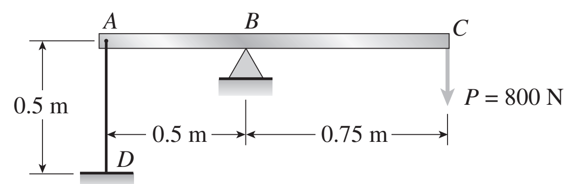
$$= \frac{PH^2}{3EI}(L + H) \longleftarrow$$

(b) MAXIMUM DEFLECTION OF BEAM AB

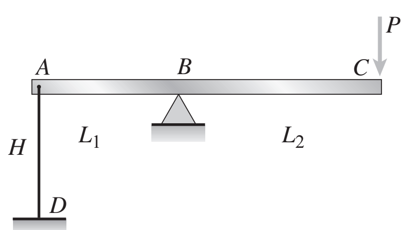
Table G-2, Case 7: $\delta_{max} = \dfrac{M_0 L^2}{9\sqrt{3}EI} = \dfrac{PHL^2}{9\sqrt{3}EI} \longleftarrow$

Problem 9.5-10 A beam ABC having flexural rigidity $EI = 75\ \text{kN}\cdot\text{m}^2$ is loaded by a force $P = 800$ N at end C and tied down at end A by a wire having axial rigidity $EA = 900$ kN (see figure).

What is the deflection at point C when the load P is applied?



Solution 9.5-10 **Beam tied down by a wire**



$EI = 75\ \text{kN}\cdot\text{m}^2$

$P = 800$ N

$EA = 900$ kN

$H = 0.5$ m $L_1 = 0.5$ m

$L_2 = 0.75$ m

CONSIDER BC AS A CANTILEVER BEAM

Table G-1, Case 4: $\delta'_C = \dfrac{PL_2^3}{3EI}$

CONSIDER AB AS A SIMPLE BEAM

$$M_0 = PL_2$$

Table G-2, Case 7: $\theta'_B = \dfrac{M_0 L_1}{3EI} = \dfrac{PL_1L_2}{3EI}$

CONSIDER THE STRETCHING OF WIRE AD

$$\delta'_A = (\text{Force in } AD)\left(\frac{H}{EA}\right) = \left(\frac{PL_2}{L_1}\right)\left(\frac{H}{EA}\right) = \frac{PL_2H}{EAL_1}$$

DEFLECTION δ_C OF POINT C

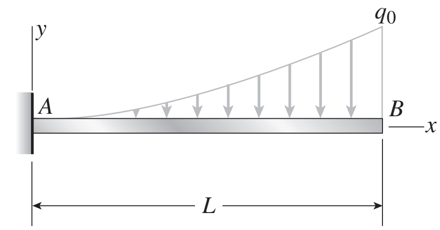
$$\delta_C = \delta'_C + \theta'_B(L_2) + \delta'_A\left(\frac{L_2}{L_1}\right)$$

$$= \frac{PL_2^3}{3EI} + \frac{PL_1L_2^2}{3EI} + \frac{PL_2^2H}{EAL_1^2} \longleftarrow$$

SUBSTITUTE NUMERICAL VALUES:

$$\delta_C = 1.50\ \text{mm} + 1.00\ \text{mm} + 1.00\ \text{mm} = 3.50\ \text{mm} \longleftarrow$$

Problem 9.5-11 Determine the angle of rotation θ_B and deflection δ_B at the free end of a cantilever beam AB supporting a parabolic load defined by the equation $q = q_0x^2/L^2$ (see figure).



Solution 9.5-11 **Cantilever beam (parabolic load)**

LOAD: $q = \dfrac{q_0x^2}{L^2}$ $\quad qdx$ = element of load

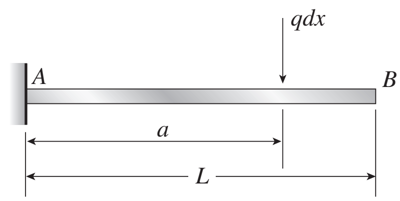


TABLE G-1, CASE 5 (Set a equal to x)

$$\theta_B = \int_0^L \frac{(qdx)(x^2)}{2EI} = \frac{1}{2EI}\int_0^L \left(\frac{q_0x^2}{L^2}\right)x^2dx$$

$$= \frac{q_0}{2EIL^2}\int_0^L x^4dx = \frac{q_0L^3}{10EI} \quad \longleftarrow$$

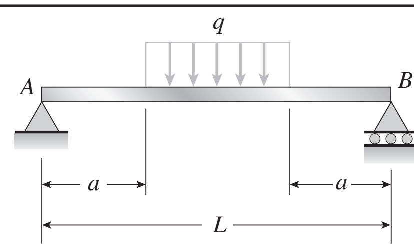
$$\delta_B = \int_0^L \frac{(qdx)(x^2)}{6EI}(3L - x)$$

$$= \frac{1}{6EI}\int_0^L \left(\frac{q_0x^2}{L^2}\right)(x^2)(3L - x)\,dx$$

$$= \frac{q_0}{6EIL^2}\int_0^L (x^4)(3L - x)\,dx = \frac{13q_0L^4}{180EI} \quad \longleftarrow$$

Problem 9.5-12 A simple beam AB supports a uniform load of intensity q acting over the middle region of the span (see figure).

Determine the angle of rotation θ_A at the left-hand support and the deflection δ_{max} at the midpoint.



Solution 9.5-12 **Simple beam (partial uniform load)**

LOAD: qdx = element of load

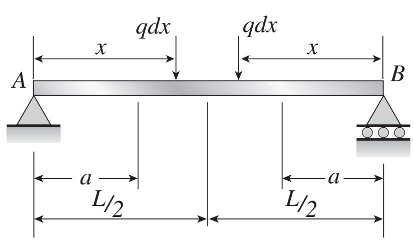


TABLE G-2, CASE 6 $\quad \theta_A = \dfrac{Pa(L-a)}{2EI}$

Replace P by qdx

Replace a by x

Integrate x from a to $L/2$

$$\theta_A = \int_a^{L/2} \frac{qdx}{2EI}(x)(L - x) = \frac{q}{2EI}\int_a^{L/2} (xL - x^2)\,dx$$

$$= \frac{q}{24EI}(L^3 - 6a^2L + 4a^3) \quad \longleftarrow$$

TABLE G-2, CASE 6 $\quad \delta_{max} = \dfrac{Pa}{24EI}(3L^2 - 4a^2)$

Replace P by qdx

Replace a by x

Integrate x from a to $L/2$

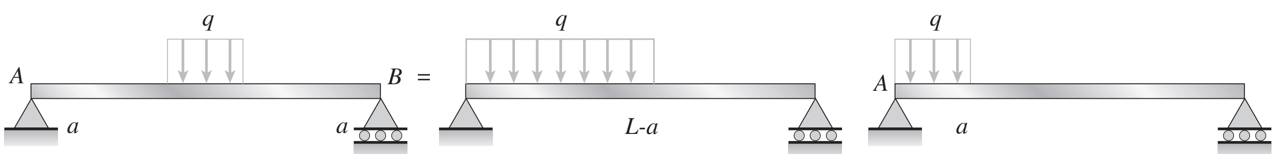
$$\delta_{\max} = \int_a^{L/2} \frac{q\,dx}{24EI}(x)(3L^2 - 4x^2)$$
$$= \frac{q}{24EI}\int_a^{L/2}(3L^2x - 4x^3)\,dx$$
$$= \frac{q}{384EI}(5L^4 - 24a^2L^2 + 16a^4) \quad \leftarrow$$

Alternate solution (not recommended; algebra is extremely lengthy)

Table G-2, Case 3

$$\theta_A = \frac{q(L-a)^2}{24LEI}[2L - (L-a)]^2 - \frac{qa^2}{24LEI}(2L-a)^2$$
$$= \frac{q}{24EI}(L^3 - 6La^2 + 4a^3) \quad \leftarrow$$

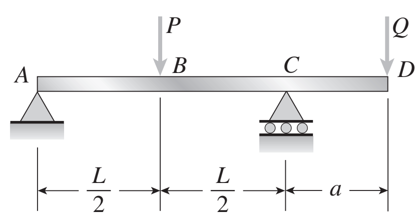
$$\delta_{\max} = \frac{q(L/2)}{24LEI}\Bigg[(L-a)^4 - 4L(L-a)^3 + 4L^2(L-a)^2 + 2(L-a)^2\left(\frac{L}{2}\right)^2 - 4L(L-a)\left(\frac{L}{2}\right)^2 + L\left(\frac{L}{2}\right)^3\Bigg]$$
$$= \frac{qa^2}{24LEI}\Bigg[-La^2 + 4L^2\left(\frac{L}{2}\right) + a^2\left(\frac{L}{2}\right) - 6L\left(\frac{L}{2}\right)^2 + 2\left(\frac{L}{2}\right)^3\Bigg]$$
$$\delta_{\max} = \frac{q}{384EI}(5L^4 - 24L^3a^2 + 16a^4) \quad \leftarrow$$



Problem 9.5-13 The overhanging beam $ABCD$ supports two concentrated loads P and Q (see figure).

(a) For what ratio P/Q will the deflection at point B be zero?

(b) For what ratio will the deflection at point D be zero?



Solution 9.5-13 Overhanging beam

(a) Deflection at point B

Table G-2, Cases 4 and 7

$$\delta_B = \frac{PL^3}{48EI} - Qa\left(\frac{L^2}{16EI}\right) = 0 \qquad \frac{P}{Q} = \frac{3a}{L} \quad \leftarrow$$

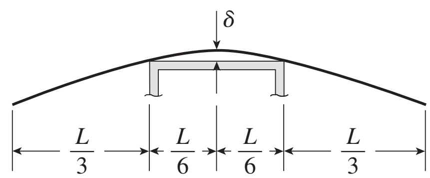
(b) Deflection at point D

Table G-2, Case 4; Table G-1, Case 4; Table G-2, Case 7

$$\delta_D = -\frac{PL^2}{16EI}(a) + \frac{Qa^3}{3EI} + Qa\left(\frac{L}{3EI}\right)(a) = 0$$
$$\frac{P}{Q} = \frac{16a(L+a)}{3L^2} \quad \leftarrow$$

Problem 9.5-14 A thin metal strip of total weight W and length L is placed across the top of a flat table of width $L/3$ as shown in the figure.

What is the clearance δ between the strip and the middle of the table? (The strip of metal has flexural rigidity EI.)



Solution 9.5-14 **Thin metal strip**

W = total weight $\quad q = \dfrac{W}{L}$

EI = flexural rigidity

FREE BODY DIAGRAM (the part of the strip above the table)

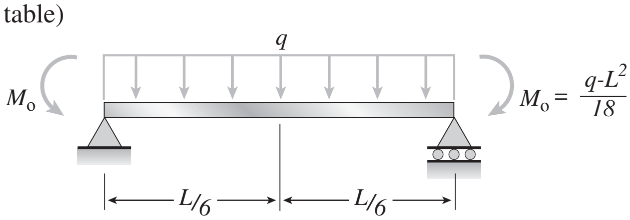


TABLE G-2, CASES 1 AND 10

$$\delta = -\frac{5q}{384EI}\left(\frac{L}{3}\right)^4 + \frac{M_0}{8EI}\left(\frac{L}{3}\right)^2$$

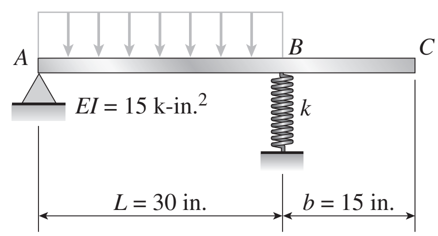
$$= -\frac{5qL^4}{31{,}104EI} + \frac{qL^4}{1296EI}$$

$$= \frac{19qL^4}{31{,}104EI}$$

But $q = \dfrac{W}{L}$: $\quad \therefore \delta = \dfrac{19WL^3}{31{,}104EI}$ ←

Problem 9.5-15 An overhanging beam ABC with flexural rigidity $EI = 15$ k-in.^{2} is supported by a pin support at A and by a spring of stiffness k at point B (see figure). Span AB has length $L = 30$ in. and carries a uniformly distributed load. The overhang BC has length $b = 15$ in.

For what stiffness k of the spring will the uniform load produce no deflection at the free end C?



Solution 9.5-15 **Overhanging beam with a spring support**

$EI = 15$ k-in.^{2} $\quad L = 30$ in. $\quad b = 15$ in.
q = intensity of uniform load

(1) Assume that point B is on a simple support

Table G-2, Case 1

$$\delta_C' = \theta_B b = \frac{qL^3}{24EI}(b) \quad \text{(upward deflection)}$$

(2) Assume that the spring shortens

$$R_B = \text{force in the spring} = \frac{qL}{2}$$

$$\delta_B = \frac{R_B}{k} = \frac{qL}{2k}$$

$$\delta_C'' = \delta_B\left(\frac{L+b}{L}\right) = \frac{q}{2k}(L+b) \quad \text{(downward deflection)}$$

(3) Deflection at point C (equal to zero)

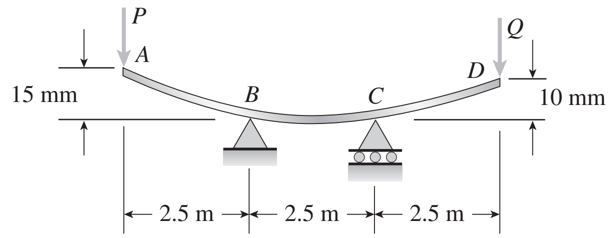
$$\delta_C = \delta_C' - \delta_C'' = \frac{qL^3 b}{24EI} - \frac{q}{2k}(L+b) = 0$$

Solve for k: $\quad k = \dfrac{12EI}{L^3}\left(1 + \dfrac{L}{b}\right)$ ←

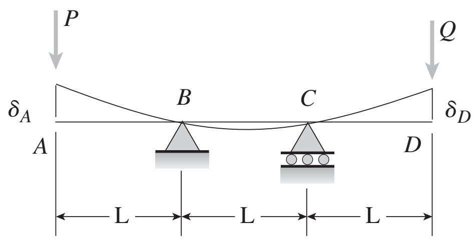
Substitute numerical values: $\quad k = 20$ lb/in. ←

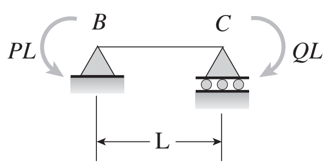
Problem 9.5-16 A beam $ABCD$ rests on simple supports at B and C (see figure). The beam has a slight initial curvature so that end A is 15 mm above the elevation of the supports and end D is 10 mm above.

What loads P and Q, acting at points A and D, respectively, will move points A and D downward to the level of the supports? (The flexural rigidity EI of the beam is $2.5 \times 10^6\ \text{N} \cdot \text{m}^2$.)



Solution 9.5-16 **Beam with initial curvature**





$\delta_A = 15$ mm
$\delta_D = 10$ mm
$EI = 2.5 \times 10^6\ \text{N} \cdot \text{m}^2$
$L = 2.5$ m

Table G-2, Case 7: $\theta_B = PL\left(\frac{L}{3EI}\right) + QL\left(\frac{L}{6EI}\right)$

$$= \frac{L^2}{6EI}(2P + Q)$$

Table G-1, Case 4: $\delta_A = \frac{PL^3}{3EI} + \theta_B L = \frac{L^3}{6EI}(4P + Q)$

$$4P + Q = \frac{6EI\delta_A}{L^3} \qquad \text{(Eq. 1)}$$

In a similar manner, $\delta_D = \frac{L^3}{6EI}(4Q + P)$

$$4Q + P = \frac{6EI\delta_D}{L^3} \qquad \text{(Eq. 2)}$$

Solve Eqs. (1) and (2):

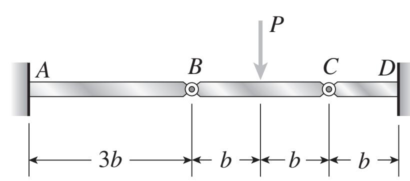
$$P = \frac{2EI}{5L^3}(4\delta_A - \delta_D) \quad Q = \frac{2EI}{5L^3}(4\delta_D - \delta_A) \leftarrow$$

Substitute numerical values:

$P = 3200$ N $\quad Q = 1600$ N $\leftarrow$

Problem 9.5-17 The compound beam $ABCD$ shown in the figure has fixed supports at ends A and D and consists of three members joined by pin connections at B and C.

Find the deflection δ under the load P.



Solution 9.5-17 **Compound beam**

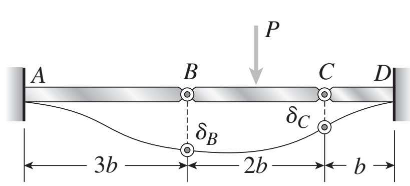


Table G-1, Case 4 and Table G-2, Case 4

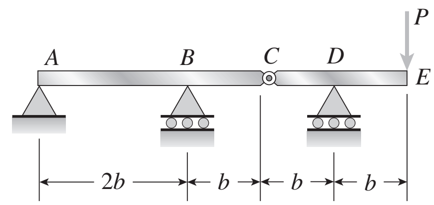
$$\delta_B = \frac{"PL^3"}{3EI} = \left(\frac{P}{2}\right)(3b)^3\left(\frac{1}{3EI}\right) = \frac{9Pb^3}{2EI}$$

$$\delta_C = \frac{"PL^3"}{3EI} = \left(\frac{P}{2}\right)(b^3)\left(\frac{1}{3EI}\right) = \frac{Pb^3}{6EI}$$

$$\delta = \frac{1}{2}(\delta_B + \delta_C) + \frac{P(2b)^3}{48EI} = \frac{5Pb^3}{2EI} \leftarrow$$

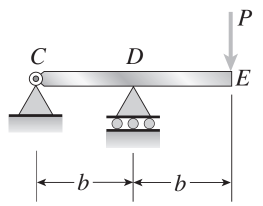
Problem 9.5-18 A compound beam *ABCDE* (see figure) consists of two parts (*ABC* and *CDE*) connected by a hinge at *C*.

Determine the deflection δ_E at the free end *E* due to the load *P* acting at that point.



Solution 9.5-18 Compound beam

BEAM *CDE* WITH A SUPPORT AT *C*

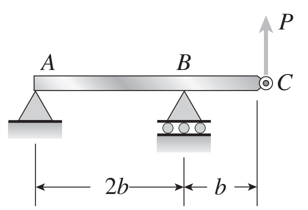


δ'_E = downward deflection of point *E*

$$\delta'_E = \frac{Pb^3}{3EI} + \theta'_D\, b = \frac{Pb^3}{3EI} + Pb\left(\frac{b}{3EI}\right)b$$

$$= \frac{2Pb^3}{3EI}$$

BEAM *ABC*



δ_C = upward deflection of point *C*

$$\delta_C = \frac{Pb^3}{3EI} + Q_B b = \frac{Pb^3}{3EI} + Pb\left(\frac{2b}{3EI}\right)b$$

$$= \frac{Pb^3}{EI}$$

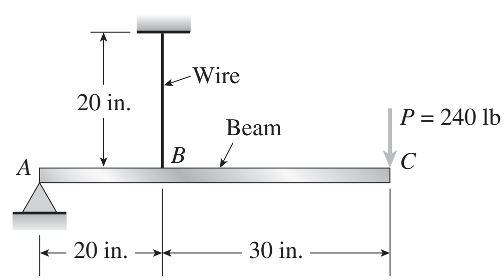
The upward deflection δ_C produces an equal downward displacement at point *E*. $\quad \therefore\ \delta''_E = \delta_C = \dfrac{Pb^3}{EI}$

DEFLECTION AT END *E*

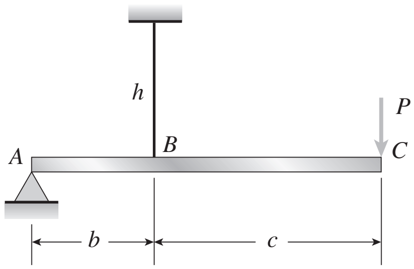
$$\delta_E = \delta'_E + \delta''_E = \frac{5Pb^3}{3EI} \quad \longleftarrow$$

Problem 9.5-19 A steel beam *ABC* is simply supported at *A* and held by a high-strength steel wire at *B* (see figure). A load $P = 240$ lb acts at the free end *C*. The wire has axial rigidity $EA = 1500 \times 10^3$ lb, and the beam has flexural rigidity $EI = 36 \times 10^6$ lb-in.^{2}

What is the deflection δ_C of point *C* due to the load *P*?



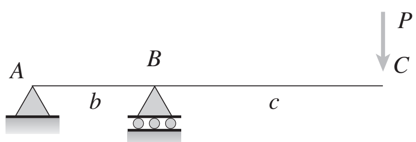
Solution 9.5-19 Beam supported by a wire



$P = 240$ lb $\quad b = 20$ in. $\quad c = 30$ in. $\quad h = 20$ in.

Beam: $EI = 36 \times 10^6$ lb-in.^{2}
Wire: $EA = 1500 \times 10^3$ lb

(1) ASSUME THAT POINT B IS ON A SIMPLE SUPPORT



$$\delta'_C = \frac{Pc^3}{3EI} + \theta'_B c = \frac{Pc^3}{3EI} + (Pc)\left(\frac{b}{3EI}\right)c$$

$$= \frac{Pc^2}{3EI}(b + c) \quad \text{(downward)}$$

(2) ASSUME THAT THE WIRE STRETCHES

T = tensile force in the wire

$$= \frac{P}{b}(b + c)$$

$$\delta_B = \frac{Th}{EA} = \frac{Ph(b + c)}{EAb}$$

$$\delta''_C = \delta_B\left(\frac{b + c}{b}\right) = \frac{Ph(b + c)^2}{EAb^2} \quad \text{(downward)}$$

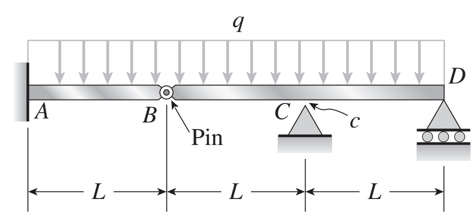
(3) DEFLECTION AT POINT C

$$\delta_C = \delta'_C + \delta''_C \quad = P(b + c)\left[\frac{c^2}{3EI} + \frac{h(b + c)}{EAb^2}\right] \longleftarrow$$

Substitute numerical values:
$\delta_C = 0.10$ in. $+ 0.02$ in. $= 0.12$ in. $\longleftarrow$

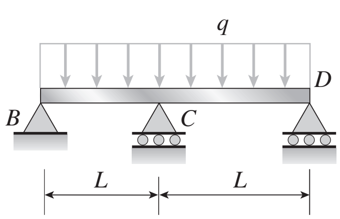
Problem 9.5-20 The compound beam shown in the figure consists of a cantilever beam AB (length L) that is pin-connected to a simple beam BD (length $2L$). After the beam is constructed, a clearance c exists between the beam and a support at C, midway between points B and D. Subsequently, a uniform load is placed along the entire length of the beam.

What intensity q of the load is needed to close the gap at C and bring the beam into contact with the support?



Solution 9.5-20 Compound beam

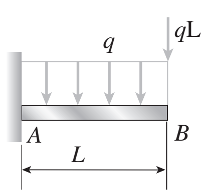
BEAM BCD WITH A SUPPORT AT B



$$\delta'_C = \frac{5q(2L)^4}{384EI}$$

$$= \frac{5qL^4}{24EI}$$

CANTILEVER BEAM AB



$$\delta_B = \frac{qL^4}{8EI} + \frac{(qL)L^3}{3EI}$$

$$= \frac{11qL^4}{24EI} \quad \text{(downward)}$$

δ''_C = downward displacement of point C due to δ_B

$$\delta''_C = \frac{1}{2}\,\delta_B = \frac{11qL^4}{48EI}$$

DOWNWARD DISPLACEMENT OF POINT C

$$\delta_C = \delta'_C + \delta''_C = \frac{5qL^4}{24EI} + \frac{11qL^4}{48EI} = \frac{7qL^4}{16EI}$$

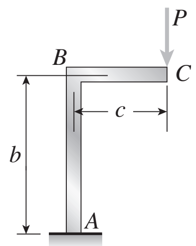
c = clearance $\qquad c = \delta_C = \dfrac{7qL^4}{16EI}$

INTENSITY OF LOAD TO CLOSE THE GAP

$$q = \frac{16EIc}{7L^4} \longleftarrow$$

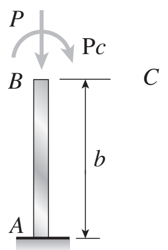
Problem 9.5-21 Find the horizontal deflection δ_h and vertical deflection δ_v at the free end C of the frame ABC shown in the figure. (The flexural rigidity EI is constant throughout the frame.)

Note: Disregard the effects of axial deformations and consider only the effects of bending due to the load P.



Solution 9.5-21 **Frame** *ABC*

MEMBER *AB*:



δ_h = horizontal deflection of point B

Table G-1, Case 6:

$$\delta_h = \frac{(Pc)b^2}{2EI} = \frac{Pcb^2}{2EI}$$

$$\theta_B = \frac{Pcb}{EI}$$

Since member BC does not change in length, δ_h is also the horizontal displacement of point C.

$$\therefore \ \delta_h = \frac{Pcb^2}{2EI} \quad \longleftarrow$$

MEMBER *BC* WITH *B* FIXED AGAINST ROTATION

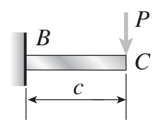


Table G-1, Case 4:

$$\delta'_C = \frac{Pc^3}{3EI}$$

VERTICAL DEFLECTION OF POINT *C*

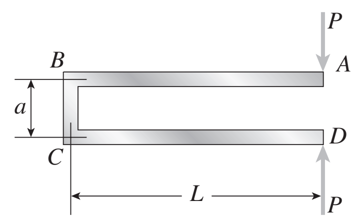
$$\delta_C = \delta_v = \delta'_C + \theta_B c = \frac{Pc^3}{3EI} + \frac{Pcb}{EI}(c)$$

$$= \frac{Pc^2}{3EI}(c + 3b)$$

$$\delta_v = \frac{Pc^2}{3EI}(c + 3b) \quad \longleftarrow$$

Problem 9.5-22 The frame $ABCD$ shown in the figure is squeezed by two collinear forces P acting at points A and D. What is the decrease δ in the distance between points A and D when the loads P are applied? (The flexural rigidity EI is constant throughout the frame.)

Note: Disregard the effects of axial deformations and consider only the effects of bending due to the loads P.



Solution 9.5-22 **Frame** *ABCD*

MEMBER *BC*:

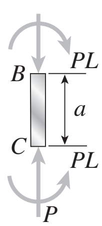


Table G-2, Case 10: $\quad \theta_B = \dfrac{(PL)a}{2EI} = \dfrac{PLa}{2EI}$

MEMBER *BA*:

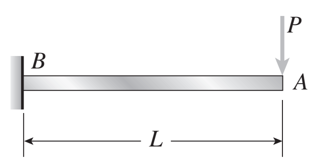


Table G-1, Case 4: $\quad \delta_A = \dfrac{PL^3}{3EI} + \theta_B L$

$$= \frac{PL^3}{3EI} + \frac{PLa}{2EI}(L)$$

$$= \frac{PL^2}{6EI}(2L + 3a)$$

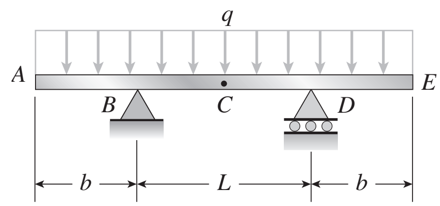
DECREASE IN DISTANCE BETWEEN POINTS *A* AND *D*

$$\delta = 2\delta_A = \frac{PL^2}{3EI}(2L + 3a) \quad \longleftarrow$$

Problem 9.5-23 A beam *ABCDE* has simple supports at *B* and *D* and symmetrical overhangs at each end (see figure). The center span has length *L* and each overhang has length *b*. *A* uniform load of intensity *q* acts on the beam.

(a) Determine the ratio b/L so that the deflection δ_C at the midpoint of the beam is equal to the deflections δ_A and δ_E at the ends.

(b) For this value of b/L, what is the deflection δ_C at the midpoint?



Solution 9.5-23 Beam with overhangs

BEAM *BCD*:

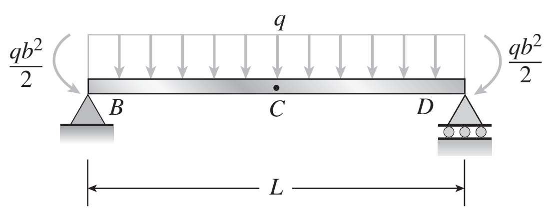


Table G-2, Case 1 and Case 10:

$$\theta_B = \frac{qL^3}{24EI} - \frac{qb^2}{2}\left(\frac{L}{2EI}\right) = \frac{qL}{24EI}(L^2 - 6b^2)$$

(clockwise is positive)

$$\delta_C = \frac{5qL^4}{384EI} - \frac{qb^2}{2}\left(\frac{L^2}{8EI}\right) = \frac{qL^2}{384EI}(5L^2 - 24b^2) \quad (1)$$

(downward is positive)

BEAM *AB:*

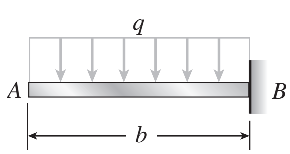


Table G-1, Case 1:

$$\delta_A = \frac{qb^4}{8EI} - \theta_B b = \frac{qb^4}{8EI} - \frac{qL}{24EI}(L^2 - 6b^2)b$$

$$= \frac{qb}{24EI}(3b^3 + 6b^2L - L^3)$$

(downward is positive)

DEFLECTION δ_C EQUALS DEFLECTION δ_A

$$\frac{qL^2}{384EI}(5L^2 - 24b^2) = \frac{qb}{24EI}(3b^3 + 6b^2L - L^3)$$

Rearrange and simplify the equation:

$$48b^4 + 96b^3L + 24b^2L^2 - 16bL^3 - 5L^4 = 0$$

or

$$48\left(\frac{b}{L}\right)^4 + 96\left(\frac{b}{L}\right)^3 + 24\left(\frac{b}{L}\right)^2 - 16\left(\frac{b}{L}\right) - 5 = 0$$

(a) RATIO $\dfrac{b}{L}$

Solve the preceding equation numerically:

$$\frac{b}{L} = 0.40301 \qquad \text{Say, } \frac{b}{L} = 0.4030 \quad \longleftarrow$$

(b) DEFLECTION δ_C (EQ. 1)

$$\delta_C = \frac{qL^2}{384EI}(5L^2 - 24b^2)$$

$$= \frac{qL^2}{384EI}[5L^2 - 24\,(0.40301\,L)^2]$$

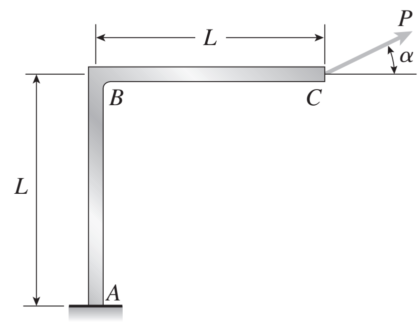
$$= 0.002870\,\frac{qL^4}{EI} \quad \longleftarrow$$

(downward deflection)

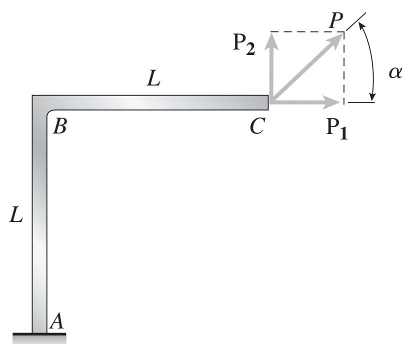
Problem 9.5-24 A frame *ABC* is loaded at point *C* by a force *P* acting at an angle α to the horizontal (see figure). Both members of the frame have the same length and the same flexural rigidity.

Determine the angle α so that the deflection of point *C* is in the same direction as the load. (Disregard the effects of axial deformations and consider only the effects of bending due to the load *P*.)

Note: A direction of loading such that the resulting deflection is in the same direction as the load is called a *principal direction*. For a given load on a planar structure, there are two principal directions, perpendicular to each other.



Solution 9.5-24 Principal directions for a frame



P_1 and P_2 are the components of the load P

$P_1 = P\cos\alpha$

$P_2 = P\sin\alpha$

IF P_1 ACTS ALONE $\quad \delta'_H = \dfrac{P_1L^3}{3EI}$ (to the right)

$$\delta'_v = \theta_B L = \left(\frac{P_1L^2}{2EI}\right)L = \frac{P_1L^3}{2EI}$$

(downward)

IF P_2 ACTS ALONE $\quad \delta''_H = \dfrac{P_2L^3}{2EI}$ (to the left)

$$\delta''_v = \frac{P_2L^3}{3EI} + \theta_B L = \frac{P_2L^3}{3EI} + \left(\frac{P_2L^2}{EI}\right)L = \frac{4P_2L^3}{3EI}$$

(upward)

DEFLECTIONS DUE TO THE LOAD P

$$\delta_H = \frac{P_1L^3}{3EI} - \frac{P_2L^3}{2EI} = \frac{L^3}{6EI}(2P_1 - 3P_2) \text{ (to the right)}$$

$$\delta_v = -\frac{P_1L^3}{2EI} + \frac{4P_2L^3}{3EI} = \frac{L^3}{6EI}(-3P_1 + 8P_2) \text{ (upward)}$$

$$\frac{\delta_v}{\delta_H} = \frac{-3P_1 + 8P_2}{2P_1 - 3P_2}$$

$$= \frac{-3P\cos\alpha + 8P\sin\alpha}{2P\cos\alpha - 3P\sin\alpha} = \frac{-3 + 8\tan\alpha}{2 - 3\tan\alpha}$$

PRINCIPAL DIRECTIONS

The deflection of point *C* is in the same direction as the load *P*.

$$\therefore \tan\alpha = \frac{P_2}{P_1} = \frac{\delta_v}{\delta_H} \quad \text{or} \quad \tan\alpha = \frac{-3 + 8\tan\alpha}{2 - 3\tan\alpha}$$

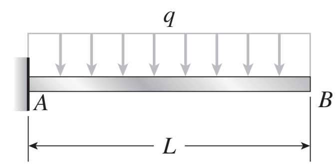
Rearrange and simplity: $\tan^2\alpha + 2\tan\alpha - 1 = 0$ (quadratic equation)

Solving, $\tan\alpha = -1 \pm \sqrt{2}$

$\alpha = 22.5°, \quad 112.5°, \quad -67.5°, \quad -157.5°,$ ⟵

Moment-Area Method

The problems for Section 9.6 are to be solved by the moment-area method. All beams have constant flexural rigidity EI.

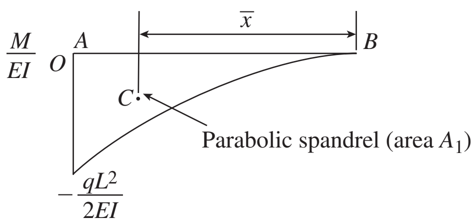


Problem 9.6-1 A cantilever beam AB is subjected to a uniform load of intensity q acting throughout its length (see figure).

Determine the angle of rotation θ_B and the deflection δ_B at the free end.

Solution 9.6-1 Cantilever beam (uniform load)

M/EI DIAGRAM:



ANGLE OF ROTATION

Use absolute values of areas.

Appendix D, Case 18: $A_1 = \dfrac{1}{3}(L)\left(\dfrac{qL^2}{2EI}\right) = \dfrac{qL^3}{6EI}$

$$\bar{x} = \frac{3L}{4}$$

$$\theta_{B/A} = \theta_B - \theta_A = A_1 = \frac{qL^3}{6EI}$$

$$\theta_A = 0 \qquad \theta_B = \frac{qL^3}{6EI} \qquad \text{(clockwise)} \longleftarrow$$

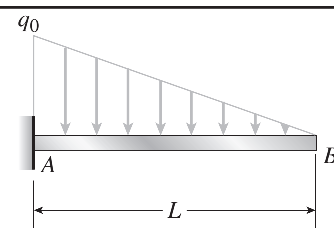
DEFLECTION

Q_1 = First moment of area A_1 with respect to B

$$Q_1 = A_1\bar{x} = \left(\frac{qL^3}{6EI}\right)\left(\frac{3L}{4}\right) = \frac{qL^4}{8EI}$$

$$\delta_B = Q_1 = \frac{qL^4}{8EI} \quad \text{(Downward)} \longleftarrow$$

(These results agree with Case 1, Table G-1.)

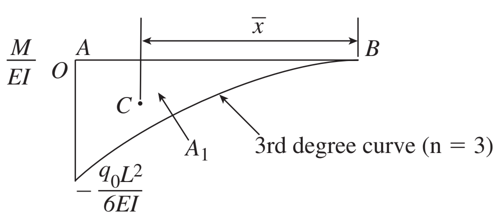


Problem 9.6-2 The load on a cantilever beam AB has a triangular distribution with maximum intensity q_0 (see figure).

Determine the angle of rotation θ_B and the deflection δ_B at the free end.

Solution 9.6-2 Cantilever beam (triangular load)

M/EI DIAGRAM



ANGLE OF ROTATION

Use absolute values of areas.

Appendix D, Case 20:

$$A_1 = \frac{bh}{n+1} = \frac{1}{4}(L)\left(\frac{q_0L^2}{6EI}\right) = \frac{q_0L^3}{24EI}$$

$$\bar{x} = \frac{b(n+1)}{n+2} = \frac{4L}{5}$$

$$\theta_{B/A} = \theta_B - \theta_A = A_1 = \frac{q_0L^3}{24\,EI}$$

$$\theta_A = 0 \qquad \theta_B = \frac{q_0L^3}{24EI} \qquad \text{(clockwise)} \longleftarrow$$

DEFLECTION

Q_1 = First moment of area A_1 with respect to B

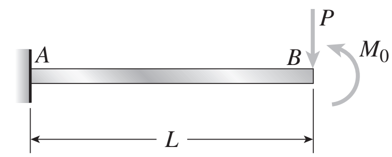
$$Q_1 = A_1\bar{x} = \left(\frac{q_0L^3}{24EI}\right)\left(\frac{4L}{5}\right) = \frac{q_0L^4}{30EI}$$

$$\delta_B = Q_1 = \frac{q_0L^4}{30EI} \quad \text{(Downward)} \longleftarrow$$

(These results agree with Case 8, Table G-1.)

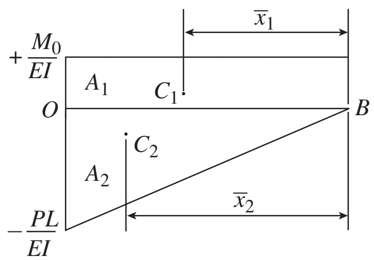
Problem 9.6-3 A cantilever beam AB is subjected to a concentrated load P and a couple M_0 acting at the free end (see figure).

Obtain formulas for the angle of rotation θ_B and the deflection δ_B at end B.



Solution 9.6-3 Cantilever beam (force P and couple M_0)

M/EI DIAGRAM



NOTE: A_1 is the *M/EI* diagram for M_0 (rectangle). A_2 is the *M/EI* diagram for P (triangle).

ANGLE OF ROTATION

Use the sign conventions for the moment-area theorems (page 628 of textbook).

$$A_1 = \frac{M_0 L}{EI} \quad \bar{x}_1 = \frac{L}{2} \quad A_2 = -\frac{PL^2}{2EI} \quad \bar{x}_2 = \frac{2L}{3}$$

$$A_0 = A_1 + A_2 = \frac{M_0 L}{EI} - \frac{PL^2}{2EI}$$

$$\theta_{B/A} = \theta_B - \theta_A = A_0 \qquad \theta_A = 0$$

$$\theta_B = A_0 = \frac{M_0 L}{EI} - \frac{PL^2}{2EI}$$

(θ_B is positive when counterclockwise)

DEFLECTION

Q = first moment of areas A_1 and A_2 with respect to point B

$$Q = A_1\bar{x}_1 + A_2\bar{x}_2 = \frac{M_0 L^2}{2EI} - \frac{PL^3}{3EI}$$

$$t_{B/A} = Q = \delta_B \qquad \delta_B = \frac{M_0 L^2}{2EI} - \frac{PL^3}{3EI}$$

(δ_B is positive when upward)

FINAL RESULTS

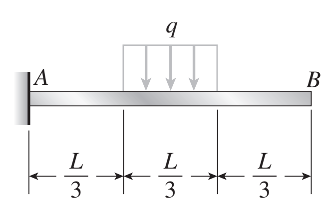
To match the sign conventions for θ_B and δ_B used in Appendix G, change the signs as follows.

$$\theta_B = \frac{PL^2}{2EI} - \frac{M_0 L}{EI} \text{ (positive clockwise)} \quad \longleftarrow$$

$$\delta_B = \frac{PL^3}{3EI} - \frac{M_0 L^2}{2EI} \text{ (positive downward)} \quad \longleftarrow$$

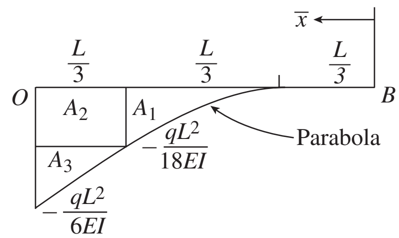
(These results agree with Cases 4 and 6, Table G-1.)

Problem 9.6-4 Determine the angle of rotation θ_B and the deflection δ_B at the free end of a cantilever beam AB with a uniform load of intensity q acting over the middle third of the length (see figure).



Solution 9.6-4 **Cantilever beam with partial uniform load**

M/EI DIAGRAM



ANGLE OF ROTATION

Use absolute values of areas. Appendix D, Cases 1, 6, and 18:

$$A_1 = \frac{1}{3}\left(\frac{L}{3}\right)\left(\frac{qL^2}{18EI}\right) = \frac{qL^3}{162EI} \qquad \bar{x}_1 = \frac{L}{3} + \frac{3}{4}\left(\frac{L}{3}\right) = \frac{7L}{12}$$

$$A_2 = \left(\frac{L}{3}\right)\left(\frac{qL^2}{18EI}\right) = \frac{qL^3}{54EI} \qquad \bar{x}_2 = \frac{2L}{3} + \frac{L}{6} = \frac{5L}{6}$$

$$A_3 = \frac{1}{2}\left(\frac{L}{3}\right)\left(\frac{qL^2}{9EI}\right) = \frac{qL^3}{54EI} \qquad \bar{x}_3 = \frac{2L}{3} + \frac{2}{3}\left(\frac{L}{3}\right) = \frac{8L}{9}$$

$$A_0 = A_1 + A_2 + A_3 = \frac{7qL^3}{162EI}$$

$$\theta_{B/A} = \theta_B - \theta_A = A_0$$

$$\theta_A = 0 \qquad \theta_B = \frac{7qL^3}{162EI} \quad \text{(clockwise)} \quad \longleftarrow$$

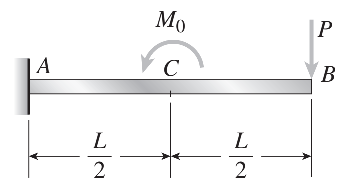
DEFLECTION

Q = first moment of area A_0 with respect to point B

$$Q = A_1\bar{x}_1 + A_2\bar{x}_2 + A_3\bar{x}_3 = \frac{23qL^4}{648EI}$$

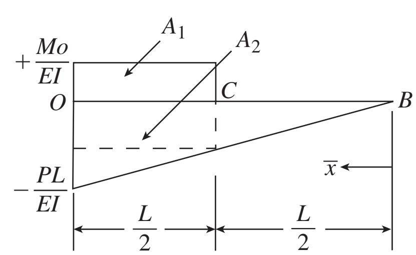
$$\delta_B = Q = \frac{23qL^4}{648EI} \quad \text{(Downward)} \quad \longleftarrow$$

Problem 9.6-5 Calculate the deflections δ_B and δ_C at points B and C, respectively, of the cantilever beam ACB shown in the figure. Assume $M_0 = 36$ k-in., $P = 3.8$ k, $L = 8$ ft, and $EI = 2.25 \times 10^9$ lb-in.^{2}



Solution 9.6-5 **Cantilever beam (force P and couple M_0)**

M/EI DIAGRAM



NOTE: A_1 is the *M/EI* diagram for M_0 (rectangle). A_2 is the *M/EI* diagram for P (triangle).

Use the sign conventions for the moment-area theorems (page 628 of textbook).

DEFLECTION δ_B

Q_B = first moment of areas A_1 and A_2 with respect to point B

$$= A_1\bar{x}_1 + A_2\bar{x}_2 = \left(\frac{M_0}{EI}\right)\left(\frac{L}{2}\right)\left(\frac{3L}{4}\right) - \frac{1}{2}\left(\frac{PL}{EI}\right)(L)\left(\frac{2L}{3}\right)$$

$$= \frac{L^2}{24EI}(9M_0 - 8PL)$$

$$t_{B/A} = Q_B = \delta_B \qquad \delta_B = \frac{L^2}{24EI}(9M_0 - 8PL)$$

(δ_B is positive when upward)

DEFLECTION δ_C

Q_C = first moment of area A_1 and left-hand part of A_2 with respect to point C

$$= \left(\frac{M_0}{EI}\right)\left(\frac{L}{2}\right)\left(\frac{L}{4}\right) - \left(\frac{PL}{2EI}\right)\left(\frac{L}{2}\right)\left(\frac{L}{4}\right) - \frac{1}{2}\left(\frac{PL}{2EI}\right)\left(\frac{L}{2}\right)\left(\frac{L}{3}\right)$$

$$= \frac{L^2}{48EI}(6M_0 - 5PL)$$

$$t_{C/A} = Q_C = \delta_C \qquad \delta_C = \frac{L^2}{48EI}(6M_0 - 5PL)$$

(δ_C is positive when upward)

ASSUME DOWNWARD DEFLECTIONS ARE POSITIVE
(change the signs of δ_B and δ_C)

$$\delta_B = \frac{L^2}{24EI}(8PL - 9M_0) \quad \longleftarrow$$

$$\delta_C = \frac{L^2}{48EI}(5PL - 6M_0) \quad \longleftarrow$$

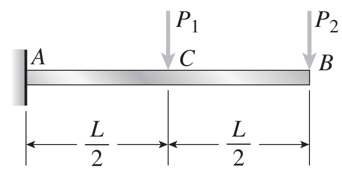
SUBSTITUTE NUMERICAL VALUES:
$M_0 = 36$ k-in. $\quad P = 3.8$ k
$L = 8$ ft $= 96$ in. $\quad EI = 2.25 \times 10^6$ k-in.^{2}

$$\delta_B = 0.4981 \text{ in.} - 0.0553 \text{ in.} = 0.443 \text{ in.} \quad \longleftarrow$$

$$\delta_C = 0.1556 \text{ in.} - 0.0184 \text{ in.} = 0.137 \text{ in.} \quad \longleftarrow$$

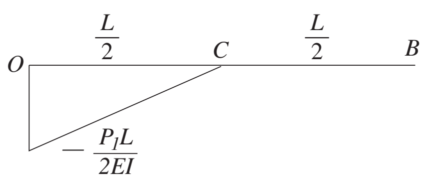
Problem 9.6-6 A cantilever beam ACB supports two concentrated loads P_1 and P_2 as shown in the figure.

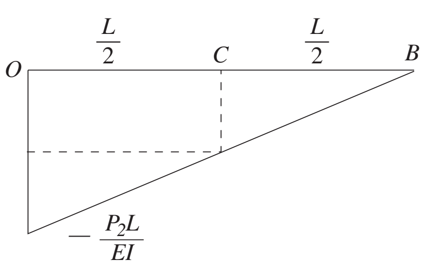
Calculate the deflections δ_B and δ_C at points B and C, respectively. Assume $P_1 = 10$ kN, $P_2 = 5$ kN, $L = 2.6$ m, $E = 200$ GPa, and $I = 20.1 \times 10^6$ mm^4.



Solution 9.6-6 **Cantilever beam (forces P_1 and P_2)**

M/EI DIAGRAMS





$P_1 = 10$ kN $\quad P_2 = 5$ kN $\quad L = 2.6$ m
$E = 200$ GPa $\quad I = 20.1 \times 10^6$ mm^4

Use absolute values of areas.

DEFLECTION δ_B

$\delta_B = t_{B/A} = Q_B =$ first moment of areas with respect to point B

$$\delta_B = \frac{1}{2}\left(\frac{P_1L}{2EI}\right)\left(\frac{L}{2}\right)\left(\frac{L}{2} + \frac{L}{3}\right) + \frac{1}{2}\left(\frac{P_2L}{EI}\right)(L)\left(\frac{2L}{3}\right)$$

$$= \frac{5P_1L^3}{48EI} + \frac{P_2L^3}{3EI} \quad \text{(downward)} \quad \longleftarrow$$

DEFLECTION δ_C

$\delta_C = t_{C/A} = Q_C =$ first moment of areas to the left of point C with respect to point C

$$\delta_c = \frac{1}{2}\left(\frac{P_1L}{2EI}\right)\left(\frac{L}{2}\right)\left(\frac{L}{3}\right) + \left(\frac{P_2L}{2EI}\right)\left(\frac{L}{2}\right)\left(\frac{L}{4}\right)$$

$$+ \frac{1}{2}\left(\frac{P_2L}{2EI}\right)\left(\frac{L}{2}\right)\left(\frac{L}{3}\right)$$

$$= \frac{P_1L^3}{24EI} + \frac{5P_2L^3}{48EI} \quad \text{(downward)} \quad \longleftarrow$$

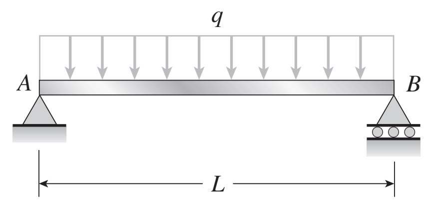
SUBSTITUTE NUMERICAL VALUES:

$$\delta_B = 4.554 \text{ mm} + 7.287 \text{ mm} = 11.84 \text{ mm} \quad \longleftarrow$$

$$\delta_C = 1.822 \text{ mm} + 2.277 \text{ mm} = 4.10 \text{ mm} \quad \longleftarrow$$

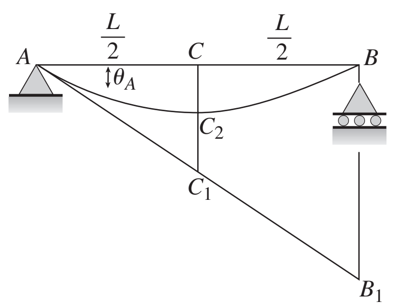
(deflections are downward)

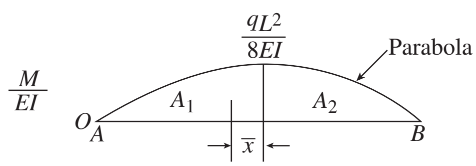
Problem 9.6-7 Obtain formulas for the angle of rotation θ_A at support A and the deflection δ_{max} at the midpoint for a simple beam AB with a uniform load of intensity q (see figure).



Solution 9.6-7 Simple beam with a uniform load

DEFLECTION CURVE AND M/EI DIAGRAM





δ_{max} = maximum deflection (distance CC_2)

Use absolute values of areas.

ANGLE OF ROTATION AT END A

Appendix D, Case 17:

$$A_1 = A_2 = \frac{2}{3}\left(\frac{L}{2}\right)\left(\frac{qL^2}{8\,EI}\right) = \frac{qL^3}{24\,EI}$$

$$\bar{x}_1 = \frac{3}{8}\left(\frac{L}{2}\right) = \frac{3L}{16}$$

$t_{B/A} = BB_1$ = first moment of areas A_1 and A_2 with respect to point B

$$= (A_1 + A_2)\left(\frac{L}{2}\right) = \frac{qL^4}{24\,EI}$$

$$\theta_A = \frac{BB_1}{L} = \frac{qL^3}{24\,EI} \text{ (clockwise)} \longleftarrow$$

DEFLECTION δ_{max} AT THE MIDPOINT C

$$\text{Distance } CC_1 = \frac{1}{2}(BB_1) = \frac{qL^4}{48\,EI}$$

$t_{C_2}/A = C_2C_1$ = first moment of area A_1 with respect to point C

$$= A_1\,\bar{x}_1 = \left(\frac{qL^3}{24EI}\right)\left(\frac{3L}{16}\right) = \frac{qL^4}{128\,EI}$$

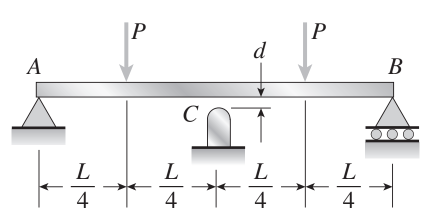
$$\delta_{max} = CC_2 = CC_1 - C_2C_1 = \frac{qL^4}{48EI} - \frac{qL^4}{128EI}$$

$$= \frac{5qL^4}{384\,EI} \text{ (downward)} \longleftarrow$$

(These results agree with Case 1 of Table G-2.)

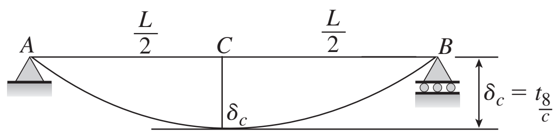
Problem 9.6-8 A simple beam AB supports two concentrated loads P at the positions shown in the figure. A support C at the midpoint of the beam is positioned at distance d below the beam before the loads are applied.

Assuming that $d = 10$ mm, $L = 6$ m, $E = 200$ GPa, and $I = 198 \times 10^6$ mm^4, calculate the magnitude of the loads P so that the beam just touches the support at C.

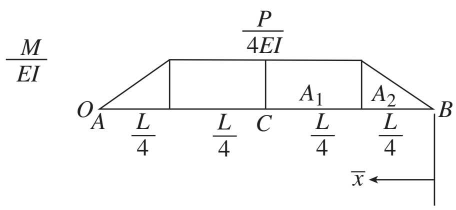


Solution 9.6-8 Simple beam with two equal loads

DEFLECTION CURVE AND M/EI DIAGRAM



δ_c = deflection at the midpoint C



$$A_1 = \frac{PL^2}{16\,EI} \qquad \bar{x}_1 = \frac{3L}{8}$$

$$A_2 = \frac{PL^2}{32EI} \qquad \bar{x}_2 = \frac{L}{6}$$

Use absolute values of areas.

DEFLECTION δ_c AT MIDPOINT OF BEAM

At point C, the deflection curve is horizontal.

$\delta_c = t_{B/C}$ = first moment of area between B and C with respect to B

$$= A_1\bar{x}_1 + A_2\bar{x}_2 = \frac{PL^2}{16EI}\left(\frac{3L}{8}\right) + \frac{PL^2}{32EI}\left(\frac{L}{6}\right)$$

$$= \frac{11PL^3}{384\,EI}$$

d = gap between the beam and the support at C

MAGNITUDE OF LOAD TO CLOSE THE GAP

$$\delta = d = \frac{11PL^3}{384EI} \qquad P = \frac{384EId}{11L^3} \quad \longleftarrow$$

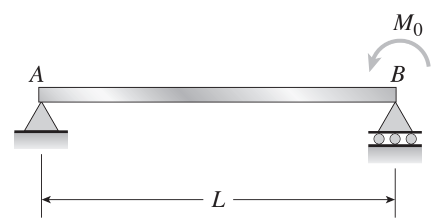
SUBSTITUTE NUMERICAL VALUES:

$d = 10$ mm $\quad L = 6$ m $\quad E = 200$ GPa

$I = 198 \times 10^6\ \text{mm}^4 \quad P = 64$ kN $\longleftarrow$

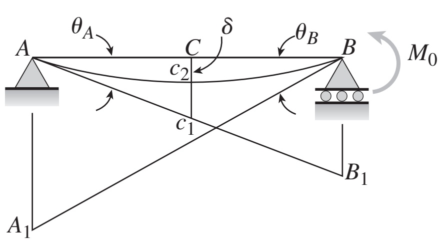
Problem 9.6-9 A simple beam AB is subjected to a load in the form of a couple M_0 acting at end B (see figure).

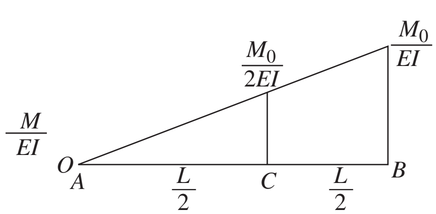
Determine the angles of rotation θ_A and θ_B at the supports and the deflection δ at the midpoint.



Solution 9.6-9 Simple beam with a couple M_0

DEFLECTION CURVE AND M/EI DIAGRAM





δ = deflection at the midpoint C

δ = distance CC_2

Use absolute values of areas.

ANGLE OF ROTATION θ_A

$t_{B/A} = BB_1$ = first moment of area between A and B with respect to B

$$= \frac{1}{2}\left(\frac{M_0}{EI}\right)(L)\left(\frac{L}{3}\right) = \frac{M_0L^2}{6EI}$$

$$\theta_A = \frac{BB_1}{L} = \frac{M_0L}{6\,EI} \text{ (clockwise)} \quad \longleftarrow$$

Angle of rotation θ_B

$t_{A/B} = AA_1$ = first moment of area between A and B with respect to A

$$= \frac{1}{2}\left(\frac{M_0}{EI}\right)(L)\left(\frac{2L}{3}\right) = \frac{M_0 L^2}{3\,EI}$$

$$\theta_B = \frac{AA_1}{L} = \frac{M_0 L}{3EI} \text{ (Counterclockwise)} \longleftarrow$$

Deflection δ at the midpoint C

Distance $\quad CC_1 = \frac{1}{2}(BB_1) = \frac{M_0 L^2}{12EI}$

$t_{c_2}/A = C_2C_1 =$ first moment of area between A and C with respect to C

$$= \frac{1}{2}\left(\frac{M_0}{2EI}\right)\left(\frac{L}{2}\right)\left(\frac{L}{6}\right) = \frac{M_0 L^2}{48\,EI}$$

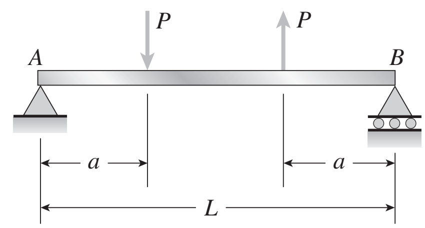
$$\delta = CC_1 - C_2C_1 = \frac{M_0 L^2}{12EI} - \frac{M_0 L^2}{48\,EI}$$

$$= \frac{M_0 L^2}{16EI} \quad \text{(Downward)} \longleftarrow$$

(These results agree with Case 7 of Table G-2.)

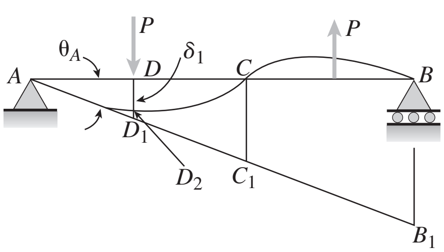
Problem 9.6-10 The simple beam AB shown in the figure supports two equal concentrated loads P, one acting downward and the other upward.

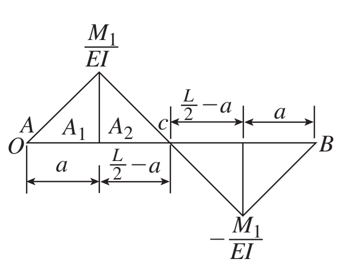
Determine the angle of rotation θ_A at the left-hand end, the deflection δ_1 under the downward load, and the deflection δ_2 at the midpoint of the beam.



Solution 9.6-10 Simple beam with two loads

Because the beam is symmetric and the load is antisymmetric, the deflection at the midpoint is zero.





$$\therefore \delta_2 = 0 \longleftarrow$$

$$\frac{M_1}{EI} = \frac{Pa(L-2a)}{LEI}$$

$$A_1 = \frac{1}{2}\left(\frac{M_1}{EI}\right)(a) = \frac{Pa^2(L-2a)}{2LEI}$$

$$A_2 = \frac{1}{2}\left(\frac{M_1}{EI}\right)\left(\frac{L}{2} - a\right) = \frac{Pa(L-2a)^2}{4LEI}$$

Angle of rotation θ_A at end A

$t_{C/A} = CC_1$ = first moment of area between A and C with respect to C

$$= A_1\left(\frac{L}{2} - a + \frac{a}{3}\right) + A_2\left(\frac{2}{3}\right)\left(\frac{L}{2} - a\right)$$

$$= \frac{Pa(L-a)(L-2a)}{12EI}$$

$$\theta_A = \frac{CC_1}{L/2} = \frac{Pa(L-a)(L-2a)}{6LEI} \text{ (clockwise)} \longleftarrow$$

DEFLECTION δ_1 UNDER THE DOWNWARD LOAD

Distance $$DD_1 = \left(\frac{a}{L/2}\right)(CC_1)$$

$$= \frac{Pa^2\,(L-a)\,(L-2a)}{6LEI}$$

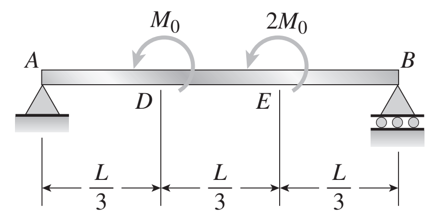
$t_{D_2}/A = D_2D_1 =$ first moment of area between A and D with respect to D

$$= A_1\left(\frac{a}{3}\right) = \frac{Pa^3(L-2a)}{6LEI}$$

$$\delta_1 = DD_1 - D_2D_1$$

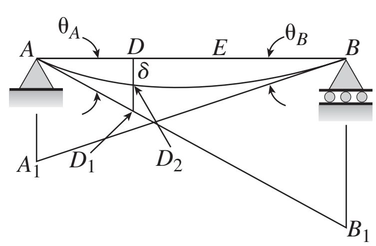
$$= \frac{Pa^2(L-2a)^2}{6LEI} \quad \text{(Downward)} \longleftarrow$$

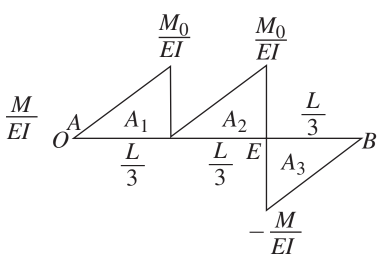
Problem 9.6-11 A simple beam AB is subjected to couples M_0 and $2M_0$ as shown in the figure. Determine the angles of rotation θ_A and θ_B at the the beam and the deflection δ at point D where the load M_0 is applied.



Solution 9.6-11 **Simple beam with two couples**

DEFLECTION CURVE AND M/EI DIAGRAM





$$A_1 = A_2 = \frac{1}{2}\left(\frac{M_0}{EI}\right)\left(\frac{L}{3}\right) = \frac{M_0L}{6EI} \qquad A_3 = -\frac{M_0L}{6EI}$$

ANGLE OF ROTATION θ_A AT END A

$t_{B/A} = BB_1 =$ first moment of area between A and B with respect to B

$$= A_1\left(\frac{2L}{3} + \frac{L}{9}\right) + A_2\left(\frac{L}{3} + \frac{L}{9}\right) + A_3\left(\frac{2L}{9}\right) = \frac{M_0L^2}{6EI}$$

$$\theta_A = \frac{BB_1}{L} = \frac{M_0L}{6EI} \quad \text{(clockwise)} \longleftarrow$$

ANGLE OF ROTATION θ_B AT END B

$t_{A/B} = AA_1 =$ first moment of area between A and B with respect to A

$$= A_1\left(\frac{2L}{9}\right) + A_2\left(\frac{L}{3} + \frac{2L}{9}\right) + A_3\left(\frac{2L}{3} + \frac{L}{9}\right) = 0$$

$$\theta_B = \frac{AA_1}{L} = 0 \longleftarrow$$

DEFLECTION δ AT POINT D

Distance $$DD_1 = \frac{1}{3}(BB_1) = \frac{M_0L^2}{18\,EI}$$

$t_{D_2}/A = D_2D_1 =$ first moment of area between A and D with respect to D

$$= A_1\left(\frac{L}{9}\right) = \frac{M_0L^2}{54EI}$$

$$\delta = DD_1 - D_2D_1 = \frac{M_0L^2}{27\,EI} \quad \text{(downward)} \longleftarrow$$

NOTE: This deflection is also the maximum deflection.

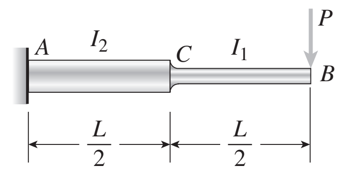
Nonprismatic Beams

Problem 9.7-1 The cantilever beam *ACB* shown in the figure has moments of inertia I_2 and I_1 in parts *AC* and *CB*, respectively.

(a) Using the method of superposition, determine the deflection δ_B at the free end due to the load *P*.

(b) Determine the ratio *r* of the deflection δ_B to the deflection δ_1 at the free end of a prismatic cantilever with moment of inertia I_1 carrying the same load.

(c) Plot a graph of the deflection ratio *r* versus the ratio I_2/I_1 of the moments of inertia. (Let I_2/I_1 vary from 1 to 5.)

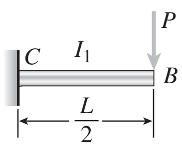


Solution 9.7-1 **Cantilever beam (nonprismatic)**

Use the method of superposition.

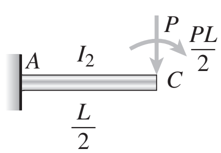
(a) Deflection δ_B at the free end

(1) Part *CB* of the beam:



$$(\delta_B)_1 = \frac{P}{3EI_1}\left(\frac{L}{2}\right)^3 = \frac{PL^3}{24EI_1}$$

(2) Part *AC* of the beam:



$$\delta_C = \frac{P(L/2)^3}{3EI_2} + \frac{(PL/2)(L/2)^2}{2EI_2} = \frac{5PL^3}{48EI_2}$$

$$\theta_C = \frac{P(L/2)^2}{2EI_2} + \frac{(PL/2)(L/2)}{EI_2} = \frac{3PL^2}{8EI_2}$$

$$(\delta_B)_2 = \delta_C + \theta_C\left(\frac{L}{2}\right) = \frac{7PL^3}{24EI_2}$$

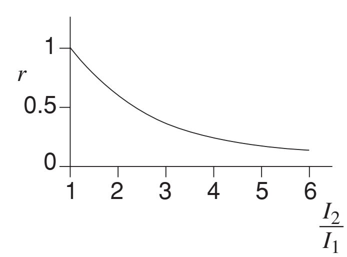
(3) Total deflection at point *B*

$$\delta_B = (\delta_B)_1 + (\delta_B)_2 = \frac{PL^3}{24EI_1}\left(1 + \frac{7I_1}{I_2}\right) \longleftarrow$$

(b) Prismatic beam $\quad \delta_1 = \dfrac{PL^3}{3\,EI_1}$

Ratio: $\quad r = \dfrac{\delta_B}{\delta_1} = \dfrac{1}{8}\left(1 + \dfrac{7I_1}{I_2}\right) \longleftarrow$

(c) Graph of ratio



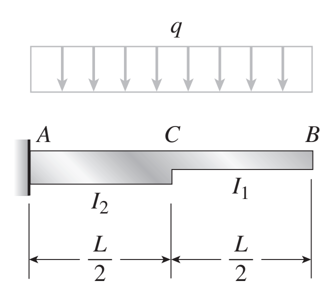
$\frac{I_2}{I_1}$	r
1	1.00
2	0.56
3	0.42
4	0.34
5	0.30

Problem 9.7-2 The cantilever beam *ACB* shown in the figure supports a uniform load of intensity *q* throughout its length. The beam has moments of inertia I_2 and I_1 in parts *AC* and *CB*, respectively.

(a) Using the method of superposition, determine the deflection δ_B at the free end due to the uniform load.

(b) Determine the ratio *r* of the deflection δ_B to the deflection δ_1 at the free end of a prismatic cantilever with moment of inertia I_1 carrying the same load.

(c) Plot a graph of the deflection ratio *r* versus the ratio I_2/I_1 of the moments of inertia. (Let I_2/I_1 vary from 1 to 5.)

