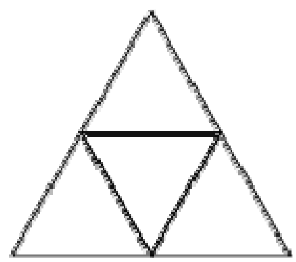
Tetrahedron

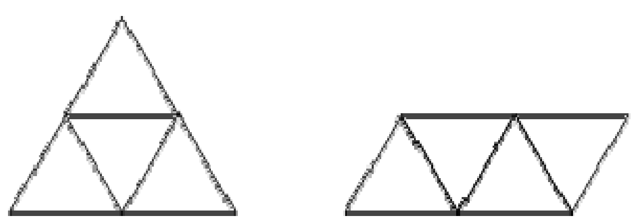
In general, a tetrahedron is a polyhedron with four sides. If all faces are congruent, the tetrahedron is known as an isosceles tetrahedron. If all faces are congruent to an equilateral triangle, then the tetrahedron is known as a regular tetrahedron.



The regular tetrahedron, often simply called "the" tetrahedron, is the Platonic solid P_5 with four polyhedron vertices, six polyhedron edges, and four equivalent equilateral triangular faces, $4\{3\}$. It is also uniform polyhedron U_1 and Wenninger model W_1. It is described by the Schläfli symbol $\{3, 3\}$ and the Wythoff symbol is $3 \mid 2\,3$. It is an isohedron.

The tetrahedron has 7 axes of symmetry: $4C_3$ (axes connecting vertices with the centers of the opposite faces) and $3C_2$ (the axes connecting the midpoints of opposite sides).

There are no other convex polyhedra other than the tetrahedron having four faces.



The tetrahedron has two distinct nets (Buekenhout and Parker 1998). Questions of polyhedron coloring of the tetrahedron can be addressed using the Pólya enumeration theorem.

The surface area of the tetrahedron is simply four times the area of a single face, so

$$S = \sqrt{3}\, a^2. \quad (1)$$

Since a tetrahedron is a pyramid with a triangular base, $V = \frac{1}{3} A_b h$, giving

$$V = \frac{1}{12} \sqrt{2}\, a^3. \quad (2)$$

The dihedral angle is

$$\alpha = \tan^{-1}\left(2\sqrt{2}\right) = 2 \sin^{-1}\left(\frac{1}{3}\sqrt{3}\right) = \cos^{-1}\left(\frac{1}{3}\right) \approx 70.53°. \quad (3)$$

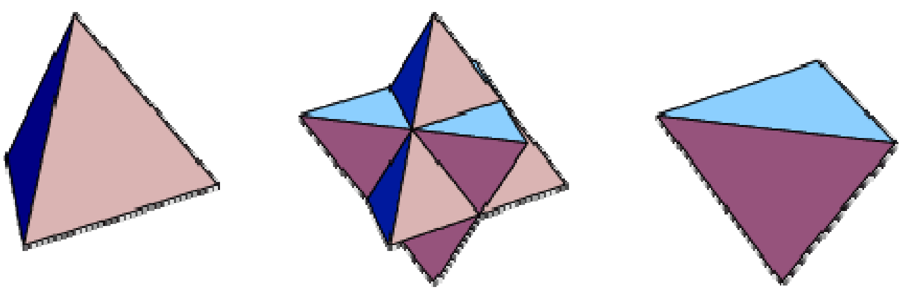
The midradius of the tetrahedron is

$$\rho = \sqrt{r^2 + d^2} = \sqrt{\frac{1}{8}}\, a = \frac{1}{4}\sqrt{2}\, a \quad (4)$$

