MATHematics illuminated

TEXTBOOK

UNIT 8

UNIT 08

GEOMETRIES BEYOND EUCLID

TEXTBOOK

UNIT OBJECTIVES

- Geometry is the mathematical study of space.
- Euclid's postulates form the basis of the geometry we learn in high school.
- Euclid's fifth postulate, also known as the parallel postulate, stood for over two thousand years before it was shown to be unnecessary in creating a self-consistent geometry.
- There are three broad categories of geometry: flat (zero curvature), spherical (positive curvature), and hyperbolic (negative curvature).
- The geometry of a space goes hand in hand with how one defines the shortest distance between two points in that space.
- Stereographic projection and other mappings allow us to visualize spaces that might be conceptually difficult.
- Einstein showed that curved geometry is a way to model gravitational attraction.
- The recently proven Geometrization Theorem states that if we live in a randomly selected universe with a uniform geometry, then it is probably a hyperbolic universe.

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It should be known that geometry enlightens the intellect and sets one’s mind right. All of its proofs are very clear and orderly. It is hardly possible for errors to enter into geometrical reasoning, because it is well arranged and orderly. Thus, the mind that constantly applies itself to geometry is not likely to fall into error. In this convenient way, the person who knows geometry acquires intelligence

IBN KHALDUN (1332-1406)

SECTION 8.1

INTRODUCTION

Math encompasses far more than the study of numbers. At its heart, it is the application of logic in the search for order in the world around us. A fundamental question in this search is how to divide up, or describe, space. The study of this problem, geometry, has been of importance for thousands of years. The geometers of Ancient Egypt established geometric concepts and rules that form the basis of a discussion that has continued into the modern age.

The Egyptians were concerned with a variety of everyday geometric challenges, from how to divide up lands that had been flooded, to the construction of the pyramids. In fact, the need to measure and divide up land helped bring the word "geometry" into existence—"geo" meaning "earth," and "meter" meaning "to measure." The meanings of both of these roots have been expanded throughout the centuries so that now, the "earth" aspect can be thought of as encompassing all of space in general, and the "measure" element can be thought of as "divide into regular sections." Thus, a more useful, modern-day definition of geometry is "the study of how to break space up into regular sections."

As with other mathematical ideas, the geometric concepts of the Egyptians did not stay confined to North Africa, but rather spread across the Mediterranean. Points, lines, circles, and planes formed the vocabulary of a new kind of thinking, one that was tied to empirical observations, and yet could exist without them. The Greeks latched onto this notion of conceptual mathematics, and soon complicated geometric ideas were being constructed with only the most basic of theoretical tools. Much of this knowledge, accumulated over centuries, was collected and expanded upon by the great mathematician Euclid of Alexandria around 300 BC. His comprehensive collection of geometric knowledge, entitled The Elements, went on to become the authoritative math book throughout the world, with over a thousand editions since its initial printing in 1482.

Of central importance to Euclid were his postulates. These were statements that could not be proven and had to be agreed upon as a starting point. His five postulates described a world of straight lines and flat planes. The shapes he focused on were idealized versions of shapes found in nature. The geometric world Euclid described was, and still is, a wondrous achievement of logical construction. It is a world that behaves self-consistently, lending credence to the idea that it is a model of the "real" world. This idea, that statements about the real world can be made on the basis of reason alone, has guided much of western thought for centuries.

SECTION 8.1

INTRODUCTION
CONTINUED

Euclid saw only part of the picture, however. Still, his geometry (which, throughout the remainder of this discussion, will be referred to as "Euclidean geometry") withstood centuries of scrutiny by the best minds of the day. It was not until the 1800s that Euclid's view of the world was shown to be inadequate as a model of the real world. The insights that have come to form the basis of the modern study of geometry do not conform to Euclid's postulates—they do, however, lead to logical ways to describe the world as we know it, and space in general. We are no longer challenged with questions of how to divide plots of land; instead, our new tools enable us to ask, and answer, bigger questions. In fact, we can use the techniques of modern, non-Euclidean, geometry to understand the very fabric of reality.

In this unit we will see how Euclid elegantly combined the mathematical knowledge of his day into a logically self-consistent system. We will then examine how the close scrutiny of one of his fundamental assumptions led to an entirely new kind of geometric thinking. From there we will explore this modern view of geometry to see how one can replace Euclid's straight lines with curves and what that means for our understanding of the universe.

SECTION 8.2

EUCLIDEAN GEOMETRY

- Euclid of Alexandria
- Axiomatic Systems
- Foundations of Geometry

EUCLID OF ALEXANDRIA

- Not much is known of Euclid's life.
- Although he was not responsible for all of the content in The Elements, Euclid broke new ground in his organization of the foundational mathematical knowledge of the day.

Euclid is perhaps the most influential figure in the history of mathematics, so it is somewhat surprising that almost nothing is known about his life. The little that is known is mainly about his work as a teacher in Alexandria during the reign of Ptolemy I, which dates to around 300 BC. This was some while after the creation of Euclid's most famous work, The Elements.

Euclid himself was known primarily for his skills as a teacher rather than for his theorizing and contributions to research. Indeed, much of the content of the thirteen volumes that make up The Elements is not original, nor is it a complete overview of the mathematics of Euclid's time. Rather, this text was intended to serve as an introduction to the mathematical concepts of the day. Its great triumph was in presenting concepts in logical order, beginning with the most basic of assumptions and using them to build a series of propositions and conclusions of increasing complexity.

AXIOMATIC SYSTEMS

- Axiomatic systems are a way of creating logical order.
- Axioms are agreed-upon first principles, which are then used to generate other statements, known as "theorems," using logical principles.
- Systems can be internally consistent or not, depending on whether or not their axioms admit contradictions.

The system that Euclid used in The Elements—beginning with the most basic assumptions and making only logically allowed steps in order to come up with propositions or theorems—is what is known today as an axiomatic system. Here is a very simple example of such a system:

SECTION 8.2

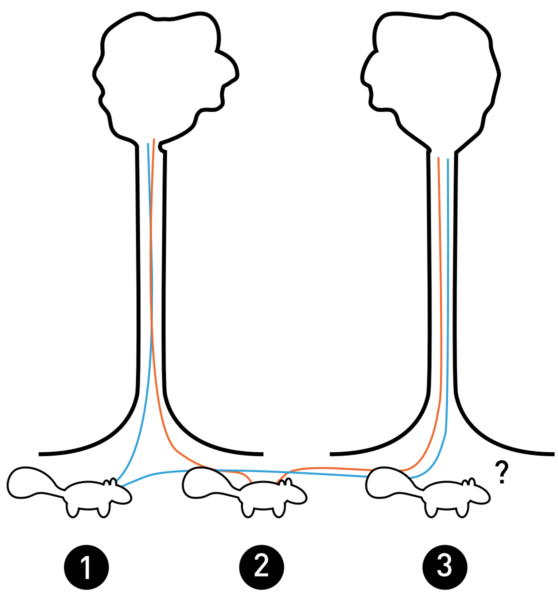
EUCLIDEAN GEOMETRY
CONTINUED

Given the things: squirrels, trees, and climbing,

1. There are exactly three squirrels.
2. Every squirrel climbs at least two trees.
3. No tree is climbed by more than two squirrels.

A logical theorem could be the statement: there must be more than two trees.

A simple picture would prove this theorem:



So, a theorem is something that can be shown to be true, given a set of basic assumptions and a series of logical steps with no contradictions introduced. Now, consider the following axiomatic system:

Given the things: cat, dog
1. A cat is not a dog.
2. A cat is a dog.

It is clear that both statements 1 and 2 cannot be true simultaneously. However, these are the basic axioms of our system, and axioms have to be assumed to be true—so, this system is clearly worthless, because it contains a logical contradiction from the start. In other words, it is not self-consistent. In this example, the contradiction presents itself directly in the axioms, but most contradictory systems are not so easy to identify.

SECTION 8.2

EUCLIDEAN GEOMETRY
CONTINUED

FOUNDATIONS OF GEOMETRY

- Euclid used five common notions and five postulates in The Elements.
- The fifth postulate, also known as the "parallel postulate," is somehow not like the others.

When Euclid laid the foundation for The Elements, he had to be careful to start with statements that would be both self-consistent and basic enough to be assumed true. He divided his initial assumptions into five postulates[1] and five common notions. They are as follows:

Common Notions:
1. Things that are equal to the same thing are also equal to one another.
2. If equals be added to equals, the wholes are equal.
3. If equals be subtracted from equals, the remainders are equal.
4. Things that coincide with one another are equal to one another.
5. The whole is greater than the part.

Postulates:
1. Any two points can be joined by a straight line.
2. Any straight line segment can be extended indefinitely in a straight line.
3. Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.
4. All right angles are congruent.
5. If two lines intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough.

That fifth postulate is a mouthful; fortunately, it can be rephrased. In the fifth century, the philosopher Proclus re-stated Euclid's fifth postulate in the following form, which has become known as the parallel postulate:

Exactly one line parallel to a given line can be drawn through any point not on the given line.

[1] Note: A postulate is not quite the same as an axiom. Axioms are general statements that can apply to different contexts, whereas postulates are applicable only in one context, geometry in this case.

SECTION 8.2

EUCLIDEAN GEOMETRY
CONTINUED

This postulate is somehow not like the other four. The first four seem to be simple and self-evident in that it seems things could be no other way, but the fifth is more complicated. Euclid, himself, likely noticed this discrepancy, as he did not use the parallel postulate until the 29th proposition (theorem) of The Elements.

Euclid's system has been incredibly long-lasting, and it is still standard fare in high school geometry classes to this day. It represents an achievement in organization and logical thought that remains as relevant today as it was 2000 years ago. That bothersome fifth postulate, however, showed a small crack in the foundation of the system. This crack was ignored for centuries until mathematicians of the 1800s, with further exploration, found it to be a doorway into a world of broader understanding.

SECTION 8.3

NON-EUCLIDEAN GEOMETRY

- Of Questioned Necessity
- Saccheri's Quadrilaterals

OF QUESTIONED NECESSITY

- There were multiple attempts to show that the parallel postulate was not necessary to form an internally consistent geometry.
- Girolamo Saccheri built upon the work of Nasir Eddin in an attempt to "clear Euclid of every flaw."

Euclid's parallel postulate bothered mathematicians for many years. Everyone agreed that the first four postulates were completely obvious, but it seemed to be asking too much to say that the fifth was equally as obvious. It was more complicated than the other postulates, and, in fact, many thought that it could actually be proved from them. This idea led to numerous attempts to show that the fifth postulate was not independent of the other four and that, therefore, all of geometry could be built upon only four fundamental ideas. One of the most famous of these attempts was undertaken by an Italian Jesuit named Girolamo Saccheri in the early eighteenth century.

Saccheri wanted to show that the parallel postulate was not necessary (i.e., that it was a derivative of the other four), and if the title of his book, "Euclid Cleared of Every Flaw," is any indication, he truly believed that he had accomplished this. He, like many others before, believed that the parallel postulate could be proved from the other four. After numerous unsuccessful attempts to find a direct proof of his claim, he tried a different tack. His renewed efforts were somewhat influenced by the work of one of Genghis Khan's numerous grandchildren, Nasir Eddin. Eddin had attempted to prove the fifth postulate almost 500 years earlier by looking at quadrilaterals and making assumptions that he hoped would lead to contradictions. We've seen examples of similar "proofs by contradiction" before, such as Euclid's proof of the infinitude of primes back in unit one.

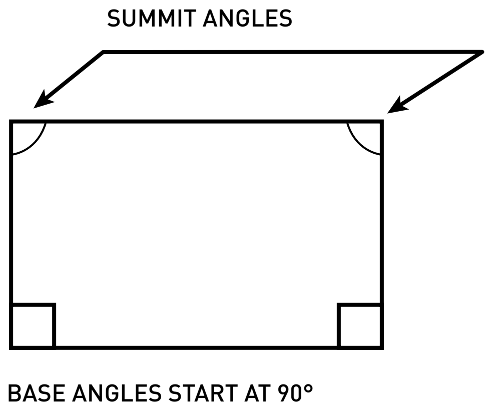
SECTION 8.3

NON-EUCLIDEAN GEOMETRY
CONTINUED

SACCHERI'S QUADRILATERALS

- Saccheri looked at three cases of a quadrilateral constructed without the aid of the fifth postulate.
- Saccheri's results, though intriguing, were misinterpreted.

Saccheri's approach was similar to that of Eddin. He began by considering a quadrilateral whose base angles are both 90 degrees:



He then showed that the summit angles must be equal to each other without using the fifth postulate. In renouncing the parallel postulate in the construction of this quadrilateral, Saccheri had to consider two possibilities—either:

1. There are no lines parallel to a given line, or
2. There are at least two lines parallel to a given line

These cases presented three optional scenarios: 1) that the summit angles are acute, (which would allow for more than one parallel line); 2) that the summit angles are right angles (indicating that there is only one parallel line); and 3) that the summit angles are obtuse (and, therefore, there are no parallel lines). He hoped to show that the only possible arrangement would be the second scenario, because it can play the role of the parallel postulate. Consequently, if it turned out to be the only case that works, then he would have shown that the parallel postulate is not necessary (recall that he arrived at this arrangement without its help).