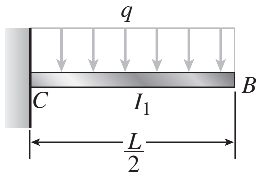


Solution 9.7-2 **Cantilever beam (nonprismatic)**

Use the method of superposition

(a) DEFLECTION δ_B AT THE FREE END

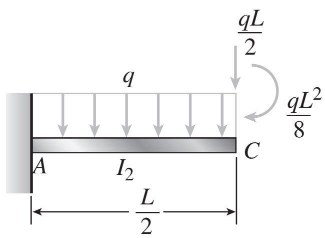
(1) Part *CB* of the beam:



$$(\delta_B)_1 = \frac{q}{8EI_1}\left(\frac{L}{2}\right)^4 = \frac{qL^4}{128EI_1}$$

(2) Part *AC* of the beam:

$$\delta_c = \frac{q(L/2)^4}{8EI_2} + \frac{\left(\frac{qL}{2}\right)(L/2)^3}{3EI_2} + \frac{\left(\frac{qL^2}{8}\right)\left(\frac{L}{2}\right)^2}{2EI_8} = \frac{17qL^4}{384EI_2}$$



$$\theta_C = \frac{q(L/2)^3}{6EI_2} + \frac{(qL/2)(L/2)^2}{2EI_2} + \frac{(qL^2/8)(L/2)}{EI_2}$$
$$= \frac{7qL^3}{48EI_2}$$

$$(\delta_B)_2 = \delta_C + \theta_C\left(\frac{L}{2}\right) = \frac{15qL^4}{128EI_2}$$

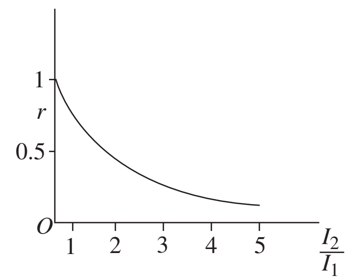
(3) Total deflection at point *B*

$$\delta_B = (\delta_B)_1 + (\delta_B)_2 = \frac{qL^4}{128EI_1}\left(1 + \frac{15I_1}{I_2}\right) \quad \longleftarrow$$

(b) PRISMATIC BEAM $\quad \delta_1 = \frac{qL^4}{8EI_1}$

Ratio: $\quad r = \frac{\delta_B}{\delta_1} = \frac{1}{16}\left(1 + \frac{15I_1}{I_2}\right) \quad \longleftarrow$

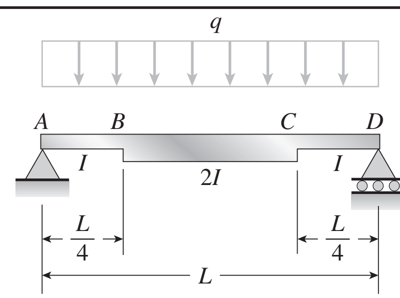
(c) GRAPH OF RATIO



$\frac{I_2}{I_1}$	r
1	1.00
2	0.53
3	0.38
4	0.30
5	0.25

Problem 9.7-3 A simple beam *ABCD* has moment of inertia *I* near the supports and moment of inertia 2*I* in the middle region, as shown in the figure. A uniform load of intensity *q* acts over the entire length of the beam.

Determine the equations of the deflection curve for the left-hand half of the beam. Also, find the angle of rotation θ_A at the left-hand support and the deflection δ_{max} at the midpoint.

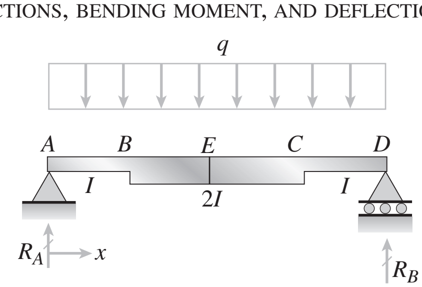


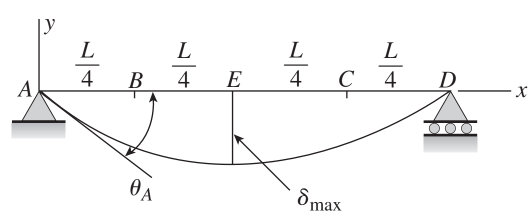
Solution 9.7-3 **Simple beam (nonprismatic)**

Use the bending-moment equation (Eq. 9-12a).

REACTIONS, BENDING MOMENT, AND DEFLECTION CURVE

$$R_A = R_B = \frac{qL}{2} \qquad M = Rx - \frac{qx^2}{2} = \frac{qLx}{2} - \frac{qx^2}{2}$$





BENDING-MOMENT EQUATIONS FOR THE LEFT-HAND HALF OF THE BEAM

$$EIv'' = M = \frac{qLx}{2} - \frac{qx^2}{2} \qquad \left(0 \le x \le \frac{L}{4}\right) \tag{1}$$

$$E(2I)v'' = M = \frac{qLx}{2} - \frac{qx^2}{2} \qquad \left(\frac{L}{4} \le x \le \frac{L}{2}\right) \tag{2}$$

INTEGRATE EACH EQUATION

$$EIv' = \frac{qLx^2}{4} - \frac{qx^3}{6} + C_1 \qquad \left(0 \le x \le \frac{L}{4}\right) \tag{3}$$

$$2EIv' = \frac{qLx^2}{4} - \frac{qx^3}{6} + C_2 \qquad \left(\frac{L}{4} \le x \le \frac{L}{2}\right) \tag{4}$$

B.C. 1 Symmetry: $v'\left(\frac{L}{2}\right) = 0$

From Eq. (4): $C_2 = -\frac{qL^3}{24}$

$$2EIv' = \frac{qLx^2}{4} - \frac{qx^3}{6} - \frac{qL^3}{24} \qquad \left(\frac{L}{4} \le x \le \frac{L}{2}\right) \tag{5}$$

SLOPE AT POINT B (FROM THE RIGHT)

Substitute $x = \frac{L}{4}$ into Eq. (5):

$$EIv'_B = -\frac{11\,qL^3}{768} \tag{6}$$

B.C. 2 CONTINUITY OF SLOPES AT POINT B

$(v'_B)_{\text{Left}} = (v'_B)_{\text{Right}}$

From Eqs. (3) and (6):

$$\frac{qL}{4}\left(\frac{L}{4}\right)^2 - \frac{q}{6}\left(\frac{L}{4}\right)^3 + C_1 = -\frac{11qL^3}{768} \qquad \therefore\ C_1 = -\frac{7qL^3}{256}$$

SLOPES OF THE BEAM (from Eqs. 3 and 5)

$$EIv' = \frac{qLx^2}{4} - \frac{qx^3}{6} - \frac{7qL^3}{256} \qquad \left(0 \le x \le \frac{L}{4}\right) \tag{7}$$

$$EIv' = \frac{qLx^2}{8} - \frac{qx^3}{12} - \frac{qL^3}{48} \qquad \left(\frac{L}{4} \le x \le \frac{L}{2}\right) \tag{8}$$

ANGLE OF ROTATION θ_A (FROM EQ. 7)

$$\theta_A = -v'(0) = \frac{7qL^3}{256EI} \text{ (positive clockwise)} \quad \longleftarrow$$

INTEGRATE EQS. (7) AND (8)

$$EIv = \frac{qLx^3}{12} - \frac{qx^4}{24} - \frac{7qL^3x}{256} + C_3 \qquad \left(0 \le x \le \frac{L}{4}\right) \tag{9}$$

$$EIv = \frac{qLx^3}{24} - \frac{qx^4}{48} - \frac{qL^3x}{48} + C_4 \qquad \left(\frac{L}{4} \le x \le \frac{L}{2}\right) \tag{10}$$

B.C. 3 Deflection at support A

$v(0) = 0$ From Eq. (9): $C_3 = 0$

DEFLECTION AT POINT B (FROM THE LEFT)

Substitute $x = \frac{L}{4}$ into Eq. (9) with $C_3 = 0$:

$$EIv_B = -\frac{35\,qL^4}{6144} \tag{11}$$

B.C. 4 Continuity of deflections at point B

$(v_B)_{\text{Right}} = (v_B)_{\text{Left}}$

From Eqs. (10) and (11):

$$\frac{qL}{24}\left(\frac{L}{4}\right)^3 - \frac{q}{48}\left(\frac{L}{4}\right)^4 - \frac{qL^3}{48}\left(\frac{L}{4}\right) + C_4 = -\frac{35qL^4}{6144}$$

$$\therefore\ C_4 = -\frac{13qL^4}{12{,}288}$$

DEFLECTIONS OF THE BEAM (FROM EQS. 9 AND 10)

$$v = -\frac{qx}{768EI}(21L^3 - 64Lx^2 + 32x^3) \qquad \left(0 \le x \le \frac{L}{4}\right) \quad \longleftarrow$$

$$v = -\frac{q}{12{,}288EI}(13L^4 + 256L^3x - 512Lx^3 + 256x^4) \qquad \left(\frac{L}{4} \le x \le \frac{L}{2}\right) \quad \longleftarrow$$

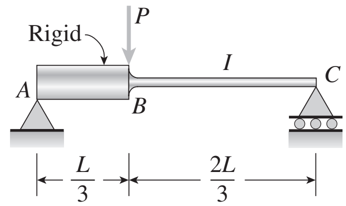
MAXIMUM DEFLECTION (AT THE MIDPOINT E)

(From the preceding equation for v.)

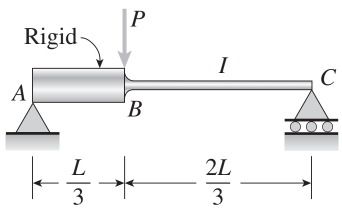
$$\delta_{\max} = -v\left(\frac{L}{2}\right) = \frac{31qL^4}{4096EI} \quad \text{(positive downward)} \quad \longleftarrow$$

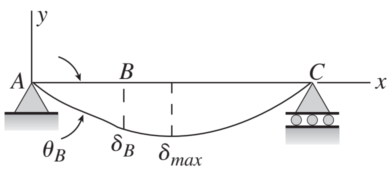
Problem 9.7-4 A beam ABC has a rigid segment from A to B and a flexible segment with moment of inertia I from B to C (see figure). A concentrated load P acts at point B.

Determine the angle of rotation θ_A of the rigid segment, the deflection δ_B at point B, and the maximum deflection δ_{max}.



Solution 9.7-4 Simple beam with a rigid segment





From A to B

$$v = -\frac{3\delta_B x}{L} \quad \left(0 \le x \le \frac{L}{3}\right) \tag{1}$$

$$v' = -\frac{3\delta_B}{L} \quad \left(0 \le x \le \frac{L}{3}\right) \tag{2}$$

From B to C

$$EIv'' = M = \frac{PL}{3} - \frac{Px}{3} \tag{3}$$

$$EIv' = \frac{PLx}{3} - \frac{Px^2}{6} + C_1$$

B.C. 1 At $x = L/3$, $\quad v' = -\dfrac{3\delta_B}{L}$

$$\therefore C_1 = -\frac{5PL^2}{54} - \frac{3EI\delta_B}{L}$$

$$EIv' = \frac{PLx}{3} - \frac{Px^2}{6} - \frac{5PL^2}{54} - \frac{3EI\delta_B}{L} \quad \left(\frac{L}{3} \le x \le L\right) \tag{4}$$

$$EIv = \frac{PLx^2}{6} - \frac{Px^3}{18} - \frac{5PL^2x}{54} - \frac{3EI\delta_B x}{L} + C_2 \quad \left(\frac{L}{3} \le x \le L\right)$$

B.C. 2 $v(L) = 0 \quad \therefore C_2 = -\dfrac{PL^3}{54} + 3EI\delta_B$

$$EIv = \frac{PLx^2}{6} - \frac{Px^3}{18} - \frac{5PL^2x}{54} - \frac{3EI\delta_B x}{L} - \frac{PL^2}{54} + 3EI\delta_B \quad \left(\frac{L}{3} \le x \le L\right) \tag{5}$$

B.C. 3 At $x = \dfrac{L}{3}$, $(v_B)_{\text{Left}} = (v_B)_{\text{Right}}$ (Eqs. 1 and 5)

$$\therefore \delta_B = \frac{8PL^3}{729EI} \quad \longleftarrow$$

$$\theta_A = \frac{\delta_B}{L/3} = \frac{8PL^2}{243EI} \quad \longleftarrow$$

Substitute for δ_B in Eq. (5) and simplify:

$$v = \frac{P}{486EI}(7L^3 - 61L^2x + 81Lx^2 - 27x^3) \quad \left(\frac{L}{3} \le x \le L\right) \tag{6}$$

Also,

$$v' = \frac{P}{486EI}(-61L^2 + 162Lx - 81x^2) \quad \left(\frac{L}{3} \le x \le L\right) \tag{7}$$

Maximum deflection

$v' = 0$ gives $x_1 = \dfrac{L}{9}(9 - 2\sqrt{5}) = 0.5031L$

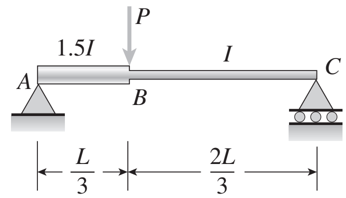
Substitute x_1 in Eq. (6) and simplify:

$$v_{max} = -\frac{40\sqrt{5}PL^3}{6561EI}$$

$$\delta_{max} = -v_{max} = \frac{40\sqrt{5}PL^3}{6561EI} = 0.01363\frac{PL^3}{EI} \quad \longleftarrow$$

Problem 9.7-5 A simple beam ABC has moment of inertia $1.5I$ from A to B and I from B to C (see figure). A concentrated load P acts at point B.

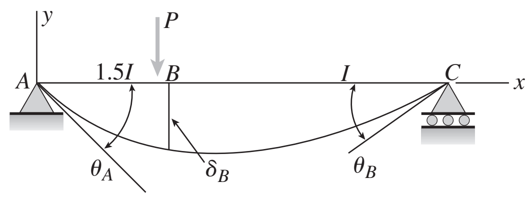
Obtain the equations of the deflection curves for both parts of the beam. From the equations, determine the angles of rotation θ_A and θ_C at the supports and the deflection δ_B at point B.



Solution 9.7-5 **Simple beam (nonprismatic)**

Use the bending-moment equation (Eq. 9-12a).

DEFLECTION CURVE



BENDING-MOMENT EQUATIONS

$$E\left(\frac{3I}{2}\right)v'' = M = \frac{2Px}{3} \quad \left(0 \le x \le \frac{L}{3}\right) \tag{1}$$

$$EIv'' = M = \frac{PL}{3} - \frac{Px}{3} \quad \left(\frac{L}{3} \le x \le L\right) \tag{2}$$

INTEGRATE EACH EQUATION

$$EIv' = \frac{4Px^2}{18} + C_1 \quad \left(0 \le x \le \frac{L}{3}\right) \tag{3}$$

$$EIv' = \frac{PLx}{3} - \frac{Px^2}{6} + C_2 \quad \left(\frac{L}{3} \le x \le L\right) \tag{4}$$

B.C. 1 Continuity of slopes at point B

$(v'_B)_{\text{Left}} = (v'_B)_{\text{Right}}$

From Eqs. (3) and (4):

$$\frac{4P}{18}\left(\frac{L}{3}\right)^2 + C_1 = \frac{PL}{3}\left(\frac{L}{3}\right) - \frac{P}{6}\left(\frac{L}{3}\right)^2 + C_2$$

$$C_2 = C_1 - \frac{11PL^2}{162} \tag{5}$$

INTEGRATE EQS. (3) AND (4)

$$EIv = \frac{4Px^3}{54} + C_1x + C_3 \quad \left(0 \le x \le \frac{L}{3}\right) \tag{6}$$

$$EIv = \frac{PLx^2}{6} - \frac{Px^3}{18} + C_2x + C_4 \quad \left(\frac{L}{3} \le x \le L\right) \tag{7}$$

B.C. 2 Deflection at support A

$$v(0) = 0 \quad \text{From Eq. (6):} \quad C_3 = 0 \tag{8}$$

B.C. 3 Deflection at support C

$$v(L) = 0 \quad \text{From Eq. (7):} \quad C_4 = -\frac{PL^3}{9} - C_2L \tag{9}$$

B.C. 4 Continuity of deflections at point B

$(v_B)_{\text{Left}} = (v_B)_{\text{Right}}$

From Eqs. (6), (8), and (7):

$$\frac{4P}{54}\left(\frac{L}{3}\right)^3 + C_1\left(\frac{L}{3}\right) = \frac{PL}{6}\left(\frac{L}{3}\right)^2 - \frac{P}{18}\left(\frac{L}{3}\right)^3 + C_2\left(\frac{L}{3}\right) + C_4$$

$$C_1L = \frac{10PL^3}{243} + C_2L + 3C_4 \tag{10}$$

SOLVE EQS (5), (8), (9), AND (10)

$$C_1 = -\frac{38PL^2}{729} \quad C_2 = -\frac{175PL^2}{1458} \quad C_3 = 0$$

$$C_4 = \frac{13PL^3}{1458}$$

SLOPES OF THE BEAM (FROM EQS. 3 AND 4)

$$v' = -\frac{2P}{729EI}(19L^2 - 81x^2) \quad \left(0 \le x \le \frac{L}{3}\right) \tag{11}$$

$$v' = -\frac{P}{1458EI}(175L^2 - 486Lx + 243x^2) \quad \left(\frac{L}{3} \le x \le L\right) \tag{12}$$

ANGLE OF ROTATION θ_A (FROM EQ. 11)

$$\theta_A = -v'(0) = \frac{38PL^2}{729EI} \quad \text{(positive clockwise)} \quad \leftarrow$$

ANGLE OF ROTATION θ_C (FROM EQ. 12)

$$\theta_C = v'(L) = \frac{34PL^2}{729EI} \quad \text{(positive counterclockwise)} \quad \leftarrow$$

DEFLECTIONS OF THE BEAM

Substitute C_1, C_2, C_3, and C_4 into Eqs. (6) and (7):

$$v = -\frac{2Px}{729EI}(19L^2 - 27x^2) \quad \left(0 \le x \le \frac{L}{3}\right) \quad \leftarrow$$

$$v = -\frac{P}{1458EI}(-13L^3 + 175L^2x - 243Lx^2 + 81x^3) \quad \left(\frac{L}{3} \le x \le L\right) \quad \leftarrow$$

DEFLECTION AT POINT B $\left(x = \frac{L}{3}\right)$

$$\delta_B = -v\left(\frac{L}{3}\right) = \frac{32PL^3}{2187\,EI} \quad \text{(positive downward)} \quad \leftarrow$$

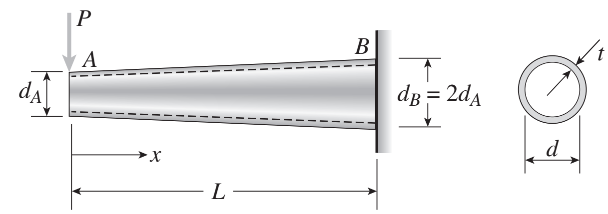
Problem 9.7-6 The tapered cantilever beam AB shown in the figure has thin-walled, hollow circular cross sections of constant thickness t. The diameters at the ends A and B are d_A and $d_B = 2d_A$, respectively. Thus, the diameter d and moment of inertia I at distance x from the free end are, respectively,

$$d = \frac{d_A}{L}(L + x)$$

$$I = \frac{\pi t d^3}{8} = \frac{\pi t d_A^3}{8L^3}(L + x)^3 = \frac{I_A}{L^3}(L + x)^3$$

in which I_A is the moment of inertia at end A of the beam.

Determine the equation of the deflection curve and the deflection δ_A at the free end of the beam due to the load P.



Solution 9.7-6 **Tapered cantilever beam**

$$M = -Px \quad EIv'' = -Px \quad I = \frac{I_A}{L^3}(L + x)^3$$

$$v'' = -\frac{Px}{EI} = -\frac{PL^3}{EI_A}\left[\frac{x}{(L + x)^3}\right] \qquad (1)$$

INTEGRATE EQ. (1)

From Appendix C: $\displaystyle\int \frac{xdx}{(L + x)^3} = -\frac{L + 2x}{2(L + x)^2}$

$$v' = \frac{PL^3}{EI_A}\left[\frac{L + 2x}{2(L + x)^2}\right] + C_1$$

B.C. 1 $v'(L) = 0 \quad \therefore C_1 = -\dfrac{3PL^2}{8EI_A}$

$$v' = \frac{PL^3}{EI_A}\left[\frac{L + 2x}{2(L + x)^2}\right] - \frac{3PL^2}{8EI_A}$$

or

$$v' = \frac{PL^3}{EI_A}\left[\frac{L}{2(L + x)^2}\right] + \frac{PL^3}{EI_A}\left[\frac{x}{(L + x)^2}\right] - \frac{3PL^2}{8EI_A} \qquad (2)$$

INTEGRATE EQ. (2)

From Appendix C:

$$\int \frac{dx}{(L + x)^2} = -\frac{1}{L + x}$$

$$\int \frac{xdx}{(L + x)^2} = \frac{L}{L + x} + ln(L + x)$$

$$v = \frac{PL^3}{EI_A}\left(\frac{L}{2}\right)\left(-\frac{1}{L + x}\right) + \frac{PL^3}{EI_A}\left[\frac{L}{L + x} + ln(L + x)\right]$$
$$-\frac{3PL^2}{8EI_A}x + C_2$$
$$= \frac{PL^3}{EI_A}\left[\frac{L}{2(L + x)} + ln(L + x) - \frac{3x}{8L}\right] + C_2 \qquad (3)$$

B.C. 2 $v(L) = 0 \quad \therefore C_2 = \dfrac{PL^3}{EI_A}\left[\dfrac{1}{8} - ln(2L)\right]$

DEFLECTION OF THE BEAM

Substitute C_2 into Eq. (3).

$$v = \frac{PL^3}{EI_A}\left[\frac{L}{2(L + x)} - \frac{3x}{8L} + \frac{1}{8} + ln\left(\frac{L + x}{2L}\right)\right] \longleftarrow$$

DEFLECTION δ_A AT END A OF THE BEAM

$$\delta_A = -v(0) = \frac{PL^3}{8EI_A}(8\,ln2 - 5) = 0.06815\frac{PL^3}{EI_A}$$

(positive downward) $\longleftarrow$

NOTE: $ln\dfrac{1}{2} = -ln\,2$

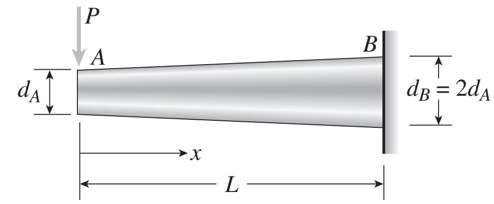
Problem 9.7-7 The tapered cantilever beam AB shown in the figure has a solid circular cross section. The diameters at the ends A and B are d_A and $d_B = 2d_A$, respectively. Thus, the diameter d and moment of inertia I at distance x from the free end are, respectively,

$$d = \frac{d_A}{L}(L + x)$$

$$I = \frac{\pi d^4}{64} = \frac{\pi d_A^4}{64L^4}(L + x)^4 = \frac{I_A}{L^4}(L + x)^4$$

in which I_A is the moment of inertia at end A of the beam.

Determine the equation of the deflection curve and the deflection δ_A at the free end of the beam due to the load P.



Solution 9.7-7 **Tapered cantilever beam**

$$M = -Px \quad EIv'' = -Px \quad I = \frac{I_A}{L^4}(L + x)^4$$

$$v'' = -\frac{Px}{EI} = -\frac{PL^4}{EI_A}\left[\frac{x}{(L + x)^4}\right] \qquad (1)$$

INTEGRATE EQ. (1)

From Appendix C: $\displaystyle\int \frac{xdx}{(L + x)^4} = -\frac{L + 3x}{6(L + x)^3}$

$$v' = \frac{PL^4}{EI_A}\left[\frac{L + 3x}{6\,(L + x)^3}\right] + C_1$$

B.C. 1 $v'(L) = 0 \quad \therefore C_1 = -\dfrac{PL^2}{12EI_A}$

$$v' = \frac{PL^4}{EI_A}\left[\frac{L + 3x}{6\,(L + x)^3}\right] - \frac{PL^2}{12EI_A}$$

or

$$v' = \frac{PL^4}{EI_A}\left[\frac{L}{6\,(L + x)^3}\right] + \frac{PL^4}{EI_A}\left[\frac{x}{2(L + x)^3}\right] - \frac{PL^2}{12EI_A} \qquad (2)$$

INTEGRATE EQ. (2)

From Appendix C: $\displaystyle\int \frac{dx}{(L + x)^3} = -\frac{1}{2(L + x)^2}$

$$\int \frac{xdx}{(L + x)^3} = \frac{-(L + 2x)}{2(L + x)^2}$$

$$v = \frac{PL^4}{EI_A}\left(\frac{L}{6}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{L + x}\right)^2 + \frac{PL^4}{EI_A}\left(\frac{1}{2}\right)\left[-\frac{L + 2x}{2(L + x)^2}\right] - \frac{PL^2}{12EI_A}x + C_2$$

$$= \frac{PL^3}{EI_A}\left[-\frac{L^2}{12(L + x)^2} - \frac{L(L + 2x)}{4(L + x)^2} - \frac{x}{12L}\right] + C_2 \qquad (3)$$

B.C. 2 $v(L) = 0 \quad \therefore C_2 = \dfrac{PL^3}{EI_A}\left(\dfrac{7}{24}\right)$

DEFLECTION OF THE BEAM

Substitute C_2 into Eq. (3).

$$v = \frac{PL^3}{24EI_A}\left[7 - \frac{4L(2L + 3x)}{(L + x)^2} - \frac{2x}{L}\right] \quad \longleftarrow$$

DEFLECTION δ_A AT END A OF THE BEAM

$$\delta_A = -v(0) = \frac{PL^3}{24EI_A} \quad \text{(positive downward)} \quad \longleftarrow$$

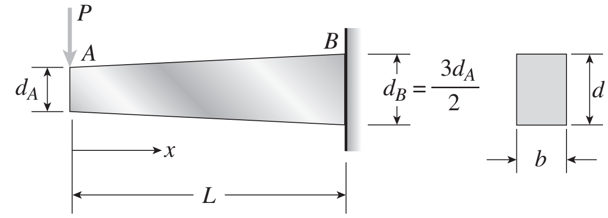
Problem 9.7-8 A tapered cantilever beam AB supports a concentrated load P at the free end (see figure). The cross sections of the beam are rectangular with constant width b, depth d_A at support A, and depth $d_B = 3d_A/2$ at the support. Thus, the depth d and moment of inertia I at distance x from the free end are, respectively,

$$d = \frac{d_A}{2L}(2L + x)$$

$$I = \frac{bd^3}{12} = \frac{bd_A^3}{96L^3}(2L + x)^3 = \frac{I_A}{8L^3}(2L + x)^3$$

in which I_A is the moment of inertia at end A of the beam.

Determine the equation of the deflection curve and the deflection δ_A at the free end of the beam due to the load P.



Solution 9.7-8 Tapered cantilever beam

$$M = -Px \quad EIv'' = -Px \quad I = \frac{I_A}{8L^3}(2L + x)^3$$

$$v'' = -\frac{Px}{EI} = -\frac{8PL^3}{EI_A}\left[\frac{x}{(2L + x)^3}\right] \qquad (1)$$

INTEGRATE EQ. (1)

From Appendix C: $\displaystyle\int \frac{xdx}{(2L + x)^3} = -\frac{2L + 2x}{2(2L + x)^2}$

$$v' = \frac{8PL^3}{EI_A}\left[\frac{L + x}{(2L + x)^2}\right] + C_1$$

B.C. 1 $v'(L) = 0 \quad \therefore C_1 = -\dfrac{16PL^2}{9EI_A}$

$$v' = \frac{8PL^3}{EI_A}\left[\frac{L + x}{(2L + x)^2}\right] - \frac{16PL^2}{9EI_A}$$

or

$$v' = \frac{8PL^3}{EI_A}\left[\frac{L}{(2L + x)^2}\right] + \frac{8PL^3}{EI_A}\left[\frac{x}{(2L + x)^2}\right] - \frac{16PL^2}{9EI_A} \qquad (2)$$

INTEGRATE EQ. (2)

From Appendix C: $\displaystyle\int \frac{dx}{(2L + x)^2} = -\frac{1}{2L + x}$

$$\int \frac{xdx}{(2L + x)^2} = \frac{2L}{2L + x} + ln(2L + x)$$

$$v = \frac{8PL^3}{EI_A}\left(-\frac{L}{2L + x}\right) + \frac{8PL^3}{EI_A}\left[\frac{2L}{2L + x} + ln(2L + x)\right] - \frac{16PL^2}{9EI_A}x + C_2$$

$$= \frac{PL^3}{EI_A}\left[\frac{8L}{2L + x} + 8\,ln(2L + x) - \frac{16x}{9L}\right] + C_2 \qquad (3)$$

B.C. 2 $v(L) = 0 \quad \therefore C_2 = -\dfrac{8PL^3}{EI_A}\left[\dfrac{1}{9} + ln(3L)\right]$

DEFLECTION OF THE BEAM

Substitute C_2 into EQ. (3).

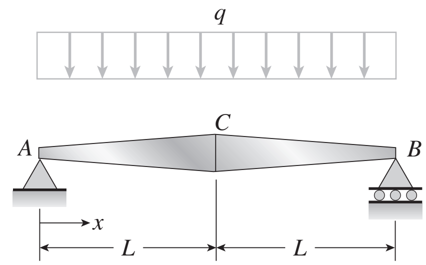
$$v = \frac{8PL^3}{EI_A}\left[\frac{L}{2L + x} - \frac{2x}{9L} - \frac{1}{9} + ln\left(\frac{2L + x}{3L}\right)\right] \leftarrow$$

DEFLECTION δ_A AT END A OF THE BEAM

????

NOTE: $ln\dfrac{2}{3} = -ln\dfrac{3}{2}$

Problem 9.7-9 A simple beam ACB is constructed with square cross sections and a double taper (see figure). The depth of the beam at the supports is d_A and at the midpoint is $d_C = 2d_A$. Each half of the beam has length L. Thus, the depth d and moment of inertia I at distance x from the left-hand end are, respectively,



$$d = \frac{d_A}{L}(L + x)$$

$$I = \frac{d^4}{12} = \frac{d_A^4}{12L^4}(L + x)^4 = \frac{I_A}{L^4}(L + x)^4$$

in which I_A is the moment of inertia at end A of the beam. (These equations are valid for x between 0 and L, that is, for the left-hand half of the beam.)

(a) Obtain equations for the slope and deflection of the left-hand half of the beam due to the uniform load.

(b) From those equations obtain formulas for the angle of rotation θ_A at support A and the deflection δ_C at the midpoint.

Solution 9.7-9 **Simple beam with a double taper**

L = length of one-half of the beam

$$I = \frac{I_A}{L^4}(L + x)^4 \quad (0 \le x \le L)$$

(x is measured from the left-hand support A)

Reactions: $R_A = R_B = qL$

Bending moment: $M = R_A x - \dfrac{qx^2}{2} = qLx - \dfrac{qx^2}{2}$

From Eq. (9-12a):

$$EIv'' = M = qLx - \frac{qx^2}{2}$$

$$v'' = \frac{qL^5x}{EI_A(L + x)^4} - \frac{qL^4x^2}{2EI_A(L + x)^4} \quad (0 \le x \le L) \quad (1)$$

INTEGRATE EQ. (1)

From Appendix C: $\displaystyle\int \frac{xdx}{(L + x)^4} = -\frac{L + 3x}{6(L + x)^3}$

$$\int \frac{x^2dx}{(L + x)^4} = -\frac{L^2 + 3Lx + 3x^2}{3(L + x)^3}$$

$$v' = \frac{qL^5}{EI_A}\left[-\frac{L + 3x}{6(L + x)^3}\right] - \frac{qL^4}{2EI_A}\left[-\frac{L^2 + 3Lx + 3x^2}{3(L + x)^3}\right] + C_1$$

$$= \frac{qL^4x^2}{2EI_A(L + x)^3} + C_1 \quad (0 \le x \le L) \quad (2)$$

B.C. 1 (symmetry) $v'(L) = 0 \quad \therefore C_1 = -\dfrac{qL^3}{16EI_A}$

SLOPE OF THE BEAM

Substitute C_1 into Eq. (2).

$$v' = \frac{qL^4x^2}{2EI_A(L + x)^3} - \frac{qL^3}{16EI_A}$$

$$= -\frac{qL^3}{16EI_A}\left[1 - \frac{8Lx^2}{(L + x)^3}\right] \quad (0 \le x \le L) \quad (3) \quad \longleftarrow$$

ANGLE OF ROTATION AT SUPPORT A

$$\theta_A = -v'(0) = \frac{qL^3}{16EI_A} \quad \text{(positive clockwise)} \quad \longleftarrow$$

INTEGRATE EQ. (3)

From Appendix C:

$$\int \frac{x^2dx}{(L + x)^3} = \frac{L(3L + 4x)}{2(L + x)^2} + ln(L + x)$$

$$v = -\frac{qL^3}{16EI_A}\left[x - \frac{8L^2(3L + 4x)}{2(L + x)^2} - 8L\,ln(L + x)\right] + C_2 \quad (0 \le x \le L) \quad (4)$$

B.C. 2 $v(0) = 0 \quad \therefore C_2 = -\dfrac{qL^4}{2EI_A}\left(\dfrac{3}{2} + ln\,L\right)$

DEFLECTION OF THE BEAM

Substitute C_2 into Eq. (4) and simplify. (The algebra is lengthy.)

$$v = -\frac{qL^4}{2EI_A}\left[\frac{(9L^2 + 14Lx + x^2)x}{8L(L + x)^2} - ln\left(1 + \frac{x}{L}\right)\right]$$

$$(0 \le x \le L) \quad \longleftarrow$$

DEFLECTION AT THE MIDPOINT C OF THE BEAM

$$\delta_C = -v(L) = \frac{qL^4}{8EI_A}(3 - 4\,ln\,2) = 0.02843\frac{qL^4}{EI_A}$$

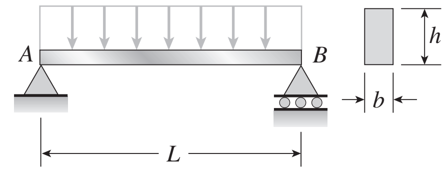
(positive downward) $\longleftarrow$

Strain Energy

The beams described in the problems for Section 9.8 have constant flexural rigidity EI.

Problem 9.8-1 A uniformly loaded simple beam AB (see figure) of span length L and rectangular cross section (b = width, h = height) has a maximum bending stress σ_{max} due to the uniform load.

Determine the strain energy U stored in the beam.



Solution 9.8-1 Simple beam with a uniform load

Given: L, b, h, σ_{max} Find: U(strain energy)

Bending moment: $M = \dfrac{qLx}{2} - \dfrac{qx^2}{2}$

Strain energy (Eq. 9-80a): $U = \displaystyle\int_0^L \frac{M^2 dx}{2EI}$

$$= \frac{q^2L^5}{240EI} \qquad (1)$$

Maximum stress: $\sigma_{max} = \dfrac{M_{max}c}{I} = \dfrac{M_{max}h}{2I}$

$M_{max} = \dfrac{qL^2}{8} \qquad \sigma_{max} = \dfrac{qL^2h}{16I}$

Solve for q: $\quad q = \dfrac{16I\sigma_{max}}{L^2h}$

Substitute q into Eq. (1):

$$U = \frac{16I\sigma_{max}^2 L}{15h^2E}$$

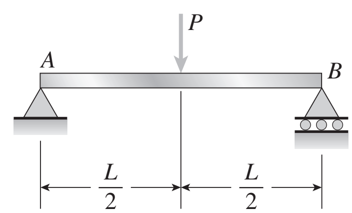
Substitute $I = \dfrac{bh^3}{12}$: $\quad U = \dfrac{4bhL\sigma_{max}^2}{45E}$ ←

Problem 9.8-2 A simple beam AB of length L supports a concentrated load P at the midpoint (see figure).

(a) Evaluate the strain energy of the beam from the bending moment in the beam.

(b) Evaluate the strain energy of the beam from the equation of the deflection curve.

(c) From the strain energy, determine the deflection δ under the load P.



Solution 9.8-2 Simple beam with a concentrated load

(a) BENDING MOMENT $\quad M = \dfrac{Px}{2} \quad \left(0 \le x \le \dfrac{L}{2}\right)$

Strain energy (Eq. 9-80a): $U = 2\displaystyle\int_0^{L/2} \frac{M^2 dx}{2EI} = \frac{P^2L^3}{96EI}$ ←

(b) DEFLECTION CURVE

From Table G-2, Case 4:

$$v = -\frac{Px}{48EI}(3L^2 - 4x^2) \quad \left(0 \le x \le \frac{L}{2}\right)$$

$$\frac{dv}{dx} = -\frac{P}{16EI}(L^2 - 4x^2) \qquad \frac{d^2v}{dx^2} = \frac{Px}{2EI}$$

Strain energy (Eq. 9-80b):

$$U = 2\int_0^{L/2} \frac{EI}{2}\left(\frac{d^2v}{dx^2}\right)^2 dx = EI\int_0^{L/2}\left(\frac{Px}{2EI}\right)^2 dx$$

$$= \frac{P^2L^3}{96EI} \quad ←$$

(c) DEFLECTION δ UNDER THE LOAD P

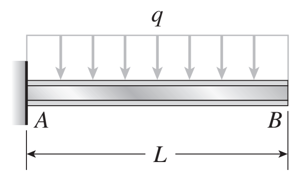
From Eq. (9-82a):

$$\delta = \frac{2U}{P} = \frac{PL^3}{48EI} \quad ←$$

Problem 9.8-3 A cantilever beam AB of length L supports a uniform load of intensity q (see figure).

(a) Evaluate the strain energy of the beam from the bending moment in the beam.

(b) Evaluate the strain energy of the beam from the equation of the deflection curve.



Solution 9.8-3 Cantilever beam with a uniform load

(a) BENDING MOMENT

Measure x from the free end B.

$$M = -\frac{qx^2}{2}$$

Strain energy (Eq. 9-80a):

$$U = \int_0^L \frac{M^2 dx}{2EI} = \int_0^L \left(\frac{1}{2EI}\right)\left(-\frac{qx^2}{2}\right)^2 dx = \frac{q^2L^5}{40EI} \longleftarrow$$

(b) DEFLECTION CURVE

Measure x from the fixed support A.
From Table G-1, Case 1:

$$v = -\frac{qx^2}{24EI}(6L^2 - 4Lx + x^2)$$

$$\frac{dv}{dx} = -\frac{q}{6EI}(3L^2x - 3Lx^2 + x^3)$$

$$\frac{d^2v}{dx^2} = -\frac{q}{2EI}(L^2 - 2Lx + x^2)$$

Strain energy (Eq. 9-80b):

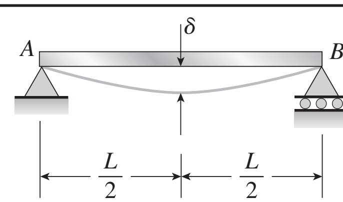
$$U = \int_0^L \frac{EI}{2}\left(\frac{d^2v}{dx^2}\right)^2 dx$$

$$= \frac{EI}{2}\int_0^L \left(-\frac{q}{2EI}\right)^2 (L^2 - 2Lx + x^2)^2 dx$$

$$= \frac{q^2L^5}{40EI} \longleftarrow$$

Problem 9.8-4 A simple beam AB of length L is subjected to loads that produce a symmetric deflection curve with maximum deflection δ at the midpoint of the span (see figure).

How much strain energy U is stored in the beam if the deflection curve is (a) a parabola, and (b) a half wave of a sine curve?



Solution 9.8-4 Simple beam (symmetric deflection curve)

GIVEN: L, EI, δ $\quad\delta$ = maximum deflection at midpoint

Determine the strain energy U.
Assume the deflection v is positive downward.

(a) DEFLECTION CURVE IS A PARABOLA

$$v = \frac{4\delta x}{L^2}(L - x) \qquad \frac{dv}{dx} = \frac{4\delta}{L^2}(L - 2x)$$

$$\frac{d^2v}{dx^2} = -\frac{8\delta}{L^2}$$

Strain energy (Eq. 9-80b):

$$U = \int_0^L \frac{EI}{2}\left(\frac{d^2v}{dx^2}\right)^2 dx = \frac{EI}{2}\int_0^L \left(-\frac{8\delta}{L^2}\right)^2 dx = \frac{32EI\delta^2}{L^3} \longleftarrow$$

(b) DEFLECTION CURVE IS A SINE CURVE

$$v = \delta \sin\frac{\pi x}{L} \qquad \frac{dv}{dx} = \frac{\pi\delta}{L}\cos\frac{\pi x}{L} \qquad \frac{d^2v}{dx^2} = -\frac{\pi^2\delta}{L^2}\sin\frac{\pi x}{L}$$

Strain energy (Eq. 9-80b):

$$U = \int_0^L \frac{EI}{2}\left(\frac{d^2v}{dx^2}\right)^2 dx = \frac{EI}{2}\int_0^L \left(-\frac{\pi^2\delta}{L^2}\right)^2 \sin^2\frac{\pi x}{L}\, dx$$

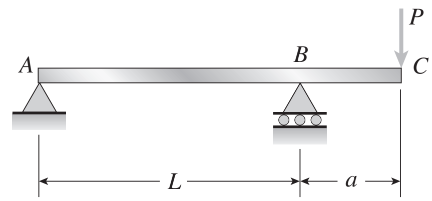
$$= \frac{\pi^4 EI\delta^2}{4L^3} \longleftarrow$$

Problem 9.8-5 A beam ABC with simple supports at A and B and an overhang BC supports a concentrated load P at the free end C (see figure).

(a) Determine the strain energy U stored in the beam due to the load P.

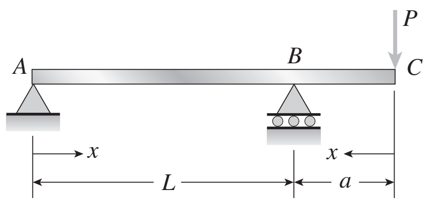
(b) From the strain energy, find the deflection δ_C under the load P.

(c) Calculate the numerical values of U and δ_C if the length L is 8 ft, the overhang length a is 3 ft, the beam is a W 10 × 12 steel wide-flange section, and the load P produces a maximum stress of 12,000 psi in the beam. (Use $E = 29 \times 10^6$ psi.)



Solution 9.8-5 **Simple beam with an overhang**

(a) Strain energy (use Eq.9-80a)



From A to B: $M = -\dfrac{Pax}{L}$

$$U_{AB} = \int \frac{M^2 dx}{2EI} = \int_0^L \frac{1}{2EI}\left(-\frac{Pax}{L}\right)^2 dx = \frac{P^2a^2L}{6EI}$$

From B to C: $M = -Px$

$$U_{BC} = \int_0^a \frac{1}{2EI}(-Px)^2 dx = \frac{P^2a^3}{6EI}$$

Total strain energy:

$$U = U_{AB} + U_{BC} = \frac{P^2a^2}{6EI}(L + a) \longleftarrow$$

(b) Deflection δ_C under the load P

From Eq. (9-82a):

$$\delta_C = \frac{2U}{P} = \frac{Pa^2}{3EI}(L + a) \longleftarrow$$

(c) Calculate U and δ_c

Data: $L = 8$ ft $= 96$ in. $\quad a = 3$ft $= 36$ in.

W 10 × 12 $\quad E = 29 \times 10^6$ psi

$\sigma_{max} = 12{,}000$ psi

$I = 53.8 \text{ in.}^4 \quad c = \dfrac{d}{2} = \dfrac{9.87}{2} = 4.935$ in.

Express load P in terms of maximum stress:

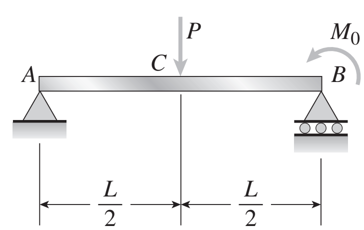
$$\sigma_{max} = \frac{Mc}{I} = \frac{M_{max}c}{I} = \frac{Pac}{I} \qquad \therefore\ P = \frac{\sigma_{max} I}{ac}$$

$$U = \frac{P^2a^2(L + a)}{6EI} = \frac{\sigma_{max}^2 I(L + a)}{6c^2E} = 241 \text{ in.-lb} \longleftarrow$$

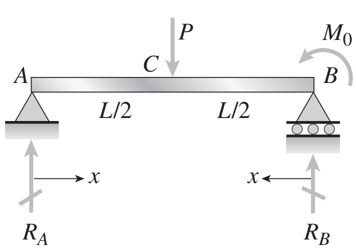
$$\delta_C = \frac{Pa^2(L + a)}{3EI} = \frac{\sigma_{max} a(L + a)}{3cE} = 0.133 \text{ in.} \longleftarrow$$

Problem 9.8-6 A simple beam ACB supporting a concentrated load P at the midpoint and a couple of moment M_0 at one end is shown in the figure.

Determine the strain energy U stored in the beam due to the force P and the couple M_0 acting simultaneously.



Solution 9.8-6 **Simple beam with two loads**



$$R_A = \frac{P}{2} + \frac{M_0}{L}$$

$$R_B = \frac{P}{2} - \frac{M_0}{L}$$

From A to C $\quad M = R_A x = \left(\dfrac{P}{2} + \dfrac{M_0}{L}\right)x$

$$U_{AC} = \int \frac{M^2 dx}{2EI} = \frac{1}{2EI}\int_0^{L/2}\left(\frac{P}{2} + \frac{M_0}{L}\right)^2 x^2 dx$$

$$= \frac{L}{192EI}(P^2L^2 + 4PLM_0 + 4M_0^2)$$

FROM B TO C $\quad M = R_B x + M_0 = \left(\frac{P}{2} - \frac{M_0}{L}\right)x + M_0$

$$U_{BC} = \int \frac{M^2 dx}{2EI} = \frac{1}{2EI}\int_0^{L/2}\left[\left(\frac{P}{2} - \frac{M_0}{L}\right)x + M_0\right]^2 dx$$

$$= \frac{L}{192EI}(P^2L^2 + 8PLM_0 + 28M_0^2)$$

STRAIN ENERGY OF THE ENTIRE BEAM

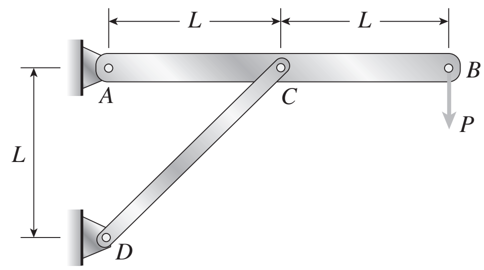
$$U = U_{AC} + U_{BC} = \frac{L}{96EI}(P^2L^2 + 6PLM_0 + 16M_0^2)$$

$$= \frac{P^2L^3}{96EI} + \frac{PM_0L^2}{16EI} + \frac{M_0^2 L}{6EI} \longleftarrow$$

Problem 9.8-7 The frame shown in the figure consists of a beam ACB supported by a strut CD. The beam has length $2L$ and is continuous through joint C. A concentrated load P acts at the free end B.

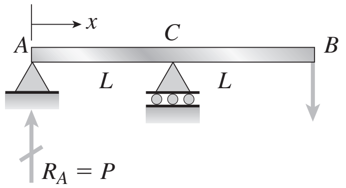
Determine the vertical deflection δ_B at point B due to the load P.

Note: Let EI denote the flexural rigidity of the beam, and let EA denote the axial rigidity of the strut. Disregard axial and shearing effects in the beam, and disregard any bending effects in the strut.



Solution 9.8-7 **Frame with beam and strut**

BEAM ACB



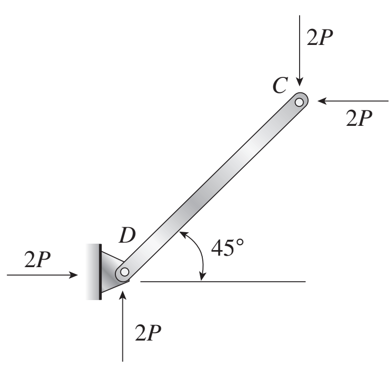
For part AC of the beam: $M = -Px$

$$U_{AC} = \int \frac{M^2 dx}{2EI} = \frac{1}{2EI}\int_0^L (-Px)^2 dx = \frac{P^2L^3}{6EI}$$

For part CB of the beam: $U_{CB} = U_{AC} = \frac{P^2L^3}{6EI}$

Entire beam: $U_{\text{BEAM}} = U_{AC} + U_{CB} = \frac{P^2L^3}{3EI}$

STRUT CD



L_{CD} = length of strut
$= \sqrt{2}L$

F = axial force in strut
$= 2\sqrt{2}P$

$$U_{\text{STRUT}} = \frac{F^2 L_{CD}}{2EA} \quad \text{(Eq. 2-37a)}$$

$$U_{\text{STRUT}} = \frac{(2\sqrt{2}P)^2(\sqrt{2}L)}{2EA} = \frac{4\sqrt{2}P^2L}{EA}$$

FRAME $\quad U = U_{\text{BEAM}} + U_{\text{STRUT}} = \frac{P^2L^3}{3EI} + \frac{4\sqrt{2}P^2L}{EA}$

DEFLECTION δ_B AT POINT B

From Eq. (9-82 a):

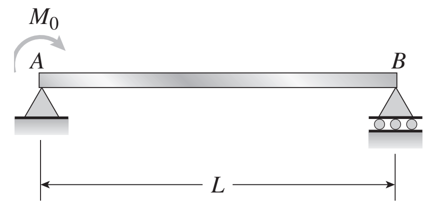
$$\delta_B = \frac{2U}{P} = \frac{2PL^3}{3EI} + \frac{8\sqrt{2}PL}{EA} \longleftarrow$$

Castigliano's Theorem

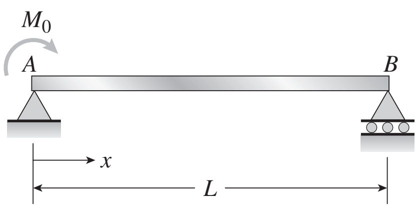
The beams described in the problems for Section 9.9 have constant flexural rigidity EI.

Problem 9.9-1 A simple beam AB of length L is loaded at the left-hand end by a couple of moment M_0 (see figure).

Determine the angle of rotation θ_A at support A. (Obtain the solution by determining the strain energy of the beam and then using Castigliano's theorem.)



Solution 9.9-1 Simple beam with couple M_0



$$R_A = \frac{M_0}{L} \quad \text{(downward)}$$

$$M = M_0 - R_A x = M_0 - \frac{M_0 x}{L}$$

$$= M_0\left(1 - \frac{x}{L}\right)$$

STRAIN ENERGY

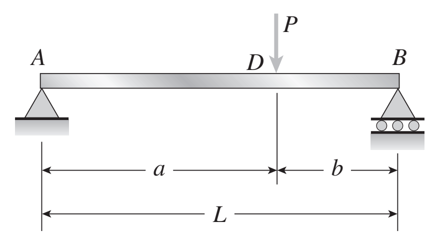
$$U = \int \frac{M^2 dx}{2EI} = \frac{M_0^2}{2EI}\int_0^L \left(1 - \frac{x}{L}\right)^2 dx = \frac{M_0^2 L}{6EI}$$

CASTIGLIANO'S THEOREM $\quad \theta_A = \dfrac{dU}{dM_0} = \dfrac{M_0 L}{3EI}$ (clockwise) ←

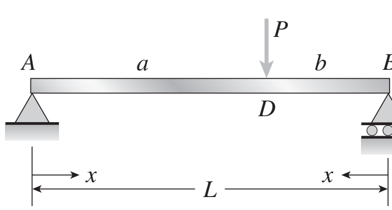
(This result agree with Case 7, Table G-2)

Problem 9.9-2 The simple beam shown in the figure supports a concentrated load P acting at distance a from the left-hand support and distance b from the right-hand support.

Determine the deflection δ_D at point D where the load is applied. (Obtain the solution by determining the strain energy of the beam and then using Castigliano's theorem.)



Solution 9.9-2 Simple beam with load P



$$R_A = \frac{Pb}{L} \qquad R_B = \frac{Pa}{L}$$

$$M_{AD} = R_A x = \frac{Pbx}{L}$$

$$M_{DB} = R_B x = \frac{Pax}{L}$$

STRAIN ENERGY $\quad U = \displaystyle\int \frac{M^2 dx}{2EI}$

$$U_{AD} = \frac{1}{2EI}\int_0^a \left(\frac{Pbx}{L}\right)^2 dx = \frac{P^2 a^3 b^2}{6EIL^2}$$

$$U_{DB} = \frac{1}{2EI}\int_0^b \left(\frac{Pax}{L}\right)^2 dx = \frac{P^2 a^2 b^3}{6EIL^2}$$

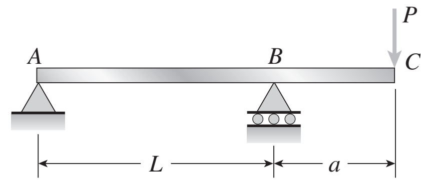
$$U = U_{AD} + U_{DB} = \frac{P^2 a^2 b^2}{6LEI}$$

CASTIGLIANO'S THEOREM

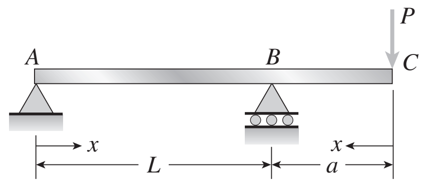
$$\delta_D = \frac{dU}{dP} = \frac{Pa^2 b^2}{3LEI} \quad \text{(downward)} \quad \leftarrow$$

Problem 9.9-3 An overhanging beam ABC supports a concentrated load P at the end of the overhang (see figure). Span AB has length L and the overhang has length a.

Determine the deflection δ_C at the end of the overhang. (Obtain the solution by determining the strain energy of the beam and then using Castigliano's theorem.)



Solution 9.9-3 Overhanging beam



$$R_A = \frac{Pa}{L} \quad \text{(downward)}$$

$$M_{AB} = -R_A x = -\frac{Pax}{L}$$

$$M_{CB} = -Px$$

STRAIN ENERGY $\quad U = \int \frac{M^2 dx}{2EI}$

$$U_{AB} = \frac{1}{2EI}\int_0^L \left(-\frac{Pax}{L}\right)^2 dx = \frac{P^2a^2L}{6EI}$$

$$U_{CB} = \frac{1}{2EI}\int_0^a (-Px)^2 dx = \frac{P^2a^3}{6EI}$$

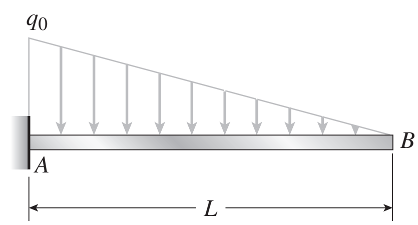
$$U = U_{AB} + U_{CB} = \frac{P^2a^2}{6EI}(L + a)$$

CASTIGLIANO'S THEOREM

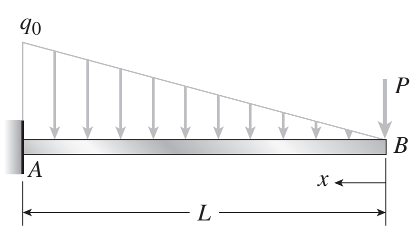
$$\delta_C = \frac{dU}{dP} = \frac{Pa^2}{3EI}(L + a) \quad \text{(downward)} \quad \longleftarrow$$

Problem 9.9-4 The cantilever beam shown in the figure supports a triangularly distributed load of maximum intensity q_0.

Determine the deflection δ_B at the free end B. (Obtain the solution by determining the strain energy of the beam and then using Castigliano's theorem.)



Solution 9.9-4 Cantilever beam with triangular load



P = fictitious load corresponding to deflection δ_B

$$M = -Px - \frac{q_0 x^3}{6L}$$

STRAIN ENERGY

$$U = \int \frac{M^2 dx}{2EI} = \frac{1}{2EI}\int_0^L \left(-Px - \frac{q_0 x^3}{6L}\right)^2 dx$$

$$= \frac{P^2L^3}{6EI} + \frac{Pq_0L^4}{30EI} + \frac{q_0^2L^5}{42EI}$$

CASTIGLIANO'S THEOREM

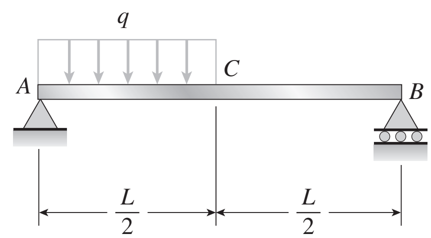
$$\delta_B = \frac{\partial U}{\partial P} = \frac{PL^3}{3EI} + \frac{q_0L^4}{30EI} \quad \text{(downward)}$$

(This result agrees with Cases 1 and 8 of Table G-1.)

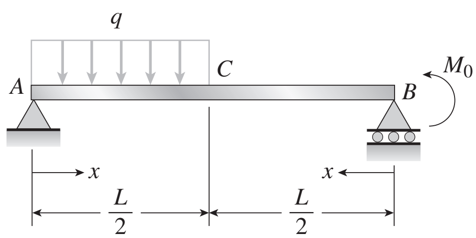
SET $P = 0$: $\quad \delta_B = \frac{q_0L^4}{30EI} \quad \longleftarrow$

Problem 9.9-5 A simple beam ACB supports a uniform load of intensity q on the left-hand half of the span (see figure).

Determine the angle of rotation θ_B at support B. (Obtain the solution by using the modified form of Castigliano's theorem.)



Solution 9.9-5 Simple beam with partial uniform load



M_0 = fictitious load corresponding to angle of rotation θ_B

$$R_A = \frac{3qL}{8} + \frac{M_0}{L} \qquad R_B = \frac{qL}{8} - \frac{M_0}{L}$$

Bending moment and partial derivative for segment AC

$$M_{AC} = R_A x - \frac{qx^2}{2} = \left(\frac{3qL}{8} + \frac{M_0}{L}\right)x - \frac{qx^2}{2}$$

$$\left(0 \le x \le \frac{L}{2}\right)$$

$$\frac{\partial M_{AC}}{\partial M_0} = \frac{x}{L}$$

Bending moment and partial derivative for segment CB

$$M_{CB} = R_B x + M_0 = \left(\frac{qL}{8} - \frac{M_0}{L}\right)x + M_0$$

$$\left(0 \le x \le \frac{L}{2}\right)$$

$$\frac{\partial M_{CB}}{\partial M_0} = -\frac{x}{L} + 1$$

Modified Castigliano's theorem (Eq. 9-88)

$$\theta_B = \int \left(\frac{M}{EI}\right)\left(\frac{\partial M}{\partial M_0}\right)dx$$

$$= \frac{1}{EI}\int_0^{L/2}\left[\left(\frac{3qL}{8} + \frac{M_0}{L}\right)x - \frac{qx^2}{2}\right]\left[\frac{x}{L}\right]dx$$

$$+ \frac{1}{EI}\int_0^{L/2}\left[\left(\frac{qL}{8} - \frac{M_0}{L}\right)x + M_0\right]\left[1 - \frac{x}{L}\right]dx$$

Set fictitious load M_0 equal to zero

$$\theta_B = \frac{1}{EI}\int_0^{L/2}\left(\frac{3qLx}{8} - \frac{qx^2}{2}\right)\left(\frac{x}{L}\right)dx$$

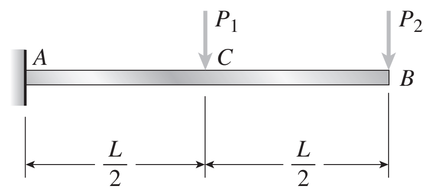
$$+ \frac{1}{EI}\int_0^{L/2}\left(\frac{qLx}{8}\right)\left(1 - \frac{x}{L}\right)dx$$

$$= \frac{qL^3}{128EI} + \frac{qL^3}{96EI} = \frac{7qL^3}{384EI} \quad \text{(counterclockwise)} \quad \longleftarrow$$

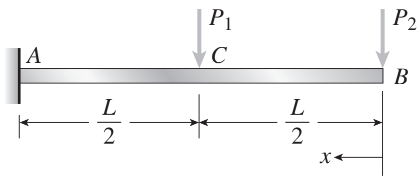
(This result agrees with Case 2, Table G-2.)

Problem 9.9-6 A cantilever beam ACB supports two concentrated loads P_1 and P_2, as shown in the figure.

Determine the deflections δ_C and δ_B at points C and B, respectively. (Obtain the solution by using the modified form of Castigliano's theorem.)



Solution 9.9-6 **Cantilever beam with loads P_1 and P_2**



BENDING MOMENT AND PARTIAL DERIVATIVES FOR SEGMENT CB

$$M_{CB} = -P_2 x \quad \left(0 \le x \le \frac{L}{2}\right)$$

$$\frac{\partial M_{CB}}{\partial P_1} = 0 \qquad \frac{\partial M_{CB}}{\partial P_2} = -x$$

BENDING MOMENT AND PARTIAL DERIVATIVES FOR SEGMENT AC

$$M_{AC} = -P_1\left(x - \frac{L}{2}\right) - P_2 x \quad \left(\frac{L}{2} \le x \le L\right)$$

$$\frac{\partial M_{AC}}{\partial P_1} = \frac{L}{2} - x \qquad \frac{\partial M_{AC}}{\partial P_2} = -x$$

MODIFIED CASTIGLIANO'S THEOREM FOR DEFLECTION δ_C

$$\delta_C = \frac{1}{EI}\int_0^{L/2} (M_{CB})\left(\frac{\partial M_{CB}}{\partial P_1}\right)dx + \frac{1}{EI}\int_{L/2}^{L} (M_{AC})\left(\frac{\partial M_{AC}}{\partial P_1}\right)dx$$

$$= 0 + \frac{1}{EI}\int_{L/2}^{L}\left[-P_1\left(x - \frac{L}{2}\right) - P_2 x\right]\left(\frac{L}{2} - x\right)dx$$

$$= \frac{L^3}{48\,EI}(2\,P_1 + 5\,P_2) \quad \longleftarrow$$

MODIFIED CASTIGLIANO'S THEOREM FOR DEFLECTION δ_B

$$\delta_B = \frac{1}{EI}\int_0^{L/2} (M_{CB})\left(\frac{\partial M_{CB}}{\partial P_2}\right)dx + \frac{1}{EI}\int_{L/2}^{L} (M_{AC})\left(\frac{\partial M_{AC}}{\partial P_2}\right)dx$$

$$= \frac{1}{EI}\int_0^{L/2} (-P_2 x)\,(-x)\,dx + \frac{1}{EI}\int_{L/2}^{L}\left[-P_1\left(x - \frac{L}{2}\right) - P_2 x\right](-x)\,dx$$

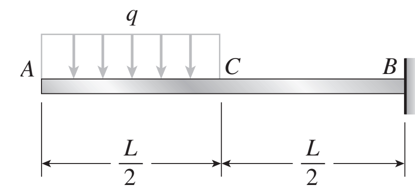
$$= \frac{P_2 L^3}{24\,EI} + \frac{L^3}{48\,EI}(5\,P_1 + 14\,P_2)$$

$$= \frac{L^3}{48\,EI}(5P_1 + 16P_2) \quad \longleftarrow$$

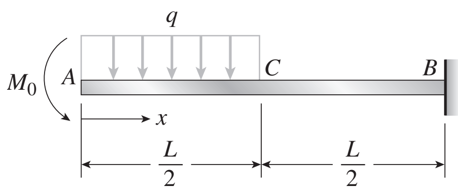
(These results can be verified with the aid of Cases 4 and 5, Table G-1.)

Problem 9.9-7 The cantilever beam ACB shown in the figure is subjected to a uniform load of intensity q acting between points A and C.

Determine the angle of rotation θ_A at the free end A. (Obtain the solution by using the modified form of Castigliano's theorem.)



Solution 9.9-7 Cantilever beam with partial uniform load



M_0 = fictitious load corresponding to the angle of rotation θ_A

Bending moment and partial derivative for segment AC

$$M_{AC} = -M_0 - \frac{qx^2}{2} \quad \left(0 \le x \le \frac{L}{2}\right)$$

$$\frac{\partial M_{AC}}{\partial M_0} = -1$$

Bending moment and partial derivative for segment CB

$$M_{CB} = -M_0 - \frac{qL}{2}\left(x - \frac{L}{4}\right) \quad \left(\frac{L}{2} \le x \le L\right)$$

$$\frac{\partial M_{CB}}{\partial M_0} = -1$$

Modified Castigliano's theorem (Eq. 9-88)

$$\theta_A = \int\left(\frac{M}{EI}\right)\left(\frac{\partial M}{\partial M_0}\right)dx$$

$$= \frac{1}{EI}\int_0^{L/2}\left(-M_0 - \frac{qx^2}{2}\right)(-1)\,dx$$

$$+ \frac{1}{EI}\int_{L/2}^{L}\left[-M_0 - \frac{qL}{2}\left(x - \frac{L}{4}\right)\right](-1)\,dx$$

Set fictitious load M_0 equal to zero

$$\theta_A = \frac{1}{EI}\int_0^{L/2}\frac{qx^2}{2}\,dx + \frac{1}{EI}\int_{L/2}^{L}\left(\frac{qL}{2}\right)\left(x - \frac{L}{4}\right)dx$$

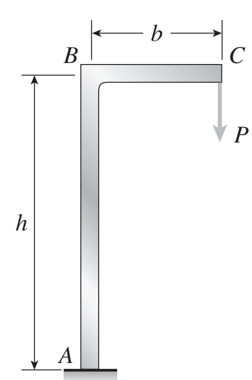
$$= \frac{qL^3}{48\,EI} + \frac{qL^3}{8\,EI}$$

$$= \frac{7qL^3}{48\,EI} \quad \text{(counterclockwise)} \quad \longleftarrow$$

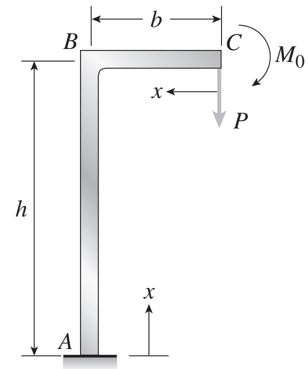
(This result can be verified with the aid of Case 3, Table G-1.)

Problem 9.9-8 The frame ABC supports a concentrated load P at point C (see figure). Members AB and BC have lengths h and b, respectively.

Determine the vertical deflection δ_C and angle of rotation θ_C at end C of the frame. (Obtain the solution by using the modified form of Castigliano's theorem.)



Solution 9.9-8 Frame with concentrated load



P = concentrated load acting at point C (corresponding to the deflection δ_C)

M_0 = fictitious moment corresponding to the angle of rotation θ_C

Bending moment and partial derivatives for member AB

$$M_{AB} = Pb + M_0 \quad (0 \leq x \leq h)$$

$$\frac{\partial M_{AB}}{\partial P} = b \qquad \frac{\partial M_{AB}}{M_0} = 1$$

Bending moment and partial derivatives for member BC

$$M_{BC} = Px + M_0 \quad (0 \leq x \leq b)$$

$$\frac{\partial M_{BC}}{\partial P} = x \qquad \frac{\partial M_{BC}}{\partial M_0} = 1$$

Modified Castigliano's theorem for deflection δ_C

$$\delta_C = \int \left(\frac{M}{EI}\right)\left(\frac{\partial M}{\partial P}\right) dx$$

$$= \frac{1}{EI}\int_0^h (Pb + M_0)(b)dx + \frac{1}{EI}\int_0^b (Px + M_0)(x)dx$$

Set $M_0 = 0$:

$$\delta_C = \frac{1}{EI}\int_0^h Pb^2 dx + \frac{1}{EI}\int_0^b Px^2 dx$$

$$= \frac{Pb^2}{3\,EI}(3h + b) \quad \text{(downward)} \quad \longleftarrow$$

Modified Castigliano's theorem for angle of rotation θ_C

$$\theta_C = \int \left(\frac{M}{EI}\right)\left(\frac{\partial M}{\partial M_0}\right) dx$$

$$= \frac{1}{EI}\int_0^h (Pb + M_0)(1)dx + \frac{1}{EI}\int_0^b (Px + M_0)(1)\,dx$$

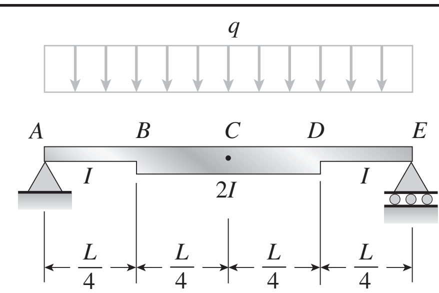
Set $M_0 = 0$:

$$\theta_C = \frac{1}{EI}\int_0^h Pb\,dx + \frac{1}{EI}\int_0^b Px\,dx$$

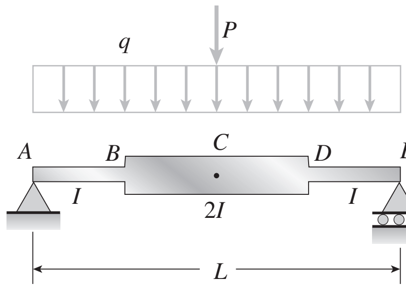
$$= \frac{Pb}{2EI}(2h + b) \quad \text{(clockwise)} \quad \longleftarrow$$

Problem 9.9-9 A simple beam $ABCDE$ supports a uniform load of intensity q (see figure). The moment of inertia in the central part of the beam (BCD) is twice the moment of inertia in the end parts (AB and DE).

