<u>Year 12 Optimisation III – Term 4</u>

TextBook (Revision)

Generate functions from worded situations.

Find maximum values of functions, prove if they are Maximum or Minimums both Mathematically and Graphically.

Include trigonometric functions in your revision.

Exercise 2C	KAPS –	Q 1 4 5 6 7 8 9		
Exercise 2D	KAPS -	Q 1 2 4		
Exercise 2D	MAPS -	Q 5 6 7 9	"B" -	Q 13, 14, 15

And do the following worksheets.

OPTIMISATION PROBLEMS T4 I

1. Find the maximum rectangular area that can be enclosed using 200 metres of moveable fencing and an existing fence as one side of the enclosure.
   Justify your solution graphically and mathematically.

2. A piece of wire, 30 cm long, is bent to form the two shorter sides of a right-angled triangle. Find the maximum area of the triangle.
   Justify your solution graphically and mathematically.

3. The profit of a small manufacturing firm is dependent on how many units (n) it produces. The profit function for producing n items per week (in dollars) is given by
   $P(n) = 0.0156n^2 - 100000 - 2\times10^{-6}n^3$.

   Determine the number of products (n) in order to maximise profit
   Also determine the profit that would be made.
   And Justify your solution both graphically and mathematically.

4. A trough with a semicircular cross-section is to be made from material totalling 6π m^2 in area. Determine the maximum volume the trough can contain.
   Justify your solution graphically and mathematically.

5. Beth takes some medicine. The amount, M, of medicine present in Beth's bloodstream t hours later is given by $M = 4t^2 - t^3$ for $0 \leq t \leq 3$. At what time is the amount of medicine in Beth's bloodstream a maximum? (justify your solution graphically and mathematically)

ANSWERS:

1. 5000 m^2
2. 112.5 cm^2
3. 5200 items with a total profit of $140,508,000
4. $2\pi\sqrt{2}$ m^3
5. 2 hours 40 minutes

OPTIMISATION PROBLEMS T4 II

1. A square-based rectangular prism has a total surface area of 2400 cm^2. What are the dimensions of the prism if its volume is a maximum?

2. A piece of wire, 16 cm long, is bent to form the two shorter sides of a right-angled triangle. Find the maximum area of the triangle.

3. A rectangular garden plot is surrounded by a fence and is divided into two equal parts by another fence parallel to the shorter side of the rectangle. If the area of the entire garden is 1200 m^2, what dimensions are required for the least amount of fencing to be used?

4. Find the greatest volume of a rectangular box with a square base where the sum of the height and the side of the base must not exceed 24 cm.

5. A rectangular field shares one side with an existing paddock and so requires no fence on one side. There is only 1000 m of fencing material to fence the remaining sides. Find the maximum possible area of the field.

6. A right cylinder of radius r metres is to be constructed to hold a total volume of 200 m^3. Express the total surface area (including the two ends), SA, in terms of r and π and hence find the minimum surface area.

ANSWERS:

1. 20 cm x 20 cm x 20 cm
2. 32 cm^2
3. 42.43 m x 28.28 m
4. 2048 cm^3
5. 125 000 m^2
6. SA=$2\pi r^2 + \frac{400}{r}$, 189.32 m^2

Optimisation T4 III

1. A farmer wants to fence an area of 1.5 million square metres in a rectangular field and then divide it in half with a fence parallel to one of the sides. How should he do this to minimize the cost of the fence.

2. A ferris wheel had a diameter of 20 metres and its lowest point it is 1 metre from the ground The wheel makes 6 revolutions per minute.
   a. Determine a function that describes the height of the bottom most carriages as a function of time.
   b. Now find an expression for the vertical rate of change of the carriage.
   c. Also determine an expression for the vertical acceleration.

3. The motion of a swing is given by $y = \sin 3t$, where y is the Horizontal displacement from the centre position., after t seconds.
   a. Determine the velocity of the 'swinger' at any time t.
   b. Mathematically, calculate when the Maximum velocity of the 'swinger' will be.

4. The depth of water in a river is modeled by the following functions, where d is in metres and t is the number of hours since noon.

$$d = 2.5\cos\left(\frac{2\pi t}{12.4} - 3\right) + 5$$

   a. Find the first three times when the depth of water is 3 metres?
   b. How fast will the water level be changing at 3pm that day?
   c. Will the tide be going in or out?
   d. When will the current in the river be at its greatest?

5. A particle moves in a straight line such that it's displacement is D cm from the equilibrium point (P) at any time (t) seconds is given by the function $D(t) = 2\sin 4t$. Determine the time when the particle first reaches its maximum displacement from P.