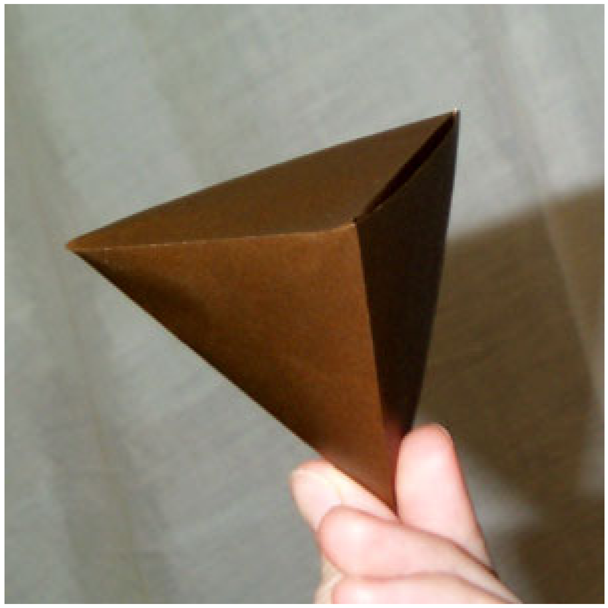
$$\approx 0.35355\, a. \quad (5)$$

Plugging in for the polyhedron vertices gives

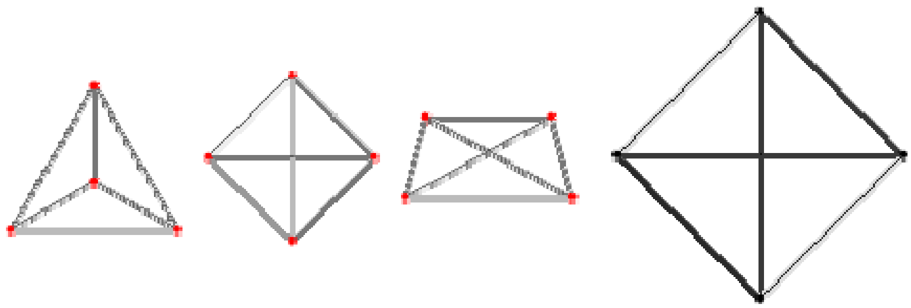
$$\left(\frac{1}{3}\sqrt{3}\, a, 0, 0\right), \left(-\frac{1}{6}\sqrt{3}\, a, \pm\frac{1}{2} a, 0\right), \text{ and } \left(0, 0, \frac{1}{3}\sqrt{6}\, a\right). \quad (6)$$



The dual polyhedron of an tetrahedron with unit edge lengths is another oppositely oriented tetrahedron with unit edge lengths.



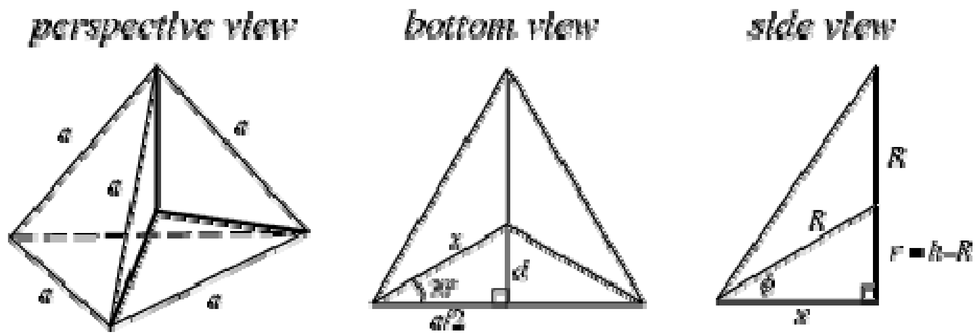
The figure above shows an origami tetrahedron constructed from a single sheet of paper (Kasahara and Takahama 1987, pp. 56-57).



It is the prototype of the tetrahedral group T_d. The connectivity of the vertices is given by the tetrahedral graph, equivalent to the circulant graph $Ci_{4,2,3}(4)$ and the complete graph K_4.

The tetrahedron is its own dual polyhedron, and therefore the centers of the faces of a tetrahedron form another tetrahedron (Steinhaus 1999, p. 201). The tetrahedron is the only simple polyhedron with no polyhedron diagonals, and it cannot be stellated. If a regular tetrahedron is cut by six planes, each passing through an edge and bisecting the opposite edge, it is sliced into 24 pieces (Gardner 1984, pp. 190 and 192; Langman 1951).