Find the deflection δ_C at the midpoint C of the beam. (Obtain the solution by using the modified form of Castigliano's theorem.)



Solution 9.9-9 **Nonprismatic beam**



$P =$ fictitious load corresponding to the deflection δ_C at the midpoint

$$R_A = \frac{qL}{2} + \frac{P}{2}$$

BENDING MOMENT AND PARTIAL DERIVATIVE FOR THE LEFT-HAND HALF OF THE BEAM (A TO C)

$$M_{AC} = \frac{qLx}{2} - \frac{qx^2}{2} + \frac{Px}{2} \quad \left(0 \le x \le \frac{L}{2}\right)$$

$$\frac{\partial M_{AC}}{\partial P} = \frac{x}{2} \quad \left(0 \le x \le \frac{L}{2}\right)$$

MODIFIED CASTIGLIANO'S THEOREM (EQ. 9-88)

Integrate from A to C and multiply by 2.

$$\delta_C = 2\int\left(\frac{M_{AC}}{EI}\right)\left(\frac{\partial M_{AC}}{\partial P}\right)dx$$

$$= 2\left(\frac{1}{EI}\right)\int_0^{L/4}\left(\frac{qLx}{2} - \frac{qx^2}{2} + \frac{Px}{2}\right)\left(\frac{x}{2}\right)dx$$

$$+ 2\left(\frac{1}{2EI}\right)\int_{L/4}^{L/2}\left(\frac{qLx}{2} - \frac{qx^2}{2} + \frac{Px}{2}\right)\left(\frac{x}{2}\right)dx$$

SET FICTITIOUS LOAD P EQUAL TO ZERO

$$\delta_C = \frac{2}{EI}\int_0^{L/4}\left(\frac{qLx}{2} - \frac{qx^2}{2}\right)\left(\frac{x}{2}\right)dx$$

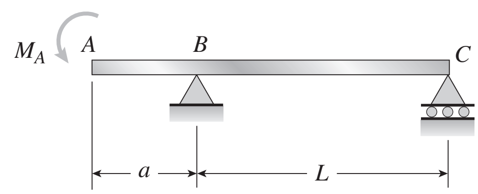
$$+ \frac{1}{EI}\int_{L/4}^{L/2}\left(\frac{qLx}{2} - \frac{qx^2}{2}\right)\left(\frac{x}{2}\right)dx$$

$$= \frac{13\,qL^4}{6{,}144\,EI} + \frac{67\,qL^4}{12{,}288\,EI}$$

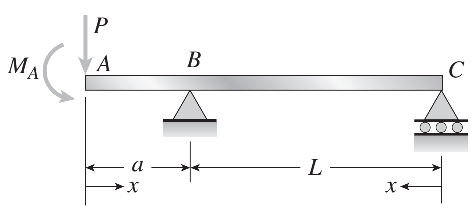
$$\delta_C = \frac{31qL^4}{4096\,EI} \quad \text{(downward)} \quad \longleftarrow$$

Problem 9.9-10 An overhanging beam ABC is subjected to a couple M_A at the free end (see figure). The lengths of the overhang and the main span are a and L, respectively.

Determine the angle of rotation θ_A and deflection δ_A at end A. (Obtain the solution by using the modified form of Castigliano's theorem.)



Solution 9.9-10 **Overhanging beam** ABC



$M_A =$ couple acting at the free end A (corresponding to the angle of rotation θ_A)

$P =$ fictitious load corresponding to the deflection δ_A

BENDING MOMENT AND PARTIAL DERIVATIVES FOR SEGMENT AB

$$M_{AB} = -M_A - Px \quad (0 \le x \le a)$$

$$\frac{\partial M_{AB}}{\partial M_A} = -1 \qquad \frac{\partial M_{AB}}{\partial P} = -x$$

BENDING MOMENT AND PARTIAL DERIVATIVES FOR SEGMENT BC

Reaction at support C: $R_C = \dfrac{M_A}{L} + \dfrac{Pa}{L}$ (downward)

$$M_{BC} = -R_C x = -\frac{M_A x}{L} - \frac{Pax}{L} \quad (0 \le x \le L)$$

$$\frac{\partial M_{BC}}{\partial M_A} = -\frac{x}{L} \qquad \frac{\partial M_{BC}}{\partial P} = -\frac{ax}{L}$$

Modified Castigliano's theorem for angle of rotation θ_A

$$\theta_A = \int \left(\frac{M}{EI}\right)\left(\frac{\partial M}{\partial M_A}\right) dx$$

$$= \frac{1}{EI}\int_0^a (-M_A - Px)(-1)\, dx + \frac{1}{EI}\int_0^L \left(-\frac{M_A x}{L} - \frac{Pax}{L}\right)\left(-\frac{x}{L}\right) dx$$

Set $P = 0$:

$$\theta_A = \frac{1}{EI}\int_0^a M_A dx + \frac{1}{EI}\int_0^L \left(\frac{M_A x}{L}\right)\left(\frac{x}{L}\right) dx$$

$$= \frac{M_A}{3\,EI}(L + 3a) \quad \text{(counterclockwise)} \longleftarrow$$

Modified Castigliano's theorem for deflection δ_A

$$\delta_A = \int \left(\frac{M}{EI}\right)\left(\frac{\partial M}{\partial P}\right) dx$$

$$= \frac{1}{EI}\int_0^a (-M_A - Px)(-x) dx + \frac{1}{EI}\int_0^L \left(-\frac{M_A x}{L} - \frac{Pax}{L}\right)\left(-\frac{ax}{L}\right) dx$$

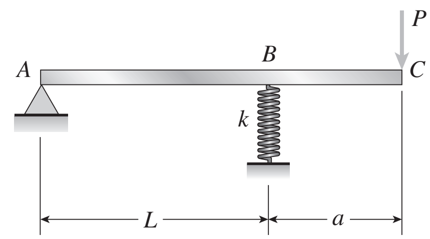
Set $P = 0$:

$$\delta_A = \frac{1}{EI}\int_0^a M_A x dx + \frac{1}{EI}\int_0^L \left(\frac{M_A x}{L}\right)\left(\frac{ax}{L}\right) dx$$

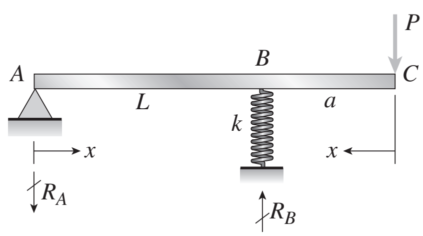
$$= \frac{M_A a}{6EI}(2L + 3a) \quad \text{(downward)} \longleftarrow$$

Problem 9.9-11 An overhanging beam ABC rests on a simple support at A and a spring support at B (see figure). A concentrated load P acts at the end of the overhang. Span AB has length L, the overhang has length a, and the spring has stiffness k.

Determine the downward displacement δ_C of the end of the overhang. (Obtain the solution by using the modified form of Castigliano's theorem.)



Solution 9.9-11 **Beam with spring support**



$$R_A = \frac{Pa}{L} \quad \text{(downward)}$$

$$R_B = \frac{P}{L}(L + a) \quad \text{(upward)}$$

Bending moment and partial derivative for segment AB

$$M_{AB} = -R_A x = -\frac{Pax}{L} \qquad \frac{dM_{AB}}{dP} = -\frac{ax}{L} \qquad (0 \le x \le L)$$

Bending moment and partial derivative for segment BC

$$M_{BC} = -Px \qquad \frac{dM_{BC}}{dP} = -x \qquad (0 \le x \le a)$$

Strain energy of the spring (Eq. 2-38a)

$$U_S = \frac{R_B^2}{2k} = \frac{P^2(L + a)^2}{2\,kL^2}$$

Strain energy of the beam (Eq. 9-80a)

$$U_B = \int \frac{M^2 dx}{2\,EI}$$

Total strain energy U

$$U = U_B + U_S = \int \frac{M^2 dx}{2\,EI} + \frac{P^2(L + a)^2}{2\,kL^2}$$

Apply Castigliano's theorem (Eq. 9-87)

$$\delta_C = \frac{dU}{dP} = \frac{d}{dP}\int \frac{M^2 dx}{2\,EI} + \frac{d}{dP}\left[\frac{P^2(L + a)^2}{2\,kL^2}\right]$$

$$= \frac{d}{dP}\int \frac{M^2 dx}{2\,EI} + \frac{P(L + a)^2}{kL^2}$$

DIFFERENTIATE UNDER THE INTEGRAL SIGN (MODIFIED CASTIGLIANO'S THEOREM)

$$\delta_C = \int\left(\frac{M}{EI}\right)\left(\frac{dM}{dP}\right)dx + \frac{P(L+a)^2}{kL^2}$$

$$= \frac{1}{EI}\int_0^L\left(-\frac{Pax}{L}\right)\left(-\frac{ax}{L}\right)dx$$

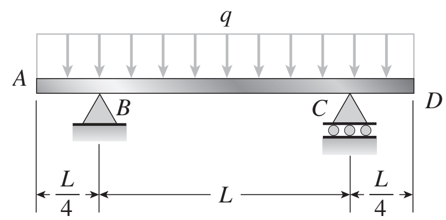
$$+ \frac{1}{EI}\int_0^a(-Px)(-x)dx + \frac{P(L+a)^2}{kL^2}$$

$$= \frac{Pa^2L}{3\,EI} + \frac{Pa^3}{3\,EI} + \frac{P(L+a)^2}{kL^2}$$

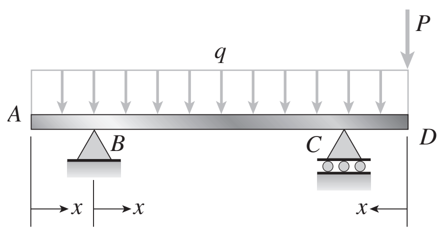
$$\delta_C = \frac{Pa^2(L+a)}{3\,EI} + \frac{P(L+a)^2}{kL^2} \longleftarrow$$

Problem 9.9-12 A symmetric beam $ABCD$ with overhangs at both ends supports a uniform load of intensity q (see figure).

Determine the deflection δ_D at the end of the overhang. (Obtain the solution by using the modified form of Castigliano's theorem.)



Solution 9.9-12 **Beam with overhangs**



q = intensity of uniform load

P = fictitious load corresponding to the deflection δ_D

$\frac{L}{4}$ = length of segments AB and CD

L = length of span BC

$$R_B = \frac{3\,qL}{4} - \frac{P}{4} \qquad R_C = \frac{3\,qL}{4} + \frac{5P}{4}$$

BENDING MOMENTS AND PARTIAL DERIVATIVES

SEGMENT AB

$$M_{AB} = -\frac{qx^2}{2} \qquad \frac{\partial M_{AB}}{\partial P} = 0 \qquad \left(0 \le x \le \frac{L}{4}\right)$$

SEGMENT BC

$$M_{BC} = -\left[q\left(x+\frac{L}{4}\right)\right]\left[\frac{1}{2}\left(x+\frac{L}{4}\right)\right] + R_Bx$$

$$= -\frac{q}{2}\left(x+\frac{L}{4}\right)^2 + \left(\frac{3\,qL}{4} - \frac{P}{4}\right)x \qquad (0 \le x \le L)$$

$$\frac{\partial M_{BC}}{\partial P} = -\frac{x}{4}$$

SEGMENT CD $\quad M_{CD} = -\frac{qx^2}{2} - Px \quad \left(0 \le x \le \frac{L}{4}\right)$

$$\frac{\partial M_{CD}}{\partial P} = -x$$

MODIFIED CASTIGLIANO'S THEOREM FOR DEFLECTION δ_D

$$\delta_D = \int\left(\frac{M}{EI}\right)\left(\frac{\partial M}{\partial P}\right)dx$$

$$= \frac{1}{EI}\int_0^{L/4}\left(-\frac{qx^2}{2}\right)(0)\,dx$$

$$+ \frac{1}{EI}\int_0^L\left[-\frac{q}{2}\left(x+\frac{L}{4}\right)^2 + \left(\frac{3qL}{4} - \frac{P}{4}\right)x\right]$$

$$\left[-\frac{x}{4}\right]dx + \frac{1}{EI}\int_0^{L/4}\left(-\frac{qx^2}{2} - Px\right)(-x)dx$$

SET $P = 0$:

$$\delta_D = \frac{1}{EI}\int_0^L\left[-\frac{q}{2}\left(x+\frac{L}{4}\right)^2 + \frac{3qL}{4}x\right]\left[-\frac{x}{4}\right]dx$$

$$+ \frac{1}{EI}\int_0^{L/4}\left(-\frac{qx^2}{2}\right)(-x)\,dx$$

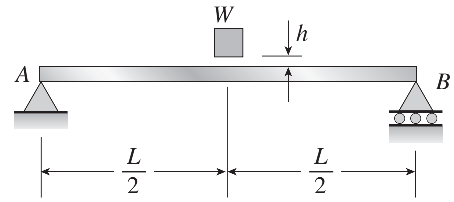
$$= -\frac{5\,qL^4}{768\,EI} + \frac{qL^4}{2048\,EI} = -\frac{37\,qL^4}{6144\,EI}$$

(Minus means the deflection is opposite in direction to the fictitious load P.)

$$\therefore\ \delta_D = \frac{37\,qL^4}{6144\,EI} \quad \text{(upward)} \longleftarrow$$

Deflections Produced by Impact

The beams described in the problems for Section 9.10 have constant flexural rigidity EI. Disregard the weights of the beams themselves, and consider only the effects of the given loads.



Problem 9.10-1 A heavy object of weight W is dropped onto the midpoint of a simple beam AB from a height h (see figure).

Obtain a formula for the maximum bending stress σ_{max} due to the falling weight in terms of h, σ_{st}, and δ_{st}, where σ_{st} is the maximum bending stress and δ_{st} is the deflection at the midpoint when the weight W acts on the beam as a statically applied load.

Plot a graph of the ratio σ_{max}/σ_{st} (that is, the ratio of the dynamic stress to the static stress) versus the ratio h/δ_{st}. (Let h/δ_{st} vary from 0 to 10.)

Solution 9.10-1 **Weight W dropping onto a simple beam**

MAXIMUM DEFLECTION (EQ. 9-94)

$$\delta_{max} = \delta_{st} + (\delta_{st}^2 + 2h\delta_{st})^{1/2}$$

MAXIMUM BENDING STRESS

For a linearly elastic beam, the bending stress σ is proportional to the deflection δ.

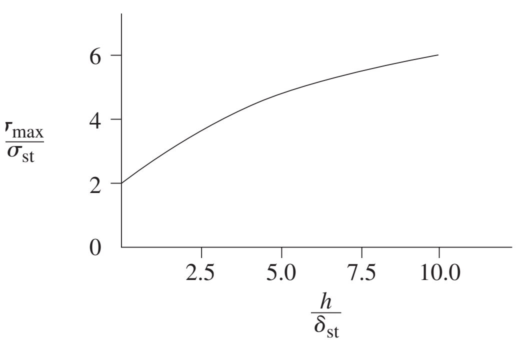
$$\therefore \quad \frac{\sigma_{max}}{\sigma_{st}} = \frac{\delta_{max}}{\delta_{st}} = 1 + \left(1 + \frac{2h}{\delta_{st}}\right)^{1/2}$$

$$\sigma_{max} = \sigma_{st}\left[1 + \left(1 + \frac{2h}{\delta_{st}}\right)^{1/2}\right] \quad \longleftarrow$$

$\frac{h}{\delta_{st}}$	$\frac{\sigma_{max}}{\sigma_{st}}$
0	2.00
2.5	3.45
5.0	4.33
7.5	5.00
10.0	5.58

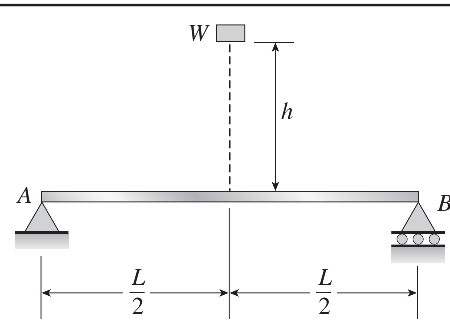
NOTE: $\delta_{st} = \dfrac{WL^3}{48EI}$ for a simple beam with a load at the midpoint.

GRAPH OF RATIO σ_{max}/σ_{st}



Problem 9.10-2 An object of weight W is dropped onto the midpoint of a simple beam AB from a height h (see figure). The beam has a rectangular cross section of area A.

Assuming that h is very large compared to the deflection of the beam when the weight W is applied statically, obtain a formula for the maximum bending stress σ_{max} in the beam due to the falling weight.



Solution 9.10-2 Weight W dropping onto a simple beam

Height h is very large.

MAXIMUM DEFLECTION (EQ. 9-95)

$$\delta_{max} = \sqrt{2h\delta_{st}}$$

MAXIMUM BENDING STRESS

For a linearly elastic beam, the bending stress σ is proportional to the deflection δ.

$$\therefore \frac{\sigma_{max}}{\sigma_{st}} = \frac{\delta_{max}}{\delta_{st}} = \sqrt{\frac{2h}{\delta_{st}}}$$

$$\sigma_{max} = \sqrt{\frac{2h\sigma_{st}^2}{\delta_{st}}} \qquad (1)$$

$$\sigma_{st} = \frac{M}{S} = \frac{WL}{4S} \qquad \sigma_{st}^2 = \frac{W^2L^2}{16S^2}$$

$$\delta_{st} = \frac{WL^3}{48EI} \qquad \frac{\sigma_{st}^2}{\delta_{st}} = \frac{3WEI}{S^2L} \qquad (2)$$

For a RECTANGULAR BEAM (with b, depth d):

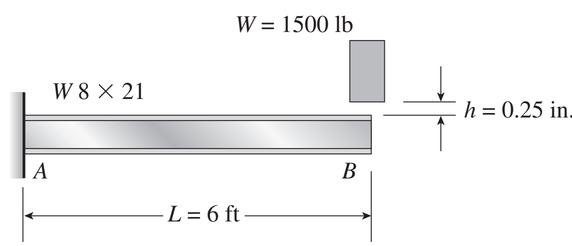
$$I = \frac{bd^3}{12} \qquad S = \frac{bd^2}{6} \qquad \frac{I}{S^2} = \frac{3}{bd} = \frac{3}{A} \qquad (3)$$

Substitute (2) and (3) into (1):

$$\sigma_{max} = \sqrt{\frac{18WhE}{AL}} \longleftarrow$$

Problem 9.10-3 A cantilever beam AB of length $L = 6$ ft is constructed of a W 8×21 wide-flange section (see figure). A weight $W = 1500$ lb falls through a height $h = 0.25$ in. onto the end of the beam.

Calculate the maximum deflection δ_{max} of the end of the beam and the maximum bending stress σ_{max} due to the falling weight. (Assume $E = 30 \times 10^6$ psi.)



Solution 9.10-3 Cantilever beam

DATA: $L = 6$ ft $= 72$ in. $\quad W = 1500$ lb
$h = 0.25$ in. $\quad E = 30 \times 10^6$ psi
W 8×21 $\quad I = 75.3$ in.^{4} $\quad S = 18.2$ in.^{3}

MAXIMUM DEFLECTION (EQ. 9-94)

Equation (9-94) may be used for any linearly elastic structure by substituting $\delta_{st} = W/k$, where k is the stiffress of the particular structure being considered. For instance: Simple beam with load at midpoint:

$$k = \frac{48EI}{L^3}$$

Cantilever beam with load at the free end: $k = \dfrac{3EI}{L^3}$ Etc.

For the cantilever beam in this problem:

$$\delta_{st} = \frac{WL^3}{3EI} = \frac{(1500 \text{ lb})(72 \text{ in.})^3}{3(30 \times 10^6 \text{ psi})(75.3 \text{ in.}^4)}$$

$$= 0.08261 \text{ in.}$$

Equation (9-94):

$$\delta_{max} = \delta_{st} + (\delta_{st}^2 + 2h\delta_{st})^{1/2} = 0.302 \textit{ in.} \longleftarrow$$

MAXIMUM BENDING STRESS

Consider a cantilever beam with load P at the free end:

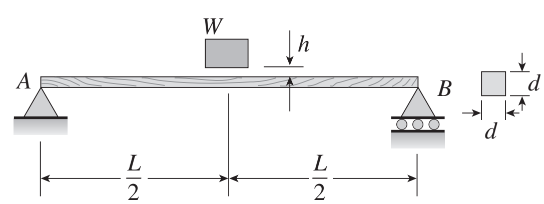
$$\sigma_{max} = \frac{M_{max}}{S} = \frac{PL}{S} \qquad \delta_{max} = \frac{PL^3}{3EI}$$

$$\text{Ratio: } \frac{\sigma_{max}}{\delta_{max}} = \frac{3EI}{SL^2}$$

$$\therefore \sigma_{max} = \frac{3EI}{SL^2}\delta_{max} = 21{,}700 \text{ psi} \longleftarrow$$

Problem 9.10-4 A weight $W = 20$ kN falls through a height $h = 1.0$ mm onto the midpoint of a simple beam of length $L = 3$ m (see figure). The beam is made of wood with square cross section (dimension d on each side) and $E = 12$ GPa.

If the allowable bending stress in the wood is $\sigma_{allow} = 10$ MPa, what is the minimum required dimension d?



Solution 9.10-4 Simple beam with falling weight W

DATA: $W = 20$ kN $h = 1.0$ mm $L = 3.0$ m
$E = 12$ GPa $\sigma_{allow} = 10$ MPa

CROSS SECTION OF BEAM (SQUARE)

d = dimension of each side

$$I = \frac{d^4}{12} \quad S = \frac{d^3}{6}$$

MAXIMUM DEFLECTION (EQ. 9-94)

$$\delta_{max} = \delta_{st} + (\delta_{st}^2 + 2h\delta_{st})^{1/2}$$

MAXIMUM BENDING STRESS

For a linearly elastic beam, the bending stress σ is proportional to the deflection δ.

$$\therefore \frac{\sigma_{max}}{\sigma_{st}} = \frac{\delta_{max}}{\delta_{st}} = 1 + \left(1 + \frac{2h}{\delta_{st}}\right)^{1/2} \quad (1)$$

STATIC TERMS σ_{st} AND δ_{st}

$$\sigma_{st} = \frac{M}{S} = \left(\frac{WL}{4}\right)\left(\frac{6}{d^3}\right) = \frac{3WL}{2d^3} \quad (2)$$

$$\delta_{st} = \frac{WL^3}{48EI} = \frac{WL^3}{48E}\left(\frac{12}{d^4}\right) = \frac{WL^3}{4Ed^4} \quad (3)$$

SUBSTITUTE (2) AND (3) INTO EQ. (1)

$$\frac{2\sigma_{max}d^3}{3WL} = 1 + \left(1 + \frac{8hEd^4}{WL^3}\right)^{1/2}$$

SUBSTITUTE NUMERICAL VALUES:

$$\frac{2(10 \text{ MPa})d^3}{3(20 \text{ kN})(3.0 \text{ m})} = 1 + \left[1 + \frac{8(1.0 \text{ mm})(12 \text{ GPa})d^4}{(20 \text{ kN})(3.0 \text{ m})^3}\right]^{1/2}$$

$$\frac{1000}{9}d^3 - 1 = \left[1 + \frac{1600}{9}d^4\right]^{1/2} \quad (d = \text{meters})$$

SQUARE BOTH SIDES, REARRANGE, AND SIMPLIFY

$$\left(\frac{1000}{9}\right)^2 d^3 - \frac{1600}{9}d - \frac{2000}{9} = 0$$

$$2500d^3 - 36d - 45 = 0 \quad (d = \text{meters})$$

SOLVE NUMERICALLY

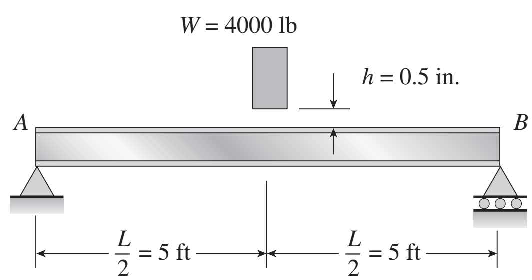
$d = 0.2804$ m $= 280.4$ mm

For minimum value, round upward.

$\therefore d = 281$ mm ←

Problem 9.10-5 A weight $W = 4000$ lb falls through a height $h = 0.5$ in. onto the midpoint of a simple beam of length $L = 10$ ft (see figure).

Assuming that the allowable bending stress in the beam is $\sigma_{allow} = 18{,}000$ psi and $E = 30 \times 10^6$ psi, select the lightest wide-flange beam listed in Table E-1 in Appendix E that will be satisfactory.



Solution 9.10-5 Simple beam of wide-flange shape

DATA: $W = 4000$ lb $h = 0.5$ in.
$L = 10$ ft $= 120$ in.
$\sigma_{allow} = 18{,}000$ psi $E = 30 \times 10^6$ psi

MAXIMUM DEFLECTION (EQ. 9-94)

$$\delta_{max} = \delta_{st} + (\delta_{st}^2 + 2h\delta_{st})^{1/2}$$

or $$\frac{\delta_{max}}{\delta_{st}} = 1 + \left(1 + \frac{2h}{\delta_{st}}\right)^{1/2}$$

MAXIMUM BENDING STRESS

For a linearly elastic beam, the bending stress σ is proportional to the deflection δ.

$$\therefore \frac{\sigma_{max}}{\sigma_{st}} = \frac{\delta_{max}}{\delta_{st}} = 1 + \left(1 + \frac{2h}{\delta_{st}}\right)^{1/2} \quad (1)$$

STATIC TERMS σ_{st} AND δ_{st}

$$\sigma_{st} = \frac{M}{S} = \frac{WL}{4S} \quad \delta_{st} = \frac{WL^3}{48EI}$$

$$\frac{\sigma_{max}}{\sigma_{st}} = \sigma_{allow}\left(\frac{4S}{WL}\right) = \frac{4\sigma_{allow}S}{WL} \quad (2)$$

$$\frac{2h}{\delta_{st}} = 2h\left(\frac{48EI}{WL^3}\right) = \frac{96hEI}{WL^3} \quad (3)$$

SUBSTITUTE (2) AND (3) INTO EQ. (1):

$$\frac{4\sigma_{allow}S}{WL} = 1 + \left(1 + \frac{96hEI}{WL^3}\right)^{1/2}$$

REQUIRED SECTION MODULUS

$$S = \frac{WL}{4\sigma_{allow}}\left[1 + \left(1 + \frac{96hEI}{WL^3}\right)^{1/2}\right]$$

SUBSTITUTE NUMERICAL VALUES

$$S = \left(\frac{20}{3}\text{ in.}^3\right)\left[1 + \left(1 + \frac{5\,I}{24}\right)^{1/2}\right] \qquad (4)$$

(S = in.^{3}; I = in.^{4})

PROCEDURE

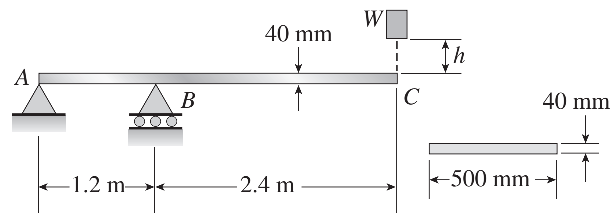
1. Select a trial beam from Table E-1.
2. Substitute I into Eq. (4) and calculate required S.
3. Compare with actual S for the beam.
4. Continue until the lightest beam is found.

Trial beam	Actual I	Actual S	Required S
W 8 × 35	127	31.2	41.6 (NG)
W 10 × 45	248	49.1	55.0 (NG)
W 10 × 60	341	66.7	63.3 (OK)
W 12 × 50	394	64.7	67.4 (NG)
W 14 × 53	541	77.8	77.8 (OK)
W 16 × 31	375	47.2	66.0 (NG)

Lightest beam is W 14 × 53 ←

Problem 9.10-6 An overhanging beam ABC of rectangular cross section has the dimensions shown in the figure. A weight W = 750 N drops onto end C of the beam.

If the allowable normal stress in bending is 45 MPa, what is the maximum height h from which the weight may be dropped? (Assume E = 12 GPa.)



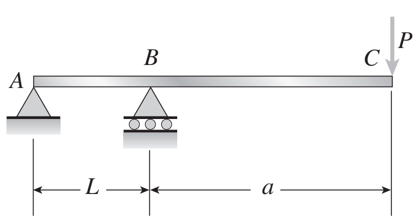
Solution 9.10-6 Overhanging beam

DATA: $W = 750$ N $\quad L_{AB} = 1.2$ in. $\quad L_{BC} = 2.4$ m
$E = 12$ GPa $\quad \sigma_{\text{allow}} = 45$ MPa

$$I = \frac{bd^3}{12} = \frac{1}{12}(500\text{ mm})(40\text{ mm})^3$$
$$= 2.6667 \times 10^6\text{ mm}^4$$
$$= 2.6667 \times 10^{-6}\text{ m}^4$$

$$S = \frac{bd^2}{6} = \frac{1}{6}(500\text{ mm})(40\text{ mm})^2$$
$$= 133.33 \times 10^3\text{ mm}^3$$
$$= 133.33 \times 10^{-6}\text{ m}^3$$

DEFLECTION δ_C AT THE END OF THE OVERHANG



P = load at end C
L = length of spear AB
a = length of overhang BC

From the answer to Prob. 9.8-5 or Prob. 9.9-3:

$$\delta_C = \frac{Pa^2(L + a)}{3\,EI}$$

Stiffness of the beam: $$k = \frac{P}{\delta_C} = \frac{3\,EI}{a^2(L + a)} \qquad (1)$$

MAXIMUM DEFLECTION (EQ. 9-94)

Equation (9-94) may be used for any linearly elastic structure by substituting $\delta_{st} = W/k$, where k is the stiffness of the particular structure being considered. For instance:

Simple beam with load at midpoint: $k = \dfrac{48\,EI}{L^3}$

Cantilever beam with load at free end: $k = \dfrac{3\,EI}{L^3}$ Etc.

For the overhanging beam in this problem (see Eq. 1):

$$\delta_{st} = \frac{W}{k} = \frac{Wa^2(L + a)}{3\,EI} \qquad (2)$$

in which $a = L_{BC}$ and $L = L_{AB}$:

$$\delta_{st} = \frac{W(L_{BC}^2)(L_{AB} + L_{BC})}{3\,EI} \qquad (3)$$

EQUATION (9-94):

$$\delta_{max} = \delta_{st} + (\delta_{st}^2 + 2\,h\,\delta_{st})^{1/2}$$

or

$$\frac{\delta_{max}}{\delta_{st}} = 1 + \left(1 + \frac{2h}{\delta_{st}}\right)^{1/2} \qquad (4)$$

MAXIMUM BENDING STRESS

For a linearly elastic beam, the bending stress σ is proportional to the deflection δ.

$$\therefore \frac{\sigma_{max}}{\sigma_{st}} = \frac{\delta_{max}}{\delta_{st}} = 1 + \left(1 + \frac{2h}{\delta_{st}}\right)^{1/2} \quad (5)$$

$$\sigma_{st} = \frac{M}{S} = \frac{WL_{BC}}{S} \quad (6)$$

MAXIMUM HEIGHT h

Solve Eq. (5) for h:

$$\frac{\sigma_{max}}{\sigma_{st}} - 1 = \left(1 + \frac{2h}{\delta_{st}}\right)^{1/2}$$

$$\left(\frac{\sigma_{max}}{\sigma_{st}}\right)^2 - 2\left(\frac{\sigma_{max}}{\sigma_{st}}\right) + 1 = 1 + \frac{2h}{\delta_{st}}$$

$$h = \frac{\delta_{st}}{2}\left(\frac{\sigma_{max}}{\sigma_{st}}\right)\left(\frac{\sigma_{max}}{\sigma_{st}} - 2\right) \quad (7)$$

Substitute δ_{st} from Eq. (3), σ_{st} from Eq. (6), and σ_{allow} for σ_{max}:

$$h = \frac{W(L_{BC}^2)(L_{AB} + L_{BC})}{6EI}\left(\frac{\sigma_{allow}S}{WL_{BC}}\right)\left(\frac{\sigma_{allow}S}{WL_{BC}} - 2\right) \quad (8)$$

SUBSTITUTE NUMERICAL VALUES INTO EQ. (8):

$$\frac{W(L_{BC}^2)\ (L_{AB} + L_{BC})}{6EI} = 0.08100 \text{ m}$$

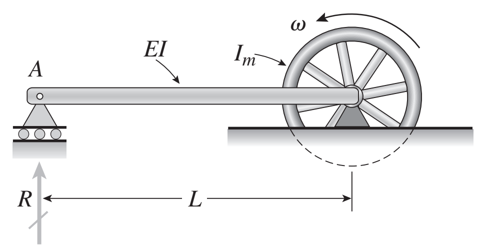
$$\frac{\sigma_{allow}S}{WL_{BC}} = \frac{10}{3} = 3.3333$$

$$h = (0.08100 \text{ m})\left(\frac{10}{3}\right)\left(\frac{10}{3} - 2\right) = 0.36 \text{ m}$$

or $h = 360$ mm ←

Problem 9.10-7 A heavy flywheel rotates at an angular speed ω (radians per second) around an axle (see figure). The axle is rigidly attached to the end of a simply supported beam of flexural rigidity EI .and length L (see figure). The flywheel has mass moment of inertia I_m about its axis of rotation.

If the flywheel suddenly freezes to the axle, what will be the reaction R at support A of the beam?



Solution 9.10-7 **Rotating flywheel**

NOTE: We will disregard the mass of the beam and all energy losses due to the sudden stopping of the rotating flywheel. Assume that *all* of the kinetic energy of the flywheel is transformed into strain energy of the beam.

KINETIC ENERGY OF ROTATING FLYWHEEL

$$kE = \frac{1}{2} I_m \omega^2$$

STRAIN ENERGY OF BEAM $U = \displaystyle\int \frac{M^2 dx}{2EI}$

$M = Rx$, where x is measured from support A.

$$U = \frac{1}{2EI}\int_{\infty}^{L} (Rx)^2 dx = \frac{R^2L^3}{6EI}$$

CONSERVATION OF ENERGY

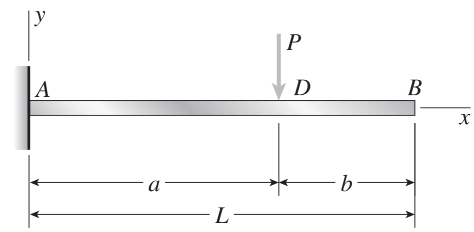
$$kE = U \qquad \frac{1}{2} I_m \omega^2 = \frac{R^2 L^3}{6EI}$$

$$R = \sqrt{\frac{3EI\, I_m \omega^2}{L^3}} \quad \leftarrow$$

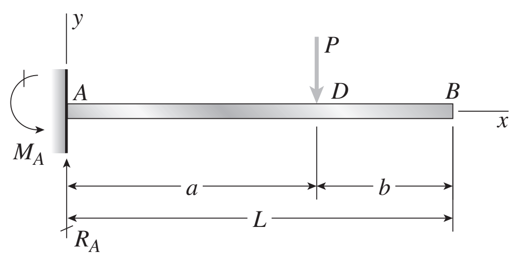
NOTE: The moment of inertia I_M has units of kg · m^2 or N · m · s^2

Representation of Loads on Beams by Discontinuity Functions

Problem 9.11-1 through 9.11-12 A beam and its loading are shown in the figure. Using discontinuity functions, write the expression for the intensity $q(x)$ of the equivalent distributed load acting on the beam (include the reactions in the expression for the equivalent load).



Solution 9.11-1 Cantilever beam



FROM EQUILIBRIUM:

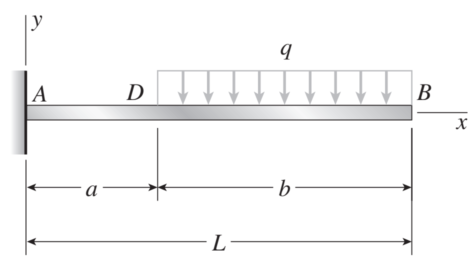
$R_A = P \qquad M_A = Pa$

USE TABLE 9-2.

$$q(x) = -R_A\langle x\rangle^{-1} + M_A\langle x\rangle^{-2} + P\langle x-a\rangle^{-1}$$

$$= -P\langle x\rangle^{-1} + Pa\langle x\rangle^{-2} + P\langle x-a\rangle^{-1} \quad \longleftarrow$$

Problem 9.11-2



Solution 9.11-2 Cantilever beam

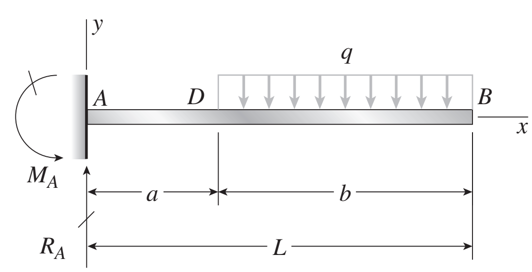
FROM EQUILIBRIUM: $R_A = qb \qquad M_A = \dfrac{qb}{2}(2a+b)$

USE TABLE 9-2.

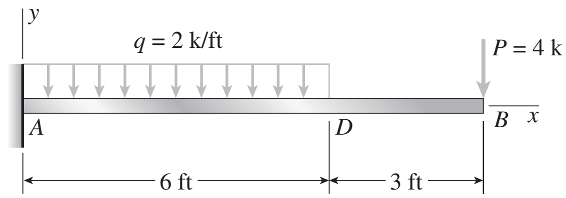
$$q(x) = -R_A\langle x\rangle^{-1} + M_A\langle x\rangle^{-2} + q\langle x-a\rangle^{0} - q\langle x-L\rangle^{0}$$

$$= -qb\langle x\rangle^{-1} + \frac{qb}{2}(2a+b)\langle x\rangle^{-2}$$

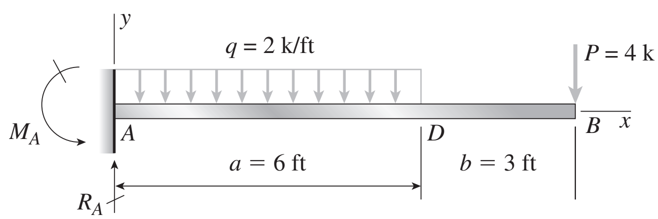
$$+ q\langle x-a\rangle^{0} - q\langle x-L\rangle^{0} \quad \longleftarrow$$



Problem 9.11-3



Solution 9.11-3 **Cantilever beam**



$q = 2\text{ k/ft} = \dfrac{1}{6}\text{ k/in.}$

$a = 6\text{ ft} = 72\text{ in.}$
$b = 3\text{ ft} = 36\text{ in.}$
$L = 9\text{ ft} = 108\text{ in.}$

FROM EQUILIBRIUM:

$R_A = 16\text{ k} \qquad M_A = 864\text{ k-in.}$

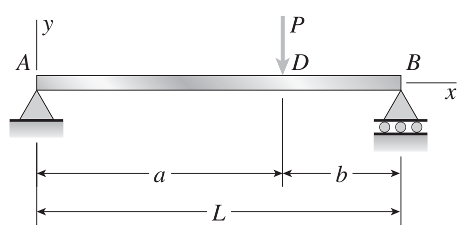
USE TABLE 9-2. Units: kips, inches

$$q(x) = -R_A\langle x\rangle^{-1} + M_A\langle x\rangle^{-2} + q\langle x\rangle^{0} - q\langle x-a\rangle^{0} + P\langle x-L\rangle^{-1}$$

$$= -16\langle x\rangle^{-1} + 864\langle x\rangle^{-2} + \frac{1}{6}\langle x\rangle^{0} - \frac{1}{6}\langle x-72\rangle^{0} + 4\langle x-108\rangle^{-1} \longleftarrow$$

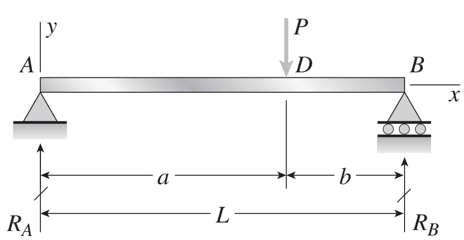
(Units: x = in., q = k/in.)

Problem 9.11-4



Solution 9.11-4 **Simple beam**

FROM EQUILIBRIUM: $R_A = \dfrac{pb}{L} \qquad R_B = \dfrac{Pa}{L}$

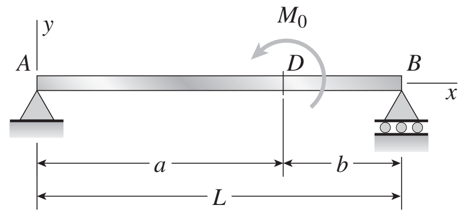


USE TABLE 9-2.

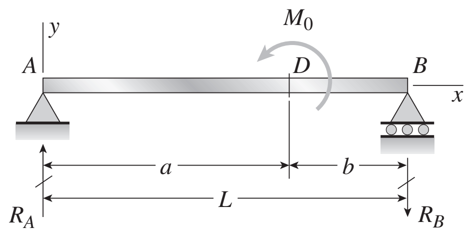
$$q(x) = -R_A\langle x\rangle^{-1} + P\langle x-a\rangle^{-1} - R_B\langle x-L\rangle^{-1}$$

$$= -\frac{Pb}{L}\langle x\rangle^{-1} + P\langle x-a\rangle^{-1} - \frac{Pa}{L}\langle x-L\rangle^{-1} \longleftarrow$$

Problem 9.11-5



Solution 9.11-5 Simple beam



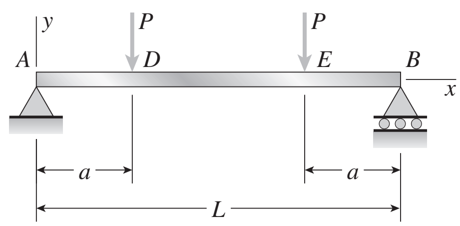
FROM EQUILIBRIUM: $R_A = \frac{M_0}{L} \quad R_B = \frac{M_0}{L}$ (downward)

USE TABLE 9-2.

$$q(x) = -R_A \langle x \rangle^{-1} + M_0 \langle x - a \rangle^{-2} + R_B \langle x - L \rangle^{-1}$$

$$= -\frac{M_0}{L} \langle x \rangle^{-1} + M_0 \langle x - a \rangle^{-2} + \frac{M_0}{L} \langle x - L \rangle^{-1} \quad \longleftarrow$$

Problem 9.11-6



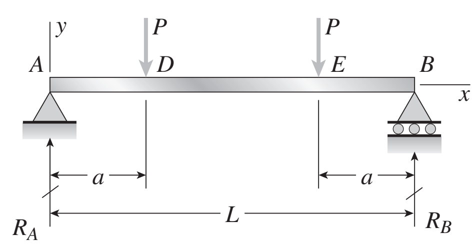
Solution 9.11-6 Simple beam

FROM EQUILIBRIUM: $R_A = R_B = P$

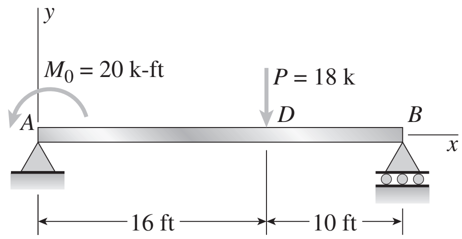
USE TABLE 9-2.

$$q(x) = -R_A \langle x \rangle^{-1} + P \langle x - a \rangle^{-1} + P \langle x - L + a \rangle^{-1} - R_B \langle x - L \rangle^{-1}$$

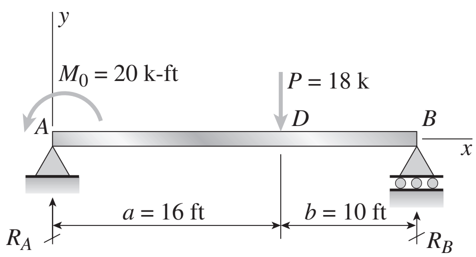
$$= -P \langle x \rangle^{-1} + P \langle x - a \rangle^{-1} + P \langle x - L + a \rangle^{-1} - P \langle x - L \rangle^{-1} \quad \longleftarrow$$



Problem 9.11-7



Solution 9.11-7 **Simple beam**



$M_0 = 20$ k-ft $= 240$ k-in. $\quad P = 18$ k
$a = 16$ ft $= 192$ in. $\quad b = 10$ ft $= 120$ in.
$L = 26$ ft $= 312$ in.

FROM EQUILIBRIUM: $R_A = 7.692$ k $\quad R_B = 10.308$ k

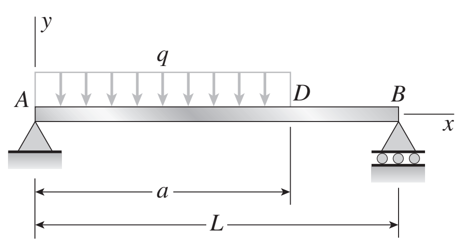
USE TABLE 9-2. Units: kips, inches

$$q(x) = -R_A \langle x \rangle^{-1} + M_0 \langle x \rangle^{-2} + P \langle x - a \rangle^{-1} - R_B \langle x - L \rangle^{-1}$$

$$= -7.692 \langle x \rangle^{-1} + 240 \langle x \rangle^{-2} + 18 \langle x - 192 \rangle^{-1} - 10.308 \langle x - 312 \rangle^{-1} \longleftarrow$$

(Units: $x =$ in., $q =$ k/in.)

Problem 9.11-8



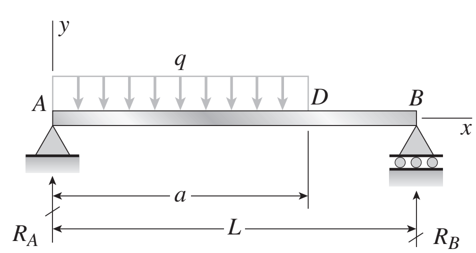
Solution 9.11-8 **Simple beam**

FROM EQUILIBRIUM: $R_A = \dfrac{qa}{2L}(2L - a) \qquad R_B = \dfrac{qa^2}{2L}$

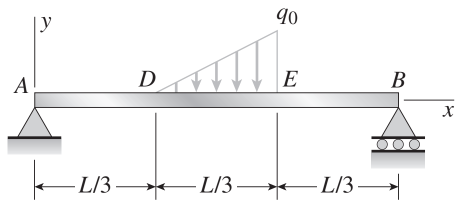
USE TABLE 9-2.

$$q(x) = -R_A \langle x \rangle^{-1} + q \langle x \rangle^{0} - q \langle x - a \rangle^{0} - R_B \langle x - L \rangle^{-1}$$

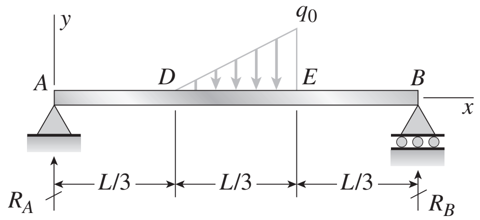
$$= -(qa/2L)(2L - a) \langle x \rangle^{-1} + q \langle x \rangle^{0} - q \langle x - a \rangle^{0} - (qa^2/2L) \langle x - L \rangle^{-1} \longleftarrow$$



Problem 9.11-9



Solution 9.11-9 Simple beam



FROM EQUILIBRIUM: $R_A = \dfrac{2q_0L}{27}$ $\quad R_B = \dfrac{5q_0L}{54}$

USE TABLE 9-2.

$$q(x) = -R_A\langle x\rangle^{-1} + \frac{3q_0}{L}\langle x - \frac{L}{3}\rangle^1 - \frac{3q_0}{L}\langle x - \frac{2L}{3}\rangle^1$$

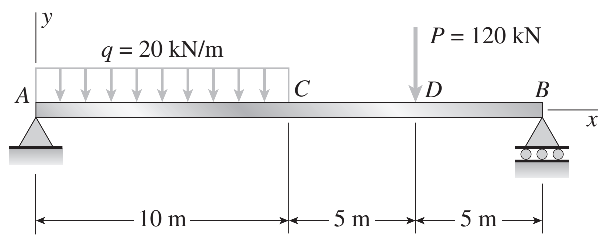
$$-q_0\langle x - \frac{2L}{3}\rangle^0 - R_B\langle x - L\rangle^{-1}$$

$$= -(2q_0L/27)\langle x\rangle^{-1} + (3q_0/L)\langle x - L/3\rangle^1$$

$$-(3q_0/L)\langle x - 2L/3\rangle^1 - q_0\langle x - 2L/3\rangle^0$$

$$-(5q_0L/54)\langle x - L\rangle^{-1} \longleftarrow$$

Problem 9.11-10



Solution 9.11-10 Simple beam

FROM EQUILIBRIUM: $R_A = 180$ kN $\quad R_B = 140$ kN

USE TABLE 9-2. Units: kilonewtons, meters

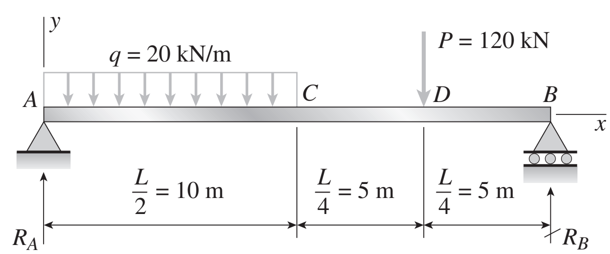
$$q(x) = -R_A\langle x\rangle^{-1} + q\langle x\rangle^0 - q\langle x - L/2\rangle^0$$

$$+ P\langle x - 3L/4\rangle^{-1} - R_B\langle x - L\rangle^{-1}$$

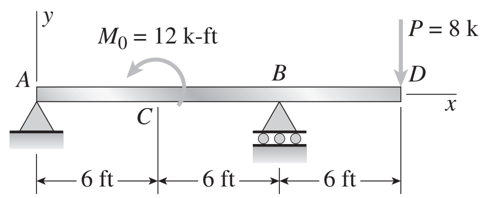
$$= -180\langle x\rangle^{-1} + 20\langle x\rangle^0 - 20\langle x - 10\rangle^0$$

$$+ 120\langle x - 15\rangle^{-1} - 140\langle x - 20\rangle^{-1}$$

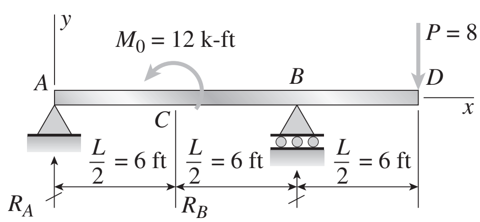
(Units: x = meters, q = kN/m)



Problem 9.11-11



Solution 9.11-11 Beam with an overhang



$M_0 = 12$ k-ft $= 144$ k-in.

$\frac{L}{2} = 6$ ft $= 72$ in.

$L = 12$ ft $= 144$ in.

FROM EQUILIBRIUM: $R_A = 3$ k (downward)
$R_B = 11$ k (upward)

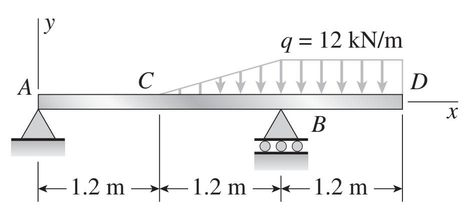
USE TABLE 9-2. Units: kips, inches

$$q(x) = R_A\langle x\rangle^{-1} + M_0\langle x - L/2\rangle^{-2} - R_B\langle x - L\rangle^{-1} + P\langle x - 3L/2\rangle^{-1}$$

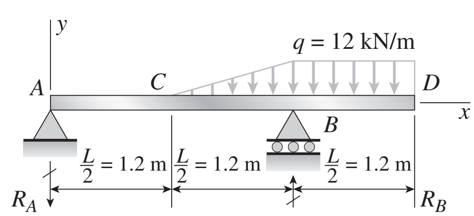
$$= 3\langle x\rangle^{-1} + 144\langle x - 72\rangle^{-2} - 11\langle x - 144\rangle^{-1} + 8\langle x - 216\rangle^{-1} \longleftarrow$$

(Units: $x =$ in., $q =$ kN/in.)

Problem 9.11-12



Solution 9.11-12 Beam with an overhang



$q = 12$ kN/m

$\frac{L}{2} = 1.2$ m

$L = 2.4$ m

FROM EQUILIBRIUM: $R_A = 2.4$ kN (downward)
$R_B = 24.0$ kN (upward)

USE TABLE 9-2. Units: kilonewtons, meters

$$q(x) = R_A\langle x\rangle^{-1} + \frac{q}{L/2}\langle x - L/2\rangle^{1} - \frac{q}{L/2}\langle x - L\rangle^{1} - q\langle x - L\rangle^{0} - R_B\langle x - L\rangle^{-1} + q\langle x - L\rangle^{0} - q\langle x - 3L/2\rangle^{0}$$

$$= 2.4\langle x\rangle^{-1} + 10\langle x - 1.2\rangle^{1} - 10\langle x - 2.4\rangle^{1} - 12\langle x - 2.4\rangle^{0} - 24\langle x - 2.4\rangle^{-1} + 12\langle x - 2.4\rangle^{0} - 12\langle x - 3.6\rangle^{0}$$

$$= 2.4\langle x\rangle^{-1} + 10\langle x - 1.2\rangle^{1} - 10\langle x - 2.4\rangle^{1} - 24\langle x - 2.4\rangle^{-1} - 12\langle x - 3.6\rangle^{0} \longleftarrow$$

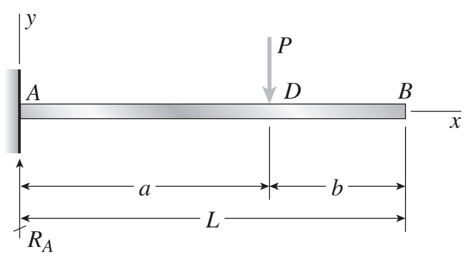
(Units: $x =$ meters, $q =$ kN/m)

Beam Deflections Using Discontinuity Functions

The problems for Section 9.12 are to be solved by using discontinuity functions. All beams have constant flexural rigidity EI. (Obtain the equations for the equivalent distributed loads from the corresponding problems in Section 9.11.)

Problem 9.12-1, 9.12-2, and 9.12-3 Determine the equation of the deflection curve for the cantilever beam *ADB* shown in the figure. Also, obtain the angle of rotation θ_B and deflection δ_B at the free end. (For the beam of Problem 9.12-3, assume $E = 10 \times 10^3$ ksi and $I = 450$ in.^{4})

Solution 9.12-1 Cantilever beam



FROM PROB: 9.11-1:

$$EIv'''' = -q(x) = P\langle x\rangle^{-1} - Pa\langle x\rangle^{-2} - P\langle x-a\rangle^{-1}$$

INTEGRATE THE EQUATION

$$EIv''' = V = P\langle x\rangle^{0} - Pa\langle x\rangle^{-1} - P\langle x-a\rangle^{0}$$

$$EIv'' = M = P\langle x\rangle^{1} - Pa\langle x\rangle^{0} - P\langle x-a\rangle^{1}$$

Note: $\langle x\rangle^1 = x$ and $\langle x\rangle^0 = 1$

$$EIv' = Px^2/2 - Pax - (P/2)\langle x-a\rangle^2 + C_1$$

B.C. $v'(0) = 0$ $\quad EI(0) = 0 - 0 - 0 + C_1$

$\therefore C_1 = 0$

$$EIv = Px^3/6 - Pax^2/2 - (P/6)\langle x-a\rangle^3 + C_2$$

B.C. $v(0) = 0$ $\quad EI(0) = 0 - 0 - 0 + C_2$

$\therefore C_2 = 0$

FINAL EQUATIONS

$$EIv' = (Px/2)(x - 2a) - (P/2)\langle x-a\rangle^2$$

$$EIv = (Px^2/6)(x - 3a) - (P/6)\langle x-a\rangle^3 \quad \longleftarrow$$

θ_B = CLOCKWISE ROTATION AT END B $(x = L)$

$$EIv'(L) = (PL/2)(L - 2a) - (P/2)\langle L-a\rangle^2$$
$$= (PL/2)(L - 2a) - (P/2)(L - a)^2$$
$$= -Pa^2/2$$

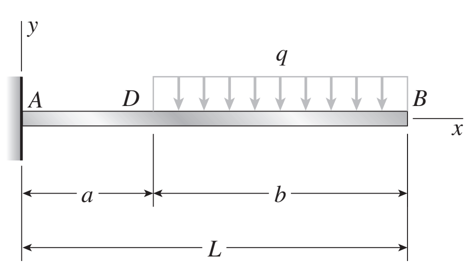
$$\theta_B = -v'(L) = \frac{Pa^2}{2EI} \quad \text{(clockwise)} \quad \longleftarrow$$

δ_B = DOWNLOAD DEFLECTION AT END B $(x = L)$

$$EIv(L) = (PL^2/6)(L - 3a) - (P/6)\langle L-a\rangle^3$$
$$= (PL^2/6)(L - 3a) - (P/6)(L - a)^3$$
$$= (Pa^2/6)(-3L + a)$$

$$\delta_B = -v(L) = \frac{Pa^2}{6EI}(3L - a) \quad \text{(downward)} \quad \longleftarrow$$

Solution 9.12-2 Cantilever beam



FROM PROB: 9.11-2:

$$EIv'''' = -q(x) = qb\langle x\rangle^{-1} - (qb/2)(2a + b)\langle x\rangle^{-2} - q\langle x-a\rangle^{0} + q\langle x-L\rangle^{0}$$

Note: $\langle x - L\rangle^0 = 0$ and may be dropped from the equation.

INTEGRATE THE EQUATION

$$EIv''' = V = qb\langle x\rangle^{0} - (qb/2)(2a + b)\langle x\rangle^{-1} - q\langle x-a\rangle^{1}$$

$$EIv'' = M = qb\langle x\rangle^{1} - (qb/2)(2a + b)\langle x\rangle^{0} - q\langle x-a\rangle^{2}/2$$

Note: $\langle x\rangle^1 = x$ and $\langle x\rangle^0 = 1$

$$EIv' = qbx^2/2 - (qb/2)(2a + b)x - (q/6)\langle x-a\rangle^3 + C_1$$

B.C. $v'(0) = 0$ $\quad EI(0) = 0 - 0 - 0 + C_1$

$\therefore C_1 = 0$

$$EIv = qbx^3/6 - (qb/2)(2a + b)(x^2/2) - (q/24)\langle x-a\rangle^4 + C_2$$

B.C. $v(0) = 0$ $\quad EI(0) = 0 - 0 - 0 + C_2$

$\therefore C_2 = 0$

FINAL EQUATIONS

$EIv' = (qbx/2)(x - L - a) - (q/6)\langle x - a\rangle^3$

$EIv = (qbx^2/12)(2x - 3a - 3L)$
$\quad - (q/24)\langle x - a\rangle^4 \longleftarrow$

θ_B = CLOCKWISE ROTATION AT END B $(x = L)$

$EIv'(L) = (qbL/2)(-a) - (q/6)\langle L - a\rangle^3$
$\quad = -qabL/2 - (q/6)(L/a)^3$
$\quad = -(q/6)(L^3 - a^3)$

$\theta_B = -v'(L) = \dfrac{q}{6EI}(L^3 - a^3)$ (clockwise) $\longleftarrow$

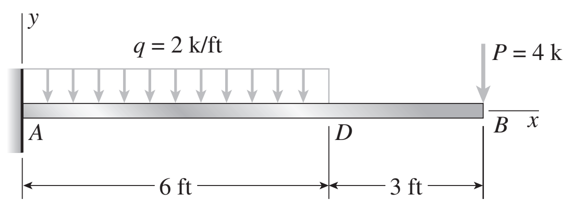
δ_B = DOWNWARD DEFLECTION AT END B $(x = L)$

$EIv(L) = (qbL^2/12)(-3a - L) - (q/24)\langle L - a\rangle^4$
$\quad = (qbL^2/12)(-3a - L) - (q/24)(L - a)^4$
$\quad = -(q/24)(3L^4 - 4a^3L + a^4)$

(After some lengthy algebra)

$\delta_B = -v(L) = \dfrac{q}{24EI}(3L^4 - 4a^3L + a^4)$ (downward) $\longleftarrow$

Solution 9.12-3 Cantilever beam



$q = 2$ k/ft $= \dfrac{1}{6}$ k/in.

$a = 72$ in. $\quad b = 36$ in.
$L = 108$ in.
$E = 10 \times 10^3$ ksi. $\quad I = 450$ in.^{4}

FROM PROB: 9.11-3 Units: kips, inches

$EIv'''' = -q(x) = 16\langle x\rangle^{-1} - 864\langle x\rangle^{-2} - (1/6)\langle x\rangle^0$
$\quad + (1/6)\langle x - 72\rangle^0 - 4\langle x - 108\rangle^{-1}$

Note: $\langle x - 108\rangle^{-1} = 0$ and may be dropped from the equation.

INTEGRATE THE EQUATION

$EIv''' = V = 16\langle x\rangle^0 - 864\langle x\rangle^{-1} - (1/6)\langle x\rangle^1$
$\quad + (1/6)\langle x - 72\rangle^1$

Note: $\langle x\rangle^0 = 1$ and $\langle x\rangle^1 = x$

$EIv'' = M = 16x - 864\langle x\rangle^0 - x^2/12 + (1/12)\langle x - 72\rangle^2$

$EIv' = 8x^2 - 864\langle x\rangle^1 - x^3/36$
$\quad + (1/36)\langle x - 72\rangle^3 + C_1$

Note: $\langle x\rangle^1 = x$

B.C. $v'(0) = 0 \quad EI(0) = 0 - 0 - 0 + 0 + C_1$
$\therefore C_1 = 0$

$EIv = 8x^3/3 - 432x^2 - x^4/144$
$\quad + (1/144)\langle x - 72\rangle^4 + C_2$

B.C. $v(0) = 0 \quad EI(0) = 0 - 0 - 0 + 0 + C_2$
$\therefore C_2 = 0$

FINAL EQUATIONS

$EIv' = (x/36)(-x^2 + 288x - 31{,}104) + (1/36)\langle x - 72\rangle^3$

$EIv = (x^2/144)(-x^2 + 384x - 62{,}208)$
$\quad + (1/144)\langle x - 72\rangle^4 \longleftarrow$

Units: E = ksi, $\quad I$ = in.^{4}, $\quad v'$ = radians,
v = in., $\quad x$ = in.

θ_B = CLOCKWISE ROTATION AT END B $(x = L = 108$ in.$)$

$\theta_B = -v'(L) = -v'(108)$

$$\theta_B = -\frac{108}{36EI}[-(108)(108) + 288(108) - 31{,}104] - \left(\frac{1}{36EI}\right)(108 - 72)^3$$
$$= \frac{108}{36EI}(11{,}664) - \frac{1}{36EI}(46{,}656) = \frac{1}{EI}(33{,}696)$$

$EI = (10 \times 10^3 \text{ ksi})(450 \text{ in.}^4) = 4.5 \times 10^6$ k-in.^{2}

$$\theta_B = \frac{33{,}696}{4.5 \times 10^6}$$
$= 0.007488$ radians (clockwise) $\longleftarrow$

δ_B = DOWNWARD DEFLECTION AT END

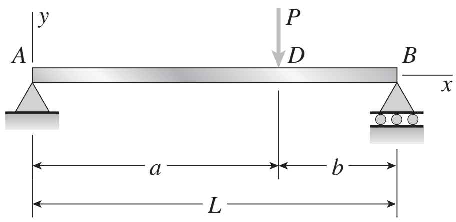
B $(x = L = 108$ in.$)$

$\delta_B = -v(L) = -v(108)$

$$\delta_B = -\frac{(108)^2}{144EI}[-(108)(108) + 384(108) - 62{,}208] - \frac{1}{144EI}(108 - 72)^4$$
$$= \frac{(108)^2}{144EI}(32{,}400) - \frac{1}{144EI}(1{,}679{,}616)$$
$$= \frac{2{,}612{,}736}{EI} = \frac{2{,}612{,}736}{4.5 \times 10^6}$$
$= 0.5806$ in. (downward) $\longleftarrow$

Problem 9.12-4, 9.12-5, and 9.12-6 Determine the equation of the deflection curve for the simple beam AB shown in the figure. Also, obtain the angle of rotation θ_A at the left-hand support and the deflection δ_D at point D.

Solution 9.12-4 Simple beam



$$EIv'''' = -q(x) = (Pb/L)\langle x\rangle^{-1} - P\,\langle x-a\rangle^{-1} + (Pa/L)\langle x-L\rangle^{-3}$$

Note: $\langle x-L\rangle^{-1} = 0$ and may be dropped from the equation.

INTEGRATE THE EQUATION

$$EIv''' = V = (Pb/L)\langle x\rangle^{0} - P\,\langle x-a\rangle^{0}$$

$$EIv'' = M = (Pb/L)\langle x\rangle^{1} - P\,\langle x-a\rangle^{1}$$

$$EIv' = (Pb/2L)\,\langle x\rangle^{2} - (P/2)\,\langle x-a\rangle^{2} + C_1$$

$$EIv = (Pb/6L)\,\langle x\rangle^{3} - (P/6)\,\langle x-a\rangle^{3} + C_1x + C_2$$

Note: $\langle x\rangle^2 = x^2$ and $\langle x\rangle^3 = x^3$

B.C. $v(0) = 0$ $\quad EI(0) = 0 - 0 + 0 + C_2 \quad \therefore C_2 = 0$

B.C. $v(L) = 0$

$$EI(0) = PbL^2/6 - (P/6)\,\langle L-a\rangle^{3} + C_1L$$

$$= PbL^2/6 - (P/6)(b^3) + C_1L$$

$$\therefore\ C_1 = -\frac{PbL}{6} + \frac{Pb^3}{6L} = -\frac{Pb}{6L}\,(L^2 - b^2)$$

FROM PROB: 9.11-4: FINAL EQUATIONS

$$EIv' = Pbx^2/2L - (P/2)\,\langle x-a\rangle^{2} - \frac{Pb}{6L}(L^2 - b^2)$$

$$= (Pb/6L)(3x^2 + b^2 - L^2) - (P/2)\,\langle x-a\rangle^{2}$$

$$EIv = (Pb/6L)(x)^3 - (P/6)\,\langle x-a\rangle^{3} - (Pbx/6L)(L^2 - b^2)$$

$$= (Pbx/6L)(x^2 + b^2 - L^2) - (P/6)\,\langle x-a\rangle^{3} \quad \longleftarrow$$

θ_A = CLOCKWISE ROTATION AT SUPPORT A $(x = 0)$

$$EIv'(0) = (Pb/6L)(b^2 - L^2) + (P/2)(0)$$

$$\theta_A = -v'(0) = (Pb/6L)(L^2 - b^2)(1/EI)$$

$$\theta_A = \frac{Pb}{6LEI}(L^2 - b^2) = \frac{Pb}{6LEI}(L-b)(L+b)$$

$$= \frac{Pab}{6LEI}(L+b) \quad \longleftarrow$$

δ_D = DOWNWARD DEFLECTION AT POINT D $(x = a)$

$$EIv(a) = (Pba/6L)(a^2 + b^2 - L^2) - (P/6)(0)$$

$$= -(Pab/6L)(L^2 - b^2 - a^2)$$

$$\delta_D = -v(a) = \frac{Pab}{6LEI}(L^2 - b^2 - a^2) = \frac{Pa^2b^2}{3LEI} \quad \longleftarrow$$

Solution 9.12-5 Simple beam

FROM PROB: 9.11-5:

$$EIv'''' = -q(x) = (M_0/L)\langle x\rangle^{-1} - M_0\,\langle x-a\rangle^{-2} - (M_0/L)\langle x-L\rangle^{-1}$$

Note: $\langle x-L\rangle^{-1} = 0$ and may be dropped from the equation.