SECTION 8.3

NON-EUCLIDEAN GEOMETRY
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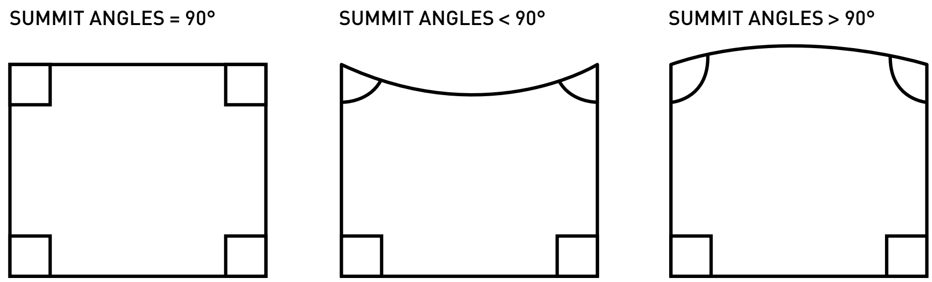

Saccheri was able to eliminate the obtuse angle scenario by assuming (implicitly) that straight lines can be extended forever. If this were not true, it would violate Euclid's second postulate--that lines may be extended indefinitely. This in turn makes the obtuse case useless as far as an indicator of the necessity of the parallel postulate. Ruling out obtuse summit angles leaves just two options—they must be either acute or right.

To show that the acute case was incorrect, Saccheri set about trying to derive the propositions found in The Elements, assuming that the summit angles were less than 90 degrees. He was hoping that this path would consistently lead to contradictions. To his chagrin, he found that he was able to derive many of Euclid's propositions using this assumption and yet still avoid contradictions. He was onto something, but at the time he was unaware that he was indeed building a logically consistent universe in which the summit angles of such a quadrilateral could each be acute.

SECTION 8.3

NON-EUCLIDEAN GEOMETRY
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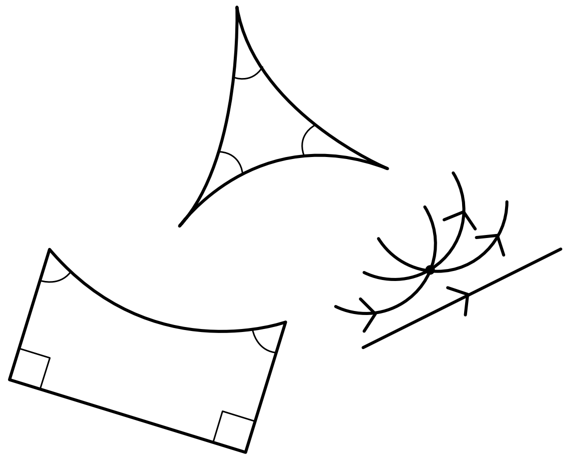


He was so sure that Euclid had to be correct, however, and that the acute angle case could not stand, that he twisted his own logic to accommodate what he had hoped to find, namely that the summit angles must be 90 degrees. Saccheri basically invented a contradiction where none existed in order to fit his preconception. Not surprisingly, his arguments were not convincing, even to the mathematicians of his day. He published another work after "Euclid Cleared of Every Flaw," attempting to clarify his so-called proof, but to no avail. Mathematicians of the time believed that Saccheri had neither proven nor disproven the necessity of Euclid's fifth postulate.

Saccheri's work was not in vain; he simply did not recognize what he had found. In his dogged effort to prove his preconception, he missed out on his claim to one of the great discoveries in geometry: that Euclid's system of geometry is not the only possible self-consistent geometry.

SECTION 8.4

SPHERICAL AND HYPERBOLIC GEOMETRY

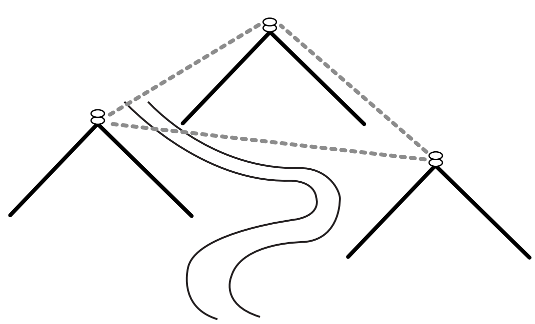
- Gauss
- Lobachevsky and Bolyai
- Spherical Geometry
- A World of Worlds

GAUSS

- Gauss realized that it is possible to construct a self-consistent geometry with the "many parallels" version of the fifth postulate.

Almost 100 years later, in the early 1800s, the great German mathematician Carl Friedrich Gauss attacked the same issue of Euclid's fifth postulate. He recognized that although Saccheri had simply dismissed the possibility of having more than one parallel line through a given point, one could construct a completely self-consistent geometry that differed from that of Euclid by using this postulate instead of the original. This new geometry describes a world in which the summit angles of Saccheri's quadrilaterals are acute.

There is a mathematical legend that Gauss, as much experimental scientist as mathematician, sought to determine whether or not this new type of geometry was actually the geometry of the real world. To do this, he supposedly constructed a great triangle using signal fires and mirrors set on mountaintops.



He then measured the angles between these points of fire and compared his measurements to the expected finding of 180 degrees total. Why would he use mountain tops? Why not just draw a large triangle on a flat space of land? As we will see soon, a shape drawn on the surface of a sphere (or near-sphere, such as Earth) is different than a shape drawn in space. By connecting the tops of mountains with rays of light, Gauss was creating a triangle using the minimal-length connections made possible in space. These lines were free to be as straight as possible, without having to bend or conform to the shape of the surface of Earth. The veracity of this story is questionable; nevertheless, it illustrates the difference between lines on a curved surface and lines in a potentially curved space.

A more-radical mathematical idea for its time would be hard to find, but Gauss did not publish his findings in this arena.[2] Some suggest that this was to avoid confrontation with the great philosopher, Kant, who espoused that the human perception of reality is Euclidean. Others suggest that Gauss was afraid that he would lose face with his contemporary mathematicians. For whatever reason, it was not Gauss who first brought these ideas to light.

LOBACHEVSKY AND BOLYAI

- Lobachevsky and Bolyai independently came to the same conclusion as Gauss.

Nicolai Lobachevsky was a Russian who, in 1829, published a version of a geometry in which, instead of just one parallel line, multiple parallel lines were possible through any given point.

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[2] It is interesting to note that Gauss did not publish many of his ideas. It is commonly thought that this was because he was a perfectionist and would only make his views known if they were above criticism. To that end, he would not provide the intuitions behind his proofs, preferring instead to give the impression that they came "out of thin air." Eric Temple Bell estimated in 1937 that, were Gauss to have been more forthcoming, mathematics would have been advanced by at least 50 years! (Here is yet another example of why students should be encouraged to show their work!)

SECTION 8.4

SPHERICAL AND HYPERBOLIC GEOMETRY
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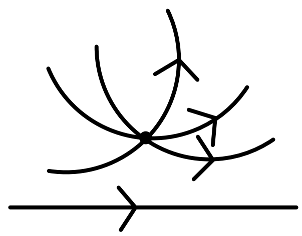
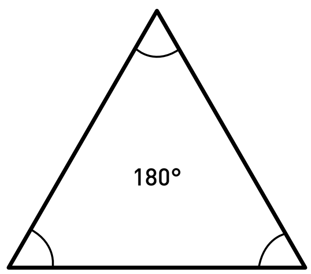
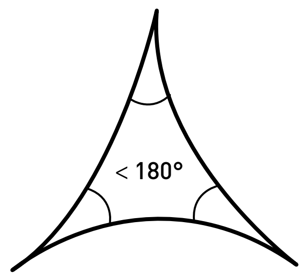


If we take Saccheri's quadrilaterals and modify them a bit, we can show that Lobachevsky's idea is equivalent to saying that the three angles of a triangle can add up to less than a Euclidean 180 degrees.)



EUCLIDEAN



HYPERBOLIC

Lobachevsky showed that this assumption would not lead to any logical contradictions and was, thus, just as valid as Euclid's geometry. Almost at the same time, in 1832, a Hungarian mathematician named János Bolyai published a similar finding after studying what he called "absolute," or neutral, geometry—that is, geometry that uses only the first four of Euclid's postulates.

So, Gauss, Lobachevsky, and Bolyai all independently found that a new and completely self-consistent geometry could be established by letting more than one line through a given point be parallel to a given line that does not include the point. Evidently this was an idea whose time had come. What of the other case, the case in which no parallel lines are possible?

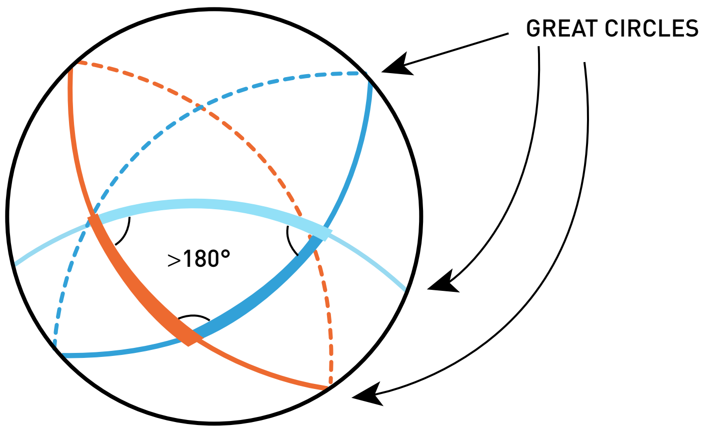
SPHERICAL GEOMETRY

- The "no parallels" flavor of Euclid's fifth postulate yields the geometry of the surface of a sphere.

SECTION 8.4

SPHERICAL AND HYPERBOLIC GEOMETRY
CONTINUED

Oddly enough, a geometry that allows no parallel lines is not as strange as the type that Gauss, Lobachevsky, and Bolyai described. We can indeed imagine a world in which lines always cross other lines. Think about the surface of a sphere, such as Earth (more or less). Lines that follow the surface will continue around the sphere and back to their beginning points to create circles. Lines such as this that are the maximum length for a given sphere (that is, lines whose length is equal to the circumference of the sphere) are called "great circles"—the equator is an example. Thus, the shortest distance between any two points on a sphere will be a part of one of these great circles. Any two such lines will always intersect in two places; hence, there can be no parallel lines in this system! People as far back as the Greeks understood this, and they understood that geometry on a sphere is different from that on a plane.



In this type of geometry, also known as "spherical geometry," Saccheri's quadrilaterals would have obtuse summit angles, and the angles of a triangle would add up to more than 180 degrees. In such a system, one has to replace the parallel postulate with a version that admits no parallel lines as well as modify Euclid's first two postulates. The first postulate's restriction that "through any two points, there is only one possible straight line" does not hold true on a sphere.

SECTION 8.4

SPHERICAL AND HYPERBOLIC GEOMETRY
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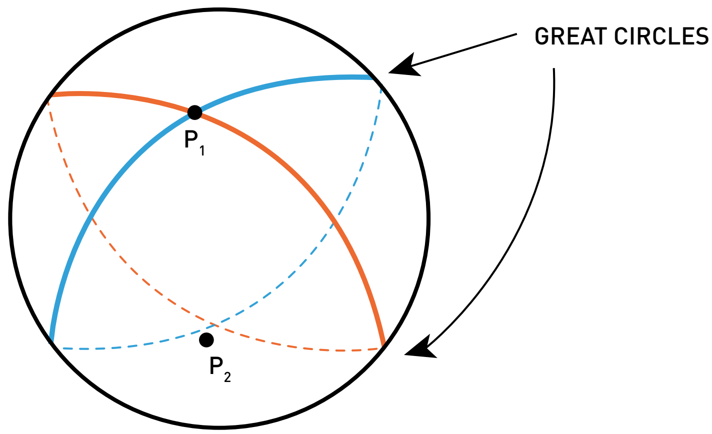
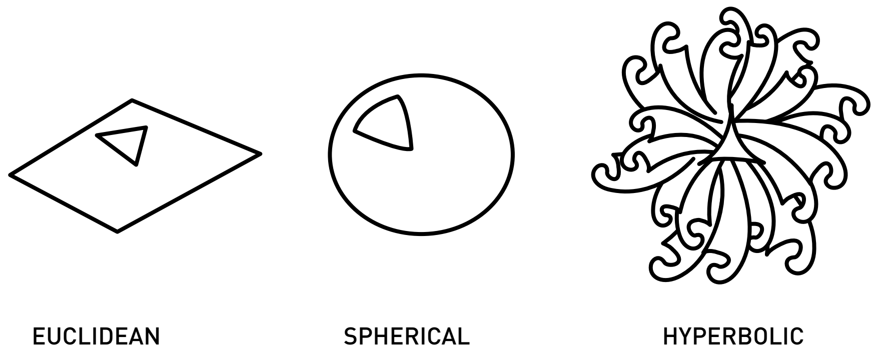


On the surface of a sphere, we must allow for more than one line through any two points. Likewise, we must modify Euclid's second postulate, because "lines" on the surface of a sphere are really circles, which have no end and no beginning—they cannot extend to infinity. We can circumvent this by requiring simply that lines be unbounded.

