# Tensors and Matrices

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Univ. Illinois at Chicago

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## Characterization of tensor in $\mathbb{C}^{4\times 4\times 4}$ of border rank 4

# Basic notions

scalar $a \in \mathbb{F}$, vector $\mathbf{x} = (x_1, \ldots, x_n)^\top \in \mathbb{F}^n$, matrix $A = [a_{ij}] \in \mathbb{F}^{m\times n}$,
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COR $\mathrm{rank}\ \mathcal{T} \le \min(mn, ml, nl)$

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PRF: 3-sat with $n$ variables $m$ clauses
satisfiable iff $\mathrm{rank}\, \mathcal{T} = 4n + 2m, \mathcal{T} \in \mathbb{F}^{(2n+3m)\times(3n)\times(3n+m)})$
otherwise rank is larger

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generic rank=border rank=typical rank of $\mathbb{F}^{m \times n \times l}$: $\mathrm{grank}_{\mathbb{F}}(m, n, l)$ -
the rank of a random tensor $\mathcal{T} \in \mathbb{F}^{m \times n \times l}$

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Conjecture $\mathrm{grank}_{\mathbb{C}}(m,n,l)=\lceil\frac{mnl}{(m+n+l-2)}\rceil$

for $2\le m\le n\le l<(m-1)(n-1)$ and $(3,n,l)\ne(3,2p+1,2p+1)$

## Generic rank of $\mathbb{F}^{m\times n\times l}$

THM: $\mathrm{grank}_{\mathbb{C}}(m,n,l) = \min(l, mn)$ for $(m-1)(n-1)+1 \le l$.

Reason: For $l = (m-1)(n-1)+1$ a generic subspace of matrices of dimension $l$ in $\mathbb{C}^{m\times n}$ intersect the variety of rank one matrices in $\mathbb{C}^{m\times n}$ at least at $l$ lines which contain $l$ linearly independent matrices

COR: $\mathrm{grank}_{\mathbb{C}}(2,n,l) = \min(l, 2n)$ for $2 \le n \le l$

Dimension count for $\mathbb{F} = \mathbb{C}$ and $2 \le m \le n \le l \le (m-1)(n-1)+1$:

$f_r : (\mathbb{C}^m \times \mathbb{C}^n \times \mathbb{C}^l)^r \to \mathbb{C}^{m\times n\times l}$, $\mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z} = (a\mathbf{x})\otimes(b\mathbf{y})\otimes((ab)^{-1}\mathbf{z})$

$\mathrm{grank}_{\mathbb{C}}(m,n,l)(m+n+l-2) \ge mnl \Rightarrow \mathrm{grank}_{\mathbb{C}}(m,n,l) \ge \lceil \frac{mnl}{(m+n+l-2)} \rceil$

Conjecture $\mathrm{grank}_{\mathbb{C}}(m,n,l) = \lceil \frac{mnl}{(m+n+l-2)} \rceil$

for $2 \le m \le n \le l < (m-1)(n-1)$ and $(3,n,l) \ne (3, 2p+1, 2p+1)$

Fact: $\mathrm{grank}_{\mathbb{C}}(3, 2p+1, 2p+1) = \lceil \frac{3(2p+1)^2}{4p+3} \rceil + 1$

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Product of two $2 \times 2$ matrices is done by 7 multiplications

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I checked the conjecture up to $m, n, l \leq 14$

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Examples [3]
$m = n \geq 2,\ l = (m-1)(n-1) + 1.$
$m = n = 4,\ l = 11, 12$

# Rank one approximations

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$\mathbb{R}^{m\times n\times l}$ IPS: $\langle \mathcal{A}, \mathcal{B}\rangle = \sum_{i=j=k}^{m,n,l} a_{i,j,k} b_{i,j,k}, \ \|\mathcal{T}\| = \sqrt{\langle \mathcal{T}, \mathcal{T}\rangle}$

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How many distinct singular values are for a generic tensor?

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Assume that $\mathcal{T} \geq 0$. Then $\mathbf{x}, \mathbf{y}, \mathbf{z} \geq 0$

For which values of $p$ we have an analog of Perron-Frobenius theorem?

Yes, for $p \geq 3$, No, for $p < 3$,
Friedland-Gauber-Han [1]

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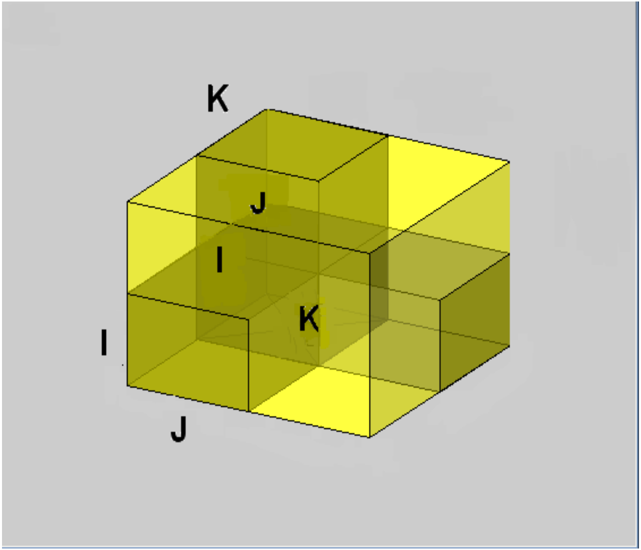
Use Newton method on Grassmannians - Eldén-Savas 2009 [1]