INTEGRATE THE EQUATION

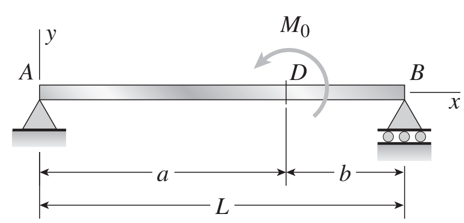
$$EIv''' = V = (M_0/L)\langle x\rangle^{0} - M_0\,\langle x-a\rangle^{1}$$

$$EIv'' = M = (M_0/L)\langle x\rangle^{1} - M_0\,\langle x-a\rangle^{0}$$

$$EIv' = (M_0/2L)\,\langle x\rangle^{2} - M_0\,\langle x-a\rangle^{1} + C_1$$

$$EIv = (M_0/6L)\,\langle x\rangle^{3} - (M_0/2)\,\langle x-a\rangle^{2} + C_1x + C_2$$

Note: $\langle x\rangle^2 = x^2$ and $\langle x\rangle^3 = x^3$



B.C. $v(0) = 0$ $\quad EI(0) = 0 - 0 + 0 + C_2$

$\therefore C_2 = 0$

B.C. $v(L) = 0$

$$EI(0) = M_0L^2/6 - (M_0/2)\,\langle L-a\rangle^{2} + C_1L$$

$$= M_0L^2/6 - (M_0/2)(L-a)^2 + C_1L$$

$$\therefore\ C_1 = -\frac{M_0}{6L} = (2L^2 - 6aL + 3a^2)$$

FINAL EQUATIONS

$$EIv' = (M_0/2L)x^2 - M_0\langle x - a\rangle^1 + (M_0/6L)(2L^2 - 6aL + 3a^2)$$
$$= (M_0/6L)(3x^2 - 6aL + 3a^2 + 2L^2) - M_0\langle x - a\rangle^1$$

$$EIv = (M_0/6L)(x)^3 - (M_0/2)\langle x - a\rangle^2 + (M_0 x/6L)(2L^2 - 6aL + 3a^2)$$
$$= (M_0 x/6L)(x^2 - 6aL + 3a^2 + 2L^2) - (M_0/2)\langle x - a\rangle^2$$

θ_A = CLOCKWISE ROTATION AT SUPPORT A $(x = 0)$

$$EIv'(0) = (M_0/6L)(-6aL + 3a^2 + 2L^2) - (M_0/2)(0)$$

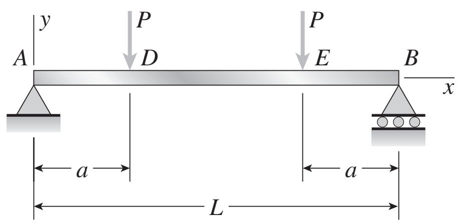
$$\theta_A = -v'(0) = \frac{M_0}{6LEI}(6aL - 3a^2 - 2L^2) \quad \text{(clockwise)}$$

δ_D = DOWNWARD DEFLECTION AT POINT D $(x = a)$

$$EIv(a) = (M_0/6L)(a^3) - (M_0/2)(0) + (M_0 a/6L)(2L^2 - 6aL + 3a^2)$$
$$= \frac{M_0 a}{6L}(a^2 + 2L^2 - 6aL + 3a^2)$$
$$= \frac{M_0 a}{6L}(L - a)(2)(L - 2a)$$
$$= \frac{M_0 ab}{3L}(L - 2a)$$

$$\delta_D = -v(a) = \frac{M_0 ab}{3LEI}(2a - L) \quad \text{(downward)}$$

Solution 9.12-6 Simple beam



FROM PROB: 9.11-6:

$$EIv'''' = -q(x) = P\langle x\rangle^{-1} - P\langle x - a\rangle^{-1} - P\langle x - L + a\rangle^{-1} + P\langle x - L\rangle^{-1}$$

Note: $\langle x - L\rangle^{-1} = 0$ and may be dropped from the equation.

INTEGRATE THE EQUATION

$$EIv''' = V = P\langle x\rangle^0 - P\langle x - a\rangle^0 - P\langle x - L + a\rangle^0$$
$$EIv'' = M = P\langle x\rangle^1 - P\langle x - a\rangle^1 - P\langle x - L + a\rangle^1$$
$$EIv' = (P/2)\langle x\rangle^2 - (P/2)\langle x - a\rangle^2 - (P/2)\langle x - L + a\rangle^2 + C_1$$

B.C. (symmetry) $EIv'(L/2) = 0$

$$0 = (P/2)(L/2)^2 - (P/2)(L/2 - a)^2 - (P/2)(0) + C_1$$

$$\therefore\ C_1 = -\frac{Pa}{2}(L - a)$$

$$EIv' = (P/2)\langle x\rangle^2 - (P/2)\langle x - a\rangle^2 - (P/2)\langle x - L + a\rangle^2 - (Pa/2)(L - a)$$

$$EIv = (P/6)\langle x\rangle^3 - (P/6)\langle x - a\rangle^3 - (P/6)\langle x - L + a\rangle^3 - (Pa/2)(L - a)\,x + C_2$$

B.C. $EIv(0) = 0 \quad 0 = 0 - 0 - 0 - 0 + C_2$

$\therefore C_2 = 0$

Note: $\langle x\rangle^2 = x^2$ and $\langle x\rangle^3 = x^3$

FINAL EQUATIONS

$$EIv' = Px^2/2 - (P/2)\langle x - a\rangle^2 - (P/2)\langle x - L + a\rangle^2 - (Pa/2)(L - a)$$
$$= (P/2)(x^2 - aL + a^2) - (P/2)\langle x - a\rangle^2 - (P/2)\langle x - L + a\rangle^2$$

$$EIv = Px^3/6 - (P/6)\langle x - a\rangle^3 - (P/6)\langle x - L + a\rangle^3 - (3Pax/6)(L - \mathrm{a})$$
$$= (Px/6)(x^2 - 3aL + 3a^2) - (P/6)\langle x - a\rangle^3 - (P/6)\langle x - L + a\rangle^3$$

θ_A = CLOCKWISE ROTATION AT SUPPORT A $(x = 0)$

$$EIv'(0) = (Pa/2)(-L + a) - (P/2)(0) - (P/2)(0)$$
$$= (Pa/2)(-L + a)$$

$$\theta_A = -v'(0) = \frac{Pa}{2EI}(L - a) \quad \text{(clockwise)}$$

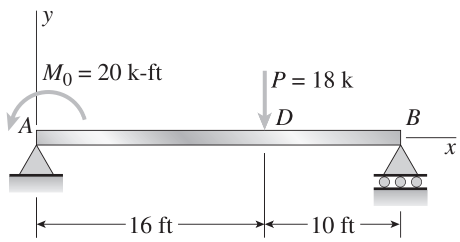
δ_D = DOWNWARD DEFLECTION AT POINT D $(x = a)$

$$EIv(a) = (Pa/6)(4a^2 - 3aL) - (P/6)(0) - (P/6)\langle -L + 2a\rangle^3$$
$$= (Pa/6)(4a^2 - 3aL) - (P/6)(0)$$
$$= (Pa^2/6)(4a - 3L)$$

$$\delta_D = -v(a) = \frac{Pa^2}{6EI}(3L - 4a) \quad \text{(downward)}$$

Problem 9.12-7 Determine the equation of the deflection curve for the simple beam ADB shown in the figure. Also, obtain the angle of rotation θ_A at the left-hand support and the deflection δ_D at point D. Assume $E = 30 \times 10^6$ psi and $I = 720$ in.^{4}

Solution 9.12-7 Simple beam



$M_0 = 20$ k-ft $= 240$ k-in.
$P = 18$ k
$a = 16$ ft $= 192$ in.
$b = 10$ ft $= 120$ in.
$L = a + b = 312$ in.
$E = 30 \times 10^3$ ksi
$I = 720$ in.^{4}

FROM PROB. 9.11-7: Units: kips, inches

$$EIv'''' = -q(x) = 7.692\langle x\rangle^{-1} - 240\langle x\rangle^{-2} - 18\langle x-192\rangle^{-1} + 10.308\langle x-312\rangle^{-1}$$

Note: $\langle x - 312\rangle^{-1} = 0$ and may be dropped from the equation.

INTEGRATE THE EQUATION

$$EIv''' = V = 7.692\langle x\rangle^{0} - 240\langle x\rangle^{-1} - 18\langle x-192\rangle^{0}$$

$$EIv'' = M = 7.692\langle x\rangle^{1} - 240\langle x\rangle^{0} - 18\langle x-192\rangle^{1}$$

$$EIv' = (7.692/2)\langle x\rangle^{2} - 240\langle x\rangle^{1} - (18/2)\langle x-192\rangle^{2} + C_1$$

Note: $\langle x\rangle^2 = x^2$ and $\langle x\rangle^1 = x$

$$EIv' = 3.846\,x^2 - 240\,x - 9\langle x-192\rangle^2 + C_1$$

$$EIv = 1.282\,x^3 - 120\,x^2 - 3\langle x-192\rangle^3 + C_1x + C_2$$

B.C. $EIv\,(0) = 0$ $\quad 0 = 0 - 0 - 0 + C_1(0) + C_2$

$\therefore C_2 = 0$

B.C. $EIv\,(312) = 0$

$0 = 1.282(312)^3 - 120(312)^2 - 3\langle 120\rangle^3 + C_1(312)$

Note: $\langle 120\rangle^3 = (120)^3$

$0 = 22{,}071 \times 10^3 + C_1(312)$ $\quad \therefore C_1 = -70{,}740$

FINAL EQUATIONS

(Note: x = in., E = ksi, I = in.^{4}, v' = rad, v = in.)

$EIv' = 3.846x^2 - 240x - 9\langle x-192\rangle^2 - 70{,}740$

$EIv = 1.282x^3 - 120x^2 - 3\langle x-192\rangle^3 - 70{,}740x$ ←

θ_A = CLOCKWISE ROTATION AT SUPPORT A ($x = 0$)

$EIv'(0) = -9\langle -192\rangle^2 - 70{,}740 = -70{,}740$

$$\theta_A = -v'(0) = \frac{70{,}740}{EI} = \frac{70{,}740}{(30 \times 10^3)(720)}$$

$= 0.00327$ rad (clockwise) ←

δ_D = DOWNWARD DEFLECTION AT POINT D ($x = 192$)

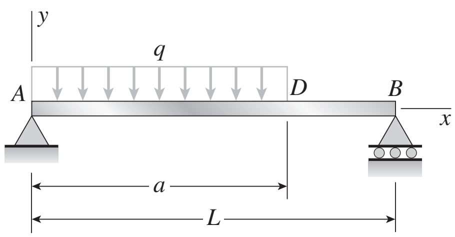
$EIv(192) = 1.282(192)^3 - 120(192)^2 - 70{,}740(192) = -8.932 \times 10^6$

$$\delta_D = -v(192) = \frac{8.932 \times 10^6}{EI} = \frac{8.932 \times 10^6}{(30 \times 10^3)(720)}$$

$= 0.414$ in. (downward) ←

Problem 9.12-8, 9.12-9, and 9.12-10 Obtain the equation of the deflection curve for the simple beam AB (see figure). Also, determine the angle of rotation θ_B at the right-hand support and the deflection δ_D at point D. (For the beam of Problem 9.12-10, assume $E = 200$ GPa and $I = 2.60 \times 10^9$ mm^4.)

Solution 9.12-8 Simple beam



FROM PROB. 9.11-8:

$$EIv'''' = -q(x) = (qa/2L)(2L - a)\langle x\rangle^{-1} - q\langle x\rangle^{0} + q\langle x - a\rangle^{0} + (qa^2/2L)\langle x - L\rangle^{-1}$$

Note: $\langle x - L\rangle^{-1} = 0$ and may be dropped from the equation

INTEGRATE THE EQUATION

$$EIv''' = V = (qa/2L)(2L - a)\langle x\rangle^{0} - q\langle x\rangle^{1} + q\langle x - a\rangle^{1}$$

$$EIv'' = M = (qa/2L)(2L - a)\langle x\rangle^{1} - (q/2)\langle x\rangle^{2} + (q/2)\langle x - a\rangle^{2}$$

$$EIv' = (qa/4L)(2L - a)\langle x\rangle^{2} - (q/6)\langle x\rangle^{3} + (q/6)\langle x - a\rangle^{3} + C_1$$

$$EIv = (qa/12L)(2L - a)\langle x\rangle^{3} - (q/24)\langle x\rangle^{4} + (q/24)\langle x - a\rangle^{4} + C_1 x + C_2$$

Note: $\langle x\rangle^2 = x^2$, $\langle x\rangle^3 = x^3$, and $\langle x\rangle^4 = x^4$

B.C. $EIv(0) = 0$ $\quad 0 = 0 - 0 + (q/24)(0) + C_1(0) + C_2$

$\therefore C_2 = 0$

B.C. $EIv(L) = 0$

$$0 = (qaL^2/12)(2L - a) - qL^4/24 + (q/24)(L - a)^4 + C_1 L$$

After lengthy algebra,

$$C_1 = -\frac{qa^2}{24L}(2L - a)^2$$

FINAL EQUATIONS

$$EIv' = (qax^2/4L)(2L - a) - qx^3/6 + (q/6)\langle x - a\rangle^3 - (qa^2/24L)(2L - a)^2$$

$$EIv = (qax^3/12L)(2L - a) - qx^4/24 + (q/24)\langle x - a\rangle^4 - (qa^2x/24L)(2L - a)^2$$
$$= qx[-a^2(2L - a)^2 + 2a(2L - a)x^2 - Lx^3]/24L + q\langle x - a\rangle^4/24 \quad \longleftarrow$$

θ_B = COUNTERCLOCKWISE ROTATION AT SUPPORT B $(x = L)$

$$EIv'(L) = (qaL/4)(2L - a) - qL^3/6 + (q/6)(L - a)^3 - (qa^2/24L)(2L - a)^2$$

After lengthy algebra,

$$EIv'(L) = (qa^2/24L)(2L^2 - a^2)$$

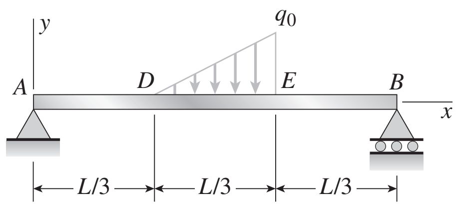
$$\theta_B = v'(L) = \frac{qa^2}{24LEI}(2L^2 - a^2) \quad \text{(counterclockwise)} \quad \longleftarrow$$

δ_D = DOWNWARD DEFLECTION AT POINT D $(x = a)$

$$EIv(a) = qa[-a^2(2L - a)^2 + 2a^3(2L - a) - a^3L]/24L + q(0)$$
$$= (qa^3/24L)[-(2L - a)^2 + 2a(2L - a) - aL]$$
$$= (qa^3/24L)(-4L^2 + 7aL - 3a^2)$$

$$\delta_D = -v(a) = \frac{qa^3}{24LEI}(4L^2 - 7aL + 3a^2) \text{(downward)} \quad \longleftarrow$$

Solution 9.12-9 Simple beam



FROM PROB. 9.11-9:

$$EIv'''' = -q(x) = (2q_0L/27)\langle x\rangle^{-1} - (3q_0/L)\langle x - L/3\rangle^1 + (3q_0/L)\langle x - 2L/3\rangle^1 + q_0\langle x - 2L/3\rangle^0 + (5q_0L/54)\langle x - L\rangle^{-1}$$

Note: $\langle x - L\rangle^{-1} = 0$ and may be dropped from the equation

INTEGRATE THE EQUATION

$$EIv''' = V = (2q_0L/27)\langle x\rangle^0 - (3q_0/2L)\langle x - L/3\rangle^2 + (3q_0/2L)\langle x - 2L/3\rangle^2 + q_0\langle x - 2L/3\rangle^1$$

Note: $\langle x\rangle^0 = 1$

$$EIv'' = M = (2q_0L/27)x - (q_0/2L)\langle x - L/3\rangle^3 + (q_0/2L)\langle x - 2L/3\rangle^3 + (q_0/2)\langle x - 2L/3\rangle^2$$

$$EIv' = (q_0L/27)x^2 - (q_0/8L)\langle x - L/3\rangle^4 + (q_0/8L)\langle x - 2L/3\rangle^4 + (q_0/6)\langle x - 2L/3\rangle^3 + C_1$$

$$EIv = (q_0L/81)x^3 - (q_0/40L)\langle x - L/3\rangle^5 + (q_0/40L)\langle x - 2L/3\rangle^5 + (q_0/24)\langle x - 2L/3\rangle^4 + C_1x + C_2$$

B.C. $EIv(0) = 0 \quad 0 = 0 - 0 + 0 + 0 + C_1(0) + C_2$

$\therefore C_2 = 0$

B.C. $EIv(L) = 0$

$$0 = q_0L^4/81 - (q_0/40L)(2L/3)^5 + (q_0/40L)(L/3)^5 + (q_0/24)(L/3)^4 + C_1L$$

$$0 = \frac{47q_0L^4}{4860} + C_1L \quad \therefore C_1 = -\frac{47q_0L^3}{4860}$$

FINAL EQUATIONS

$$EIv' = (q_0L/27)x^2 - (q_0/8L)\langle x - L/3\rangle^4 + (q_0/8L)\langle x - 2L/3\rangle^4 + (q_0/6)\langle x - 2L/3\rangle^3 - 47q_0L^3/4860$$

$$EIv = (q_0L/81)x^3 - (q_0/40L)\langle x - L/3\rangle^5 + (q_0/40L)\langle x - 2L/3\rangle^5 + (q_0/24)\langle x - 2L/3\rangle^4 - 47q_0L^3x/4860 \longleftarrow$$

θ_B = COUNTERCLOCKWISE ROTATION AT SUPPORT B $(x = L)$

$$EIv'(L) = q_0L^3/27 - (q_0/8L)(2L/3)^4 + (q_0/8L)(L/3)^4 + (q_0/6)(L/3)^3 - 47q_0L^3/4860 = 101q_0L^3/9720$$

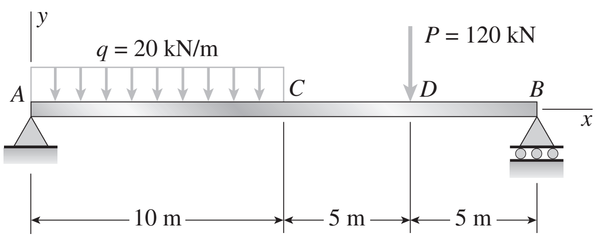
$$\theta_B = v'(L) = \frac{101q_0L^3}{9720EI} \quad \text{(counterclockwise)} \longleftarrow$$

δ_D = DOWNWARD DEFLECTION AT POINT D $(x = L/3)$

$$EIv(L/3) = (q_0L/81)(L/3)^3 - (q_0/40L)(0) + (q_0/40L)(0) + (q_0/24)(0) - 47q_0L^3(L/3)/4860 = -121q_0L^4/43{,}740$$

$$\delta_D = -v\left(\frac{L}{3}\right) = \frac{121q_0L^4}{43{,}740EI} \quad \text{(downward)} \longleftarrow$$

Solution 9.12-10 Simple beam



$q = 20$ kN/m
$P = 120$ kN
$\dfrac{L}{2} = 10$ m
$L = 20$ m
$E = 200$ GPa
$I = 2.60 \times 10^{-3}$ m^4

FROM PROB. 9.11-10: Units: kilonewtons, meters

$$EIv'''' = -g(x) = 180\langle x\rangle^{-1} - 20\langle x\rangle^0 + 20\langle x-10\rangle^0 - 120\langle x-15\rangle^{-1} + 140\langle x-20\rangle^{-1}$$

Note: $\langle x-20\rangle^{-1} = 0$ and may be dropped from the equation

INTEGRATE THE EQUATION

$$EIv''' = V = 180\langle x\rangle^0 - 20\langle x\rangle^1 + 20\langle x-10\rangle^1 - 120\langle x-15\rangle^0$$

Note: $\langle x\rangle^0 = 1$ and $\langle x\rangle^1 = x$

$$EIv'' = M = 180x - 20(x^2/2) + (20/2)\langle x-10\rangle^2 - 120\langle x-15\rangle^1$$

$$EIv' = 180(x^2/2) - 20(x^3/6) + (10/3)\langle x-10\rangle^3 - 60\langle x-15\rangle^2 + C_1$$

$$EIv = 30x^3 - (5/6)x^4 + (5/6)\langle x-10\rangle^4 - 20\langle x-15\rangle^3 + C_1x + C_2$$

B.C. $EIv(0) = 0 \quad 0 = 0 - 0 + 0 - 0 + C_1(0) + C_2$

$\therefore C_2 = 0$

B.C. $EIv(20) = 0$

$$0 = 30(20)^3 - (5/6)(20)^4 + (5/6)(10)^4 - 20(5)^3 + C_1(20)$$

$$0 = 112{,}500 + 20C_1 \quad \therefore C_1 = -5625$$

FINAL EQUATIONS

$$EIv' = 90x^2 - (10/3)x^3 + (10/3)\langle x-10\rangle^3 - 60\langle x-15\rangle^2 - 5625$$

$$EIv = 30x^3 - (5/6)x^4 + (5/6)\langle x-10\rangle^4 - 20\langle x-15\rangle^3 - 5625x \quad \longleftarrow$$

(x = meters, v = meters, v' = radians, E = kilopascals, I = meters^4)

θ_B = COUNTERCLOCKWISE ROTATION AT SUPPORT B ($x = 20$)

$$Eiv'(20) = 90(20)^2 - (10/3)(20)^3 + (10/3)(10)^3 - 60(5)^2 - 5625 = 5541.67$$

$$\theta_B = v'(20) = \frac{5541.67}{EI} = \frac{5541.67}{(200 \times 10^6 \text{ kPa})(2.60 \times 10^{-3} \text{ m})}$$

$$= 0.01066 \text{ rad} \quad \text{(counterclockwise)} \quad \longleftarrow$$

δ_D = DOWNWARD DEFLECTION AT POINT D ($x = 15$)

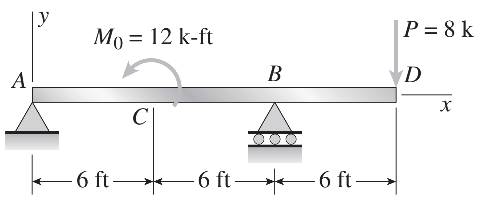
$$EIv(15) = 30(15)^3 - (5/6)(15)^4 + (5/6)(5)^4 - 20(0) - 5625(15) = -24{,}791.7$$

$$\delta_D = -v(15) = \frac{24{,}791.7}{EI} = \frac{24{,}791.7}{(200 \times 10^6 \text{ kPa})(2.60 \times 10^{-3} \text{ m})}$$

$$= 0.04768 \text{ m} = 47.68 \text{ mm} \quad \text{(downward)} \quad \longleftarrow$$

Problem 9.12-11 A beam *ACBD* with simple supports at *A* and *B* and an overhang *BD* is shown in the figure. (a) Obtain the equation of the deflection curve for the beam. (b) Calculate the deflections δ_C and δ_D at points *C* and *D*, respectively. (Assume $E = 30 \times 10^6$ psi and $I = 280$ in.^{4})

Solution 9.12-11 **Beam with an overhang**



$M_0 = 144$ k-in.

$\dfrac{L}{2} = 72$ in.

$L = L_{AB} = 144$ in.

$\dfrac{3L}{2} = 216$ in.

$E = 30 \times 10^3$ ksi

$I = 280$ in.^{4}

FROM PROB. 9.11-11: Units: kips, inches

$$EIv'''' = -q(x) = -3\langle x\rangle^{-1} - 144\langle x - 72\rangle^{-2} + 11\langle x - 144\rangle^{-1} - 8\langle x - 216\rangle^{-1}$$

Note: $\langle x - 216\rangle^{-1} = 0$ and may be dropped from the equation.

INTEGRATE THE EQUATION

$$EIv''' = V = -3\langle x\rangle^{0} - 144\langle x - 72\rangle^{-1} + 11\langle x - 144\rangle^{0}$$

$$EIv'' = M = -3\langle x\rangle^{1} - 144\langle x - 72\rangle^{0} + 11\langle x - 144\rangle^{1}$$

$$EIv' = -(3/2)\langle x\rangle^{2} - 144\langle x - 72\rangle^{1} + (11/2)\langle x - 144\rangle^{2} + C_1$$

$$EIv = -(1/2)\langle x\rangle^{3} - (144/2)\langle x - 72\rangle^{2} + (11/6)\langle x - 144\rangle^{3} + C_1 x + C_2$$

B.C. $EIv(0) = 0 \quad 0 = 0 - 0 + 0 + C_1(0) + C_2$

$\therefore C_2 = 0$

B.C. $EIv(144) = 0 \quad 0 = -(1/2)(144)^3 - (72)(72)^2 + (11/6)(0) + C_1(144)$

$0 = -1{,}866{,}240 + 144\, C_1$

$\therefore C_1 = 12{,}960$

FINAL EQUATIONS

$$EIv' = -3x^2/2 - 144\langle x - 72\rangle^{1} + (11/2)\langle x - 144\rangle^{2} + 12{,}960$$

$$EIv = -x^3/2 - 72\langle x - 72\rangle^{2} + (11/6)\langle x - 144\rangle^{3} + 12{,}960\, x \quad \longleftarrow$$

(x = in., v = in., v' = rad, $E = 30 \times 10^3$ ksi, $I = 280$ in.^{4})

δ_C = UPWARD DEFLECTION AT POINT C ($x = 72$)

$$EIv(15) = -(72)^3/2 - 72(0) + (11/6)(0) + 12{,}960(72) = 746{,}496$$

$$\delta_C = v(15) = \frac{746{,}496}{EI} = \frac{746{,}496}{(30 \times 10^3)(280)} = 0.08887 \text{ in.} \quad \text{(upward)} \quad \longleftarrow$$

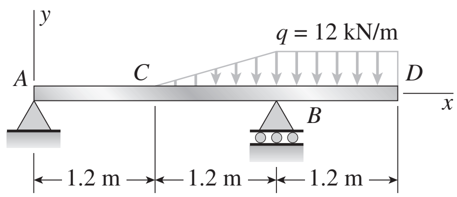
δ_D = DOWNWARD DEFLECTION AT POINT D ($x = 216$)

$$EIv(216) = -(216)^3/2 - 72(144)^2 + (11/6)(72)^3 + 12{,}960(216) = -3{,}048{,}192$$

$$\delta_D = -v(216) = \frac{3{,}048{,}192}{EI} = \frac{3{,}048{,}192}{(30 \times 10^3)(280)} = 0.3629 \text{ in.} \quad \text{(downward)} \quad \longleftarrow$$

Problem 9.12-12 The overhanging beam $ACBD$ shown in the figure is simply supported at A and B. Obtain the equation of the deflection curve and the deflections δ_C and δ_D at points C and D, respectively. (Assume $E = 200$ GPa and $I = 15 \times 10^6$ mm^4.)

Solution 9.12-12 Beam with an overhang



$q = 12$ kn/m

$\dfrac{L}{2} = 1.2$ m

$L = L_{AB} = 2.4$ m

$E = 200$ GPa

$I = 15 \times 10^{-6}$ mm^4

FROM PROB. 9.11-12: Units: kilometers, meters

$$EIv'''' = -q(x) = -2.4\langle x\rangle^{-1} - 10\langle x - 1.2\rangle^{1}$$
$$+ 10\langle x - 2.4\rangle^{1}$$
$$+ 24\langle x - 2.4\rangle^{-1} + 12\langle x - 3.6\rangle^{0}$$

Note: $\langle x - 3.6\rangle^0 = 0$ and may be dropped from the equation.

INTEGRATE THE EQUATION

$$EIv''' = v = -2.4\langle x\rangle^{0} - (10/2)\langle x - 1.2\rangle^{2}$$
$$+ (10/2)\langle x - 2.4\rangle^{2} + 24\langle x - 2.42\rangle^{0}$$
$$EIv'' = M = -2.4\langle x\rangle' - (5/3)\langle x - 1.2\rangle^{3}$$
$$+ (5/3)\langle x - 2.4\rangle^{3} + 24\langle x - 2.4\rangle'$$

Note: $\langle x\rangle' = x$

$$EIv' = -1.2x^2 - (5/12)\langle x - 1.2\rangle^4 + (5/12)\langle x - 2.4\rangle^4$$
$$+ 12\langle x - 2.4\rangle^2 + C_1$$
$$EIv = -0.4x^3 - (1/12)\langle x - 1.2\rangle^5 + (1/12)\langle x - 2.4\rangle^5$$
$$+ 4\langle x - 2.4\rangle^3 + C_1x + C_2$$

B.C. $EIv(0) = 0 \quad 0 = 0 - 0 + 0 + 0 + C_1(0) + C_2$

$\therefore C_2 = 0$

B.C. $EIv(2.4) = 0$

$$0 = -0.4(2.4)^3 - (1/12)(1.2)^5 + (1/12)(0) + 4(0)$$
$$+ 2.4\,C_1$$
$$0 = -5.73696 + 2.4\,C_1$$
$$\therefore C_1 = 2.3904$$

FINAL EQUATION

$$EIv' = -1.2x^2 - (5/12)\langle x - 1.2\rangle^4 + (5/12)\langle x - 2.4\rangle^4$$
$$+ 12\langle x - 2.4\rangle^2 + 2.3904$$
$$EIv = -0.4x^3 - (1/12)\langle x - 1.2\rangle^5 + (1/12)\langle x - 2.4\rangle^5$$
$$+ 4\langle x - 2.4\rangle^3 + 2.3904x$$

(x = meters, v = meters, v' = radians,
$E = 200 \times 10^6$ kPa, $I = 15 \times 10^{-6}$ m^4)

δ_C = UPWARD DEFLECTION AT POINT C ($x = 1.2$)

$$EIv(1.2) = -0.4(1.2)^3 - (1/12)\,(0) + (1/12)\,(0)$$
$$+ 4\,(0) + 2.3904\,(1.2) = 2.17728$$
$$\delta_C = v(1.2) = \frac{2.17728}{EI} = \frac{2.17728}{(200 \times 10^6)(15 \times 10^{-6})}$$
$$= 0.00072576 \text{ m} = 0.7258 \text{ mm (upward)} \longleftarrow$$

δ_D = DOWNWARD DEFLECTION AT POINT D ($x = 3.6$)

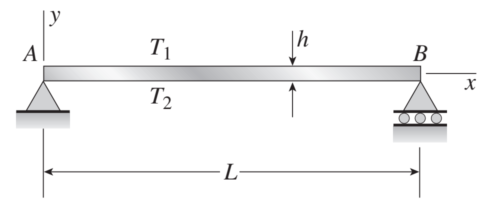
$$EIv(3.6) = -0.4(3.6)^3 - (1/12)\,(2.4)^5$$
$$+ (1/12)\,(1.2)^5$$
$$+ 4\,(1.2)^3 + 2.3904\,(3.6)$$
$$= -9.57312$$
$$\delta_D = -v(3.6) = \frac{9.57312}{EI} = \frac{9.57312}{(200 \times 10^6)(15 \times 10^{-6})}$$
$$= 0.00319104 \text{ m} = 3.191 \text{ mm (downward)} \longleftarrow$$

Temperature Effects

The beams described in the problems for Section 9.13 have constant flexural rigidity EI. In every problem, the temperature varies linearly between the top and bottom of the beam.

Problem 9.13-1 A simple beam AB of length L and height h undergoes a temperature change such that the bottom of the beam is at temperature T_2 and the top of the beam is at temperature T_1 (see figure).

Determine the equation of the deflection curve of the beam, the angle of rotation θ_A at the left-hand support, and the deflection δ_{max} at the midpoint.



Solution 9.13-1 Simple beam with temperature differential

Eq. (9-147): $v'' = \dfrac{d^2v}{dx^2} = \dfrac{\alpha(T_2 - T_1)}{h}$

$v' = \dfrac{dv}{dx} = \dfrac{\alpha(T_2 - T_1)x}{h} + C_1$

B.C. 1 (Symmetry) $v'\left(\dfrac{L}{2}\right) = 0$

$\therefore C_1 = -\dfrac{\alpha L(T_2 - T_1)}{2h}$

$v = \dfrac{\alpha(T_2 - T_1)x^2}{2h} - \dfrac{\alpha L(T_2 - T_1)x}{2h} + C_2$

B.C. 2 $v(0) = 0$ $\therefore C_2 = 0$

$v = -\dfrac{\alpha(T_2 - T_1)(x)(L - x)}{2h}$ ←

(positive v is upward deflection)

$v' = -\dfrac{\alpha(T_2 - T_1)(L - 2x)}{2h}$

$\theta_A = -v'(0) = \dfrac{\alpha L(T_2 - T_1)}{2h}$ ←

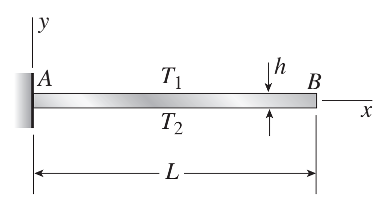
(positive θ_A is clockwise rotation)

$\delta_{max} = -v\left(\dfrac{L}{2}\right) = \dfrac{\alpha L^2(T_2 - T_1)}{8h}$ ←

(positive δ_{max} is downward deflection)

Problem 9.13-2 A cantilever beam AB of length L and height h (see figure) is subjected to a temperature change such that the temperature at the top is T_1 and at the bottom is T_2.

Determine the equation of the deflection curve of the beam, the angle of rotation θ_B at end B, and the deflection δ_B at end B.



Solution 9.13-2 Cantilever beam with temperature differential

Eq. (9-147): $v'' = \dfrac{d^2v}{dx^2} = \dfrac{\alpha(T_2 - T_1)}{h}$

$v' = \dfrac{dv}{dx} = \dfrac{\alpha(T_2 - T_1)}{h}x + C_1$

B.C. 1 $v'(0) = 0$ $\therefore C_1 = 0$

$v' = \dfrac{\alpha(T_2 - T_1)}{h}x$

$v = \dfrac{\alpha(T_2 - T_1)}{h}\left(\dfrac{x^2}{2}\right) + C_2$

B.C. 2 $v(0) = 0$ $\therefore C_2 = 0$

$v = \dfrac{\alpha(T_2 - T_1)x^2}{2h}$ ←

(positive v is upward deflection)

$\theta_B = v'(L) = \dfrac{\alpha L(T_2 - T_1)}{h}$ ←

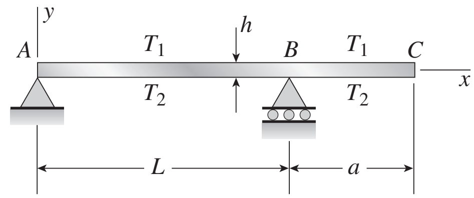
(positive θ_B is counterclockwise rotation)

$\delta_B = v(L) = \dfrac{\alpha L^2(T_2 - T_1)}{2h}$ ←

(positive δ_B is upward deflection)

Problem 9.13-3 An overhanging beam ABC of height h is heated to a temperature T_1 on the top and T_2 on the bottom (see figure).

Determine the equation of the deflection curve of the beam, the angle of rotation θ_C at end C, and the deflection δ_C at end C.



Solution 9.13-3 Overhanging beam with temperature differential

Eq. (9-147): $v'' = \dfrac{d^2v}{dx^2} = \dfrac{\alpha(T_2 - T_1)}{h}$

(This equation is valid for the entire length of the beam.)

$$v' = \frac{\alpha(T_2 - T_1)x}{h} + C_1$$

$$v = \frac{\alpha(T_2 - T_1)x^2}{2h} + C_1x + C_2$$

B.C. 1 $v(0) = 0 \quad \therefore C_2 = 0$

B.C. 2 $v(L) = 0 \quad \therefore C_1 = -\dfrac{\alpha(T_2 - T_1)L}{2h}$

$$v = \frac{\alpha(T_2 - T_1)}{2h}(x^2 - Lx) \quad \longleftarrow$$

(positive v is upward deflection)

$$v' = \frac{\alpha(T_2 - T_1)}{2h}(2x - L)$$

$$\theta_C = v'(L + a) = \frac{\alpha(T_2 - T_1)}{2h}(L + 2a) \quad \longleftarrow$$

(positive θ_C is counterclockwise rotation)

$$\delta_C = v(L + a) = \frac{\alpha(T_2 - T_1)\,(L + a)\,(a)}{2h} \quad \longleftarrow$$

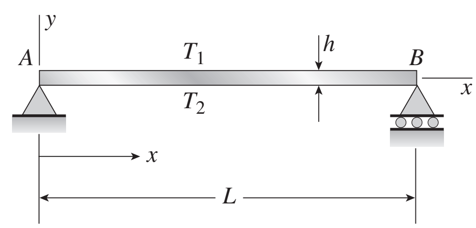
(positive δ_C is upward deflection)

Problem 9.13-4 A simple beam AB of length L and height h (see figure) is heated in such a manner that the temperature difference $T_2 - T_1$ between the bottom and top of the beam is proportional to the distance from support A; that is,

$$T_2 - T_1 = T_0 x$$

in which T_0 is a constant having units of temperature (degrees) per unit distance.

Determine the maximum deflection δ_{max} of the beam.



Solution 9.13-4 Simple beam with temperature differential proportional to distance x

$$T_2 - T_1 = T_0 x$$

Eq. (9-147): $v'' = \dfrac{d^2v}{dx^2} = \dfrac{\alpha(T_2 - T_1)}{h} = \dfrac{\alpha T_0 x}{h}$

$$v' = \frac{dv}{dx} = \frac{\alpha T_0 x^2}{2h} + C_1$$

$$v = \frac{\alpha T_0 x^3}{6h} + C_1x + C_2$$

B.C. 1 $v\,(0) = 0 \quad \therefore C_2 = 0$

B.C. 2 $v\,(L) = 0 \quad \therefore C_1 = -\dfrac{\alpha T_0 L^2}{6h}$

$$v = -\frac{\alpha T_0 x}{6h}(L^2 - x^2)$$

(positive v is upward deflection)

$$v' = -\frac{\alpha T_0}{6h}(L^2 - 3x^2)$$

(positive v' is upward to the right)

Maximum deflection

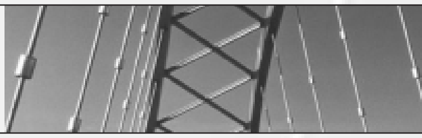
Set $v' = 0$ and solve for x.

$$L^2 - 3x^2 = 0 \qquad x_1 = \frac{L}{\sqrt{3}}$$

$$v_{max} = v(x_1) = -\frac{\alpha T_0 L^3}{9\sqrt{3}h}$$

$$\delta_{max} = -v_{max} = \frac{\alpha T_0 L^3}{9\sqrt{3}h} \quad \longleftarrow$$

(positive δ_{max} is downward)



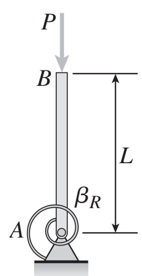
11

Columns

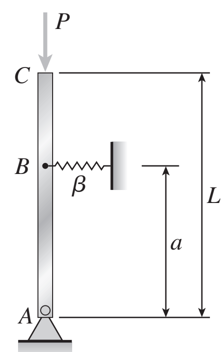
Idealized Buckling Models

Problem 11.2-1 through 11.2-4 The figure shows an idealized structure consisting of one or more **rigid bars** with pinned connections and linearly elastic springs. Rotational stiffness is denoted β_R and translational stiffness is denoted β.

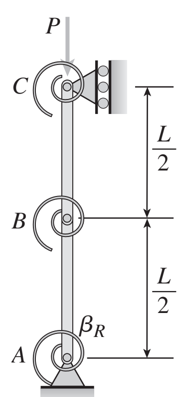
Determine the critical load P_{cr} for the structure.



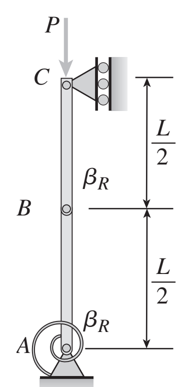
Prob. 11.2-1



Prob. 11.2-2

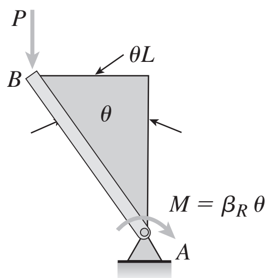


Prob. 11.2-3



Prob. 11.2-4

Solution 11.2-1 Rigid bar AB

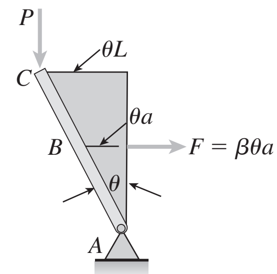


$$\sum M_A = 0$$

$$P(\theta L) - \beta_R \theta = 0$$

$$P_{\text{cr}} = \frac{\beta_R}{L} \quad \longleftarrow$$

Solution 11.2-2 Rigid bar *ABC*

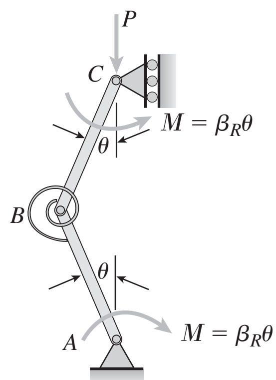


$$\sum M_A = 0$$

$$P\theta L - \beta\theta a^2 = 0$$

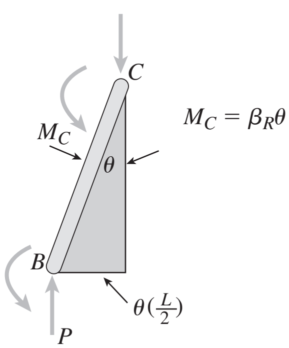
$$P_{\text{cr}} = \frac{\beta a^2}{L} \quad \longleftarrow$$

Solution 11.2-3 Two rigid bars with a pin connection



$\sum M_A = 0$ Shows that there are no horizontal reactions at the supports.

FREE-BODY DIAGRAM OF BAR *BC*



$$M_C = \beta_R\theta$$

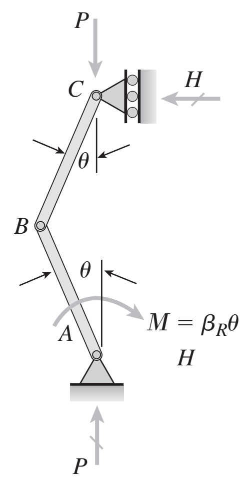
$$M_B = \beta_R(2\theta)$$

$$\sum M_B = 0 \quad M_B + M_C - P\theta\left(\frac{L}{2}\right) = 0$$

$$\beta_R(2\theta) + \beta_R\theta = \frac{PL\theta}{2}$$

$$P_{\text{cr}} = \frac{6\beta_R}{L} \quad \longleftarrow$$

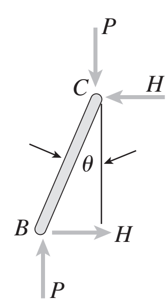
Solution 11.2-4 Two rigid bars with a pin connection



$$\sum M_A = 0 \quad HL - \beta_R\theta = 0$$

$$H = \frac{\beta_R\theta}{L}$$

FREE-BODY DIAGRAM OF BAR *BC*



$$\sum M_B = 0 \quad H\left(\frac{L}{2}\right) - P\left(\frac{\theta L}{2}\right) = 0$$

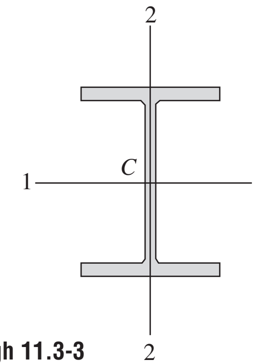
$$P_{\text{cr}} = \frac{H}{\theta} = \frac{\beta_R}{L} \quad \longleftarrow$$

Critical Loads of Columns with Pinned Supports

The problems for Section 11.3 are to be solved using the assumptions of ideal, slender, prismatic, linearly elastic columns (Euler buckling). Buckling occurs in the plane of the figure unless stated otherwise.

Problem 11.3-1 Calculate the critical load P_{cr} for a W 8 × 35 steel column (see figure) having length $L = 24$ ft and $E = 30 \times 10^6$ psi under the following conditions:

(a) The column buckles by bending about its strong axis (axis 1-1), and (b) the column buckles by bending about its weak axis (axis 2-2). In both cases, assume that the column has pinned ends.



Probs. 11.3-1 through 11.3-3

Solution 11.3-1 Column with pinned supports

W 8 × 35 steel column

$L = 24 \text{ ft} = 288 \text{ in.} \quad E = 30 \times 10^6 \text{ psi}$

$I_1 = 127 \text{ in.}^4 \quad I_2 = 42.6 \text{ in.}^4 \quad A = 10.3 \text{ in.}^2$

(a) BUCKLING ABOUT STRONG AXIS

$$P_{cr} = \frac{\pi^2 EI_1}{L^2} = 453 \text{ k} \quad \longleftarrow$$

(b) BUCKLING ABOUT WEAK AXIS

$$P_{cr} = \frac{\pi^2 EI_2}{L^2} = 152 \text{ k} \quad \longleftarrow$$

NOTE: $\sigma_{cr} = \dfrac{P_{cr}}{A} = \dfrac{453 \text{ k}}{10.3 \text{ in.}^2} = 44 \text{ ksi}$

$\therefore$ Solution is satisfactory if $\sigma_{PL} \geq 44$ ksi

Problem 11.3-2 Solve the preceding problem for a W 10 × 60 steel column having length $L = 30$ ft.

Solution 11.3-2 Column with pinned supports

W 10 × 60 steel column

$L = 30 \text{ ft} = 360 \text{ in.} \quad E = 30 \times 10^6 \text{ psi}$

$I_1 = 341 \text{ in.}^4 \quad I_2 = 116 \text{ in.}^4 \quad A = 17.6 \text{ in.}^2$

(a) BUCKLING ABOUT STRONG AXIS

$$P_{cr} = \frac{\pi^2 EI_1}{L^2} = 779 \text{ k} \quad \longleftarrow$$

(b) BUCKLING ABOUT WEAK AXIS

$$P_{cr} = \frac{\pi^2 EI_2}{L^2} = 265 \text{ k} \quad \longleftarrow$$

NOTE: $\sigma_{cr} = \dfrac{P_{cr}}{A} = \dfrac{779 \text{ k}}{17.6 \text{ in.}^2} = 44 \text{ ksi}$

$\therefore$ Solution is satisfactory if $\sigma_{PL} \geq 44$ ksi

Problem 11.3-3 Solve Problem 11.3-1 for a W 10 × 45 steel column having length $L = 28$ ft.

Solution 11.3-3 Column with pinned supports

W 10 × 45 steel column

$L = 28 \text{ ft} = 336 \text{ in.} \quad E = 30 \times 10^6 \text{ psi}$

$I_1 = 248 \text{ in.}^4 \quad I_2 = 53.4 \text{ in.}^4 \quad A = 13.3 \text{ in.}^2$

(a) BUCKLING ABOUT STRONG AXIS

$$P_{cr} = \frac{\pi^2 EI_1}{L^2} = 650 \text{ k} \quad \longleftarrow$$

(b) BUCKLING ABOUT WEAK AXIS

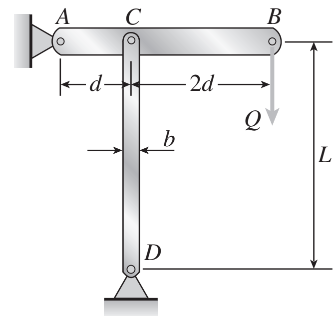
$$P_{cr} = \frac{\pi^2 EI_2}{L^2} = 140 \text{ k} \quad \longleftarrow$$

NOTE: $\sigma_{cr} = \dfrac{P_{CR}}{A} = \dfrac{650 \text{ k}}{13.3 \text{ in.}^2} = 49 \text{ ksi}$

$\therefore$ Solution is satisfactory if $\sigma_{PL} \geq 49$ ksi

Problem 11.3-4 A horizontal beam AB is pin-supported at end A and carries a load Q at end B, as shown in the figure. The beam is supported at C by a pinned-end column. The column is a solid steel bar ($E = 200$ GPa) of square cross section having length $L = 1.8$ m and side dimensions $b = 60$ mm.

Based upon the critical load of the column, determine the allowable load Q if the factor of safety with respect to buckling is $n = 2.0$.



Probs. 11.3-4 and 11.3-5

Solution 11.3-4 Beam supported by a column

COLUMN CD (STEEL)

$E = 200$ GPa $\quad L = 1.8$ m

Square cross section: $b = 60$ mm

Factor of safety: $n = 2.0$

$$I = \frac{b^4}{12} = 1.08 \times 10^6 \text{ mm}^4$$

$$P_{cr} = \frac{\pi^2 EI}{L^2} = 657.97 \text{ kN}$$

BEAM ACB $\quad \sum M_A = 0 \quad Q = \frac{P}{3}$

$$Q_{allow} = \frac{P_{allow}}{3} = \frac{P_{cr}}{3n} = \frac{P_{cr}}{6.0} = 109.7 \text{ kN} \quad \longleftarrow$$

Problem 11.3-5 Solve the preceding problem if the column is aluminum ($E = 10 \times 10^6$ psi), the length $L = 30$ in., the side dimension $b = 1.5$ in., and the factor of safety $n = 1.8$.

Solution 11.3-5 Beam supported by a column

COLUMN CD (STEEL)

$E = 10 \times 10^6$ psi $\quad L = 30$ in.

Square cross section: $b = 1.5$ in.

Factor of safety: $n = 1.8$

$$I = \frac{b^4}{12} = 0.42188 \text{ in.}^4$$

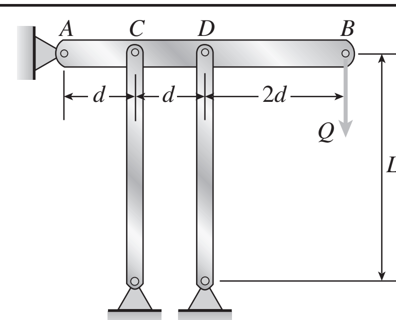
$$P_{cr} = \frac{\pi^2 EI}{L^2} = 46.264 \text{ k}$$

BEAM ACB $\quad \sum M_A = 0 \quad Q = \frac{P}{3}$

$$Q_{allow} = \frac{P_{allow}}{3} = \frac{P_{cr}}{3n} = \frac{P_{cr}}{5.4} = 8.57 \text{ k} \quad \longleftarrow$$

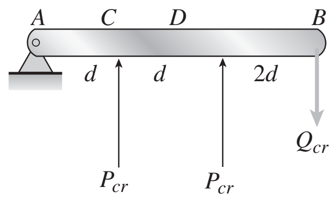
Problem 11.3-6 A horizontal beam AB is pin-supported at end A and carries a load Q at end B, as shown in the figure. The beam is supported at C and D by two identical pinned-end columns of length L. Each column has flexural rigidity EI.

What is the critical load Q_{cr}? (In other words, at what load Q_{cr} does the system collapse because of Euler buckling of the columns?)



Solution 11.3-6 Beam supported by two columns

Collapse occurs when both columns reach the critical load.

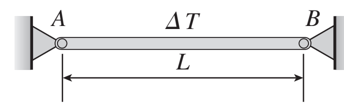


$$\sum M_A = 0 \qquad Q_{cr} = \frac{3P_{cr}}{4}$$

$$P_{cr} = \frac{\pi^2 EI}{L^2} \qquad \therefore\ Q_{cr} = \frac{3\pi^2 EI}{4L^2} \quad \longleftarrow$$

Problem 11.3-7 A slender bar AB with pinned ends and length L is held between immovable supports (see figure).

What increase ΔT in the temperature of the bar will produce buckling at the Euler load?



Solution 11.3-7 Bar with immovable pin supports

L = length $\quad A$ = cross-sectional area

I = moment of inertia $\quad E$ = modulus of elasticity

α = coefficient of thermal expansion

ΔT = uniform increase in temperature

AXIAL COMPRESSIVE FORCE IN BAR (EQ. 2-17)

$$P = EA\alpha(\Delta T)$$

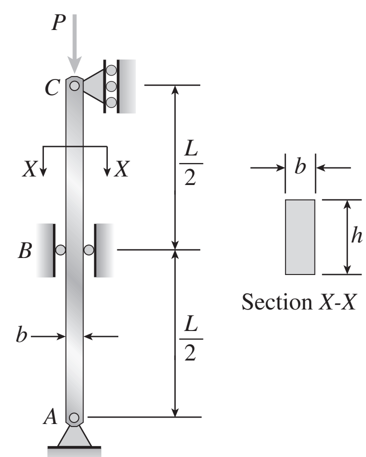
EULER LOAD $\quad P_{cr} = \dfrac{\pi^2 EI}{L^2}$

INCREASE IN TEMPERATURE TO PRODUCE BUCKLING

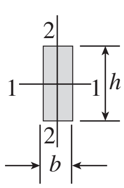
$$P = P_{cr} \qquad EA\alpha(\Delta T) = \frac{\pi^2 EI}{L^2} \qquad \Delta T = \frac{\pi^2 I}{\alpha A L^2} \quad \longleftarrow$$

Problem 11.3-8 A rectangular column with cross-sectional dimensions b and h is pin-supported at ends A and C (see figure). At midheight, the column is restrained in the plane of the figure but is free to deflect perpendicular to the plane of the figure.

Determine the ratio h/b such that the critical load is the same for buckling in the two principal planes of the column.



Solution 11.3-8 Column with restraint at midheight



Critical loads for buckling about axes 1-1 and 2-2:

$$P_1 = \frac{\pi^2 EI_1}{L^2} \qquad P_2 = \frac{\pi^2 EI_2}{(L/2)^2} = \frac{4\pi^2 EI_2}{L^2}$$

FOR EQUAL CRITICAL LOADS

$$P_1 = P_2 \qquad \therefore\ I_1 = 4I_2$$

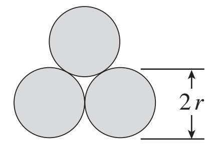
$$I_1 = \frac{bh^3}{12} \qquad I_2 = \frac{hb^3}{12}$$

$$bh^3 = 4hb^3 \qquad \frac{h}{b} = 2 \quad \longleftarrow$$

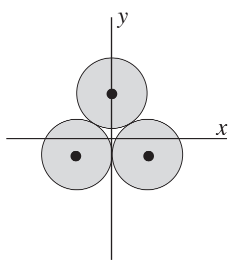
Problem 11.3-9 Three identical, solid circular rods, each of radius r and length L, are placed together to form a compression member (see the cross section shown in the figure).

Assuming pinned-end conditions, determine the critical load P_{cr} as follows: (a) The rods act independently as individual columns, and (b) the rods are bonded by epoxy throughout their lengths so that they function as a single member.

What is the effect on the critical load when the rods act as a single member?



Solution 11.3-9 **Three solid circular rods**



R = Radius L = Length

(a) Rods act independently

$$P_{cr} = \frac{\pi^2 EI}{L^2}(3) \qquad I = \frac{\pi r^4}{4}$$

$$P_{cr} = \frac{3\pi^3 E r^4}{4L^2} \longleftarrow$$

(b) Rods are bonded together

The x and y axes have their origin at the centroid of the cross section. Because there are three different centroidal axes of symmetry, all centroidal axes are principal axes and all centroidal moments of inertia are equal (see Section 12.9).

From Case 9, Appendix D:

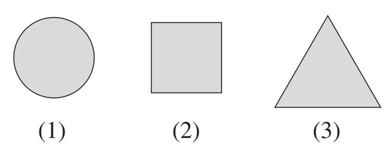
$$I = I_Y = \frac{\pi r^4}{4} + 2\left(\frac{5\pi r^4}{4}\right) = \frac{11\pi r^4}{4}$$

$$P_{cr} = \frac{\pi^2 EI}{L^2} = \frac{11\pi^3 E r^4}{4L^2} \longleftarrow$$

Note: Joining the rods so that they act as a single member increases the critical load by a factor of 11/3, or 3.67. $\longleftarrow$

Problem 11.3-10 Three pinned-end columns of the same material have the same length and the same cross-sectional area (see figure). The columns are free to buckle in any direction. The columns have cross sections as follows: (1) a circle, (2) a square, and (3) an equilateral triangle.

Determine the ratios $P_1 : P_2 : P_3$ of the critical loads for these columns.



Solution 11.3-10 **Three pinned-end columns**

E, L, and A are the same for all three columns.

$$P_{cr} = \frac{\pi^2 EI}{L^2} \qquad \therefore P_1 : P_2 : P_3 = I_1 : I_2 : I_3$$

(1) Circle Case 9, Appendix D

$$I = \frac{\pi d^4}{64} \qquad A = \frac{\pi d^2}{4} \qquad \therefore I_1 = \frac{A^2}{4\pi}$$

(2) Square Case 1, Appendix D

$$I = \frac{b^4}{12} \qquad A = b^2 \qquad \therefore I_2 = \frac{A^2}{12}$$

(3) Equilateral triangle Case 5, Appendix D

$$I = \frac{b^4\sqrt{3}}{96} \qquad A = \frac{b^2\sqrt{3}}{4} \qquad \therefore I_3 = \frac{A^2\sqrt{3}}{18}$$

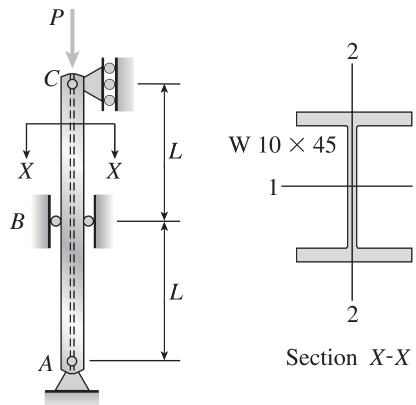
$$P_1 : P_2 : P_3 = I_1 : I_2 : I_3 = 1 : \frac{\pi}{3} : \frac{2\pi\sqrt{3}}{9}$$

$$= 1.000 : 1.047 : 1.209 \longleftarrow$$

Note: For each of the above cross sections, every centroidal axis has the same moment of inertia (see Section 12.9).

Problem 11.3-11 A long slender column *ABC* is pinned at ends *A* and *C* and compressed by an axial force *P* (see figure). At the midpoint *B*, lateral support is provided to prevent deflection in the plane of the figure. The column is a steel wide-flange section (W 10 × 45) with $E = 30 \times 10^6$ psi. The distance between lateral supports is $L = 18$ ft.

Calculate the allowable load *P* using a factor of safety $n = 2.4$, taking into account the possibility of Euler buckling about either principal centroidal axis (i.e., axis 1-1 or axis 2-2).



Solution 11.3-11 **Column with restraint at midheight**

W 10 × 45 $\quad E = 30 \times 10^6$ psi

$L = 18$ ft $= 216$ in. $\quad I_1 = 248$ in.^{4} $\quad I_2 = 53.4$ in.^{4}

$n = 2.4$

BUCKLING ABOUT AXIS 1-1

$$P_{cr} = \frac{\pi^2 E I_1}{(2L)^2} = 393.5\,\text{k}$$

BUCKLING ABOUT AXIS 2-2

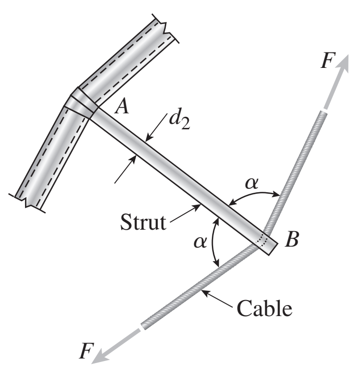
$$P_{cr} = \frac{\pi^2 E I_2}{L^2} = 338.9\,\text{k}$$

ALLOWABLE LOAD

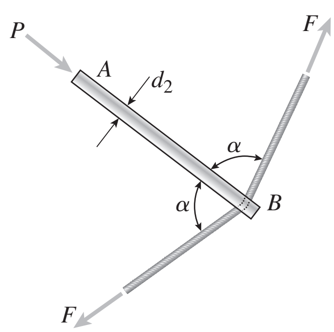
$$P_{allow} = \frac{P_{cr}}{n} = \frac{338.9\,\text{k}}{2.4} = 141\,\text{k} \longleftarrow$$

Problem 11.3-12 The multifaceted glass roof over the lobby of a museum building is supported by the use of pretensioned cables. At a typical joint in the roof structure, a strut *AB* is compressed by the action of tensile forces *F* in a cable that makes an angle $\alpha = 75°$ with the strut (see figure). The strut is a circular tube of aluminum ($E = 72$ GPa) with outer diameter $d_2 = 50$ mm and inner diameter $d_1 = 40$ mm. The strut is 1.0 m long and is assumed to be pin-connected at both ends.

Using a factor of safety $n = 2.5$ with respect to the critical load, determine the allowable force *F* in the cable.



Solution 11.3-12 **Strut and cable**



P = compressive force in strut

F = tensile force in cable

α = angle between strut and cable
$= 75°$

PROPERTIES OF STRUT $\quad E = 72$ GPa

$d_2 = 50$ mm $\quad d_1 = 40$ mm $\quad L = 1.0$ m

$$I = \frac{\pi}{64}(d_2^4 - d_1^4) = 181.13 \times 10^3\,\text{mm}^4$$

$$P_{cr} = \frac{\pi^2 EI}{L^2} = 128.71\,\text{kN}$$

$$P_{allow} = \frac{P_{cr}}{n} = \frac{128.71\,\text{kN}}{2.5} = 51.49\,\text{kN}$$

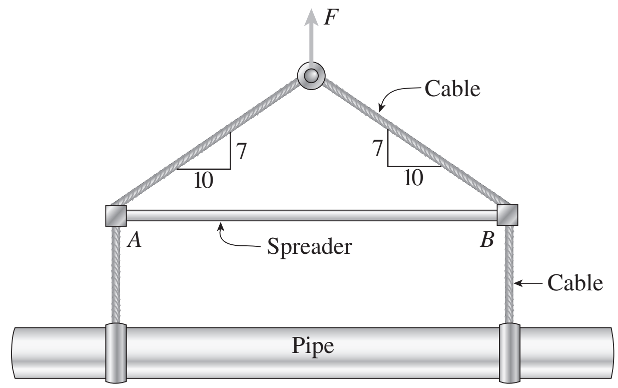
EQUILIBRIUM OF JOINT *B*

$P = 2F \cos 75°$

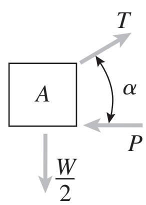
$$\therefore\ F_{allow} = \frac{P_{allow}}{2\cos 75°} = 99.5\,\text{kN} \longleftarrow$$

Problem 11.3-13 The hoisting arrangement for lifting a large pipe is shown in the figure. The spreader is a steel tubular section with outer diameter 2.75 in. and inner diameter 2.25 in. Its length is 8.5 ft and its modulus of elasticity is 29×10^6 psi.

Based upon a factor of safety of 2.25 with respect to Euler buckling of the spreader, what is the maximum weight of pipe that can be lifted? (Assume pinned conditions at the ends of the spreader.)



Solution 11.3-13 **Hoisting arrangement for a pipe**



T = tensile force in cable
P = compressive force in spreader
W = weight of pipe

$$\tan\alpha = \frac{7}{10}$$

Properties of spreader $E = 29 \times 10^6$ psi

$d_2 = 2.75$ in. $d_1 = 2.25$ in. $L = 8.5$ ft $= 102$ in.

$$I = \frac{\pi}{64}(d_2^4 - d_1^4) = 1.549 \text{ in.}^4$$

$$P_{cr} = \frac{\pi^2 EI}{L^2} = 42.61 \text{ k}$$

$$P_{allow} = \frac{P_{cr}}{n} = \frac{42.61 \text{ k}}{2.25} = 18.94 \text{ k}$$

Equilibrium of joint A

$$\sum F_{horiz} = 0 \qquad -P + T\cos\alpha = 0$$

$$\sum F_{vert} = 0 \qquad T\sin\alpha - \frac{w}{2} = 0$$

Solve the equation

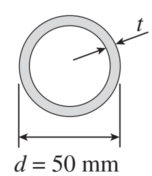
$$W = 2P\tan\alpha$$

Maximum weight of pipe

$$W_{max} = 2P_{allow}\tan\alpha = 2(18.94 \text{ k})(0.7)$$
$$= 26.5 \text{ k} \longleftarrow$$

Problem 11.3-14 A pinned-end strut of aluminum ($E = 72$ GPa) with length $L = 1.8$ m is constructed of circular tubing with outside diameter $d = 50$ mm (see figure). The strut must resist an axial load $P = 18$ kN with a factor of safety $n = 2.0$ with respect to the critical load.

Determine the required thickness t of the tube.



Solution 11.3-14 **Aluminum strut**

$E = 72$ GPa $\quad L = 1.8$ m

Outer diameter $d = 50$ mm

t = thickness

Inner diameter $= d - 2t$

$P = 18$ kN $\quad n = 2.0$

Critical load $\quad P_{cr} = nP = (2.0)(18\text{ kN}) = 36$ kN

$$P_{cr} = \frac{\pi^2 EI}{L^2} \qquad \therefore\ I = \frac{P_{cr}L^2}{\pi^2 E} = 164.14 \times 10^3\,\text{mm}^4$$

Moment of inertia

$$I = \frac{\pi}{64}[d^4\ 2\ (d\ 2\ 2t)^4] = 164.14\ 3\ 10^3\,\text{mm}^4$$

Required thickness

$d^4 - (d - 2t)^4 = 3.3438 \times 10^6$ mm^4

$(d - 2t)^4 = (50\text{ mm})^4 - 3.3438 \times 10^6$ mm^4

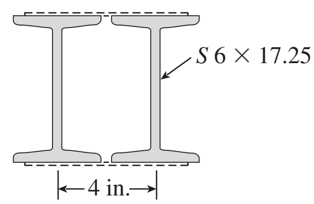
$= 2.9062 \times 10^6$ mm^4

$d - 2t = 41.289$ mm

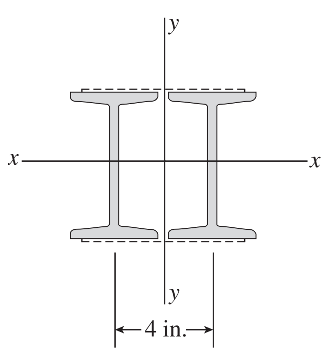
$2t = 50$ mm $- 41.289$ mm $= 8.711$ mm

$t_{min} = 4.36$ mm ⟵

Problem 11.3-15 The cross section of a column built up of two steel I-beams (S 6 × 17.25 sections) is shown in the figure on the next page. The beams are connected by spacer bars, or *lacing*, to ensure that they act together as a single column. (The lacing is represented by dashed lines in the figure.) The column is assumed to have pinned ends and may buckle in any direction. Assuming $E = 30 \times 10^6$ psi and $L = 27.5$ ft, calculate the critical load P_{cr} for the column.



Solution 11.3-15 **Column of two steel beams**



S 6 × 17.25

$E = 30 \times 10^6$ psi

$L = 27.5$ ft $= 330$ in.

$I_1 = 26.3$ in.^{4}

$I_2 = 2.31$ in.^{4}

$A = 5.07$ in.^{2}

Composite column $\quad I_x = 2I_1 = 52.6$ in.^{4}

$$I_y = 2(I_2 + Ad^2) \qquad d = \frac{4\text{ in.}}{2} = 2\text{ in.}$$

$I_y = 2[2.31\text{ in.}^4 + (5.07\text{ in.}^2)(2\text{ in.})^2]$

$= 45.18$ in.$^4 \qquad I_y < I_x$

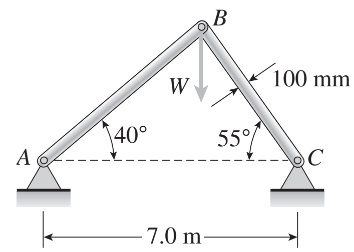
$\therefore$ Buckling occurs about the y axis.

Critical load

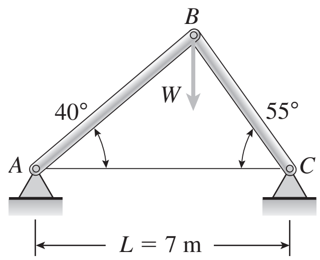
$$P_{cr} = \frac{\pi^2 EI_y}{L^2} = 123\,\text{k}$$ ⟵

Problem 11.3-16 The truss *ABC* shown in the figure supports a vertical load W at joint B. Each member is a slender circular steel pipe ($E = 200$ GPa) with outside diameter 100 mm and wall thickness 6.0 mm. The distance between supports is 7.0 m. Joint B is restrained against displacement perpendicular to the plane of the truss.

Determine the critical value W_{cr} of the load.



Solution 11.3-16 **Truss *ABC* with load *W***



STEEL PIPES *AB* AND *BC*

$E = 200$ GPa $\quad L = 7.0$ m

$d_2 = 100$ mm $\quad t = 6.0$ mm

$d_1 = d_2 - 2t = 88$ mm

$$I = \frac{\pi}{64}(d_2^4 - d_1^4) = 1.965 \times 10^6 \text{ mm}^4$$

LENGTHS OF MEMBERS *AB* AND *BC*

use the law of sines (see Appendix C)

$$L_{AB} = L\left(\frac{\sin 55°}{\sin 85°}\right) = 5.756 \text{ m}$$

$$L_{BC} = L\left(\frac{\sin 40°}{\sin 85°}\right) = 4.517 \text{ m}$$

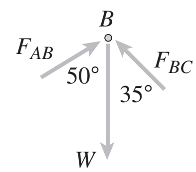
Buckling occurs when either member reaches its critical load.