Answers

1. 1000 X 1500
2. a. $h = 11 - 10\cos 12\pi t$, b. $\frac{dh}{dt} = 120\pi \sin 12\pi t$, c. $1440\pi^2 \cos 12\pi t$
3. a. $v = \frac{dy}{dx} = 3\cos 3t$ b. $t = 0 = \frac{\pi}{3} = \frac{2\pi}{3}$...
4. b. 1.26 m/h c. coming in
5. $\frac{\pi}{8} = 0.39$ seconds

Optimisation IV

1. If $L = 2x + \frac{50}{X^2}$
   a. Find $\frac{dL}{dx}$
   b. Find x when $\frac{dL}{dx} = 0$
   c. Find the minimum value of L

2. A small manufacturer can produce x fittings per day. The costs in productions are:
   i. \$1000 per day for the workers
   ii. \$2 per day per fitting
   iii. $\$\frac{5000}{x}$ per day for running costs and maintenance

   How many fittings should be produced to Minimise costs?

3. An open box is to be made by cutting equal squares out of the corners of a 36 cm by 36 cm square sheet of tin plate.
   a. Determine a function of the volume of the box
   b. How big should the square cut outs be?
   c. Calculate the maximum volume of the box
   d. Verify this mathematically.
   e. Justify your solution by sketching an appropriate graph.

4. Find the maximum volume of a right circular cylinder that can be inscribed in a cone of height 12cm and base radius of 4cm, given that the axes of the cylinder and cone coincide.

5. A settling pond is to be constructed to hold 324 m^3 of waste. The pond has a square base and four vertical sides, all made of concrete, and a square top made of steel. If the steel costs twice as much per unit area as the concrete, determine the dimensions of the cistern that will minimise the cost of construction.

6. A firm sells all units it produces at \$8 per unit. The firm's total cost, C, for producing x units is given by
$$C = 5000 + 0.7x + 0.0002x^2$$

   How many items should be produced so that the profit is a maximum?

Answers:

1. a. $\frac{dL}{dx} = \frac{2x^3 - 100}{x^3}$ b. $\sqrt[3]{50}$ *or* 3.6843 c. 11.052
2. 50
3. a. $V = 4x^3 - 144x^2 + 1296x$ b. 6cm X 6cm c. 3456 cm^3 d. $V''(6) = 24 \times (6) - 288 = -ve$ $\therefore$ Maximum
4. 89.36 cm^3
5. 6m x 6m x 9m
6. 18250 units to be produced with a profit of \$61,612.50

Optimisation T4 V

Question 1. (KAPS)

Find two positive numbers whose sum is 15, such that the product of one with the square of the other is maximised.

Question 2. (C level)

A large transport company executive has requested you investigate the efficiency of his fleet. Although revenue increases with speed of the trucks, expenses also increase. Research into the company reveals that profit per trip (P) can be described in terms of velocity (v):

$$P(v) = \frac{-60}{361}v^2 + \frac{600}{19}v$$

a) Find the velocity that brings the company the greatest profit.
b) What Profit should the company make per trip?

Question 3. (KAPS/C level))

A farmer needs to fence off a rectangular paddock for some lambs. He has 250 metres of fencing and wants to maximise the area of grass upon which the lambs can feed. What dimensions should he make the enclosure? Prove this mathematically.

Question 4. (B level)

A chain supermarket is looking to increase profits by reducing its packaging costs. Your department has been asked to verify the optimal dimensions of the 500ml juice can. You are told that the steel circular end pieces cost 1 cent per square centimetre, while the remaining steel costs 0.5 cent per square centimetre.

a) What is the optimal dimensions of the can?
b) What will each can cost?

Question 5. (B level)

A major Internet Service Provider is looking to increase their profit from their dominant market share of 2.5 million customers. Their market research reveals that for every extra dollar per month, over the current $50 per month fee, 45 000 customers would cancel their service. Your office is responsible for identifying the optimal fee increase to maximise Revenue.

*** Fully worked solutions follow **

Optimisation V - Solutions

Question 1.

Find two positive numbers whose sum is 15, such that the product of one with the square of the other is maximised.