A WORLD OF WORLDS

- Modern geometry comes in three varieties, and each can be visualized via a model.
- A geodesic is, in the local view, the shortest distance between two points in whichever geometry you choose to use. In a global view, a geodesic is the path a particle would follow were it free of the influence of all forces.
- The surface of a sphere models a geometry that admits no parallel lines.
- The surface of a pseudosphere models a geometry that admits many parallel lines through a given point.

With the inclusion of spherical geometry, mathematicians now had three broad types of geometry with which to study and measure shapes and space: hyperbolic (Lobachevsky), spherical (Riemann), and flat (Euclid).

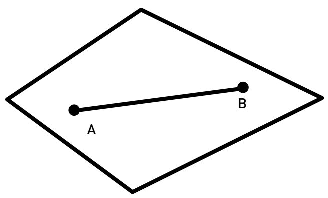


SECTION 8.4

SPHERICAL AND HYPERBOLIC GEOMETRY
CONTINUED

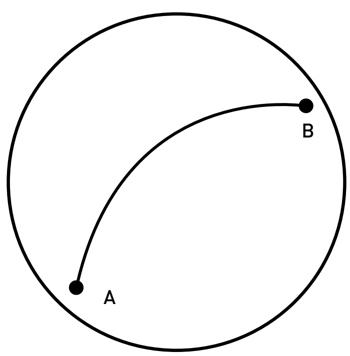
Obviously, the type of world you "live in" and its geometry depend on which flavor" of the fifth postulate you choose. If you choose to allow only one line parallel to a given line through a given point, you are choosing to inhabit Euclid's world of flat geometry. If you choose to allow many parallel lines through the same point, you are in Lobachevsky's world of hyperbolic geometry. If, on the other hand, you choose to allow no parallel lines, you are in the world of spherical geometry.

EUCLIDEAN



We can envision Euclid's universe as a flat plane; this is the universe that we learn about in high school, and its concepts make intuitive sense. To get from one point to another point on the plane, we all know that a straight line will be the shortest distance. This connection of minimal length can be generalized into the notion of a "geodesic." A geodesic, in the local view, is simply the shortest distance between two points in whichever geometry you choose to use. When looking at the geometry of an object on a global level, a geodesic is the path that a particle would take were it free from the influence of any forces.

SPHERICAL



A geodesic, the shortest distance from a to b is curved in spherical geometry.

To envision the spherical universe, we earlier used a sphere as a model. We said that the shortest distance between two points on the sphere, a "geodesic" by our new terminology, is a part of a great circle. In other words, a straight line in spherical geometry is actually a curve, when viewed from the Euclidean perspective.

Lobachevsky's universe is a bit harder to visualize. Eugenio Beltrami, an Italian mathematician working in Bologna, Pisa, and Rome, found a shape, analogous to a sphere, but with a surface that obeys Lobachevsky's geometry. Imagine a tractrix (the path something takes when you drag it by a leash horizontally) rotated around its long axis to generate a shape not unlike two trumpets glued bell to bell.

SECTION 8.4

SPHERICAL AND HYPERBOLIC GEOMETRY
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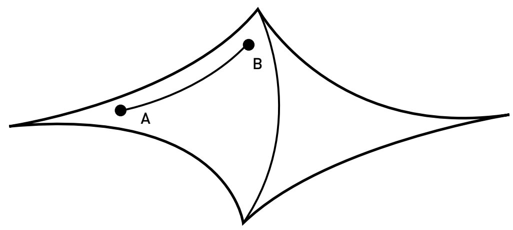


This is called a “pseudosphere,” and it can be thought of as the opposite of a sphere—or, if you are feeling adventurous, as a sphere of imaginary radius. The surface of a psuedosphere behaves according to the rules of hyperbolic geometry. The geodesic of a pseudosphere, the minimal-length connection between two points, is again a curve, but not a section of a great circle, as it was in the world of spherical geometry.

In this discussion, we have seen how basic axioms can define a mental world and that, by changing the axioms, we change the characteristic behavior, or reality, of this mental world. Axioms are verbal statements; to visualize the worlds that they create, we need visual models. We saw earlier that one way to create such models is to embed them in some sort of space. This is what we are doing when we look at a sphere or a pseudosphere. This method has proven to be handy, because it preserves all the geometric properties, such as length and angle, that are determined by our axioms.

However, we don't always have the option of looking at spatial models of the sphere and pseudosphere. Consequently, we have developed another method of visualizing these spaces: maps. Maps are handy because they can be drawn on a flat piece of paper. Unlike our spatial embeddings, however, maps necessarily distort the picture in some way. They can, nevertheless, be of great help as we try to understand all of these strange geometries, and so it is to maps that we next turn our attention.

SECTION 8.5

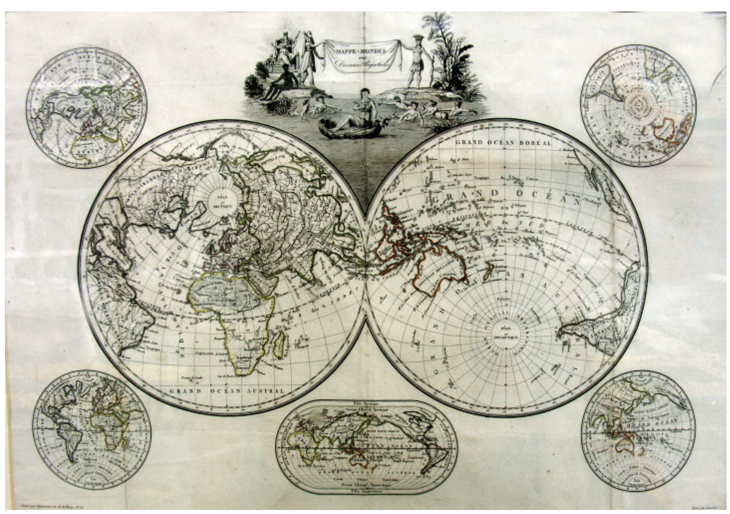
SHAPES ON A PLANE: MAKING MAPS

- Distortion
- Projecting the Sphere
- Poincaré's Disk

DISTORTION

- Representing curved geometries on a flat surface requires distortion.
- The type of distortion depends on the technique used to project a surface onto a map.

If you were to take a globe and compare it to a flat map of Earth, you would find some great discrepancies. Greenland, for instance, appears much larger on a map than on a globe. The reason for this is that whoever designed the map was faced with the tricky task of using a flat surface to portray a piece of the surface of a sphere. Anyone who has ever tried to wrap an unboxed basketball or soccer ball as a gift will have encountered a similar problem—it's difficult to take a flat surface and attempt to form it into a sphere.



Item 2877 / Giraldon, MAPPE-MONDES SUR DIVERSES PROJECTIONS (WORLD MAP)(Ca. 1815). Courtesy of Kathleen Cohen.

For our purposes, a map is a flat representation of a curved space.

SECTION 8.5

SHAPES ON A PLANE: MAKING MAPS
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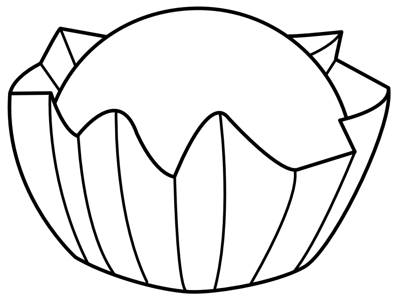
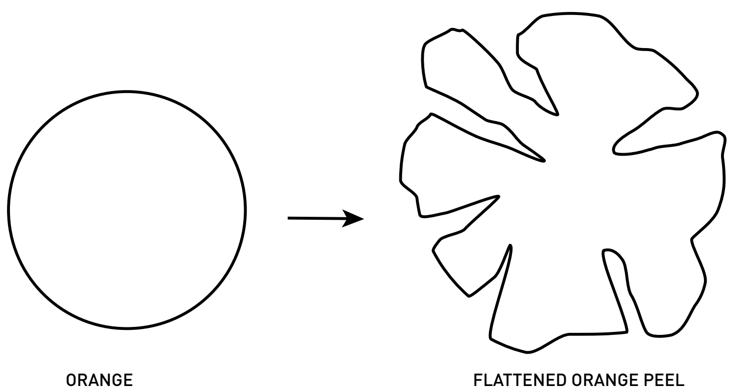


Likewise, if you've ever peeled an orange, especially if you've accomplished this with just one strand of peel, you've found that you can't lay it on a flat surface without tearing it. This is actually just the opposite problem of gift-wrapping the ball—here we're trying to take a spherical surface and turn it into something flat.



This is also exactly the problem that the mapmaker faces. Fortunately, there are a variety of techniques for translating an image of a curved surface onto a flat piece of paper. What's even better is that, once we have a general technique, we can use it to make maps not only of spheres, but also of pseudospheres and other curved surfaces that defy our intuition. The technique that we will focus on in this discussion is called a "stereographic projection."

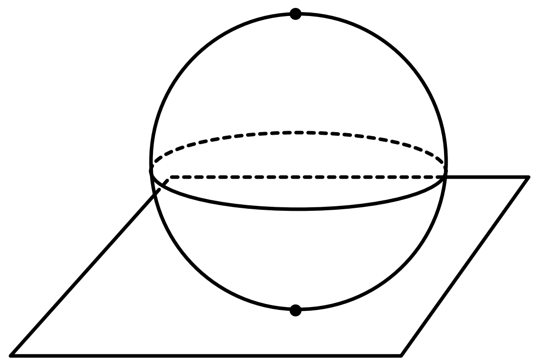
SECTION 8.5

SHAPES ON A PLANE: MAKING MAPS
CONTINUED

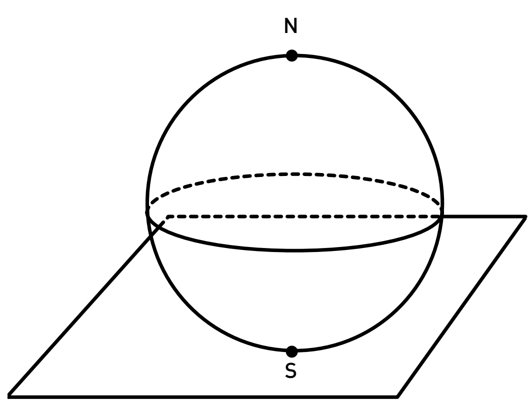
PROJECTING THE SPHERE

- The stereographic projection of a sphere yields a map with increasing distortion of features near its north pole.
- The stereographic projection preserves angles, but not length.
- Geodesics are portions of a circle in stereographic projections of the sphere.

Let's first see how stereographic projection works for a sphere.



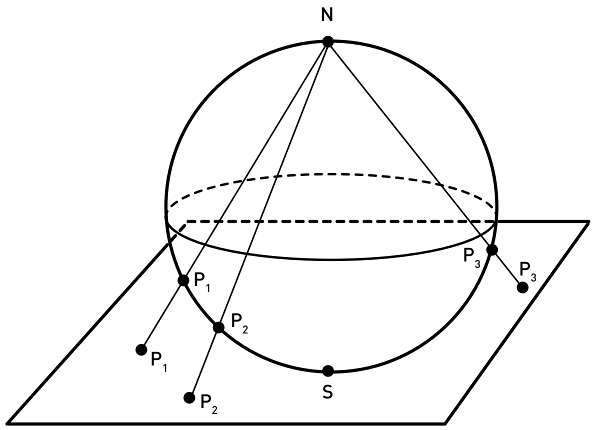
Take a sphere and place it on a plane. Let's call the exact point where the sphere touches the plane the south pole. The plane is tangent to this point. The north pole would then be the point antipodal to the south pole, or in other words, the point directly across the interior of the sphere.



To perform a stereographic projection, we draw straight lines from the north pole to the plane, rather like staking down a tarp or a tent.