CRITICAL LOADS

$$(P_{cr})_{AB} = \frac{\pi^2 EI}{L_{AB}^2} = 117.1 \text{ kN}$$

$$(P_{cr})_{BC} = \frac{\pi^2 EI}{L_{BC}^2} = 190.1 \text{ kN}$$

FREE-BODY DIAGRAM OF JOINT *B*



$$\sum F_{horiz} = 0 \qquad F_{AB} \sin 50° - F_{BC} \sin 35° = 0$$

$$\sum F_{vert} = 0 \qquad F_{AB} \cos 50° - F_{BC} \cos 35° - W = 0$$

SOLVE THE TWO EQUATIONS

$W = 1.7368\, F_{AB} \qquad W = 1.3004\, F_{BC}$

CRITICAL VALUE OF THE LOAD *W*

Based on member *AB*: $W_{cr} = 1.7368\,(P_{cr})_{AB}$
$= 203$ kN

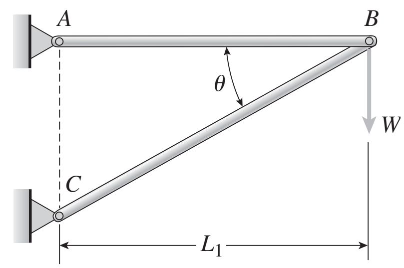
Based on member *BC*: $W_{cr} = 1.3004\,(P_{cr})_{BC}$
$= 247$ kN

lower load governs. Member *AB* buckler.

$W_{cr} = 203$ kN ←

Problem 11.3-17 A truss ABC supports a load W at joint B, as shown in the figure. The length L_1 of member AB is fixed, but the length of strut BC varies as the angle θ is changed. Strut BC has a solid circular cross section. Joint B is restrained against displacement perpendicular to the plane of the truss.

Assuming that collapse occurs by Euler buckling of the strut, determine the angle θ for minimum weight of the strut.



Solution 11.3-17 Truss ABC (minimum weight)

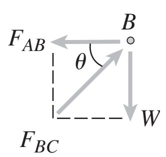
LENGTHS OF MEMBERS

$L_{AB} = L_1$ (a constant)

$$L_{BC} = \frac{L_1}{\cos\theta} \text{(angle } \theta \text{ is variable)}$$

Strut BC may buckle.

FREE-BODY DIAGRAM OF JOINT B



$$\sum F_{\text{vert}} = 0 \qquad F_{BC}\sin\theta - W = 0$$

$$F_{BC} = \frac{W}{\sin\theta}$$

STRUT BC (SOLID CIRCULAR BAR)

$$A = \frac{\pi d^2}{4} \qquad I = \frac{\pi d^4}{64} \qquad \therefore\ I = \frac{A^2}{4\pi}$$

$$P_{\text{cr}} = \frac{\pi^2 EI}{L_{BC}^2} = \frac{\pi E A^2 \cos^2\theta}{4L_1^2}$$

$$F_{BC} = P_{\text{cr}} \quad \text{or} \quad \frac{W}{\sin\theta} = \frac{\pi E A^2 \cos^2\theta}{4L_1^2}$$

Solve for area A: $A = \dfrac{2L_1}{\cos\theta}\left(\dfrac{W}{\pi E\sin\theta}\right)^{1/2}$

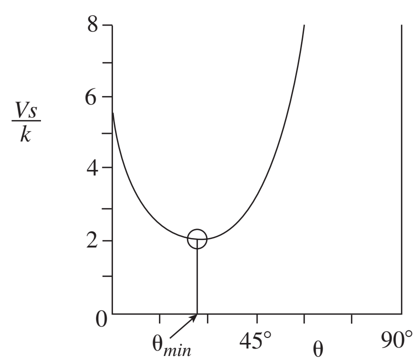
For minimum weight, the volume V_S of the strut must be a minimum.

$$V_S = AL_{BC} = \frac{AL_1}{\cos\theta} = \frac{2L_1^2}{\cos^2\theta}\left(\frac{W}{\pi E\sin\theta}\right)^{1/2}$$

All terms are constants except $\cos\theta$ and $\sin\theta$. Therefore, we can write V_S in the following form:

$$V_S = \frac{k}{\cos^2\theta\sqrt{\sin\theta}} \quad \text{where } k \text{ is a constant.}$$

GRAPH OF $\dfrac{V_S}{k}$



θ_{min} = angle for minimum volume (and minimum weight)

For minimum weight, the term $\cos^2\theta\sqrt{\sin\theta}$ must be a maximum.

For a maximum value, the derivative with respect to θ equals zero.

Therefore, $\dfrac{d}{d\theta}(\cos^2\theta\sqrt{\sin\theta}) = 0$

Taking the derivative and simplifying, we get $\cos^2\theta - 4\sin^2\theta = 0$

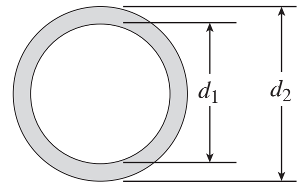
or $1 - 4\tan^2\theta = 0$ and $\tan\theta = \dfrac{1}{2}$

Columns with Other Support Conditions

The problems for Section 11.4 are to be solved using the assumptions of ideal, slender, prismatic, linearly elastic columns (Euler buckling). Buckling occurs in the plane of the figure unless stated otherwise.

Problem 11.4-1 An aluminum pipe column ($E = 10{,}400$ ksi) with length $L = 10.0$ ft has inside and outside diameters $d_1 = 5.0$ in. and $d_2 = 6.0$ in., respectively (see figure). The column is supported only at the ends and may buckle in any direction.

Calculate the critical load P_{cr} for the following end conditions: (1) pinned-pinned, (2) fixed-free, (3) fixed-pinned, and (4) fixed-fixed.



Probs. 11.4-1 and 11.4-2

Solution 11.4-1 **Aluminum pipe column**

$d_2 = 6.0$ in. $\quad d_1 = 5.0$ in. $\quad E = 10{,}400$ ksi

$$I = \frac{\pi}{64}(d_2^4 - d_1^4) = 32.94 \text{ in.}^4$$

$L = 10.0$ ft $= 120$ in.

(1) Pinned-pinned

$$P_{cr} = \frac{\pi^2 EI}{L^2} = \frac{\pi^2(10{,}400 \text{ ksi})(32.94 \text{ in.}^4)}{(120 \text{ in.})^2}$$
$$= 235 \text{ k} \longleftarrow$$

(2) Fixed-free $\quad P_{cr} = \dfrac{\pi^2 EI}{4L^2} = 58.7 \text{ k} \longleftarrow$

(3) Fixed-pinned $\quad P_{cr} = \dfrac{2.046\,\pi^2 EI}{L^2} = 480 \text{ k} \longleftarrow$

(4) Fixed-fixed $\quad P_{cr} = \dfrac{4\pi^2 EI}{L^2} = 939 \text{ k} \longleftarrow$

Problem 11.4-2 Solve the preceding problem for a steel pipe column ($E = 210$ GPa) with length $L = 1.2$ m, inner diameter $d_1 = 36$ mm, and outer diameter $d_2 = 40$ mm.

Solution 11.4-2 **Steel pipe column**

$d_2 = 40$ mm $\quad d_1 = 36$ mm $\quad E = 210$ GPa

$$I = \frac{\pi}{64}(d_2^4 - d_1^4) = 43.22 \times 10^3 \text{ mm}^4 \quad L = 1.2 \text{ m}$$

(1) Pinned-pinned $\quad P_{cr} = \dfrac{\pi^2 EI}{L^2} = 62.2 \text{ kN} \longleftarrow$

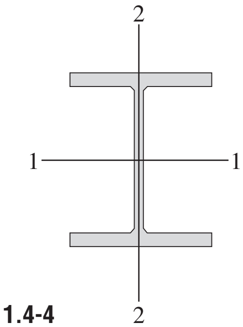
(2) Fixed-free $\quad P_{cr} = \dfrac{\pi^2 EI}{4L^2} = 15.6 \text{ kN} \longleftarrow$

(3) Fixed-pinned $\quad P_{cr} = \dfrac{2.046\,\pi^2 EI}{L^2} = 127 \text{ kN} \longleftarrow$

(4) Fixed-fixed $\quad P_{cr} = \dfrac{4\pi^2 EI}{L^2} = 249 \text{ kN} \longleftarrow$

Problem 11.4-3 A wide-flange steel column ($E = 30 \times 10^6$ psi) of W 12 × 87 shape (see figure) has length $L = 28$ ft. It is supported only at the ends and may buckle in any direction.

Calculate the allowable load P_{allow} based upon the critical load with a factor of safety $n = 2.5$. Consider the following end conditions: (1) pinned-pinned, (2) fixed-free, (3) fixed-pinned, and (4) fixed-fixed.



Probs. 11.4-3 and 11.4-4

Solution 11.4-3 Wide-flange column

W 12 × 87 $\quad E = 30 \times 10^6$ psi

$L = 28$ ft $= 336$ in. $\quad n = 2.5 \quad I_2 = 241$ in.^{4}

(1) Pinned-pinned

$$P_{allow} = \frac{P_{cr}}{n} = \frac{\pi^2 EI_2}{nL^2} = 253 \text{ k} \longleftarrow$$

(2) Fixed-free

$$P_{allow} = \frac{\pi^2 EI_2}{4nL^2} = 63.2 \text{ k} \longleftarrow$$

(3) Fixed-pinned

$$P_{allow} = \frac{2.046\pi^2 EI_2}{nL^2} = 517 \text{ k} \longleftarrow$$

(4) Fixed-fixed

$$P_{allow} = \frac{4\pi^2 EI_2}{nL^2} = 1011 \text{ k} \longleftarrow$$

Problem 11.4-4 Solve the preceding problem for a W 10 × 60 shape with length $L = 24$ ft.

Solution 11.4-4 Wide-flange column

W 10 × 60 $\quad E = 30 \times 10^6$ psi

$L = 24$ ft $= 288$ in. $\quad n = 2.5 \quad I_2 = 116$ in.^{4}

(1) Pinned-pinned

$$P_{allow} = \frac{P_{cr}}{n} = \frac{\pi^2 EI_2}{nL^2} = 166 \text{ k} \longleftarrow$$

(2) Fixed-free

$$P_{allow} = \frac{\pi^2 EI_2}{4nL^2} = 41.4 \text{ k} \longleftarrow$$

(3) Fixed-pinned

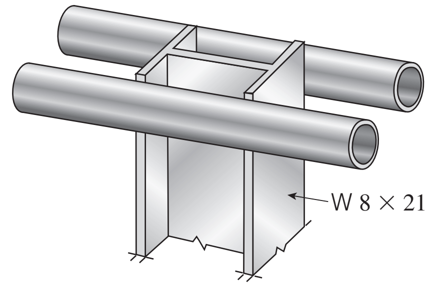
$$P_{allow} = \frac{2.046\pi^2 EI_2}{nL^2} = 339 \text{ k} \longleftarrow$$

(4) Fixed-fixed

$$P_{allow} = \frac{4\pi^2 EI_2}{nL^2} = 663 \text{ k} \longleftarrow$$

Problem 11.4-5 The upper end of a W 8 × 21 wide-flange steel column ($E = 30 \times 10^3$ ksi) is supported laterally between two pipes (see figure). The pipes are not attached to the column, and friction between the pipes and the column is unreliable. The base of the column provides a fixed support, and the column is 13 ft long.

Determine the critical load for the column, considering Euler buckling in the plane of the web and also perpendicular to the plane of the web.

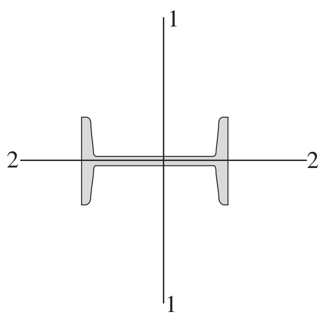


Solution 11.4-5 **Wide-flange steel column**

W 8 × 21 $E = 30 \times 10^3$ ksi

$L = 13$ ft $= 156$ in. $I_1 = 75.3$ in.^{4}

$I_2 = 9.77$ in.^{4}



AXIS 1-1 (FIXED-FREE)

$$P_{cr} = \frac{\pi^2 EI_1}{4L^2} = 229\,\text{k}$$

AXIS 2-2 (FIXED-PINNED)

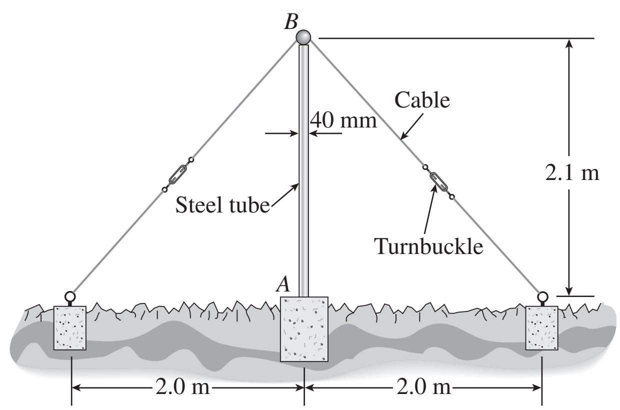
$$P_{cr} = \frac{2.046\pi^2 EI_2}{L^2} = 243\,\text{k}$$

Buckling about axis 1-1 governs.

$P_{cr} = 229$ k ←

Problem 11.4-6 A vertical post AB is embedded in a concrete foundation and held at the top by two cables (see figure). The post is a hollow steel tube with modulus of elasticity 200 GPa, outer diameter 40 mm, and thickness 5 mm. The cables are tightened equally by turnbuckles.

If a factor of safety of 3.0 against Euler buckling in the plane of the figure is desired, what is the maximum allowable tensile force T_{allow} in the cables?



Solution 11.4-6 **Steel tube**

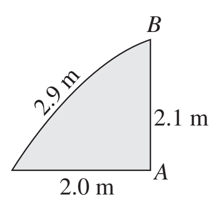
$E = 200$ GPa $d_2 = 40$ mm $d_1 = 30$ mm
$L = 2.1$ m $n = 3.0$

$$I = \frac{\pi}{64}(d_2^4 - d_1^4) = 85{,}903\,\text{mm}^4$$

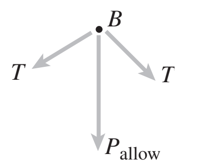
Buckling in the plane of the figure means fixed-pinned end conditions.

$$P_{cr} = \frac{2.046\pi^2 EI}{L^2} = 78.67\text{ kN}$$

$$P_{allow} = \frac{P_{cr}}{n} = \frac{78.67\,\text{kN}}{3.0} = 26.22\,\text{kN}$$



FREE-BODY DIAGRAM OF JOINT B



T = tensile force in each cable
P_{allow} = compressive force in tube

EQUILIBRIUM

$$\sum F_{vert} = 0 \quad P_{allow} - 2T\left(\frac{2.1\text{ m}}{2.9\text{ m}}\right) = 0$$

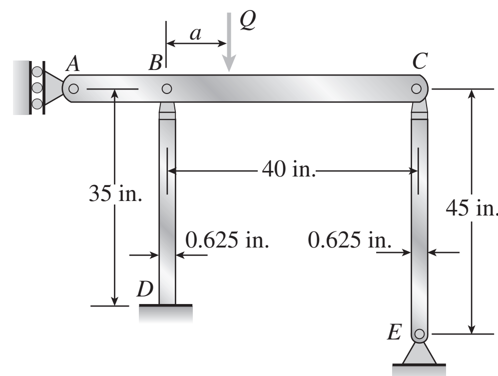
ALLOWABLE FORCE IN CABLES

$$T_{allow} = (P_{allow})\left(\frac{1}{2}\right)\left(\frac{2.9\text{ m}}{2.1\text{ m}}\right) = 18.1\text{ kN} \quad \longleftarrow$$

Problem 11.4-7 The horizontal beam ABC shown in the figure is supported by columns BD and CE. The beam is prevented from moving horizontally by the roller support at end A, but vertical displacement at end A is free to occur. Each column is pinned at its upper end to the beam, but at the lower ends, support D is fixed and support E is pinned. Both columns are solid steel bars ($E = 30 \times 10^6$ psi) of square cross section with width equal to 0.625 in. A load Q acts at distance a from column BD.

(a) If the distance $a = 12$ in., what is the critical value Q_{cr} of the load?

(b) If the distance a can be varied between 0 and 40 in., what is the maximum possible value of Q_{cr}? What is the corresponding value of the distance a?



Solution 11.4-7 Beam supported by two columns

COLUMN BD $\quad E = 30 \times 10^6$ psi $\quad L = 35$ in.

$$b = 0.625\text{ in.} \quad I = \frac{b^4}{12} = 0.012716\text{ in.}^4$$

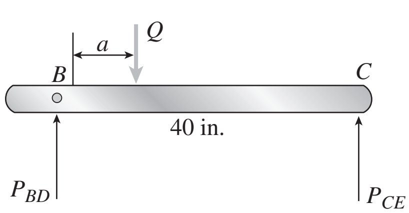
$$P_{cr} = \frac{2.046\,\pi^2 EI}{L^2} = 6288\text{ lb}$$

COLUMN CE $\quad E = 30 \times 10^6$ psi $\quad L = 45$ in.

$$b = 0.625\text{ in.} \quad I = \frac{b^4}{12} = 0.012716\text{ in.}^4$$

$$P_{CR} = \frac{\pi^2 EI}{L^2} = 1859\text{ lb}$$

(a) FIND Q_{cr} IF $a = 12$ in.



$$P_{BD} = \frac{28}{40}Q = \frac{7}{10}Q \quad Q = \frac{10}{7}P_{BD}$$

$$P_{CE} = \frac{12}{40}Q = \frac{3}{10}Q \quad Q = \frac{10}{3}P_{CE}$$

If column BD buckles: $\quad Q = \dfrac{10}{7}(6288\text{ lb}) = 8980\text{ lb}$

If column CE buckles: $\quad Q = \dfrac{10}{3}(1859\text{ lb}) = 6200\text{ lb}$

$$\therefore\ Q_{cr} = 6200\text{ lb} \quad \longleftarrow$$

(b) MAXIMUM VALUE OF Q_{CR}

Both columns buckle simultaneously.
$P_{BD} = 6288$ lb $\quad P_{CE} = 1859$ lb

$$\sum F_{vert} = 0 \quad Q_{CR} = P_{BD} + P_{CE} = 8150\text{ lb} \quad \longleftarrow$$

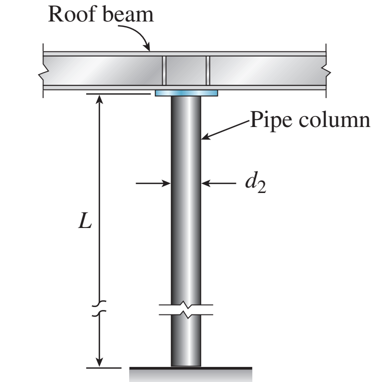
$$\sum M_B = 0 \quad Q_{CR}(a) = P_{CE}(40\text{ in.})$$

$$a = \frac{P_{CE}(40\text{ in.})}{Q_{cr}} = \frac{(1859\text{ lb})(40\text{ in.})}{P_{BD} + P_{CE}}$$

$$= \frac{(1859\text{ lb})(40\text{ in.})}{6288\text{ lb} + 1859\text{ lb}} = 9.13\text{ in.} \quad \longleftarrow$$

Problem 11.4-8 The roof beams of a warehouse are supported by pipe columns (see figure on the next page) having outer diameter $d_2 = 100$ mm and inner diameter $d_1 = 90$ mm. The columns have length $L = 4.0$ m, modulus $E = 210$ GPa, and fixed supports at the base.

Calculate the critical load P_{cr} of one of the columns using the following assumptions: (1) the upper end is pinned and the beam prevents horizontal displacement; (2) the upper end is fixed against rotation and the beam prevents horizontal displacement; (3) the upper end is pinned but the beam is free to move horizontally; and (4) the upper end is fixed against rotation but the beam is free to move horizontally.



Solution 11.4-8 **Pipe column (with fixed base)**

$E = 210$ GPa $\quad L = 4.0$ m

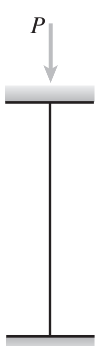
$d_2 = 100$ mm $\quad I = \dfrac{\pi}{64}(d_2^4 - d_1^4) = 1688 \times 10^3\ \text{mm}^4$

$d_1 = 90$ mm

(1) Upper end is pinned (with no horizontal displacement)

$$P_{cr} = \frac{2.046\pi^2 EI}{L^2} = 447\ \text{kN} \quad \longleftarrow$$

(2) Upper end is fixed (with no horizontal displacement)

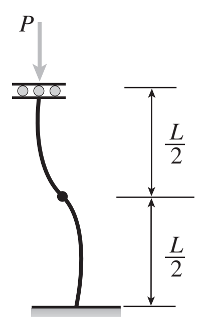


$$P_{cr} = \frac{4\pi^2 EI}{L^2} = 875\ \text{kN} \quad \longleftarrow$$

(3) Upper end is pinned (but no horizontal restraint)

$$P_{cr} = \frac{\pi^2 EI}{4L^2} = 54.7\ \text{kN} \quad \longleftarrow$$

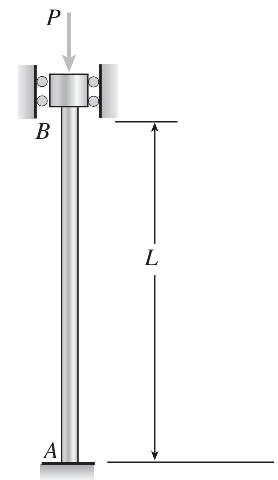
(4) Upper end is guided
(no rotation; no horizontal restraint)



The lower half of the column is in the same condition as Case (3) above.

$$P_{cr} = \frac{\pi^2 EI}{4(L/2)^2} = \frac{\pi^2 EI}{L^2} = 219\ \text{kN} \quad \longleftarrow$$

Problem 11.4-9 Determine the critical load P_{cr} and the equation of the buckled shape for an ideal column with ends fixed against rotation (see figure) by solving the differential equation of the deflection curve. (See also Fig. 11-17.)



Solution 11.4-9 Fixed-end column

v = deflection in the y direction

DIFFERENTIAL EQUATION (EQ.11-3)

$$EIv'' = M = M_0 - Pv \quad k^2 = \frac{P}{EI}$$

$$v'' + k^2 v = \frac{M_0}{EI}$$

GENERAL SOLUTION

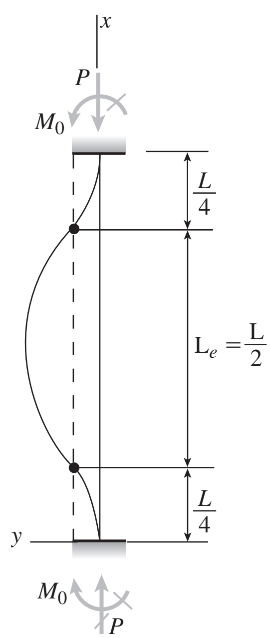
$$v = C_1 \sin kx + C_2 \cos kx + \frac{M_0}{P}$$

B.C. 1 $v(0) = 0 \quad \therefore C_2 = -\frac{M_0}{P}$

$$v' = C_1 k \cos kx - C_2 k \sin kx$$

B.C. 2 $v'(0) = 0 \quad \therefore C_1 = 0$

$$v = \frac{M_0}{P}(1 - \cos kx)$$



BUCKLING EQUATION

B.C. 3 $v(L) = 0 \quad 0 = \frac{M_0}{P}(1 - \cos kL)$

$\therefore \cos kL = 1 \quad$ and $\quad kL = 2\pi$

CRITICAL LOAD

$$k^2 = \left(\frac{2\pi}{L}\right)^2 = \frac{4\pi^2}{L^2} \quad \frac{P}{EI} = \frac{4\pi^2}{L^2}$$

$$P_{cr} = \frac{4\pi^2 EI}{L^2} \quad \longleftarrow$$

BUCKLED MODE SHAPE

Let δ = deflection at midpoint $\left(x = \frac{L}{2}\right)$

$$v\left(\frac{L}{2}\right) = \delta = \frac{M_0}{P}\left(1 - \cos\frac{kL}{2}\right)$$

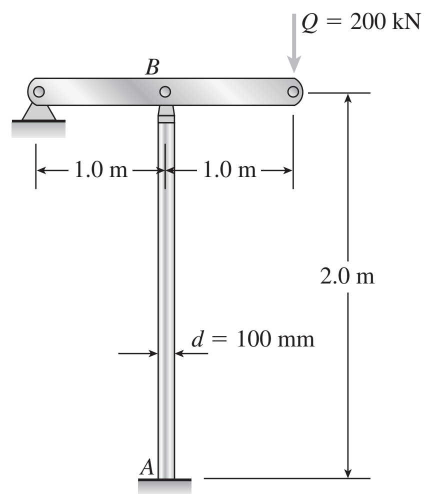
$$\frac{kL}{2} = \pi \quad \therefore \quad \delta = \frac{M_0}{P}(1 - \cos\pi)$$

$$= \frac{2M_0}{P} \quad \frac{M_0}{P} = \frac{\delta}{2}$$

$$v = \frac{\delta}{2}\left(1 - \cos\frac{2\pi x}{L}\right) \quad \longleftarrow$$

Problem 11.4-10 An aluminum tube AB of circular cross section is fixed at the base and pinned at the top to a horizontal beam supporting a load $Q = 200$ kN (see figure).

Determine the required thickness t of the tube if its outside diameter d is 100 mm and the desired factor of safety with respect to Euler buckling is $n = 3.0$. (Assume $E = 72$ GPa.)



Solution 11.4-10 Aluminum tube

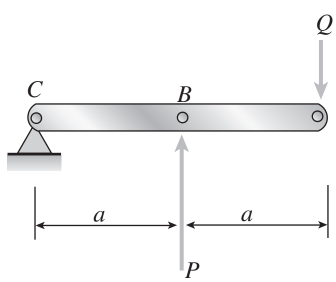
End conditions: Fixed-pinned

$E = 72$ GPa $\quad L = 2.0$ m $\quad n = 3.0$
$d_2 = 100$ mm $\quad t =$ thickness (mm)
$d_1 = 100 \text{ mm} - 2t$

MOMENT OF INERTIA (mm^4)

$$I = \frac{\pi}{64}(d_2^4 - d_1^4)$$
$$= \frac{\pi}{64}[(100)^4 - (100 - 2t)^4] \qquad (1)$$

HORIZONTAL BEAM



$Q = 200$ kN
$P =$ compressive force in tube
$\sum M_c = 0 \quad \text{Pa} - 2Qa = 0$

$$Q = \frac{P}{2} \qquad \therefore\ P = 2Q = 400 \text{ kN}$$

ALLOWABLE FORCE P

$$P_{\text{allow}} = \frac{P_{\text{cr}}}{n} = \frac{2.046\pi^2 EI}{nL^2} \qquad (2)$$

MOMENT OF INERTIA

$$I = \frac{nL^2 P_{\text{allow}}}{2.046\pi^2 E} = \frac{(3.0)(2.0\,\text{m})^2(400\,\text{kN})}{(2.046)(\pi^2)(72\,\text{GPa})}$$
$$= 3.301 \times 10^{-6}\ \text{m}^4 = 3.301 \times 10^6\ \text{mm}^4 \qquad (3)$$

EQUATE (1) AND (3):

$$\frac{\pi}{64}[(100)^4 - (100 - 2t)^4] = 3.301 \times 10^6$$
$$(100 - 2t)^4 = 32.74 \times 10^6\ \text{mm}^4$$
$$100 - 2t = 75.64\ \text{mm} \qquad t_{\min} = 12.2\ \text{mm} \longleftarrow$$

Problem 11.4-11 The frame ABC consists of two members AB and BC that are rigidly connected at joint B, as shown in part (a) of the figure. The frame has pin supports at A and C. A concentrated load P acts at joint B, thereby placing member AB in direct compression.

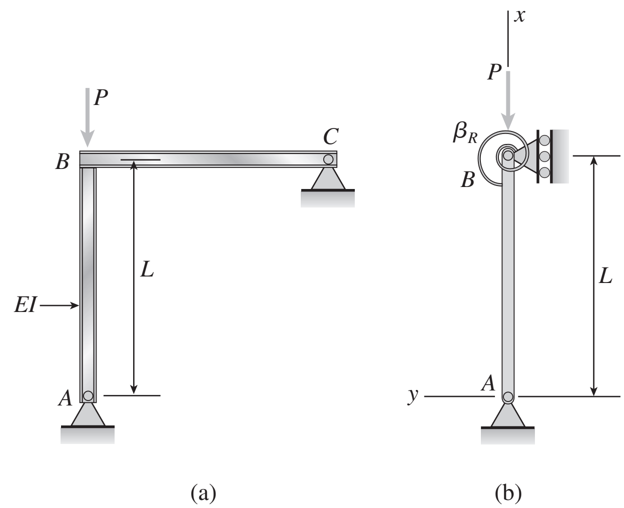
To assist in determining the buckling load for member AB, we represent it as a pinned-end column, as shown in part (b) of the figure. At the top of the column, a rotational spring of stiffness β_R represents the restraining action of the horizontal beam BC on the column (note that the horizontal beam provides resistance to rotation of joint B when the column buckles). Also, consider only bending effects in the analysis (i.e., disregard the effects of axial deformations).

(a) By solving the differential equation of the deflection curve, derive the following buckling equation for this column:

$$\frac{\beta_R L}{EI}(kL \cot kL - 1) - k^2L^2 = 0$$

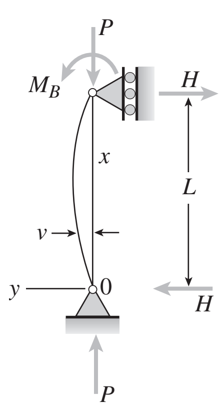
in which L is the length of the column and EI is its flexural rigidity.

(b) For the particular case when member BC is identical to member AB, the rotational stiffness β_R equals $3EI/L$ (see Case 7, Table G-2, Appendix G). For this special case, determine the critical load P_{cr}.



Solution 11.4-11 **Column AB with elastic support at B**

FREE-BODY DIAGRAM OF COLUMN



v = deflection in the y direction

M_B = moment at end B

θ_B = angle of rotation at end B (positive clockwise)

$M_B = \beta_R\theta_B$

H = horizontal reactions at ends A and B

EQUILIBRIUM

$$\sum M_0 = \sum M_A = 0 \quad M_B - HL = 0$$

$$H = \frac{M_B}{L} = \frac{\beta_R\theta_B}{L}$$

DIFFERENTIAL EQUATION (EQ. 11-3)

$$EIv'' = M = Hx - Pv \quad k^2 = \frac{P}{EI}$$

$$v'' + k^2v = \frac{\beta_R\theta_B}{LEI}x$$

GENERAL SOLUTION

$$v = C_1 \sin kx + C_2 \cos kx + \frac{\beta_R\theta_B}{PL}x$$

B.C. 1 $\quad v(0) = 0 \quad \therefore\ C_2 = 0$

B.C. 2 $\quad v(L) = 0 \quad \therefore\ C_1 = -\frac{\beta_R\theta_B}{P\sin kL}$

$$v = C_1 \sin kx + \frac{\beta_R\theta_B}{PL}x$$

$$v' = C_1 k \cos kx + \frac{\beta_R\theta_B}{PL}$$

(a) Buckling equation

B.C. 3 $v'(L) = -\theta_B$

$$\therefore\ -\theta_B = -\frac{\beta_R\theta_B}{P\sin kL}(k\cos kL) + \frac{\beta_R\theta_B}{PL}$$

Cancel θ_B and multiply by PL:

$$-PL = -\beta_R\, kL \cot kL + \beta_R$$

Substitute $P = k^2EI$ and rearrange:

$$\frac{\beta_R L}{EI}(kL\cot kL - 1) - k^2L^2 = 0 \quad \longleftarrow$$

(b) Critical load for $\beta_R = 3EI/L$

$$3(kL\cot kL - 1) - (kL)^2 = 0$$

Solve numerically for kL: $kL = 3.7264$

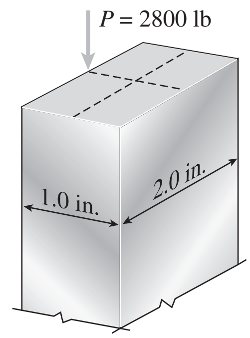
$$P_{cr} = k^2EI = (kL)^2\left(\frac{EI}{L^2}\right) = 13.89\frac{EI}{L^2} \quad \longleftarrow$$

Columns with Eccentric Axial Loads

When solving the problems for Section 11.5, assume that bending occurs in the principal plane containing the eccentric axial load.

Problem 11.5-1 An aluminum bar having a rectangular cross section (2.0 in. × 1.0 in.) and length $L = 30$ in. is compressed by axial loads that have a resultant $P = 2800$ lb acting at the midpoint of the long side of the cross section (see figure).

Assuming that the modulus of elasticity E is equal to 10×10^6 psi and that the ends of the bar are pinned, calculate the maximum deflection δ and the maximum bending moment M_{max}.



Solution 11.5-1 **Bar with rectangular cross section**

$b = 2.0$ in. $h = 1.0$ in. $L = 30$ in.
$P = 2800$ lb $e = 0.5$ in. $E = 10 \times 10^6$ psi

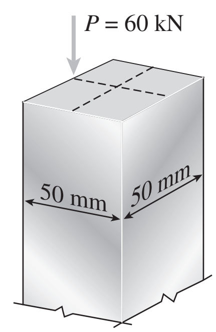
$$I = \frac{bh^3}{12} = 0.1667\text{ in.}^4 \quad kL = L\sqrt{\frac{P}{EI}} = 1.230 \quad \longleftarrow$$

Eq. (11-51): $$\delta = e\left(\sec\frac{kL}{2} - 1\right) = 0.112\text{ in.} \quad \longleftarrow$$

Eq. (11-56): $$M_{max} = Pe\sec\frac{kL}{2} = 1710\text{ lb-in.} \quad \longleftarrow$$

Problem 11.5-2 A steel bar having a square cross section (50 mm × 50 mm) and length $L = 2.0$ m is compressed by axial loads that have a resultant $P = 60$ kN acting at the midpoint of one side of the cross section (see figure).

Assuming that the modulus of elasticity E is equal to 210 GPa and that the ends of the bar are pinned, calculate the maximum deflection δ and the maximum bending moment M_{max}.



Solution 11.5-2 **Bar with square cross section**

$b = 50$ mm. $L = 2$ m. $P = 60$ kN $e = 25$ mm

$$E = 210\text{ GPa} \quad I = \frac{b^4}{12} = 520.8 \times 10^3\ \text{mm}^4$$

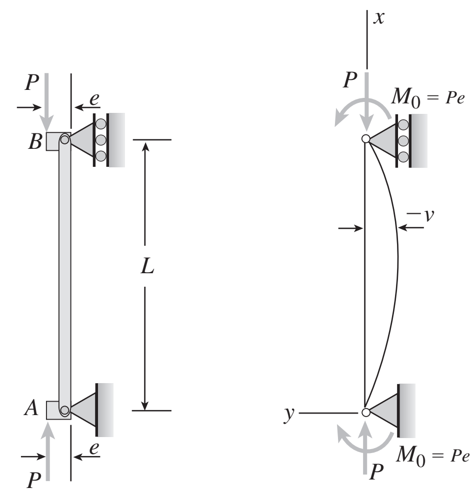
$$kL = L\sqrt{\frac{P}{EI}} = 1.481$$

Eq. (11-51): $$\delta = e\left(\sec\frac{kL}{2} - 1\right) = 8.87\text{ mm} \quad \longleftarrow$$

Eq. (11-56): $$M_{max} = Pe\sec\frac{kL}{2} = 2.03\text{ kN}\cdot\text{m} \quad \longleftarrow$$

Problem 11.5-3 Determine the bending moment M in the pinned-end column with eccentric axial loads shown in the figure. Then plot the bending-moment diagram for an axial load $P = 0.3P_{cr}$.

Note: Express the moment as a function of the distance x from the end of the column, and plot the diagram in nondimensional form with M/Pe as ordinate and x/L as abscissa.



Probs. 11.5-3, 11.5-4, and 11.5-5

Solution 11.5-3 **Column with eccentric loads**

Column has pinned ends.

Use Eq. (11-49):

$$v = -e\left(\tan\frac{kL}{2}\sin kx + \cos kx - 1\right)$$

From Eq. (11-45): $M = Pe - Pv$

$$\therefore M = Pe\left(\tan\frac{kL}{2}\sin kx + \cos kx\right) \longleftarrow$$

For $P = 0.3\,P_{cr}$:

From Eq. (11-52): $kL = \pi\sqrt{\dfrac{P}{P_{cr}}} = \pi\sqrt{0.3}$

$= 1.7207$

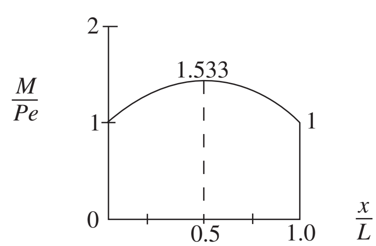
$$\frac{M}{Pe} = \left(\tan\frac{1.7207}{2}\right)\left(\sin 1.7207\frac{x}{L}\right) + \cos 1.7207\frac{x}{L}$$

or

$$\frac{M}{Pe} = 1.162\left(\sin 1.721\frac{x}{L}\right) + \cos 1.721\frac{x}{L} \longleftarrow$$

(Note: kL and kx are in radians)

Bending-moment diagram for $P = 0.3\,P_{cr}$



Problem 11.5-4 Plot the load-deflection diagram for a pinned-end column with eccentric axial loads (see figure) if the eccentricity e of the load is 5 mm and the column has length $L = 3.6$ m, moment of inertia $I = 9.0 \times 10^6$ mm^4, and modulus of elasticity $E = 210$ GPa.

Note: Plot the axial load as ordinate and the deflection at the midpoint as abscissa.

Solution 11.5-4 **Column with eccentric loads**

Column has pinned ends.

Use Eq. (11-54) for the deflection at the midpoint (maximum deflection):

$$\delta = e\left[\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) - 1\right] \qquad (1)$$

Data

$e = 5.0$ mm $\quad L = 3.6$ m $\quad E = 210$ GPa
$I = 9.0 \times 10^6$ mm^4

Critical load

$$P_{cr} = \frac{\pi^2 EI}{L^2} = 1439.3\,\text{kN}$$

MAXIMUM DEFLECTION (FROM EQ. 1)

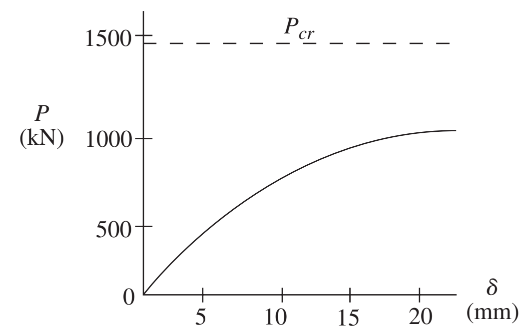
$$\delta = (5.0)[\sec(0.041404\sqrt{P}) - 1] \quad (2)$$

Units: P = kN δ = mm
angles are in radians.

SOLVE EQ. (2) FOR P:

$$P = 583.3\left[\arccos\left(\frac{5.0}{5.0 + \delta}\right)\right]^2 \quad \longleftarrow$$

LOAD-DEFLECTION DIAGRAM



Problem 11.5-5 Solve the preceding problem for a column with $e = 0.20$ in., $L = 12$ ft, $I = 21.7$ in.^{4}, and $E = 30 \times 10^6$ psi.

Solution 11.5-5 Column with eccentric loads

Column has pinned ends
Use Eq. (11-54) for the deflection at the midpoint (maximum deflection):

$$\delta = e\left[\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) - 1\right] \quad (1)$$

DATA

$e = 0.20$ in. $L = 12$ ft $= 144$ in.
$E = 30 \times 10^6$ psi
$I = 21.7$ in.^{4}

CRITICAL LOAD

$$P_{cr} = \frac{\pi^2 EI}{L^2} = 309.9\,k$$

MAXIMUM DEFLECTION (FROM EQ. 1)

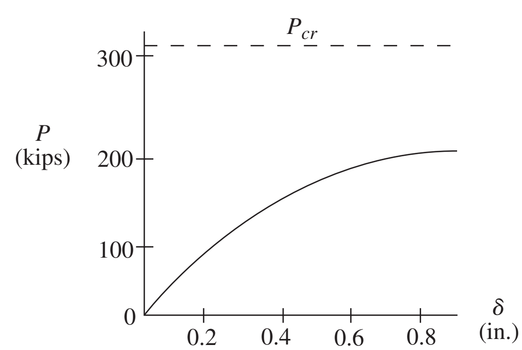
$$\delta = (0.20)[\sec(0.08924\sqrt{P}) - 1] \quad (2)$$

Units: P = kips δ = inches
Angles are in radians.

SOLVE EQ. (2) FOR P:

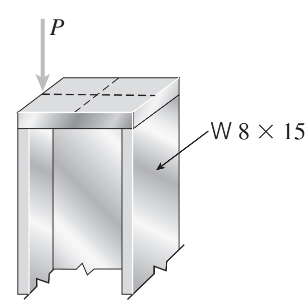
$$P = 125.6\left[\arccos\left(\frac{0.2}{0.2 + \delta}\right)\right]^2 \quad \longleftarrow$$

LOAD-DEFLECTION DIAGRAM



Problem 11.5-6 A wide-flange member (W 8 × 15) is compressed by axial loads that have a resultant P acting at the point shown in the figure. The member has modulus of elasticity $E = 29{,}000$ ksi and pinned conditions at the ends. Lateral supports prevent any bending about the weak axis of the cross section.

If the length of the member is 20 ft and the deflection is limited to 1/4 inch, what is the maximum allowable load P_{allow}?



Solution 11.5-6 Column with eccentric axial load

Wide-flange member: W 8 $\times$ 15
$E = 29{,}000$ psi $L = 20$ ft $= 240$ in.
Maximum allowable deflection $= 0.25$ in. $(= \delta)$
Pinned-end conditions
Bending occurs about the strong axis (axis 1-1)

From Table E-1: $I = 48.0$ in.^{4}

$$e = \frac{8.11\text{ in.}}{2} = 4.055\text{ in.}$$

CRITICAL LOAD

$$P_{cr} = \frac{\pi^2 EI}{L^2} = 238{,}500\text{ lb}$$

MAXIMUM DEFLECTION (EQ. 11-54)

$$\delta_{max} = e\left[\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) - 1\right]$$

$0.25\text{ in.} = (4.055\text{ in.})[\sec(0.003216\sqrt{P}) - 1]$

Rearrange terms and simplify:

$\cos(0.003216\sqrt{P}) = 0.9419$

$0.003216\sqrt{P} = \arccos 0.9419 = 0.3426$

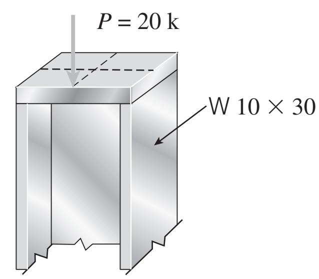
(Note: Angles are in radians)

Solve for P: $P = 11{,}300$ lb

ALLOWABLE LOAD

$P_{allow} = 11{,}300$ lb ⟵

Problem 11.5-7 A wide-flange member (W 10 $\times$ 30) is compressed by axial loads that have a resultant $P = 20$ k acting at the point shown in the figure. The material is steel with modulus of elasticity $E = 29{,}000$ ksi. Assuming pinned-end conditions, determine the maximum permissible length L_{max} if the deflection is not to exceed 1/400th of the length.



Solution 11.5-7 Column with eccentric axial load

Wide-flange member: W 10 $\times$ 30
Pinned-end conditions.
Bending occurs about the weak axis (axis 2-2).
$P = 20$ k $E = 29{,}000$ ksi L = length (inches)

Maximum allowable deflection $= \dfrac{L}{400}$ $(= \delta)$

From Table E-1: $I = 16.7$ in.^{4}

$$e = \frac{5.810\text{ in.}}{2} = 2.905\text{ in.}$$

$$k = \sqrt{\frac{P}{EI}} = 0.006426\text{ in.}^{-1}$$

DEFLECTION AT MIDPOINT (EQ. 11-51)

$$\delta = e\left(\sec\frac{kL}{2} - 1\right)$$

$$\frac{L}{400} = (2.905\text{ in.})[\sec(0.003213\,L) - 1]$$

Rearrange terms and simplify:

$$\sec(0.003213\,L) - 1 - \frac{L}{1162\text{ in.}} = 0$$

(Note: angles are in radians)

Solve the equation numerically for the length L:
$L = 150.5$ in.

MAXIMUM ALLOWABLE LENGTH

$L_{max} = 150.5$ in. $= 12.5$ ft ⟵

Problem 11.5-8 Solve the preceding problem (W 10 × 30) if the resultant force P equals 25 k.

Solution 11.5-8 Column with eccentric axial load

Wide-flange member: W 10 × 30
Pinned-end conditions
Bending occurs about the weak axis (axis 2-2)
$P = 25$ k $E = 29{,}000$ ksi L = length (inches)

Maximum allowable deflection $= \dfrac{L}{400}\ (= \delta)$

From Table E-1: $I = 16.7\text{ in.}^4$

$$e = \frac{5.810\text{ in.}}{2} = 2.905\text{ in.}$$

$$k = \sqrt{\frac{P}{EI}} = 0.007185\text{ in.}^{-1}$$

DEFLECTION AT MIDPOINT (EQ. 11-51)

$$\delta = e\left(\sec\frac{kL}{2} - 1\right)$$

$$\frac{L}{400} = (2.905\text{ in.})[\sec(0.003592\,\mathrm{L}) - 1]$$

Rearrange terms and simplify:

$$\sec(0.003592\,\mathrm{L}) - 1 - \frac{L}{1162\text{ in.}} = 0$$

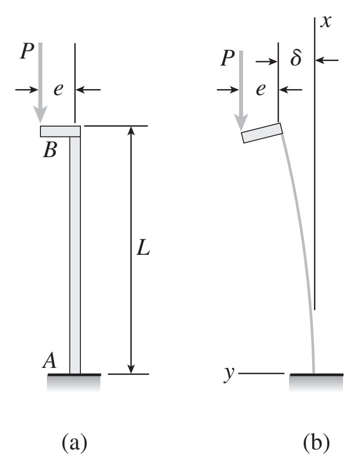
(Note: angles are in radians)
Solve the equation numerically for the length L:
$L = 122.6$ in.

MAXIMUM ALLOWABLE LENGTH

$L_{max} = 122.6\text{ in.} = 10.2\text{ ft}$ ⟵

Problem 11.5-9 The column shown in the figure is fixed at the base and free at the upper end. A compressive load P acts at the top of the column with an eccentricity e from the axis of the column.

Beginning with the differential equation of the deflection curve, derive formulas for the maximum deflection δ of the column and the maximum bending moment M_{max} in the column.



Solution 11.5-9 Fixed-free column

e = eccentricity of load P
δ = deflection at the end of the column
v = deflection of the column at distance x from the base

DIFFERENTIAL EQUATION (EQ. 11.3)

$$EIv'' = M = P(e + \delta - v) \qquad k^2 = \frac{P}{EI}$$

$$v'' = k^2(e + \delta - v)$$

$$v'' + k^2 v = k^2(e + \delta)$$

GENERAL SOLUTION

$$v = C_1 \sin kx + C_2 \cos kx + e + \delta$$

$$v' = C_1 k \cos kx - C_2 k \sin kx$$

B.C. 1 $v(0) = 0 \quad \therefore C_2 = -e - \delta$
B.C. 2 $v'(0) = 0 \quad \therefore C_1 = 0$
$v = (e + \delta)(1 - \cos kx)$
B.C. 3 $v(\mathrm{L}) = \delta \quad \therefore \delta = (e + \delta)(1 - \cos kL)$
or $\delta = e(\sec kL - 1)$

MAXIMUM DEFLECTION $\delta = e(\sec kL - 1)$ ⟵

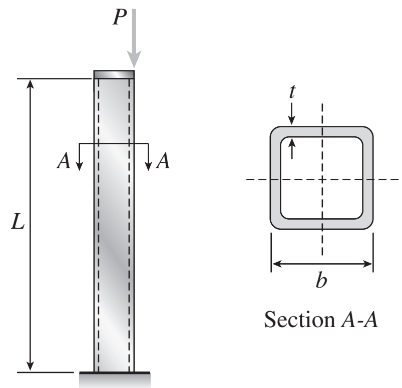
MAXIMUM LENDING MOMENT (AT BASE OF COLUMN)

$M_{max} = P(e + \delta) = Pe\sec kL$ ⟵

NOTE: $v = (e + \delta)(1 - \cos kx)$
$= e(\sec kL)(1 - \cos kx)$

Problem 11.5-10 An aluminum box column of square cross section is fixed at the base and free at the top (see figure). The outside dimension b of each side is 100 mm and the thickness t of the wall is 8 mm. The resultant of the compressive loads acting on the top of the column is a force $P = 50$ kN acting at the outer edge of the column at the midpoint of one side.

What is the longest permissible length L_{max} of the column if the deflection at the top is not to exceed 30 mm? (Assume $E = 73$ GPa.)



Probs. 11.5-10 and 11.5-11

Solution 11.5-10 **Fixed-free column**

δ = deflection at the top

Use Eq. (11-51) with $L/2$ replaced by L:

$$\delta = e(\sec kL - 1) \quad (1)$$

(This same equation is obtained in Prob. 11.5-9.)

SOLVE FOR L FROM EQ. (1)

$$\sec kL = 1 + \frac{\delta}{e} = \frac{e + \delta}{e}$$

$$\cos kL = \frac{e}{e + \delta} \qquad kL = \arccos\frac{e}{e + \delta}$$

$$L = \frac{1}{k}\arccos\frac{e}{e + \delta} \qquad k = \sqrt{\frac{P}{EI}}$$

$$L = \sqrt{\frac{EI}{P}}\arccos\frac{e}{e + \delta} \quad (2)$$

NUMERICAL DATA

$E = 73$ GPa $\quad b = 100$ mm $\quad t = 8$ mm

$P = 50$ kN $\quad \delta = 30$ mm $\quad e = \frac{b}{2} = 50\,\text{mm}$

$$I = \frac{1}{12}[b^4 - (b - 2t)^4] = 4.1844 \times 10^6\,\text{mm}^4$$

MAXIMUM ALLOWABLE LENGTH

Substitute numerical data into Eq. (2).

$$\sqrt{\frac{EI}{P}} = 2.4717\,\text{m} \qquad \frac{e}{e + \delta} = 0.625$$

$$\arccos\frac{e}{e + \delta} = 0.89566\,\text{radians}$$

$$L_{max} = (2.4717\text{ m})(0.89566) = 2.21\text{ m} \longleftarrow$$

Problem 11.5-11 Solve the preceding problem for an aluminum column with $b = 6.0$ in., $t = 0.5$ in., $P = 30$ k, and $E = 10.6 \times 10^3$ ksi. The deflection at the top is limited to 2.0 in.

Solution 11.5-11 **Fixed-free column**

δ = deflection at the top

Use Eq. (11-51) with $L/2$ replaced by L:

$$\delta = e(\sec kL - 1) \quad (1)$$

(This same equation is obtained in Prob. 11.5-9.)

SOLVE FOR L FROM EQ. (1)

$$\sec kL = 1 + \frac{\delta}{e} = \frac{e + \delta}{e}$$

$$\cos kL = \frac{e}{e + \delta} \qquad kL = \arccos\frac{e}{e + \delta}$$

$$L = \frac{1}{k}\arccos\frac{e}{e + \delta} \qquad k = \sqrt{\frac{P}{EI}}$$

$$L = \sqrt{\frac{EI}{P}}\arccos\frac{e}{e + \delta} \quad (2)$$

NUMERICAL DATA

$E = 10.6 \times 10^3$ ksi $\quad b = 6.0$ in. $\quad t = 0.5$ in.

$P = 30$ k $\quad \delta = 2.0$ in. $\quad e = \frac{b}{2} = 3.0\,\text{in.}$

$$I = \frac{1}{12}[b^4 - (b - 2t)^4] = 55.917\,\text{in.}^4$$

MAXIMUM ALLOWABLE LENGTH

Substitute numerical data into Eq. (2).

$$\sqrt{\frac{EI}{P}} = 140.56\text{ in.} \qquad \frac{e}{e+\delta} = 0.60$$

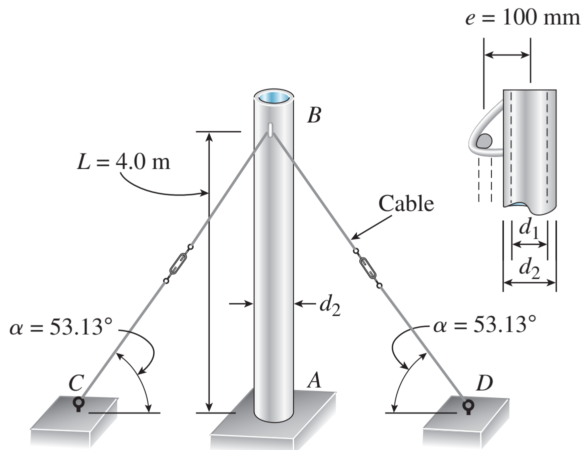
$$\arccos\frac{e}{e+\delta} = 0.92730\text{ radians}$$

$$L_{max} = (140.56\text{ in.})(0.92730)$$
$$= 130.3\text{ in.} = 10.9\text{ ft} \longleftarrow$$

Problem 11.5-12 A steel post *AB* of hollow circular cross section is fixed at the base and free at the top (see figure). The inner and outer diameters are $d_1 = 96$ mm and $d_2 = 110$ mm, respectively, and the length $L = 4.0$ m.

A cable *CBD* passes through a fitting that is welded to the side of the post. The distance between the plane of the cable (plane *CBD*) and the axis of the post is $e = 100$ mm, and the angles between the cable and the ground are $\alpha = 53.13°$. The cable is pretensioned by tightening the turnbuckles.

If the deflection at the top of the post is limited to $\delta = 20$ mm, what is the maximum allowable tensile force T in the cable? (Assume $E = 205$ GPa.)



Solution 11.5-12 Fixed-free column

δ = deflection at the top

P = compressive force in post $\qquad k = \sqrt{\frac{P}{EI}}$

Use Eq. (11-51) with $L/2$ replaced by L:

$$\delta = e(\sec kL - 1) \qquad (1)$$

(This same equation in obtained in Prob. 11.5-9.)

SOLVE FOR P FROM EQ.(1)

$$\sec kL = 1 + \frac{\delta}{e} = \frac{e+\delta}{e}$$

$$\cos kL = \frac{e}{e+\delta} \qquad kL = \arccos\frac{e}{e+\delta}$$

$$kL = \sqrt{\frac{PL^2}{EI}} \qquad \sqrt{\frac{PL^2}{EI}} = \arccos\frac{e}{e+\delta}$$

Square both sides and solve for P:

$$P = \frac{EI}{L^2}\left(\arccos\frac{e}{e+\delta}\right)^2 \qquad (2)$$

NUMERICAL DATA

$E = 205$ GPa $\quad L = 4.0$ m $\quad e = 100$ mm

$\delta = 20$ mm $\quad d_2 = 110$ mm $\quad d_1 = 96$ mm

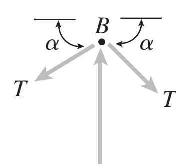
$$I = \frac{\pi}{64}(d_2^4 - d_1^4) = 3.0177 \times 10^6\text{ mm}^4$$

MAXIMUM ALLOWABLE COMPRESSIVE FORCE P

Substitute numerical data into Eq. (2).

$$P_{allow} = 13{,}263\text{ N} = 13{,}263\text{ kN}$$

MAXIMUM ALLOWABLE TENSILE FORCE T IN THE CABLE



Free-body diagram of joint B:

$\alpha = 53.13°$

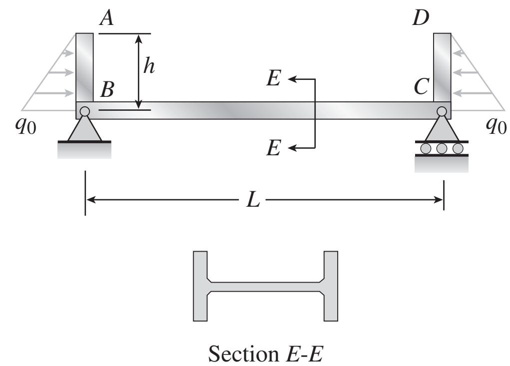
$$\sum F_{vert} = 0 \qquad P - 2T\sin\alpha = 0$$

$$T = \frac{P}{2\sin\alpha} = \frac{5P}{8} = 8289\text{ N}$$

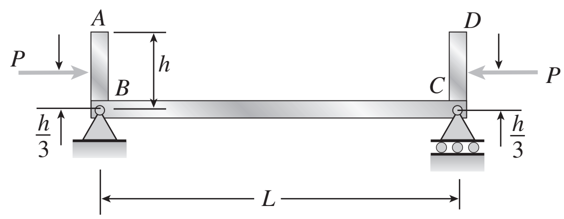
Problem 11.5-13 A frame *ABCD* is constructed of steel wide-flange members (W 8 × 21; $E = 30 \times 10^6$ psi) and subjected to triangularly distributed loads of maximum intensity q_0 acting along the vertical members (see figure). The distance between supports is $L = 20$ ft and the height of the frame is $h = 4$ ft. The members are rigidly connected at B and C.

(a) Calculate the intensity of load q_0 required to produce a maximum bending moment of 80 k-in. in the horizontal member *BC*.

(b) If the load q_0 is reduced to one-half of the value calculated in part (a), what is the maximum bending moment in member *BC*? What is the ratio of this moment to the moment of 80 k-in. in part (a)?



Solution 11.5-13 **Frame with triangular loads**



P = resultant force
e = eccentricity

$$P = \frac{q_0 h}{2} \qquad e = \frac{h}{3}$$

MAXIMUM BENDING MOMENT IN BEAM *BC*

From Eq. (11-56): $\quad M_{max} = Pe \sec\dfrac{kL}{2}$

$$k = \sqrt{\frac{P}{EI}} \qquad \therefore\ M_{max} = Pe\sec\sqrt{\frac{PL^2}{4EI}} \qquad (1)$$

NUMERICAL DATA

W 8 × 21 $\quad I = I_2 = 9.77$ in.^{4} (from Table E-1)
$E = 30 \times 10^6$ psi $\quad L = 20$ ft $= 240$ in.
$h = 4$ ft $= 48$ in.

$$e = \frac{h}{3} = 16\,\text{in.}$$

(a) LOAD q_0 TO PRODUCE $M_{max} = 80$ k-in.

Substitute numerical values into Eq. (1).
Units: pounds and inches

$$M_{max} = 80{,}000\,\text{lb-in.}\ \sqrt{\frac{PL^2}{4EI}}$$
$$= 0.0070093\sqrt{P} \qquad \text{(radians)}$$
$$80{,}000 = P(16\,\text{in.})\,[\sec(0.0070093\sqrt{P})]$$
$$5{,}000 = P\sec(0.0070093\sqrt{P})$$
$$P - 5{,}000\,[\cos(0.0070093\sqrt{P})] = 0 \qquad (2)$$

SOLVE EQ. (2) NUMERICALLY

$P = 4461.9$ lb

$$q_0 = \frac{2P}{h} = 186\,\text{lb/in.} = 2230\,\text{lb/ft} \quad \longleftarrow$$

(b) LOAD q_0 IS REDUCED TO ONE-HALF ITS VALUE

$\therefore$ P is reduced to one-half its value.

$$P = \frac{1}{2}(4461.9\,\text{lb}) = 2231.0\,\text{lb}$$

Substitute numerical values into Eq. (1) and solve for M_{max}.

$M_{max} = 37.75$ k-in. $\longleftarrow$

Ratio: $\dfrac{M_{max}}{80\,\text{k-in.}}\ 5\ \dfrac{37.7}{80}\ 5\ 0.47 \quad \longleftarrow$

This result shows that the bending moment varies nonlinearly with the load.

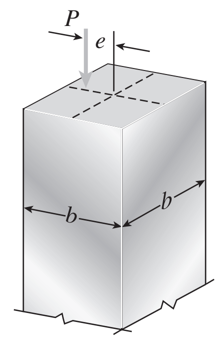
The Secant Formula

When solving the problems for Section 11.6, assume that bending occurs in the principal plane containing the eccentric axial load.

Problem 11.6-1 A steel bar has a square cross section of width $b = 2.0$ in. (see figure). The bar has pinned supports at the ends and is 3.0 ft long. The axial forces acting at the end of the bar have a resultant $P = 20$ k located at distance $e = 0.75$ in. from the center of the cross section. Also, the modulus of elasticity of the steel is 29,000 ksi.

(a) Determine the maximum compressive stress σ_{max} in the bar.

(b) If the allowable stress in the steel is 18,000 psi, what is the maximum permissible length L_{max} of the bar?



Probs. 11.6-1 through 11.6-3

Solution 11.6-1 Bar with square cross section

Pinned supports.

DATA

$b = 2.0$ in. $L = 3.0$ ft $= 36$ in. $P = 20$ k
$e = 0.75$ in. $E = 29{,}000$ ksi

(a) MAXIMUM COMPRESSIVE STRESS

Secant formula (Eq. 11-59):

$$\sigma_{max} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{L}{2r}\sqrt{\frac{P}{EA}}\right)\right] \quad (1)$$

$$\frac{P}{A} = \frac{P}{b^2} = 5.0\,\text{ksi} \qquad c = \frac{b}{2} = 1.0\,\text{in.}$$

$$I = \frac{b^4}{12} = 1.333\,\text{in.}^4 \qquad r^2 = \frac{I}{A} = 0.3333\,\text{in.}^2$$

$$\frac{ec}{r^2} = 2.25 \qquad \frac{L}{r} = 62.354 \qquad \frac{P}{EA} = 0.00017241$$

Substitute into Eq. (1):

$\sigma_{max} = 17.3$ ksi ←

(b) MAXIMUM PERMISSIBLE LENGTH

$\sigma_{allow} = 18{,}000$ psi

Solve Eq. (1) for the length L:

$$L = 2\sqrt{\frac{EI}{P}}\arccos\left[\frac{P(ec/r^2)}{\sigma_{max}A - P}\right] \quad (2)$$

Substitute numerical values:

$L_{max} = 46.2$ in. ←

Problem 11.6-2 A brass bar ($E = 100$ GPa) with a square cross section is subjected to axial forces having a resultant P acting at distance e from the center (see figure). The bar is pin supported at the ends and is 0.6 m in length. The side dimension b of the bar is 30 mm and the eccentricity e of the load is 10 mm.

If the allowable stress in the brass is 150 MPa, what is the allowable axial force P_{allow}?

Solution 11.6-2 Bar with square cross section

Pinned supports.

DATA

$b = 30$ mm $L = 0.6$ m $\sigma_{allow} = 150$ MPa
$e = 10$ mm $E = 100$ GPa

SECANT FORMULA (Eq. 11-59):

$$\sigma_{max} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{L}{2r}\sqrt{\frac{P}{EA}}\right)\right] \quad (1)$$

Units: Newtons and meters

$\sigma_{max} = 150 \times 10^6\ \text{N/m}^2$

$A = b^2 = 900 \times 10^{-6}\ \text{m}^2$

$$c = \frac{b}{2} = 0.015\,\text{m} \qquad r^2 = \frac{I}{A} = \frac{b^2}{12} = 75 \times 10^{-6}\,\text{m}^2$$

$$\frac{ec}{r^2} = 2.0 \qquad P = \text{newtons} \qquad \frac{L}{2r}\sqrt{\frac{P}{EA}} = 0.0036515\sqrt{P}$$

Substitute numerical values into Eq. (1):

$$150 \times 10^6 = \frac{P}{900 \times 10^{-6}}\,[1 + 2\sec(0.0036515\sqrt{P})]$$

or

$$P[1 + 2\sec(0.0036515\sqrt{P})] - 135{,}000 = 0 \qquad (2)$$

Solve Eq. (2) numerically:

$P_{allow} = 37{,}200 \quad N = 37.2\ \text{kN}$ ←

Problem 11.6-3 A square aluminum bar with pinned ends carries a load $P = 25$ k acting at distance $e = 2.0$ in. from the center (see figure on the previous page). The bar has length $L = 54$ in. and modulus of elasticity $E = 10{,}600$ ksi.

If the stress in the bar is not to exceed 6 ksi, what is the minimum permissible width b_{min} of the bar?

Solution 11.6-3 Square aluminum bar

Pinned ends

Data

Units: pounds and inches

$P = 25\ \text{k} = 25{,}000\ \text{psi} \qquad e = 2.0\ \text{in.}$

$L = 54\ \text{in.} \qquad E = 10{,}600\ \text{ksi} = 10{,}600{,}000\ \text{psi}$

$\sigma_{max} = 6.0\ \text{ksi} = 6{,}000\ \text{psi}$

Secant formula (Eq. 11-59)

$$\sigma_{max} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{L}{2r}\sqrt{\frac{P}{EA}}\right)\right] \qquad (1)$$

$$A = b^2 \qquad c = \frac{b}{2} \qquad r^2 = \frac{I}{A} = \frac{b^2}{12}$$

$$\frac{ec}{r^2} = \frac{12}{b} \qquad \frac{L}{2r}\sqrt{\frac{P}{EA}} = \frac{4.5423}{b^2}$$

Substitute terms into Eq. (1):

$$6{,}000 = \frac{25{,}000}{b^2}\left[1 + \frac{12}{b}\sec\left(\frac{4.5423}{b^2}\right)\right]$$

or

$$1 + \frac{12}{b}\sec\left(\frac{4.5423}{b^2}\right) - 0.24\,b^2 = 0 \qquad (2)$$

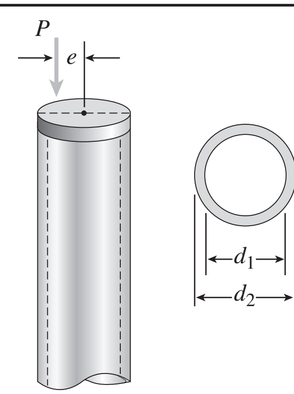
Solve Eq. (2) numerically:

$b_{min} = 4.10\ \text{in.}$ ←

Problem 11.6-4 A pinned-end column of length $L = 2.1$ m is constructed of steel pipe ($E = 210$ GPa) having inside diameter $d_1 = 60$ mm and outside diameter $d_2 = 68$ mm (see figure). A compressive load $P = 10$ kN acts with eccentricity $e = 30$ mm.

(a) What is the maximum compressive stress σ_{max} in the column?

(b) If the allowable stress in the steel is 50 MPa, what is the maximum permissible length L_{max} of the column?



Probs. 11.6-4 through 11.6-6

Solution 11.6-4 **Steel pipe column**

Pinned ends.

DATA Units: Newtons and meters

$L = 2.1$ m $E = 210$ GPa $= 210 \times 10^9$ N/m^2
$d_1 = 60$ mm $= 0.06$ m $d_2 = 68$ mm $= 0.068$ m
$P = 10$ kN $= 10{,}000$ N $e = 30$ mm $= 0.03$ m

TUBULAR CROSS SECTION

$$A = \frac{\pi}{4}(d_2^2 - d_1^2) = 804.25 \times 10^{-6}\text{m}^2$$

$$I = \frac{\pi}{64}(d_2^4 - d_1^4) = 413.38 \times 10^{-9}\text{m}^4$$

(a) MAXIMUM COMPRESSIVE STRESS

Secant formula (Eq. 11-59):

$$\sigma_{\max} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{L}{2r}\sqrt{\frac{P}{EA}}\right)\right] \qquad (1)$$

$$\frac{P}{A} = 12.434 \times 10^6\,\text{N/m}^2$$

$$r^2 = \frac{I}{A} = 513.99 \times 10^{-6}\,m^2$$

$$r = 22.671 \times 10^{-3}\,\text{m} \qquad c = \frac{d_2}{2} = 0.034\,\text{m}$$

$$\frac{ec}{r^2} = 1.9845 \qquad \frac{L}{2r}\sqrt{\frac{P}{EA}} = 0.35638$$

Substitute into Eq. (1):

$\sigma_{\max} = 38.8 \times 10^6$ N/m^2 $= 38.8$ MPa ←

(b) MAXIMUM PERMISSIBLE LENGTH

$\sigma_{\text{allow}} = 50$ MPa

Solve Eq. (1) for the length L:

$$L = 2\sqrt{\frac{EI}{P}}\arccos\left[\frac{P(ec/r^2)}{\sigma_{\max}A - P}\right] \qquad (2)$$

Substitute numerical values:

$L_{\max} = 5.03$ m ←

Problem 11.6-5 A pinned-end strut of length $L = 5.2$ ft is constructed of steel pipe ($E = 30 \times 10^3$ ksi) having inside diameter $d_1 = 2.0$ in. and outside diameter $d_2 = 2.2$ in. (see figure). A compressive load $P = 2.0$ k is applied with eccentricity $e = 1.0$ in.