Solution:

Let p + v = 15 then v = 15 – p

$$f(product) = p^2 v$$

becomes $$f(p) = p^2(15 - p) = -p^3 + 15p^2$$

Max of function happens at a stationary point where gradient = 0

$$f'(x) = -3p^2 + 30p$$

at grad 0 … $$0 = -3p^2 + 30p = -3p\left(p - 10\right)$$

therefore p=0 (not appropriate as f(0)=0

or p=10

By looking to the second derivative we can confirm this is a Maximum value.

$$f''(x) = -6p + 30$$

and

Therefore this is a maximum,

And is p = 10 then v = 5

Maximum happens with the where the number 10 is squared and then multiplied by the number 5.

Question 2.

A large transport company executive has requested you investigate the efficiency of his fleet. Although revenue increases with speed of the trucks, expenses also increase. Research into the company reveals that profit per trip in dollars (P) can be described in terms of velocity in km/hr (v):

c) Find the velocity that brings the company the greatest profit.
d) What Profit should the company make per trip?

Solution:

Greatest Profit will happen when function at a stationary point (maximum). Here the derivative, which gives the gradient of the function will equal will equal 0.

To confirm this is a Maximum Profit, we can look to the second derivative:

As the second derivative is negative, this confirms that speed 95 km/hr represents a maximum profit.

At speed 95, from the initial function, the profit per trip can be equated as:

Therefore, the most efficient speed for the vehicles to travel is 95 km/hr, and at this speed, the profit per trip will be $1500.

Question 3.

A farmer needs to fence off a paddock for some lambs. He has 250 metres of fencing and wants to maximise the area of grass upon which the lambs can feed. What dimensions should he make the enclosure?

Solution:

Let the sides of the paddock be 'l' and 'w'.

Area stands at

We know 2l + 2w = 250 so w = 125 – l

So we can now say

Maximum Area will happen when the function is at a maximum value. Here gradient will be 0.

let derivative = 0

Further we can confirm this is a maximum by looking to the second derivative:

so this is a maximum value.

if then,

As the second derivative is negative, we confirm this to be a maximum value for the Area, so we can conclude the lamb enclosure should be a square with side length equal to 62.5 metres.

Question 4.

A chain supermarket is looking to increase profits by reducing its packaging costs. Your department has been asked to verify the optimal dimensions of the 500ml juice cylindrical can. You are told that the steel circular end pieces cost 1 cent per square centimetre, while the remaining steel costs 0.5 cents per square centimetre.

c) What is the optimal dimensions of the can?
d) What will each can cost?

Solution:

so if then

Cost of Tin = cost of end pieces x 1 cents + cost of body x 0.5 cents

Cost as a function of 'r':

Cost as a function of the radius becomes $f(r) = 2\pi r^2 + \frac{500}{r}$

To obtain a minimum cost, look for a stationary point, where gradient and the derivative equals 0.

$$f'(r) = 4\pi r - \frac{500}{r^2}$$

set gradient to 0 $\quad 0 = 4\pi r - \frac{500}{r^2}$

so $\quad r = 3.413920316$

To check this is a minimum, check the second derivative:

$$f''(r) = 4\pi + \frac{500}{r^3} = +ve \text{ for all values of r}$$

As the second derivative is positive, this is a minimum cost.

a) the dimensions of the cheapest can would be,
   - radius of 3.413920316 cm
   - height of $h = \frac{500}{\pi r^2} = \frac{500}{\pi \times 3.413920316^2} = 13.65568945\,\text{cm}$
b) cost of can would be

$$f(3.41) = 2\pi 3.413920316^2 + \frac{500}{3.413920316} = 219.6887831 \text{ cents}$$

Question 5.

A major Internet Service Provider is looking to increase their profit from their dominant market share of 2.5 million customers. Their market research reveals that for every extra dollar per month, over the current \$50 per month fee, 45 000 customers would cancel their service. Your office is responsible for identifying the optimal fee increase to maximise Revenue.

Solution:

Monthly Revenue = customers X monthly fee

Customers = 2 500 000 – 45 000X

Monthly fee = 50 + X

Revenue as a function of fee increase becomes,

$$f(x) = (2500000 - 45000x)(50 + x) = -45000x^2 + 250000x + 125000000$$

To find the maximum Revenue, we look to a stationary point, where gradient is 0.

$$f'(x) = -90000x + 250000$$

where gradient is 0, we have the derivative = 0:

$$0 = -90000x + 250000 \qquad \text{so} \qquad x = \pm 2.77$$

To check this is a maximum revenue, we can check the second derivative,

$$f''(x) = -90000 = -ve \qquad \text{so we know this is a Maximum.}$$

We have confirmed that an increase from \$50 pm, to \$52.77 would result in the maximum revenue to the company.

(although the company would lose 124650 customers, monthly revenue would increase by \$347 219 per month)