Alexander Graham Bell was a proponent of use of the tetrahedron in framework structures, including kites (Bell 1903; Lesage 1956, Gardner 1984, pp. 184-185). The opposite edges of a regular tetrahedron are perpendicular, and so can form a universal coupling if hinged appropriately. Eight regular tetrahedra can be placed in a ring which rotates freely, and the number can be reduced to six for squashed irregular tetrahedra (Wells 1975, 1991).



Let a tetrahedron be length a on a side, and let its base lie in the plane $z = 0$ with one vertex lying along the positive x-axis. The polyhedron vertices of this tetrahedron are then located at (x, 0, 0), ($-d$, $\pm a/2$, 0), and (0, 0, h), where

$$x = \frac{\frac{a}{2}}{\cos\left(\frac{\pi}{6}\right)} = \frac{1}{3}\sqrt{3}\,a. \tag{7}$$

d is then

$$d = \sqrt{x^2 - \left(\frac{1}{2}a\right)^2} = \frac{1}{6}\sqrt{3}\,a. \tag{8}$$

This gives the area of the base as

$$A = \frac{1}{2}a(R + x) = \frac{1}{4}\sqrt{3}\,a^2. \tag{9}$$

The height is

$$h = \sqrt{a^2 - x^2} = \frac{1}{3}\sqrt{6}\,a. \tag{10}$$

The circumradius R is found from

$$x^2 + (h - R)^2 = R^2 \tag{11}$$
$$x^2 + h^2 - 2hR + R^2 = R^2. \tag{12}$$

Solving gives

$$R = \frac{x^2 + h^2}{2h} \quad (13)$$
$$= \frac{1}{4}\sqrt{6}\,a \quad (14)$$
$$\approx 0.61237\,a. \quad (15)$$

The inradius r is

$$r \equiv h - R = \frac{1}{12}\sqrt{6}\,a \quad (16)$$
$$\approx 0.20412\,a, \quad (17)$$

which is also

$$r = \frac{1}{4}h \quad (18)$$
$$= \frac{1}{3}R. \quad (19)$$

The angle between the bottom plane and center is then given by

$$\phi = \tan^{-1}\left(\frac{r}{x}\right) \quad (20)$$
$$= \tan^{-1}\left(\frac{1}{4}\sqrt{2}\right) \quad (21)$$
$$= \cot^{-1}\left(2\sqrt{2}\right) \quad (22)$$
$$\approx 19.47°. \quad (23)$$

Given a tetrahedron of edge length a situated with vertical apex and with the origin of coordinate system at the geometric centroid of the vertices, the four polyhedron vertices are located at $(x, 0, -r)$, $(-d, \pm a/2, -r)$, $(0, 0, R)$, with, as shown above

$$x = \frac{1}{3}\sqrt{3}\,a \quad (24)$$
$$r = \frac{1}{12}\sqrt{6}\,a \quad (25)$$
$$R = \frac{1}{4}\sqrt{6}\,a \quad (26)$$
$$d = \frac{1}{6}\sqrt{3}\,a. \quad (27)$$

The vertices of a tetrahedron of side length $\sqrt{2}$ can also be given by a particularly simple form when the vertices are taken as corners of a cube (Gardner 1984, pp. 192-194). One such tetrahedron for a cube of side length 1 gives the tetrahedron of side length $\sqrt{2}$ having vertices (0, 0, 0), (0, 1, 1), (1, 0, 1), (1, 1, 0), and satisfies the inequalities

$$x + y + z \le 2 \quad (28)$$
$$x - y - z \le 0 \quad (29)$$

$$-x+y-z \leq 0 \quad (30)$$

$$-x-y+z \leq 0. \quad (31)$$

The following table gives polyhedra which can be constructed by cumulation of a tetrahedron by pyramids of given heights h.

h	$(r+h)/h$	result
$\frac{1}{15}\sqrt{6}$	$\frac{7}{5}$	triakis tetrahedron
$\frac{1}{6}\sqrt{6}$	2	cube
$\frac{1}{3}\sqrt{6}$	3	9-faced star deltahedron

Connecting opposite pairs of edges with equally spaced lines gives a configuration like that shown above which divides the tetrahedron into eight regions: four open and four closed (Steinhaus 1999, p. 246).