(a) What is the maximum compressive stress $\sigma_{\max}$ in the strut?

(b) What is the allowable load P_{allow} if a factor of safety $n = 2$ with respect to yielding is required? (Assume that the yield stress σ_Y of the steel is 42 ksi.)

Solution 11.6-5 **Pinned-end strut**

Steel pipe.

DATA Units: kips and inches

$L = 5.2$ ft $= 62.4$ in. $E = 30 \times 10^3$ ksi
$d_1 = 2.0$ in. $d_2 = 2.2$ in.
$P = 2.0$ k $e = 1.0$ in.

TUBULAR CROSS SECTION

$$A = \frac{\pi}{4}(d_2^2 - d_1^2) = 0.65973\,\text{in.}^2$$

$$I = \frac{\pi}{64}(d_2^4 - d_1^4) = 0.36450\,\text{in.}^4$$

(a) MAXIMUM COMPRESSIVE STRESS

Secant formula (Eq. 11-59):

$$\sigma_{\max} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{L}{2r}\sqrt{\frac{P}{EA}}\right)\right] \qquad (1)$$

$$\frac{P}{A} = 3.0315\,\text{ksi} \qquad c = \frac{d_2}{2} = 1.1\,\text{in.}$$

$$r^2 = \frac{I}{A} = 0.55250\,\text{in.}^2 \qquad \frac{ec}{r^2} = 1.9910$$

$$r = 0.74330\,\text{in.} \qquad \frac{L}{2r}\sqrt{\frac{P}{EA}} = 0.42195$$

Substitute into Eq. (1):

$\sigma_{\max} = 9.65$ ksi ←

(b) ALLOWABLE LOAD

$\sigma_Y = 42$ ksi $n = 2$ find P_{allow}

Substitute numerical values into Eq. (1):

$$42 = \frac{P}{0.65973}[1 + 1.9910\sec(0.29836\sqrt{P})] \qquad (2)$$

Solve Eq. (2) numerically: $P = P_Y = 7.184$ k

$$P_{\text{allow}} = \frac{P_Y}{n} = 3.59\,\text{k}$$ ←

Problem 11.6-6 A circular aluminum tube with pinned ends supports a load $P = 18$ kN acting at distance $e = 50$ mm from the center (see figure). The length of the tube is 3.5 m and its modulus of elasticity is 73 GPa.

If the maximum permissible stress in the tube is 20 MPa, what is the required outer diameter d_2 if the ratio of diameters is to be $d_1/d_2 = 0.9$?

Solution 11.6-6 **Aluminum tube**

Pinned ends.

DATA $P = 18$ kN $e = 50$ mm

$L = 3.5$ m $E = 73$ GPa

$\sigma_{max} = 20$ MPa $d_1/d_2 = 0.9$

SECANT FORMULA (EQ. 11-59)

$$\sigma_{max} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{L}{2r}\sqrt{\frac{P}{EA}}\right)\right] \quad (1)$$

$$A = \frac{\pi}{4}(d_2^2 - d_1^2) = \frac{\pi}{4}[d_2^2 - (0.9\,d_2)^2] = 0.14923\,d_2^2$$

(d_2 = mm; A = mm^2)

$$\frac{P}{A} = \frac{18{,}000\,\text{N}}{0.14923\,d_2^2} = \frac{120{,}620}{d_2^2}\left(\frac{P}{A} = \text{MPa}\right)$$

$$I = \frac{\pi}{64}(d_2^4 - d_1^4) = \frac{\pi}{64}[d_2^4 - (0.9\,d_2)^4] = 0.016881\,d_2^4$$

(d_2 = mm; I = mm^4)

$$r^2 = \frac{I}{A} = 0.11313\,d_2^2 \quad (d_2 = \text{mm}; r^2 = \text{mm}^2)$$

$r = 0.33634\,d_2$ (r = mm)

$$c = \frac{d_2}{2} \qquad \frac{ec}{r^2} = \frac{(50\,\text{mm})(d_2/2)}{0.11313\,d_2^2} = \frac{220.99}{d_2}$$

$$\frac{L}{2r} = \frac{3500\,\text{mm}}{2(0.33634\,d_2)} = \frac{5{,}203.1}{d_2}$$

$$\frac{P}{EA} = \frac{18{,}000\,\text{N}}{(73{,}000\,\text{N/mm}^2)(0.14923\,d_2^2)} = \frac{1.6524}{d_2^2}$$

$$\frac{L}{2r}\sqrt{\frac{P}{EA}} = \frac{5{,}203.1}{d_2}\sqrt{\frac{1.6524}{d_2^2}} = \frac{6688.2}{d_2^2}$$

SUBSTITUTE THE ABOVE EXPRESSIONS INTO EQ. (1):

$$\sigma_{max} = 20\,\text{MPa} = \frac{120{,}620}{d_2^2} + \left[1 + \frac{220.99}{d_2}\sec\left(\frac{6688.2}{d_2^2}\right)\right] \quad (2)$$

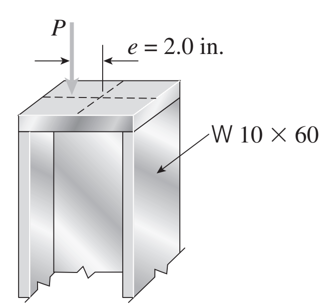
SOLVE EQ. (2) NUMERICALLY:

$d_2 = 131$ mm ⟵

Problem 11.6-7 A steel column ($E = 30 \times 10^3$ ksi) with pinned ends is constructed of a W 10×60 wide-flange shape (see figure). The column is 24 ft long. The resultant of the axial loads acting on the column is a force P acting with an eccentricity $e = 2.0$ in.

(a) If $P = 120$ k, determine the maximum compressive stress σ_{max} in the column.

(b) Determine the allowable load P_{allow} if the yield stress is $\sigma_Y = 42$ ksi and the factor of safety with respect to yielding of the material is $n = 2.5$.



Solution 11.6-7 Steel column with pinned ends

$E = 30 \times 10^3$ ksi $\quad L = 24$ ft $= 288$ in.
$e = 2.0$ in.
W 10×60 wide-flange shape
$A = 17.6$ in.^{2} $\quad I = 341$ in.^{4} $\quad d = 10.22$ in.

$$r^2 = \frac{I}{A} = 19.38\text{ in.}^2 \quad r = 4.402\text{ in.} \quad c = \frac{d}{2} = 5.11\text{ in.}$$

$$\frac{L}{r} = 65.42 \qquad \frac{ec}{r^2} = 0.5273$$

(a) Maximum compressive stress ($P = 120$ k)

Secant formula (Eq. 11-59):

$$\sigma_{\max} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{L}{2r}\sqrt{\frac{P}{EA}}\right)\right] \qquad (1)$$

$$\frac{P}{A} = 6.818\text{ ksi} \quad \frac{L}{2r}\sqrt{\frac{P}{EA}} = 0.4931$$

Substitute into Eq. (1): $\quad \sigma_{\max} = 10.9$ ksi ←

(b) Allowable load

$\sigma_Y = 42$ ksi $\quad n = 2.5 \quad$ find P_{allow}

Substitute into Eq. (1):

$$42 = \frac{P}{17.6}[1 + 0.5273\sec(0.04502\sqrt{P})]$$

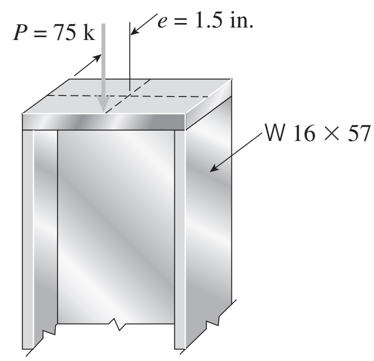
Solve numerically: $P = P_Y = 399.9$ k

$P_{allow} = P_Y/n = 160$ k ←

Problem 11.6-8 A W 16×57 steel column is compressed by a force $P = 75$ k acting with an eccentricity $e = 1.5$ in., as shown in the figure. The column has pinned ends and length L. Also, the steel has modulus of elasticity $E = 30 \times 10^3$ ksi and yield stress $\sigma_Y = 36$ ksi.

(a) If the length $L = 10$ ft, what is the maximum compressive stress $\sigma_{\max}$ in the column?

(b) If a factor of safety $n = 2.0$ is required with respect to yielding, what is the longest permissible length $L_{\max}$ of the column?



Solution 11.6-8 Steel column with pinned ends

W $16 \times 57 \quad A = 16.8$ in.^{2} $\quad I = I_2 = 43.1$ in.^{4}
$b = 7.120$ in.
$c = b/2 = 3.560$ in.

$$e = 1.5\text{ in.} \qquad r^2 = \frac{I}{A} = 2.565\text{ in.}^2$$

$$\frac{ec}{r^2} = 2.082 \qquad r = 1.602\text{ in.}$$

$$P = 75\text{ k} \quad E = 30 \times 10^3\text{ ksi} \quad \frac{P}{EA} = 148.8 \times 10^{-6}$$

(a) Maximum compressive stress

Secant formula (Eq. 11-59):

$$\sigma_{\max} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{L}{2r}\sqrt{\frac{P}{EA}}\right)\right] \qquad (1)$$

$L = 10$ ft $= 120$ in.

$$\frac{P}{A} = 4.464\text{ ksi} \qquad \frac{L}{2r}\sqrt{\frac{P}{EA}} = 0.4569$$

Substitute into Eq. (1):

$\sigma_{\max} = 4.464\,[1 + 2.082\sec(0.4569)]$
$= 14.8$ ksi ←

(b) Maximum length

Solve Eq. (1) for the length L:

$$L = 2\sqrt{\frac{EI}{P}}\arccos\left[\frac{P(ec/r^2)}{\sigma_{\max}A - P}\right] \qquad (2)$$

$\sigma_Y = 36$ ksi $\quad n = 2.0 \quad P_Y = n \quad P = 150$ k

Substitute P_Y for P and σ_Y for $\sigma_{\max}$ in Eq. (2):

$$L_{\max} = 2\sqrt{\frac{EI}{P_Y}}\arccos B\frac{P_Y(ec/r^2)}{\sigma_Y A - P_Y}\Big] \qquad (3)$$

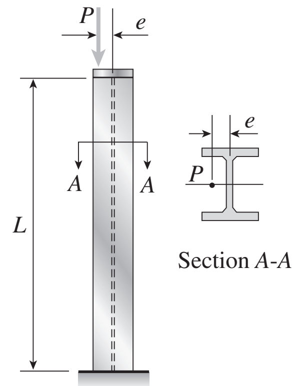
Substitute numerical values in Eq. (3) and solve for $L_{\max}$:

$L_{\max} = 151.1$ in. $= 12.6$ ft ←

Problem 11.6-9 A steel column ($E = 30 \times 10^3$ ksi) that is fixed at the base and free at the top is constructed of a W 8 × 35 wide-flange member (see figure). The column is 9.0 ft long. The force P acting at the top of the column has an eccentricity $e = 1.25$ in.

(a) If $P = 40$ k, what is the maximum compressive stress in the column?

(b) If the yield stress is 36 ksi and the required factor of safety with respect to yielding is 2.1, what is the allowable load P_{allow}?



Probs. 11.6-9 and 11.6-10

Solution 11.6-9 **Steel column (fixed-free)**

$E = 30 \times 10^3$ ksi $\quad e = 1.25$ in.

$Le = 2L = 2\,(9.0 \text{ ft}) = 18 \text{ ft} = 216$ in.

W 8 × 35 WIDE-FLANGE SHAPE

$A = 10.3 \text{ in.}^2 \quad I = I_2 = 42.6 \text{ in.}^4 \quad b = 8.020$ in.

$$r^2 = \frac{I}{A} = 4.136 \text{ in.}^2 \quad r = 2.034 \text{ in.}$$

$$c = \frac{b}{2} = 4.010 \text{ in.} \quad \frac{Le}{r} = 106.2 \quad \frac{ec}{r^2} = 1.212$$

(a) MAXIMUM COMPRESSIVE STRESS ($P = 40$ k)

Secant formula (Eq. 11-59):

$$\sigma_{max} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{Le}{2r}\sqrt{\frac{P}{EA}}\right)\right] \qquad (1)$$

$$\frac{P}{A} = 3.883 \text{ ksi} \quad \frac{Le}{2r}\sqrt{\frac{P}{EA}} = 0.6042$$

Substitute into Eq. (1): $\quad \sigma_{max} = 9.60$ ksi ←

(b) ALLOWABLE LOAD

$\sigma_Y = 36$ ksi $\quad n = 2.1 \quad$ find P_{allow}

Substitute into Eq. (1):

$$36 = \frac{P}{10.3}[1 + 1.212\sec(0.09552\sqrt{P})]$$

Solve numerically: $\quad P = P_Y = 112.6$ k

$P_{allow} = P_Y/n = 53.6$ k ←

Problem 11.6-10 A W 12 × 50 wide-flange steel column with length $L = 12.5$ ft is fixed at the base and free at the top (see figure). The load P acting on the column is intended to be centrally applied, but because of unavoidable discrepancies in construction, an eccentricity ratio of 0.25 is specified. Also, the following data are supplied: $E = 30 \times 10^3$ ksi, $\sigma_Y = 42$ ksi, and $P = 70$ k.

(a) What is the maximum compressive stress σ_{max} in the column?

(b) What is the factor of safety n with respect to yielding of the steel?

Solution 11.6-10 **Steel column (fixed-free)**

$$E = 30 \times 10^3 \text{ ksi} \quad \frac{ec}{r^2} = 0.25$$

$Le = 2L = 2\,(12.5 \text{ ft}) = 25 \text{ ft} = 300$ in.

W 12 × 50 WIDE-FLANGE SHAPE

$A = 14.7 \text{ in.}^2 \quad I = I_2 = 56.3 \text{ in.}^4$

$$r^2 = \frac{I}{A} = 3.830 \text{ in.}^2 \quad r = 1.957 \text{ in.}$$

(a) MAXIMUM COMPRESSIVE STRESS ($P = 70$ k)

Secant formula (Eq. 11-59):

$$\sigma_{max} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{L_e}{2r}\sqrt{\frac{P}{EA}}\right)\right] \qquad (1)$$

$$\frac{P}{A} = 4.762 \text{ ksi} \quad \frac{L_e}{2r}\sqrt{\frac{P}{EA}} = 0.9657$$

Substitute into Eq. (1): $\sigma_{max} = 6.85$ ksi ⟵

(b) Factor of safety with respect to yielding

$\sigma_Y = 42$ psi

Substitute into Eq. (1) with $\sigma_{max} = \sigma_Y$ and $P = P_Y$:

$$42 = \frac{P_Y}{A}[1 + 0.25\sec(0.1154\sqrt{P_Y})]$$

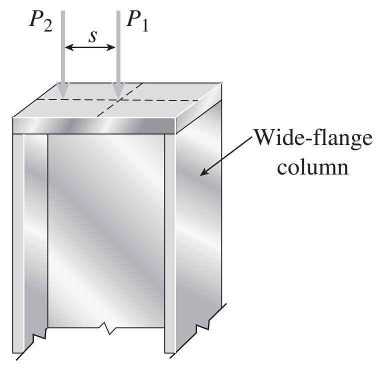
Solve numerically: $P_Y = 164.5$ k

$$P = 70\text{ k} \quad n = \frac{P_Y}{P} = \frac{164.5\text{ k}}{70\text{ k}} = 2.35 \longleftarrow$$

Problem 11.6-11 A pinned-end column with length $L = 18$ ft is constructed from a W 12 × 87 wide-flange shape (see figure). The column is subjected to a centrally applied load $P_1 = 180$ k and an eccentrically applied load $P_2 = 75$ k. The load P_2 acts at distance $s = 5.0$ in. from the centroid of the cross section. The properties of the steel are $E = 29{,}000$ ksi and $\sigma_Y = 36$ ksi.

(a) Calculate the maximum compressive stress in the column.

(b) Determine the factor of safety with respect to yielding.



Probs. 11.6.11 and 11.6.12

Solution 11.6-11 Column with two loads

Pinned-end column. W 12 × 87

Data

$L = 18$ ft $= 216$ in.

$P_1 = 180$ k $\quad P_2 = 75$ k $\quad s = 5.0$ in.

$E = 29{,}000$ ksi $\quad \sigma_Y = 36$ ksi

$$P = P_1 + P_2 = 255\text{ k} \quad e = \frac{P_2 s}{P} = 1.471\text{ in.}$$

$A = 25.6$ in.^{2} $\quad I = I_1 = 740$ in.^{4} $\quad d = 12.53$ in.

$$r^2 = \frac{I}{A} = 28.91\text{ in.}^2 \quad r = 5.376\text{ in.}$$

$$c = \frac{d}{2} = 6.265\text{ in.} \quad \frac{ec}{r^2} = 0.3188$$

$$\frac{P}{A} = 9.961\text{ ksi} \quad \frac{L}{2r}\sqrt{\frac{P}{EA}} = 0.3723$$

(a) Maximum compressive stress

Secant formula (Eq. 11-59):

$$\sigma_{max} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{L}{2r}\sqrt{\frac{P}{EA}}\right)\right] \tag{1}$$

Substitute into Eq. (1): $\sigma_{max} = 13.4$ ksi ⟵

(b) Factor of safety with respect to yielding

$\sigma_{max} = \sigma_Y = 36$ ksi $\quad P = P_Y$

Substitute into Eq. (1):

$$36 = \frac{P_Y}{25.6}[1 + 0.3188\sec(0.02332\sqrt{P_Y})]$$

Solve numerically: $P_Y = 664.7$ k

$$P = 2.55\text{ k} \quad n = \frac{P_Y}{P} = \frac{664.7\text{ k}}{255\text{ k}} = 2.61 \longleftarrow$$

Problem 11.6-12 The wide-flange pinned-end column shown in the figure carries two loads, a force $P_1 = 100$ k acting at the centroid and a force $P_2 = 60$ k acting at distance $s = 4.0$ in. from the centroid. The column is a W 10 × 45 shape with $L = 13.5$ ft, $E = 29 \times 10^3$ ksi, and $\sigma_Y = 42$ ksi.

(a) What is the maximum compressive stress in the column?

(b) If the load P_1 remains at 100 k, what is the largest permissible value of the load P_2 in order to maintain a factor of safety of 2.0 with respect to yielding?

Solution 11.6-12 Column with two loads

Pinned-end column. W 10 × 45

DATA

$L = 13.5 \text{ ft} = 162 \text{ in.}$

$P_1 = 100 \text{ k} \quad P_2 = 60 \text{ k} \quad s = 4.0 \text{ in.}$

$E = 29{,}000 \text{ ksi} \quad \sigma_Y = 42 \text{ ksi}$

$$P = P_1 + P_2 = 160\,\text{k} \qquad e = \frac{P_2 s}{P} = 1.50\,\text{in.}$$

$$A = 13.3 \text{ in.}^2 \qquad I = I_1 = 248 \text{ in.}^4 \qquad d = 10.10 \text{ in.}$$

$$r^2 = \frac{I}{A} = 18.65\,\text{in.}^2 \qquad r = 4.318\,\text{in.}$$

$$c = \frac{d}{2} = 5.05\,\text{in.} \qquad \frac{ec}{r^2} = 0.4062$$

$$\frac{P}{A} = 12.03\,\text{ksi} \qquad \frac{L}{2r}\sqrt{\frac{P}{EA}} = 0.3821$$

(a) MAXIMUM COMPRESSIVE STRESS

Secant formula (Eq. 11-59):

$$\sigma_{max} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{L}{2r}\sqrt{\frac{P}{EA}}\right)\right] \qquad (1)$$

Substitute into Eq. (1): $\sigma_{max} = 17.3$ ksi ⟵

(b) LARGEST VALUE OF LOAD P_2

$P_1 = 100$ k (no change)

$n = 2.0$ with respect to yielding

Units: kips, inches

$P = P_1 + P_2 = 100 + P_2$

$$e = \frac{P_2 s}{P} = \frac{P_2(4.0)}{100 + P_2} \qquad \frac{ec}{r^2} = \frac{1.0831\,P_2}{100 + P_2}$$

$$\sigma_{max} = \sigma_Y = 42 \text{ ksi} \qquad P_Y = n\,P = 2.0\,(100 + P_2)$$

Use Eq. (1) with σ_{max} replaced by σ_Y and P replaced by P_Y:

$$\sigma_Y = \frac{P_Y}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{L}{2r}\sqrt{\frac{P_Y}{EA}}\right)\right] \qquad (2)$$

Substitute into Eq. (2):

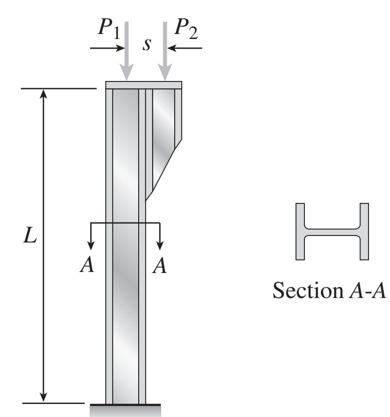
$$42 = \frac{2.0(100 + P_2)}{13.3}\left[1 + \frac{1.0831\,P_2}{100 + P_2}\sec\left(0.04272\sqrt{100 + P_2}\right)\right]$$

Solve numerically: $P_2 = 78.4$ k ⟵

Problem 11.6-13 A W 14 × 53 wide-flange column of length $L = 15$ ft is fixed at the base and free at the top (see figure). The column supports a centrally applied load $P_1 = 120$ k and a load $P_2 = 40$ k supported on a bracket. The distance from the centroid of the column to the load P_2 is $s = 12$ in. Also, the modulus of elasticity is $E = 29{,}000$ ksi and the yield stress is $\sigma_Y = 36$ ksi.

(a) Calculate the maximum compressive stress in the column.

(b) Determine the factor of safety with respect to yielding.



Probs. 11.6-13 and 11.6-14

Solution 11.6-13 Column with two loads

Fixed-free column. W 14 × 53

DATA

$L = 15 \text{ ft} = 180 \text{ in.} \quad L_e = 2L = 360 \text{ in.}$

$P_1 = 120 \text{ k} \quad P_2 = 40 \text{ k} \quad s = 12 \text{ in.}$

$E = 29{,}000 \text{ ksi} \quad \sigma_Y = 36 \text{ ksi}$

$$P = P_1 + P_2 = 160 \text{ k} \quad e = \frac{P_2 s}{P} = 3.0 \text{ in.}$$

$A = 15.6 \text{ in.}^2 \quad I = I_1 = 541 \text{ in.}^4 \quad d = 13.92 \text{ in.}$

$$r^2 = \frac{I}{A} = 34.68 \text{ in.}^2 \quad r = 5.889 \text{ in.}$$

$$c = \frac{d}{2} = 6.960 \text{ in.} \quad \frac{ec}{r^2} = 0.6021$$

$$\frac{P}{A} = 10.26 \text{ ksi} \quad \frac{L_e}{2r}\sqrt{\frac{P}{EA}} = 0.5748$$

(a) MAXIMUM COMPRESSIVE STRESS

Secant formula (Eq. 11-59):

$$\sigma_{\max} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{L_e}{2r}\sqrt{\frac{P}{EA}}\right)\right] \quad (1)$$

Substitute into Eq. (1): $\sigma_{\max} = 17.6 \text{ ksi}$ ←

(b) FACTOR OF SAFETY WITH RESPECT TO YIELDING

$\sigma_{\max} = \sigma_Y = 36 \text{ ksi} \quad P = P_Y$

Substitute into Eq. (1):

$$36 = \frac{P_Y}{15.6}\left[1 + 0.6021 \sec(0.04544\sqrt{P_Y})\right]$$

Solve numerically: $P_Y = 302.6 \text{ k}$

$$P = 160 \text{ k} \quad n = \frac{P_Y}{P} = \frac{302.6 \text{ k}}{160 \text{ k}} = 1.89 \quad \leftarrow$$

Problem 11.6-14 A wide-flange column with a bracket is fixed at the base and free at the top (see figure on the preceding page). The column supports a load $P_1 = 75$ k acting at the centroid and a load $P_2 = 25$ k acting on the bracket at distance $s = 10.0$ in. from the load P_1. The column is a W 12 × 35 shape with $L = 16$ ft, $E = 29 \times 10^3$ ksi, and $\sigma_Y = 42$ ksi.

(a) What is the maximum compressive stress in the column?

(b) If the load P_1 remains at 75 k, what is the largest permissible value of the load P_2 in order to maintain a factor of safety of 1.8 with respect to yielding?

Solution 11.6-14 Column with two loads

Fixed-free column. W 12 × 35

DATA

$L = 16 \text{ ft} = 192 \text{ in.} \quad L_e = 2L = 384 \text{ in.}$

$P_1 = 75 \text{ k} \quad P_2 = 25 \text{ k} \quad s = 10.0 \text{ in.}$

$E = 29{,}000 \text{ ksi} \quad \sigma_Y = 42 \text{ ksi}$

$$P = P_1 + P_2 = 100 \text{ k} \quad e = \frac{P_2 s}{P} = 2.5 \text{ in.}$$

$A = 10.3 \text{ in.}^2 \quad I = I_1 = 285 \text{ in.}^4 \quad d = 12.50 \text{ in.}$

$$r^2 = \frac{I}{A} = 27.67 \text{ in.}^2 \quad r = 5.260 \text{ in.}$$

$$c = \frac{d}{2} = 6.25 \text{ in.} \quad \frac{ec}{r^2} = 0.5647$$

$$\frac{P}{A} = 9.709 \text{ ksi} \quad \frac{L_e}{2r}\sqrt{\frac{P}{EA}} = 0.6679$$

(a) MAXIMUM COMPRESSIVE STRESS

Secant formula (Eq. 11-59):

$$\sigma_{\max} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{L_e}{2r}\sqrt{\frac{P}{EA}}\right)\right] \quad (1)$$

Substitute into Eq. (1): $\sigma_{\max} = 16.7 \text{ ksi}$ ←

(b) LARGEST VALUE OF LOAD P_2

$P_1 = 75$ k (no change)

$m = 1.8$ with respect to yielding

Units: kips, inches

$P = P_1 + P_2 = 75 + P_2$

$$e = \frac{P_2 s}{P} = \frac{P_2(10.0)}{75 + P_2} \quad \frac{ec}{r^2} = \frac{2.259 P_2}{75 + P_2}$$

$\sigma_{\max} = \sigma_Y = 42 \text{ ksi} \quad P_Y = n\,P = 1.8\,(75 + P_2)$

Use Eq. (1) with $\sigma_{\max}$ replaced by σ_Y and P replaced by P_Y:

$$\sigma_Y = \frac{P_Y}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{L_e}{2r}\sqrt{\frac{P_Y}{EA}}\right)\right] \quad (2)$$

Substitute into Eq. (2):

$$42 = \frac{1.8(75 + P_2)}{10.3}\left[1 + \frac{2.259 P_2}{75 + P_2}\sec\left(0.08961\sqrt{75 + P_2}\right)\right]$$

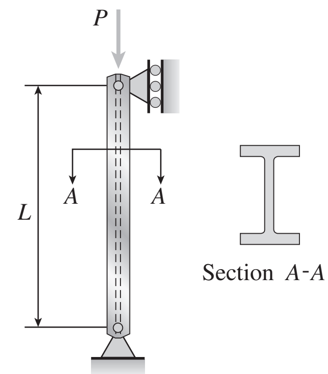
Solve numerically: $P_2 = 34.3 \text{ k}$ ←

Design Formulas for Columns

The problems for Section 11.9 are to be solved assuming that the axial loads are centrally applied at the ends of the columns. Unless otherwise stated, the columns may buckle in any direction.

STEEL COLUMNS

Problem 11.9-1 Determine the allowable axial load P_{allow} for a W 10 × 45 steel wide-flange column with pinned ends (see figure) for each of the following lengths: $L = 8$ ft, 16 ft, 24 ft, and 32 ft. (Assume $E = 29{,}000$ ksi and $\sigma_Y = 36$ ksi.)



Probs. 11.9-1 through 11.9-6

Solution 11.9-1 **Steel wide-flange column**

Pinned ends ($K = 1$).
Buckling about axis 2-2 (see Table E-1).
Use AISC formulas.

W 10 × 45 $A = 13.3 \text{ in.}^2$ $r_2 = 2.01$ in.

$E = 29{,}000$ ksi $\sigma_Y = 36$ ksi $\left(\frac{L}{r}\right)_{max} = 200$

Eq. (11-76): $\left(\frac{L}{r}\right)_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = 126.1$

$L_c = 126.1\, r = 253.5$ in. $= 21.1$ ft

L	8 ft	16 ft	24 ft	32 ft
L/r	47.76	95.52	143.3	191.0
n_1 (Eq. 11-79)	1.802	1.896	–	–
n_2 (Eq. 11-80)	–	–	1.917	1.917
σ_{allow}/σ_Y (Eq. 11-81)	0.5152	0.3760	–	–
σ_{allow}/σ_Y (Eq. 11-82)	–	–	0.2020	0.1137
σ_{allow} (ksi)	18.55	13.54	7.274	4.091
$P_{allow} = A\,\sigma_{allow}$	247 k	180 k	96.7 k	54.4 k

Problem 11.9-2 Determine the allowable axial load P_{allow} for a W 12 × 87 steel wide-flange column with pinned ends (see figure) for each of the following lengths: $L = 10$ ft, 20 ft, 30 ft, and 40 ft. (Assume $E = 29{,}000$ ksi and $\sigma_Y = 50$ ksi.)

Solution 11.9-2 **Steel wide-flange column**

Pinned ends ($K = 1$).
Buckling about axis 2-2 (see Table E-1).
Use AISC formulas.

W 12 × 87 $A = 25.6 \text{ in.}^2$ $r_2 = 3.07$ in.

$E = 29{,}000$ ksi $\sigma_Y = 50$ ksi $\left(\frac{L}{r}\right)_{max} = 200$

Eq. (11-76): $\left(\frac{L}{r}\right)_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = 107.0$

$L_c = 1.070\, r = 328.5$ in. $= 27.4$ ft

L	10 ft	20 ft	30 ft	40 ft
L/r	39.09	78.18	117.3	156.4
n_1 (Eq. 11-79)	1.798	1.892	–	–
n_2 (Eq. 11-80)	–	–	1.917	1.917
σ_{allow}/σ_Y (Eq. 11-81)	0.5192	0.3875	–	–
σ_{allow}/σ_Y (Eq. 11-82)	–	–	0.2172	0.1222
σ_{allow} (ksi)	25.96	19.37	10.86	6.11
$P_{allow} = A\,\sigma_{allow}$	665 k	496 k	278 k	156 k

Problem 11.9-3 Determine the allowable axial load P_{allow} for a W 10 × 60 steel wide-flange column with pinned ends (see figure) for each of the following lengths: $L = 10$ ft, 20 ft, 30 ft, and 40 ft. (Assume $E = 29{,}000$ ksi and $\sigma_Y = 36$ ksi.)

Solution 11.9-3 Steel wide-flange column

Pinned ends ($K = 1$).
Buckling about axis 2-2 (see Table E-1).
Use AISC formulas.
W 10 × 60 $A = 17.6\text{ in.}^2$ $r_2 = 2.57$ in.

$E = 29{,}000$ ksi $\sigma_Y = 36$ ksi $\left(\frac{L}{r}\right)_{max} = 200$

$$\text{Eq. (11-76): } \left(\frac{L}{r}\right)_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = 126.1$$

$L_c = 126.1\, r = 324.1$ in. $= 27.0$ ft

L	10 ft	20 ft	30 ft	40 ft
L/r	46.69	93.39	140.1	186.8
n_1 (Eq. 11-79)	1.799	1.894	–	–
n_2 (Eq. 11-80)	–	–	1.917	1.917
σ_{allow}/σ_Y (Eq. 11-81)	0.5177	0.3833	–	–
σ_{allow}/σ_Y (Eq. 11-82)	–	–	0.2114	0.1189
σ_{allow} (ksi)	18.64	13.80	7.610	4.281
$P_{allow} = A\,\sigma_{allow}$	328 k	243 k	134 k	75.3 k

Problem 11.9-4 Select a steel wide-flange column of nominal depth 10 in. (W 10 shape) to support an axial load $P = 180$ k (see figure). The column has pinned ends and length $L = 14$ ft. Assume $E = 29{,}000$ ksi and $\sigma_Y = 36$ ksi. (*Note:* The selection of columns is limited to those listed in Table E-1, Appendix E.)

Solution 11.9-4 Select a column of W10 shape

$P = 180$ k $L = 14$ ft $= 168$ in. $K = 1$
$\sigma_Y = 36$ ksi
$E = 29{,}000$ ksi

$$\text{Eq. (11-76): } \left(\frac{L}{r}\right)_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = 126.1$$

(1) Trial value of σ_{allow}

Upper limit: use Eq. (11-81) with $L/r = 0$

$$\text{max. } \sigma_{allow} = \frac{\sigma_Y}{n_1} = \frac{\sigma_Y}{5/3} = 21.6\text{ ksi}$$

Try $\sigma_{allow} = 16$ ksi

(2) Trial value of area

$$A = \frac{P}{\sigma_{allow}} = \frac{180\text{ k}}{16\text{ ksi}} = 11.25\text{ in.}^2$$

(3) Trial column W 10 × 45

$A = 13.3\text{ in.}^2$ $r = 2.01$ in.

(4) Allowable stress for trial column

$$\frac{L}{r} = \frac{168\text{ in.}}{2.01\text{ in.}} = 83.58 \quad \frac{L}{r} < \left(\frac{L}{r}\right)_c$$

Eqs. (11-79) and (11-81): $n_1 = 1.879$

$$\frac{\sigma_{allow}}{\sigma_Y} = 0.4153 \quad \sigma_{allow} = 14.95\text{ ksi}$$

(5) Allowable load for trial column

$P_{allow} = \sigma_{allow} A = 199\text{ k} > 180\text{ k}$ (ok)
(W 10 × 45)

(6) Next smaller size column

W 10 × 30 $A = 8.84\text{ in.}^2$ $r = 1.37$ in.

$$\frac{L}{r} = 122.6 \quad < \left(\frac{L}{r}\right)_c$$

$n = 1.916$ $\sigma_{allow} = 9.903$ ksi
$P_{allow} = 88\text{ k} < P = 180\text{ k}$ (Not satisfactory)

Problem 11.9-5 Select a steel wide-flange column of nominal depth 12 in. (W 12 shape) to support an axial load $P = 175$ k (see figure). The column has pinned ends and length $L = 35$ ft. Assume $E = 29{,}000$ ksi and $\sigma_Y = 36$ ksi. (*Note:* The selection of columns is limited to those listed in Table E-1, Appendix E.)

Solution 11.9-5 Select a column of W12 shape

$P = 175$ k $L = 35$ ft $= 420$ in. $K = 1$

$\sigma_Y = 36$ ksi $E = 29{,}000$ ksi

Eq. (11-76): $\left(\dfrac{L}{r}\right)_c = \sqrt{\dfrac{2\pi^2 E}{\sigma_Y}} = 126.1$

(1) TRIAL VALUE OF σ_{allow}

Upper limit: use Eq. (11-81) with $L/r = 0$

$$\text{max. } \sigma_{allow} = \frac{\sigma_Y}{n_1} = \frac{\sigma_Y}{5/3} = 21.6\,\text{ksi}$$

Try $\sigma_{allow} = 8$ ksi (Because column is very long)

(2) TRIAL VALUE OF AREA

$$A = \frac{P}{\sigma_{allow}} = \frac{175\,\text{k}}{8\,\text{ksi}} = 22\,\text{in.}^2$$

(3) TRIAL COLUMN W 12 × 87

$A = 25.6$ in.^{2} $r = 3.07$ in.

(4) ALLOWABLE STRESS FOR TRIAL COLUMN

$$\frac{L}{r} = \frac{4.20\,\text{in.}}{3.07\,\text{in.}} = 136.8 \qquad \frac{L}{r} > \left(\frac{L}{r}\right)_c$$

Eqs. (11-80) and (11-82): $n_2 = 1.917$

$$\frac{\sigma_{allow}}{\sigma_Y} = 0.2216 \qquad \sigma_{allow} = 7.979\,\text{ksi}$$

(5) ALLOWABLE LOAD FOR TRIAL COLUMN

$P_{allow} = \sigma_{allow} A = 204\text{ k} > 175\text{ k}$ (ok)

(6) NEXT SMALLER SIZE COLUMN

W 12 × 50 $A = 14.7$ in.^{2} $r = 1.96$ in.

$\dfrac{L}{r} = 214$ Since the maximum permissible value of L/r is 200, this section is not satisfactory.

Select W 12 × 87 ←

Problem 11.9-6 Select a steel wide-flange column of nominal depth 14 in. (W 14 shape) to support an axial load $P = 250$ k (see figure). The column has pinned ends and length $L = 20$ ft. Assume $E = 29{,}000$ ksi and $\sigma_Y = 50$ ksi. (*Note:* The selection of columns is limited to those listed in Table E-1, Appendix E.)

Solution 11.9-6 Select a column of W14 shape

$P = 250$ k $L = 20$ ft $= 240$ in. $K = 1$

$\sigma_Y = 50$ ksi

$E = 29{,}000$ ksi

Eq. (11-76): $\left(\dfrac{L}{r}\right)_c = \sqrt{\dfrac{2\pi^2 E}{\sigma_Y}} = 107.0$

(1) TRIAL VALUE OF σ_{allow}

Upper limit: use Eq. (11-81) with $L/r = 0$

$$\text{max. } \sigma_{allow} = \frac{\sigma_Y}{n_1} = \frac{\sigma_Y}{5/3} = 30\,\text{ksi}$$

Try $\sigma_{allow} = 12$ ksi

(2) TRIAL VALUE OF AREA

$$A = \frac{P}{\sigma_{allow}} = \frac{250\,\text{k}}{12\,\text{ksi}} = 21\,\text{in.}^2$$

(3) TRIAL COLUMN W 14 × 82

$A = 24.1$ in.^{2} $r = 2.48$ in.

(4) ALLOWABLE STRESS FOR TRIAL COLUMN

$$\frac{L}{r} = \frac{240\,\text{in.}}{2.48\,\text{in.}} = 96.77 \qquad \frac{L}{r} < \left(\frac{L}{r}\right)_c$$

Eqs. (11-79) and (11-81): $n_1 = 1.913$

$$\frac{\sigma_{allow}}{\sigma_Y} = 0.3089 \qquad \sigma_{allow} = 15.44\,\text{ksi}$$

(5) Allowable load for trial column

$P_{allow} = \sigma_{allow} A = 372 \text{ k} > 250 \text{ k}$ (ok)
(W 14 × 82)

(6) Next smaller size column

W 14 × 53 $A = 15.6 \text{ in.}^2$ $r = 1.92 \text{ in.}$

$\frac{L}{r} = 125.0 > \left(\frac{L}{r}\right)_c$

$n = 1.917$ $\sigma_{allow} = 9.557 \text{ ksi}$

$P_{allow} = 149 \text{ k} < P = 250 \text{ k}$ (Not satisfactory)

Select W 14 × 82 ⟵

Problem 11.9-7 Determine the allowable axial load P_{allow} for a steel *pipe column with pinned ends* having an outside diameter of 4.5 in. and wall thickness of 0.237 in. for each of the following lengths: $L = 6$ ft, 12 ft, 18 ft, and 24 ft. (Assume $E = 29{,}000$ ksi and $\sigma_Y = 36$ ksi.)

Solution 11.9-7 Steel pipe column

Pinned ends ($K = 1$).
Use AISC formulas.

$d_2 = 4.5 \text{ in.}$ $t = 0.237 \text{ in.}$ $d_1 = 4.026 \text{ in.}$

$$A = \frac{\pi}{4}(d_2^2 - d_1^2) = 3.1740 \text{ in.}^2$$

$$I = \frac{\pi}{64}(d_2^4 - d_1^4) = 7.2326 \text{ in.}^4$$

$$r = \sqrt{\frac{I}{A}} = 1.5095 \text{ in.} \qquad \left(\frac{L}{r}\right)_{max} = 200$$

$E = 29{,}000 \text{ ksi}$ $\sigma_Y = 36 \text{ ksi}$

Eq.(11-76): $$\left(\frac{L}{r}\right)_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = 126.1$$

$L_c = 126.1\, r = 190.4 \text{ in.} = 15.9 \text{ ft}$

L	6 ft	12 ft	18 ft	24 ft
L/r	47.70	95.39	143.1	190.8
n_1 (Eq. 11-79)	1.802	1.896	–	–
n_2 (Eq. 11-80)	–	–	1.917	1.917
σ_{allow}/σ_Y (Eq. 11-81)	0.5153	0.3765	–	–
σ_{allow}/σ_Y (Eq. 11-82)	–	–	0.2026	0.1140
σ_{allow} (ksi)	18.55	13.55	7.293	4.102
$P_{allow} = A\,\sigma_{allow}$	58.9 k	43.0 k	23.1 k	13.0 k

Problem 11.9-8 Determine the allowable axial load P_{allow} for a steel *pipe column with pinned ends* having an outside diameter of 220 mm and wall thickness of 12 mm for each of the following lengths: $L = 2.5$ m, 5 m, 7.5 m, and 10 m. (Assume $E = 200$ GPa and $\sigma_Y = 250$ MPa.)

Solution 11.9-8 Steel pipe column

Pinned ends ($K = 1$).
Use AISC formulas.

$d_2 = 220 \text{ mm}$ $t = 12 \text{ mm}$ $d_1 = 196 \text{ mm}$

$$A = \frac{\pi}{4}(d_2^2 - d_1^2) = 7841.4 \text{ mm}^2$$

$$I = \frac{\pi}{64}(d_2^4 - d_1^4) = 42.548 \times 10^6 \text{ mm}^4$$

$$r = \sqrt{\frac{I}{A}} = 73.661 \text{ mm} \qquad \left(\frac{L}{r}\right)_{max} = 200$$

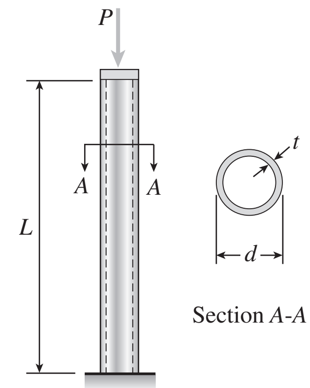
$E = 200 \text{ GPa}$ $\sigma_Y = 250 \text{ MPa}$

Eq.(11-76): $$\left(\frac{L}{r}\right)_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = 125.7$$

$L_c = 125.7\, r = 9257 \text{ mm} = 9.26 \text{ m}$

L	2.5 m	5.0 m	7.5 m	10.0 m
L/r	33.94	67.88	101.8	135.8
n_1 (Eq. 11-79)	1.765	1.850	1.904	–
n_2 (Eq. 11-80)	–	–	–	1.917
σ_{allow}/σ_Y (Eq. 11-81)	0.5458	0.4618	0.3528	–
σ_{allow}/σ_Y (Eq. 11-82)	–	–	–	0.2235
σ_{allow} (MPa)	136.4	115.5	88.20	55.89
$P_{allow} = A\,\sigma_{allow}$	1070 kN	905 kN	692 kN	438 kN

Problem 11.9-9 Determine the allowable axial load P_{allow} for a steel pipe column that is fixed at the base and free at the top (see figure) for each of the following lengths: $L = 6$ ft, 9 ft, 12 ft, and 15 ft. The column has outside diameter $d = 6.625$ in. and wall thickness $t = 0.280$ in. (Assume $E = 29{,}000$ ksi and $\sigma_Y = 36$ ksi.)



Probs. 11.9-9 through 11.9-12

Solution 11.9-9 Steel pipe column

Fixed-free column ($K = 2$).

Use AISC formulas.

$d_2 = 6.625$ in. $\quad t = 0.280$ in. $\quad d_1 = 6.065$ in.

$$A = \frac{\pi}{4}(d_2^2 - d_1^2) = 5.5814\ \text{in.}^2$$

$$I = \frac{\pi}{64}(d_2^4 - d_1^4) = 28.142\ \text{in.}^4$$

$$r = \sqrt{\frac{I}{A}} = 2.2455 \qquad \left(\frac{KL}{r}\right)_{max} = 200$$

$$E = 29{,}000 \text{ ksi} \qquad \sigma_Y = 36 \text{ ksi}$$

Eq.(11-76): $\left(\frac{KL}{r}\right)_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = 126.1$

$L_c = 126.1\,\frac{r}{k} = 141.6 \text{ in.} = 11.8 \text{ ft}$

L	6 ft	9 ft	12 ft	15 ft
KL/r	64.13	96.19	128.3	160.3
n_1 (Eq. 11-79)	1.841	1.897	–	–
n_2 (Eq. 11-80)	–	–	1.917	1.917
σ_{allow}/σ_Y (Eq. 11-81)	0.4730	0.3737	–	–
σ_{allow}/σ_Y (Eq. 11-82)	–	–	0.2519	0.1614
σ_{allow} (ksi)	17.03	13.45	9.078	5.810
$P_{allow} = A\,\sigma_{allow}$	95.0 k	75.1 k	50.7 k	32.4 k

Problem 11.9-10 Determine the allowable axial load P_{allow} for a steel pipe column that is fixed at the base and free at the top (see figure) for each of the following lengths: $L = 2.6$ m, 2.8 m, 3.0 m, and 3.2 m. The column has outside diameter $d = 140$ mm and wall thickness $t = 7$ mm. (Assume $E = 200$ GPa and $\sigma_Y = 250$ MPa.)

Solution 11.9-10 Steel pipe column

Fixed-free column ($K = 2$).

Use AISC formulas.

$d_2 = 140$ mm $\quad t = 7.0$ mm $\quad d_1 = 126$ mm

$$A = \frac{\pi}{4}(d_2^2 - d_1^2) = 2924.8\ \text{mm}^2$$

$$I = \frac{\pi}{64}(d_2^4 - d_1^4) = 6.4851 \times 10^6\ \text{mm}^4$$

$$r = \sqrt{\frac{I}{A}} = 47.09 \text{ mm} \qquad \left(\frac{KL}{r}\right)_{max} = 200$$

$$E = 200 \text{ GPa} \qquad \sigma_Y = 250 \text{ MPa}$$

Eq.(11-76): $\left(\frac{KL}{r}\right)_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = 125.7$

$L_c = 125.7\,\frac{r}{K} = 2959 \text{ mm} = 2.959 \text{ m}$

L	2.6 m	2.8 m	3.0 m	3.2 m
KL/r	110.4	118.9	127.4	135.9
n_1 (Eq. 11-79)	1.911	1.916	–	–
n_2 (Eq. 11-80)	–	–	1.917	1.917
σ_{allow}/σ_Y (Eq. 11-81)	0.3212	0.2882	–	–
σ_{allow}/σ_Y (Eq. 11-82)	–	–	0.2537	0.2230
σ_{allow} (MPa)	80.29	72.06	63.43	55.75
$P_{allow} = A\,\sigma_{allow}$	235 kN	211 kN	186 kN	163 kN

Problem 11.9-11 Determine the maximum permissible length L_{max} for a steel pipe column that is fixed at the base and free at the top and must support an axial load $P = 40$ k (see figure). The column has outside diameter $d = 4.0$ in., wall thickness $t = 0.226$ in., $E = 29{,}000$ ksi, and $\sigma_Y = 42$ ksi.

Solution 11.9-11 **Steel pipe column**

Fixed-free column ($K = 2$). $P = 40$ k

Use AISC formulas.

$d_2 = 4.0$ in. $t = 0.226$ in. $d_1 = 3.548$ in.

$$A = \frac{\pi}{4}(d_2^2 - d_1^2) = 2.6795\ \text{in.}$$

$$I = \frac{\pi}{64}(d_2^4 - d_1^4) = 4.7877\ \text{in.}^4$$

$$r = \sqrt{\frac{I}{A}} = 1.3367 \qquad \left(\frac{KL}{r}\right)_{max} = 200$$

$E = 29{,}000$ ksi $\sigma_Y = 42$ ksi

$$\text{Eq.(11-76):} \quad \left(\frac{KL}{r}\right)_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = 116.7$$

$$L_c = 116.7\frac{r}{K} = 78.03\ \text{in.} = 6.502\ \text{ft}$$

Select trial values of the length L and calculate the corresponding values of P_{allow} (see table). Interpolate between the trial values to obtain the value of L that produces $P_{allow} = P$.

Note: If $L < L_c$, use Eqs. (11-79) and (11-81).

If $L > L_c$, use Eqs. (11-80) and (11-82).

L(ft)	5.20	5.25	5.23
KL/r	93.86	94.26	93.90
n_1 (Eq. 11-79)	1.903	1.904	1.903
n_2 (Eq. 11-80)	–	–	–
σ_{allow}/σ_Y (Eq. 11-81)	0.3575	0.3541	0.3555
σ_{allow}/σ_Y (Eq. 11-82)	–	–	–
σ_{allow} (ksi)	15.02	14.87	14.93
$P_{allow} = A\,\sigma_{allow}$	40.2 k	39.8 k	40.0 k

For $P = 40$ k, $L_{max} = 5.23$ ft ⟵

Problem 11.9-12 Determine the maximum permissible length L_{max} for a steel pipe column that is fixed at the base and free at the top and must support an axial load $P = 500$ kN (see figure). The column has outside diameter $d = 200$ mm, wall thickness $t = 10$ mm, $E = 200$ GPa, and $\sigma_Y = 250$ MPa.

Solution 11.9-12 **Steel pipe column**

Fixed-free column ($K = 2$). $P = 500$ kN

Use AISC formulas.

$d_2 = 200$ mm $t = 10$ mm $d_1 = 180$ mm

$$A = \frac{\pi}{4}(d_2^2 - d_1^2) = 5{,}969.0\ \text{mm}^2$$

$$I = \frac{\pi}{64}(d_2^4 - d_1^4) = 27.010 \times 10^6\ \text{mm}^4$$

$$r = \sqrt{\frac{I}{A}} = 67.27\ \text{mm} \quad \left(\frac{KL}{r}\right)_{max} = 200$$

$E = 200$ GPa $\sigma_Y = 250$ MPa

$$\text{Eq. (11-76):} \left(\frac{KL}{r}\right)_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = 125.7$$

$$L_c = 125.7\frac{r}{K} = 4.226\ \text{m}$$

Select trial values of the length L and calculate the corresponding values of P_{allow} (see table). Interpolate between the trial values to obtain the value of L that produces $P_{allow} = P$.

Note: If $L < L_c$, use Eqs. (11-79) and (11-81).

If $L > L_c$, use Eqs. (11-80) and (11-82).

L(m)	3.55	3.60	3.59
KL/r	105.5	107.0	106.7
n_1 (Eq. 11-79)	1.908	1.909	1.909
n_2 (Eq. 11-80)	–	–	–
σ_{allow}/σ_Y (Eq. 11-81)	0.3393	0.3338	0.3349
σ_{allow}/σ_Y (Eq. 11-82)	–	–	–
σ_{allow} (MPa)	84.83	83.46	83.74
$P_{allow} = A\,\sigma_{allow}$	506 kN	498 kN	500 kN

For $P = 500$ kN, $L = 3.59$ m ⟵

Problem 11.9-13 A steel pipe column with *pinned ends* supports an axial load $P = 21$ k. The pipe has outside and inside diameters of 3.5 in. and 2.9 in., respectively. What is the maximum permissible length L_{max} of the column if $E = 29{,}000$ ksi and $\sigma_Y = 36$ ksi?

Solution 11.9-13 Steel pipe column

Pinned ends ($K = 1$). $P = 21$ k

Use AISC formulas.

$d_2 = 3.5$ in. $t = 0.3$ in. $d_1 = 2.9$ in.

$$A = \frac{\pi}{4}(d_2^2 - d_1^2) = 3.0159 \text{ in.}^2$$

$$I = \frac{\pi}{64}(d_2^4 - d_1^4) = 3.8943 \text{ in.}^4$$

$$r = \sqrt{\frac{I}{A}} = 1.1363 \text{ in.} \qquad \left(\frac{L}{r}\right)_{max} = 200$$

$E = 29{,}000$ ksi $\sigma_Y = 36$ ksi

$$\text{Eq. (11-76): } \left(\frac{L}{r}\right)_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = 126.1$$

$L_c = 126.1\ r = 143.3$ in. $= 11.9$ ft

Select trial values of the length L and calculate the corresponding values of P_{allow} (see table). Interpolate between the trial values to obtain the value of L that produces $P_{allow} = P$.

Note: If $L < L_c$, use Eqs. (11-79) and (11-81).

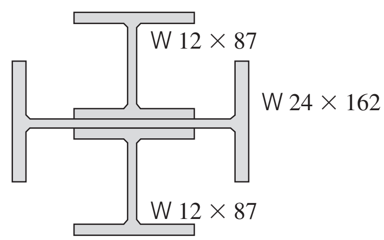
If $L > L_c$, use Eqs. (11-80) and (11-82).

L(ft)	13.8	13.9	14.0
L/r	145.7	146.8	147.8
n_1 (Eq. 11-79)	–	–	–
n_2 (Eq. 11-80)	1.917	1.917	1.917
σ_{allow}/σ_Y (Eq. 11-81)	–	–	–
σ_{allow}/σ_Y (Eq. 11-82)	0.1953	0.1925	0.1898
σ_{allow} (ksi)	7.031	6.931	6.832
$P_{allow} = A\ \sigma_{allow}$	21.2 k	20.9 k	20.6 k

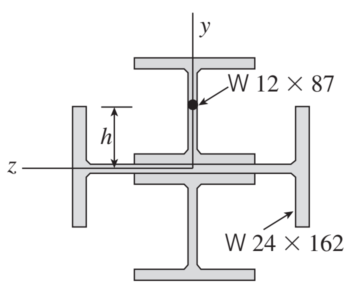
For $P = 21$ k, $L = 13.9$ ft ⟵

Problem 11.9-14 The steel columns used in a college recreation center are 55 ft long and are formed by welding three wide-flange sections (see figure). The columns are pin-supported at the ends and may buckle in any direction.

Calculate the allowable load P_{allow} for one column, assuming $E = 29{,}000$ ksi and $\sigma_Y = 36$ ksi.



Solution 11.9-14 Pinned-end column ($K = 1$)



$L = 55$ ft $= 660$ in.

$E = 29{,}000$ ksi

$\sigma_Y = 36$ ksi

W 12 × 87

$A = 25.6$ in.^{2} $d = 12.53$ in.

$I_1 = 740$ in.^{4} $I_2 = 241$ in.^{4}

W 24 × 162

$A = 47.7$ in.^{2} $t_w = 0.705$ in.

$I_1 = 5170$ in.^{4} $I_2 = 443$ in.^{4}

For the entire cross section

$A = 2\,(25.6) + 47.7 = 98.9$ in.^{2}

$I_Y = 2\,(241) + 5170 = 5652$ in.^{4}

$h = d/2 + t_w/2 = 6.6175$ in.

$I_z = 443 + 2\,[740 + (25.6)(6.6175)^2] = 4165$ in.^{4}

$$\text{min. } r = \sqrt{\frac{I_z}{A}} = \sqrt{\frac{4165}{98.9}} = 6.489 \text{ in.}$$

$$\text{Eq. (11-76): } \left(\frac{L}{r}\right)_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = 126.1$$

$$\frac{L}{r} = \frac{660 \text{ in.}}{6.489 \text{ in.}} = 101.7 \qquad \frac{L}{r} < \left(\frac{L}{r}\right)_c$$

$\therefore$ Use Eqs. (11-79) and (11-81).

From Eq. (11-79): $n_1 = 1.904$

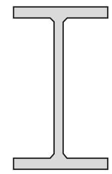
From Eq. (11-81): $\sigma_{allow}/\sigma_Y = 0.3544$

$\sigma_{allow} = 0.3544\,\sigma_Y = 12.76$ ksi

$P_{allow} = \sigma_{allow} A = (12.76 \text{ ksi})(98.9 \text{ in.}^2)$

$= 1260$ k ←

Problem 11.9-15 A W 8 × 28 steel wide-flange column with pinned ends carries an axial load P. What is the maximum permissible length L_{max} of the column if (a) $P = 50$ k, and (b) $P = 100$ k? (Assume $E = 29{,}000$ ksi and $\sigma_Y = 36$ ksi.)



Probs. 11.9-15 and 11.9-16

Solution 11.9-15 **Steel wide-flange column**

Pinned ends ($K = 1$).

Buckling about axis 2-2 (see Table E-1).

Use AISC formulas.

W 8 × 28 $\quad A = 8.25 \text{ in.}^2 \quad r_2 = 1.62$ in.

$$E = 29{,}000 \text{ ksi} \qquad \sigma_Y = 36 \text{ ksi} \qquad \left(\frac{L}{r}\right)_{max} = 200$$

$$\text{Eq. (11-76): } \left(\frac{L}{r}\right)_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = 126.1$$

$L_c = 126.1\,r = 204.3$ in. $= 17.0$ ft

For each load P, select trial values of the length L and calculate the corresponding values of P_{allow} (see table). Interpolate between the trial values to obtain the value of L that produces $P_{allow} = P$.

Note: If $L < L_c$, use Eqs. (11-79) and (11-81).

If $L > L_c$, use Eqs. (11-80) and (11-82).

L (ft)	21.0	21.5	21.2
L/r	155.6	159.3	157.0
n_1 (Eq. 11-79)	–	–	–
n_2 (Eq. 11-80)	1.917	1.917	1.917
σ_{allow}/σ_Y (Eq. 11-81)	–	–	–
σ_{allow}/σ_Y (Eq. 11-82)	0.1714	0.1635	0.1682
σ_{allow} (ksi)	6.171	5.888	6.056
$P_{allow} = A\,\sigma_{allow}$	50.9 k	48.6 k	50.0 k

(a) $P = 50$ k

For $P = 50$ k, $\quad L_{max} = 21.2$ ft ←

(b) $P = 100$ k

L (ft)	14.3	14.4	14.5	
L/r	105.9	106.7	107.4	
n_1 (Eq. 11-79)	1.908	1.908	1.909	
n_2 (Eq. 11-80)	–	–	–	
σ_{allow}/σ_Y (Eq. 11-81)	0.3393	0.3366	0.3338	
σ_{allow}/σ_Y (Eq. 11-82)	–	–	–	
σ_{allow} (ksi)	12.21	12.12	12.02	
$P_{allow} = A\,\sigma_{allow}$	100.8 k	100.0 k	99.2 k	k

For $P = 100$ k, $\quad L_{max} = 14.4$ ft ←

Problem 11.9-16 A W 10 × 45 steel wide-flange column with pinned ends carries an axial load P. What is the maximum permissible length L_{max} of the column if (a) P = 125 k, and (b) P = 200 k? (Assume E = 29,000 ksi and σ_Y = 42 ksi.)

Solution 11.9-16 **Steel wide-flange column**

Pinned ends (K = 1).
Buckling about axis 2-2 (see Table E-1).
Use AISC formulas.
W 10 × 45 A = 13.3 in.^{2} r_2 = 2.01 in.

$$E = 29{,}000 \text{ ksi} \qquad \sigma_Y = 42 \text{ ksi} \qquad \left(\frac{L}{r}\right)_{max} = 200$$

$$\text{Eq. (11-76): } \left(\frac{L}{r}\right)_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = 116.7$$

L_c = 116.7 r = 235 in. = 19.6 ft

For each load P, select trial values of the length L and calculate the corresponding values of P_{allow} (see table). Interpolate between the trial values to obtain the value of L that produces $P_{allow} = P$.

Note: If $L < L_c$, use Eqs. (11-79) and (11-81).
If $L > L_c$, use Eqs. (11-80) and (11-82).

(a) P = 125 k

L (ft)	21.0	21.1	21.2
L/r	125.4	126.0	126.6
n_1 (Eq. 11-79)	–	–	–
n_2 (Eq. 11-80)	1.917	1.917	1.917
σ_{allow}/σ_Y (Eq. 11-81)	–	–	–
σ_{allow}/σ_Y (Eq. 11-82)	0.2202	0.2241	0.2220
σ_{allow} (ksi)	9.500	9.411	9.322
$P_{allow} = A\,\sigma_{allow}$	126.4 k	125.2 k	124.0 k

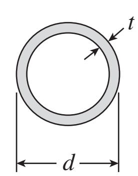
For P = 125 k, L_{max} = 21.1 ft ⟵

(b) P = 200 k

L(ft)	15.5	15.6	15.7
L/r	92.54	93.13	93.73
n_1 (Eq. 11-79)	1.902	1.902	1.903
n_2 (Eq. 11-80)	–	–	–
σ_{allow}/σ_Y (Eq. 11-81)	0.3607	0.3584	0.3561
σ_{allow}/σ_Y (Eq. 11-82)	–	–	–
σ_{allow} (ksi)	15.15	15.05	14.96
$P_{allow} = A\,\sigma_{allow}$	201.5 k	200.2 k	198.9 k

For P = 200 k, L_{max} = 15.6 ft ⟵

Problem 11.9-17 Find the required outside diameter d for a steel pipe column (see figure) of length L = 20 ft that is pinned at both ends and must support an axial load P = 25 k. Assume that the wall thickness t is equal to d/20. (Use E = 29,000 ksi and σ_Y = 36 ksi.)



Probs. 11.9-17 through 11.9-20

Solution 11.9-17 **Pipe column**

Pinned ends (K = 1).

L = 20 ft = 240 in. P = 25 k

d = outside diameter t = d/20

E = 29,000 ksi σ_Y = 36 ksi

$$A = \frac{\pi}{4}[d^2 - (d - 2t)^2] = 0.14923\, d^2$$

$$I = \frac{\pi}{64}[d^4 - (d - 2t)^4] = 0.016881\, d^4$$

$$r = \sqrt{\frac{I}{A}} = 0.33634\, d$$

$$\left(\frac{L}{r}\right)_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = 126.1 \qquad L_c = (126.1)r$$

Select various values of diameter d until we obtain $P_{allow} = P$.

If $L \le L_c$, Use Eqs. (11-79) and (11-81).

If $L \ge L_c$, Use Eqs. (11-80) and (11-82).

For $P = 25$ k, $d = 4.89$ in. ←

d (in.)	4.80	4.90	5.00
A (in.^{2})	3.438	3.583	3.731
I (in.^{4})	8.961	9.732	10.551
r (in.)	1.614	1.648	1.682
L_c (in.)	204	208	212
L/r	148.7	145.6	142.7
n_2 (Eq. 11-80)	23/12	23/12	23/12
σ_{allow}/σ_Y (Eq. 11-82)	0.1876	0.1957	0.2037
σ_{allow} (ksi)	6.754	7.044	7.333
$P_{allow} = A\,\sigma_{allow}$	23.2 k	25.2 k	27.4 k

Problem 11.9-18 Find the required outside diameter d for a steel pipe column (see figure) of length $L = 3.5$ m that is pinned at both ends and must support an axial load $P = 130$ kN. Assume that the wall thickness t is equal to $d/20$. (Use $E = 200$ GPa and $\sigma_Y = 275$ MPa).

Solution 11.9-18 Pipe column

Pinned ends ($K = 1$).

$L = 3.5$ m $\quad P = 130$ kN

d = outside diameter $\quad t = d/20$

$E = 200$ GPa $\quad \sigma_Y = 275$ MPa

$$A = \frac{\pi}{4}[d^2 - (d - 2t)^2] = 0.14923\, d^2$$

$$I = \frac{\pi}{64}[d^4 - (d - 2t)^4] = 0.016881\, d^4$$

$$r = \sqrt{\frac{I}{A}} = 0.33634\, d$$

$$\left(\frac{L}{r}\right)_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = 119.8 \qquad L_c = (119.8)r$$

Select various values of diameter d until we obtain $P_{allow} = P$.

If $L \le L_c$, Use Eqs. (11-79) and (11-81).

If $L \ge L_c$, Use Eqs. (11-80) and (11-82).

d (mm)	98	99	100
A (mm^2)	1433	1463	1492
I (mm^4)	1557×10^3	1622×10^3	1688×10^3
r (mm)	32.96	33.30	33.64
L_c (mm)	3950	3989	4030
L/r	106.2	105.1	104.0
n_1 (Eq. 11-79)	1.912	1.911	1.910
σ_{allow}/σ_Y (Eq. 11-81)	0.3175	0.3219	0.3263
σ_{allow} (MPa)	87.32	88.53	89.73
$P_{allow} = A\,\sigma_{allow}$	125.1 kN	129.5 kN	133.9 kN

For $P = 130$ kN, $d = 99$ mm ←

Problem 11.9-19 Find the required outside diameter d for a steel pipe column (see figure) of length $L = 11.5$ ft that is pinned at both ends and must support an axial load $P = 80$ k. Assume that the wall thickness t is 0.30 in. (Use $E = 29{,}000$ ksi and $\sigma_Y = 42$ ksi.)

Solution 11.9-19 Pipe column

Pinned ends ($K = 1$).

$L = 11.5$ ft $= 138$ in. $\quad P = 80$ k

d = outside diameter $\quad t = 0.30$ in.

$E = 29{,}000$ ksi $\quad \sigma_Y = 42$ ksi

$$A = \frac{\pi}{4}[d^2 - (d - 2t)^2]$$

$$I = \frac{\pi}{64}[d^4 - (d - 2t)^4] \qquad r = \sqrt{\frac{I}{A}}$$

$$\left(\frac{L}{r}\right)_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = 116.7 \qquad L_c = (116.7)r$$

Select various values of diameter d until we obtain $P_{allow} = P$.

If $L \leq L_c$, Use Eqs. (11-79) and (11-81).
If $L \geq L_c$, Use Eqs. (11-80) and (11-82).

For $P = 80$ k, $d = 5.23$ in. ⟵

d (in.)	5.20	5.25	5.30
A (in.^{2})	4.618	4.665	4.712
I (in.^{4})	13.91	14.34	14.78
r (in.)	1.736	1.753	1.771
L_c (in.)	203	205	207
L/r	79.49	78.72	77.92
n_1 (Eq. 11-79)	1.883	1.881	1.880
σ_{allow}/σ_Y (Eq. 11-81)	0.4079	0.4107	0.4133
σ_{allow} (ksi)	17.13	17.25	17.36
$P_{allow} = A\,\sigma_{allow}$	79.1 k	80.5 k	81.8 k

Problem 11.9-20 Find the required outside diameter d for a steel pipe column (see figure) of length $L = 3.0$ m that is pinned at both ends and must support an axial load $P = 800$ kN. Assume that the wall thickness t is 9 mm. (Use $E = 200$ GPa and $\sigma_Y = 300$ MPa.)

Solution 11.9-20 **Pipe column**

Pinned ends ($K = 1$).
$L = 3.0$ m $P = 800$ kN
d = outside diameter $t = 9.0$ mm
$E = 200$ GPa $\sigma_Y = 300$ MPa

$$A = \frac{\pi}{4}[d^2 - (d - 2t)^2]$$

$$I = \frac{\pi}{64}[d^4 - (d - 2t)^4] \qquad r = \sqrt{\frac{I}{A}}$$

$$\left(\frac{L}{r}\right)_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = 114.7 \qquad L_c = (114.7)r$$

Select various values of diameter d until we obtain $P_{allow} = P$.

If $L \leq L_c$, Use Eqs. (11-79) and (11-81).
If $L \geq L_c$, Use Eqs. (11-80) and (11-82).

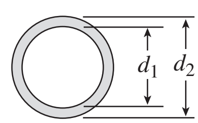
d (mm)	193	194	195
A (mm^2)	5202	5231	5259
I (mm^4)	20.08×10^6	22.43×10^6	22.80×10^6
r (mm)	65.13	65.48	65.84
L_c (mm)	7470	7510	7550
L/r	46.06	45.82	45.57
n_1 (Eq. 11-79)	1.809	1.809	1.808
σ_{allow}/σ_Y (Eq. 11-81)	0.5082	0.5087	0.5094
σ_{allow} (MPa)	152.5	152.6	152.8
$P_{allow} = A\,\sigma_{allow}$	793.1 kN	798.3 kN	803.8 kN

For $P = 800$ kn, $d = 194$ mm ⟵

Aluminum Columns

Problem 11.9-21 An aluminum pipe column (alloy 2014-T6) with pinned ends has outside diameter $d_2 = 5.60$ in. and inside diameter $d_1 = 4.80$ in. (see figure).