SECTION 8.5

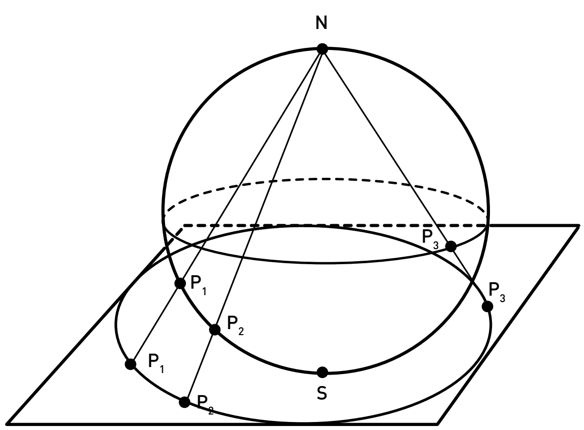
SHAPES ON A PLANE: MAKING MAPS
CONTINUED



Now, each of these lines will intersect the sphere and continue on to the plane. This is how we will map each unique point on the sphere to a unique point on the plane.

Notice that every point on the sphere will be uniquely mapped onto the plane except for the north pole. Where should it go? If we notice that points arbitrarily close to the north pole get mapped further and further out on the plane, it makes sense to define the north pole to be mapped to infinity.

What happens to geodesics in this mapping? To answer this question, we can start by looking at the equator, which is a special case of geodesic. Notice that the equator gets mapped to a circle on the plane.

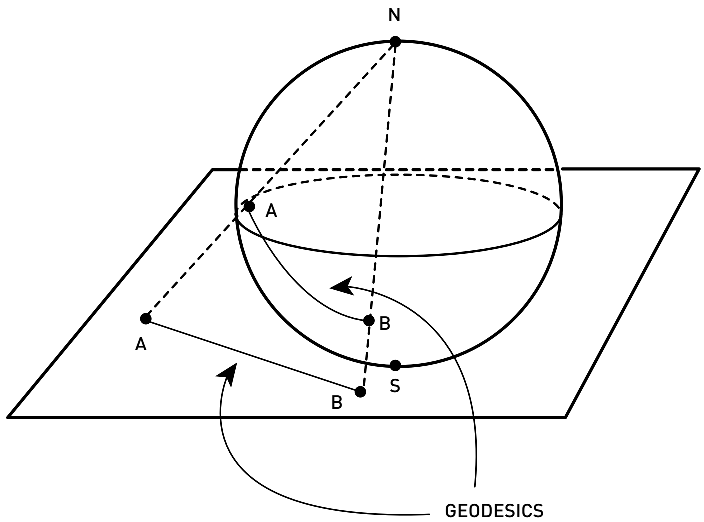


We call this the "equatorial circle." All other geodesics on the sphere get mapped to circles and lines in the plane that intersect the equatorial circle at two opposite (called "antipodal") points.

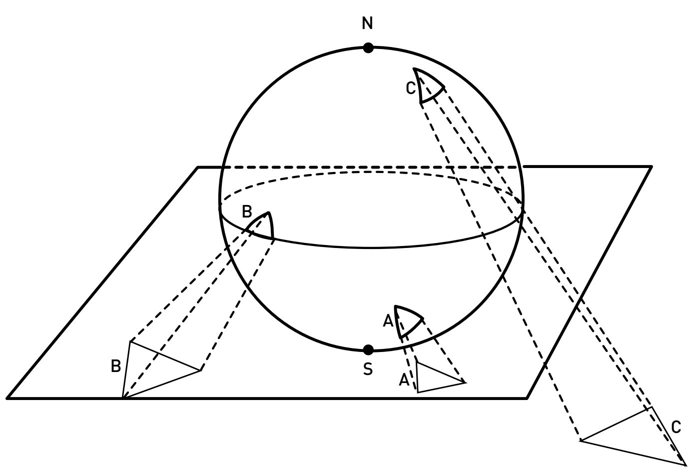
SECTION 8.5

SHAPES ON A PLANE: MAKING MAPS
CONTINUED

What determines whether or not a geodesic on the sphere gets mapped to a circle or a line on the plane? Recall that the north pole gets mapped to infinity, so any geodesic on the sphere that passes through the north pole will, when mapped to the plane, extend to infinity, forming what we normally think of as a line.



A great thing about the stereographic projection is that it is conformal, which means that it preserves the angles between geodesics. In other words, the shape of an object, such as a triangle, is preserved because its angles are, but its size is not preserved. This is because in order to preserve angle, we must distort lengths.



SECTION 8.5

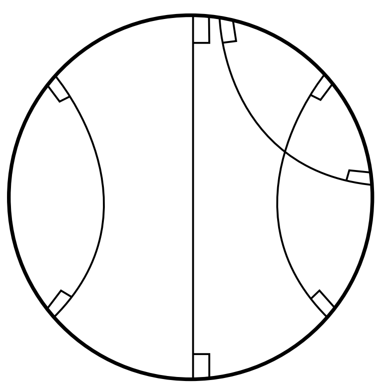
SHAPES ON A PLANE: MAKING MAPS
CONTINUED

Notice that triangle A, in the southern hemisphere, looks more or less the same size in its projection, whereas triangles B and C, each located progressively further north on the sphere, get larger and larger on the plane. The triangles are still three-sided figures, but their sizes have changed. Also, notice that they appear "fat" on this map. This is a consequence of the fact that the three angles of a triangle can add up to more than 180 degrees in spherical geometry. The only way to represent this on the map is to replace the straight geodesics of the plane with the curved geodesics of the sphere.

THE POINCARÉ DISK

- Hyperbolic space can be modeled using the Poincaré disk.
- The boundary of the Poincaré disk represents infinity.
- Most geodesics map as semi-circles that form right angles with the boundary of the disk.
- Triangles on the disk are "skinny."

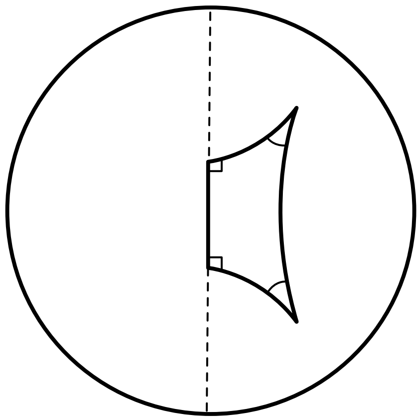
Now that we have seen a map of spherical space, let's look at a map of hyperbolic space. We saw earlier that the pseudosphere is a good model of hyperbolic space. In a process similar to the one we used with the sphere, we can make a map of the pseudosphere.



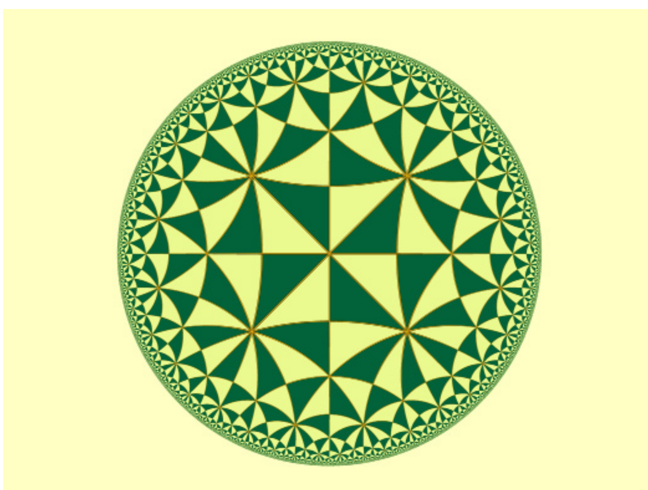
Multiple parallel lines through a given point. Note that the edges of the disk represent infinity.

This map is called a Poincaré disk, in honor of Henri Poincaré, the great French mathematician, who was its initial creator. The boundary of the disk is mapped to infinity. Most geodesics are represented on this map as semicircles that make right angles with the boundary, signifying that lines in hyperbolic space both "begin" and "end" at infinity. Geodesics that pass through the center of the disk, however, are represented as straight lines, connecting antipodal points.

Like the spherical map we just saw, this map is conformal—it preserves the angles between geodesics. We can see quite easily how Saccheri's quadrilateral, if mapped on a Poincaré disk, would have acute summit angles.



Triangles in hyperbolic space appear, on this map, to be "skinny" or "cuspy," showing that their angles add up to less than 180 degrees.



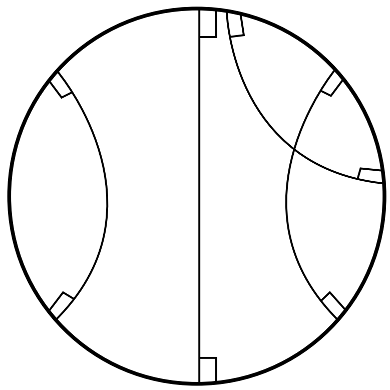
Item 1725 / Jos Leys, HYP023 (2002). Courtesy of Jos Leys.

A flap map of hyperbolic space tiled with quadrilaterals.

We can see that the "many-parallels" version of Euclid's fifth postulate is obeyed. If we draw a geodesic, we will get a rainbow shape. If we then choose a point not on that line, we will be able to draw as many parallel lines as we choose. In other words, this proves that we do indeed have a map of hyperbolic space.

SECTION 8.5

SHAPES ON A PLANE: MAKING MAPS
CONTINUED



We see from these examples that it is possible to make maps of the different geometries discussed so far. It should be evident from exploring the nature of these geometries and their maps that, to be comfortable "getting around" in these new multi-dimensional realms, we are going to have to understand curves as well as we understand straight lines. To further that understanding, we now turn to the subject of how to measure curvature.

SECTION 8.6

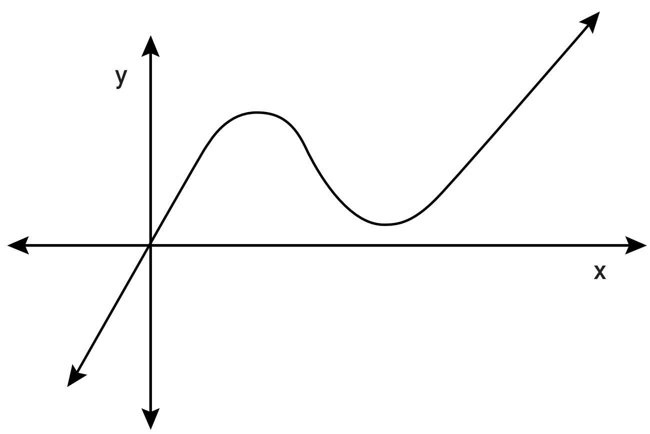
CURVATURE

- The Kissing Circle
- Curved Surfaces
- Pi Doesn't Have To Be 3.14159...

THE KISSING CIRCLE

- We can measure the curvature of a curve in the plane extrinsically via an osculating circle.
- Intrinsic measurements of curvature are impossible in one dimension.

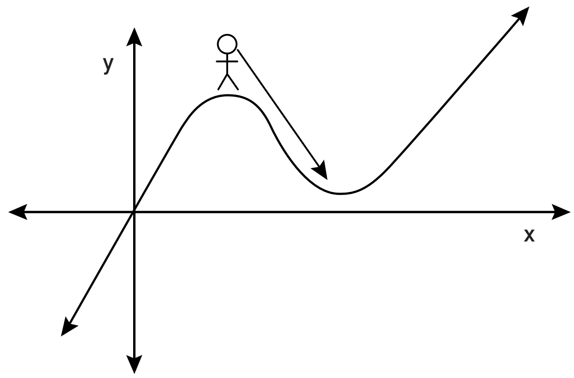
In looking at both the sphere and the pseudosphere, we see that they are unlike the plane in that they are both curved surfaces. Furthermore, we saw that "straight" lines (lines of the shortest length, in other words) are not straight at all on these surfaces; rather, they are curves. In order to explore these surfaces, and others that do not obey Euclid's fifth postulate, we need to be able to discuss curves meaningfully. Let's start with a simple curve in a plane:



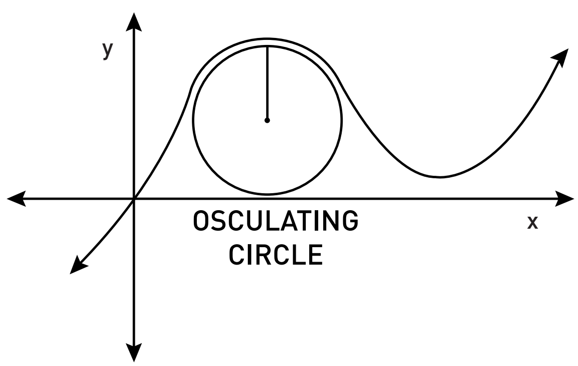
How can we describe this curve's curviness? We could compare it to a circle, a "perfect curve" in some respects, but it is evident that this curve is not really even close to being a circle. It has regions that seem more tightly curved than others, and it even has regions that curve in opposite directions. When we look at a curve in this way, in the broader context of the plane, we are viewing it extrinsically. By contrast, viewing a curve intrinsically, that is from the point of view of someone on the curve, yields a different perspective and different possible measurements.

SECTION 8.6

CURVATURE
CONTINUED



So, perhaps a good thing to do would be, instead of talking about the curvature of the whole thing right away, to talk about the curvature at each point along the curve. As theoretical travelers along the curve, we could stop at each point and ask, "What size circle would define the curve in the immediate vicinity of this point?"