Michigan artist David Barr designed his "Four Corners Project" in 1976. It is an Earth-sized regular tetrahedron that spans the planet, with just the tips of its four corners protruding. These visible portions are four-inch tetrahedra, which protrude from the globe at Easter Island, Greenland, New Guinea, and the Kalahari Desert. Barr traveled to these locations and was able to permanently install the four aligned marble tetrahedra between 1981 and 1985 (G. Hart, pers. comm.; Arlinghaus and Nystuen 1986).

By slicing a tetrahedron as shown above, a square can be obtained. This cut divides the tetrahedron into two congruent solids rotated by 90°. The projection of a tetrahedron can be an equilateral triangle or a square (Steinhaus 1999, pp. 191-192).

Now consider a general (not necessarily regular) tetrahedron, defined as a convex polyhedron consisting of four (not necessarily identical) triangular faces. Let the tetrahedron be specified by its polyhedron vertices at (x_i, y_i, z_i) where $i=1$, ..., 4. Then the volume is given by

$$V = \frac{1}{3!}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}. \quad (32)$$

Specifying the tetrahedron by the three polyhedron edge vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ from a given polyhedron vertex, the volume is

$$V = \frac{1}{3!} |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|. \tag{33}$$

If the faces are congruent and the sides have lengths a, b, and c, then

$$V = \sqrt{\frac{(a^2 + b^2 - c^2)(a^2 + c^2 - b^2)(b^2 + c^2 - a^2)}{72}} \tag{34}$$

(Klee and Wagon 1991, p. 205). In general, if the edge between vertices i and j is of length d_{ij}, then the volume V is given by the Cayley-Menger determinant

$$288\, V^2 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & d_{12}^2 & d_{13}^2 & d_{14}^2 \\ 1 & d_{21}^2 & 0 & d_{23}^2 & d_{24}^2 \\ 1 & d_{31}^2 & d_{32}^2 & 0 & d_{34}^2 \\ 1 & d_{41}^2 & d_{42}^2 & d_{43}^2 & 0 \end{vmatrix}. \tag{35}$$

Consider an arbitrary tetrahedron $A_1 A_2 A_3 A_4$ with triangles $T_1 = \Delta A_2 A_3 A_4$, $T_2 = \Delta A_1 A_3 A_4$, $T_3 = \Delta A_1 A_2 A_4$, and $T_4 = \Delta A_1 A_2 A_3$. Let the areas of these triangles be s_1, s_2, s_3, and s_4, respectively, and denote the dihedral angle with respect to T_i and T_j for $i \neq j = 1, 2, 3, 4$ by θ_{ij}. Then the four face areas are connected by

$$s_k^2 = \sum_{\substack{j \neq k \\ 1 \leq j \leq 4}} s_j^2 - 2 \sum_{\substack{i, j \neq k \\ 1 \leq i < j \leq 4}} s_i s_j \cos \theta_{ij} \tag{36}$$

involving the six dihedral angles (Dostor 1905, pp. 252-293; Lee 1997). This is a generalization of the law of cosines to the tetrahedron. Furthermore, for any $i \neq j = 1, 2, 3, 4$,

$$V = \frac{2}{3\, l_{ij}} s_i s_j \sin \theta_{ij}, \tag{37}$$

where l_{ij} is the length of the common edge of T_i and T_j (Lee 1997).

Given a right-angled tetrahedron with one apex where all the edges meet orthogonally and where the face opposite this apex is denoted s_k, then

$$s_k^2 = \sum_{\substack{i \neq k \\ 1 \leq i \leq 4}} s_i^2. \qquad (38)$$

This is a generalisation of Pythagoras's theorem which also applies to higher dimensional simplices (F. M. Jackson, pers. comm., Feb. 20, 2006).