Determine the allowable axial load P_{allow} for each of the following lengths: $L = 6$ ft, 8 ft, 10 ft, and 12 ft.



Probs. 11.9-21 through 11.9-24

Solution 11.9-21 **Aluminum pipe column**

Alloy 2014-T6
Pinned ends ($K = 1$).
$d_2 = 5.60$ in.
$d_1 = 4.80$ in.

$$A = \frac{\pi}{4}(d_2^2 - d_1^2) = 6.535 \text{ in.}^2$$

$$I = \frac{\pi}{64}(d_2^4 - d_1^4) = 22.22 \text{ in.}^4$$

$$r = \sqrt{\frac{I}{A}} = 1.844 \text{ in.}$$

Use Eqs. (11-84 a and b):

$\sigma_{allow} = 30.7 - 0.23\,(L/r)$ ksi $\quad L/r \leq 55$

$\sigma_{allow} = 54{,}000/(L/r)^2$ ksi $\quad L/r \geq 55$

L (ft)	6 ft	8 ft	10 ft	12 ft
L/r	39.05	52.06	65.08	78.09
σ_{allow} (ksi)	21.72	18.73	12.75	8.86
$P_{allow} = \sigma_{allow} A$	142 k	122 k	83 k	58 k

Problem 11.9-22 An aluminum pipe column (alloy 2014-T6) with pinned ends has outside diameter $d_2 = 120$ mm and inside diameter $d_1 = 110$ mm (see figure).

Determine the allowable axial load P_{allow} for each of the following lengths: $L = 1.0$ m, 2.0 m, 3.0 m, and 4.0 m.

(*Hint:* Convert the given data to USCS units, determine the required quantities, and then convert back to SI units.)

Solution 11.9-22 Aluminum pipe column

Alloy 2014-T6

Pinned ends ($K = 1$).

$d_2 = 120$ mm $= 4.7244$ in.

$d_1 = 110$ mm $= 4.3307$ in.

$$A = \frac{\pi}{4}(d_2^2 - d_1^2) = 2.800 \text{ in.}^2$$

$$I = \frac{\pi}{64}(d_2^2 - d_1^2) = 7.188 \text{ in.}^4$$

$$r = \sqrt{\frac{I}{A}} = 40.697 \text{ mm} = 1.6022 \text{ in.}$$

Use Eqs. (11-84 a and b):

$\sigma_{allow} = 30.7 - 0.23\,(L/r)$ ksi $\quad L/r \leq 55$

$\sigma_{allow} = 54{,}000/(L/r)^2$ ksi $\quad L/r \geq 55$

L (m)	1.0 m	2.0 m	3.0 m	4.0 m
L (in.)	39.37	78.74	118.1	157.5
L/r	24.58	49.15	73.73	98.30
σ_{allow} (ksi)	25.05	19.40	9.934	5.588
$P_{allow} = \sigma_{allow} A$	70.14 k	54.31 k	27.81 k	15.65 k
P_{allow} (kN)	312 kN	242 kN	124 kN	70 kN

Problem 11.9-23 An aluminum pipe column (alloy 6061-T6) that is fixed at the base and free at the top has outside diameter $d_2 = 3.25$ in. and inside diameter $d_1 = 3.00$ in. (see figure).

Determine the allowable axial load P_{allow} for each of the following lengths: $L = 2$ ft, 3 ft, 4 ft, and 5 ft.

Solution 11.9-23 Aluminum pipe column

Alloy 6061-T6

Fixed-free ends ($K = 2$).

$d_2 = 3.25$ in.

$d_1 = 3.00$ in.

$$A = \frac{\pi}{4}(d_2^2 - d_1^2) = 1.227 \text{ in.}^2$$

$$I = \frac{\pi}{64}(d_2^2 - d_1^2) = 1.500 \text{ in.}^4$$

$$r = \sqrt{\frac{I}{A}} = 1.106 \text{ in.}$$

Use Eqs. (11-85 a and b):

$\sigma_{allow} = 20.2 - 0.126\,(KL/r)$ ksi $\quad KL/r \leq 66$

$\sigma_{allow} = 51{,}000/(KL/r)^2$ ksi $\quad KL/r \geq 66$

L (ft)	2 ft	3 ft	4 ft	5 ft
KL/r	43.40	65.10	86.80	108.5
σ_{allow} (ksi)	14.73	12.00	6.77	4.33
$P_{allow} = \sigma_{allow} A$	18.1 k	14.7 k	8.3 k	5.3 k

Problem 11.9-24 An aluminum pipe column (alloy 6061-T6) that is fixed at the base and free at the top has outside diameter $d_2 = 80$ mm and inside diameter $d_1 = 72$ mm (see figure).

Determine the allowable axial load P_{allow} for each of the following lengths: $L = 0.6$ m, 0.8 m, 1.0 m, and 1.2 m.

(*Hint:* Convert the given data to USCS units, determine the required quantities, and then convert back to SI units.)

Solution 11.9-24 **Aluminum pipe column**

Alloy 6061-T6

Fixed-free ends ($K = 2$).

$d_2 = 80 \text{ mm} = 3.1496 \text{ in.}$

$d_1 = 72 \text{ mm} = 2.8346 \text{ in.}$

$$A = \frac{\pi}{4}(d_2^2 - d_1^2) = 1.480 \text{ in.}^2$$

$$I = \frac{\pi}{64}(d_2^2 - d_1^2) = 1.661 \text{ in.}^4$$

$$r = \sqrt{\frac{I}{A}} = 26.907 \text{ mm} = 1.059 \text{ in.}$$

Use Eqs. (11-85 *a* and *b*):

$\sigma_{allow} = 20.2 - 0.126\,(KL/r)$ ksi $\quad KL/r \le 66$

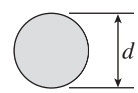
$\sigma_{allow} = 51{,}000/(KL/r)^2$ ksi $\quad KL/r \ge 66$

L (m)	0.6 m	0.8 m	1.0 m	1.2 m
KL (in.)	47.24	62.99	78.74	94.49
KL/r	44.61	59.48	74.35	89.23
σ_{allow} (ksi)	14.58	12.71	9.226	6.405
$P_{allow} = \sigma_{allow} A$	21.58 k	18.81 k	13.65 k	9.48 k
P_{allow} (kN)	96 kN	84 kN	61 kN	42 kN

Problem 11.9-25 A solid round bar of aluminum having diameter d (see figure) is compressed by an axial force $P = 60$ k. The bar has pinned supports and is made of alloy 2014-T6.

(a) If the diameter $d = 2.0$ in., what is the maximum allowable length L_{max} of the bar?

(b) If the length $L = 30$ in., what is the minimum required diameter d_{min}?



Probs. 11.9-25 through 11.9-28

Solution 11.9-25 **Aluminum bar**

Alloy 2014-T6

Pinned supports ($K = 1$). $\quad P = 60$ k

(a) FIND L_{max} IF $d = 2.0$ IN.

$$A = \frac{\pi d^2}{4} = 3.142 \text{ in.}^2 \qquad I = \frac{\pi d^4}{64}$$

$$r = \sqrt{\frac{I}{A}} = \frac{d}{4} = 0.5 \text{ in.}$$

$$\sigma_{allow} = \frac{P}{A} = \frac{60 \text{ k}}{3.142 \text{ in.}^2} = 19.10 \text{ ksi}$$

Assume L/r is less than 55:

Eq. (11-84*a*): $\sigma_{allow} = 30.7 - 0.23\,(L/r)$ ksi

or $19.10 = 30.7 - 0.23\,(L/r)$

Solve for L/r: $\dfrac{L}{r} = 50.43 \qquad \dfrac{L}{r} < 55 \qquad \therefore$ ok

$L_{max} = (50.43) \quad r = 25.2$ in. ←

(b) FIND d_{min} IF $L = 30$ IN.

$$A = \frac{\pi d^2}{4} \qquad r = \frac{d}{4} \qquad \frac{L}{r} = \frac{30 \text{ in.}}{d/4} = \frac{120 \text{ in.}}{d}$$

$$\sigma_{allow} = \frac{P}{A} = \frac{60 \text{ k}}{\pi d^2/4} = \frac{76.39}{d^2} \text{ (ksi)}$$

Assume L/r is greater than 55:

Eq. (11-84*b*): $\sigma_{allow} = \dfrac{54{,}000 \text{ ksi}}{(L/r)^2}$

or $\quad \dfrac{76.39}{d^2} = \dfrac{54{,}000}{(120/d)^2}$

$d^4 = 20.37 \text{ in.}^4 \qquad d_{min} = 2.12$ in. ←

$L/r = 120/d = 120/2.12 = 56.6 > 55 \quad \therefore$ ok

Problem 11.9-26 A solid round bar of aluminum having diameter d (see figure) is compressed by an axial force $P = 175$ kN. The bar has pinned supports and is made of alloy 2014-T6.

(a) If the diameter $d = 40$ mm, what is the maximum allowable length L_{max} of the bar?

(b) If the length $L = 0.6$ m, what is the minimum required diameter d_{min}?

(*Hint:* Convert the given data to USCS units, determine the required quantities, and then convert back to SI units.)

Solution 11.9-26 Aluminum bar

Alloy 2014-T6

Pinned supports ($K = 1$). $P = 175$ kN $= 39.34$ k

(a) FIND L_{max} IF $d = 40$ MM $= 1.575$ IN.

$$A = \frac{\pi d^2}{4} = 1.948\text{ in.}^2 \qquad I = \frac{\pi d^4}{64}$$

$$r = \sqrt{\frac{I}{A}} = \frac{d}{4} = 0.3938\text{ in.}$$

$$\sigma_{allow} = \frac{P}{A} = \frac{39.34\text{ k}}{1.948\text{ in.}^2} = 20.20\text{ ksi}$$

Assume L/r is less than 55:

Eq. (11-84a): $\sigma_{allow} = 30.7 - 0.23\,(L/r)$ ksi

or $20.20 = 30.7 - 0.23\,(L/r)$

Solve for L/r: $\frac{L}{r} = 45.65 \qquad \frac{L}{r} < 55 \qquad \therefore$ ok

$L_{max} = (45.65)\,r = 17.98$ in. $= 457$ mm ←

(b) FIND d_{min} IF $L = 0.6$ m $= 23.62$ in.

$$A = \frac{\pi d^2}{4} \qquad r = \frac{d}{4} \qquad \frac{L}{r} = \frac{23.62\text{ in.}}{d/4} = \frac{94.48\text{ in.}}{d}$$

$$\sigma_{allow} = \frac{P}{A} = \frac{39.34\text{ k}}{\pi d^2/4} = \frac{50.09}{d^2} \quad \text{(ksi)}$$

Assume L/r is greater than 55:

Eq. (11-84b): $\sigma_{allow} = \dfrac{54{,}000\text{ ksi}}{(L/r)^2}$

or $\dfrac{50.09}{d^2} = \dfrac{54{,}000}{(94.48/d)^2}$

$d^4 = 8.280$ in.^{4} $\quad d_{min} = 1.696$ in. $= 43.1$ mm ←

$L/r = 94.48/d = 94.48/1.696 = 55.7 > 55 \quad \therefore$ ok

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Problem 11.9-27 A solid round bar of aluminum having diameter d (see figure) is compressed by an axial force $P = 10$ k. The bar has pinned supports and is made of alloy 6061-T6.

(a) If the diameter $d = 1.0$ in., what is the maximum allowable length L_{max} of the bar?

(b) If the length $L = 20$ in., what is the minimum required diameter d_{min}?

Solution 11.9-27 Aluminum bar

Alloy 6061-T6

Pinned Supports ($K = 1$). $P = 10$ k

(a) FIND L_{max} IF $d = 1.0$ IN.

$$A = \frac{\pi d^2}{4} = 0.7854\text{ in.}^2 \qquad I = \frac{\pi d^4}{64}$$

$$r = \sqrt{\frac{I}{A}} = \frac{d}{4} = 0.2500\text{ in.}$$

$$\sigma_{allow} = \frac{P}{A} = \frac{10\text{ k}}{0.7854\text{ in.}^2} = 12.73\text{ ksi}$$

Assume L/r is less than 66:

Eq. (11-85a): $\sigma_{allow} = 20.2 - 0.126\,(L/r)$ ksi

or $12.73 = 20.2 - 0.126\,(L/r)$

Solve For L/r: $\frac{L}{r} = 59.29 \qquad \frac{L}{r} < 66 \qquad \therefore$ ok

$L_{max} = (59.29)r = 14.8$ in. ←

(b) FIND d_{min} IF $L = 20$ in.

$$A = \frac{\pi d^2}{4} \qquad r = \frac{d}{4} \qquad \frac{L}{r} = \frac{20\text{ in.}}{d/4} = \frac{80\text{ in.}}{d}$$

$$\sigma_{allow} = \frac{P}{A} = \frac{10\text{ k}}{\pi d^2/4} = \frac{12.73}{d^2}\text{ (ksi)}$$

Assume L/r is Greater than 66:

Eq. (11-85b): $\sigma_{allow} = \dfrac{51{,}000\text{ ksi}}{(L/r)^2}$

or $\dfrac{12.73}{d^2} = \dfrac{51{,}000}{(80/d)^2}$

$d^4 = 1.597\text{ in.}^4 \qquad d_{min} = 1.12$ in. ←

$L/r = 80/\text{d} = 80/1.12 = 71 > 66 \qquad \therefore$ ok

Problem 11.9-28 A solid round bar of aluminum having diameter d (see figure) is compressed by an axial force $P = 60$ kN. The bar has pinned supports and is made of alloy 6061-T6.

(a) If the diameter $d = 30$ mm, what is the maximum allowable length L_{max} of the bar?

(b) If the length $L = 0.6$ m, what is the minimum required diameter d_{min}?

(*Hint:* Convert the given data to USCS units, determine the required quantities, and then convert back to SI units.)

Solution 11.9-28 Aluminum bar

Alloy 6061-T6

Pinned Supports ($K = 1$). $\quad P = 60\text{ kN} = 13.49$ k

(a) FIND L_{max} IF $d = 30$ MM $= 1.181$ IN.

$$A = \frac{\pi d^2}{4} = 1.095\text{ in.}^2 \qquad I = \frac{\pi d^4}{64}$$

$$r = \sqrt{\frac{I}{A}} = \frac{d}{4} = 0.2953\text{ in.}$$

$$\sigma_{allow} = \frac{P}{A} = \frac{13.49\text{ k}}{1.095\text{ in.}^2} = 12.32\text{ ksi}$$

Assume L/r is less than 66:

Eq. (11-85a): $\sigma_{allow} = 20.2 - 0.126\,(L/r)$ ksi

or $12.32 = 20.2 - 0.126\,(L/r)$

Solve For L/r: $\dfrac{L}{r} = 62.54 \qquad \dfrac{L}{r} < 66 \qquad \therefore$ ok

$L_{max} = (62.54)r = 18.47\text{ in.} = 469$ mm ←

(b) FIND d_{min} IF $L = 0.6$ M $= 23.62$ IN.

$$A = \frac{\pi d^2}{4} \qquad r = \frac{d}{4} \qquad \frac{L}{r} = \frac{23.62\text{ in.}}{d/4} = \frac{94.48\text{ in.}}{d}$$

$$\sigma_{allow} = \frac{P}{A} = \frac{13.48\text{ k}}{\pi d^2/4} = \frac{17.18}{d^2}\text{ (ksi)}$$

Assume L/r is Greater than 66:

Eq. (11-85b): $\sigma_{allow} = \dfrac{51{,}000\text{ ksi}}{(L/r)^2}$

or $\dfrac{17.18}{d^2} = \dfrac{51{,}000}{(94.48/d)^2}$

$d^4 = 3.007\text{ in.}^4 \quad d_{min} = 1.317\text{ in.} = 33.4$ mm ←

$L/r = 94.48/\text{d} = 94.48/1.317 = 72 > 66 \qquad \therefore$ ok

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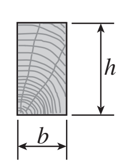
Wood Columns

When solving the problems for wood columns, assume that the columns are constructed of sawn lumber (c = 0.8 and K_{cE} = 0.3) and have pinned-end conditions. Also, buckling may occur about either principal axis of the cross section.

Problem 11.9-29 A wood post of rectangular cross section (see figure) is constructed of 4 in. × 6 in. structural grade, Douglas fir lumber ($F_c = 2{,}000$ psi, $E = 1{,}800{,}00$ psi). The net cross-sectional dimensions of the post are $b = 3.5$ in. and $h = 5.5$ in. (see Appendix F).

Determine the allowable axial load P_{allow} for each of the following lengths: $L = 5.0$ ft, 7.5 ft, and 10.0 ft.



Probs. 11.9-29 through 11.9-32

Solution 11.9-29 Wood post (rectangular cross section)

$F_c = 2{,}000$ psi $E = 1{,}800{,}000$ psi $c = 0.8$

$K_{cE} = 0.3$ $b = 3.5$ in. $h = 5.5$ in. $d = b$

Find P_{allow}

Eq. (11-94): $\phi = \dfrac{K_{cE}E}{F_c(L_e/d)^2}$

Eq. (11-95): $C_P = \dfrac{1+\phi}{2c} - \sqrt{\left[\dfrac{1+\phi}{2c}\right]^2 - \dfrac{\phi}{c}}$

Eq. (11-92): $P_{\text{allow}} = F_cC_PA = F_cC_Pbh$

L_e	5 ft	7.5 ft	10.0 ft
L_e/d	17.14	25.71	34.29
ϕ	0.9188	0.4083	0.2297
C_P	0.6610	0.3661	0.2176
P_{allow}	25.4 k	14.1 k	8.4 k

Problem 11.9-30 A wood post of rectangular cross section (see figure) is constructed of structural grade, southern pine lumber ($F_c = 14$ MPa, $E = 12$ GPa). The cross-sectional dimensions of the post (actual dimensions) are $b = 100$ mm and $h = 150$ mm.

Determine the allowable axial load P_{allow} for each of the following lengths: $L = 1.5$ m, 2.0 m, and 2.5 m.

Solution 11.9-30 Wood post (rectangular cross section)

$F_c = 14$ MPa $E = 12$ GPa $c = 0.8$ $K_{cE} = 0.3$

$b = 100$ mm $h = 150$ mm $d = b$

Find P_{allow}

Eq. (11-94): $\phi = \dfrac{K_{cE}E}{F_c(L_e/d)^2}$

Eq. (11-95): $C_P = \dfrac{1+\phi}{2c} - \sqrt{\left[\dfrac{1+\phi}{2c}\right]^2 - \dfrac{\phi}{c}}$

Eq. (11-92): $P_{\text{allow}} = F_cC_PA = F_cC_Pbh$

L_e	1.5 m	2.0 m	2.5 m
L_e/d	15	20	25
ϕ	1.1429	0.6429	0.4114
C_P	0.7350	0.5261	0.3684
P_{allow}	154 kN	110 kN	77 kN ←

Problem 11.9-31 A wood column of rectangular cross section (see figure) is constructed of 4 in. × 8 in. construction grade, western hemlock lumber ($F_c = 1{,}000$ psi, $E = 1{,}300{,}000$ psi). The net cross-sectional dimensions of the column are $b = 3.5$ in. and $h = 7.25$ in. (see Appendix F).

Determine the allowable axial load P_{allow} for each of the following lengths: $L = 6$ ft, 8 ft, and 10 ft.

Solution 11.9-31 Wood column (rectangular cross section)

$F_c = 1{,}000$ psi $E = 1{,}300{,}000$ psi $c = 0.8$

$K_{cE} = 0.3$ $b = 3.5$ in. $h = 7.25$ in. $d = b$

Find P_{allow}

Eq. (11-94): $\phi = \dfrac{K_{cE}E}{F_c(L_e/d)^2}$

Eq. (11-95): $C_P = \dfrac{1+\phi}{2c} - \sqrt{\left[\dfrac{1+\phi}{2c}\right]^2 - \dfrac{\phi}{c}}$

Eq. (11-92): $P_{\text{allow}} = F_cC_FA = F_cC_Pbh$

L_e	6 ft	8 ft	10 ft
L_e/d	20.57	27.43	34.29
ϕ	0.9216	0.5184	0.3318
C_P	0.6621	0.4464	0.3050
P_{allow}	16.8 k	11.3 k	7.7 k ←

Problem 11.9-32 A wood column of rectangular cross section (see figure) is constructed of structural grade, Douglas fir lumber (F_c = 12 MPa, E = 10 GPa). The cross-sectional dimensions of the column (actual dimensions) are b = 140 mm and h = 210 mm.

Determine the allowable axial load P_{allow} for each of the following lengths: L = 2.5 m, 3.5 m, and 4.5 m.

Solution 11.9-32 **Wood column (rectangular cross section)**

$F_c = 12$ MPa $\quad E = 10$ GPa $\quad c = 0.8 \quad K_{cE} = 0.3$
$b = 140$ mm $\quad h = 210$ mm $\quad d = b$

Find P_{allow}

Eq. (11-94): $\quad \phi = \dfrac{K_{cE}E}{F_c(L_e/d)^2}$

Eq. (11-95): $\quad C_P = \dfrac{1+\phi}{2c} - \sqrt{\left[\dfrac{1+\phi}{2c}\right]^2 - \dfrac{\phi}{c}}$

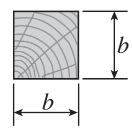
Eq. (11-92): $\quad P_{allow} = F_cC_PA = F_cC_Pbh$

L_e	2.5 m	3.5 m	4.5 m
L_e/d	17.86	25.00	32.14
ϕ	0.7840	0.4000	0.2420
C_P	0.6019	0.3596	0.2284
P_{allow}	212 kN	127 kN	81 kN ←

Problem 11.9-33 A square wood column with side dimensions b (see figure) is constructed of a structural grade of Douglas fir for which F_c = 1,700 psi and E = 1,400,000 psi. An axial force P = 40 k acts on the column.

(a) If the dimension b = 5.5 in., what is the maximum allowable length L_{max} of the column?

(b) If the length L = 11 ft, what is the minimum required dimension b_{min}?



Probs. 11.9-33 through 11.9-36

Solution 11.9-33 **Wood column (square cross section)**

$F_c = 1{,}700$ psi $\quad E = 1{,}400{,}000$ psi $\quad c = 0.8$
$K_{cE} = 0.3 \quad P = 40$ k

(a) MAXIMUM LENGTH L_{max} FOR $b = d = 5.5$ IN.

From Eq. (11-92): $\quad C_P = \dfrac{P}{F_cb^2} = 0.77783$

From Eq. (11-95):

$$C_P = 0.77783 = \frac{1+\phi}{1.6} - \sqrt{\left[\frac{1+\phi}{1.6}\right]^2 - \frac{\phi}{0.8}}$$

Trial and error: $\quad \phi = 1.3225$

From Eq. (11-94): $\quad \dfrac{L}{d} = \sqrt{\dfrac{K_{cE}E}{\phi F_c}} = 13.67$

$\therefore\ L_{max} = 13.67\ d = (13.67)(5.5 \text{ in.})$
$= 75.2$ in. ←

(b) MINIMUM DIMENSION b_{min} FOR $L = 11$ ft

Trial and error: $\quad \dfrac{L}{d} = \dfrac{L}{b} \quad \phi = \dfrac{K_{cE}E}{F_c(L/d)^2}$

$$C_P = \frac{1+\phi}{1.6} - \sqrt{\left[\frac{1+\phi}{1.6}\right]^2 - \frac{\phi}{0.8}} \qquad P = F_cC_Pb^2$$

Given load: $\quad P = 40$ k

Trial b (in.)	$\frac{L}{d} = \frac{L}{b}$	ϕ	C_P	P (kips)
6.50	20.308	0.59907	0.49942	35.87
6.70	19.701	0.63651	0.52230	39.86
6.71	19.672	0.63841	0.52343	40.06

$\therefore\ b_{min} = 6.71$ in. ←

Problem 11.9-34 A square wood column with side dimensions b (see figure) is constructed of a structural grade of southern pine for which $F_c = 10.5$ MPa and $E = 12$ GPa. An axial force $P = 200$ kN acts on the column.

(a) If the dimension $b = 150$ mm, what is the maximum allowable length L_{max} of the column?

(b) If the length $L = 4.0$ m, what is the minimum required dimension b_{min}?

Solution 11.9-34 **Wood column (square cross section)**

$F_c = 10.5$ MPa $\quad E = 12$ GPa $\quad c = 0.8$

$K_{cE} = 0.3 \quad P = 200$ kN

(a) MAXIMUM LENGTH L_{max} FOR $b = d = 150$ mm

From Eq. (11-92): $C_P = \dfrac{P}{F_c b^2} = 0.84656$

From Eq. (11-95):

$$C_P = 0.84656 = \frac{1+\phi}{1.6} - \sqrt{\left[\frac{1+\phi}{1.6}\right]^2 - \frac{\phi}{0.8}}$$

Trial and error: $\phi = 1.7807$

From Eq. (11-94): $\dfrac{L}{d} = \sqrt{\dfrac{K_{cE}E}{\phi F_c}} = 13.876$

$\therefore L_{max} = 13.876\, d = (13.876)(150 \text{ mm})$
$= 2.08 \text{ m} \leftarrow$

(b) MINIMUM DIMENSION b_{min} FOR $L = 4.0$ M

Trial and error: $\dfrac{L}{d} = \dfrac{L}{b} \quad \phi = \dfrac{K_{cE}E}{F_c(L/d)^2}$

$$C_P = \frac{1+\phi}{1.6} - \sqrt{\left[\frac{1+\phi}{1.6}\right]^2 - \frac{\phi}{0.8}} \quad P = F_c C_P b^2$$

Given load: $P = 200$ kN

Trial b (mm)	$\dfrac{L}{d} = \dfrac{L}{b}$	ϕ	C_P	P (kN)
180	22.22	0.69429	0.55547	189.0
182	21.98	0.70980	0.56394	196.1
183	21.86	0.71762	0.56814	199.8
184	21.74	0.72549	0.57231	203.5

$\therefore b_{min} = 184 \text{ mm} \leftarrow$

Problem 11.9-35 A square wood column with side dimensions b (see figure) is constructed of a structural grade of spruce for which $F_c = 900$ psi and $E = 1{,}500{,}000$ psi. An axial force $P = 8.0$ k acts on the column.

(a) If the dimension $b = 3.5$ in., what is the maximum allowable length L_{max} of the column?

(b) If the length $L = 10$ ft, what is the minimum required dimension b_{min}?

Solution 11.9-35 **Wood column (square cross section)**

$F_c = 900$ psi $\quad E = 1{,}500{,}000$ psi $\quad c = 0.8$

$K_{cE} = 0.3 \quad P = 8.0$ k

(a) MAXIMUM LENGTH L_{max} FOR $b = d = 3.5$ IN.

From Eq. (11-92): $C_P = \dfrac{P}{F_c b^2} = 0.72562$

From Eq. (11-95):

$$C_P = 0.72562 = \frac{1+\phi}{1.6} - \sqrt{\left[\frac{1+\phi}{1.6}\right]^2 - \frac{\phi}{0.8}}$$

Trial and error: $\phi = 1.1094$

From Eq. (11-94): $\dfrac{L}{d} = \sqrt{\dfrac{K_{cE}E}{\phi F_c}} = 21.23$

$\therefore L_{max} = 21.23\, d = (21.23)(3.5 \text{ in.}) = 74.3 \text{ in.} \leftarrow$

(b) MINIMUM DIMENSION b_{min} FOR $L = 10$ FT

Trial and error. $\frac{L}{d} = \frac{L}{b}$ $\quad \phi = \frac{K_{cE}E}{F_c(L/d)^2}$

$$C_P = \frac{1+\phi}{1.6} - \sqrt{\left[\frac{1+\phi}{1.6}\right]^2 - \frac{\phi}{0.8}} \qquad P = F_c C_P b^2$$

Given load: $P = 8000$ lb

$\therefore b_{min} = 4.20$ in. ⟵

Trial b (in.)	$\frac{L}{d} = \frac{L}{b}$	ϕ	C_P	P (lb)
4.00	30.00	0.55556	0.47145	6789
4.20	28.57	0.61250	0.50775	8061
4.19	28.64	0.60959	0.50596	7994

Problem 11.9-36 A square wood column with side dimensions b (see figure) is constructed of a structural grade of eastern white pine for which $F_c = 8.0$ MPa and $E = 8.5$ GPa. An axial force $P = 100$ kN acts on the column.

(a) If the dimension $b = 120$ mm, what is the maximum allowable length L_{max} of the column?

(b) If the length $L = 4.0$ m, what is the minimum required dimension b_{min}?

Solution 11.9-36 Wood column (square cross section)

$F_c = 8.0$ MPa $\quad E = 8.5$ GPa $\quad c = 0.8$

$K_{cE} = 0.3 \quad P = 100$ kN

(a) MAXIMUM LENGTH L_{max} FOR $b = d = 120$ mm

From Eq. (11-92): $C_P = \frac{P}{F_c b^2} = 0.86806$

From Eq. (11-95):

$$C_P = 0.86806 = \frac{1+\phi}{1.6} - \sqrt{\left[\frac{1+\phi}{1.6}\right]^2 - \frac{\phi}{0.8}}$$

Trial and error: $\phi = 2.0102$

From Eq. (11-94): $\frac{L}{d} = \sqrt{\frac{K_{cE}E}{\phi F_c}} = 12.592$

$\therefore L_{max} = 12.592\, d = (12.592)(120 \text{ mm})$

$= 1.51$ m ⟵

(b) MINIMUM DIMENSION b_{min} FOR $L = 4.0$ m

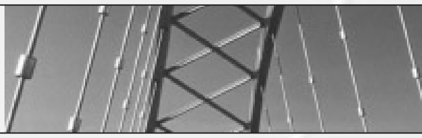
Trial and error. $\frac{L}{d} = \frac{L}{b}$ $\quad \phi = \frac{K_{cE}E}{F_c(L/d)^2}$

$$C_P = \frac{1+\phi}{1.6} - \sqrt{\left[\frac{1+\phi}{1.6}\right]^2 - \frac{\phi}{0.8}} \qquad P = F_c C_P b^2$$

Given load: $P = 100$ kN

Trial b (mm)	$\frac{L}{d} = \frac{L}{b}$	ϕ	C_P	P (kN)
160	25.00	0.51000	0.44060	90.23
164	24.39	0.53582	0.45828	98.61
165	24.24	0.54237	0.46269	100.77

$\therefore b_{min} = 165$ mm ⟵



12

Review of Centroids and Moments of Inertia

Differential Equations of the Deflection Curve

The problems for Section 12.2 are to be solved by integration.

Problem 12.2-1 Determine the distances $\bar{x}$ and $\bar{y}$ to the centroid C of a right triangle having base b and altitude h (see Case 6, Appendix D).

Solution 12.2-1 **Centroid of a right triangle**

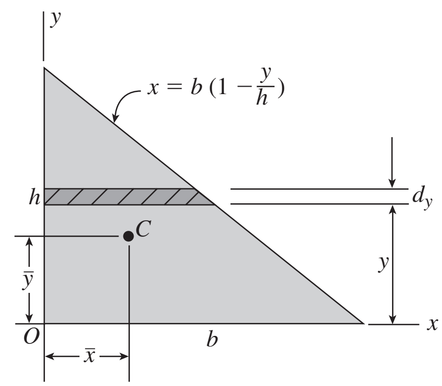
$dA = x\,dy = b(1 - y/h)\,dy$

$$A = \int dA = \int_0^h b(1 - y/h)\,dy \quad = \frac{bh}{2}$$

$$Q_x = \int y\,dA = \int_0^h yb(1 - y/h)\,dy \quad = \frac{bh^2}{6}$$

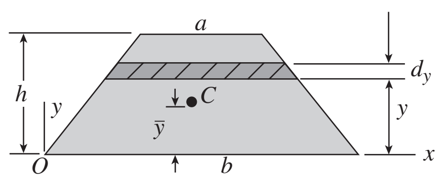
$$\bar{y} = \frac{Q_x}{A} = \frac{h}{3} \quad \longleftarrow$$

$$\text{Similarly, } \bar{x} = \frac{b}{3} \quad \longleftarrow$$



Problem 12.2-2 Determine the distance $\bar{y}$ to the centroid C of a trapezoid having bases a and b and altitude h (see Case 8, Appendix D).

Solution 12.2-2 **Centroid of a trapezoid**



Width of element $= b + (a - b)y/h$

$dA = [b + (a - b)y/h]\,dy$

$$A = \int dA = \int_0^h [b + (a - b)y/h]\,dy \quad = \frac{h(a + b)}{2}$$

$$Q_x = \int y\,dA = \int_0^h y[b + (a - b)\,y/h]\,dy$$

$$= \frac{h^2}{6}(2a + b)$$

$$\bar{y} = \frac{Qx}{A} = \frac{h(2a + b)}{3(a + b)} \quad \longleftarrow$$

Problem 12.2-3 Determine the distance $\bar{y}$ to the centroid C of a semicircle of radius r (see Case 10, Appendix D).

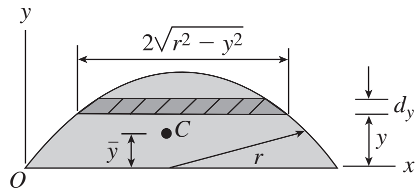
Solution 12.2-3 Centroid of a semicircle

$$dA = 2\sqrt{r^2 - y^2}\,dy$$

$$A = \int dA = \int_0^r 2\sqrt{r^2 - y^2}\,dy = \frac{\pi r^2}{2}$$

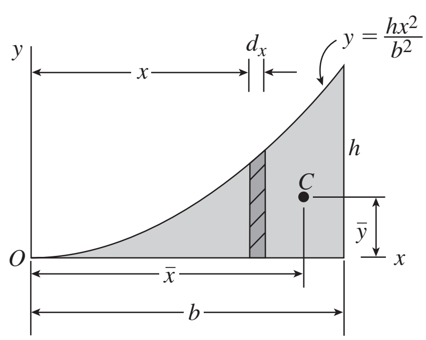
$$Q_x = \int y\,dA = \int_0^r 2y\sqrt{r^2 - y^2}\,dy = \frac{2r^3}{3}$$

$$\bar{y} = \frac{Q_x}{A} = \frac{4r}{3\pi} \quad \longleftarrow$$



Problem 12.2-4 Determine the distances $\bar{x}$ and $\bar{y}$ to the centroid C of a parabolic spandrel of base b and height h (see Case 18, Appendix D).

Solution 12.2-4 Centroid of a parabolic spandrel



$$dA = y\,dx = \frac{hx^2\,dx}{b^2}$$

$$A = \int dA = \int_0^b \frac{hx^2}{b^2}\,dx = \frac{bh}{3}$$

$$Q_y = \int x\,dA = \int_0^b \frac{hx^3}{b^2}\,dx = \frac{b^2 h}{4}$$

$$\bar{x} = \frac{Q_y}{A} = \frac{3b}{4} \quad \longleftarrow$$

$$Q_x = \int y/2\,dA = \int_0^b \frac{1}{2}\left(\frac{hx^2}{b^2}\right)\left(\frac{hx^2}{b^2}\right)dx = \frac{bh^2}{10}$$

$$\bar{y} = \frac{Q_x}{A} = \frac{3h}{10} \quad \longleftarrow$$

Problem 12.2-5 Determine the distances $\bar{x}$ and $\bar{y}$ to the centroid C of a semisegment of nth degree having base b and height h (see Case 19, Appendix D).

Solution 12.2-5 Centroid of a semisegment of *n*th degree

$$dA = y\,dx = h\left(1 - \frac{x^n}{b^n}\right)dx$$

$$A = \int dA = \int_0^b h\left(1 - \frac{x^n}{b^n}\right)dx = bh\left(\frac{n}{n+1}\right)$$

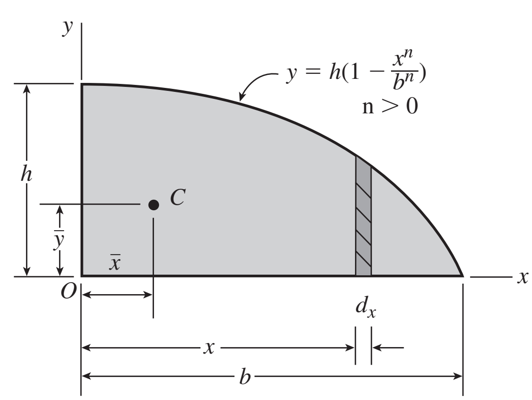
$$Q_y = \int x\,dA = \int_0^b xh\left(1 - \frac{x^n}{b^n}\right)dx = \frac{hb^2}{2}\left(\frac{n}{n+2}\right)$$

$$\bar{x} = \frac{Q_y}{A} = \frac{b(n+1)}{2(n+2)} \quad \longleftarrow$$

$$Q_x = \int \frac{y}{2}\,dA = \int_0^b \frac{1}{2}h\left(1 - \frac{x^n}{b^n}\right)(h)\left(1 - \frac{x^n}{b^n}\right)dx$$

$$= bh^2\left[\frac{n^2}{(n+1)(2n+1)}\right]$$

$$\bar{y} = \frac{Q_x}{A} = \frac{hn}{2n+1} \quad \longleftarrow$$

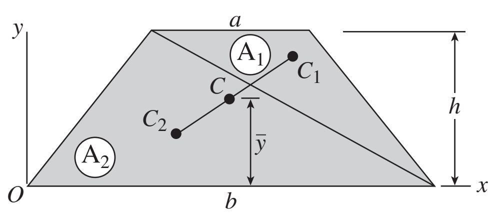


Centroids of Composite Areas

The problems for Section 12.3 are to be solved by using the formulas for composite areas.

Problem 12.3-1 Determine the distance $\bar{y}$ to the centroid C of a trapezoid having bases a and b and altitude h (see Case 8, Appendix D) by dividing the trapezoid into two triangles.

Solution 12.3-1 **Centroid of a trapezoid**



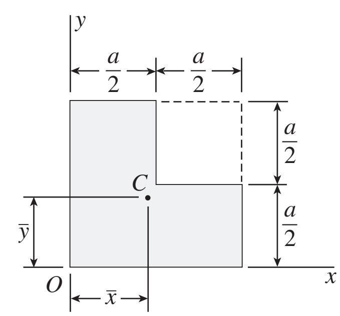
$$A_1 = \frac{ah}{2} \qquad \bar{y}_1 = \frac{2h}{3} \qquad A_2 = \frac{bh}{2} \qquad \bar{y}_2 = \frac{h}{3}$$

$$A = \sum A_i = \frac{ah}{2} + \frac{bh}{2} = \frac{h}{2}(a + b)$$

$$Q_x = \sum \bar{y}_i A_i = \frac{2h}{3}\left(\frac{ah}{2}\right) + \frac{h}{3}\left(\frac{bh}{2}\right) = \frac{h^2}{6}(2a + b)$$

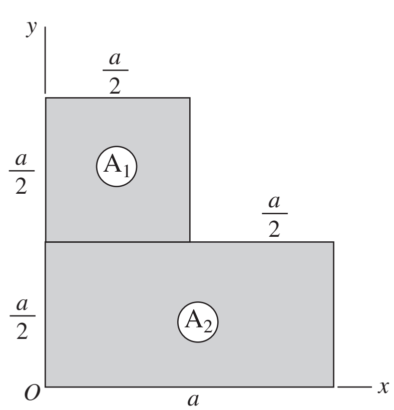
$$\bar{y} = \frac{Q_x}{A} = \frac{h(2a + b)}{3(a + b)} \quad \longleftarrow$$

Problem 12.3-2 One quarter of a square of side a is removed (see figure). What are the coordinates $\bar{x}$ and $\bar{y}$ of the centroid C of the remaining area?



PROBS. 12.3-2 and 12.5-2

Solution 12.3-2 **Centroid of a composite area**



$$A_1 = \frac{a^2}{4} \qquad \bar{y}_1 = \frac{3a}{4}$$

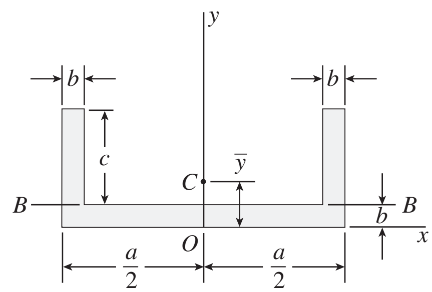
$$A_2 = \frac{a^2}{2} \qquad \bar{y}_2 = \frac{a}{4}$$

$$A = \sum A_i = \frac{3a^2}{4}$$

$$Q_x = \sum \bar{y}_i A_i = \frac{3a}{4}\left(\frac{a^2}{4}\right) + \frac{a}{4}\left(\frac{a^2}{2}\right) = \frac{5a^3}{16}$$

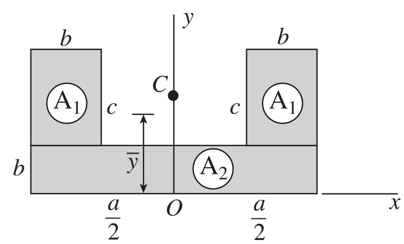
$$\bar{x} = \bar{y} = \frac{Qx}{A} = \frac{5a}{12} \quad \longleftarrow$$

Problem 12.3-3 Calculate the distance $\bar{y}$ to the centroid C of the channel section shown in the figure if $a = 6$ in., $b = 1$ in., and $c = 2$ in.



PROBS. 12.3-3, 12.3-4, and 12.5-3

Solution 12.3-3 **Centroid of a channel section**



$a = 6$ in. $\quad b = 1$ in. $\quad c = 2$ in.

$$A_1 = bc = 2 \text{ in.}^2 \qquad \bar{y}_1 = b + c/2 = 2 \text{ in.}$$

$$A_2 = ab = 6 \text{ in.}^2 \qquad \bar{y}_2 = \frac{b}{2} = 0.5 \text{ in.}$$

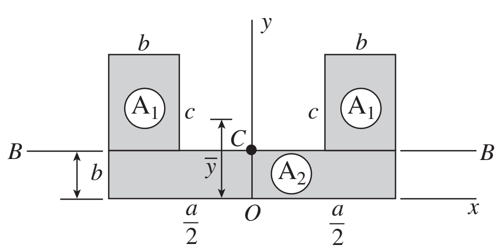
$$A = \sum A_i = 2A_1 + A_2 = 10 \text{ in.}^2$$

$$Q_x = \sum \bar{y}_i A_i = 2\bar{y}_1 A_1 + \bar{y}_2 A_2 = 11.0 \text{ in.}^3$$

$$\bar{y} = \frac{Q_x}{A} = 1.10 \text{ in.} \quad \longleftarrow$$

Problem 12.3-4 What must be the relationship between the dimensions a, b, and c of the channel section shown in the figure in order that the centroid C will lie on line BB?

Solution 12.3-4 **Dimensions of channel section**



$$A_1 = bc \qquad \bar{y}_1 = b + c/2$$

$$A_2 = ab \qquad \bar{y}_2 = b/2$$

$$A = \sum A_i = 2A_1 + A_2 = b(2c + a)$$

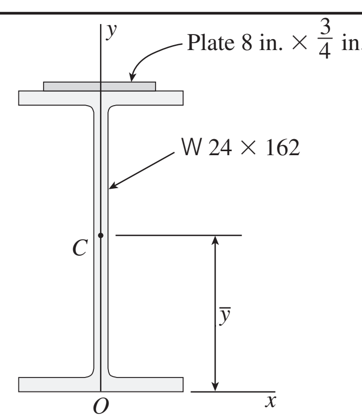
$$Q_x = \sum \bar{y}_i A_i = 2\bar{y}_1 A_1 + \bar{y}_2 A_2 = b/2(4bc + 2c^2 + ab)$$

$$\bar{y} = \frac{Q_x}{A} = \frac{4bc + 2c^2 + ab}{2(2c + a)}$$

Set $\bar{y} = b$ and solve: $2c^2 = ab \quad \longleftarrow$

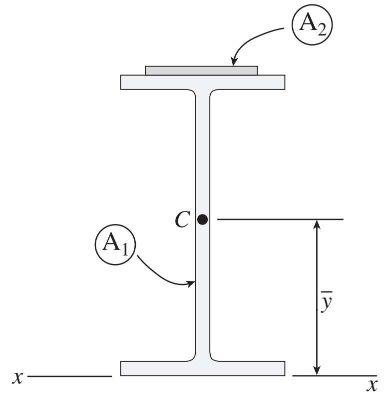
Problem 12.3-5 The cross section of a beam constructed of a W 24 × 162 wide-flange section with an 8 in. × 3/4 in. cover plate welded to the top flange is shown in the figure.

Determine the distance $\bar{y}$ from the base of the beam to the centroid C of the cross-sectional area.



PROBS. 12.3-5 and 12.5-5

Solution 12.3-5 **Centroid of beam cross section**



W 24 × 162 $A_1 = 47.7 \text{ in.}^2$ $d = 25.00$ in.

$\bar{y}_1 = d/2 = 12.5$ in.

PLATE: 8.0 × 0.75 in. $A_2 = (8.0)(0.75) = 6.0 \text{ in.}^2$

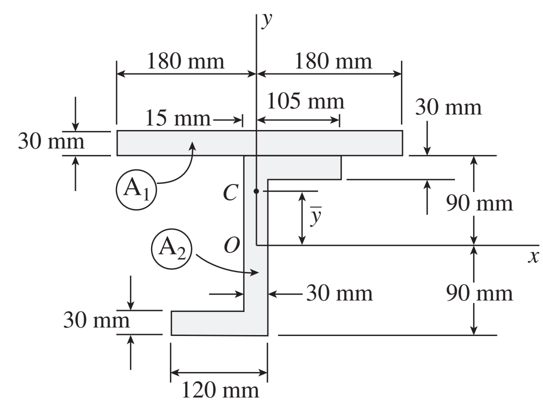
$\bar{y}_2 = 25.00 + 0.75/2 = 25.375$ in.

$A = \sum A_i = A_1 + A_2 = 53.70 \text{ in.}^2$

$Q_x = \sum \bar{y}_i A_i = \bar{y}_1 A_1 + \bar{y}_2 A_2 = 748.5 \text{ in.}^3$

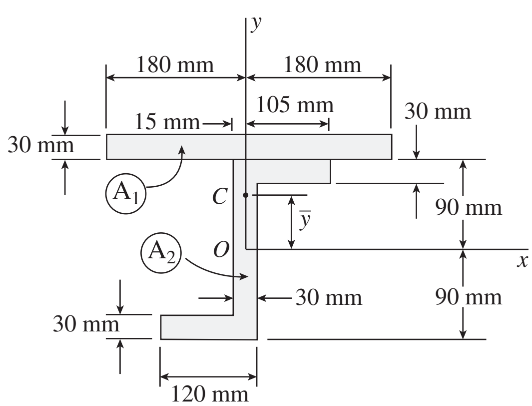
$\bar{y} = \dfrac{Q_x}{A} = 13.94$ in. ←

Problem 12.3-6 Determine the distance $\bar{y}$ to the centroid C of the composite area shown in the figure.



PROBS. 12.3-6, 12.5-6 and 12.7-6

Solution 12.3-6 **Centroid of composite area**



$A_1 = (360)(30) = 10{,}800 \text{ mm}^2$

$\bar{y}_1 = 105$ mm

$A_2 = 2(120)(30) + (120)(30) = 10{,}800 \text{ mm}^2$

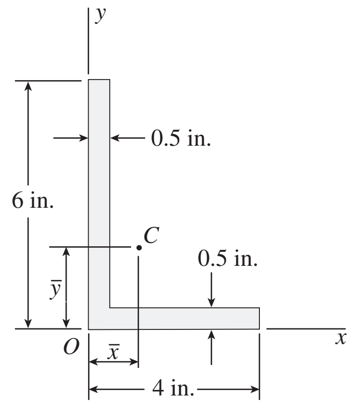
$\bar{y}_2 = 0$

$A = \sum A_i = A_1 + A_2 = 21{,}600 \text{ mm}^2$

$Q_x = \sum \bar{y}_i A_i = \bar{y}_1 A_1 + \bar{y}_2 A_2 = 1.134 \times 10^6 \text{ mm}^3$

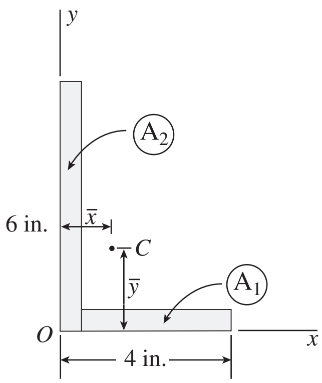
$\bar{y} = \dfrac{Q_x}{A} = 52.5$ mm ←

Problem 12.3-7 Determine the coordinates $\bar{x}$ and $\bar{y}$ of the centroid C of the L-shaped area shown in the figure.



PROBS. 12.3-7, 12.4-7, 12.5-7 and 12.7-7

Solution 12.3-7 **Centroid of L-shaped area**



$A_1 = (3.5)(0.5) = 1.75 \text{ in.}^2$

$\bar{y}_1 = 0.25 \text{ in.} \qquad \bar{x}_1 = 2.25 \text{ in.}$

$A_2 = (6)(0.5) = 3.0 \text{ in.}^2$

$\bar{y}_2 = 3.0 \text{ in.} \qquad \bar{x}_2 = 0.25 \text{ in.}$

$$A = \sum A_i = A_1 + A_2 = 4.75 \text{ in.}^2$$

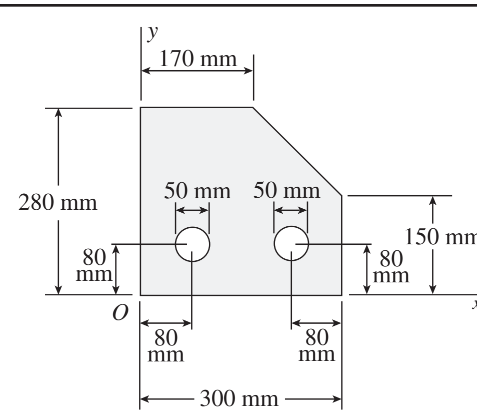
$$Q_y = \sum \bar{x}_i A_i = \bar{x}_1 A_1 + \bar{x}_2 A_2 = 4.688 \text{ in.}^3$$

$$\bar{x} = \frac{Q_y}{A} = 0.99 \text{ in.} \quad \longleftarrow$$

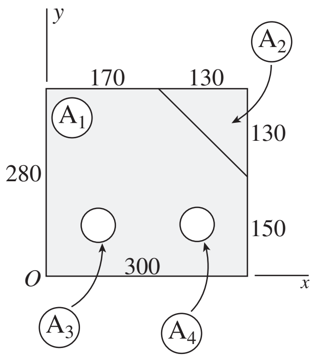
$$Q_x = \sum \bar{y}_i A_i = \bar{y}_1 A_1 + \bar{y}_2 A_2 = 9.438 \text{ in.}^3$$

$$\bar{y} = \frac{Q_x}{A} = 1.99 \text{ in.} \quad \longleftarrow$$

Problem 12.3-8 Determine the coordinates $\bar{x}$ and $\bar{y}$ of the centroid C of the area shown in the figure.



Solution 12.3-8 Centroid of composite area



A_1 = large rectangle
A_2 = triangular cutout
$A_3 = A_4$ = circular holes
All dimensions are in millimeters.
Diameter of holes = 50 mm
Centers of holes are 80 mm from edges.

$A_1 = (280)(300) = 84{,}000 \text{ mm}^2$

$\bar{x}_1 = 150 \text{ mm} \qquad \bar{y}_1 = 140 \text{ mm}$

$A_2 = 1/2(130)^2 = 8450 \text{ mm}^2$

$\bar{x}_2 = 300 - 130/3 = 256.7 \text{ mm}$

$\bar{y}_2 = 280 - 130/3 = 236.7 \text{ mm}$

$$A_3 = \frac{\pi d^2}{4} = \frac{\pi}{4}(50)^2 = 1963 \text{ mm}^2$$

$\bar{x}_3 = 80 \text{ mm} \qquad \bar{y}_3 = 80 \text{ mm}$

$A_4 = 1963 \text{ mm}^2 \qquad \bar{x}_4 = 220 \text{ mm} \qquad \bar{y}_4 = 80 \text{ mm}$

$$A = \sum A_i = A_1 - A_2 - A_3 - A_4 = 71{,}620 \text{ mm}^2$$

$$Q_y = \sum \bar{x}_i A_i = \bar{x}_1 A_1 - \bar{x}_2 A_2 - \bar{x}_3 A_3 - \bar{x}_4 A_4$$
$$= 9.842 \times 10^6 \text{ mm}^3$$

$$\bar{x} = \frac{Q_y}{A} = \frac{9.842 \times 10^6}{71{,}620} = 137 \text{ mm} \longleftarrow$$

$$Q_x = \sum \bar{y}_i A_i = \bar{y}_1 A_1 - \bar{y}_2 A_2 - \bar{y}_3 A_3 - \bar{y}_4 A_4$$
$$= 9.446 \times 10^6 \text{ mm}^3$$

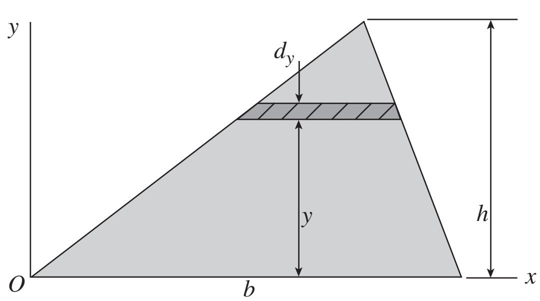
$$\bar{y} = \frac{Q_x}{A} = \frac{9.446 \times 10^6}{71{,}620} = 132 \text{ mm} \longleftarrow$$

Moments of Inertia

Problems 12.4-1 through 12.4-4 are to be solved by integration.

Problem 12.4-1 Determine the moment of inertia I_x of a triangle of base b and altitude h with respect to its base (see Case 4, Appendix D).

Solution 12.4-1 Moment of inertia of a triangle



Width of element

$$= b\left(\frac{h - y}{h}\right)$$

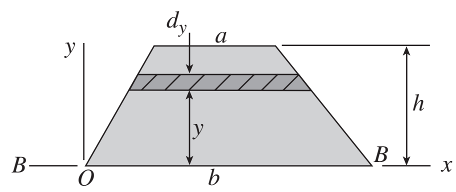
$$dA = \frac{b(h - y)}{h} dy$$

$$I_x = \int y^2 dA = \int_0^h y^2 b \frac{(h - y)}{h} dy$$

$$= \frac{bh^3}{12} \longleftarrow$$

Problem 12.4-2 Determine the moment of inertia I_{BB} of a trapezoid having bases a and b and altitude h with respect to its base (see Case 8, Appendix D).

Solution 12.4-2 Moment of inertia of a trapezoid



Width of element

$$= a + (b - a)\left(\frac{h - y}{h}\right)$$

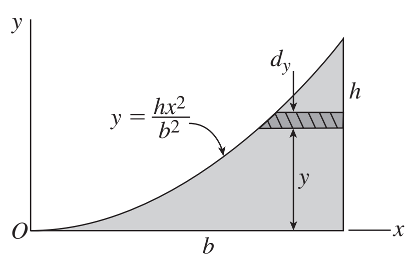
$$dA = \left[a + (b - a)\left(\frac{h - y}{h}\right)\right]dy$$

$$I_{BB} = \int y^2 dA = \int_0^h y^2\left[a + (b - a)\left(\frac{h - y}{h}\right)\right]dy$$

$$= \frac{h^3(3a + b)}{12} \longleftarrow$$

Problem 12.4-3 Determine the moment of inertia I_x of a parabolic spandrel of base b and height h with respect to its base (see Case 18, Appendix D).

Solution 12.4-3 Moment of inertia of a parabolic spandrel



Width of element

$$= b - x = b - b\sqrt{\frac{y}{h}}$$

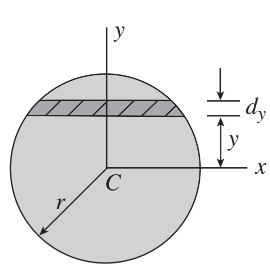
$$= b(1 - \sqrt{y/h})$$

$$dA = b(1 - \sqrt{y/h})\,dy$$

$$I_x = \int y^2 dA = \int_0^h y^2 b\,(1 - \sqrt{y/h})\,dy = \frac{bh^3}{21} \longleftarrow$$

Problem 12.4-4 Determine the moment of inertia I_x of a circle of radius r with respect to a diameter (see Case 9, Appendix D).

Solution 12.4-4 Moment of inertia of a circle



$$\text{Width of element} = 2\sqrt{r^2 - y^2}$$

$$dA = 2\sqrt{r^2 - y^2}\,dy$$

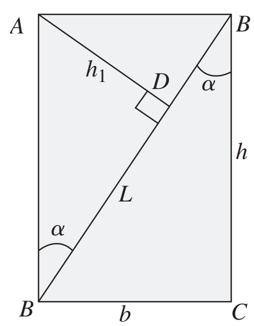
$$I_x = \int y^2 dA = \int_{-r}^{r} y^2(2\sqrt{r^2 - y^2})\,dy$$

$$= \frac{\pi r^4}{4} \longleftarrow$$

Problems 12.4-5 through 12.4-9 are to be solved by considering the area to be a composite area.

Problem 12.4-5 Determine the moment of inertia I_{BB} of a rectangle having sides of lengths b and h with respect to a diagonal of the rectangle (see Case 2, Appendix D).

Solution 12.4-5 Moment of inertia of a rectangle with respect to a diagonal



L = length of diagonal BB

$L = \sqrt{b^2 + h^2}$

h_1 = distance from A to diagonal BB triangle BBC:

$$\sin\alpha = \frac{b}{L}$$

Triangle ADB: $\sin\alpha = \dfrac{h_1}{h}$ $\quad h_1 = h\sin\alpha = \dfrac{bh}{L}$

I_1 = moment of inertia of triangle ABB with respect to its base BB

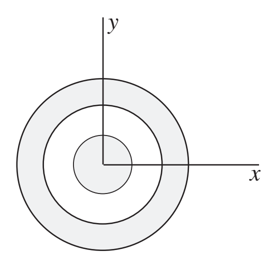
From Case 4, Appendix D:

$$I_1 = \frac{Lh_1^3}{12} = \frac{L}{12}\left(\frac{bh}{L}\right)^3 = \frac{b^3h^3}{12L^2}$$

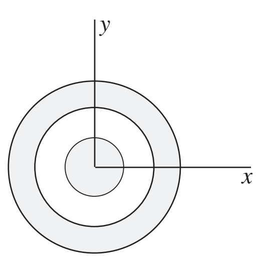
For the rectangle:

$$I_{BB} = 2I_1 = \frac{b^3h^3}{6(b^2 + h^2)} \quad \longleftarrow$$

Problem 12.4-6 Calculate the moment of inertia I_x for the composite circular area shown in the figure. The origin of the axes is at the center of the concentric circles, and the three diameters are 20, 40, and 60 mm.



Solution 12.4-6 Moment of inertia of composite area



Diameters = 20, 40, and 60 mm

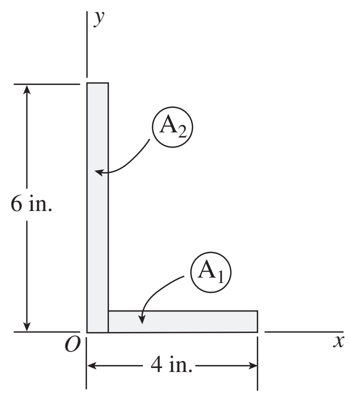
$$I_x = \frac{\pi d^4}{64} \text{ (for a circle)}$$

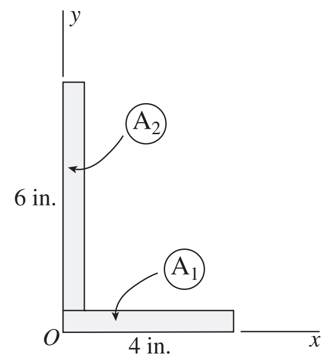
$$I_x = \frac{\pi}{64}[(60)^4 - (40)^4 + (20)^4]$$

$$I_x = 518 \times 10^3 \text{ mm}^4 \quad \longleftarrow$$

Problem 12.4-7 Calculate the moments of inertia I_x and I_y with respect to the x and y axes for the L-shaped area shown in the figure for Prob. 12.3-7.

Solution 12.4-7 Moments of inertia of composite area

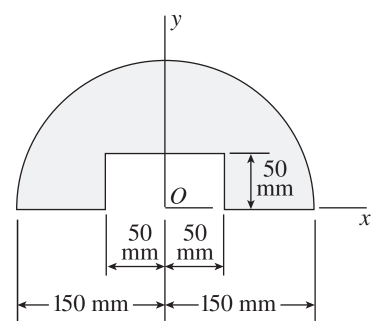




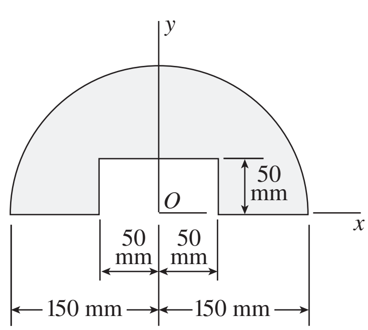
$$I_x = I_1 + I_2$$
$$= \frac{1}{3}(3.5)(0.5)^3 + \frac{1}{3}(0.5)(6)^3$$
$$= 36.1 \text{ in.}^4 \longleftarrow$$
$$I_y = I_3 + I_4$$
$$= \frac{1}{3}(0.5)(4)^3 + \frac{1}{3}(5.5)(0.5)^3$$
$$= 10.9 \text{ in.}^4 \longleftarrow$$

Problem 12.4-8 A semicircular area of radius 150 mm has a rectangular cutout of dimensions 50 mm × 100 mm (see figure).

Calculate the moments of inertia I_x and I_y with respect to the x and y axes. Also, calculate the corresponding radii of gyration r_x and r_y.



Solution 12.4-8 Moments of inertia of composite area



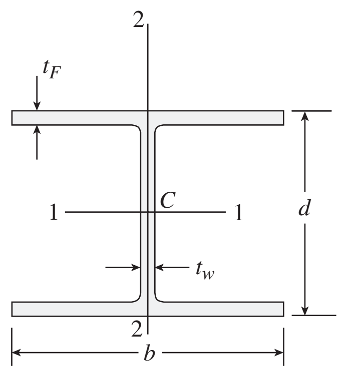
All dimensions in millimeters

$r = 150$ mm $\quad b = 100$ mm $\quad h = 50$ mm

$$I_x = (I_x)_{\text{semicircle}} - (I_x)_{\text{rectangle}} = \frac{\pi r^4}{8} - \frac{bh^3}{3}$$
$$= 194.6 \times 10^6 \text{ mm}^4 \longleftarrow$$
$$I_y = I_x \longleftarrow$$
$$A = \frac{\pi r^2}{2} - bh = 30.34 \times 10^3 \text{ mm}^2$$
$$r_x = \sqrt{I_x/A} = 80.1 \text{ mm} \longleftarrow$$
$$r_y = r_x \longleftarrow$$

Problem 12.4-9 Calculate the moments of inertia I_1 and I_2 of a W 16 × 100 wide-flange section using the cross-sectional dimensions given in Table E-l, Appendix E. (Disregard the cross-sectional areas of the fillets.) Also, calculate the corresponding radii of gyration r_1 and r_2, respectively.

Solution 12.4-9 **Moments of inertia of a wide-flange section**



W 16 × 100 $d = 16.97$ in.

$t_w = t_{web} = 0.585$ in.

$b = 10.425$ in.

$t_F = t_{Flange} = 0.985$ in.

All dimensions in inches.

$$I_1 = \frac{1}{12}bd^3 - \frac{1}{12}(b - t_w)(d - 2t_F)^3$$

$$= \frac{1}{12}(10.425)(16.97)^3 - \frac{1}{12}(9.840)(15.00)^3$$

$$= 1478 \text{ in.}^4 \qquad \text{say,} \quad I_1 = 1480 \text{ in.}^4 \longleftarrow$$

$$I_2 = 2\left(\frac{1}{12}\right)t_F b^3 + \frac{1}{12}(d - 2t_F)t_w^3$$

$$= \frac{1}{6}(0.985)(10.425)^3 + \frac{1}{12}(15.00)(0.585)^3$$

$$= 186.3 \text{ in.}^4 \qquad \text{say,} \quad I_2 = 186 \text{ in.}^4 \longleftarrow$$

$$A = 2(bt_F) + (d - 2t_F)t_w$$

$$= 2(10.425)(0.985) + (15.00)(0.585)$$

$$= 29.31 \text{ in.}^2$$

$$r_1 = \sqrt{I_1/A} = 7.10 \text{ in.} \longleftarrow$$

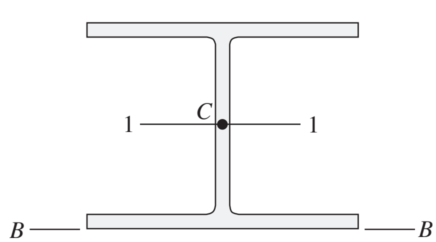
$$r_2 = \sqrt{I_2/A} = 2.52 \text{ in.} \longleftarrow$$

Note that these results are in close agreement with the tabulated values.

Parallel-Axis Theorem

Problem 12.5-1 Calculate the moment of inertia I_b of a W 12 × 50 wide-flange section with respect to its base. (Use data from Table E-l, Appendix E.)

Solution 12.5-1 **Moment of inertia**



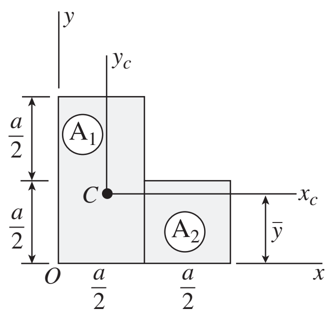
W 12 × 50 $I_1 = 394$ in.^{4} $A = 14.7$ in.^{2}
$d = 12.19$ in.

$$I_b = I_1 + A\left(\frac{d}{2}\right)^2$$

$$= 394 + 14.7(6.095)^2 = 940 \text{ in.}^4 \longleftarrow$$

Problem 12.5-2 Determine the moment of inertia I_c with respect to an axis through the centroid C and parallel to the x axis for the geometric figure described in Prob. 12.3-2.

Solution 12.5-2 Moment of inertia



From Prob. 12.3-2:

$A = 3a^2/4$

$\bar{y} = 5a/12$

$$I_x = \frac{1}{3}\left(\frac{a}{2}\right)(a^3) + \frac{1}{3}\left(\frac{a}{2}\right)\left(\frac{a}{2}\right)^3 = \frac{3a^4}{16}$$

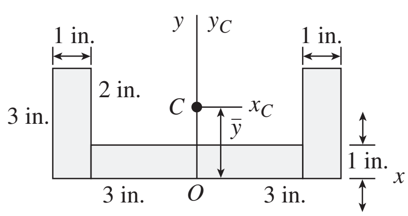
$$I_x = I_{x_C} + A\bar{y}^2$$

$$I_c = I_{x_C} = I_x - A\bar{y}^2 = \frac{3a^4}{16} - \frac{3a^2}{4}\left(\frac{5a}{12}\right)^2$$

$$= \frac{11a^4}{192} \longleftarrow$$

Problem 12.5-3 For the channel section described in Prob. 12.3-3, calculate the moment of inertia I_{x_c} with respect to an axis through the centroid C and parallel to the x axis.

Solution 12.5-3 Moment of inertia



From Prob. 12.3-3:

$A = 10.0\text{ in.}^2$

$\bar{y} = 1.10\text{ in.}$

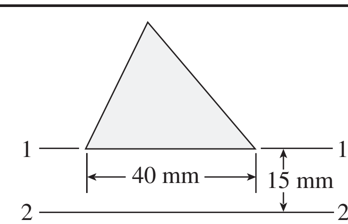
$I_x = 1/3(4)(1)^3 + 2(1/3)(1)(3)^3 = 19.33\text{ in.}^4$

$I_x = I_{x_C} + A\bar{y}^2$

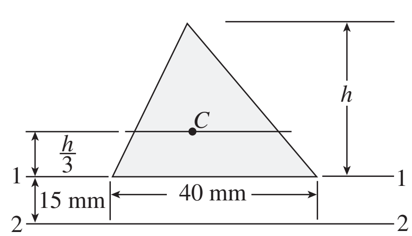
$$I_{x_C} = I_x - A\bar{y}^2 = 19.33 - (10.0)(1.10)^2$$

$$= 7.23\text{ in.}^4 \longleftarrow$$

Problem 12.5-4 The moment of inertia with respect to axis 1-1 of the scalene triangle shown in the figure is $90 \times 10^3\text{ mm}^4$. Calculate its moment of inertia I_2 with respect to axis 2-2.