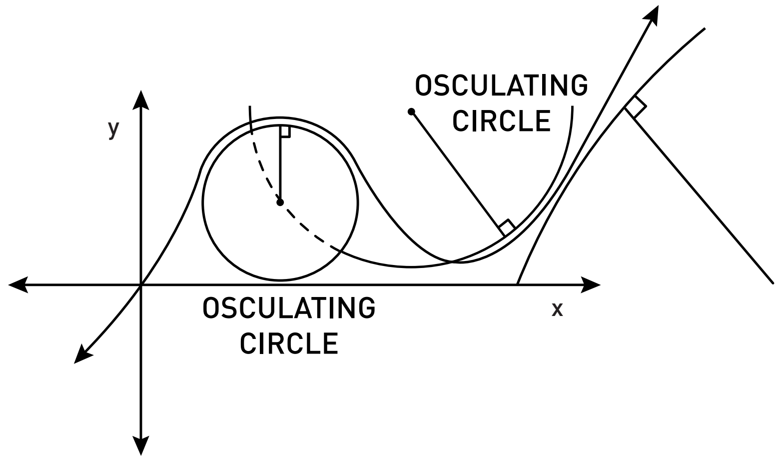


First, let's think of the tangent line at this point. This is a line that intersects our curve only at this one point (locally, that is—it's possible that the line might also intersect the curve at other more-distant points). We can also think of the tangent as the one straight line that best approximates our curve at this particular point. Let's then draw a line from this tangent point, perpendicular to the tangent line.

Let this line, called a "normal line" or just a "normal," be of a length that, if it were the radius of a circle, that circle would be the biggest possible circle that still touches the curve in only one place. In other words, the normal line should be the radius of the circle that best approximates the curve at this particular point. Such a circle is called an "osculating circle," which literally means "kissing" circle, because it just barely touches, or "kisses," the curve at this one point.

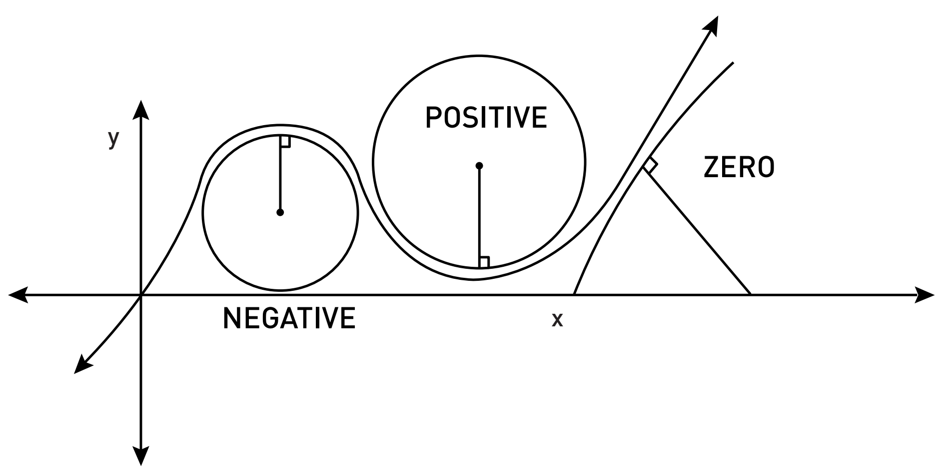
SECTION 8.6

CURVATURE
CONTINUED



We can define the curvature of a curve at any particular point to be the reciprocal of the radius of the osculating circle that fits the curve at that point. Let's refer to this curvature as "k" from here on out.

What should we do, though, about the fact that some parts of the curve open upward, whereas other parts of the curve open down? If we designate that the normal always points to the same side of the curve—let's choose upward for our case—then when the normal happens to be on the same side as the osculating circle, we'll call this negative curvature, and when the normal happens to be on the opposite side from the osculating circle, we'll call this positive curvature. At any point where the line is flat (i.e., straight), we don't need an osculating circle, and we'll call this zero curvature. The choice of defining which curvature to consider positive and which to consider negative is completely arbitrary. The method chosen for this example is nice because, if we think of our planar curve as a landscape, then the positively curved areas are the hills and the negatively curved areas are the valleys.



SECTION 8.6

CURVATURE
CONTINUED

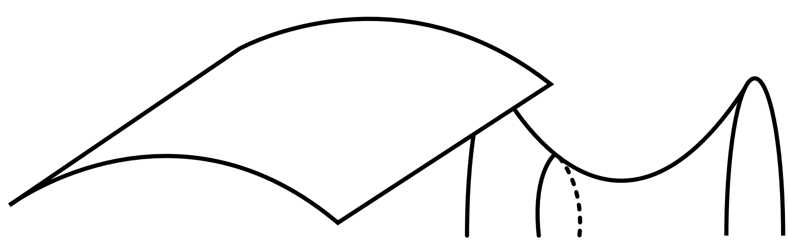
An interesting feature of looking at a curve in this way is that, were we one-dimensional beings living on the curve, we would not notice that it is curved at all. This is called an intrinsic view. The only thing that can be measured intrinsically on a curve is its length, and length alone tells us nothing about how curvy a one-dimensional object is.

Remember that the way we quantified the curvature of this curve was to compare it to a circle in the plane. Now, as one-dimensional beings, this requires envisioning one more dimension than would be available to our perception. The curvature becomes apparent only when the curve is viewed by an observer not on the curve itself—that is, one who can see it extrinsically in two dimensions.

Using this system, we can meaningfully talk about any curve in a plane, and we know from previous discussions that once we understand something in a lower-dimensional setting, we can generalize our thinking to a higher-dimensional setting. In this case, instead of talking about plane curves, we will return to our curved surfaces, such as the sphere and the pseudosphere.

CURVED SURFACES

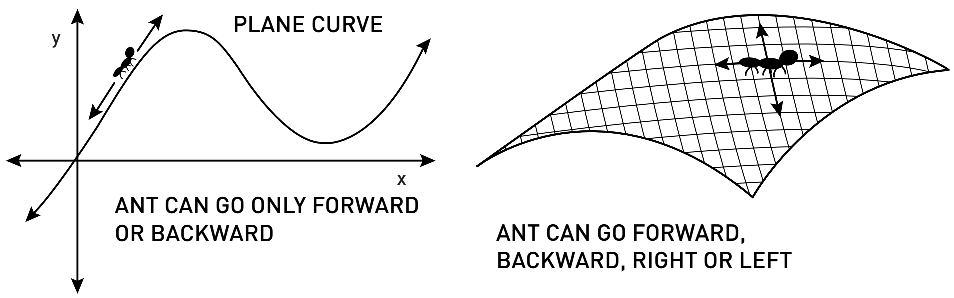
- One extrinsic way to measure the curvature of a two-dimensional surface is through principal curvatures.
- Principal curvatures cannot be used intrinsically to measure curvature.



Let's take a moment to compare and contrast our plane curve and our curved surface. Our plane curve, though drawn in a two-dimensional plane, is actually only a one-dimensional object. This is because, if you were an ant living on this curve, you would only have the option of traveling forward or backward. Because of this, you wouldn't even really know that your line was curved.

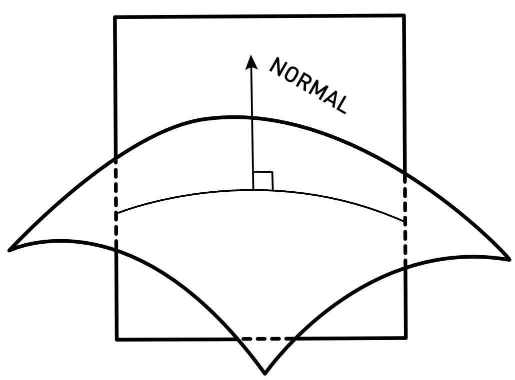
SECTION 8.6

CURVATURE
CONTINUED



A curved surface, on the other hand, is two-dimensional. If you were an ant living on it, you could move forward, backward, right, or left. As you can see, however, a curved surface requires a third dimension to represent it extrinsically, just as a one-dimensional planar curve requires a second dimension for its extrinsic representation. We said that an ant on a plane curve cannot experience this second dimension and, thus, has no idea that his world is curved. Is the same true for an ant on the surface, however? It can't experience the third dimension, but might it still be able to find out if its world is curved?

To resolve this, we need to find a way to apply our concept of the osculating circle to a curved 2-D surface. Actually, we can begin the same way as before.

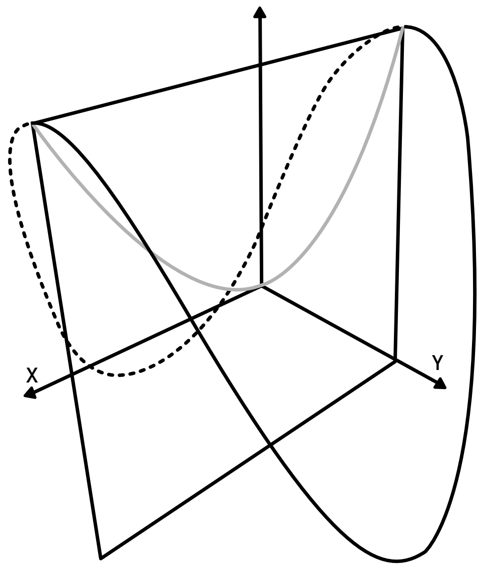


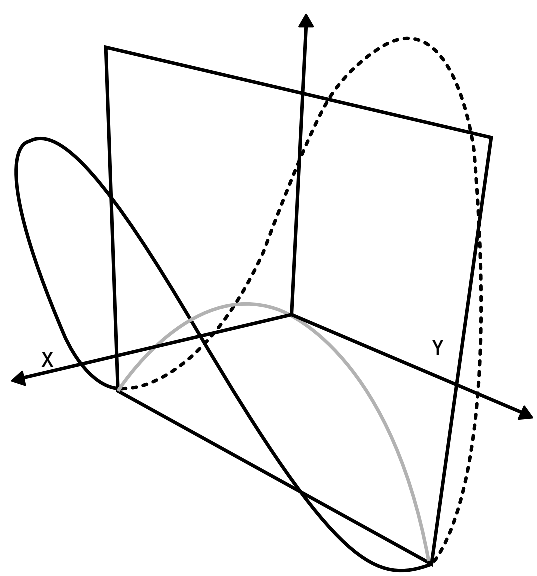
Normal plane slicing a curved surface.

Let's pick a point on our surface and define the normal (remember, that's a line that is exactly perpendicular to the surface at this point). If it helps, imagine the plane that is tangent at this one point as a flat meadow, and envision the normal as a tree growing straight up in the middle of the meadow. Now that we have both our point and our normal set, we can look at slices of the surface that contain both the point and the normal.

SECTION 8.6

CURVATURE
CONTINUED



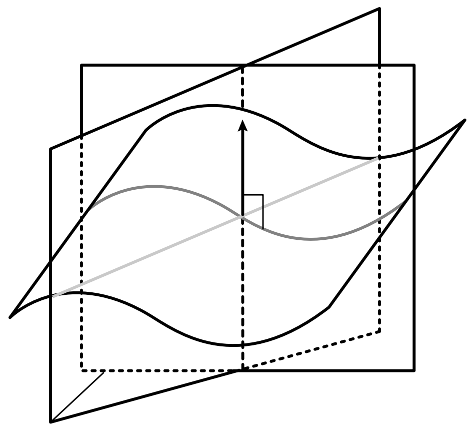


It is clear that each of these slices through the surface will show a slightly different curve, yet all of them contain our point of interest. So, which of these slices is "the" curvature at this point? We have so many possibilities to choose from!

One path toward a solution involves considering the extreme values—in other words, the curve that is most positively curved and the curve that is most negatively curved. We call these the "principal curvatures." If we were then to take the average of these two quantities, we would have a mean curvature for this point.

SECTION 8.6

CURVATURE
CONTINUED



Would an ant on this surface be able to find, or develop an awareness of, these principle curvatures? To do so, it would have to have some idea of a plane that is perpendicular to the plane of his current existence. The complicating factor here is that the ant has no idea that another perpendicular direction can even exist! It will have a great deal of difficulty trying to figure out curves that can only be seen with the aid of a perspective that it can't have.

All hope is not lost, however. Again, we can turn to Gauss. His Theorem Egregium says that there is a type of curvature that is intrinsic to a surface. That is, it can be perceived by one who lives on the surface. Usually, this curvature, called the Gaussian curvature, is simply defined as the product of the two principal curvatures. For our example here, however, that will not be good enough, because our ant can't even know the principal curvatures!

