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Volume 108

Tensor Spaces and Exterior Algebra

Takeo Yokonuma

American Mathematical Society

Tensor Spaces and Exterior Algebra

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Translated by
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American Mathematical Society
Providence, Rhode Island

テンソル空間と外積代数

TENSORU KUUKAN TO GAISEKIDAISUU
(Tensor Spaces and Exterior Algebra)
by Takeo Yokonuma

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ABSTRACT. This book provides an introduction to tensors and related topics. The book begins with definitions of the basic concepts of the theory; tensor products of vector spaces, tensors, tensor algebras, and exterior algebra. Their properties are then studied and applications given.

Algebraic systems with bilinear multiplication are introduced in the final chapter. In particular, the theory of replicas of Chevalley and several properties of Lie algebras that follow from this theory are presented.

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Preface to the English Edition

This is a translation of my work originally published by Iwanami Shoten, as a volume in their Lecture Series on linear algebra.

We assume, therefore, that readers are familiar with several fundamental concepts of linear algebra such as vector spaces, matrices, determinants, etc., though we review some of these concepts in the text. We have made some changes in the references.

On this occasion, I would like to express my gratitude to Professor Nagayoshi Iwahori for giving me encouragement and many valuable suggestions during the preparation of the original work and also to Mr. Hideo Arai of Iwanami Shoten for continued support.

I am grateful to the American Mathematical Society and the staff for their effort in publishing this English edition. I also thank Kazunari Noda and Nami Yokonuma for their excellent typing.

Takeo Yokonuma
December 1991

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Preface

The subject matter of the present book is generally called multilinear algebra; we shall discuss mainly tensors and related concepts. The readers may recall several important tensors that are used in differential geometry, mechanics, electromagnetics, and so on. On the other hand, it is generally said that tensors are difficult to understand. One of the reasons for this is that formerly tensors and tensor fields (mappings whose values are tensors) were not distinguished, and tensor fields were discussed without defining tensors in advance.([1]) In fact, readers should be aware that sometimes tensor fields are simply called tensors in the literature. In any case, it is important to understand clearly what tensors are. Our purpose is to explain tensors to the readers as clearly as possible. Tensor fields are defined by means of tensors. The analytic theory of tensor fields is known as tensor calculus. Needless to say, it is beyond the scope of linear algebra and we shall not discuss it here. The notions of tensor density and pseudotensor are also important in applications of tensors. Since they are related to representations of linear groups, we shall mention only their definitions (§II.6).

As stated above, the notion of tensor seems to have originated from what are now called tensor fields. Research has been done, since the end of last century, with the study of vector calculus and the theory of invariants, on systems of functions which satisfy certain transformation laws. Ricci and Levi-Civita founded tensor calculus. It became well known after being used by Einstein to describe the theory of relativity. When we try to describe a physical phenomenon, we use a coordinate system. Though the description depends on the coordinate system, the phenomenon itself does not. To explain the situation, systems of functions that satisfy a certain transformation law play an important role. Depending on the type of transformation law, several kinds of tensor (field) have been defined. By the way, it seems that the word "tensor" was derived from tension. Thus, a tensor is classically defined as a system (or an array) of numbers which satisfies a certain transformation law (see §II.2). We can consider that this system describes an "object" whose existence does not depend on coordinate systems. This point of view

([1])See R. Godement: Cours d'algèbre, Hermann, p. 269.

is sufficient for computation but it does not explain what the object is.

In order to define tensors, we begin in Chapter I by constructing tensor products of vector spaces. Then, in Chapter II, we define tensors and study their properties. In many branches of mathematics the construction of tensor products is used as a powerful tool to construct new objects from known ones. For example, we will describe the extension of a field of scalars (§I.8b). The tensor product of representations is another example.

In Chapter III, we discuss the notion of exterior algebra. Exterior algebras are basic to the theory of differential forms. In Chapter IV, to show another aspect of the theory of tensor products, we discuss algebraic systems with bilinear multiplication. In particular, we discuss Lie algebras.

References for the English Edition

1. W. H. Greub, *Multilinear algebra*, Grundlehren Math. Wiss., Bd. 136, Springer-Verlag, Berlin and New York, 1967.

In this book, tensor algebras are discussed at great length. This is the second volume of Greub's text books on linear algebra; the first one is, *Linear Algebra*, 3rd ed., Grundlehren Math. Wiss., Bd. 97, Springer-Verlag, Berlin and New York, 1967.

In Bourbaki's series of Éléments de Mathématique, tensor algebra is treated in:

2. N. Bourbaki, *Algèbre*, Hermann, Paris, 1970, Chapters 2 and 3.

Though there are no books in Japanese which are written about the same topics as the present volume, there are several books on linear algebra or algebra, parts of which are devoted to tensor algebras, e.g.:

3. I. Satake, *Linear algebra*, Marcel Dekker, New York, 1975.

The Japanese edition was published by Shokabo in 1973. A large part of the content of the present volume is explained neatly in Chapter V. A description of group representations is also included.

4. N. Iwahori, *Vector calculus*, Shokabo, Tokyo, 1960. (Japanese)

Tensors of type (p, q) $(p+q \leq 2)$ on $\mathbf{R}^3$ are explained in detail. There are many examples and exercises.

In some books on differential geometry, tensor algebra and its relation to geometry are explained as preliminaries.

5. S. Sternberg, *Lectures on differential geometry*, Prentice Hall, Englewood Cliffs, NJ, 1964 (republished by Chelsea in 1983).

6. S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, vol. 1, John Wiley & Sons, New York, 1963.

We consulted the above books during the preparation of the present volume.

The following books appeared more recently.

7. M. Marcus, *Finite dimensional multilinear algebra*, Parts I, II, Marcel Dekker, New York, 1973 (Part I) and 1975 (Part II).

8. L. Schwartz, *Les tenseurs*, Actualités scientifiques et industrielles, 1376, Hermann, Paris, 1975.

9. S. Lang, *Algebra*, 2nd ed., Addison-Wesley, Reading, MA, 1984.

Section 4.3 contains only the beginning of the theory of Lie algebras. For details, see e.g.

10. Y. Matsushima, *Theory of Lie algebras*, Kyoritsu Shuppan, Tokyo, 1956. (Japanese)

11. N. Iwahori, *Theory of Lie groups* II, Iwanami Shoten, Tokyo, 1957.

12. N. Jacobson, *Lie Algebras*, Interscience, New York, 1962 (republished by Dover).

13. J. E. Humphreys, *Introduction to Lie algebras and representation theory*, 2nd printing, revised. Graduated Texts in Math., vol. 9, Springer-Verlag, New York, 1972.

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