Solution 12.5-4 Moment of inertia



$b = 40\text{ mm} \qquad I_1 = 90 \times 10^3\text{ mm}^4 \qquad I_1 = bh^3/12$

$$h = \sqrt[3]{\frac{12I_1}{b}} = 30\text{ mm}$$

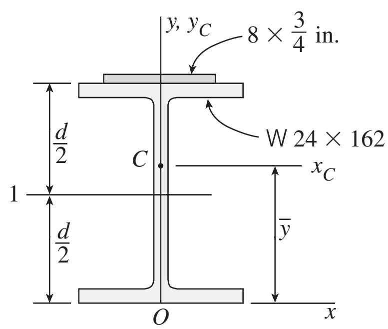
$I_c = bh^3/36 = 30 \times 10^3\text{ mm}^4$

$$I_2 = I_c + Ad^2 = I_c + (bh/2)\,d^2 = 30 \times 10^3$$

$$+ \frac{1}{2}(40)\ (30)\ (25)^2 = 405 \times 10^3\text{ mm}^4 \longleftarrow$$

Problem 12.5-5 For the beam cross section described in Prob. 12.3-5, calculate the centroidal moments of inertia I_{x_c} and I_{y_c} with respect to axes through the centroid C such that the x_c axis is parallel to the x axis and the y_c axis coincides with the y axis.

Solution 12.5-5 Moment of inertia



From Prob. 12.3-5:

$\bar{y} = 13.94$ in.

W 24 × 162 $\quad d = 25.00$ in. $\quad d/2 = 12.5$ in.

$I_1 = 5170 \text{ in.}^4 \quad A = 47.7 \text{ in.}^2$

$I_2 = I_y = 443 \text{ in.}^4$

$I'_{x_c} = I_1 + A(\bar{y} - d/2)^2 = 5170 + (47.7)(1.44)^2$
$= 5269 \text{ in.}^4$

$I'_{y_c} = I_2 = 443 \text{ in.}^4$

PLATE

$I''_{x_c} = 1/12(8)(3/4)^3 + (8)(3/4)(d + 3/8 - \bar{y})^2$
$= 0.2813 + 6(25.00 + 0.375 - 13.94)^2$
$= 0.2813 + 6(11.44)^2 = 785 \text{ in.}^4$

$I''_{y_c} = 1/12(3/4)(8)^3 = 32.0 \text{ in.}^4$

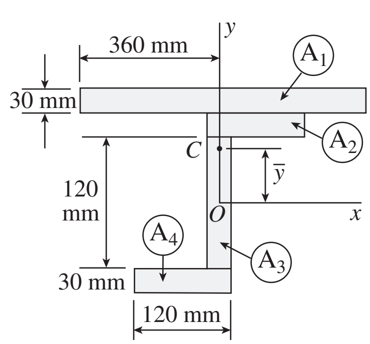
ENTIRE CROSS SECTION

$I_{x_c} = I'_{x_c} + I''_{x_c} = 5269 + 785 = 6050 \text{ in.}^4$ ←

$I_{y_c} = I'_{y_c} + I''_{y_c} = 443 + 32 = 475 \text{ in.}^4$ ←

Problem 12.5-6 Calculate the moment of inertia I_{x_c} with respect to an axis through the centroid C and parallel to the x axis for the composite area shown in the figure for Prob. 12.3-6.

Solution 12.5-6 Moment of inertia



From Prob. 12.3-6:

$\bar{y} = 52.50$ mm $\quad t = 30$ mm $\quad A = 21{,}600 \text{ mm}^2$

A_1: $I_x = 1/12(360)(30)^3 + (360)(30)(105)^2$
$= 119.9 \times 10^6 \text{ mm}^4$

A_2: $I_x = 1/12(120)(30)^3 + (120)(30)(75)^2$
$= 20.52 \times 10^6 \text{ mm}^4$

A_3: $I_x = 1/12(30)(120)^3 = 4.32 \times 10^6 \text{ mm}^4$

A_4: $I_x = 20.52 \times 10^6 \text{ mm}^4$

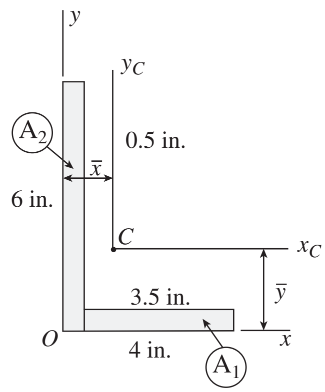
ENTIRE AREA:

$I_x = \sum I_x = 165.26 \times 10^6 \text{ mm}^4$

$I_{x_c} = I_x - A\bar{y}^2 = 165.26 \times 10^6 - (21{,}600)(52.50)^2$
$= 106 \times 10^6 \text{ mm}^4$ ←

Problem 12.5-7 Calculate the centroidal moments of inertia I_{x_c} and I_{y_c} with respect to axes through the centroid C and parallel to the x and y axes, respectively, for the L-shaped area shown in the figure for Prob. 12.3-7.

Solution 12.5-7 Moments of inertia



From Prob. 12.3-7:

$t = 0.5$ in. $\quad A = 4.75$ in.^{2}
$\bar{y} = 1.987$ in.

$\bar{x} = 0.9869$ in.

From Problem 12.4-7:

$I_x = 36.15$ in.^{4}

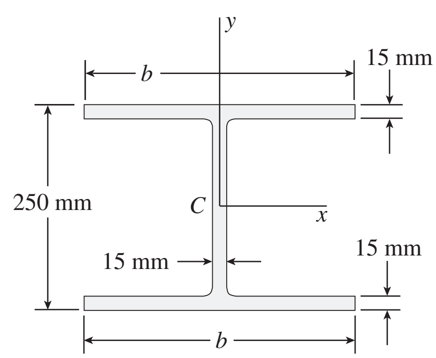
$I_y = 10.90$ in.^{4}

$I_{x_c} = I_x - A\bar{y}^2 = 36.15 - (4.75)\,(1.987)^2$
$= 17.40$ in.^{4} ←

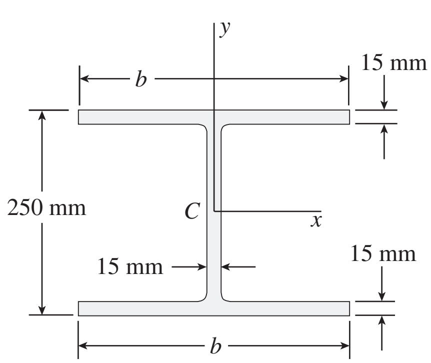
$I_{y_c} = I_y - A\bar{x}^2 = 10.90 - (4.75)(0.9869)^2$
$= 6.27$ in.^{4} ←

Problem 12.5-8 The wide-flange beam section shown in the figure has a total height of 250 mm and a constant thickness of 15 mm.

Determine the flange width b if it is required that the centroidal moments of inertia I_x and I_y be in the ratio 3 to 1, respectively.



Solution 12.5-8 Wide-flange beam



$t = 15$ mm $\qquad b =$ flange width

All dimensions in millimeters.

$$I_x = \frac{1}{12}(b)(250)^3 - \frac{1}{12}(b - 15)(220)^3$$
$$= 0.4147 \times 10^6 b + 13.31 \times 10^6 \text{ (mm)}^4$$

$$I_y = 2\left(\frac{1}{12}\right)(15)(b)^3 + \frac{1}{12}(220)(15)^3$$
$$= 25b^3 + 61{,}880 \text{ (mm}^4\text{)}$$

Equate I_x to $3I_y$ and rearrange:

$7.5b^3 - 0.4147 \times 10^6 b - 13.12 \times 10^5 = 0$

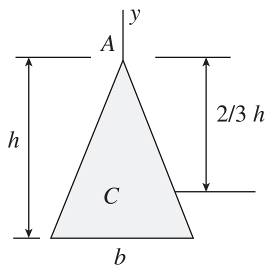
Solve numerically:

$b = 250$ mm ←

Polar Moments of Inertia

Problem 12.6-1 Determine the polar moment of inertia I_P of an isosceles triangle of base b and altitude h with respect to its apex (see Case 5, Appendix D)

Solution 12.6-1 Polar moment of inertia



POINT C (CENTROID) FROM CASE 5:

$$(I_P)_c = \frac{bh}{144}(4h^2 + 3b^2)$$

POINT A (APEX):

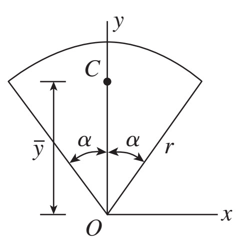
$$I_P = (I_P)_c + A\left(\frac{2h}{3}\right)^2$$

$$= \frac{bh}{144}(4h^2 + 3b^2) + \frac{bh}{2}\left(\frac{2h}{3}\right)^2$$

$$I_P = \frac{bh}{48}(b^2 + 12h^2) \longleftarrow$$

Problem 12.6-2 Determine the polar moment of inertia $(I_P)_C$ with respect to the centroid C for a circular sector (see Case 13, Appendix D).

Solution 12.6-2 Polar moment of inertia



POINT O (ORIGIN) FROM CASE 13:

$$(I_P)_o = \frac{\alpha r^4}{2} \qquad (\alpha = \text{radians})$$

$$A = \alpha r^2$$

$$\bar{y} = \frac{2r \sin \alpha}{3\alpha}$$

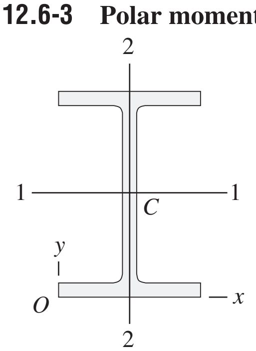
POINT C (CENTROID):

$$(I_P)_C = (I_P)_O - A\bar{y}^2 = \frac{\alpha r^4}{2} - \alpha r^2\left(\frac{2r \sin \alpha}{3\alpha}\right)^2$$

$$= \frac{r^4}{18\alpha}(9\alpha^2 - 8 \sin^2\alpha) \longleftarrow$$

Problem 12.6-3 Determine the polar moment of inertia I_P for a W 8 × 21 wide-flange section with respect to one of its outermost corners.

Solution 12.6-3 Polar moment of inertia



W 8 × 21 $\quad I_1 = 75.3 \text{ in.}^4 \quad I_2 = 9.77 \text{ in.}^4$

$A = 6.16 \text{ in.}^2$

Depth $d = 8.28$ in.

Width $b = 5.27$ in.

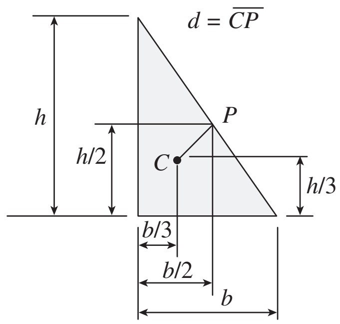
$$I_x = I_1 + A(d/2)^2 = 75.3 + 6.16(4.14)^2 = 180.9 \text{ in.}^4$$

$$I_y = I_2 + A(b/2)^2 = 9.77 + 6.16(2.635)^2 = 52.5 \text{ in.}^4$$

$$I_P = I_x + I_y = 233 \text{ in.}^4 \longleftarrow$$

Problem 12.6-4 Obtain a formula for the polar moment of inertia I_P with respect to the midpoint of the hypotenuse for a right triangle of base b and height h (see Case 6, Appendix D).

Solution 12.6-4 Polar moment of inertia



POINT C FROM CASE 6:

$$(I_P)_c = \frac{bh}{36}(h^2 + b^2)$$

POINT P:

$$I_P = (I_P)_c + Ad^2$$

$$A = \frac{bh}{2}$$

$$d^2 = \left(\frac{b}{2} - \frac{b}{3}\right)^2 + \left(\frac{h}{2} - \frac{h}{3}\right)^2$$

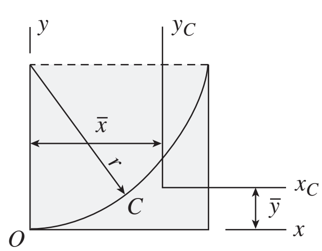
$$= \frac{b^2}{36} + \frac{h^2}{36} = \frac{b^2 + h^2}{36}$$

$$I_P = \frac{bh}{36}(h^2 + b^2) + \frac{bh}{2}\left(\frac{b^2 + h^2}{36}\right)$$

$$= \frac{bh}{24}(b^2 + h^2) \quad \longleftarrow$$

Problem 12.6-5 Determine the polar moment of inertia $(I_P)_C$ with respect to the centroid C for a quarter-circular spandrel (see Case 12, Appendix D).

Solution 12.6-5 Polar moment of inertia



POINT O FROM CASE 12:

$$I_x = \left(1 - \frac{5\pi}{16}\right)r^4$$

$$\bar{y} = \frac{(10 - 3\pi)r}{3(4 - \pi)}$$

$$A = \left(1 - \frac{\pi}{4}\right)r^2$$

POINT C (CENTROID):

$$Ix_c = I_x - A\bar{y}^2 = \left(1 - \frac{5\pi}{16}\right)r^4$$

$$-\left(1 - \frac{\pi}{4}\right)(r^2)\left[\frac{(10 - 3\pi)r}{3(4 - \pi)}\right]^2$$

COLLECT TERMS AND SIMPLIFY:

$$I_{x_C} = \frac{r^4}{144}\left(\frac{176 - 84\pi + 9\pi^2}{4 - \pi}\right)$$

$$I_{y_C} = I_{x_C} \quad \text{(by symmetry)}$$

$$(I_P)_c = 2I_{x_C} = \frac{r^4}{72}\left(\frac{176 - 84\pi + 9\pi^2}{4 - \pi}\right) \quad \longleftarrow$$

Products of Inertia

Problem 12.7-1 Using integration, determine the product of inertia I_{xy} for the parabolic semisegment shown in Fig. 12-5 (see also Case 17 in Appendix D).

Solution 12.7-1 Product of inertia

Product of inertia of element dA with respect to axes through its own centroid equals zero.

$$dA = y\,dx = h\left(1 - \frac{x^2}{b^2}\right)dx$$

dI_{xy} = product of inertia of element dA with respect to xy axes

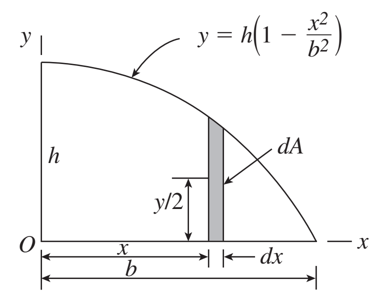
$d_1 = x \quad d_2 = y/2$

Parallel-axis theorem applied to element dA:

$$dI_{xy} = 0 + (dA)(d_1 d_2) = (y\,dx)(x)(y/2)$$

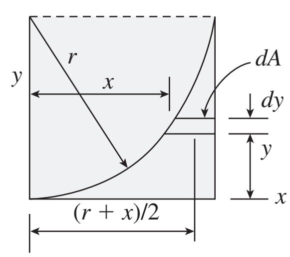
$$= \frac{h^2 x}{2}\left(1 - \frac{x^2}{b^2}\right)^2 dx$$

$$I_{xy} = \int dI_{xy} = \frac{h^2}{2}\int_0^b x\left(1 - \frac{x^2}{b^2}\right)^2 dx = \frac{b^2h^2}{12} \quad \longleftarrow$$



Problem 12.7-2 Using integration, determine the product of inertia I_{xy} for the quarter-circular spandrel shown in Case 12, Appendix D.

Solution 12.7-2 Product of inertia



EQUATION OF CIRCLE:

$x^2 + (y - r)^2 = r^2$

or $r^2 - x^2 = (y - r)^2$

ELEMENT dA:

d_1 = distance to its centroid in x direction $= (r + x)/2$

d_2 = distance to its centroid in y direction $= y$

dA = area of element $= (r - x)\,dy$

Product of inertia of element dA with respect to axes through its own centroid equals zero.

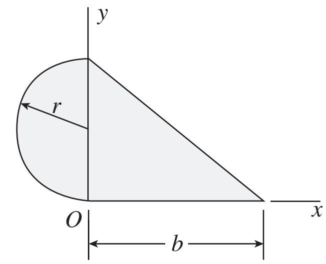
Parallel-axis theorem applied to element dA:

$$dI_{xy} = 0 + (dA)(d_1 d_2) = (r - x)(dy)\left(\frac{r + x}{2}\right)(y)$$

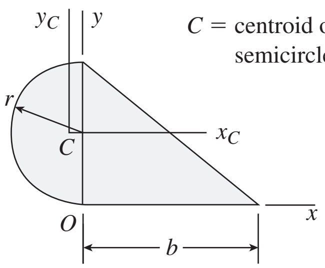
$$= \frac{1}{2}(r^2 - x^2)\,y\,dy = \frac{1}{2}(y - r)^2 y\,dy$$

$$I_{xy} = 1/2\int_0^r y(y - r)^2\,dy = \frac{r^4}{24} \quad \longleftarrow$$

Problem 12.7-3 Find the relationship between the radius r and the distance b for the composite area shown in the figure in order that the product of inertia I_{xy} will be zero.



Solution 12.7-3 Product of inertia



TRIANGLE (CASE 7):

$$I_{xy} = \frac{b^2h^2}{24} = \frac{b^2(2r)^2}{24} = \frac{b^2r^2}{6}$$

SEMICIRCLE (CASE 10):

$$I_{xy} = I_{x_cy_c} + Ad_1d_2$$

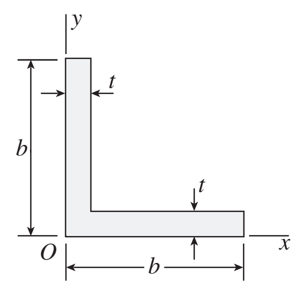
$$I_{x_cy_c} = 0 \qquad A = \frac{\pi r^2}{2} \qquad d_1 = r \qquad d_2 = -\frac{4r}{3\pi}$$

$$I_{xy} = 0 + \left(\frac{\pi r^2}{2}\right)(r)\left(-\frac{4r}{3\pi}\right) = -\frac{2r^4}{3}$$

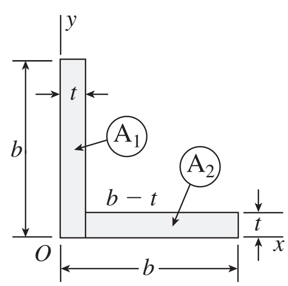
COMPOSITE AREA ($I_{xy} = 0$)

$$I_{xy} = \frac{b^2r^2}{6} - \frac{2r^4}{3} = 0 \qquad \therefore b = 2r \quad \longleftarrow$$

Problem 12.7-4 Obtain a formula for the product of inertia I_{xy} of the symmetrical L-shaped area shown in the figure.



Solution 12.7-4 Product of inertia



AREA 1:

$$(I_{xy})_1 = \frac{t^2b^2}{4}$$

AREA 2:

$$(I_{xy})_2 = I_{x_cy_c} + A_2d_1d_2$$

$$= 0 + (b - t)(t)(t/2)\left(\frac{b+t}{2}\right)$$

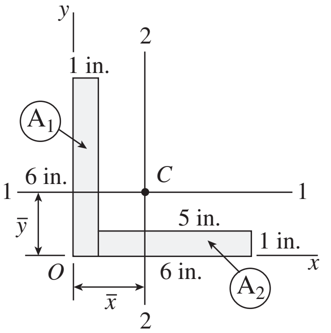
$$= \frac{t^2}{4}(b^2 - t^2)$$

COMPOSITE AREA:

$$I_{xy} = (I_{xy})_1 + (I_{xy})_2 = \frac{t^2}{4}(2b^2 - t^2) \quad \longleftarrow$$

Problem 12.7-5 Calculate the product of inertia I_{12} with respect to the centroidal axes 1-1 and 2-2 for an L 6 × 6 × 1 in. angle section (see Table E-4, Appendix E). (Disregard the cross-sectional areas of the fillet and rounded corners.)

Solution 12.7-5 Product of inertia



All dimensions in inches.

$A_1 = (6)(1) = 6.0 \text{ in.}^2$
$A_2 = (5)(1) = 5.0 \text{ in.}^2$
$A = A_1 + A_2 = 11.0 \text{ in.}^2$

With respect to the x axis:

$$Q_1 = (6.0 \text{ in.}^2)\left(\frac{6 \text{ in.}}{2}\right) = 18.0 \text{ in.}^3$$

$$Q_2 = (5.0 \text{ in.}^2)\left(\frac{1.0 \text{ in.}}{2}\right) = 2.5 \text{ in.}^3$$

$$\bar{y} = \frac{Q_1 + Q_2}{A} = \frac{20.5 \text{ in.}^3}{11.0 \text{ in.}^2} = 1.8636 \text{ in.}$$

$\bar{x} = \bar{y} = 1.8636$ in.

Coordinates of centroid of aera A_1 with respect to 1–2 axes:

$d_1 = -(\bar{x} - 0.5) = -1.3636$ in.

$d_2 = 3.0 - \bar{y} = 1.1364$ in.

Product of inertia of area A_1 with respect to 1-2 axes:

$I'_{12} = 0 + A_1 d_1 d_2$

$= (6.0 \text{ in.}^2)(-1.3636 \text{ in.})(1.1364 \text{ in.}) = -9.2976 \text{ in.}^4$

Coordinates of centroid of area A_2 with respect to 1–2 axes:

$d_1 = 3.5 - \bar{x} = 1.6364$ in.

$d_2 = -(\bar{y} - 0.5) = -1.3636$ in.

Product of inertia of area A_2 with respect to 1-2 axes:

$I''_{12} = 0 + A_2 d_1 d_2$

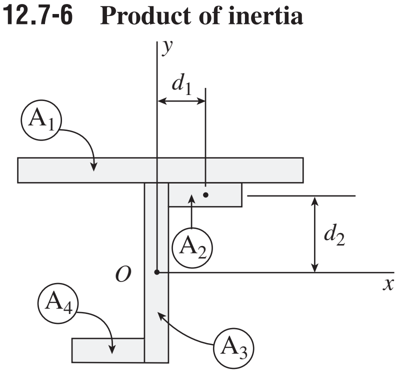
$= (5.0 \text{ in.}^2)(1.6364 \text{ in.})(-1.3636 \text{ in.})$

$= -11.1573 \text{ in.}^4$

ANGLE SECTION: $I_{12} = I'_{12} + I''_{12} = -20.5 \text{ in.}^4$ ←

Problem 12.7-6 Calculate the product of inertia I_{xy} for the composite area shown in Prob. 12.3-6.

Solution 12.7-6 Product of inertia



All dimensions in millimeters

$A_1 = 360 \times 30$ mm $\quad A_2 = 90 \times 30$ mm
$A_3 = 180 \times 30$ mm $\quad A_3 = 90 \times 30$ mm
$d_1 = 60$ mm $\quad d_2 = 75$ mm

AREA A_1: $(I_{xy})_1 = 0$ (By symmetry)

AREA A_2: $(I_{xy})_2 = 0 + A_2 d_1 d_2 = (90 \times 30)(60)(75)$
$= 12.15 \times 10^6 \text{ mm}^4$

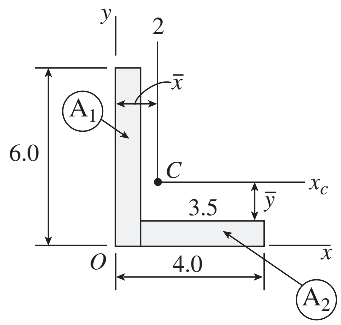
AREA A_3: $(I_{xy})_3 = 0$ (By symmetry)

AREA A_4: $(I_{xy})_4 = (I_{xy})_2 = 12.15 \times 10^6 \text{ mm}^4$

$I_{xy} = (I_{xy})_1 + (I_{xy})_2 + (I_{xy})_3 + (I_{xy})_4$
$= (2)(12.15 \times 10^6 \text{ mm}^4)$
$= 24.3 \times 10^6 \text{ mm}^4$ ←

Problem 12.7-7 Determine the product of inertia $I_{x_c y_c}$ with respect to centroidal axes x_c and y_c parallel to the x and y axes, respectively, for the L-shaped area shown in Prob. 12.3-7.

Solution 12.7-7 Product of inertia



All dimensions in inches.

$A_1 = (6.0)(0.5) = 3.0 \text{ in.}^2$

$A_2 = (3.5)(0.5) = 1.75 \text{ in.}^2$

$A = A_1 + A_2 = 4.75 \text{ in.}^2$

With respect to the x axis:

$Q_1 = A_1 \bar{y}_1 = (3.0 \text{ in.}^2)(3.0 \text{ in.}) = 9.0 \text{ in.}^3$

$Q_2 = A_2 \bar{y}_2 = (1.75 \text{ in.}^2)(0.25 \text{ in.}) = 0.4375 \text{ in.}^3$

$$\bar{y} = \frac{Q_1 + Q_2}{A} = \frac{9.4375 \text{ in.}^3}{4.75 \text{ in.}^2} = 1.9868 \text{ in.}$$

With respect to the y axis:

$Q_1 = A_1 \bar{x}_1 = (3.0 \text{ in.}^2)(0.25 \text{ in.}) = 0.75 \text{ in.}^3$

$Q_2 = A_2 \bar{x}_2 = (1.75 \text{ in.}^2)(2.25 \text{ in.}) = 3.9375 \text{ in.}^3$

$$\bar{x} = \frac{Q_1 + Q_2}{A} = \frac{4.6875 \text{ in.}^3}{4.75 \text{ in.}^2} = 0.98684 \text{ in.}$$

Product of inertia of area A_1 with respect to xy axes:

$$(I_{xy})_1 = (I_{xy})_{\text{centroid}} + A_1 d_1 d_2$$
$$= 0 + (3.0 \text{ in.}^2)(0.25 \text{ in.})(3.0 \text{ in.}) = 2.25 \text{ in.}^4$$

Product of inertia of area A_2 with respect to xy axes:

$$(I_{xy})_2 = (I_{xy})_{\text{centroid}} + A_2 d_1 d_2$$
$$= 0 + (1.75 \text{ in.}^2)(2.25 \text{ in.})(0.25 \text{ in.}) = 0.98438 \text{ in.}^4$$

Angle section

$$I_{xy} = (I_{xy})_1 + (I_{xy})_2 = 3.2344 \text{ in.}^4$$

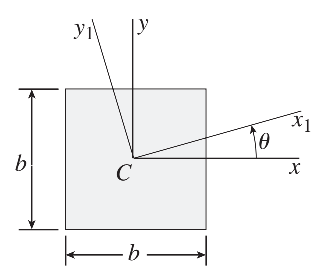
Centroidal axes

$$I_{x_c y_c} = I_{xy} - A\bar{x}\,\bar{y}$$
$$= 3.2344 \text{ in.}^4 - (4.75 \text{ in.}^2)(0.98684 \text{ in.})(1.9868 \text{ in.})$$
$$= -6.079 \text{ in.}^4 \quad \longleftarrow$$

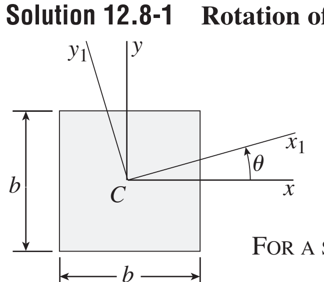
Rotation of Axes

The problems for Section 12.8 are to be solved by using the transformation equations for moments and products of inertia.

Problem 12.8-1 Determine the moments of inertia I_{x_1} and I_{y_1} and the product of inertia $I_{x_1 y_1}$ for a square with sides b, as shown in the figure. (Note that the $x_1 y_1$ axes are centroidal axes rotated through an angle θ with respect to the xy axes.)



Solution 12.8-1 Rotation of axes



For a square:

$$I_x = I_y = -\frac{b^4}{12} \qquad I_{xy} = 0$$

Eq. (12-25):

$$I_{x_1} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta$$
$$= \frac{I_x + I_y}{2} + 0 - 0 = \frac{b^4}{12} \quad \longleftarrow$$

Eq. (12-29):

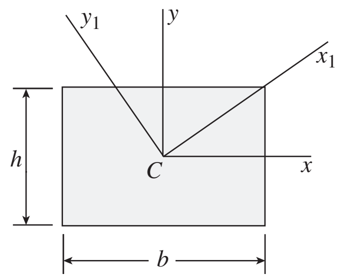
$$I_{x_1} + I_{y_1} = I_x + I_y \qquad \therefore I_{y_1} = \frac{b^4}{12} \quad \longleftarrow$$

Eq. (12-27):

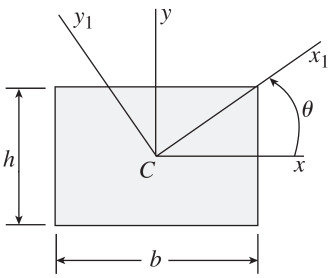
$$I_{x_1 y_1} = \frac{I_x - I_y}{2}\sin 2\theta + I_{xy}\cos 2\theta = 0 \quad \longleftarrow$$

Since θ may be any angle, we see that all moments of inertia are the same and the product of inertia is always zero (for axes through the centroid C).

Problem 12.8-2 Determine the moments and product of inertia with respect to the x_1y_1 axes for the rectangle shown in the figure. (Note that the x_1 axis is a diagonal of the rectangle.)



Solution 12.8-2 **Rotation of axes (rectangle)**



CASE 1:

$$I_x = \frac{bh^3}{12} \qquad I_y = \frac{hb^3}{12} \qquad I_{xy} = 0$$

ANGLE OF ROTATION:

$$\cos\theta = \frac{b}{\sqrt{b^2 + h^2}} \qquad \sin\theta = \frac{h}{\sqrt{b^2 + h^2}}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = \frac{b^2 - h^2}{b^2 + h^2}$$

$$\sin 2\theta = 2\sin\theta\cos\theta = \frac{2bh}{b^2 + h^2}$$

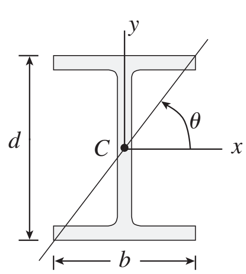
SUBSTITUTE INTO EQS. (12-25), (12-29), AND (12-27) AND SIMPLIFY:

$$I_{x_1} = \frac{b^3h^3}{6(b^2 + h^2)} \longleftarrow \qquad I_{y_1} = \frac{bh(b^4 + h^4)}{12(b^2 + h^2)} \longleftarrow$$

$$I_{x_1y_1} = \frac{b^2h^2(h^2 - b^2)}{12(b^2 + h^2)} \longleftarrow$$

Problem 12.8-3 Calculate the moment of inertia I_d for a W 12 × 50 wide-flange section with respect to a diagonal passing through the centroid and two outside corners of the flanges. (Use the dimensions and properties given in Table E-1.)

Solution 12.8-3 **Rotation of axes**



W 12 × 50 $\quad I_x = 394 \text{ in.}^4$
$I_y = 56.3 \text{ in.}^4 \quad I_{xy} = 0$
Depth $d = 12.19$ in.
Width $b = 8.080$ in.

$$\text{Tan}\,\theta = \frac{d}{b} = \frac{12.19}{8.080} = 1.509$$

$$\theta = 56.46^\circ \qquad 2\theta = 112.92^\circ$$

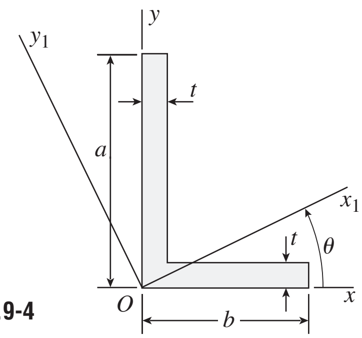
EQ. (12-25):

$$I_d = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta$$

$$= \frac{394 + 56.3}{2} + \frac{394 - 56.3}{2}\cos(112.92^\circ) - 0$$

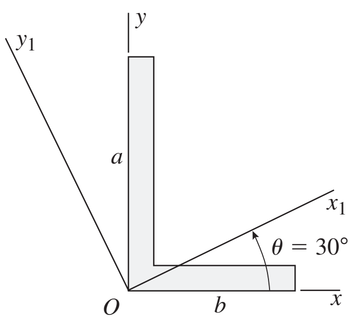
$$= 225 \text{ in.}^4 - 66 \text{ in.}^4 = 159 \text{ in.}^4 \longleftarrow$$

Problem 12.8-4 Calculate the moments of inertia I_{x_1} and I_{y_1} and the product of inertia $I_{x_1y_1}$ with respect to the x_1y_1 axes for the L-shaped area shown in the figure if $a = 150$ mm, $b = 100$ mm, $t = 15$ mm, and $\theta = 30°$.



Probs. 12.8-4 and 12.9-4

Solution 12.8-4 **Rotation of axes**



All dimensions in millimeters.

$a = 150$ mm $\quad b = 100$ mm

$t = 15$ mm $\quad \theta = 30°$

$$I_x = \frac{1}{3}ta^3 + \frac{1}{3}(b - t)\,t^3$$

$$= \frac{1}{3}(15)(150)^3 + \frac{1}{3}(85)(15)^3$$

$$= 16.971 \times 10^6 \text{ mm}^4$$

$$I_y = \frac{1}{3}(a - t)\,t^3 + \frac{1}{3}tb^3$$

$$= \frac{1}{3}(135)(15)^3 + \frac{1}{3}(15)(100)^3$$

$$= 5.152 \times 10^6 \text{ mm}^4$$

$$I_{xy} = \frac{1}{4}t^2a^2 + Ad_1d_2 \qquad A = (b-t)(t)$$

$$d_1 = t + \frac{b - t}{2} \qquad d_2 = \frac{t}{2}$$

$$I_{xy} = \frac{1}{4}(15)^2(150)^2 + (85)(15)(57.5)(7.5)$$

$$= 1.815 \times 10^6 \text{ mm}^4$$

SUBSTITUTE into Eq. (12-25) with $\theta = 30°$:

$$I_{x_1} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta$$

$$= 12.44 \times 10^6 \text{ mm}^4 \longleftarrow$$

SUBSTITUTE into Eq. (12-25) with $\theta = 120°$:

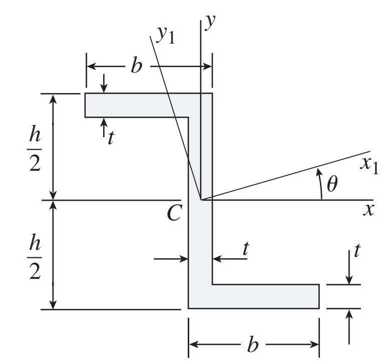
$$I_{y_1} = 9.68 \times 10^6 \text{ mm}^4 \longleftarrow$$

SUBSTITUTE into Eq. (12-27) with $\theta = 30°$:

$$I_{x_1y_1} = \frac{I_x - I_y}{2}\sin 2\theta + I_{xy}\cos 2\theta$$

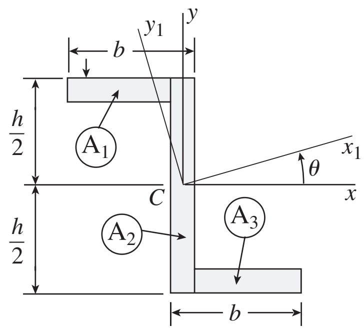
$$= 6.03 \times 10^6 \text{ mm}^4 \longleftarrow$$

Problem 12.8-5 Calculate the moments of inertia I_{x_1} and I_{y_1} and the product of inertia $I_{x_1y_1}$ with respect to the x_1y_1 axes for the Z-section shown in the figure if $b = 3$ in., $h = 4$ in., $t = 0.5$ in., and $\theta = 60°$.



Probs. 12.8-5, 12.8-6, 12.9-5 and 12.9-6

Solution 12.8-5 Rotation of axes



All dimensions in inches.

$b = 3.0$ in. $h = 4.0$ in. $t = 0.5$ in. $\theta = 60°$

Moment of inertia I_x

Area A_1: $I'_x = \frac{1}{12}(b-t)(t^3) + (b-t)(t)\left(\frac{h}{2} - \frac{t}{2}\right)^2$

$= 3.8542 \text{ in.}^4$

Area A_2: $I''_x = \frac{1}{12}(t)(h^3) = 2.6667 \text{ in.}^4$

Area A_3: $I'''_x = I'_x = 3.8542 \text{ in.}^4$

$I_x = I'_x + I''_x + I'''_x = 10.3751 \text{ in.}^4$

Moment of inertia I_y

Area A_1: $I'_y = \frac{1}{12}(t)(b-t)^3 + (b-t)(t)\left(\frac{b}{2}\right)^2$

$= 3.4635 \text{ in.}^4$

Area A_2: $I''_y = \frac{1}{12}(h)(t^3) = 0.0417 \text{ in.}^4$

Area A_3: $I'''_y = I'_y = 3.4635 \text{ in.}^4$

$I_y = I'_y + I''_y + I'''_y = 6.9688 \text{ in.}^4$

Product of inertia I_{xy}

Area A_1: $I'_{xy} = 0 + (b-t)(t)\left(-\frac{b}{2}\right)\left(\frac{h}{2} - \frac{t}{2}\right)$

$= -\frac{1}{4}(bt)(b-t)(h-t) = -3.2813 \text{ in.}^4$

Area A_2: $I''_{xy} = 0$ Area A_3: $I'''_{xy} = I'_{xy}$

$I_{xy} = I'_{xy} + I''_{xy} + I'''_{xy} = -6.5625 \text{ in.}^4$

Substitute into Eq. (12-25) with $\theta = 60°$:

$$I_{x_1} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta$$

$= 13.50 \text{ in.}^4$ ←

Substitute into Eq. (12-25) with $\theta = 150°$:

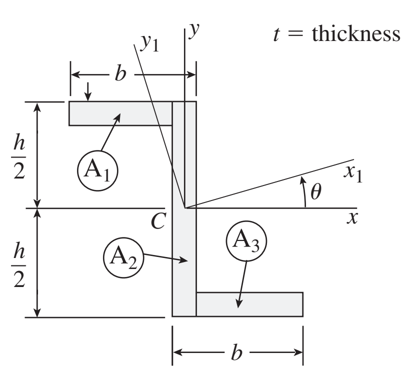
$I_{y_1} = 3.84 \text{ in.}^4$ ←

Substitute into Eq. (12-27) with $\theta = 60°$:

$$I_{x_1 y_1} = \frac{I_x - I_y}{2}\sin 2\theta + I_{xy}\cos 2\theta = 4.76 \text{ in.}^4$$ ←

Problem 12.8-6 Solve the preceding problem if $b = 80$ mm, $h = 120$ mm, $t = 12$ mm, and $\theta = 30°$.

Solution 12.8-6 Rotation of axes



t = thickness

All dimensions in millimeters.

$b = 80$ mm $h = 120$ mm

$t = 12$ mm $\theta = 30°$

Moment of inertia I_x

Area A_1: $I'_x = \frac{1}{12}(b-t)(t^3) + (b-t)(t)\left(\frac{h}{2} - \frac{t}{2}\right)^2$

$= 2.3892 \times 10^6 \text{ mm}^4$

Area A_2: $I''_x = \frac{1}{12}(t)(h^3) = 1.7280 \times 10^6 \text{ mm}^4$

Area A_3: $I'''_x = I'_x = 2.3892 \times 10^6 \text{ mm}^4$

$I_x = I'_x + I''_x + I'''_x = 6.5065 \times 10^6 \text{ mm}^4$

MOMENT OF INERTIA I_y

Area A_1: $I'_y = \frac{1}{12}(t)(b-t)^3 + (b-t)(t)\left(\frac{b}{2}\right)^2$

$= 1.6200 \times 10^6 \text{ mm}^4$

Area A_2: $I''_y = \frac{1}{12}(h)(t^3) = 0.01728 \times 10^6 \text{ mm}^4$

Area A_3: $I'''_y = I'_y = 1.6200 \times 10^6 \text{ mm}^4$

$I_y = I'_y + I''_y + I'''_y = 3.2573 \times 10^6 \text{ mm}^4$

PRODUCT OF INERTIA I_{xy}

Area A_1: $I'_{xy} = 0 + (b-t)(t)\left(-\frac{b}{2}\right)\left(\frac{h}{2} - \frac{t}{2}\right)$

$= -\frac{1}{4}(bt)(b-t)(h-t) =$

Area A_2: $I''_{xy} = 0$ Area A_3: $I'''_{xy} = I'_{xy}$

$I_{xy} = I'_{xy} + I''_{xy} + I'''_{xy} = -3.5251 \times 10^6 \text{ mm}^4$

SUBSTITUTE into Eq. (12-25) with $\theta = 30º$:

$$I_{x_1} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta$$

$= 8.75 \times 10^6 \text{ mm}^4$ ←

SUBSTITUTE into Eq. (12-25) with $\theta = 120º$:

$I_{y_1} = 1.02 \times 10^6 \text{ mm}^4$ ←

SUBSTITUTE into Eq. (12-27) with $\theta = 30º$:

$$I_{x_1y_1} = \frac{I_x - I_y}{2}\sin 2\theta + I_{xy}\cos 2\theta$$

$= -0.356 \times 10^6 \text{ mm}^4$ ←

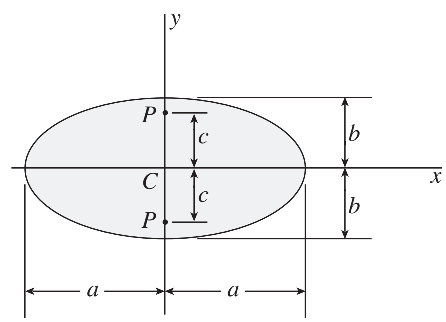
Principal Axes, Principal Points, and Principal Moments of Inertia

Problem 12.9-1 An ellipse with major axis of length $2a$ and minor axis of length $2b$ is shown in the figure.

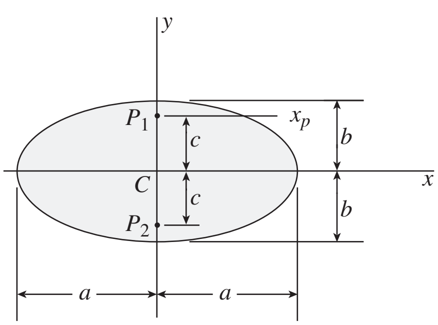
(a) Determine the distance c from the centroid C of the ellipse to the principal points P on the minor axis (y axis).

(b) For what ratio a/b do the principal points lie on the circumference of the ellipse?

(c) For what ratios do they lie inside the ellipse?



Solution 12.9-1 **Principal points of an ellipse**



(a) LOCATION OF PRINCIPAL POINTS

At a principal point, all moments of inertia are equal.

At point P_1: $I_{x_p} = I_y$ Eq. (1)

From Case 16: $I_y = \frac{\pi ba^3}{4}$

$I_x = \frac{\pi ab^3}{4}$ $A = \pi ab$

Parallal-axis theorem:

$I_{x_p} = I_x + Ac^2 = \frac{\pi ab^3}{4} + \pi abc^2$

Substitute into Eq. (1):

$\frac{\pi ab^3}{4} + \pi abc^2 = \frac{\pi ba^3}{4}$

Solve for c: $c = \frac{1}{2}\sqrt{a^2 - b^2}$ ←

(b) PRINCIPAL POINTS ON THE CIRCUMFERENCE

$\therefore c = b$ and $b = \dfrac{1}{2}\sqrt{a^2 - b^2}$

Solve for ratio $\dfrac{a}{b}$: $\quad \dfrac{a}{b} = \sqrt{5}$ ←

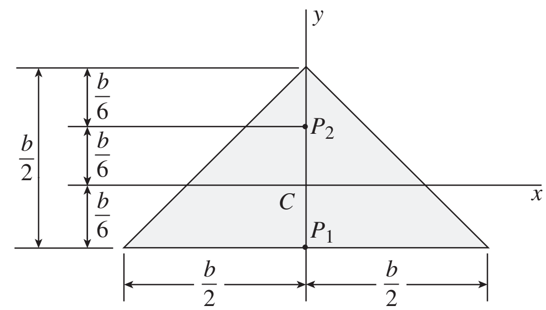
(c) PRINCIPAL POINTS INSIDE THE ELLIPSE

$\therefore 0 \le c < b \qquad$ For $c = 0$: $\quad a = b$ and $\dfrac{a}{b} = 1$

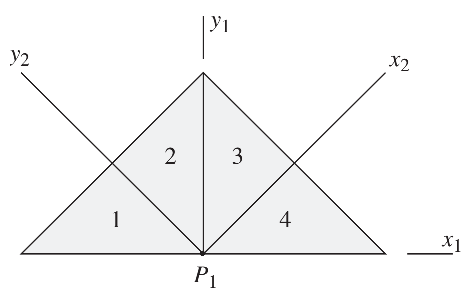
For $c = b$: $\quad \dfrac{a}{b} = \sqrt{5}$

$\therefore 1 \le \dfrac{a}{b} < \sqrt{5}$ ←

Problem 12.9-2 Demonstrate that the two points P_1 and P_2, located as shown in the figure, are the principal points of the isosceles right triangle.



Solution 12.9-2 Principal points of an isosceles right triangle



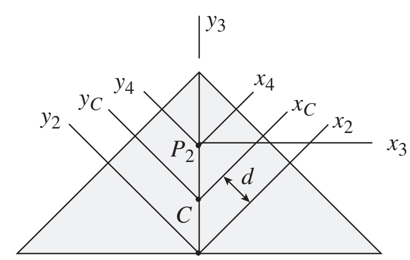
CONSIDER POINT P_1:

$I_{x_1y_1} = 0$ because y_1 is an axis of symmetry.

$I_{x_2y_2} = 0$ because areas 1 and 2 are symmetrical about the y_2 axis and areas 3 and 4 are symmetrical about the x_2 axis.

Two different sets of principal axes exist at point P_1.

$\therefore P_1$ is a principal point ←



CONSIDER POINT P_2:

$I_{x_3y_3} = 0$ because y_2 is an axis of symmetry.

$I_{x_2y_2} = 0$ (see above).

Parallel-axis theorem:

$$I_{x_2y_2} = I_{x_cy_c} + Ad_1d_2 \qquad A = \frac{b^2}{4} \quad d = d_1 = d_2 = \frac{b}{6\sqrt{2}}$$

$$I_{x_cy_c} = -\left(\frac{b^2}{4}\right)\left(\frac{b}{6\sqrt{2}}\right)^2 = -\frac{b^4}{288}$$

Parallel-axis theorem:

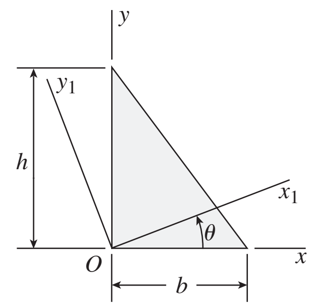
$$I_{x_4y_4} = I_{x_cy_c} + Ad_1d_2 \qquad d_1 = d_2 = -\frac{b}{6\sqrt{2}}$$

$$I_{x_4y_4} = -\frac{b^4}{288} + \frac{b^2}{4}\left(-\frac{b}{6\sqrt{2}}\right)^2 = 0$$

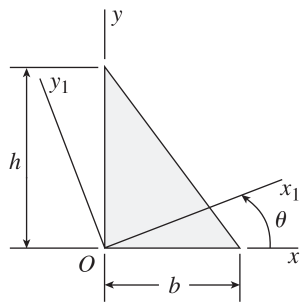
Two different sets of principal axes (x_3y_3 and x_4y_4) exist at point P_2.

$\therefore P_2$ is a principal point ←

Problem 12.9-3 Determine the angles θ_{p_1} and θ_{p_2} defining the orientations of the principal axes through the origin O for the right triangle shown in the figure if $b = 6$ in. and $h = 8$ in. Also, calculate the corresponding principal moments of inertia I_1 and I_2.



Solution 12.9-3 **Principal axes**



RIGHT TRIANGLE

$b = 6.0$ in. $\quad h = 8.0$ in.

CASE 7:

$$I_x = \frac{bh^3}{12} = 256\text{ in.}^4$$

$$I_y = \frac{hb^3}{12} = 144\text{ in.}^4$$

$$I_{xy} = \frac{b^2h^2}{24} = 96\text{ in.}^4$$

EQ. (12-30): $\tan 2\theta_p = -\dfrac{2I_{xy}}{I_x - I_y} = -1.71429$

$2\theta_p = -59.744°$ and $120.256°$

$\theta_p = -29.872°$ and $60.128°$

SUBSTITUTE into Eq. (12-25) with $\theta = -29.872°$:

$I_{x_1} = 311.1\text{ in.}^4$

SUBSTITUTE into Eq. (12-25) with $\theta = 60.128°$:

$I_{x_1} = 88.9\text{ in.}^4$

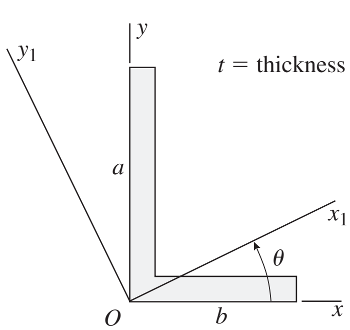
THEREFORE, $I_1 = 311.1\text{ in.}^4 \quad \theta_{p_1} = -29.87°$

$I_2 = 88.9\text{ in.}^4 \quad \theta_{p_2} = 60.13°$

NOTE: The principal moments of inertia can be verified with Eqs. (12-33a and b) and Eq. (12-29).

Problem 12.9-4 Determine the angles θ_{p_1} and θ_{p_2} defining the orientations of the principal axes through the origin O and the corresponding principal moments of inertia I_1 and I_2 for the L-shaped area described in Prob. 12.8-4 ($a = 150$ mm, $b = 100$ mm, and $t = 15$ mm).

Solution 12.9-4 **Principal axes**



ANGLE SECTION

$a = 150$ mm $\quad b = 100$ mm $\quad t = 15$ mm

FROM PROB. 12.8-4:

$I_x = 16.971 \times 10^6\text{ mm}^4$

$I_y = 5.152 \times 10^6\text{ mm}^4 \quad I_{xy} = 1.815 \times 10^6\text{ mm}^4$

EQ. (12-30): $\tan 2\theta_p = -\dfrac{2I_{xy}}{I_x - I_y} = -0.3071$

$2\theta_p = -17.07°$ and $162.93°$

$\theta_p = -8.54°$ and $81.46°$

SUBSTITUTE into Eq. (12-25) with $\theta = -8.54°$:

$I_{x_1} = 17.24 \times 10^6\,\text{mm}^4$

SUBSTITUTE into Eq. (12-25) with $\theta = 81.46°$:

$I_{x_1} = 4.88 \times 10^6\,\text{mm}^4$

THEREFORE,

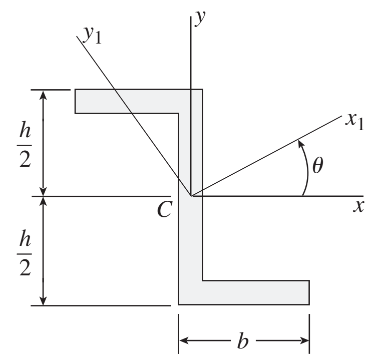
$I_1 = 17.24 \times 10^6\ \text{mm}^4 \quad \theta_{p_1} = -8.54°$

$I_2 = 4.88 \times 10^6\ \text{mm}^4 \quad \theta_{p_2} = -81.46°$

NOTE: The principal moments of inertia I_1 and I_2 can be verified with Eqs. (12-33*a* and *b*) and Eq. (12-29).

Problem 12.9-5 Determine the angles θ_{p_1} and θ_{p_2} defining the orientations of the principal axes through the centroid C and the corresponding principal centroidal moments of inertia I_1 and I_2 for the Z-section described in Prob. 12.8-5 (b = 3 in., h = 4 in., and t = 0.5 in.).

Solution 12.9-5 Principal axes



Z-SECTION

t = thickness = 0.5 in.
b = 3.0 in $\quad h$ = 4.0 in

FROM PROB. 12.8-5:

$I_x = 10.3751\ \text{in.}^4 \quad I_y = 6.9688\ \text{in.}^4$
$I_{xy} = -6.5625\ \text{in.}^4$

EQ. (12-30): $\quad \tan 2\theta_p = -\dfrac{2I_{xy}}{I_x - I_y} = 3.8538$

$2\theta_p = 75.451°$ and $255.451°$

$\theta_p = 37.726°$ and $127.726°$

SUBSTITUTE into Eq. (12-25) with $\theta = 37.726°$:

$I_{x_1} = 15.452\,\text{in.}^4$

SUBSTITUTE into Eq. (12-25) with $\theta = 127.726°$:

$I_{x_1} = 1.892\,\text{in.}^4$

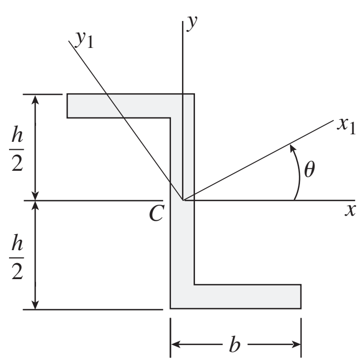
THEREFORE, $I_1 = 15.45\ \text{in.}^4 \quad \theta_{p_1} = 37.73°$

$I_2 = 1.89\ \text{in.}^4 \quad \theta_{p_2} = 127.73°$

NOTE: The principal moments of inertia I_1 and I_2 can be verified with Eqs. (12-33*a* and *b*) and Eq. (12-29).

Problem 12.9-6 Solve the preceding problem for the Z-section described in Prob. 12.8-6 (b = 80 mm, h = 120 mm, and t = 12 mm).

Solution 12.9-6 Principal axes



Z-SECTION

t = thickness
= 12 mm
b = 80 mm
h = 120 mm

FROM PROB. 12.8-6:

$I_x = 6.5065 \times 10^6\ \text{mm}^4 \quad I_y = 3.2573 \times 10^6\ \text{mm}^4$
$I_{xy} = -3.5251 \times 10^6\ \text{mm}^4$

Eq. (12-30): $\tan 2\theta_p = -\dfrac{2I_{xy}}{I_x - I_y} = 2.1698$

$2\theta_p = 65.257^\circ$ and 245.257°
$\theta_p = 32.628^\circ$ and 122.628°

SUBSTITUTE into EQ. (12-25) with $\theta = 32.628^\circ$:

$I_{x_1} = 8.763 \times 10^6\,\text{mm}^4$

SUBSTITUTE into Eq. (12-25) with $\theta = 122.628^\circ$:

$I_{x_1} = 1.000 \times 10^6\,\text{mm}^4$

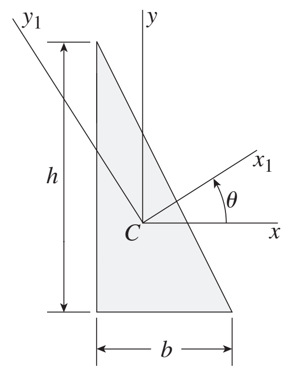
THEREFORE,

$I_1 = 8.76 \times 10^6\ \text{mm}^4 \quad \theta_{p_1} = 32.63^\circ$

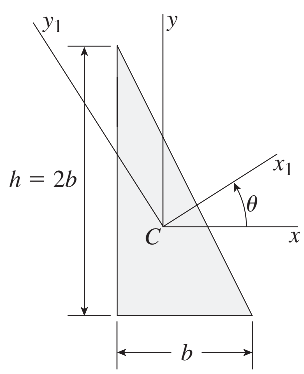
$I_2 = 1.00 \times 10^6\ \text{mm}^4 \quad \theta_{p_2} = 122.63^\circ$

NOTE: The principal moments of inertia I_1 and I_2 can be verified with Eqs. (12-33*a* and *b*) and Eq. (12-29).

Problem 12.9-7 Determine the angles θ_{p1} and θ_{p2} defining the orientations of the principal axes through the centroid C for the right triangle shown in the figure if $h = 2b$. Also, determine the corresponding principal centroidal moments of inertia I_1 and I_2.



Solution 12.9-7 Principal axes



RIGHT TRIANGLE

$h = 2b$

CASE 6

$I_x = \dfrac{bh^3}{36} = \dfrac{2b^4}{9}$

$I_y = \dfrac{hb^3}{36} = \dfrac{b^4}{18}$

$I_{xy} = -\dfrac{b^2h^2}{72} = -\dfrac{b^4}{18}$

EQ. (12-30): $\tan 2\theta_p = -\dfrac{2I_{xy}}{I_x - I_y} = \dfrac{2}{3}$

$2\theta_p = 33.6901^\circ$ and 213.6901°
$\theta_p = 16.8450^\circ$ and 106.8450°

SUBSTITUTE into Eq. (12-25) with $\theta = 16.8450^\circ$:

$I_{x_1} = 0.23904\,\text{b}^4$

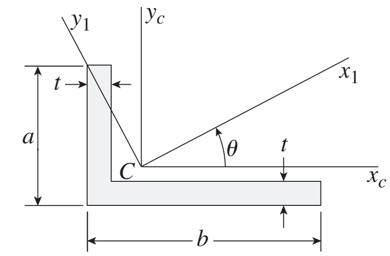
SUBSTITUTE into Eq. (12-25) with $\theta = 106.8450^\circ$:

$I_{x_1} = 0.03873\,\text{b}^4$

THEREFORE, $I_1 = 0.2390\,b^4 \quad \theta_{p_1} = 16.85^\circ$
$I_2 = 0.0387\,b^4 \quad \theta_{p_2} = 106.85^\circ$

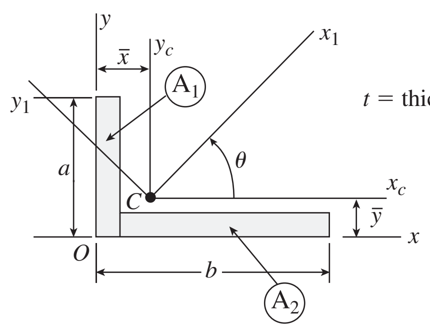
NOTE: The principal moments of inertia I_1 and I_2 can be verified with Eqs. (12-33*a* and *b*) and Eq. (12-29).

Problem 12.9-8 Determine the angles θ_{p_1} and θ_{p_2} defining the orientations of the principal centroidal axes and the corresponding principal moments of inertia I_1 and I_2 for the L-shaped area shown in the figure if $a = 80$ mm, $b = 150$ mm, and $t = 16$ mm.



Probs. 12.9-8 and 12.9-9

Solution 12.9-8 **Principal axes (angle section)**



$a = 80$ mm $\quad b = 150$ mm $\quad t = 16$ mm

$A_1 = at = 1280\ \text{mm}^2$

$A_2 = (b - t)(t) = 2144\ \text{mm}^2$

$A = A_1 + A_2 = t(a + b - t) = 3424\ \text{mm}^2$

LOCATION OF CENTROID C

$$Q_x = \sum A_i \bar{y}_2 = (at)\left(\frac{a}{2}\right) + (b - t)(t)\left(\frac{t}{2}\right)$$

$$= 68{,}352\ \text{mm}^3$$

$$\bar{y} = \frac{Q_x}{A} = \frac{68{,}352\ \text{mm}^3}{3{,}424\ \text{mm}^2} = 19.9626\ \text{mm}$$

$$Q_y = \sum A_i \bar{x}_i = (at)\left(\frac{t}{2}\right) + (b - t)(t)\left(\frac{b + t}{2}\right)$$

$$= 188{,}192\ \text{mm}^3$$

$$\bar{x} = \frac{Q_y}{A} = \frac{188{,}192\ \text{mm}^3}{3{,}424\ \text{mm}^2} = 54.9626\ \text{mm}$$

MOMENTS OF INERTIA (xy AXES)

Use parallel-axis theorem.

$$I_x = \frac{1}{12}(t)(a^3) + A_1\left(\frac{a}{2}\right)^2 + \frac{1}{12}(b - t)(t^3) + A_2\left(\frac{t}{2}\right)^2$$

$$= \frac{1}{12}(16)(80)^3 + (1280)(40)^2 + \frac{1}{12}(134)(16)^3 + (2144)(8)^2$$

$$= 2.91362 \times 10^6\ \text{mm}^4$$

$$I_y = \frac{1}{12}(a)(t^3) + A_1\left(\frac{t}{2}\right)^2 + \frac{1}{12}(t)(b - t^3) + A_2\left(\frac{b + t}{2}\right)^2$$

$$= \frac{1}{12}(80)(16)^3 + (1280)(8)^2 + \frac{1}{12}(16)(134)^3 + (2144)\left(\frac{166}{2}\right)^2$$

$$= 18.08738 \times 10^6\ \text{mm}^4$$

MOMENTS OF INERTIA ($x_c y_c$ AXES)

Use parallel-axis theorem.

$$I_{x_c} = I_x - A\bar{y}^2 = 2.91362 \times 10^6 - (3424)(19.9626)^2$$

$$= 1.54914 \times 10^6\ \text{mm}^4$$

$$I_{y_c} = I_y - A\bar{x}^2 = 18.08738 \times 10^6 - (3424)(54.9626)^2$$

$$= 7.74386 \times 10^6\ \text{mm}^4$$

PRODUCT OF INERTIA

Use parallel-axis theorem: $\quad I_{xy} = I_{\text{centroid}} + A\,d_1 d_2$

$$\text{Area } A_1\text{: } I'_{x_c y_c} = 0 + A_1\left[-\left(\bar{x} - \frac{t}{2}\right)\right]\left[\frac{e}{2} - \bar{y}\right]$$

$$= (1280)(8 - 54.9626)(40 - 19.9626)$$

$$= -1.20449 \times 10^6\ \text{mm}^4$$

$$\text{Area } A_2\text{: } I''_{x_c y_c} = 0 + A_2\left[\frac{b + t}{2} - \bar{x}\right]\left[-\left(\bar{y} - \frac{t}{2}\right)\right]$$

$$= (2144)(83 - 54.9626)(8 - 19.9626)$$

$$= -0.71910 \times 10^6\ \text{mm}^4$$

$$I_{x_c y_c} = I'_{x_c y_c} + I''_{x_c y_c} = -1.92359 \times 10^6\ \text{mm}^4$$

SUMMARY

$I_{x_c} = 1.54914 \times 10^6\ \text{mm}^4 \qquad I_{y_c} = 7.74386 \times 10^6\ \text{mm}^4$

$I_{x_c y_c} = -1.92359 \times 10^6\ \text{mm}^4$

PRINCIPAL AXES

Eq. (12-30): $\tan 2\theta_p = -\dfrac{2I_{xy}}{I_x - I_y} = -0.621041$

$2\theta_p = -31.8420°$ and $148.1580°$

$\theta_p = -15.9210°$ and $74.0790°$

SUBSTITUTE into Eq. (12-25) with $\theta = -15.9210°$

$I_{x_1} = 1.0004 \times 10^6\,\text{mm}^4$

SUBSTITUTE into Eq. (12-25) with $\theta = 74.0790°$

$I_{x_1} = 8.2926 \times 10^6\,\text{mm}^4$

THEREFORE,

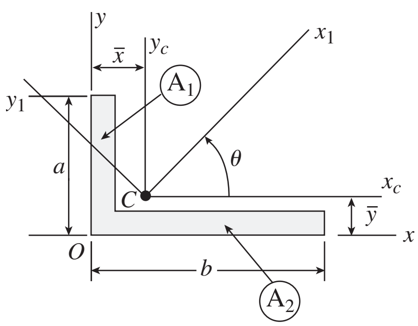
$I_1 = 8.29 \times 10^6\ \text{mm}^4 \quad \theta_{p_1} = 74.08°$

$I_2 = 1.00 \times 10^6\ \text{mm}^4 \quad \theta_{p_2} = -15.92°$

NOTE: The principal moments of inertia I_1 and I_2 can be verified with Eqs. (12-33a and b) and Eq. (12-29).

Problem 12.9-9 Solve the preceding problem if $a = 3$ in., $b = 6$ in., and $t = 5/8$ in.

Solution 12.9-9 Principal axes (angle section)



$a = 3.0$ in.

$b = 6.0$ in.

$t = 5/8$ in.

$A_1 = at = 1.875\text{ in.}^2$

$A_2 = (b - t)(t) = 3.35938\text{ in.}^2$

$A = A_1 + A_2 = t\,(a + b - t) = 5.23438\text{ in.}^2$

LOCATION OF CENTROID C

$$Q_x = \sum A_i \bar{y}_2 = (at)\left(\frac{a}{2}\right) + (b - t)(t)\left(\frac{t}{2}\right)$$

$$= 3.86230\text{ in.}^3$$

$$\bar{y} = \frac{Q_x}{A} = \frac{3.86230\text{ in.}^3}{5.23438\text{ in.}^2} = 0.73787\text{ in.}$$

$$Q_y = \sum A_i \bar{x}_i = (at)\left(\frac{t}{2}\right) + (b - t)(t)\left(\frac{b + t}{2}\right)$$

$$= 11.71387\text{ in.}^3$$

$$\bar{x} = \frac{Q_y}{A} = \frac{11.71387\text{ in.}^3}{5.23438\text{ in.}^2} = 2.23787\text{ in.}$$

MOMENTS OF INERTIA (xy AXES)

Use parallel-axis theorem.

$$I_x = \frac{1}{12}(t)(a^3) + A_1\left(\frac{a}{2}\right)^2 + \frac{1}{12}(b - t)(t^3) + A_2\left(\frac{t}{2}\right)^2$$

$$= \frac{1}{12}\left(\frac{5}{8}\right)(3.0)^3 + (1.875)(1.5)^2 + \frac{1}{12}(5.375)\left(\frac{5}{8}\right)^3$$

$$+ (3.35938)\left(\frac{5}{16}\right)^2$$

$$= 6.06242\text{ in.}^4$$

$$I_y = \frac{1}{12}(a)(t^3) + A_1\left(\frac{t}{2}\right)^2 + \frac{1}{12}(t)(b - t^3)$$

$$+ A_2\left(\frac{b + t}{2}\right)^2$$

$$= \frac{1}{12}(3.0)\left(\frac{5}{8}\right)^3 + (1.875)\left(\frac{5}{16}\right)^2 + \frac{1}{12}\left(\frac{5}{8}\right)(5.375)^3$$

$$+ (3.35938)\left(\frac{6.625}{2}\right)^2$$

$$= 45.1933\text{ in.}^4$$

MOMENTS OF INERTIA ($x_c y_c$ AXES)

Use parallel-axis theorem.

$$I_{x_c} = I_x - A\bar{y}^2 = 6.06242 - (5.23438)(0.73787)^2$$

$$= 3.21255\text{ in.}^4$$

$$I_{y_c} = I_y - A\bar{x}^2 = 45.1933 - (5.23438)(2.23787)^2$$

$$= 18.97923\text{ in.}^4$$

PRODUCT OF INERTIA

Use parallel-axis theorem: $I_{xy} = I_{\text{centroid}} + A\,d_1 d_2$

$$\text{Area } A_1\text{: } I'_{x_c y_c} = 0 + A_1\left[-\left(\bar{x} - \frac{t}{2}\right)\right]\left[\frac{a}{2} - \bar{y}\right]$$
$$= (1.875)(-1.92537)(0.76213)$$
$$= -2.75134 \text{ in.}^4$$

$$\text{Area } A_2\text{: } I''_{x_c y_c} = 0 + A_2\left[\frac{b+t}{2} - \bar{x}\right]\left[-\left(\bar{y} - \frac{t}{2}\right)\right]$$
$$= (3.35938)(1.07463)(-0.42537)$$
$$= -1.53562 \text{ in.}^4$$

$$I_{x_c y_c} = I'_{x_c y_c} + I''_{x_c y_c} = -4.28696 \text{ in.}^4$$

SUMMARY

$I_{x_c} = 3.21255 \text{ in.}^4$ $I_{y_c} = 18.97923 \text{ in.}^4$

$I_{x_c y_c} = -4.28696 \text{ in.}^4$

PRINCIPAL AXES

EQ. (12-30): $$\tan 2\theta_p = -\frac{2I_{xy}}{I_x - I_y} = -0.54380$$

$2\theta_p = -28.5374°$ and $151.4626°$
$\theta_p = -14.2687°$ and $75.7313°$

SUBSTITUTE into Eq. (12-25) with $\theta = -14.2687°$

$I_{x_1} = 2.1223 \text{ in.}^4$

SUBSTITUTE into Eq. (12-25) with $\theta = 75.7313°$

$I_{x_1} = 20.0695 \text{ in.}^4$

THEREFORE,

$I_1 = 20.07 \text{ in.}^4$ $\theta_{p_1} = 75.73°$
$I_2 = 2.12 \text{ in.}^4$ $\theta_{p_2} = -14.27°$

NOTE: The principal moments of inertia I_1 and I_2 can be verified with Eqs. (12-33a and b) and Eq. (12-29).