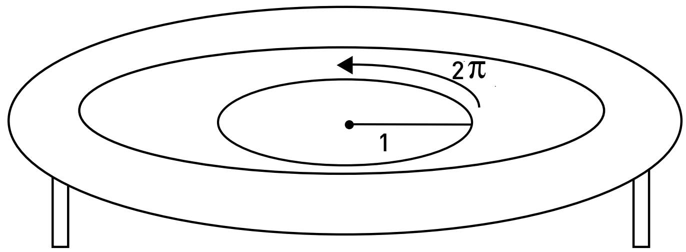
PI DOESN'T HAVE TO BE 3.14159...

- One can measure the intrinsic curvature of a surface by drawing circles and comparing their circumferences to their radii.
- Positive curvature yields a smaller circumference than we would expect for a given radius.
- Negative curvature yields a larger circumference than we would expect for a given radius.

Instead of trying to find the principal curvatures, the ant can draw a circle on his surface and look at the ratio of the circumference to the diameter. This ratio is often known as π, and in flat space it is about 3.14159. We usually consider π to be a universal constant, and it can be, but that depends on which universe we are talking about. In a Euclidean universe, π is indeed constant. In non-Euclidean universes, however, the value of π depends on where exactly the circle is drawn—it's not a constant at all!

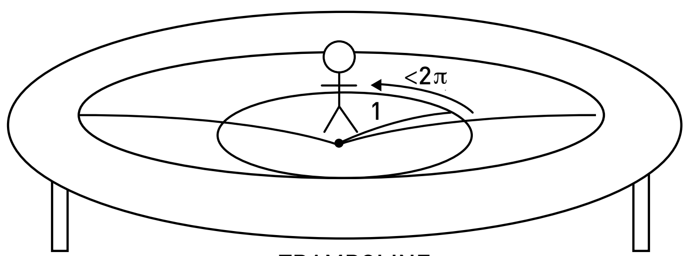
SECTION 8.6

CURVATURE
CONTINUED



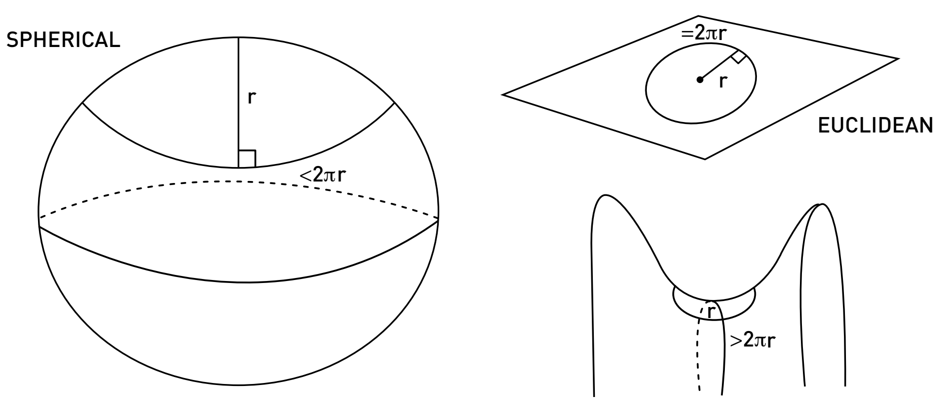
TRAMPOLINE

Consider a circular trampoline. The circumference of this trampoline is fixed, but the webbing in the center is flexible and can be thought of as a surface. When no one is standing on the trampoline, the ratio of its circumference to its diameter is indeed π. Now, consider what happens when someone stands in the middle of the trampoline: the fabric stretches and the diameter, as measured on the surface, increases.



TRAMPOLINE

The circumference, however, remains unchanged as the surface is stretched. This causes the ratio of the circumference to the diameter, π , to decrease. Our ant could indeed detect such a distortion! This would be positive curvature.

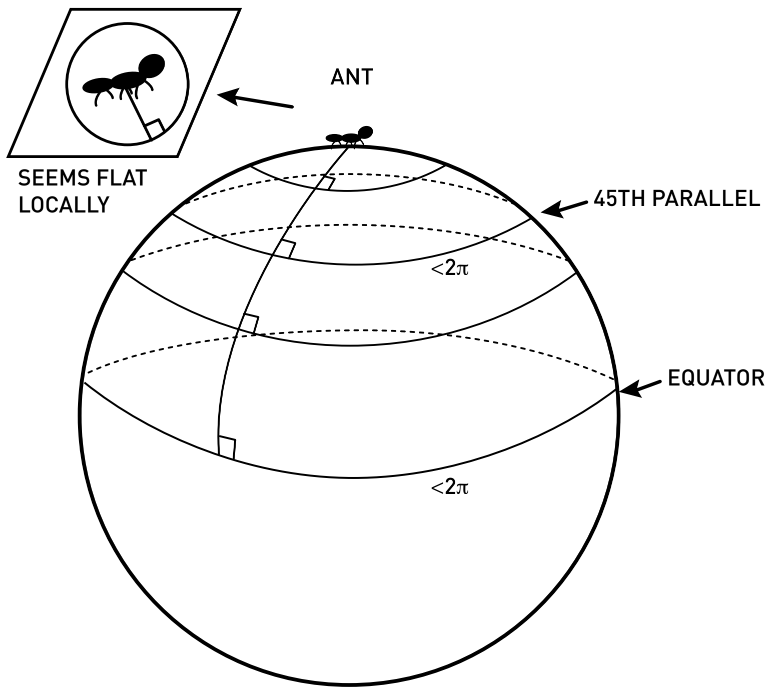


3 circles, 3 geometries, 3 different ratios of circumference to diameter

SECTION 8.6

CURVATURE
CONTINUED

For an example of how an ant could detect the curvature of a surface, consider a globe. If we draw a small circle near the north pole, it will be more or less indistinguishable from circles drawn on a flat plane. Now draw a circle that is a bit bigger, say at the 45th parallel. This line is halfway between the north pole and the equator. Its radius, as measured on the surface, will be considerably longer in proportion to the circumference of the circle than was the case for the small polar circle. Therefore, the ratio of circumference to diameter will be smaller—in other words, larger diameter and smaller circumference.



Now consider the circle represented by the equator. The diameter of this circle, as measured on the surface, will be half the length of the circle! This would mean that for the equator, π is equal to 2. This is quite a discrepancy from the customary 3.14... value, and it indicates that we must be on a curved surface.

Negative curvature can be visualized as a saddle. Such a surface has more circumference for a given radius (and, hence, diameter) than we would expect with either flat or positive curvature.

Gaussian curvature is not as concerned with determining specific values of π as it is with measuring how π changes as the radius changes. The more curved a surface is, the faster π will change for circles of increasing radius.

SECTION 8.6

CURVATURE
CONTINUED

This idea, that there are certain properties that can be measured regardless of how our curve sits in space, was important in our topology unit and, as we have just seen, it plays a significant role in our discussion of curvature as well. These intrinsic properties of a surface—or the generalization of a surface, a manifold—are definable and measurable without regard to any external frame of reference. The Gaussian curvature is such a property, but the principal curvatures are not.

Recall that to find the principal curvatures, one must take perpendicular slices, which requires that our surface sit in some higher-dimensional space. This is an extrinsic view. The fact that the Gaussian curvature of a surface, as computed by the principal curvatures, yields an intrinsic quantity is quite remarkable. In fact, it is known to this day as "Gauss's Theorema Egregium," meaning "Gauss's Remarkable Theorem."

So, a natural question to ask might be: what kind of surface do we live on? We must have an intrinsic view of whatever space we inhabit—indeed, we have no way to get outside of it! A bit of thought, though, will lead to the realization that, unlike ants, we perceive a third dimension, so whatever this is that we are all living on, it is not a 2-D surface, but rather a 3-D manifold. A manifold can be thought of as a higher-dimensional surface, or it might help to think of it as a collection of points that sits in some larger collection of points.

Furthermore, our everyday experience includes a fourth degree of freedom, time. If we consider time to be part and parcel of our reality, then we are really living in a 4-D manifold called "spacetime." So, is our spacetime the 4-D equivalent of a flat, Euclidean plane, or is our reality curved in some spherical or hyperbolic way? For help in exploring this question, we'll turn to the ideas of a certain former patent clerk whose theories permanently altered the way in which we view our universe. First, however, we need to consider what happens when our surface is not as "nice" as a simple, smooth sphere or pseudosphere.

SECTION 8.7

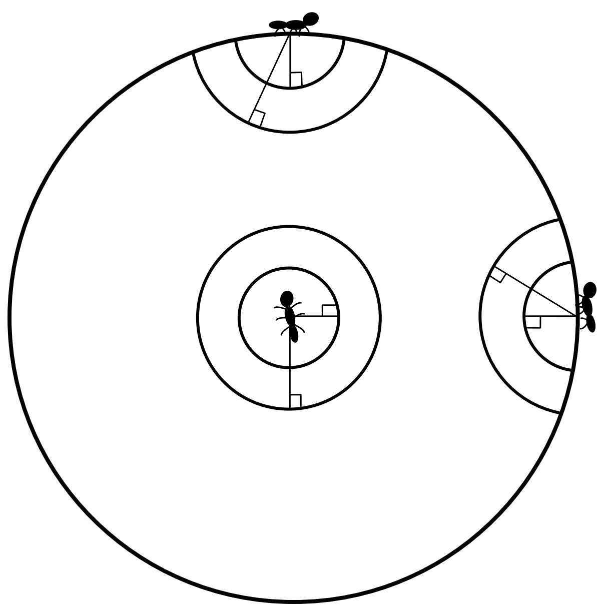
GENERAL RELATIVITY

- Riemann's Hills and Valleys
- Einstein

RIEMANN'S HILLS AND VALLEYS

- Riemann described how to deal with geometries that contain regions of positive and negative curvature.
- The concept of the curvature of 3-D manifolds paved the way for Einstein's work.

We saw earlier that an ant on a globe can find the curvature of his world by drawing circles of differing sizes and seeing how the value of π changes. The globe is a surface of constant positive curvature, which simply means that no matter where our ant decides to begin drawing circles, he'll find that π changes in the same way.

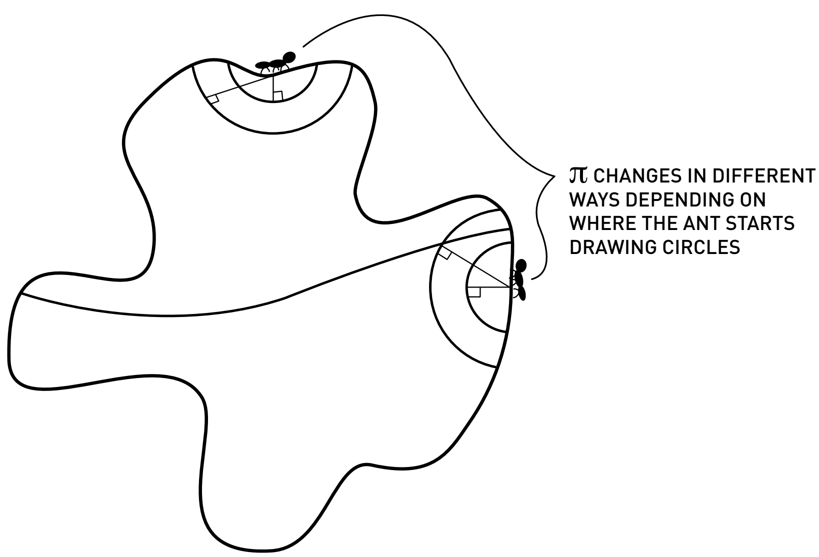


However, what if the surface that the ant draws on were not so uniform? Let's say that, just like the real surface of Earth, the ant's surface has hills and valleys. Dimensionally, this would mean that some parts of his world are more curved than others. Riemann studied surfaces like this and came up with a generalized way to deal with such surfaces and manifolds—ones with so-called, non-constant curvature.

SECTION 8.7

GENERAL RELATIVITY
CONTINUED

While the details of this method are beyond our scope, there are a couple of key features worth noting. The first condition is that all the hills and valleys must be smooth—in other words, there is a way to get from point to point without encountering any cliffs or walls. As long as this condition is met, Riemann said that it is possible to use the local geometry to find the lengths of curves. So, our ant would basically draw circles, as it did before, but it would see that π changes in different ways, depending on the specific placements of the circles.



Using this method, the ant can measure the intrinsic curvature of the different regions of its world. We find this idea of a non-constantly-curved surface easy to visualize because we have a vantage point from the third dimension.

There is no reason, however, to think that only 2-D manifolds can be curved. In fact, it would be surprising if that were true, because we have seen that math concepts often generalize to higher dimensions. What would curvature in a 3-D manifold look like? This question puts us in a bit of a bind, for, just like the ant, we have no higher-dimensional viewpoint from which we can observe this curved space. We are stuck in it! Also, we must consider that our space is not of uniform curvature. That is, the ant's surface can have mountains and valleys, so our space might have the 3-D equivalent. Thus, there might be pockets of our space that are more curved than others, and there might be places that are more or less flat.