Let A be the set of edges of a tetrahedron and $P(A)$ the power set of A. Write $\bar{t}$ for the complement in A of an element $t \in P(A)$. Let F be the set of triples $\{x, y, z\} \in P(A)$ such that x, y, z span a face of the tetrahedron, and let G be the set of $(e \cap f) \cup (\overline{e \cup f}) \in P(A)$, so that $e, f \in F$ and $e \neq f$. In G, there are therefore three elements which are the pairs of opposite edges. Now define D, which associates to an edge x of length L the quantity $\left(L/\sqrt[3]{12}\right)^2$, p, which associates to an element $t \in P(A)$ the product of $D(x)$ for all $x \in t$, and s, which associates to t the sum of $D(x)$ for all $x \in t$. Then the volume of a tetrahedron is given by

$$\sqrt{\sum_{t \in G} \left(s(\bar{t}) - s(t)\right) p(t) - \sum_{t \in F} p(t)} \qquad (39)$$

(P. Kaeser, pers. comm.).

The analog of Gauss's circle problem can be asked for tetrahedra: how many lattice points lie within a tetrahedron centered at the origin with a given inradius (Lehmer 1940, Granville 1991, Xu and Yau 1992, Guy 1994).

There are a number of interesting and unexpected theorems on the properties of general (i.e., not necessarily regular) tetrahedron (Altshiller-Court 1979). If a plane divides two opposite edges of a tetrahedron in a given ratio, then it divides the volume of the tetrahedron in the same ratio (Altshiller-Court 1979, p. 89). It follows that any plane passing through a bimedian of a tetrahedron bisects the volume of the tetrahedron (Altshiller-Court 1979, p. 90).

Let the vertices of a tetrahedron be denoted A, B, C, and D, and denote the side lengths $BC = a$, $CA = b$, $AB = c$, $DA = a'$, $DB = b'$, and $DC = c'$. Then if Δ denotes the area of the triangle with sides of lengths $a a'$, $b b'$, and $c c'$, the volume and circumradius of the tetrahedron are related by the beautiful formula

$$6 R V = \Delta \qquad (40)$$

(Crelle 1821, p. 117; von Staudt 1860; Rouché and Comberousse 1922, pp. 568-576 and 643-664; Altshiller-Court 1979, p. 250).

Let Δ_i be the area of the spherical triangle formed by the ith face of a tetrahedron in a sphere of radius R, and let ϵ_i be the angle subtended by edge i. Then

$$\sum_{i=1}^{4} \Delta_i = \left[2\left(\sum_{i=1}^{6} \theta_i\right) - 4\pi\right] R^2, \qquad (41)$$

as shown by J.-P. Gua de Malves around 1740 or 1783 (Hopf 1940). The above formula provides the means to calculate the solid angle subtended by the vertex of a regular tetrahedron by substituting $\theta_i = \cos^{-1}(1/3)$ (the dihedral angle). Consequently,

$$\Omega = \frac{\Delta_i}{R^2} = 3\cos^{-1}\left(\frac{1}{3}\right) - \pi, \qquad (42)$$

or approximately 0.55129 steradians.

SEE ALSO: Augmented Truncated Tetrahedron, Bang's Theorem, Cube Tetrahedron Picking, Ehrhart Polynomial, Heronian Tetrahedron, Hilbert's 3rd Problem, Isosceles Tetrahedron, Pentatope, Polyhedron Coloring, Reuleaux Tetrahedron, Sierpiński Tetrahedron, Sphere Tetrahedron Picking, Stella Octangula, Tangent Spheres, Tangential Tetrahedron, Tetrahedron 4-Compound, Tetrahedron 5-Compound, Tetrahedron 6-Compound, Tetrahedron 10-Compound, Trirectangular Tetrahedron, Truncated Tetrahedron

Portions of this entry contributed by Frank Jackson

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