# Fast low rank approximation I:



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$\min_{\mathcal{U}\in\mathbb{C}^{p\times q\times r}} \|\mathcal{A} - \mathcal{U} \times F \times E \times G\|_F$ achieved for $\mathcal{U} = \mathcal{A} \times E^\dagger \times F^\dagger \times G^\dagger$

# Fast low rank approximations II:

Approximate $A \in \mathbb{R}^{m\times n}$ by $CUR$ where $C \in \mathbb{R}^{m\times p}$, $R \in \mathbb{R}^{q\times n}$ for some submatrices of $A$.

$\min_{U\in\mathbb{C}^{p\times q}} \|A - CUR\|_F$ achieved for $U = C^\dagger A R^\dagger$

Faster choice: $U = A[I, J]^\dagger$
(corresponds to best $CUR$ approximation on the entries read)

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$CUR$ approximation of $\mathcal{A}$ obtained by choosing $E, F, G$ submatrices of unfolded $\mathcal{A}$ in the mode $1, 2, 3$.

# List of applications

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Face recognition

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Video tracking

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Factor analysis

# Scaling of nonnegative tensors to tensors with given row, column and depth sums

$0 \leq \mathcal{T} = [t_{i,j,k}] \in \mathbb{R}^{m \times n \times l}$ has given row, column and depth sums:
$\mathbf{r} = (r_1, \ldots, r_m)^\top, \mathbf{c} = (c_1, \ldots, c_n)^\top, \mathbf{d} = (d_1, \ldots, d_l)^\top > \mathbf{0}$:

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$f_{\mathcal{T}}$ is strictly convex implies $\mathcal{T}$ is not decomposable: $\mathcal{T} \ne \mathcal{T}_1 \oplus \mathcal{T}_2$.

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# Characterization of tensor in $\mathbb{C}^{4\times 4\times 4}$ of border rank 4

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phylogenic trees and their invariants [1]

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Friedland [5] one needs a equations of degree 16

# References I


- E.S. Allman and J.A. Rhodes, Phylogenic ideals and varieties for general Markov model, *Advances in Appl. Math.*, 40 (2008) 127-148.
- R.B. Bapat $D_1 AD_2$ theorems for multidimensional matrices, *Linear Algebra Appl.* 48 (1982), 437-442.
- R.B. Bapat and T.E.S. Raghavan, An extension of a theorem of Darroch and Ratcliff in loglinear models and its application to scaling multidimensional matrices, *Linear Algebra Appl.* 114/115 (1989), 705-715.
- R.A. Brualdi, Convex sets of nonnegative matrices, *Canad. J. Math* 20 (1968), 144-157.
- R.A. Brualdi, S.V. Parter and H. Schneider, The diagonal equivalence of a nonnegative matrix to a stochastic matrix, *J. Math. Anal. Appl.* 16 (1966), 31–50.

# References II


- Lar. Eldén and B. Savas, A Newton Grassmann method for computing the Best Multi-Linear Rank-(r1; r2; r3) Approximation of a Tensor, *SIAM J. Matrix Anal. Appl.* 31 (2009), 248–271.
- J. Franklin and J. Lorenz, On the scaling of multidimensional matrices, *Linear Algebra Appl.* 114/115 (1989), 717-735.
- S. Friedland, On the generic rank of 3-tensors, arXiv: 0805.3777v2.
- S. Friedland, Positive diagonal scaling of a nonnegative tensor to one with prescribed slice sums, to appear in *Linear Algebra Appl.*, arXiv:0908.2368v1, http://arxiv.org/abs/0908.2368v2.
- S. Friedland,On tensors of border rank $l$ in $\mathbb{C}^{m\times n\times l}$, arXiv:1003.1968v1.

# References III

- S. Friedland, S. Gauber and L. Han, Perron-Frobenius theorem for nonnegative multilinear forms, *arXiv:0905.1626*.
- S. Friedland, V. Mehrmann, A. Miedlar, and M. Nkengla, Fast low rank approximations of matrices and tensors, submitted, www.matheon.de/preprints/4903.
- S. Friedland and V. Mehrmann, Best subspace tensor approximations, arXiv:0805.4220v1, http://arxiv.org/abs/0805.4220v1 .
- S.A. Goreinov, E.E. Tyrtyshnikov, N.L. Zmarashkin, A theory of pseudo-skeleton approximations of matrices, *Linear Algebra Appl.* 261 (1997), 1-21.
- L.H. Lim, Singular values and eigenvalues of tensors: a variational approach, *CAMSAP* 05, 1 (2005), 129-132.

# References IV


- M.W. Mahoney and P. Drineas, CUR matrix decompositions for improved data analysis, *PNAS* 106, (2009), 697-702.
- M.V. Menon, Matrix links, an extremisation problem and the reduction of a nonnegative matrix to one with with prescribed row and column sums, *Canad. J. Math* 20 (1968), 225-232.
- V. Strassen, Rank and optimal computations of generic tensors, *Linear Algebra Appl.* 52/53: 645-685, 1983.