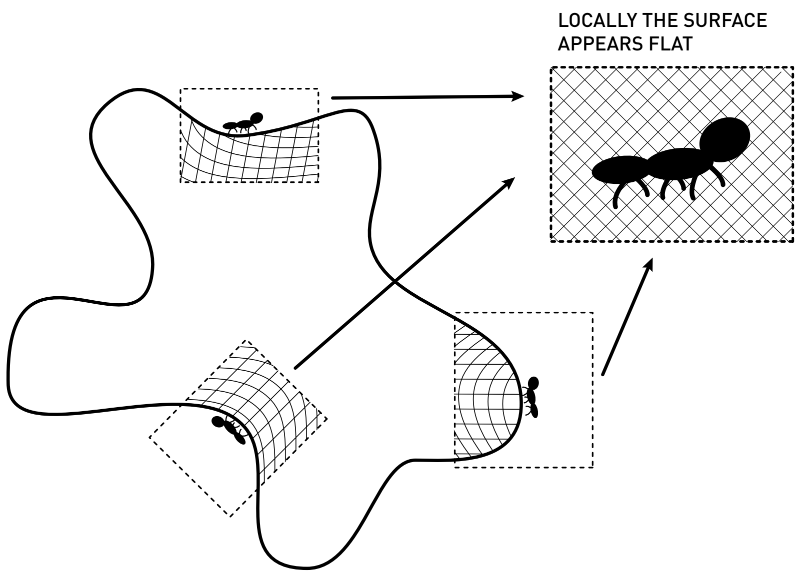
SECTION 8.7

GENERAL RELATIVITY
CONTINUED

EINSTEIN

- Einstein said that space and time curve around massive objects, creating gravity.
- Events follow geodesics in this curved spacetime.
- The General Theory of Relativity has been experimentally verified.

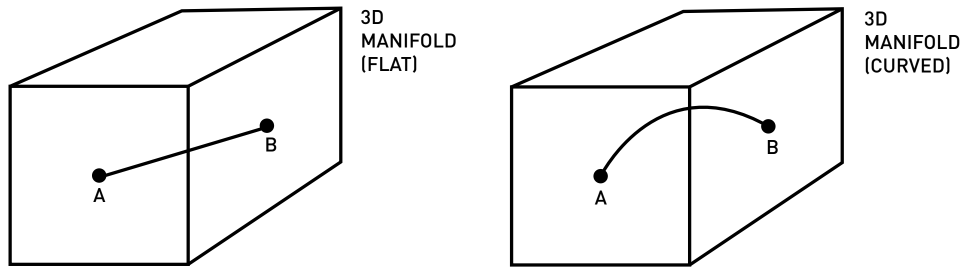
Einstein shed light on these questions with his General Theory of Relativity, which he developed in 1915-1916. He used the notion that some areas of a surface are more curved than others. All that is required is that, if we look closely enough at the surface, it appears to be flat.



Likewise, if you look at a planar curve closely enough, it will appear to be a straight line. A straight line is, of course, the geodesic of a Euclidean plane. So, there are regions, however small, of any line that will appear as a geodesic of the flat plane. A geodesic in a 3-D manifold will also be a line. If that manifold has curvature, then the geodesic line will be curved also. Nonetheless, just as with the planar curve, we can look at this curve in our 3-D manifold closely enough for it to seem straight.

SECTION 8.7

GENERAL RELATIVITY
CONTINUED



What would a geodesic in spacetime look like? Following the definition, it would be the minimal-length connection between one time-and-place and another time-and-place. Another way to think about a minimal-length connection is to think of the path of least resistance. A geodesic can then be considered to be the path that requires the least amount of energy. If we place an object at an arbitrary point in spacetime, whatever it does naturally can be considered to be the geodesic of that spacetime. If there happens to be a massive object, such as a planet, nearby, we would expect our test object to "fall" toward the planet. This suggests that the state of free fall is a minimal-length connection "in action" in spacetime. In other words, falling is like following a geodesic.

Einstein noticed that an object in free fall "feels" no force of gravity.[3] This is analogous to an ant looking very closely at a curved surface and seeing it to be flat—it doesn't see the curvature. Likewise, if one thinks of gravity as the curvature of our 4-D manifold of spacetime, then being in free fall, in which the effects of gravity are not noticed, is like looking closely at the curved surface. In other words, it is a perspective from which geodesics appear to be flat (straight lines). Free fall is just a straight-line geodesic through spacetime.

Using reasoning such as this as a starting point, Einstein re-interpreted gravity to be a result of the geometry of a curved spacetime. He said that it is not a force in the Newtonian sense; rather, it is the effect of living in a curved manifold. So, in other words, whereas the ant must draw circles to experience the curvature of his world, we experience the curvature of our world though gravity.

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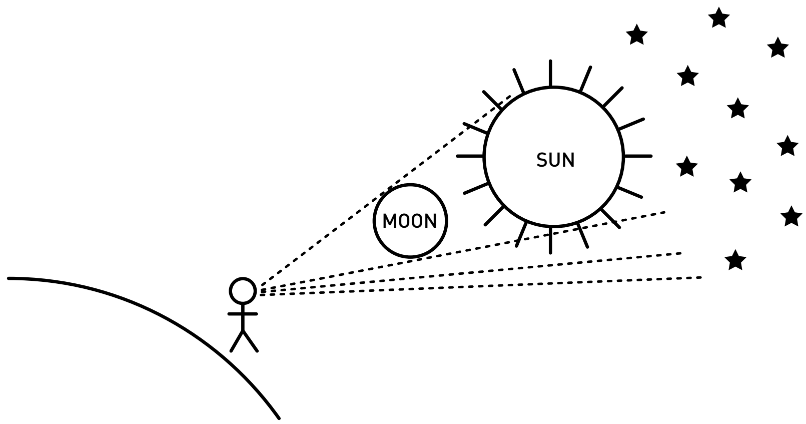
[3] Supposedly, this was after interviewing a painter who had fallen off of a house.

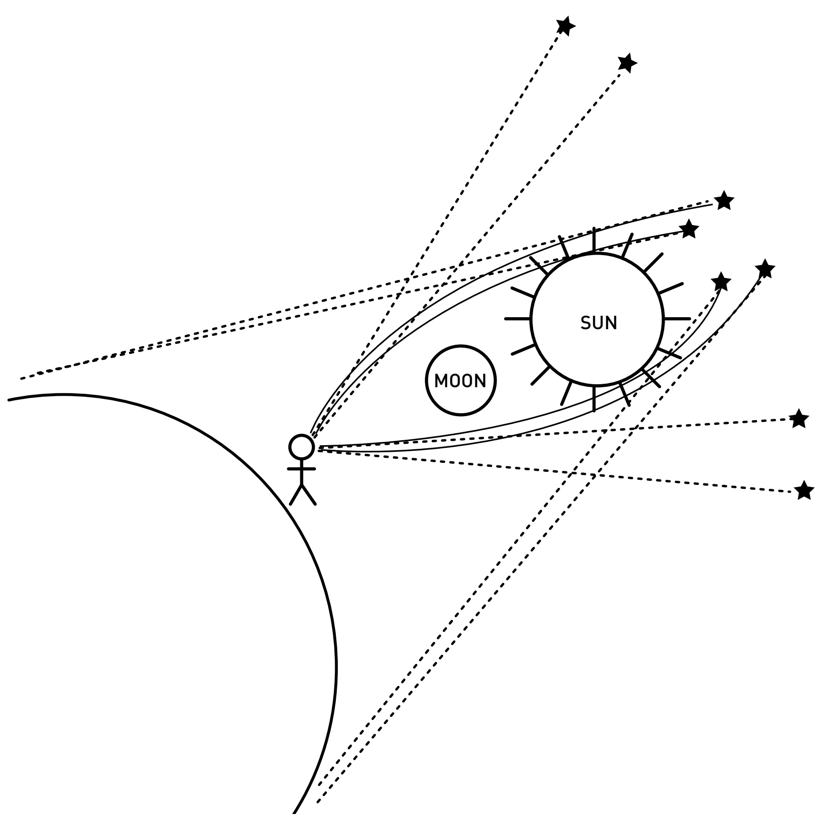
SECTION 8.7

GENERAL RELATIVITY
CONTINUED

This interpretation of gravity has been experimentally verified in numerous ways. The first such verification occurred during a solar eclipse in 1919. Arthur Eddington found that certain stars, which, according to calculations, were behind the sun at the time, were visible during the eclipse.



The only way to interpret this phenomenon is to say that the light of the hidden stars "bent" around the sun, like so:



SECTION 8.7

GENERAL RELATIVITY
CONTINUED

This was an early verification that light, which should normally travel in straight lines, actually travels in a curved path in the presence of a massive object.[4] This implied that the geodesics in the vicinity of the sun are curved, the cause being the mass of the sun.

Since this initial verification, other experiments also have shown the predictions of general relativity to be accurate. Einstein's theory that massive objects cause the spacetime in their vicinity to warp and that we experience this effect as gravity was a breakthrough in our understanding of physics. It was also a beautiful example of a mathematical idea that at first had little real-world application (i.e., the Riemannian geometry of non-constant curvature) turning out to be at the heart of one of the most fundamental phenomena in our human experience, gravity.

The General Theory of Relativity describes how mass causes the geometry of spacetime to curve locally. One can extend this thinking from considering the masses and motions of planets around a star to stars around a galaxy, galaxies around each other, clusters of galaxies around other clusters, and eventually to the large-scale curvature of the universe itself. All the mass in the universe must surely curve spacetime into some shape, and most probably that shape exhibits non-constant curvature.

We know how mass causes local curvature of spacetime, but would spacetime be flat were the mass not present? If so, would all of spacetime be flat without mass, or would it just appear to be flat because we are like the ant looking too closely at his surface to see any curvature? One of the most intriguing questions in both mathematics and physics is the question of the underlying geometry of reality.

[4] Light paths are also geodesics of spacetime; we cannot conceive of something that would find a more efficient path from point A to point B.

SECTION 8.8

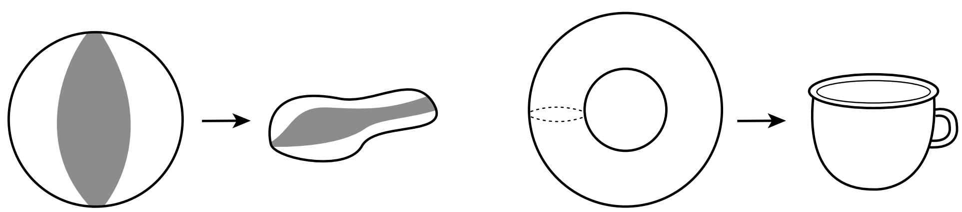
GEOMETRIZATION CONJECTURE

- Many Ways To Be Hyperbolic
- Hyperbolic Space

MANY WAYS TO BE HYPERBOLIC

- We can examine different topological surfaces via the types of geometry they admit.
- The polygons that can completely tile a given surface help to determine the type of global geometry in which that surface exists.
- Most 2-D surfaces are hyperbolic.

Recall that in our unit on topology, we learned about the classification of 2-D surfaces. This classification was supported by looking at different 2-D surfaces under various deformations and seeing that some surfaces really are just like other surfaces. For example, both inflated and deflated beach balls are really just spheres. The theorem that we explored in that earlier discussion was that every 2-D surface can be turned into either a sphere or a torus with varying numbers of holes. We will be concerned only with orientable surfaces in this present discussion.

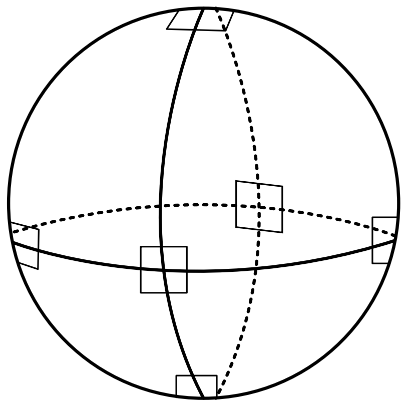


Inflated and deflated beach balls are topologically equivalent. Donut and coffee cup are topologically equivalent.

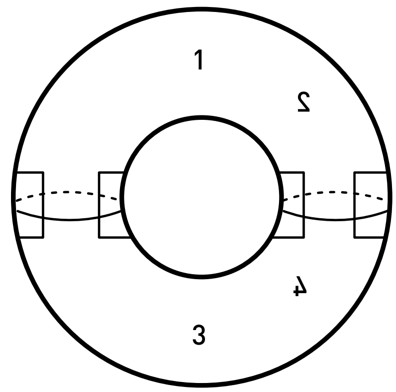
We can characterize the global curvature of these surfaces by examining what sort of shapes would be necessary to cover the surfaces completely. These tiles should have 90-degree angles at all vertices so that whenever four vertices meet, the angular sum is 360 degrees. This requirement ensures that every surface appears flat when viewed up close, one of the key criteria of a manifold. For a sphere, the only tiles that will satisfy this condition are equilateral triangles in which each angle is 90 degrees.

SECTION 8.8

GEOMETRIZATION CONJECTURE
CONTINUED



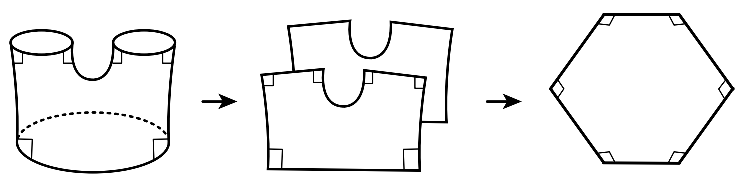
Eight triangles can cover a sphere.

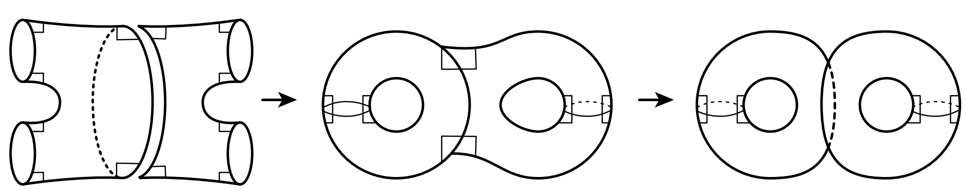


A torus convered by four rectangles.

Notice that the angles of each of these triangles sum to 270 degrees, a bit more than the Euclidean 180 degrees that we would expect. This reminds us that the surface of a sphere has positive curvature.

Notice that a single-holed torus can be completely tiled with four quadrilaterals (rectangles). The angle sum, as Saccheri would no doubt recognize, is 360 degrees, which is consistent with flat, Euclidean geometry.





Notice that the two-holed torus is tiled by hexagons whose angles are all equivalent to 90 degrees. A hexagon in flat space has an angle sum of 720 degrees. Looking at our hexagons, we can see that this is not true on the surface of our two-holed torus. The angle sum of one of our two-holed-torus-tiling hexagons is 540 degrees, decidedly less than sum expected from a Euclidean-based geometry. This means that our surface admits only hyperbolic geometry.

SECTION 8.8

GEOMETRIZATION CONJECTURE
CONTINUED

We can extend this thinking to tori having any number of holes. Hexagons can be used to make any other genus of torus. This means that although there are many ways to construct a hyperbolic 2-D surface, there is only one way to construct a flat surface, and only one way to construct a spherical surface. If we want to identify the large-scale geometry of any multi-dimensional surface, without any clues we would do well to guess "hyperbolic."—the odds of being right would be in our favor.

We have seen that the vast majority of surfaces that our ant can inhabit, the 2-D universes so to speak, are hyperbolic. What about 3-D manifolds? What are the possible geometries of the space that we inhabit? This is related to a long-standing question, asked first by Poincaré at the turn of the nineteenth century and resolved only in the first few years of the twenty-first century.

HYPERBOLIC SPACE

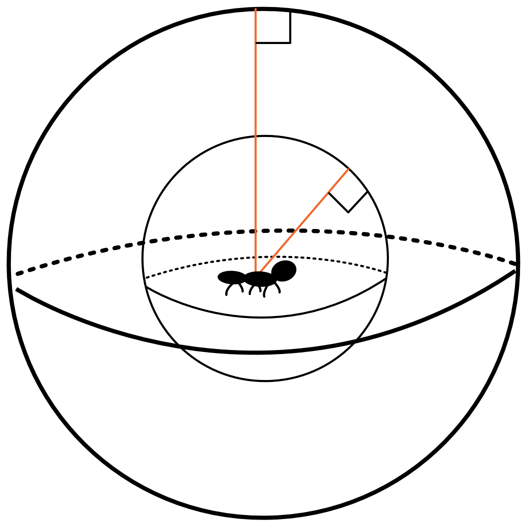
- If we were to remove all the mass of the universe, the fabric of space would most likely be hyperbolic.

As you might expect, 3-D manifolds can be curved in analogous ways to the 2-D surfaces we have seen—that is, they can be spherical, flat, or hyperbolic. Spherical space behaves like a spherical surface in that if you travel in a straight line far enough away from your starting point, you will always return to where you started, without having to turn around. This implies that the space is bounded, like the surface of a sphere. This obeys the "no-parallels" version of Euclid's fifth postulate. Furthermore, in analogy to the ant's circles, if we create a sphere and compare its surface area to its radius, we will get a smaller number than we would expect in flat space. As we create larger and larger spheres, this ratio shrinks.

SECTION 8.8

GEOMETRIZATION CONJECTURE
CONTINUED

Flat space behaves in a nice, Euclidean way. It obeys all five postulates; there is only one parallel line through a given point; lines extend to infinity. This is probably how most of us envision space. Since it is unbounded, we can think of it as larger than spherical space. Spheres of all sizes exhibit the same ratio of surface area to radius.

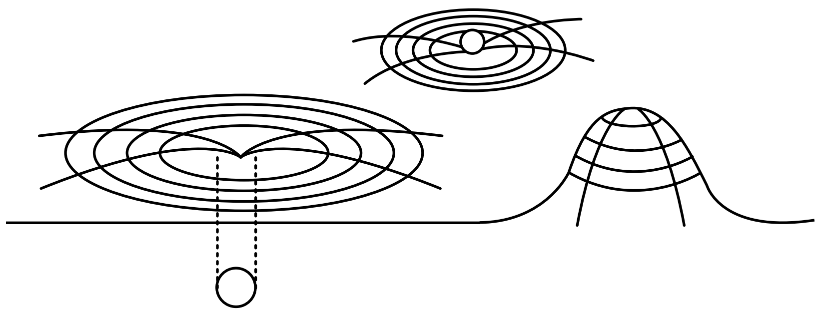


Hyperbolic space is a strange place, indeed. It can be thought of as much larger than both spherical and flat space. If we were to make spheres of a given radius, we would find that they have much larger surface areas than we would expect. Furthermore, the larger the sphere, the greater the discrepancy we would find.

SECTION 8.8

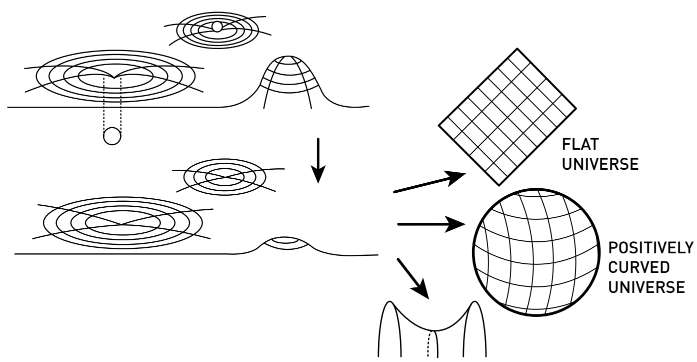
GEOMETRIZATION CONJECTURE
CONTINUED

Note that we said earlier that we actually live in a 4-D manifold called spacetime. So, to be clear about what we are talking about when we ask about the geometry of space, start by imagining all of space and all of time. Now, imagine taking what's called a "time-slice" of this spacetime; in other words, let's take a snapshot by looking at just one particular moment in time.



Planets Curving Spacetime

By taking this time slice, we have basically frozen the entire universe in time. People stop walking mid-step without falling; planets stop in their orbits around stars; galaxies cease their rotation. Geometrically, we now have a pure 3-D manifold with pockets of more or less curvature, depending on the mass present. Now, let's get rid of all the mass.



Remove planets and spacetime returns to its "natural" shape.

With no mass, we are left with "pure" space. This is a 3-D manifold that has some sort of geometry. With our now homogeneous 3-D manifold (the inhomogeneous curving effects of mass have been mitigated), the possible geometries are analogous to the geometries of the surfaces of the 2-D objects that we examined before. Now, however, instead of being polygons, the tiles will be polyhedra, and the type or types of polyhedra that will tile or "pack" a particular space are determined by the specific geometry of that space.

SECTION 8.8

GEOMETRIZATION CONJECTURE
CONTINUED

So, the question remains: what are the possible geometries? In 1982 William Thurston, one of the most influential modern geometers and topologists, proposed that there are eight possible geometries, Euclidean, spherical, hyperbolic, and five other systems. In the early and mid-2000s Perelman proved Thurston's claims to be correct. This result also proved the Poincaré conjecture, which considered only spherical 3-D manifolds.

Now that Thurston's geometrization conjecture has been proven to be correct—and has earned the title of "theorem"--we have essentially a complete list of possibilities for the fundamental geometry of our space. The task now is to determine which geometry actually governs the real world we live in. This is essentially what Gauss tried to do on his mountaintops so many years ago. The problem, as we have seen, is that to reach a definitive answer, we need to be able to look at extremely large shapes, much larger than anything on Earth or even in our galaxy, perhaps.

So we are, indeed, much like the ant on its surface: we know what is happening with the local curvature, but we are looking too closely to be able to discern much about the large-scale geometry of our system. If we had to guess the specific geometry of our space, we, like the ant, would do well to guess hyperbolic. Indeed, Thurston's Geometrization Theorem confirms that most spaces are spaces that obey the "many parallels" version of Euclid's fifth postulate.

SECTION 8.2

EUCLIDEAN GEOMETRY

- Not much is known of Euclid's life.
- Although he was not responsible for all of the content in The Elements, Euclid broke new ground in his organization of the foundational mathematical knowledge of the day.
- Axiomatic systems are a way of creating logical order.
- Axioms are agreed-upon first principles, which are then used to generate other statements, known as "theorems," using logical principles.
- Systems can be internally consistent or not, depending on whether or not their axioms admit contradictions.
- Euclid used five common notions and five postulates in The Elements.
- The fifth postulate, also known as the "parallel postulate," is somehow not like the others.

SECTION 8.3

NON-EUCLIDEAN GEOMETRY

- There were multiple attempts to show that the parallel postulate was not necessary to form an internally consistent geometry.
- Girolamo Saccheri built upon the work of Nasir Eddin in an attempt to "clear Euclid of every flaw."
- Saccheri looked at three cases of a quadrilateral constructed without the aid of the fifth postulate.
- Saccheri's results, though intriguing, were misinterpreted.

SECTION 8.4

SPHERICAL AND HYPERBOLIC GEOMETRY

- Gauss realized that it is possible to construct a self-consistent geometry with the "many parallels" version of the fifth postulate.
- He did not publish his findings.
- Lobachevsky and Bolyai independently came to the same conclusion as Gauss.
- The "no parallels" flavor of Euclid's fifth postulate yields the geometry of the surface of a sphere.
- Modern geometry comes in three varieties, and each can be visualized via a model.
- A geodesic is, in the local view, the shortest distance between two points in whichever geometry you choose to use. In a global view, a geodesic is the path a particle would follow were it free of the influence of all forces.
- The surface of a sphere models a geometry that admits no parallel lines.
- The surface of a pseudosphere models a geometry that admits many parallel lines through a given point.

SECTION 8.5

SHAPES ON A PLANE: MAKING MAPS

- Representing curved geometries on a flat surface requires distortion.
- The type of distortion depends on the technique used to project a surface onto a map.
- The stereographic projection of a sphere yields a map with increasing distortion of features near its north pole.
- The stereographic projection preserves angles, but not length.
- Geodesics are circles in stereographic projections of the sphere.
- Hyperbolic space can be modeled using the Poincaré disk.
- The boundary of the Poincaré disk represents infinity.
- Most geodesics map as semi-circles that form right angles with the boundary of the disk.
- Triangles on the disk are "skinny."

SECTION 8.6

CURVATURE

- We can measure the curvature of a curve in the plane extrinsically via an osculating circle.
- Intrinsic measurements of curvature are impossible in one dimension.
- One extrinsic way to measure the curvature of a two-dimensional surface is through principle curvatures.
- Principle curvatures cannot be used intrinsically to measure curvature.
- One can measure the intrinsic curvature of a surface by drawing circles and comparing their circumferences to their radii.
- Positive curvature yields a smaller circumference than we would expect for a given radius.
- Negative curvature yields a larger circumference than we would expect for a given radius.

SECTION 8.7

GENERAL RELATIVITY

- Riemann described how to deal with geometries that contain regions of positive and negative curvature.
- The concept of the curvature of 3-D manifolds paved the way for Einstein's work.
- Einstein said that space and time curve around massive objects, creating gravity.
- Events follow geodesics in this curved spacetime.
- The General Theory of Relativity has been experimentally verified.

SECTION 8.8

GEOMETRIZATION CONJECTURE

- We can examine different topological surfaces via the types of geometry they admit.
- The polygons that can completely tile a given surface help to determine the type of global geometry in which that surface exists.
- Most 2-D surfaces are hyperbolic.
- If we were to remove all the mass of the universe, the fabric of space would most likely be hyperbolic.

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